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UNILATERAL PROBLEMS FOR THE p-LAPLACE OPERATOR IN PERFORATED MEDIA INVOLVING LARGE PARAMETERS *

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Abstract. We address homogenization problems for variational inequalities issue from unilateral constraints for the p-Laplacian posed in perforated domains of $\mathbb{R}^n$, with $n \geq 3$ and $p \in [2, n]$. ε is a small parameter which measures the periodicity of the structure while $a_\varepsilon \ll \varepsilon$ measures the size of the perforations. We impose constraints for solutions and their fluxes (associated with the p-Laplacian) on the boundary of the perforations. These constraints imply that the solution is positive and that the flux is bounded from above by a negative, nonlinear monotonic function of the solution multiplied by a parameter β_ε which may be very large, namely, $\beta_\varepsilon \to \infty$ as $\varepsilon \to 0$. We first consider the case where $p < n$ and the domains periodically perforated by tiny balls and we obtain homogenized problems depending on the relations between the different parameters of the problem: p, n, ε, a_ε and β_ε. Critical relations for parameters are obtained which mark important changes in the behavior of the solutions. Correctors which provide improved convergence are also computed. Then, we extend the results for $p = n$ and the case of non periodically distributed isoperimetric perforations. We make it clear that in the averaged constants of the problem, the perimeter of the perforations appears for any shape.

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1. Introduction

Homogenization problems in perforated media for the p-Laplace operator have been considered in the literature over the last decades. We mention [8, 28, 32] for Dirichlet boundary conditions, [14] for Neumann conditions, [36] for Signorini conditions, [38–40] for some generalized Robin type boundary conditions, [15] for perforations along a manifold, [26, 41] for obstacles in perforated domains, and [3, 4, 12] for different abstract frameworks involving perforated media: see also references therein. Different assumptions on the geometry and the distribution of the perforations are made in the above-mentioned papers; also different assumptions on p are considered. See [13, 29, 43] and references therein in connection with models related to p-Laplacian, for different values of p, arising e.g. in glaciology, torsional creep and flows through porous media.

Keywords and phrases: nonlinear homogenization, perforated media, variational inequalities, critical relations for parameters

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The problems under consideration in this paper are different from those in previous papers. We consider the p-Laplace operator in a perforated domain by “tiny cavities”, with constraints for the solution in a general framework (cf., e.g., [42]), and further specifying, constraints for the solution and its normal derivative (associated with the p-Laplacian) on the boundary of the perforations, involving a nonlinear function of the solution, σ, and a parameter. This parameter may depend on the period and it can be either a very large parameter or a very small parameter; also σ can be a quite general monotonic increasing function (cf. (2.1)–(2.3) and Section 8).

We focus on obtaining critical sizes of perforations and critical relations between parameters which give rise to a *strange term* in the homogenized problem, at the same time as we describe all the possible homogenized media depending on the parameters of the problem. In addition, we construct correcting terms which provide strong convergence in the corresponding Sobolev spaces and obtain precise bounds for convergence rates. Strange terms issue *simultaneously* from the constraint for the solution and the constraint for the normal derivative on the boundary of the cavities, and correctors for solutions, are obtained for the first time in homogenization theory of the p-Laplace operator. These strange terms can appear in a boundary value problem or in an obstacle problem; their nonlinear character being very different. As one might expect, considering all the possible relations between parameters and the improved convergence provided by the correctors may restrict the geometrical configuration of the problem as well as the properties of the nonlinear function σ which however seems to be optimal to obtain all the results (cf. Section 8).

More precisely, we consider the domain Ω_ε which is obtained by removing small domains G_ε, *the cavities/perforations*, of diameter $2a_\varepsilon \ll \varepsilon$, from a fixed domain Ω of $\mathbb{R}^n$ (see Figure 1). The cavities are distributed over the whole domain at a distance $O(\varepsilon)$ between them, ε being a small parameter that we shall make to go to 0. We denote by S_ε the boundary of the cavities, namely, $S_\varepsilon \equiv \partial G_\varepsilon$. For $n \geq 3$ and $p \in [2, n]$, we study the asymptotic behavior of the solution u_ε, as $\varepsilon \to 0$, of the following problem:

$$\begin{cases} -\Delta_p u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ u_\varepsilon \geq 0,\ \partial_{\nu_p} u_\varepsilon \geq -\beta_\varepsilon \sigma(x, u_\varepsilon),\ u_\varepsilon(\partial_{\nu_p} u_\varepsilon + \beta_\varepsilon \sigma(x, u_\varepsilon)) = 0 & \text{for } x \in S_\varepsilon, \end{cases} \tag{1.1}$$

where $f \in L^q(\Omega)$ with $q = p/(p-1)$, $\Delta_p u \equiv div(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν denotes the unit outward normal to Ω_ε on S_ε, $\beta_\varepsilon > 0$ is a ε-dependent constant, $\sigma = \sigma(x, u)$ is a continuously differentiable function defined in $\overline{\Omega} \times \mathbb{R}$, strongly monotone with respect to u (cf. (2.1)–(2.3)). Note that β_ε is referred to as the *adsorption parameter*, and the variational formulation of (1.1) is (2.5).

We distinguish two ranges of p: $p \in [2, n)$ and $p = n$. For $2 \leq p < n$ we assume that the adsorption parameter takes the value $\varepsilon^{-\gamma}$, with $\gamma \in \mathbb{R}$, and that the cavities G_ε are balls of radius $a_\varepsilon = C_0\varepsilon^\alpha$ (with $\alpha > 1$ and $C_0 > 0$), which are periodically distributed over Ω. It should be emphasized that this geometrical configuration is essential over all for the relations $\alpha = n/(n-p)$ and $\gamma = n(p-1)/(n-p) = \alpha(p-1)$ (see the intersection point in Figures 2 and 3) since the solution of the local problem obtained from the microstructure of the model is somewhat related with the fundamental solution of the p-Laplace operator. The solution of the local problem can be computed via a nonlinear equation that recalls the functional equation (1.3) (cf. Section 8). We relax the above geometrical configuration for the case where $p = n$, a case where a certain non periodically distribution of the cavities is allowed while they can have arbitrary shapes with a fixed perimeter (see the different cells in Figure 1 and the functional equation (1.10)–(1.11)): a comparison result makes it difficult to extend the result to $p \in [2, n)$.

Below, cf. Section 1.1, we relate all the homogenized problems and the main results that we obtain as well as the structure of the paper.

1.1. The homogenized problems

For $p \in [2, n)$, we obtain the homogenized problem, as $\varepsilon \to 0$, for different relations between α and γ (see a sketch of all the possible situations in Figure 2). Among these relations, two of them provide asymptotic relations between adsorption, size and periodicity parameters which are related to as *critical size* and *critical*

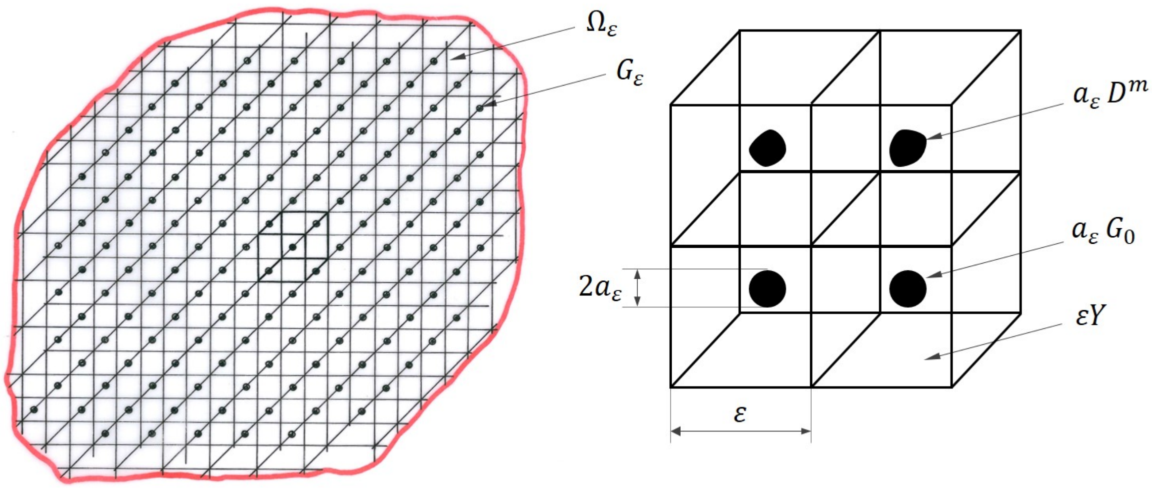

Figure 1. The geometrical configuration of Ω_ε and the periodicity cell.

relation for the adsorption. Let us explain this in further detail. By comparison with the p-Laplace operator in perforated media, the *classical critical size* for the perforations with the p-Laplacian and Dirichlet boundary conditions on S_ε (see [28]) is given by $\alpha = n/(n-p)$. For $\alpha > n/(n-p)$ the cavities are very small and they, as well as the constraints with any adsorption parameter, do not influence the average process (cf. the region in green color in Figure 2, problem (1.8), Theorem 6.4, and Theorem 6.5 for improved convergence). A *strange term* appears for $\alpha = n/(n-p)$.

It should be mentioned that the terminology of strange term here used appears in [9] for linear problems with Dirichlet boundary conditions on the perforations; see the reference to the original work introducing this terminology and further references in [9]; see also [31] in connection with the above-mentioned term in these linear problems; see [28] for Dirichlet conditions with the p-Laplacian, and [21, 24] for the Laplacian with nonlinear Robin boundary conditions on the perforations. In our problem, it happens that this strange term also depends on the adsorption parameter and it ranges from a classical reaction term associated with the p-Laplacian, namely of the type $|u|^{p-2}u$ or $|u^-|^{p-2}u^-$, to the reaction term $\sigma(x,u)$ by multiplicative constants of the problem or a reaction term given by a function implicitly defined in a functional equation of the type (2.19). Also the character of the homogenized problem can change including boundary value problems (cf. (1.2), (1.4), (1.5) and (1.8)) and obstacle problems (cf. (1.6) and (1.7)). In fact, for each value of α, $1 < \alpha \leq n/(n-p)$, the relation $\gamma = \alpha(n-1) - n$ provides the so-called *critical relation for the adsorption parameter* which implies that the total area of the perforations multiplied by the adsorption parameter is of order $O(1)$.

In order to make more comprehensible the entire results for $p \in [2,n)$, which we summarize in Figures 2 and 3, we introduce here a table with all the possible limit situations:

I. When $\alpha = n/(n-p)$ and $\gamma = n(p-1)/(n-p)$, the homogenized problem is:

$$\begin{cases} -\Delta_p u + \mathcal{A}_{n,p}\Big(|H(x,u^+)|^{p-2}H(x,u^+) + |u^-|^{p-2}u^-\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{A}_{n,p} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$ and, for every $(x,\tau) \in \Omega \times \mathbb{R}$, $H(x,\tau)$ is the solution of the functional equation

$$\mathcal{B}_{n,p}|H|^{p-2}H = \sigma(x, \tau - H), \tag{1.3}$$

with $\mathcal{B}_{n,p} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$.

II. When $\alpha = n/(n-p)$ and $\gamma < n(p-1)/(n-p)$, the homogenized problem is

$$\begin{cases} -\Delta_p u + \mathcal{A}_{n,p}|u^-|^{p-2}u^- = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\mathcal{A}_{n,p} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$.

III. When $\alpha = n/(n-p)$ and $\gamma > n(p-1)/(n-p)$, the homogenized problem is

$$\begin{cases} -\Delta_p u + \mathcal{A}_{n,p}|u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\mathcal{A}_{n,p} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$.

