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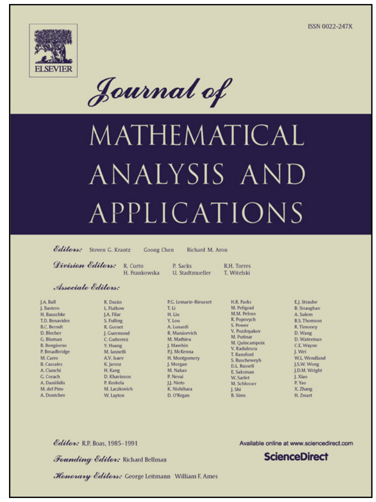

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Formulas for cubic partition with 3-cores

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Abstract

Let $C_3(n)$ denote the number of cubic partitions of n with 3-cores. In this paper, we establish the arithmetic properties and formulas for $C_3(n)$ by employing Bailey's ${}_6\psi_6$ formula and theta function identities.

Keywords: Cubic partition, 3-cores, Bailey's ${}_6\psi_6$ formula
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1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n, i.e.,

$$n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_k,$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_k \geq 1$.
For example, $7 = 3 + 2 + 1 + 1$ and $\lambda = (3, 2, 1, 1)$ is a partition of 7. Let $t \geq 1$ be a positive integer. Any partition λ of n whose Ferrers graph have no hook numbers divisible by t is known as a t-core partition of n. We denote the number of t-core partitions of n by $c_t(n)$. Garvan et al. [6, Eq. (2.1)] showed that the generating function for $c_t(n)$ is

$$\sum_{n=0}^{\infty} c_t(n) q^n = \frac{f_t^t}{f_1}. \tag{1}$$

Throughout this paper, we use

$$f_t := (q^t; q^k)_\infty, \quad \text{if} \quad t = k,$$

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$$f_{t;k} := (q^t; q^k)_\infty, \quad \text{if} \quad t \neq k,$$

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1 - aq^i),$$

and

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad (-\infty < n < \infty).$$

For convenience, we use customary notation

$$f_{a_1,a_2,\ldots,a_n;k} := f_{a_1;k} f_{a_2;k} \ldots f_{a_n;k}.$$

Using the theory of modular forms, Granville and Ono [7] proved that

$$c_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \tag{2}$$

where $d_{r,3}(n)$ denote the number of positive divisors of n congruent to r modulo 3. Hirschhorn and Sellers [9] gave an elementary proof of (2).

Let $u(n)$ denote the number of representations of a nonnegative integer n in the form x^2+3y^2 with $x, y \in \mathbb{Z}$. Using the Ramanujan's theta function identities, Baruah and Nath [2] proved that $u(12n+4) = 6c_3(n)$ and then (2) with the help of classical Lorentz identity.

If the sum of all parts of the partition k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is n, then we say that $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition k-tuple of n. For example, $(\{3,2\},\{1\})$ is a partition pair of 6 and $(\{3,1\},\{1\},\{1\})$ is a partition triple of 6. A partition k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is t-core if each λ_i is t-core. Let $c_{t,k}(n)$ denote the number of partition k-tuple of n with t-core. From (1), we see that the generating function for $c_{t,k}(n)$ is

$$\sum_{n=0}^{\infty} c_{t,k}(n) q^n = \frac{f_t^{tk}}{f_1^k}. \tag{3}$$

Here we observe that $c_{t,1}(n) = c_t(n)$.

Wang [11] established infinite families of arithmetic identities for $c_{3,2}(n)$ and $c_{3,3}(n)$. More importantly, he found the formula

$$c_{3,2}(n) = \frac{1}{3}\sigma(3n+2). \tag{4}$$

From which he obtained the following theorem:

Theorem 1.1. *Let p be a prime, and let k, n be a nonnegative integers.*

(1) If $p \equiv 1 \pmod 3$, *we have*

$$c_{3,2}\left(p^k n + \frac{2p^k-2}{3}\right) = \frac{p^k-1}{p-1} c_{3,2}\left(pn + \frac{2p-2}{3}\right) - \frac{p^k-p}{p-1} c_{3,2}(n).$$

