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Priority queueing model with balking and reneging

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Abstract: This investigation deals with two levels, single server preemptive priority queueing model with discouragement behaviour (balking and reneging) of customers. Arrivals to each level are assumed to follow a Poisson process and service times are exponentially distributed. The decision to balk / renege is made on the basis of queue length only. Two specific forms of balking behaviour are considered. The system under consideration is solved by using a finite difference equation approach for solving the governing balance equations of the queueing model, with infinite population of level 1 customer. The steady state probability distribution of the number of customers in the system is obtained.

Keywords: Priority queue, Balking, Reneging, Finite difference, Queue size distribution

INTRODUCTION

Performance is one of the key factors that is involved in the design, development, configuration and running of real time systems. The user's objective is to maximize his consumer surplus, which is equal to the benefit minus the cost of the service. Priority queues occur in many congestion situations of everyday life, particularly in situations where pre-frontal treatment is granted to certain kind of individuals.

Moreover, stochastic modeling of queueing problems via birth-death approach offers a very promising avenue for pursuing a comprehensive investigation involving a more diverse set of scenarios of natural science problems related to epidemiology and can be widely deployed for use by health practitioners (for detail see Vawter *et al.,* 2007). Sometime priority is given to those who can be benefited by proportionately smaller amounts of the resource or shorter length-of-use time. In some other circumstances, priority is given to those who can purchase it by some means. The queueing model with priority can be employed for planning large-scale emergency dispensing clinics to respond to biological threats and infectious disease outbreaks. For example, such investigation may be helpful to explore facility layout and staffing scenarios for smallpox vaccination and for an actual anthrax-treatment dispensing exercise and post event analysis.

In practical queueing situations, the arriving customers may be discouraged due to long queue. Such queueing models involve the concept of balking and reneging. Balking is not only a common feature in queues arising in day-to-day life activities, but also finds applications in many epidemiological problems. The concepts of prioritization and balking involved in queueing systems may provide insights to the development and deployment of emergency responses for identifying the ethical bases for allocation of scarce, life-saving medical resources under circumstances of pandemic influenza. The priority will be given to those for whom treatment has the highest probability of medical success. The main concept of priority queue was included by Jaiswal (1968). The relevant paper on priority was discussed by Subha Rao (1967) regarding the $(M_1 + M_2)/G_i/1$ model operating under preemptive resume service with balking and reneging. The $(M_1 + M_2)/M_i/1$ preemptive and non-preemptive priority model was explained by Miller (1981). Neuts (1980) extended the matrix analytic theory to blocks of infinite size. The solution technique of difference equations in queueing system was elaborated in Cox (1955). Jordan (1965) extended the technique by using generating functions.

In many real life situations, if a server inoperative for periods of time may increase the likelihood of customer losses due to balking and reneging. In these situations, the arriving customers may be discouraged due to long queue or due to other discouragement factors. Such queueing models involve the concept of balking and reneging and have been studied by several researchers. Kao and Narayanan (1990) have analysed the non-preemptive multi server version of the $M/M/c$ queue with two classes of customers. Jain and Singh (1998) derived the expressions for discouragement model of finite capacity priority queue. Brouns and Wal (2006) have described the optimal threshold policies in queueing model with preemptive priority. A time limited service priority queueing system was analysed by Katayama (2007).

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Many real world queueing systems have their customers divided into classes. It is only reasonable to distinguish between express and regular mail, rush and ordinary jobs, important and less important customers. The use of priority discipline improves the measures of performance of the higher priority classes at the expense of the lower priority classes. The problems of Hybrid Petri Nets can be analysed as priority queueing service systems, where Hybrid Petri Nets can be investigated by concealing queue in special places. The Petri Nets approach can be presented as appropriate representation of Markov process and queues. Another important application of priority queueing model can be observed in High Performance Computing Bioinformatics. In which, phylogenic (evolutionary) trees can be computed more accurately.