IV. When $\alpha \in (1, n/(n-p))$ and $\gamma = \alpha(n-1) - n$, the homogenized problem is

$$\begin{cases} -\Delta_p u + \mathcal{D}_n\sigma(x,u) - f \geq 0, \ u \geq 0, \ (-\Delta_p u + \mathcal{D}_n\sigma(x,u) - f)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\mathcal{D}_n = C_0^{n-1}\omega_n$.

V. When $\alpha \in (1, n/(n-p))$ and $\gamma < \alpha(n-1) - n$, the homogenized problem is

$$\begin{cases} -\Delta_p u - f \geq 0, \ u \geq 0, \ (-\Delta_p u - f)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

VI. When $\alpha \in (1, n/(n-p))$ and $\gamma > \alpha(n-1) - n$, $u \equiv 0$, that is, as $\varepsilon \to 0$, the solution u_ε vanishes asymptotically in the whole Ω.

VII. When $\alpha > n/(n-p)$ and $\gamma \in \mathbb{R}$, the homogenized problem is

$$-\Delta_p u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.8}$$

Above, and throughout the paper ω_n denotes the area of the unit sphere in $\mathbb{R}^n$, and u^+ and u^- denote $u^+ = \sup(u(x), 0)$ and $u^- = u - u^+$. The existence and uniqueness of the solution for all the homogenized problems holds as does that for the ε-dependent problem (1.1) (cf. Theorem 2.1).

Point I is referred to as the *most critical case*, where we have the critical size of perforations and the critical relation for the adsorption parameter. Points II and III deal with the the classical critical size of perforations. Case IV fits into the case of the critical relation for the adsorption. This case is of great interest, since for each size of the holes (namely, for each α) we have a critical relation for the adsorption (namely, of γ) giving rise to the strange term. Of course the role of α and γ inverts (see the red discontinuous line in Figure 2). In points V-VII some extreme relations for parameters hold.

Hence, the most critical relation between parameters is provided by the intersection, in the plane $\alpha\gamma$, cf. Figure 2, of the lines $\alpha = n/(n-p)$ and $\gamma = \alpha(n-1) - n$. The intersection point (*the big point* in Figure 2) has coordinates $\alpha = n/(n-p)$ and $\gamma = n(p-1)/(n-p) = \alpha(p-1)$. In this case, the ε-dependent problem, which is a variational inequality (cf. (2.8)), asymptotically transforms into a boundary value problem (cf. (1.2)) with a strange term defined implicitly from a functional equation (cf. (1.3)) issued from the microstructure of the problem. The function defining this new term satisfies the same properties of monotonicity as σ and it is in good agreement with the existing results in the literature for $p = 2$ (cf. [23]). As a matter of fact, the strange term is the sum of two terms related to the contribution both of the constraints $u_\varepsilon \geq 0$ and $\partial_{\nu_p} u_\varepsilon \geq -\varepsilon^{-\gamma}\sigma(x, u_\varepsilon)$ on the boundary of the perforations. The first term is given by a somewhat classical reaction term $|u^-|^{p-2}u^-$ multiplied by averaged constants; the other one involves a nonlinear function of u, $H(x,u)$ implicitly defined from

(1.3): see [28], [40] and [41] to compare with strange terms when we have a Dirichlet condition or a generalized Robin boundary condition with a more restrictive datum σ. See (8.1) for some explicit computation of H.

We also note that the above mentioned fact (on the double contribution for the strange term) has already been detected in [17,23] for variational inequalities for the Laplacian ($p = 2$) in perforated media depending on whether the perforations are placed over the whole domain or along a manifold. We mention [17] for an extensive bibliography on variational inequalities in homogenization problems. Also, [21] should be mentioned as the first work in the literature where a nonlinear strange term appears defined implicitly via a functional equation, and [24,25] as works which consider for the first time homogenization problems for the Laplace operator and semilinear boundary conditions leaving as an open question the most critical case (namely, the one homologous to the big point in Figure 2 when $p = 2$), problem which remained unsolved for a long time even for the Laplace operator (cf. Section 8 in this connection).

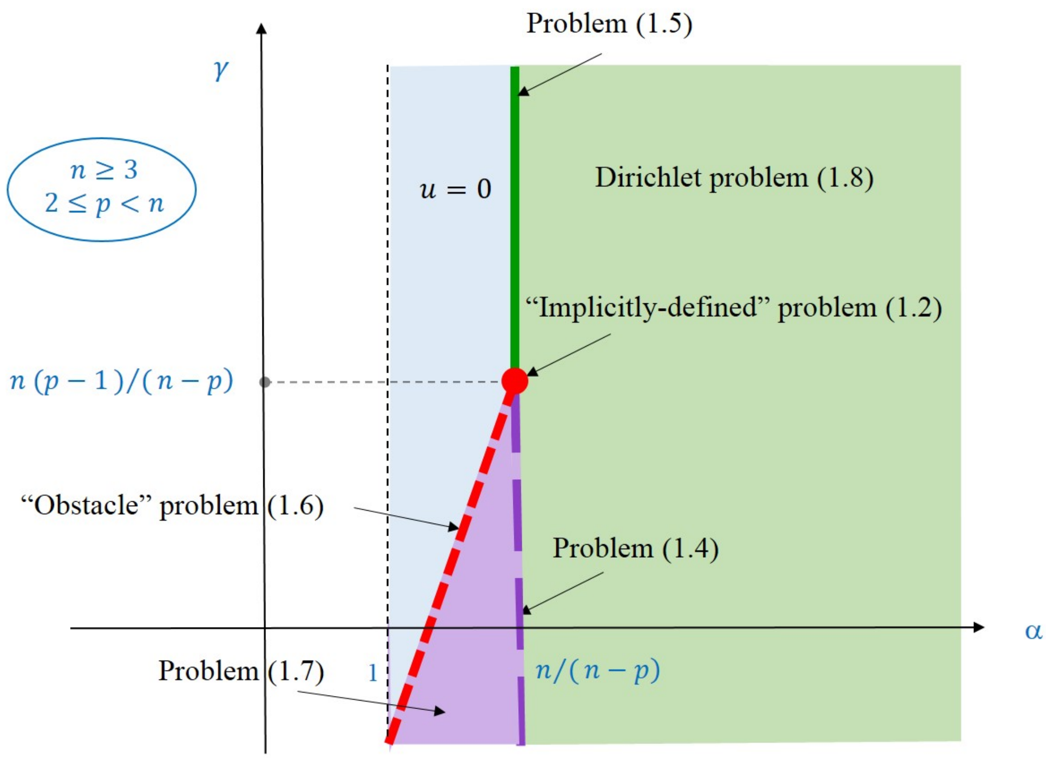

FIGURE 2. Sketch of homogenized problems depending on the relations between α and γ (n and p fixed).

For $\alpha = n/(n-p)$ and $\gamma = \alpha(p-1)$ (see the intersection point in Figures 2 and 3), we show the convergence result for the solution u_ε of (1.1) with $\beta_\varepsilon = \varepsilon^{-\gamma}$ (cf. Theorem 3.1) towards that of (1.2), where the nonlinear function H is defined via the functional equation (1.3). The existence and uniqueness of solution of (1.3) is a consequence of a general result (cf. Proposition 2.2). Also, we obtain the corrector term $W_\varepsilon(H(x,u^+) + u^-)$ which provides improved convergence (cf. Theorem 3.2 and definition (2.15)).

For the same value of α and different values of γ, namely, $\alpha = n/(n-p)$ and $\gamma \neq \alpha(p-1)$, the nonlinear strange term arising in the homogenized problem (cf. problems (1.4) and (1.5)) is provided by the reaction term $|u|^{p-2}u$ (for $\gamma > \alpha(p-1)$) or $|u^-|^{p-2}u^-$ (for $\gamma < \alpha(p-1)$ multiplied by constants of the problem obtained in the average process (cf. capacity constant and scaling constants from sizes of the cavities). Convergence and correctors are in Theorems 4.1–4.4: see line $\alpha = n/(n-p)$ in Figures 2 and 3. In both cases the homogenized problem seem to ignore the function σ of the ε-dependent problem.

In the case of the critical relation for the adsorption parameter $\gamma = \alpha(n-1) - n$ and α smaller than for the critical size, the nonlinear strange term is $\sigma(x, u)$ multiplied by some averaged constants. It accompanies the p-Laplacian in Ω and the homogenized problem is now an obstacle problem; namely, it is an obstacle problem associated to the corresponding homogenized medium (cf. (1.6) and the discontinuous red line in Figure 2). The convergence and bounds for convergence rates are in Theorems 5.1 and 5.2.

Finally, for the extreme relations, that is, the very large size of perforations and very large adsorption parameters, the solution of the ε-dependent problem is approached by 0 (cf. Theorem 6.3, and the blue region in Figure 2) as if the adsorption parameter becomes a small parameter accompanying the normal derivative, and therefore, as if Dirichlet conditions are imposed on very big perforations (see [28]). This is quite in contrast with the case of large size of perforations and small adsorption where we obtain an obstacle problem for the p-Laplace operator in Ω which ignores both the nonlinear term σ and the adsorption parameter: see problem (1.7), Theorems 6.1 and 6.2 and the region below the red discontinuous line in Figures 2 and 3.

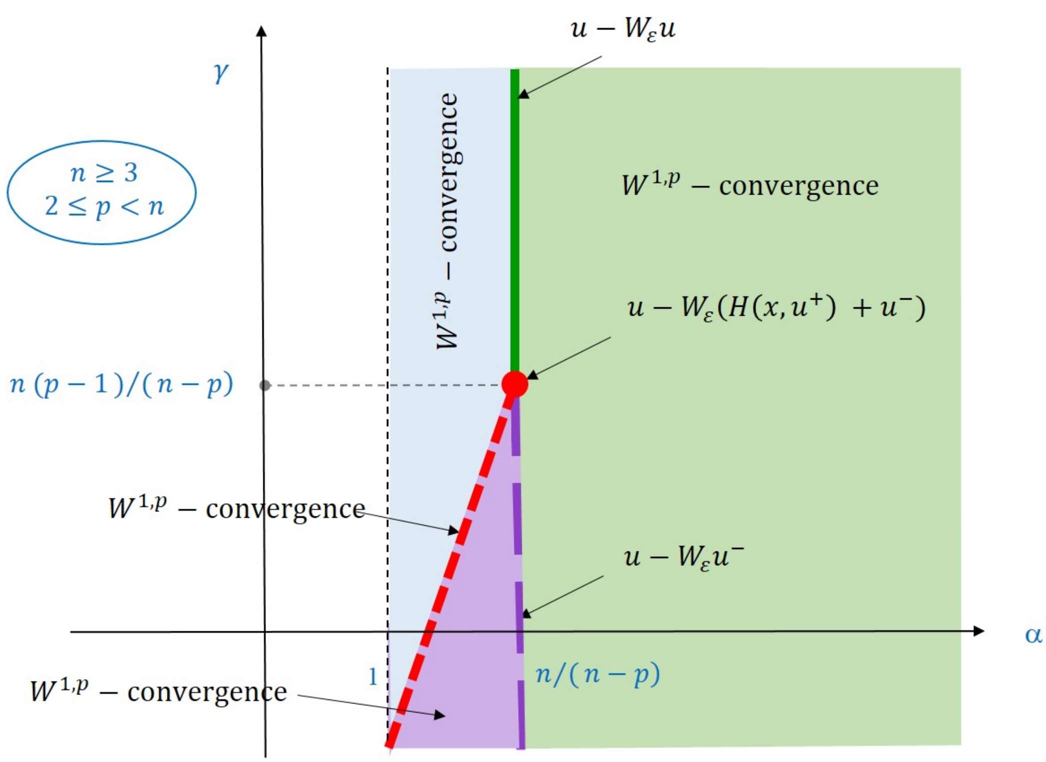

FIGURE 3. Sketch of correctors and improved convergences.