(1) If $p \equiv 2 \pmod 3$, *we have*

$$c_{3,2}\left(p^{2k} n + \frac{2p^{2k}-2}{3}\right) = \frac{p^{2k}-1}{p^2-1} c_{3,2}\left(p^2 n + \frac{2p^2-2}{3}\right) - \frac{p^{2k}-p^2}{p^2-1} c_{3,2}(n).$$

Wang [11] also found explicit formula for $c_{3,3}(n)$, that is,

$$c_{3,3}(n) = \sum_{\substack{d \mid n+1 \\ d \equiv 1 \pmod 3}} \left(\frac{n+1}{d}\right)^2 - \sum_{\substack{d \mid n+1 \\ d \equiv 2 \pmod 3}} \left(\frac{n+1}{d}\right)^2. \tag{5}$$

From which he obtained the following theorem:

Theorem 1.2. *Let* p *be a prime, and let* k, n *be a nonnegative integers.*

(1) If $p \equiv 1 \pmod 3$, *we have*

$$c_{3,3}\left(p^k n + p^k - 1\right) = \frac{p^{2k}-1}{p^2-1} c_{3,3}\left(pn + p - 1\right) - \frac{p^{2k}-p^2}{p^2-1} c_{3,3}(n).$$

(1) If $p \equiv 2 \pmod 3$, *we have*

$$c_{3,3}\left(p^k n + p^k - 1\right) = \frac{p^{2k}-(-1)^k}{p^2+1} c_{3,3}\left(pn + p - 1\right) + \frac{p^{2k}+(-1)^k p^2}{p^2+1} c_{3,3}(n).$$

Chern [5] extended the work of Wang [11] and established formulas for $c_{t,k}(n)$ for some values of t and k by employing the method of modular forms.

Chan [4] studied the cubic partition function denoted by $C(n)$ whose generating function is

$$\sum_{n=0}^{\infty} C(n) q^n = \frac{1}{f_1 f_2}.$$

In this paper, we restrict the cubic partition function $C(n)$ to 3-core which is denoted by $C_3(n)$ and the generating function of $C_3(n)$ is given by

$$\sum_{n=0}^{\infty} C_3(n) q^n = \frac{f_3^3 f_6^3}{f_1 f_2}. \tag{6}$$

Let $w(n)$ denote the number of representations of a nonnegative integer n in the form $x^2 + 2y^2 + 3z^2 + 6r^2$ with $x, y, z, r \in \mathbb{Z}$. In Section 3, we find the arithmetic properties of $C_3(n)$, and relation connecting $w(n)$ and $C_3(n)$ from which together with Alaca et al. [1] identity, we find the formula for $C_3(n)$. We also find the formula for $C_3(n)$ by employing Bailey's ${}_6\psi_6$ formula. Using formulas of $C_3(n)$, we establish the arithmetic properties of $C_3(n)$.

2. Preliminaries

In this section, we list identities which are useful in proving our main results.

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{7}$$

Using Jacobi's triple product identity [3, Entry 19, p.35], (7) becomes

$$f(a,b) = (-a, ab)_\infty (-b, ab)_\infty (ab, ab)_\infty .$$

The most important special cases of $f(a, b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{8}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f_2^2}{f_1}, \tag{9}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1. \tag{10}$$

The Ramanuja's cubic continued fraction denoted by $V(q)$ is defined by

$$V(q) := \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \, \cdots .$$

Lemma 2.1 (Bailey's ${}_6\psi_6$ formula)**.** *For $|qa^2/(bcde)| < 1$,*

$$ {}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix} ; q, \frac{qa^2}{bcde} \right] $$
$$ = \frac{(aq, aq/(bc), aq/(bd), aq/(be), aq/(cd), aq/(ce), aq/(de), q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/(bcde); q)_\infty}, \tag{11}$$

where the ${}_6\psi_6$ function is defined as

$$ {}_6\psi_6 \left[\begin{matrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6 \end{matrix} ; q, z \right] := \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, a_3, a_4, a_5, a_6; q)_n}{(b_1, b_2, b_3, b_4, b_5, b_6; q)_n} z^n. $$

Lemma 2.2. *Let $x(q) = q^{-1/3} V(q)$. Then*

$$f_1 f_2 = f_9 f_{18} \left(\frac{1}{x(q^3)} - q - 2q^2 x(q^3) \right). \tag{12}$$