Motivated by the above facts, in the present investigation, a single server queueing model with two levels of customers by incorporating discouragement factors due to balking and reneging behaviour of the customers is considered. The customers are served on a first-come, first-served basis within their own line. However, level 1 has preemptive priority over level 2, implying that level 2 customers in service would be preempted by an arriving level 1 customer. The customers with low priority will be more susceptible to balking behaviour than high priority customers as their queue become longer. It is assumed that the behaviour of arrival of customers follow the balking characteristics with different modes and constant service rate with reneging parameter. If the service rate and arrival rate are constants, the system described reduced to the classic form on the line of Saaty (1961) and Jaiswal (1968). The Section 2 describes the mathematical formulation of the model and notations used in the set of simultaneous difference equations. In Section 3, the unknown probability distribution as a function of two independent variables is obtained. Finally, in Section 4, the discussion is drawn.

THE MATHEMATICAL MODEL

In this section, a single server queueing model which has two levels of customers, each having its own respective line is formulated. The arrival rate of customers for both levels is assumed to follow poisson process and service times are exponentially distributed. The population of level 1 customer is infinite and level 2 customers are finite. It is assumed that level 2 customers arrive with constant rate until the queue size reaches the threshold level *L*. The customers of level 2 balk with probability b_{n_2} when there are $n_2 > L$ customers of level 2 present in the system. There are following notations used to formulate the model:

$n_1(n_2)$	The number of customers at level 1(2).
N_2	The system capacity for level 2 customers.
l_1	Constant arrival rate of level 1 customer
m_1	Constant service rate of level 1 customer.
a	Reneging parameter.
$p(n_1, n_2)$	Unknown probability distribution (function of two independent variables n_1 and n_2

In fact, low priority customers will be more sensitive to balking (reneging) behaviour than high priority customers. To signify the discouragement behaviour,on line of Drekic and Woolford (2005), we consider as a series of monotonically decreasing function of level 2 with size. n_2. $l_{2,n_2} = \begin{cases} l_2; n_2 \le L \\ l_2 b_{n_2}; n_2 > L \end{cases}$

(1)

Where b_{n_2} discouragements function as distributed below:

$$b_{n_2} = \begin{cases} \frac{1}{n_2+1}; & \text{Fractional} \\ e^{-bn_2}, b > 0; & \text{Exponential} \end{cases}$$

The service rate of level 2 customers is $m_{2,n_2} = m_2 + (n_2 - 1)a$

The system of balance equations governing the model is as follow:

$$m_1 p(n_1+1, n_2) - [l_1 + l_2 + m_1] p(n_1, n_2) + l_1 p(n_1 - 1, n_2) + l_2 p(n_1, n_2 - 1) = 0;$$
$$0 \le n_2 \le L; n_1 = 1,2,..... \quad (2.1)$$

$$m_1 p(n_1+1, n_2) - [l_1 + l_2 b_{n_2} + m_1] p(n_1, n_2) + l_1 p(n_1 - 1, n_2) + l_2 b_{n_2-1} p(n_1, n_2 - 1) = 0;$$
$$L < n_2 \le N_2 - 1; n_1 = 1,2,..... \quad (2.2)$$

$$m_1 p(n_1+1, N_2) - (l_1 + m_1) p(n_1, N_2) + l_1 p(n_1 - 1, N_2) + l_2 b_{N_2-1} p(n_1, N_2 - 1) = 0;$$
$$n_2 = N_2 : n_1 = 1,2,..... \quad (2.3)$$

$$m_1 p(n_1+1, 0) - [l_1 + l_2 + m_1] p(n_1, 0) + l_1 p(n_1 - 1, 0) = 0; n_1 = 1,2,... \quad (2.4)$$

$$m_1 p(1,0) - [l_1 + l_2 + m_1] p(0,0) + m_2 p(0,1) = 0. \quad (2.5)$$

$$m_1 p(1,n_2) - [l_1 + l_2 + m_2 + (n_2 - 1)a]\,p(0,n_2) + l_2 p(0,n_2 - 1) + (m_2 + n_2 a)\,p(0,n_2 + 1) = 0;\quad 1 \le n_2 \le L. \tag{2.6}$$

$$m_1 p(1,n_2) - [l_1 + l_2 b_{n_2} + m_2 + (n_2 - 1)a]\,p(0,n_2) + l_2 b_{n_2 - 1} p(0,n_2 - 1) + (m_2 + n_2 a)\,p(0,n_2 + 1) = 0;\quad L \le n_2 \le N_2 - 1. \tag{2.7}$$