In the case where $p = n$ (cf. Section 7), we consider the most critical situation which is somewhat homologous to that of the big point in Figure 2. More specifically, for the geometry of the cavities described by (7.1) and the relations between sizes of cavities a_ε and adsorption parameter β_ε described by (7.2), the homogenized problem reads:

$$\begin{cases} -\Delta_n u + \widetilde{\mathcal{A}}_n \Big(|H(x,u^+)|^{n-2} H(x,u^+) + |u^-|^{n-2} u^- \Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $\widetilde{\mathcal{A}_n} = \omega_n \widetilde{\alpha}^{-2(n-1)}$, and for every $(x,\tau) \in \Omega \times \mathbb{R}$, $H(x,\tau)$ is the solution of the functional equation

$$\widetilde{\mathcal{B}_n} |H|^{n-2} H = \sigma(x, u - H), \tag{1.10}$$

with

$$\widetilde{\mathcal{B}_n} = \omega_n \widetilde{\alpha}^{-2(n-1)} \widetilde{C}_0^{-2(n-1)} l^{-1} \text{ and } |\partial D^m| = l; \tag{1.11}$$

l, $\widetilde{\alpha}^2$ and $\widetilde{C}_0^2$ are positive constants (see their precise definitions in (7.1) and (7.2) and (1.12)), *l the perimeter of the cavities* (cf. the different cells in Figure 1).

For brevity, in order to outline the extra difficulties when dealing with cavities which are not balls, in Section 7 we consider only the above homogenized problem (1.9) when $p = n$, leaving the whole map of possible limit situations and proofs to be published in a forthcoming publication by the authors. However, it should be noted that for $p = n$, due to the fact that logarithmic scale appears (cf. (7.2)), a graphic of the type of Figure 2 summarizing all the possible homogenized problem becomes more complicated (even unthinkable), and, as occurs in [18] for the Laplacian and perforation by tubes, the graphics should be performed for well defined dependence of a_ε and β_ε in terms of ε. In this respect, as a sample, we outline that (1.9)–(1.11) provide the homogenized problem of (1.1) when we have the following relations:

$$a_\varepsilon = \widetilde{C}_0^2 \varepsilon^k e^{-\widetilde{\alpha}^2/\varepsilon^\iota}, \quad \beta_\varepsilon = \varepsilon^{n-(n-1)j} e^{\widetilde{\alpha}^2 (n-1)/\varepsilon^\iota}, \tag{1.12}$$

with $k = j \geq 0$, $\widetilde{C}_0^2 > 0$, $\widetilde{\alpha}^2 > 0$, and $\iota = n/(n-1)$. However, other choices of order functions for a_ε and β_ε could lead to the same homogenized problem.

As happens for $p \in [2,n)$, in the most critical case, the homogenized problem (1.9) is a boundary value problem in Ω containing the strange term in the partial differential equation which is the sum of two terms as a consequence of a double contribution (cf. [28], [41] and [39] to compare with other boundary conditions). The contribution due to the constraint for the flux leads to the nonlinear function $|H|^{n-2}H$ in the strange term, H being implicitly defined by (1.10), where the perimeter l of the cavities arises now in the averaged constant (1.11), due exclusively to the influence of the adsorption parameter independently of the shape.

Note that in the case where the nonlinear function σ is the classical one arising in the Robin boundary condition, namely $\sigma = b(x)|u|^{p-2}u$ with $p \in [2,n]$, H can be defined explicitly in terms of $b(x)$ and u and we observe that H depends on $b(x)$ in a quite unusual way (see (8.1)).

As regards the technique, we mainly use the energy method to show the convergence of the solutions. Nevertheless, since we are dealing with homogenization of variational inequalities, and constraints involving nonlinear functions on the boundary of the perforations, proofs rely on monotonic operator theory, on extension operators, on suitable transformations of certain surface integrals on S_ε into volume integrals, on convergence of measures, and on the appropriate choice of test functions which allows us to pass to the limit in the weak formulations. These choices imply introducing auxiliary problems in the periodicity cell (cf. (2.13), (5.8), (7.6), (7.8) and (7.26)). As a matter of fact, somehow five auxiliary functions are used in the process depending on the range of p and on the relations between the parameters β_ε and a_ε. Functions w_ε^j and w_ε^j deal with the classical test functions used in the literature when the perforations are balls; both functions can be explicitly constructed. q_ε^j deal with the test functions for more general geometries; also, the sets of functions $\{M_\varepsilon^j\}$ and $\{m_\varepsilon^j\}$ $(j \in \mathbb{Z}^n)$, which are solutions of the non-homogeneous Neumann problems for the p-Laplacian, (5.8) and (7.26) respectively, become crucial in the identification of certain homogenized problems.

For the most critical case, we construct the test functions (cf. (3.2) and (7.15)) using w_ε^j and q_ε^j and the function H arising in the strange term (see (1.3) and (1.10) depending on p). To show the improved convergence for solutions, we construct correctors using the auxiliary functions, the implicitly defined function H, and some intermediate singularly perturbed problem (cf. (5.16)): under the assumption of $W^{1,\infty}$-smoothness of the solution of the homogenized problems, allows us to obtain precise bounds for convergence rates in the $W^{1,p}$-norm (see a map of the different situations in Figure 3).

As regards the structure of the paper: Sections 2–6 are devoted to the case where $p \in [2,n)$ and Section 7 contains the case $p = n$. Figure 2 summarizes the cluster of possible homogenized problems for different relations between the parameters α and γ, once we set p and n for $p \in [2, n)$. Figure 3 provides a sketch of the corrector terms and improved convergence for $p \in [2, n)$. The proofs are distributed in the paper as follows. Section 3 contains results for the most critical case (cf. the big point in Figures 2 and 3, and case I of the table). Section 4 contains results for the critical size of the perforations (cf. the vertical half-lines $\alpha = n/(n-p)$ in Figures 2 and 3, and cases II and III in the table). Section 5 addresses the critical relation for the adsorption parameter (cf. discontinuous red line in Figures 2 and 3, and case IV of the table). Section 6 addresses the rest of extreme relations (see colors blue, purple and green in Figures 2 and 3; cases V-VII of the table). Section 2 deals with the setting of the problem and some preliminary results useful for proofs throughout Sections 3–6; some technical proofs of these results are in the Appendix. Section 8 contains some final remarks on our results and on possible extensions to this paper.

Finally, in short, we emphasize that this paper provides a very general framework for variational inequalities with the p-Laplacian and constraints on the boundary of the perforations. The entire results imply improving and extending results in former papers in the literature (cf. Remark 8.1): only the results in Theorems 3.1 and 7.4 have been stated in [16, 19] under stronger restrictions on σ, we provide here their complete proofs. Also, we extend the results in [23] for the Laplace operator; namely, in [23] only items I and IV of the table for $p = 2$ have been addressed. We consider a more general σ at the same time that cover the rest of the cases for $p = 2$ and the rest of p. In all the cases we provide correctors or improved convergence with precise bounds for convergence rates. We also note that depending on the situation in the general map of Figures 2 and 3, the result obtained can be extended to more general geometries of the cavities and other nonlinear data arising in the constraints (see Section 8 in this connection).

2. The homogenization problem and preliminaries

In this section we introduce the variational inequality for the p-Laplace operator associated with (1.1), and the precise geometry and notations used throughout Sections 3–6 for $p \in [2, n)$. Each section or subsection contains different relations between parameters. We extend notations and the geometry of the problem in Section 7 for $p = n$.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$ and $Y = (-1/2, 1/2)^n$. Let ε be a small positive parameter that we shall make converge towards zero. We set $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$ where ρ denotes the distance.

We denote by G_0 the ball of radius 1 centered at the origin of coordinates. Let ω_n be the area of the unit sphere in $\mathbb{R}^n$, that is, $\omega_n = |\partial G_0|$. For a domain B and for $\delta > 0$, we define the sets $\delta B = \{x \,|\, \delta^{-1}x \in B\}$. We set

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $a_\varepsilon \ll \varepsilon$, and $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$; $\mathbb{Z}^n$ is the set of vectors z with integer coordinates (see Figure 1). Note that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ with $d > 0$. We define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$ where $j \in \Upsilon_\varepsilon$.

In what follows, we set

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ S_\varepsilon = \partial G_\varepsilon,\ \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Also we consider the space $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ ($W^{1,p}(\Omega, \partial\Omega)$, respect.) to be the completion with respect to $W^{1,p}(\Omega_\varepsilon)$-norm ($W^{1,p}(\Omega)$-norm, respect.) of the set of infinitely differentiable functions in $\overline{\Omega}_\varepsilon$ ($\overline{\Omega}$, respect.), vanishing in a neighborhood of $\partial\Omega$. For a function u in $W^{1,p}(\Omega)$, u^+ and u^- denote $u^+ = \sup(u(x), 0)$ and $u^- = u - u^+$ respectively.

Let us consider $\sigma(x, u)$ a continuously differentiable function of variables $(x, u) \in \overline{\Omega} \times \mathbb{R}$ satisfying:

$$\sigma(x, 0) = 0, \tag{2.1}$$

$$(\sigma(x, u) - \sigma(x, v))(u - v) \geq k_1 |u - v|^p \tag{2.2}$$

and

$$|\sigma(x,u)| \le k_2[|u|^{p-1} + |u|^{\delta}] \tag{2.3}$$

for all $x \in \overline{\Omega}$, $u, v \in \mathbb{R}$, and certain constants $k_1 > 0$, $k_2 > 0$ and $\delta \in [p-1, (p-1)n/(n-p)]$ if $p \in [2, n)$, and $\delta \in [p-1, \infty)$ if $p = n$. Note that (2.1)–(2.2) imply

$$\sigma(x,u) \ge 0 \text{ if } u \ge 0 \quad \text{and} \quad \sigma(x,u) \le 0 \text{ if } u \le 0, \quad \forall x \in \overline{\Omega}. \tag{2.4}$$

The variational formulation of problem (1.1) is: find $u_\varepsilon \in K_\varepsilon$ satisfying

$$\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla(\psi - u_\varepsilon)\, dx + \beta_\varepsilon \int\limits_{S_\varepsilon} \sigma(x,u_\varepsilon)(\psi - u_\varepsilon)\, ds \ge \int\limits_{\Omega_\varepsilon} f(\psi - u_\varepsilon)\, dx, \quad \forall \psi \in K_\varepsilon, \tag{2.5}$$

where the set K_ε is defined by

$$K_\varepsilon = \{g \in W^{1,p}(\Omega_\varepsilon, \partial\Omega) : g \ge 0 \text{ a.e. on } S_\varepsilon\}. \tag{2.6}$$

For $p \in [2, n)$ we set the values

$$a_\varepsilon = C_0 \varepsilon^{\alpha}, \quad \text{with } \alpha > 1 \text{ and } C_0 > 0, \quad \text{and} \quad \beta_\varepsilon = \varepsilon^{-\gamma}, \quad \text{with } \gamma \in \mathbb{R}. \tag{2.7}$$

We have the following result:

Theorem 2.1. *Let $\varepsilon > 0$, $f \in L^q(\Omega)$ with $q = p/(p-1)$, $p \in [2, n)$, and a_ε and β_ε given by (2.7). For fixed ε, problem (2.5)–(2.6) has a unique solution $u_\varepsilon \in K_\varepsilon$ which also satisfies the inequality*

$$\int\limits_{\Omega_\varepsilon} |\nabla \psi|^{p-2} \nabla \psi \nabla(\psi - u_\varepsilon)\, dx + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} \sigma(x,\psi)(\psi - u_\varepsilon)\, ds \ge \int\limits_{\Omega_\varepsilon} f(\psi - u_\varepsilon)\, dx, \quad \forall \psi \in K_\varepsilon. \tag{2.8}$$

In addition, for u_ε the solution of (2.5)–(2.6), there exists $\mathcal{P}_\varepsilon u_\varepsilon$ an extension of u_ε to Ω, $\mathcal{P}_\varepsilon u_\varepsilon \in W^{1,p}(\Omega, \partial\Omega)$ with the following properties

$$\|\mathcal{P}_\varepsilon u_\varepsilon\|_{W^{1,p}(\Omega)} \le K \|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)}, \qquad \|\nabla \mathcal{P}_\varepsilon u_\varepsilon\|_{L^p(\Omega)} \le K \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}, \tag{2.9}$$

and

$$\|\mathcal{P}_\varepsilon u_\varepsilon\|^p_{W^{1,p}(\Omega)} + \varepsilon^{-\gamma} \|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \le K \|f\|^q_{L^q(\Omega_\varepsilon)}. \tag{2.10}$$

In all the estimates above, $K > 0$ denotes a constant independent of ε.