Proof. Due to Chan [4, Theorem 2], we have

$$\frac{(q^{1/3};q^{1/3})_\infty (q^{2/3};q^{2/3})_\infty}{(q^3;q^3)_\infty (q^6;q^6)_\infty} = \frac{1}{x(q)} - q^{1/3} - 2q^{2/3}x(q). \tag{13}$$

Replacing q by q^3 in the above equation, we obtain

$$\frac{f_1 f_2}{f_9 f_{18}} = \frac{1}{x(q^3)} - q - 2q^2 x(q^3). \tag{14}$$

Multiplying throughout by $f_9 f_{18}$, we arrive at (12). □

Lemma 2.3. *Let* $x(q) = q^{-1/3}V(q)$. *Then*

$$\frac{f_3^4 f_6^4}{f_9^4 f_{18}^4} = \frac{1}{x(q^3)^3} - 7q^3 - 8q^6 x(q^3)^3. \tag{15}$$

Proof. Due to Naika *et al.* [10, Theorem 3.1], we have

$$8V^3(-e^{-\pi\sqrt{M/3}}) - \frac{1}{V^3(-e^{-\pi\sqrt{M/3}})} + 7 = \frac{9}{a_{M,3}^2}, \tag{16}$$

where

$$a_{M,3} = \frac{3e^{-\pi/2\sqrt{M/3}}\psi^2(e^{-3\pi\sqrt{M/3}})\phi^2(-e^{-6\pi\sqrt{M/3}})}{\psi^2(e^{-\pi\sqrt{M/3}})\phi^2(-e^{-2\pi\sqrt{M/3}})}.$$

If $q = e^{-\pi\sqrt{M/3}}$, then (16) can be written as

$$8V^3(-q) - \frac{1}{V^3(-q)} + 7 = \frac{\psi^4(q)\phi^4(-q^2)}{q\psi^4(q^3)\phi^4(-q^6)}. \tag{17}$$

Replacing q by $-q$ and then multiplying throughout by $-q$, we obtain

$$-8qV^3(q) + q\frac{1}{V^3(q)} - 7q = \frac{\psi^4(-q)\phi^4(-q^2)}{\psi^4(-q^3)\phi^4(-q^6)}. \tag{18}$$

That is

$$-8q^6x^3(q^3) + \frac{1}{x^3(q^3)} - 7q^3 = \frac{\psi^4(-q^3)\phi^4(-q^6)}{\psi^4(-q^9)\phi^4(-q^{18})}. \tag{19}$$

Using $\phi(-q) = \dfrac{f_1^2}{f_2}$ and $\psi(-q) = \dfrac{f_1 f_4}{f_2}$ in the above equation, we arrive at (15). □

Lemma 2.4. *[3, p.49] we have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}). \tag{20}$$

Lemma 2.5. *[8] The following 2-dissection is true:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{21}$$

Lemma 2.6. *[2, Eq. 3.17] We have*

$$\varphi(q)f(q,q^5) = \psi(q^2)f(q^2,q^4) + 3q\frac{f_{12}^3}{f_4}. \tag{22}$$

Lemma 2.7. *[1, Theorem. 1.15] Let $w(n)$ denote the number of representations of a nonnegative integer n in the form $x^2+2y^2+3z^2+6r^2$ with $x,y,z,r \in \mathbb{Z}$. Set $n = 2^\alpha 3^\beta N$, where $\alpha, \beta \geq 0$, $N \geq 1$ and $\gcd(6,N)=1$. Then*

$$w(n) = \begin{cases} (3^{\beta+1}-2)\sigma(N) + a(n) & \text{if} \quad n \equiv 1 \pmod 2, \\ 2(3^{\beta+1}-2)\sigma(N) & \text{if} \quad n \equiv 2 \pmod 4, \\ 6(3^{\beta+1}-2)\sigma(N) & \text{if} \quad n \equiv 0 \pmod 4, \end{cases} \tag{23}$$

where $\sum_{n=0}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n})(1-q^{6n})(1-q^{12n})$.

3. Main results

In this section, we establish arithmetic properties and formulas for $C_3(n)$.