$$m_1 p(1,N_2) - [l_1 + m_2 + (N_2 - 1)a]\,p(0,N_2) + l_2 b_{N_2 - 1} p(0,N_2 - 1) = 0;\ n_1 = 0 \tag{2.8}$$

with normalizing condition

$$\sum_{n_1,n_2} p(n_1,n_2) = 1. \tag{2.9}$$

THE SOLUTION TECHNIQUE

In this section, the unknown probability distribution $p(n_1,n_2)$ for two discouragement functions namely fractional and exponential functions is determined. It is assumed that unknown function of two variables as a set of functions of one variable. Let $p(n_1,n_2) = p_{n_2}(n_1)$ $\forall n_1 = 0,1,2,...;\ n_2 = 0,1,2,...,N_2$ the difference equations for $p(n_1,n_2)$ can be considered as a set of simultaneous difference equations and can be solved by simply eliminating all the functions except one and solving the resulting difference equations for that function. As equation (2.4) involves, without any elimination, only one function $p_o(n_1)$ which may easily be solved, since with respect to our independent variable n_1, its coefficients are constant. Then $p_{n_2}(n_1)$ for n_2=1,2,... may be computed using (2.1) – (2.3), since it involves only the functions $p_{n_2}(n_1)$ and $p_{n_2-1}(n_1)$ which are already known.

By using shift operator E and difference operator such that $Ef(x) = f(x+1)$ and $Df(x) = f(x+1) - f(x)$. Let $Df(x) = f(x)$ then $f(x) = \Delta^{-1} f(x)$. Now, equations (2.1) – (2.4) can be written as

Case I: Fractional function

$$y_{n_2}(E)\,p_{n_2}(n_1) + \frac{l_2}{n_2} E p_{n_2-1}(n_1) = 0;\ n_1 = 0,1,...;\ n_2 = 0,1,...,N_2.$$

Case II: Exponential function

$$y_{n_2}(E)\,p_{n_2}(n_1) + l_2 e^{-b(n_2-1)} E p_{n_2-1}(n_1) = 0;\ n_1 = 0,1,...;\ n_2 = 0,1,...,N_2.$$

Where $y_{n_2}(E) = m_1 E^2 - (l_1 + l_2 b_{n_2} + m_1) + l_1$

For, n_2= 0, $y_0(E)\,p_0(n_1) = 0;\ n_1 = 0,1,...$ (3.1)

In general, the solution of Homogeneous difference equation $y_{n_2}(E)\,p_{n_2}(n_1) = 0$ (3.2)

may be obtained as the roots of characteristic equation $y_{n_2}(r) = 0$. The roots of the quadratic equation can be calculated as

Case I: Fractional function

$$r(n_2) = \begin{cases} \dfrac{(l_1 + l_2 + m_1) \pm \sqrt{(l_1 + l_2 + m_1)^2 - 4 l_1 m_1}}{2m_1}; \\ n_1 = 0,1,...;1 \le n_2 \le L \\ \dfrac{\left(l_1 + \dfrac{l_2}{n_2+1} + m_1\right) \pm \sqrt{\left(l_1 + \dfrac{l_2}{n_2+1} + m_1\right)^2 - 4 l_1 m_1}}{2m_1}; \\ n_1 = 0,1,...;L \le n_2 \le N_2 - 1 \\ \dfrac{(l_1 + m_1) \pm \sqrt{(l_1 + m_1)^2 - 4 l_1 m_1}}{2m_1}; n_1 = 0,1,...; n_2 = N_2. \end{cases}$$

Case II: Exponential function

$$r(n_2) = \begin{cases} \dfrac{(l_1 + l_2 + m_1) \pm \sqrt{(l_1 + l_2 + m_1)^2 - 4 l_1 m_1}}{2m_1}; \\ n_1 = 0,1,...;1 \le n_2 \le L \\ \dfrac{(l_1 + l_2 e^{-bn_2} + m_1) \pm \sqrt{(l_1 + l_2 e^{-bn_2} + m_1)^2 - 4 l_1 m_1}}{2m_1}; \\ n_1 = 0,1,...;L \le n_2 \le N_2 - 1 \\ \dfrac{(l_1 + m_1) \pm \sqrt{(l_1 + m_1)^2 - 4 l_1 m_1}}{2m_1}; n_1 = 0,1,...; n_2 = N_2. \end{cases}$$

The general solution of equation (3.2) may be expressed as $\sum_{i=1}^{2} d_i r_i^{n_i}$ where d_i is an arbitrary constant. Since $p_0(n_1)$ must tend to zero as $n_1 \to \infty$, we reject $r_1(0)$>1, and set d_i to zero when $r_2(0)$<1.