Proof. First, we show that the integral on the boundary $\int_{S_\varepsilon} \sigma(x,u_\varepsilon)(\psi - u_\varepsilon)\, ds$ is well defined for $\psi \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. To do this, we take into account (2.3), the Hölder inequality and the continuous embedding of $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ into $L^r(S_\varepsilon)$ for $p \le r \le p(n-1)/(n-p)$, and we can write

$$\begin{aligned} \left| \int\limits_{S_\varepsilon} \sigma(x,u_\varepsilon)(\psi - u_\varepsilon)\, ds \right| &\le C_\varepsilon [\|u_\varepsilon\|^{p-1}_{L^p(S_\varepsilon)} \|\psi - u_\varepsilon\|_{L^p(S_\varepsilon)} + \|u_\varepsilon\|^{\delta}_{L^{\delta r'}(S_\varepsilon)} \|\psi - u_\varepsilon\|_{L^r(S_\varepsilon)}] \\ &\le C_\varepsilon [\|u_\varepsilon\|^{p-1}_{W^{1,p}(\Omega_\varepsilon)} + \|u_\varepsilon\|^{\delta}_{W^{1,p}(\Omega_\varepsilon)}] \|\psi - u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} \end{aligned} \tag{2.11}$$

where $r = p(n-1)/(n-p)$ and $r' = p(n-1)/n(p-1)$.

The existence and uniqueness of the solution $u_\varepsilon \in K_\varepsilon$ of problem (2.5)–(2.6) follows from the monotonicity of the function $|\lambda|^{p-2}\lambda$ (cf. (2.18)) and from the monotonicity of the function $\sigma(x,u)$ with respect to u (cf.

(2.2)): see, e.g., Section II.8.2 in [30] and Section III.1 in [27]. Moreover, applying Minty Lemma (see, e.g., Theorem 8.4 in Section II.8.2 of [30]), the integral inequality (2.5) for u_ε amounts to (2.8).

The existence of a function $\mathcal{P}_\varepsilon u_\varepsilon \in W^{1,p}(\Omega, \partial\Omega)$ which extends u_ε to Ω and satisfies properties (2.9) is a consequence of Lemma 2.7 (see below).

Let us show estimate (2.10). Setting $\psi \equiv 0$ in (2.5) and $v \equiv 0$ in (2.2), we have

$$\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq \|f\|_{L^q(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}.$$

Then, from the Poincaré inequality for the elements $W^{1,p}(\Omega, \partial\Omega)$ and (2.9), we obtain the estimates

$$\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} \leq K\|f\|^q_{L^q(\Omega_\varepsilon)}, \; \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq K\|f\|^q_{L^q(\Omega_\varepsilon)},$$

$$\|u_\varepsilon\|^p_{W^{1,p}(\Omega_\varepsilon)} \leq K\|f\|^q_{L^q(\Omega_\varepsilon)}, \; \|\mathcal{P}_\varepsilon u_\varepsilon\|^p_{W^{1,p}(\Omega)} \leq K\|f\|^q_{L^q(\Omega)}.$$

Therefore, (2.10) follows and the estimates above conclude the proof of the theorem. □

Considering (2.10), there is a subsequence (still denoted by ε) such that, as $\varepsilon \to 0$,

$$\mathcal{P}_\varepsilon u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega, \partial\Omega) - \text{weak} \quad \text{and} \quad \mathcal{P}_\varepsilon u_\varepsilon \to u \text{ in } L^p(\Omega), \tag{2.12}$$

for a certain function u which, once identified, provides the convergences (2.12) for the whole sequence of ε. Note that such an extension provides a bound of the Poincaré constant independent of ε.

Throughout Sections 3–6, we show that this homogenized function u is the unique solution of a homogenized problem which depends on the relation between the parameters α, γ, p and n. That is, depending on the dimension of the space, the value of p, and the different relations between the ε-dependent parameters (the radius of the cavities $O(\varepsilon^\alpha)$ and the adsorption parameter $O(\varepsilon^{-\gamma})$), we have very different limit behaviors for the solution of problem (1.1). For fixed $n \geq 3$ and $p \in [2, n)$, Figure 2 shows a graph of γ versus α in such a way that for each $\alpha \in (1, n/(n-p)]$, the values of γ above, below or equal to $\alpha(n-1) - n$ provide different homogenized problems. In the case where $\alpha > n/(n-p)$, the size of the cavities is very small and the solution u_ε ignores asymptotically their influence. In addition, depending on the relations between the parameters α, γ, p and n, we also construct different correctors which provide estimates for convergence rates of solutions (cf. Figure 3).

2.1. Preliminary results

In this section, we introduce results which we shall use throughout Sections 3–6. We provide either precise references for their proof or a detailed proof in the Appendix. First, we introduce a function, related to the solution of the microscopic problem, which allows us to construct the test functions to pass to the limit in (2.8), as $\varepsilon \to 0$. Also, we obtain certain estimates that we need for proofs in Sections 3–6. Here and in what follows, K denotes a constant independent of ε.

Let us denote by P^j_ε the center of the ball G^j_ε, $j \in \Upsilon_\varepsilon$. We denote by $T^j_{\varepsilon/4}$ the ball of radius $\varepsilon/4$ with center P^j_ε. Let w^j_ε be the solution of the following problem

$$\begin{cases} \Delta_p w^j_\varepsilon = 0 & \text{in } T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ w^j_\varepsilon = 1 & \text{on } \partial G^j_\varepsilon, \\ w^j_\varepsilon = 0 & \text{on } \partial T^j_{\varepsilon/4}. \end{cases} \tag{2.13}$$

It can be easily verified that for $p \in [2, n)$ we have

$$w^j_\varepsilon(x) = \frac{|x - P^j_\varepsilon|^{(p-n)/(p-1)} - \left(\frac{\varepsilon}{4}\right)^{(p-n)/(p-1)}}{a_\varepsilon^{(p-n)/(p-1)} - \left(\frac{\varepsilon}{4}\right)^{(p-n)/(p-1)}}. \tag{2.14}$$

We define the function $W_\varepsilon \in W^{1,p}(\Omega, \partial\Omega)$ by setting

$$W_\varepsilon(x) = w_\varepsilon^j(x), \qquad x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \tag{2.15}$$

extended by 1 inside G^j_ε, $j \in \Upsilon_\varepsilon$, and by 0 in $\mathbb{R}^n \setminus \bigcup_{j\in\Upsilon_\varepsilon} \overline{T^j_{\varepsilon/4}}$. Thus, we compute

$$\|\nabla W_\varepsilon\|^p_{L^p(\Omega)} \leq K\varepsilon^{\alpha(n-p)-n} \tag{2.16}$$

and, consequently, as $\varepsilon \to 0$, we conclude that

$$\begin{array}{ll} W_\varepsilon \rightharpoonup 0 \text{ in } W^{1,p}(\Omega) - \text{weak} & \text{if } \alpha = n/(n-p), \\ W_\varepsilon \to 0 \text{ in } W^{1,p}(\Omega) & \text{if } \alpha > n/(n-p). \end{array} \tag{2.17}$$

Next, it will prove useful to introduce a well-known result on the monotonicity of the function $|\lambda|^{p-2}\lambda$ with respect to $\lambda \in \mathbb{R}^n$ for $p \geq 2$: there exists a constant $k_3 > 0$ such that

$$(\,|\lambda_1|^{p-2}\lambda_1 - |\lambda_2|^{p-2}\lambda_2\,)(\,\lambda_1 - \lambda_2\,) \geq k_3\,|\lambda_1 - \lambda_2|^p, \qquad \forall \lambda_1, \lambda_2 \in \mathbb{R}^n, \tag{2.18}$$

(cf., e.g., [6]). Note that here and throughout the paper we write $\lambda_1\lambda_2$ as the scalar product in $\mathbb{R}^n$.

Using this result, we introduce a proposition which provides existence and uniqueness of solution of the functional equation arising in the homogenized problem (1.2): see the Appendix for its proof. Also, see for example [20] and references therein for different functional equations when $p = 2$.

Proposition 2.2. *Let p be $p \geq 2$. Let ϱ be a strictly positive constant and let σ be the function $\sigma(x, u)$ defined from $\overline{\Omega} \times \mathbb{R}$ into $\mathbb{R}$ which is assumed to be a continuously differentiable function in $\overline{\Omega} \times \mathbb{R}$ satisfying* (2.1)–(2.2). *Then, the equation*

$$|H|^{p-2}H = \varrho\,\sigma(x, \tau - H) \tag{2.19}$$

has a unique solution $H(x, \tau)$ which is a continuously differentiable function in $\overline{\Omega} \times (\mathbb{R} \setminus \{0\})$ and continuous in $\overline{\Omega} \times \mathbb{R}$, and satisfies $H(x, 0) = 0$ and

$$(|H(x, u)|^{p-2}H(x, u) - |H(x, v)|^{p-2}H(x, v))(u - v) \geq \widetilde{k}_1|u - v|^p, \tag{2.20}$$

$$|H(x, u)| \leq |u|, \tag{2.21}$$

for all $x \in \overline{\Omega}$, $u, v \in \mathbb{R}$ and a certain constant $\widetilde{k}_1 > 0$. Consequently,

$$H(x, u) \geq 0 \text{ if } u \geq 0, \qquad H(x, u) \leq 0 \text{ if } u \leq 0, \quad \forall x \in \overline{\Omega}. \tag{2.22}$$

The following result simplifies the computations throughout the paper: see the Appendix for its proof.