Theorem 3.1. *For each $k, n \geq 0$,*

$$C_3\left(3^k n + 3^k - 1\right) = 3^k C_3\left(2^k n + 2^k - 1\right) = 3^k C_3(n) \tag{24}$$

and that

$$C_3\left(3^k n + 3^k - 1\right) \equiv 0 \pmod{3^k}. \tag{25}$$

Proof. Consider

$$\xi = \frac{f_1 f_2}{q f_9 f_{18}}, \quad \mu = \frac{1}{qx(q^3)}, \quad R = \frac{f_3^4 f_6^4}{q^3 f_9^4 f_{18}^4}. \tag{26}$$

Then, from (12), (15) and (26),

$$\xi = \frac{f_1 f_2}{q f_9 f_{18}} = \mu - 1 - \frac{2}{\mu} \tag{27}$$

and

$$R = \mu^3 - 7 - \frac{8}{\mu^3}. \tag{28}$$

From (27) and (28), we have

$$\begin{aligned}\xi^3 &= \mu^3 - 3\mu^2 - 3\mu + 11 + \frac{6}{\mu} - \frac{12}{\mu^2} - \frac{8}{\mu^3} \\ &= R + 18 - 3\mu^2 - 3\mu + \frac{6}{\mu} - \frac{12}{\mu^2} \\ &= R + 9 - 3\xi^2 - 9\mu + \frac{18}{\mu} \\ &= R - 9\xi - 3\xi^2. \end{aligned} \tag{29}$$

It follows from (29) that

$$\xi^3 + 3\xi^2 + 9\xi = R. \tag{30}$$

We can write (30)

$$\frac{1}{\xi} = \frac{1}{R}(9 + 3\xi + \xi^2). \tag{31}$$

Now let **H** be the "huffing" operator that extracts powers of q which is congruent to 0 modulo 3. If we apply **H** to (31), we find

$$\mathbf{H}\left(\frac{1}{\xi}\right) = \frac{1}{R}\left(9\mathbf{H}(1) + 3\mathbf{H}(\xi) + \mathbf{H}(\xi^2)\right). \tag{32}$$

Now,

$$\mathbf{H}\left(\xi^2\right) = \mathbf{H}\left(\mu^2 - 2\mu - 3 + \frac{4}{\mu} + \frac{4}{\mu^2}\right) = -3, \tag{33}$$

$$\mathbf{H}\left(\xi\right) = \mathbf{H}\left(\mu - 1 - \frac{2}{\mu}\right) = -1, \tag{34}$$

$$\mathbf{H}\left(1\right) = 1. \tag{35}$$

From (32)–(35), we find

$$\mathbf{H}\left(\frac{1}{\xi}\right) = \frac{3}{R}. \tag{36}$$

Using (26), (6) can be expressed as

$$\sum_{n=0}^{\infty} C_3(n) q^{n-2} = \frac{f_3^3 f_6^3}{q^3 f_9 f_{18}} \times \frac{1}{\xi} \tag{37}$$

Applying the operator **H** on both sides of (37) and then using (36), we find that

$$\sum_{n=0}^{\infty} C_3(3n+2) q^{3n} = \frac{f_3^3 f_6^3}{q^3 f_9 f_{18}} \times \frac{3}{R}. \tag{38}$$

Using (26) in (38) and then replacing q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(3n+2)q^n &= 3\frac{f_3^3 f_6^3}{f_1 f_2}\\ &= 3\sum_{n=0}^{\infty} C_3(n)q^n.\end{aligned}$$

Equating coefficients of q^n on both sides of the above equation, we obtain

$$C_3(3n+2) = 3C_3(n). \tag{39}$$

Now substituting (21) into (6), we find that

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{f_4^3 f_6^5}{f_2^3 f_{12}} + q\frac{f_6^3 f_{12}^3}{f_2 f_4}, \tag{40}$$

which implies that

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(2n+1)q^n &= \frac{f_3^3 f_6^3}{f_1 f_2}\\ &= \sum_{n=0}^{\infty} C_3(n)q^n.\end{aligned}$$

Equating coefficients of q^n on both sides of the above equation, we obtain

$$C_3(2n+1) = C_3(n). \tag{41}$$

In view of (39), (41) and by induction, we arrive at (24). Congruence (25) follows from (24). □