Case I: Fractional function

$$\frac{\left(l_1 + \dfrac{l_2}{n_2+1} + m_1\right) - \sqrt{\left(l_1 + \dfrac{l_2}{n_2+1} + m_1\right)^2 - 4 l_1 m_1}}{2m_1} < 1; n_2 < N_2.$$

Case II: Exponential function

$$\frac{(l_1 + l_2 e^{-bn_2} + m_1) - \sqrt{(l_1 + l_2 e^{-bn_2} + m_1)^2 - 4 l_1 m_1}}{2m_1} < 1; n_2 < N_2.$$

The solution of equation (3.1) may be expressed as $p_0(n_i) = C_0[r_2(0)]^{n_1}$, where is an arbitrary constant to be determined from equation (2.5)-(2.8) and normalizing condition (2.9). The solution of equation (3.1) is obtained as a sum of the general solution of a homogeneous equation and a particular solution of equation . Such a particular solution may be obtained as a result of the operation

$$-l_2 b_{n_2-1} E / y_{n_2}(E) p_{n_2-1}(n_1); 1 \le n_2 \le N_2 \quad (3.3)$$

When the function $p_{n_2-1}(n_1)$ is of the form, $\sum_j g_j a_j^{n_1}$ where g_j and a_j are constants, equation (3.3) becomes

$$\sum_j g_j[(-l_2 b_{n_2-1})a_j]/y_{n_2}(a_j)a_j^{n_1}; 1 \le n_2 \le N_2$$

Using equation and the general solution of the equation, it is clear that is $p_{n_2-1}(n_1)$ of the form considered. Since $y_k(r_2(k)) = 0$, we have

$$y_{n_2}(r_2(k)) = r_2 k[l_2 b_k - l_2 b_{n_2}]; 1 \le n_2 \le N_2$$

Case I: Fractional function

$$p_{n_2}(n_1) = \sum_{i=0}^{N_2} C_i [r_2(i)]^{n_1} \left[\prod_{j=1}^{n_2-1} l_2 \frac{j+2}{j+1} - l_2 b_i] \right]$$

$$n_1 = 0,1,2...; n_2 = 0,1,2,..., k.$$

Case II Exponential function

$$p_{n_2}(n_1) = \sum_{i=0}^{N_2} C_i [r_2(i)]^{n_1} \left[\prod_{j=1}^{n_2-1} l_2 \frac{e^{-bj}}{e^{-b(j+1)}} - l_2 b_i] \right]$$

$$; n_1 = 0,1,2...; n_2 = 0,1,2,..., k.\backslash$$

The solution exists only, if $r_2(n_2) < 1 \forall n_2$ ....(3.4)
It can be easily shown that (3.4) is equivalent to simple condition $l_1 < m_1$. The constants $C_{n_2}; n_2 = 0,1,..., N_2$ can be determined using equations (2.5)-(2.8) and the normalization condition (2.9), which may impose yet other conditions for the existence of the steady state probabilities. For each value of n_2 conditional probability $p(n_1/n_2)$ must sum to unity.

DISCUSSION

In this paper, the steady state probability for two level customers in terms of two discouragement factors for balking and reneging is obtained. By using the matrix method technique, the behavior of level 2 customers is analysed. For level 1, finite difference approach is employed for solution purpose. Based on queue theoretic methodology, the most cost-effective dispensing (lowest labor/throughput value), and the smoothest operations (shortest average wait time, average queue length, equalized utilization rate) can be proposed which may be helpful to the emergency response departments, for further fine-tuning and development of the real time systems to address different biological attacks and infectious disease outbreaks, and to ensure its practicality and usability. The queue size distribution determined can be helpful in evaluating various performance indices, which can be further employed for providing better quality of service (QOS) of such systems

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