Proposition 2.3. *Let $p > 2$. Let $v \in W^{1,\infty}(\Omega)$, $\varphi \in W^{1,p}(\Omega, \partial\Omega)$ and $\eta_\varepsilon \in W^{1,p}(\Omega, \partial\Omega)$ such that $\|\nabla\eta_\varepsilon\|_{L^m(\Omega)}$ tends to 0, as $\varepsilon \to 0$, for $m \in [1, p)$. Then,*

$$\int_{\Omega_\varepsilon} \Big(|\nabla(v + \eta_\varepsilon)|^{p-2}\nabla(v + \eta_\varepsilon) - |\nabla v|^{p-2}\nabla v\Big)\nabla\varphi\,dx = \int_{\Omega_\varepsilon} |\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\nabla\varphi\,dx + R_\varepsilon, \tag{2.23}$$

where $|R_\varepsilon| \to 0$ as $\varepsilon \to 0$ and

$$|R_\varepsilon| \leq K[\|\nabla\eta_\varepsilon\|^{p-2}_{L^{(p-2)p/(p-1)}(\Omega)} + \|\nabla\eta_\varepsilon\|_{L^{p/(p-1)}(\Omega)}]\|\nabla\varphi\|_{L^p(\Omega)}\,. \tag{2.24}$$

Moreover, if $\|\nabla\eta_\varepsilon\|_{L^p(\Omega)} \to 0$*, as* $\varepsilon \to 0$*, then*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\Big(|\nabla(v+\eta_\varepsilon)|^{p-2}\nabla(v+\eta_\varepsilon) - |\nabla v|^{p-2}\nabla v\Big)\nabla\varphi\,dx = 0. \tag{2.25}$$

In addition, (2.23)–(2.25) *also hold in the case where* φ *depends on* ε*, namely* $\varphi \equiv \varphi_\varepsilon$*, with* $\|\nabla\varphi_\varepsilon\|_{L^p(\Omega)}$ *bounded independently of* ε*.*

Finally, we introduce Lemmas 2.4–2.8 which we need for the proofs throughout Sections 3–6. Applying the technique in Lemmas 1 and 2 in [33], and Lemma 3 in [34] for $p = 2$ we obtain Lemmas 2.4, 2.5 and 2.6 respectively (see also [40] in this connection). See Theorem 1 of [39] and references therein for the proof of Lemma 2.7 (cf. also in this connection [1] and [38], and [10] and [33] when $p = 2$). We refer to Lemma 1 in [44] for the proof of Lemma 2.8. In these lemmas, the constant K does not depend on ε nor on the functions φ appearing in their statements.

Lemma 2.4. *Let* $\widetilde{Y}_\varepsilon = \varepsilon(-1/2,1/2)^n \setminus a_\varepsilon\overline{G}_0$ *where* G_0 *is the ball of radius* 1 *with center the origin of coordinates and* a_ε *is a positive constant such that* $a_\varepsilon\overline{G}_0 \subset \varepsilon(-1/2,1/2)^n$*. If* $\varphi \in W^{1,p}(\widetilde{Y}_\varepsilon)$ *and* $\int_{\widetilde{Y}_\varepsilon} w\,dx = 0$*,* $2 \le p < n$*, then*

$$\|\varphi\|_{L^p(\widetilde{Y}_\varepsilon)} \le K\varepsilon\|\nabla\varphi\|_{L^p(\widetilde{Y}_\varepsilon)}.$$

Lemma 2.5. *Let* $\widetilde{Y}_\varepsilon$ *be the domain defined in Lemma* 2.4*. Let* $\varphi \in W^{1,p}(\widetilde{Y}_\varepsilon)$*,* $2 \le p < n$*. Then,*

$$\|\varphi\|^p_{L^p(a_\varepsilon\partial G_0)} \le K[a_\varepsilon^{n-1}\varepsilon^{-n}\|\varphi\|^p_{L^p(\widetilde{Y}_\varepsilon)} + a_\varepsilon^{p-1}\|\nabla\varphi\|^p_{L^p(\widetilde{Y}_\varepsilon)}].$$

Lemma 2.6. *Let* $\widetilde{Y}_\varepsilon$ *be the domain defined in Lemma* 2.4 *and let* $\widetilde{\widetilde{Y}}_\varepsilon$ *denote the domain* $2\varepsilon(-1/2,1/2)^n \setminus a_\varepsilon\overline{G}_0$*. Let* $\varphi \in W^{1,p}(\widetilde{Y}_\varepsilon)$*,* $2 \le p < n$*. Then,*

$$\|\varphi\|^p_{L^p(\widetilde{Y}_\varepsilon)} \le K[a_\varepsilon^{1-n}\varepsilon^n\|\varphi\|^p_{L^p(a_\varepsilon\partial G_0)} + a_\varepsilon^{p-n}\varepsilon^n\|\nabla\varphi\|^p_{L^p(\widetilde{\widetilde{Y}}_\varepsilon)}].$$

Lemma 2.7. *Let* $p > 1$*. There exists an operator* $\mathcal{P}_\varepsilon$ *from* $W^{1,p}(\Omega_\varepsilon,\partial\Omega)$ *into* $W^{1,p}(\Omega,\partial\Omega)$*, such that for any* $\varphi \in W^{1,p}(\Omega_\varepsilon,\partial\Omega)$*,*

$$\|\mathcal{P}_\varepsilon\varphi\|_{W^{1,p}(\Omega)} \le K\|\varphi\|_{W^{1,p}(\Omega_\varepsilon)} \quad \text{and} \quad \|\nabla\mathcal{P}_\varepsilon\varphi\|_{L^p(\Omega)} \le K\|\nabla\varphi\|_{L^p(\Omega_\varepsilon)}. \tag{2.26}$$

Lemma 2.8. *Let* $h_\varepsilon \in H^1_0(\Omega)$ *and* $h_\varepsilon \rightharpoonup h_0$ *in* $H^1(\Omega)$*-weak as* $\varepsilon \to 0$*. Let* $T^j_{\varepsilon/4}$ *be the ball of radius* $\varepsilon/4$ *with center* P^j_ε*. Then, as* $\varepsilon \to 0$*,*

$$\sum_{j\in\Upsilon_\varepsilon} 2^{2n-2}\varepsilon\int_{\partial T^j_{\varepsilon/4}} h_\varepsilon\,ds \to \omega_n\int_\Omega h_0\,dx,$$

where ω_n *is the area of the unit sphere in* $\mathbb{R}^n$*.*

3. The most critical case for $p \in [2,n)$: $\alpha = \frac{n}{n-p}$ and $\gamma = \frac{n(p-1)}{n-p}$

In this case, the homogenized problem is the boundary value problem (1.2). We show that the nonlinear function arising in the strange term is defined through a functional equation (cf. the reaction term in (1.2) and (1.3)). The properties of this function allow us to obtain a corrector: see (3.2) for $u = v$, and the point intersection of all the lines in Figures 2 and 3. The convergence and the corrector results are in Theorem 3.1 and 3.2 respectively.

Theorem 3.1. *Let* $\alpha = n/(n-p)$*,* $\gamma = n(p-1)/(n-p)$ *for* $p \in [2,n)$*, and let* u_ε *be the weak solution of* (1.1)*. Then, the limit function* u *of the extension of* u_ε*, defined by* (2.12)*, is the weak solution of problem* (1.2)*.*

Proof. First, let us note that on account of Proposition 2.2, equation (1.3) has a unique solution and therefore, the function $H(x,u)$ arising in (1.2) is a well defined function satisfying $H(x,0)=0$, (2.20), (2.21) and (2.22) for all $x \in \overline{\Omega}$, $u,v \in \mathbb{R}$ and a certain constant $\widetilde{k}_1 > 0$. The variational formulation of (1.2) reads: find $u \in W^{1,p}(\Omega,\partial\Omega)$ such that

$$\int\limits_{\Omega} |\nabla u|^{p-2}\nabla u \nabla\phi\, dx + \mathcal{A}_{n,p}\int\limits_{\Omega}\Big(|H(x,u^+)|^{p-2}H(x,u^+) + |u^-|^{p-2}u^-\Big)\phi\, dx = \int\limits_{\Omega} f\phi\, dx, \qquad \forall \phi \in W^{1,p}(\Omega,\partial\Omega). \tag{3.1}$$

From the monotonicity of the function $|\lambda|^{p-2}\lambda$ (see (2.18)) and of the function $|H(x,z)|^{p-2}H(x,z)$ with respect to z (see (2.20) and (2.22)), $p \geq 2$, the existence and uniqueness of solution of (3.1) holds: see, e.g., Section II.8.2 in [30] (cf. also [17], [23] and [40], for related problems).

Let us consider the function

$$\psi = v - W_\varepsilon(H(x,v^+) + v^-), \tag{3.2}$$

where $v \in C_0^\infty(\Omega)$, W_ε is the function defined by (2.15) and $H(x,\tau)$ is the solution of the functional equation (1.3). Let us prove that $\psi \geq 0$ on S_ε, and thus it belongs to K_ε. Suppose that for some point $x_0 \in S_\varepsilon$ we have $\psi(x_0) < 0$. Then, we get $v^+(x_0) - H(x_0, v^+(x_0)) < 0$ and $\sigma(x_0, v^+(x_0) - H(x_0, v^+(x_0))) < 0$. However, $0 \leq \mathcal{B}_{n,p}|H(x_0,v^+(x_0))|^{p-2}H(x_0,v^+(x_0)) = \sigma(x_0, v^+(x_0) - H(x_0,v^+(x_0)))$. Thus, we obtain a contradiction.

We now take ψ defined by (3.2) as a test function in (2.8); since $W_\varepsilon = 1$ in $\overline{G}_\varepsilon$ we obtain

$$\begin{aligned} &\int\limits_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon(H(x,v^+)+v^-))|^{p-2}\nabla(v - W_\varepsilon(H(x,v^+)+v^-))\nabla(v - W_\varepsilon(H(x,v^+)+v^-) - u_\varepsilon)\, dx \\ &+ \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x, v^+ - H(x,v^+))(v^+ - H(x,v^+) - u_\varepsilon)\, ds \geq \int\limits_{\Omega_\varepsilon} f(v - W_\varepsilon(H(x,v^+)+v^-) - u_\varepsilon)\, dx \end{aligned} \tag{3.3}$$

and we pass to the limit when $\varepsilon \to 0$.

We denote by L_ε the left hand side of (3.3). Let us show that

$$\lim_{\varepsilon\to 0} L_\varepsilon \leq \int\limits_{\Omega} |\nabla v|^{p-2}\nabla v\nabla(v-u)\, dx + \mathcal{A}_{n,p}\int\limits_{\Omega}\Big(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-\Big)(v-u)\, dx. \tag{3.4}$$

In order to do that, we take into account that

$$\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^m dx \leq K\varepsilon^{n(p-m)/(n-p)}, \quad \text{for } m \in [1,p], \tag{3.5}$$

which is obtained from formula (2.14). Then, we apply Proposition 2.3 with

$$\eta_\varepsilon \equiv -W_\varepsilon(H(x,v^+)+v^-) \text{ and } \varphi = \varphi_\varepsilon \equiv v - W_\varepsilon(H(x,v^+)+v^-) - \mathcal{P}_\varepsilon u_\varepsilon,$$

where $\mathcal{P}_\varepsilon u_\varepsilon$ is the extension defined in Theorem 2.1. This is possible since on account of (2.10), (3.5), (2.12) and (2.17), we can check that $\|\nabla\varphi_\varepsilon\|_{L^p(\Omega)}$ is bounded independent of ε and

$$\varphi_\varepsilon \equiv v - W_\varepsilon(H(x,v^+)+v^-) - \mathcal{P}_\varepsilon u_\varepsilon \rightharpoonup v - u \text{ in } W^{1,p}(\Omega) - \text{weak as } \varepsilon \to 0. \tag{3.6}$$

Thus, we obtain

$$\lim_{\varepsilon\to 0} L_\varepsilon = \lim_{\varepsilon\to 0}(L_\varepsilon^1 + L_\varepsilon^2 + L_\varepsilon^3) \tag{3.7}$$

where

$$L_\varepsilon^1 \equiv \int\limits_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \nabla (v - W_\varepsilon(H(x,v^+)+v^-) - u_\varepsilon)\,dx,$$

$$L_\varepsilon^2 \equiv -\int\limits_{\Omega_\varepsilon} |\nabla (W_\varepsilon(H(x,v^+)+v^-))|^{p-2}\nabla (W_\varepsilon(H(x,v^+)+v^-))\nabla (v - W_\varepsilon(H(x,v^+)+v^-) - u_\varepsilon)\,dx$$

and

$$L_\varepsilon^3 \equiv \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x, v^+ - H(x,v^+))(v^+ - H(x,v^+) - u_\varepsilon)\,ds.$$

On account of (3.6) and the fact that $|G_\varepsilon| \to 0$, we have

$$\lim_{\varepsilon\to 0} L_\varepsilon^1 = \int\limits_\Omega |\nabla v|^{p-2}\nabla v \nabla (v-u)\,dx. \tag{3.8}$$

We study the limit of $L_\varepsilon^2 + L_\varepsilon^3$ when $\varepsilon \to 0$.