Theorem 3.2. *Let $\sigma_{odd}(n)$ denote the sum of odd divisors d of n. Then for all $n, k \geq 0$,*

$$C_3\left(2^k n + 2^k - 1\right) = \underset{3\nmid\frac{n+1}{d}}{\sigma_{odd}}(n+1) \tag{42}$$

and

$$C_3\left(3^k n + 3^k - 1\right) = 3^k \underset{3\nmid\frac{n+1}{d}}{\sigma_{odd}}(n+1). \tag{43}$$

Proof. Replacing $(a, b, c, d, e, q) \to (q^4, q^2, q^2, q^2, q^5, q^6)$ in (11) and then multiplying both sides by $\dfrac{q(1+q^2)}{(1-q^2)^2}$, we obtain

$$\frac{f_{3;6}^2 f_6^4}{f_{1,5;6} f_{2,4;6}^2} \cdot q = \sum_{m=-\infty}^{\infty} \frac{q^{3m+1}(1+q^{2(3m+1)})}{(1-q^{2(3m+1)})^2}. \tag{44}$$

Note that $f_1 = f_{1,2,3,\ldots,k-1;k} f_k$ and from (6), we deduce that

$$\sum_{n=0}^{\infty} C_3(n) q^n = \frac{f_3^3 f_6^3}{f_1 f_2} \tag{45}$$

$$= \frac{f_{3;6}^2 f_6^4}{f_{1,5;6} f_{2,4;6}^2}. \tag{46}$$

Combining (44) and (46), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} C_3(n) q^{n+1} &= \sum_{m=-\infty}^{\infty} \frac{q^{3m+1}(1+q^{2(3m+1)})}{(1-q^{2(3m+1)})^2} \\
&= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{2(3m+1)})}{(1-q^{2(3m+1)})^2} + \sum_{m=-\infty}^{-1} \frac{q^{3m+1}(1+q^{2(3m+1)})}{(1-q^{2(3m+1)})^2} \\
&= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{2(3m+1)})}{(1-q^{2(3m+1)})^2} + \sum_{m=0}^{\infty} \frac{q^{-(3m+2)}(1+q^{-2(3m+2)})}{(1-q^{-2(3m+2)})^2} \\
&= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{2(3m+1)})}{(1-q^{2(3m+1)})^2} + \sum_{m=0}^{\infty} \frac{q^{3m+2}(1+q^{2(3m+2)})}{(1-q^{2(3m+2)})^2}, \qquad (47)
\end{aligned}$$

where the third equality follows by replacing m by $-m-1$ in the second sum.

It is known that

$$\log(1-x^2) = -\sum_{k=1}^{\infty} \frac{x^{2k}}{k}, \quad |x| < 1.$$

Differentiating with respect to x twice and then multiplying throughout by x, we find that

$$\frac{x(1+x^2)}{(1-x^2)^2} = \sum_{k=0}^{\infty} (2k+1) x^{2k+1}, \quad |x| < 1.$$

In view of the above identity, (47) can be written as

$$\sum_{n=0}^{\infty} C_3(n) q^{n+1} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (2k+1) q^{(2k+1)(3m+1)} + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (2k+1) q^{(2k+1)(3m+2)}. \tag{48}$$

Comparing the coefficients of q^{n+1} on both sides of (48), we obtain

$$C_3(n) = \underset{3 \nmid \frac{n+1}{d}}{\sigma_{odd}} (n+1). \tag{49}$$

Invoking (24) and (49), we arrive at desired results. □

Theorem 3.3. *Let p be a prime, and k, n be nonnegative integers.*

(1) If $p=2$, then

$$C_3(2^k-1)=1. \tag{50}$$

(2) If $p=3$, then

$$C_3(3^k-1)=3^k. \tag{51}$$

(3) If $p\geq 5$, then

$$C_3(p^kn+p^k-1)=\frac{p^k-1}{p-1}C_3(pn+p-1)-\frac{p^k-p}{p-1}C_3(n). \tag{52}$$