From (3.5), (3.6) and (2.10), it follows

$$\begin{aligned} -\lim_{\varepsilon\to 0} L_\varepsilon^2 &= \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |H(x,v^+)+v^-|^{p-2}|\nabla W_\varepsilon|^{p-2}\nabla (W_\varepsilon(H(x,v^+)+v^-))\nabla \varphi_\varepsilon dx \\ &= \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon \nabla\Big(|H(x,v^+)+v^-|^{p-2}(H(x,v^+)+v^-)\,\varphi_\varepsilon\Big)dx. \end{aligned} \tag{3.9}$$

Moreover, by the properties of $H(x,z)$, we have $H(x,v^+)v^- = 0$ and, hence,

$$-\lim_{\varepsilon\to 0} L_\varepsilon^2 = \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon \nabla\Big((|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-)\,\varphi_\varepsilon\Big)dx.$$

Thus, using the definition of W_ε and the Green formula, we get

$$-\lim_{\varepsilon\to 0} L_\varepsilon^2 = \lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}\cup\partial G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon\,(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-)\varphi_\varepsilon\,ds. \tag{3.10}$$

In order to compute (3.10), we use the explicit form of the normal derivatives of the auxiliary functions w^j_ε given by

$$|\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon\Big|_{\partial G^j_\varepsilon} = \varepsilon^{-\frac{n\cdot(p-1)}{n-p}}\left(\frac{n-p}{p-1}\right)^{p-1}\frac{1}{C_0^{p-1}(1-\alpha_\varepsilon)^{p-1}}, \tag{3.11}$$

$$|\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon\Big|_{\partial T^j_{\varepsilon/4}} = -\varepsilon\Big(\frac{n-p}{p-1}\Big)^{p-1} C_0^{n-p}\frac{2^{2n-2}}{(1-\alpha_\varepsilon)^{p-1}}. \tag{3.12}$$

where $\alpha_\varepsilon = a_\varepsilon^{\frac{n-p}{p-1}}\varepsilon^{\frac{p-n}{p-1}}2^{\frac{2n-2p}{p-1}} = \varepsilon^{(\alpha-1)\frac{n-p}{p-1}}2^{\frac{2n-2p}{p-1}} \to 0$ as $\varepsilon \to 0$.

By the definition of φ_ε and W_ε and the fact that $v^-(v^+ - H(x,v^+)) = 0$, $u_\varepsilon \geq 0$ on ∂G_ε^j and (3.11), we obtain

$$- \lim_{\varepsilon\to 0} L_\varepsilon^2 \geq \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j \, |H(x,v^+)|^{p-2}H(x,v^+)(v^+ - H(x,v^+) - u_\varepsilon)\, ds$$
$$+ \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} |\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j \, (|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-)(v - u_\varepsilon)\, ds.$$

In addition, from (3.11)–(3.12), it follows

$$\begin{aligned} - \lim_{\varepsilon\to 0} L_\varepsilon^2 \geq& \lim_{\varepsilon\to 0} \frac{\varepsilon^{-\gamma}\mathcal{B}_{n,p}}{(1-\alpha_\varepsilon)^{p-1}} \int_{S_\varepsilon} |H(x,v^+)|^{p-2}H(x,v^+)(v^+ - H(x,v^+) - u_\varepsilon)\, ds \\ &- \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} \frac{\varepsilon\, 2^{2n-2}\mathcal{A}_{n,p}}{\omega_n(1-\alpha_\varepsilon)^{p-1}} \Big(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-\Big)(v - u_\varepsilon)\, ds. \end{aligned} \tag{3.13}$$

Now, taking into account that H is the solution of the equation (1.3) and using the Hölder inequality, (2.10) and the size of S_ε we get

$$\Big| L_\varepsilon^3 - \frac{\varepsilon^{-\gamma}\mathcal{B}_{n,p}}{(1-\alpha_\varepsilon)^{p-1}} \int_{S_\varepsilon} |H(x,v^+)|^{p-2}H(x,v^+)(v^+ - H(x,v^+) - u_\varepsilon)\, ds \Big|$$
$$\leq K\alpha_\varepsilon \varepsilon^{-\gamma} \int_{S_\varepsilon} |v^+ - H(x,v^+) - u_\varepsilon|\, ds \leq K\alpha_\varepsilon\varepsilon^{-\gamma}[|S_\varepsilon| + |S_\varepsilon|^{(p-1)/p}\|u_\varepsilon\|_{L^p(S_\varepsilon)}] \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, on account of (3.6), we apply Lemma 2.8 and have

$$\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\varepsilon/4}^j} \frac{\varepsilon\, 2^{2n-2}\mathcal{A}_{n,p}}{\omega_n(1-\alpha_\varepsilon)^{p-1}} \Big(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-\Big)(v - u_\varepsilon) ds$$
$$= \mathcal{A}_{n,p} \int_\Omega \Big(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-\Big)(v - u)\, dx,$$

and, consequently,

$$\lim_{\varepsilon\to 0} (L_\varepsilon^2 + L_\varepsilon^3) \leq \mathcal{A}_{n,p} \int_\Omega \Big(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-\Big)(v - u)\, dx. \tag{3.14}$$

Now, gathering (3.7), (3.8) and (3.14) yields (3.4).

Finally, we use (3.4) and (3.6) to pass to the limit in (3.3), as $\varepsilon \to 0$, and obtain that the limit function u satisfies the following inequality

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v-u)\, dx + \mathcal{A}_{n,p} \int_\Omega \Big(|H(x,v^+)|^{p-2}H(x,v^+) + |v^-|^{p-2}v^-\Big)(v-u)\, dx \geq \int_\Omega f(v-u)\, dx, \tag{3.15}$$

for all $v \in W^{1,p}(\Omega, \partial\Omega)$. As usual, taking $v = u \pm \lambda\phi$ in (3.15) where $\phi \in W^{1,p}(\Omega,\partial\Omega)$ and passing to the limit as $\lambda \to +0$, we obtain that u satisfies the integral identity (3.1), which concludes the proof. □

Theorem 3.2. *Let $\alpha = n/(n-p)$, $\gamma = n(p-1)/(n-p)$ and $p \in [2, n)$. Let u_ε be the weak solution of (1.1), $u \in W^{1,p}(\Omega, \partial\Omega)$ the weak solution of the boundary value problem (1.2) with the additional regularity $u \in W^{1,\infty}(\Omega)$, and W_ε defined by (2.15). Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u + W_\varepsilon(H(x,u^+) + u^-)\|^p_{W^{1,p}(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon - u^+ + H(x,u^+)\|^p_{L^p(S_\varepsilon)} \to 0. \tag{3.16}$$

Proof. Let us consider problems (2.5) and (3.1) and take as test functions $\psi = u - W_\varepsilon(H(x,u^+) + u^-)$ and $\phi = u - W_\varepsilon(H(x,u^+) + u^-) - \mathcal{P}_\varepsilon u_\varepsilon$, respectively, for $\mathcal{P}_\varepsilon u_\varepsilon$ arising in (2.12). Subtracting both expressions and taking into account the definition of W_ε on $\overline{G}_\varepsilon$, we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon)\,dx - \varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma(x,u_\varepsilon)(u^+ - H(x,u^+) - u_\varepsilon)\,ds \\
&\leq -\mathcal{A}_{n,p}\int_\Omega (|H(x,u^+)|^{p-2}H(x,u^+) + |u^-|^{p-2}u^-)(u - W_\varepsilon(H(x,u^+) + u^-) - \mathcal{P}_\varepsilon u_\varepsilon)dx \\
&\quad + \int_{G_\varepsilon} f(u^+ - H(x,u^+) - \mathcal{P}_\varepsilon u_\varepsilon)\,dx - \int_{G_\varepsilon} |\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon(H(x,u^+) + u^-) - \mathcal{P}_\varepsilon u_\varepsilon)\,dx.
\end{aligned} \tag{3.17}$$

Let us denote by A^1_ε the first integral on the left hand side of (3.17) and by $\widehat{H}$ the function $H(x,u^+) + u^-$. Then, we can rewrite it in the following way

$$\begin{aligned}
A^1_\varepsilon = &\int_{\Omega_\varepsilon} \Big(|\nabla(u - W_\varepsilon\widehat{H})|^{p-2}\nabla(u - W_\varepsilon\widehat{H}) - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\Big)\nabla(u - W_\varepsilon\widehat{H} - u_\varepsilon)\,dx \\
&+ \int_{\Omega_\varepsilon} \Big(|\nabla u|^{p-2}\nabla u - |\nabla(u - W_\varepsilon\widehat{H})|^{p-2}\nabla(u - W_\varepsilon\widehat{H})\Big)\nabla(u - W_\varepsilon\widehat{H} - u_\varepsilon)\,dx.
\end{aligned} \tag{3.18}$$

Using the monotonicity of the functions $\sigma(x,u)$ and $|\lambda|^{p-2}\lambda$ (see (2.2) and (2.18)), from (3.18) and (3.17), we deduce

$$\begin{aligned}
&K(\|\nabla(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u^+ - H(x,u^+) - u_\varepsilon\|^p_{L^p(S_\varepsilon)}) \\
&\leq \int_{\Omega_\varepsilon} \Big(|\nabla(u - W_\varepsilon\widehat{H})|^{p-2}\nabla(u - W_\varepsilon\widehat{H}) - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\Big)\nabla(u - W_\varepsilon\widehat{H} - u_\varepsilon)\,dx \\
&\quad + \varepsilon^{-\gamma}\int_{S_\varepsilon} (\sigma(x,u^+ - H(x,u^+)) - \sigma(x,u_\varepsilon))(u^+ - H(x,u^+) - u_\varepsilon)\,ds \leq I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon
\end{aligned} \tag{3.19}$$

where

$$I^1_\varepsilon \equiv \int_{\Omega_\varepsilon} \Big(|\nabla(u - W_\varepsilon\widehat{H})|^{p-2}\nabla(u - W_\varepsilon\widehat{H}) - |\nabla u|^{p-2}\nabla u\Big)\nabla(u - W_\varepsilon\widehat{H} - u_\varepsilon)\,dx,$$

$$\begin{aligned}
I^2_\varepsilon \equiv &-\mathcal{A}_{n,p}\int_\Omega (|H(x,u^+)|^{p-2}H(x,u^+) + |u^-|^{p-2}u^-)(u - W_\varepsilon\widehat{H} - \mathcal{P}_\varepsilon u_\varepsilon)\,dx \\
&+ \varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma(x,u^+ - H(x,u^+))(u^+ - H(x,u^+) - u_\varepsilon)\,ds,
\end{aligned}$$

and

$$I_\varepsilon^3 \equiv \int\limits_{G_\varepsilon} f(u^+ - H(x,u^+) - \mathcal{P}_\varepsilon u_\varepsilon)\, dx - \int\limits_{G_\varepsilon} |\nabla u|^{p-2}\nabla u \nabla(u - W_\varepsilon \widehat{H} - \mathcal{P}_\varepsilon u_\varepsilon)\, dx.$$

We study the limit of $I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3$ when $\varepsilon \to 0$.

From (2.10) and (3.5), we apply Proposition 2.3 with $\eta_\varepsilon \equiv -W_\varepsilon \widehat{H}$ and $\varphi = \varphi_\varepsilon \equiv u - W_\varepsilon \widehat{H} - \mathcal{P}_\varepsilon u_\varepsilon$, and have that

$$\lim_{\varepsilon\to 0} I_\varepsilon^1 = -\lim_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon} |\nabla(W_\varepsilon \widehat{H})|^{p-2}\nabla(W_\varepsilon \widehat{H})\nabla(u - W_\varepsilon \widehat{H} - u_\varepsilon)\, dx.$$

Moreover, rewriting the computations (3.9)–(3.13) with minor modifications, we have

$$\lim_{\varepsilon\to 0} I_\varepsilon^1 \le -\lim_{\varepsilon\to 0} \frac{\varepsilon^{-\gamma}\mathcal{B}_{n,p}}{(1-\alpha_\varepsilon)^{p-1}} \int\limits_{S_\varepsilon} |H(x,u^+)|^{p-2} H(x,u^+)(u^+ - H(x,u^+) - u_\varepsilon) ds$$
$$+ \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} \frac{\varepsilon\, 2^{2n-2}\mathcal{A}_{n,p}}{\omega_n (1-\alpha_\varepsilon)^{p-1}} \Big\{ |H(x,u^+)|^{p-2}H(x,u^+) + |u^-|^{p-2}u^- \Big\} (u - u_\varepsilon) ds.$$

Thus, using the definition of H, (2.10), Lemma 2.8, (2.12) and (2.17), we obtain

$$\lim_{\varepsilon\to 0} (I_\varepsilon^1 + I_\varepsilon^2) \le 0 \tag{3.20}$$

(see the reasoning for the proof of (3.14)). Besides, since $u - W_\varepsilon \widehat{H} - \mathcal{P}_\varepsilon u_\varepsilon$ is bounded in $W^{1,p}(\Omega)$ and $|G_\varepsilon| \to 0$ as $\varepsilon \to 0$, we derive

$$\lim_{\varepsilon\to 0} I_\varepsilon^3 = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} (I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3) \le 0. \tag{3.21}$$

Finally, gathering (3.19), (3.20) and (3.21), we obtain, as $\varepsilon \to 0$,

$$\|\nabla(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u^+ - H(x,u^+) - u_\varepsilon\|^p_{L^p(S_\varepsilon)} \to 0. \tag{3.22}$$

To get (3.16) from (3.22), we consider the Poincaré inequality for the $W^{1,p}$–extension of $u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon$ in Lemma 2.7, namely for $\mathcal{P}_\varepsilon(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon) \in W^{1,p}(\Omega, \partial\Omega)$, which satisfies

$$\|\nabla(\mathcal{P}_\varepsilon(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon))\|^p_{L^p(\Omega)} \le K \|\nabla(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)},$$

and consequently, we have

$$\|u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} \le K \|\nabla(u - W_\varepsilon(H(x,u^+) + u^-) - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)}$$

and (3.16) also holds. Thus, Theorem 3.2 is proved. □

4. Critical size for perforations when $p \in [2,n)$ and $\gamma \neq \frac{n(p-1)}{n-p}$

When $\alpha = n/(n-p)$ and $\gamma \neq n(p-1)/(n-p)$, we show that the homogenized problem does not depend on σ although its properties are somewhat present in the homogenization process. For a very small (large, respectively) adsorption the asymptotic behavior of the solution of (1.1) is the same as if Signorini (Dirichlet, respectively) conditions had been imposed on the boundary of the cavities (cf. [11] when $p = 2$, and [28], respectively). Correctors are given by $W_\varepsilon u^-$ and $W_\varepsilon u$ depending on whether we have small or large adsorption (see line $\alpha = n/(n-p)$ in Figures 2 and 3). The results for small adsorption are in Section 4.1 whereas those for large adsorption are in Section 4.2.

4.1. The case $\alpha = \frac{n}{n-p}$ and $\gamma < \frac{n(p-1)}{n-p}$

Theorem 4.1. *Let $\alpha = n/(n-p)$, $\gamma < n(p-1)/(n-p)$, $p \in [2, n)$, and let u_ε be the weak solution of* (1.1). *Then, the limit function u of the extension of u_ε, defined by* (2.12)*, is the weak solution of problem* (1.4).

Proof. The variational formulation of (1.4) reads: find $u \in W^{1,p}(\Omega, \partial\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \, dx + \mathcal{A}_{n,p}\int_\Omega |u^-|^{p-2}u^-\phi \, dx = \int_\Omega f\phi \, dx, \qquad \forall \phi \in W^{1,p}(\Omega, \partial\Omega). \tag{4.1}$$

From the monotonicity of the function $|\lambda|^{p-2}\lambda$, the existence and uniqueness of solution of (4.1) holds (cf., e.g. Section II.8.2 in [30]).

Let us take in (2.8) the test function $\psi = v - W_\varepsilon v^- \in K_\varepsilon$ where $v \in C_0^\infty(\Omega)$ and W_ε is the function defined by (2.15). Since $W_\varepsilon = 1$ in $\overline{G}_\varepsilon$ we obtain

$$\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v^-)|^{p-2}\nabla(v - W_\varepsilon v^-)\nabla(v - W_\varepsilon v^- - u_\varepsilon)\, dx + \varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma(x, v^+)(v^+ - u_\varepsilon)\, ds \ge \int_{\Omega_\varepsilon} f(v - W_\varepsilon v^- - u_\varepsilon)\, dx \tag{4.2}$$

and we pass to the limit when $\varepsilon \to 0$.

Using (2.3) and (2.10) and computing $|S_\varepsilon|$, it follows

$$\left|\varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma(x, v^+)(v^+ - u_\varepsilon)\, ds\right| \le K\varepsilon^{-\gamma}[|S_\varepsilon| + |S_\varepsilon|^{(p-1)/p}\|u_\varepsilon\|_{L^p(S_\varepsilon)}] \le K[\varepsilon^{\alpha(n-1)-n-\gamma} + \varepsilon^{(\alpha(n-1)-n-\gamma)(p-1)/p}], \tag{4.3}$$

which converges towards zero as $\varepsilon \to 0$. Moreover, on account of (2.12) and (2.17), we deduce

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(v - W_\varepsilon v^- - u_\varepsilon)\, dx = \int_\Omega f(v - u)\, dx. \tag{4.4}$$

Let us show that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v^-)|^{p-2}\nabla(v - W_\varepsilon v^-)\nabla(v - W_\varepsilon v^- - u_\varepsilon)\, dx \\ &\le \int_\Omega |\nabla v|^{p-2}\nabla v\nabla(v - u)\, dx + \mathcal{A}_{n,p}\int_\Omega |v^-|^{p-2}v^-(v - u)\, dx. \end{aligned} \tag{4.5}$$

On account of (3.5), we apply Proposition 2.3 with $\eta_\varepsilon \equiv -W_\varepsilon v^-$ and $\varphi = \varphi_\varepsilon \equiv v - W_\varepsilon v^- - \mathcal{P}_\varepsilon u_\varepsilon$ since

$$\varphi_\varepsilon \equiv v - W_\varepsilon v^- - \mathcal{P}_\varepsilon u_\varepsilon \rightharpoonup v - u \quad \text{in } W^{1,p}(\Omega) - \text{weak as } \varepsilon \to 0 \tag{4.6}$$

(cf. (3.6) for $H \equiv 0$). Thus, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v^-)|^{p-2}\nabla(v - W_\varepsilon v^-)\nabla(v - W_\varepsilon v^- - u_\varepsilon)\, dx = \lim_{\varepsilon\to 0}(\mathcal{L}_\varepsilon^1 + \mathcal{L}_\varepsilon^2) \tag{4.7}$$

where

$$\mathcal{L}_\varepsilon^1 \equiv \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\nabla(v - W_\varepsilon v^- - u_\varepsilon)\, dx \quad \text{and} \quad \mathcal{L}_\varepsilon^2 \equiv -\int_{\Omega_\varepsilon} |\nabla(W_\varepsilon v^-)|^{p-2}\nabla(W_\varepsilon v^-)\nabla(v - W_\varepsilon v^- - u_\varepsilon)\, dx.$$

By (4.6) and the fact that $|G_\varepsilon| \to 0$, we have

$$\lim_{\varepsilon\to 0} \mathcal{L}^1_\varepsilon = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v-u)\, dx. \tag{4.8}$$

Moreover, using (3.5), (4.6), (2.10), the definition of W_ε and the Green formula, we get

$$\begin{aligned} -\lim_{\varepsilon\to 0} \mathcal{L}^2_\varepsilon &= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |v^-|^{p-2} |\nabla W_\varepsilon|^{p-2} \nabla(W_\varepsilon v^-)\nabla(v - W_\varepsilon v^- - u_\varepsilon)\, dx \\ &= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon \nabla\Big(|v^-|^{p-2} v^- \,(v - W_\varepsilon v^- - u_\varepsilon)\Big)\, dx \\ &= \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}\cup \partial G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon \, |v^-|^{p-2} v^-(v - W_\varepsilon v^- - u_\varepsilon)\, ds. \end{aligned} \tag{4.9}$$

Now, by the definition of W_ε and the fact that $v^- v^+ = 0$, $u_\varepsilon \geq 0$ on ∂G^j_ε and (3.11), we obtain

$$\sum_{j\in\Upsilon_\varepsilon} \int_{\partial G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon \, |v^-|^{p-2} v^-(v - W_\varepsilon v^- - u_\varepsilon)\, ds \geq 0. \tag{4.10}$$

Besides, from (3.12), Lemma 2.8 and (4.6), we have

$$\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon \, |v^-|^{p-2} v^-(v - W_\varepsilon v^- - u_\varepsilon)\, ds = -\mathcal{A}_{n,p} \int_\Omega |v^-|^{p-2} v^-(v-u)\, dx. \tag{4.11}$$

Gathering (4.7), (4.8), (4.9), (4.10) and (4.11) yields (4.5).