Proof. (1) If we write $n+1=2^k$, then by using (24) and (42), we deduce that

$$C_3(2^k-1)=\underset{3\nmid\frac{2^k}{d}}{\sigma_{odd}}(2^k)=1.$$

(2) If we write $n+1=3^k$, then by using (24) and (42), we deduce that

$$C_3(3^k-1)=\underset{3\nmid\frac{3^k}{d}}{\sigma_{odd}}(3^k)=3^k.$$

(3) Let $p\geq 5$ is a prime. If we write $n+1=p^mN$, where N is an integer not divisible by p, then by using (24) and (42), we deduce that

$$C_3(n)=\underset{3\nmid\frac{p^mN}{d}}{\sigma_{odd}}(p^mN)=\underset{3\nmid\frac{p^m}{d}}{\sigma_{odd}}(p^m)\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N)=\frac{p^{m+1}-1}{p-1}\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N). \tag{53}$$

In a similar fashion, we find that

$$C_3(pn+p-1)=\underset{3\nmid\frac{p^{m+1}N}{d}}{\sigma_{odd}}(p^{m+1}N)=\underset{3\nmid\frac{p^{m+1}}{d}}{\sigma_{odd}}(p^{m+1})\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N)=\frac{p^{m+2}-1}{p-1}\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N) \tag{54}$$

and

$$C_3(p^kn+p^k-1)=\underset{3\nmid\frac{p^{k+m}N}{d}}{\sigma_{odd}}(p^{k+m}N)=\underset{3\nmid\frac{p^{k+m}}{d}}{\sigma_{odd}}(p^{k+m})\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N)=\frac{p^{k+m+1}-1}{p-1}\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N). \tag{55}$$

Now consider

$$\frac{p^{k+m+1}-1}{p-1}\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N)=\left(\frac{p^{m+2}-1}{p-1}c_1+\frac{p^{m+1}-1}{p-1}c_2\right)\underset{3\nmid\frac{N}{d}}{\sigma_{odd}}(N), \tag{56}$$

where c_1 and c_2 are constants that has to be find.

Equation (56) can be written as

$$p^{k+m+1} - 1 = p^{k+m+1}(p^{1-k}c_1 + p^{-k}c_2) - (c_1 + c_2). \tag{57}$$

From which we obtain

$$p^{1-k}c_1 + p^{-k}c_2 = 1$$

and

$$c_1 + c_2 = 1.$$

Solving the above two equations, we obtain that

$$c_1 = \frac{p^k - 1}{p - 1} \text{ and } c_2 = -\frac{p^k - p}{p - 1}. \tag{58}$$

Substituting (58) together with (53)–(55) into (56), we arrive at desired result. □

Theorem 3.4. *Let $p \geq 5$ be a prime and n be the nonnegative integer such that $p \nmid n + 1$. Then for each $k \geq 0$,*

$$C_3(p^k n + p^k - 1) = \frac{p^{k+1} - 1}{p - 1} C_3(n). \tag{59}$$

Proof. From (24) and (42), we deduce that

$$\begin{aligned} C_3\left(p^k n + p^k - 1\right) &= \underset{3 \nmid \frac{p^k(n+1)}{d}}{\sigma_{odd}} \left(p^k(n+1)\right) \\ &= \underset{3 \nmid \frac{p^k}{d}}{\sigma_{odd}}(p^k)\, \underset{3 \nmid \frac{n+1}{d}}{\sigma_{odd}}(n+1) \\ &= \frac{p^{k+1} - 1}{p - 1} \underset{3 \nmid \frac{n+1}{d}}{\sigma_{odd}}(n+1) \\ &= \frac{p^{k+1} - 1}{p - 1} C_3(n), \end{aligned}$$

which is same as (59). □

Theorem 3.5. *If $w(n)$ denote the number of representations of a nonnegative integer n in the form $x^2 + 2y^2 + 3z^2 + 6r^2$ with $x, y, z, r \in \mathbb{Z}$, and $C_3(n)$ is the number of 3-core cubic partitions of n, then*

$$36C_3(n) = w(12n + 12) - w(4n + 4). \tag{60}$$

Proof. We have

$$\sum_{n=0}^{\infty} w(n)q^n = \varphi(q)\varphi(q^2)\varphi(q^3)\varphi(q^6). \tag{61}$$

Substituting (20) into (61), we find that

$$\sum_{n=0}^{\infty} w(n)q^n = \varphi(q^3)\varphi(q^6)\left(\varphi(q^9)+2qf(q^3,q^{15})\right)\left(\varphi(q^{18})+2q^2f(q^6,f^{30})\right). \tag{62}$$

Extracting the terms in which powers of q is congruent to 0 modulo 3 from (62) and replacing q^3 by q, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} w(3n)q^n &= \varphi(q)\varphi(q^2)\varphi(q^3)\varphi(q^6) + 4q\varphi(q)\varphi(q^2)f(q,q^5)f(q^2,q^{10})\\ &= \sum_{n=0}^{\infty} w(n)q^n + 4q\varphi(q)\varphi(q^2)f(q,q^5)f(q^2,q^{10}).\end{aligned} \tag{63}$$

Substituting (22) into (63) and then extracting terms of q^{4n} in the resulting equation, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} w(12n)q^n &= \sum_{n=0}^{\infty} w(4n)q^n + 36q\frac{f_3^3f_6^3}{f_1f_2}\\ &= \sum_{n=0}^{\infty} w(4n)q^n + 36\sum_{n=0}^{\infty} C_3(n)q^{n+1}.\end{aligned} \tag{64}$$

Equating coefficients of q^{n+1} on both sides of the above equation, we obtain

$$36C_3(n) = w(12n+12) - w(4n+4),$$

which is (60). □

Theorem 3.6. *Let $n,k \geq 0$. Set $4n+4 = 2^\alpha 3^\beta N$, where $\alpha,\beta \geq 0$, $N \geq 1$ and $\gcd(6,N) = 1$. Then*

$$C_3\left(3^kn+3^k-1\right) = 3^{\beta+k}\sigma(N) \tag{65}$$

and

$$C_3\left(2^kn+2^k-1\right) = 3^{\beta}\sigma(N). \tag{66}$$

Proof. Since $4n+4$ and $12n+12$ are congruent to 0 modulo 4, we can make use of case 3 of (23). Therefore, in view of (23) and (60), we see that

$$\begin{aligned}36C_3(n) &= 6(3^{\beta+2}-2)\sigma(N) - 6(3^{\beta+1}-2)\sigma(N)\\ &= 6(3^{\beta+2}-3^{\beta+1})\sigma(N)\\ &= 4\cdot 3^{\beta+2}\sigma(N),\end{aligned} \tag{67}$$

which implies that

$$C_3(n) = 3^{\beta}\sigma(N). \tag{68}$$

Invoking (24) and (68), we arrive at desired results. □

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[1] A. Alaca, S. Alaca, M.F. Lemire and K.S. Williams, Nineteen quaternary quadratic forms, Acta arith. 130.3 (2007).

[2] N.D. Baruah, K. Nath, Some results on 3-cores, Proc. Amer. Math. Soc. 142 (2014) 441–448.

[3] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[4] H.C. Chan, Ramanujan's cubic continued fraction and an analog of his "most beautiful identity", Int. J. Number Theory 6(3) (2010) 673–680.

[5] S. Chern, Formulas for partition k-tuples with t-cores, J. Math. Anal. Appl. 437(2) (2016) 841–852.

[6] F. Garvan, D. Kim, D. Stanton, Cranks and t-cores, Invent. Math. 101(1) (1990) 1–17.

[7] A. Granville, K. Ono, Defect zero p-blocks for finite simple groups, Trans. Amer. Math. Soc. 348(1) (1996) 331–347.

[8] M.D. Hirschhorn, F. Garvan, J. Borwein. Cubic analogs of the Jacobian cubic theta function $\theta(z;q)$. Canad. J. Math. 45 (1993), pp. 673–694.

[9] M.D. Hirschhorn, J.A. Sellers, Elementary proofs of various facts about 3-cores, Bull. Aust. Math. Soc. 79(3) (2009) 507–512.

[10] M.S.M. Naika, M.C. Maheshkumar, K.S. Bairy. General formulas for explicit evaluations of Ramanujan's cubic continued fraction. Kyungpook Math. J. 48 (2009), pp. 435–450.

[11] L. Wang, Explicit formulas for partition pairs and triples with 3-cores, J. Math. Anal. Appl. 434(2) (2016) 1053–1064.