Finally, we use (4.3), (4.4) and (4.5) to pass to the limit in (4.2), as $\varepsilon \to 0$, and obtain that the limit function u satisfies the following inequality

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v-u)\, dx + \mathcal{A}_{n,p} \int_\Omega |v^-|^{p-2} v^-(v-u)\, dx \geq \int_\Omega f(v-u) dx, \qquad \forall v \in W^{1,p}(\Omega, \partial\Omega). \tag{4.12}$$

As usual, taking $v = u \pm \lambda\phi$ in (4.12) where $\phi \in W^{1,p}(\Omega, \partial\Omega)$ and passing to the limit as $\lambda \to +0$, we obtain that u satisfies the integral identity (4.1), which concludes the proof. □

Theorem 4.2. *Let $\alpha = n/(n-p)$, $\gamma < n(p-1)/(n-p)$ and $p \in [2, n)$. Let u_ε be the weak solution of (1.1), $u \in W^{1,p}(\Omega, \partial\Omega)$ the weak solution of the boundary value problem (1.4) with the additional regularity $u \in W^{1,\infty}(\Omega)$, and W_ε defined by (2.15). Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u + W_\varepsilon u^-\|^p_{W^{1,p}(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \to 0. \tag{4.13}$$

Proof. Let us consider problems (2.5) and (4.1) and take as test functions $\psi = u - W_\varepsilon u^-$ and $\phi = u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon$, respectively, for $\mathcal{P}_\varepsilon u_\varepsilon$ arising in (2.12). Subtracting both expressions and using that $W_\varepsilon = 1$ in $\overline{G}_\varepsilon$, we obtain

$$\int\limits_{\Omega_\varepsilon} (|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon u^- - u_\varepsilon)\, dx - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x,u_\varepsilon)(u^+ - u_\varepsilon)\, ds$$
$$\leq -\mathcal{A}_{n,p}\int\limits_{\Omega} |u^-|^{p-2}u^-(u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon)\, dx + \int\limits_{G_\varepsilon} f(u^+ - \mathcal{P}_\varepsilon u_\varepsilon)\, dx - \int\limits_{G_\varepsilon} |\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon)\, dx. \tag{4.14}$$

Besides, from (2.2), (2.18) and (4.14), we deduce

$$\begin{aligned} &K(\|\nabla(u - W_\varepsilon u^- - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u^+ - u_\varepsilon\|^p_{L^p(S_\varepsilon)}) \\ &\leq \int\limits_{\Omega_\varepsilon} (|\nabla(u - W_\varepsilon u^-)|^{p-2}\nabla(u - W_\varepsilon u^-) - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon u^- - u_\varepsilon)dx \\ &\quad + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} (\sigma(x,u^+) - \sigma(x,u_\varepsilon))(u^+ - u_\varepsilon)\, ds \leq \mathcal{I}^1_\varepsilon + \mathcal{I}^2_\varepsilon + \mathcal{I}^3_\varepsilon \end{aligned} \tag{4.15}$$

where

$$\mathcal{I}^1_\varepsilon \equiv \int\limits_{\Omega_\varepsilon} (|\nabla(u - W_\varepsilon u^-)|^{p-2}\nabla(u - W_\varepsilon u^-) - |\nabla u|^{p-2}\nabla u)\nabla(u - W_\varepsilon u^- - u_\varepsilon)dx,$$
$$\mathcal{I}^2_\varepsilon \equiv -\mathcal{A}_{n,p}\int\limits_{\Omega} |u^-|^{p-2}u^-(u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon)\, dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x,u^+)(u^+ - u_\varepsilon)\, ds,$$

and

$$\mathcal{I}^3_\varepsilon \equiv \int\limits_{G_\varepsilon} f(u^+ - \mathcal{P}_\varepsilon u_\varepsilon)\, dx - \int\limits_{G_\varepsilon} |\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon)\, dx.$$

Next, we show that the limit of $\mathcal{I}^1_\varepsilon + \mathcal{I}^2_\varepsilon + \mathcal{I}^3_\varepsilon$ is less than or equal to zero when $\varepsilon \to 0$. Indeed, from (2.10) and (3.5), we apply Proposition 2.3 with $\eta_\varepsilon \equiv -W_\varepsilon u^-$ and $\varphi = \varphi_\varepsilon \equiv u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon$, and have that

$$\lim_{\varepsilon\to 0}\mathcal{I}^1_\varepsilon = -\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla(W_\varepsilon u^-)|^{p-2}\nabla(W_\varepsilon u^-)\nabla(u - W_\varepsilon u^- - u_\varepsilon)\, dx.$$

Moreover, rewriting the computations (4.9)–(4.10) with minor modifications and using (3.12), we have

$$\lim_{\varepsilon\to 0}\mathcal{I}^1_\varepsilon \leq \lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} \frac{\varepsilon\, 2^{2n-2}\mathcal{A}_{n,p}}{\omega_n(1-\alpha_\varepsilon)^{p-1}}|u^-|^{p-2}u^-(u - W_\varepsilon u^- - u_\varepsilon)\, ds.$$

Thus, using Lemma 2.8, (2.10), (2.17) and (4.3), we obtain

$$\lim_{\varepsilon\to 0}(\mathcal{I}^1_\varepsilon + \mathcal{I}^2_\varepsilon) \leq \lim_{\varepsilon\to 0}\Big(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x,u^+)(u^+ - u_\varepsilon)\, ds\Big) = 0. \tag{4.16}$$

Besides, since $u - W_\varepsilon u^- - \mathcal{P}_\varepsilon u_\varepsilon$ is bounded in $W^{1,p}(\Omega)$ and $|G_\varepsilon| \to 0$ as $\varepsilon \to 0$, we derive

$$\lim_{\varepsilon\to 0}\mathcal{I}^3_\varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}(\mathcal{I}^1_\varepsilon + \mathcal{I}^2_\varepsilon + \mathcal{I}^3_\varepsilon) \leq 0. \tag{4.17}$$

Finally, gathering (4.15), (4.16) and (4.17), we obtain, as $\varepsilon \to 0$,

$$\|\nabla(u - W_\varepsilon u^- - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u^+ - u_\varepsilon\|^p_{L^p(S_\varepsilon)} \to 0.$$

Moreover, since $|S_\varepsilon| \le K\varepsilon^{\alpha(n-1)-n}$, we also have

$$\|\nabla(u - W_\varepsilon u^- - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \to 0. \tag{4.18}$$

To get (4.13) from (4.18), we apply the Poincaré inequality for the extension $\mathcal{P}_\varepsilon(u - W_\varepsilon u^- - u_\varepsilon) \in W^{1,p}(\Omega, \partial\Omega)$ as in Theorem 3.2, and the theorem is proved. □

4.2. The case $\alpha = \frac{n}{n-p}$ and $\gamma > \frac{n(p-1)}{n-p}$

Theorem 4.3. *Let $\alpha = n/(n-p)$, $\gamma > n(p-1)/(n-p)$, $p \in [2, n)$, and let u_ε be the weak solution of* (1.1). *Then, the limit function u of the extension of u_ε, defined by* (2.12), *is the weak solution of problem* (1.5).

Proof. The variational formulation of (1.5) reads: find $u \in W^{1,p}(\Omega, \partial\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \, dx + \mathcal{A}_{n,p}\int_\Omega |u|^{p-2}u\phi \, dx = \int_\Omega f\phi \, dx, \qquad \forall \phi \in W^{1,p}(\Omega, \partial\Omega). \tag{4.19}$$

From the monotonicity of the function $|\lambda|^{p-2}\lambda$, the existence and uniqueness of solution of (4.19) holds (cf., e.g., Section II.8.2 in [30]).

Let us take in (2.8) the test function $\psi = v - W_\varepsilon v \in K_\varepsilon$ where $v \in C_0^\infty(\Omega)$ and W_ε is the function defined by (2.15); since $W_\varepsilon = 1$ in $\overline{G}_\varepsilon$, we obtain

$$\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v)|^{p-2}\nabla(v - W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)dx \ge \int_{\Omega_\varepsilon} f(v - W_\varepsilon v - u_\varepsilon)dx, \tag{4.20}$$

and we pass to the limit when $\varepsilon \to 0$. On account of (2.12) and (2.17), we deduce

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(v - W_\varepsilon v - u_\varepsilon)\, dx = \int_\Omega f(v - u)\, dx. \tag{4.21}$$

Let us show that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v)|^{p-2}\nabla(v - W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)\, dx\\ &= \int_\Omega |\nabla v|^{p-2}\nabla v\nabla(v - u)\, dx + \mathcal{A}_{n,p}\int_\Omega |v|^{p-2}v(v - u)\, dx.\end{aligned} \tag{4.22}$$

Using (3.5), we apply Proposition 2.3 with $\eta_\varepsilon \equiv -W_\varepsilon v$ and $\varphi = \varphi_\varepsilon \equiv v - W_\varepsilon v - \mathcal{P}_\varepsilon u_\varepsilon$ since

$$\varphi_\varepsilon \equiv v - W_\varepsilon v - \mathcal{P}_\varepsilon u_\varepsilon \rightharpoonup v - u \quad \text{in } W^{1,p}(\Omega) - \text{weak as } \varepsilon \to 0, \tag{4.23}$$

which is obtained rewriting the proof for (3.6). Thus, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v)|^{p-2}\nabla(v - W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)dx = \lim_{\varepsilon\to 0}(\mathfrak{L}^1_\varepsilon + \mathfrak{L}^2_\varepsilon) \tag{4.24}$$

where

$$\mathfrak{L}^1_\varepsilon \equiv \int\limits_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \nabla(v - W_\varepsilon v - u_\varepsilon)\,dx \quad \text{and} \quad \mathfrak{L}^2_\varepsilon \equiv -\int\limits_{\Omega_\varepsilon} |\nabla(W_\varepsilon v)|^{p-2}\nabla(W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)\,dx.$$

By (4.23) and the fact that $|G_\varepsilon| \to 0$, we have

$$\lim_{\varepsilon\to 0} \mathfrak{L}^1_\varepsilon = \int\limits_{\Omega} |\nabla v|^{p-2}\nabla v \nabla(v-u)\,dx. \tag{4.25}$$

Moreover, using (3.5), (4.23), (2.10), the definition of W_ε and the Green formula, we get

$$\begin{aligned} -\lim_{\varepsilon\to 0}\mathfrak{L}^2_\varepsilon &= \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |v|^{p-2}|\nabla W_\varepsilon|^{p-2}\nabla(W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)\,dx \\ &= \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon \nabla\Big(|v|^{p-2}v\,(v - W_\varepsilon v - u_\varepsilon)\Big)\,dx \\ &= \lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}\cup\partial G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon\,|v|^{p-2}v(v - W_\varepsilon v - u_\varepsilon)\,ds. \end{aligned} \tag{4.26}$$

Now, by (2.15), (3.11) and (2.10), and the fact that $|S_\varepsilon| \le K\varepsilon^{\alpha(n-1)-n}$, it follows that

$$\begin{aligned} &\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon\,|v|^{p-2}v(v - W_\varepsilon v - u_\varepsilon)\,ds\Big| = \Big|\Big(\frac{n-p}{p-1}\Big)^{p-1}\frac{\varepsilon^{-\alpha(p-1)}}{C_0^{p-1}(1-\alpha_\varepsilon)^{p-1}}\int\limits_{S_\varepsilon}|v|^{p-2}vu_\varepsilon\,ds\Big| \\ &\le K\varepsilon^{-\alpha(p-1)}|S_\varepsilon|^{(p-1)/p}\|u_\varepsilon\|_{L^p(S_\varepsilon)} \le K\varepsilon^{[\gamma-\alpha(p-1)]/p} \to 0 \qquad \text{as } \varepsilon\to 0. \end{aligned} \tag{4.27}$$

Besides, from (3.12), Lemma 2.8 and (4.23), we have

$$\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} |\nabla w^j_\varepsilon|^{p-2}\partial_\nu w^j_\varepsilon|v|^{p-2}v(v - W_\varepsilon v - u_\varepsilon)\,ds = -\mathcal{A}_{n,p}\int\limits_{\Omega}|v|^{p-2}v(v-u)\,dx. \tag{4.28}$$

Gathering (4.24), (4.25), (4.26), (4.27) and (4.28) yields (4.22).

Finally, we use (4.21) and (4.22) to pass to the limit in (4.20), as $\varepsilon \to 0$, and obtain that the limit function u satisfies the following inequality

$$\int\limits_{\Omega} |\nabla v|^{p-2}\nabla v\nabla(v-u)\,dx + \mathcal{A}_{n,p}\int\limits_{\Omega}|v|^{p-2}v(v-u)\,dx \ge \int\limits_{\Omega} f(v-u)dx, \qquad \forall v\in W^{1,p}(\Omega,\partial\Omega). \tag{4.29}$$

As usual, taking $v = u \pm \lambda\phi$ in (4.29) where $\phi \in W^{1,p}(\Omega,\partial\Omega)$ and passing to the limit as $\lambda \to +0$, we obtain that u satisfies the integral identity (4.19), which concludes the proof. □

Theorem 4.4. *Let $\alpha = n/(n-p)$, $\gamma > n(p-1)/(n-p)$ with $p \in [2,n)$. Let u_ε be the weak solution of (1.1), $u \in W^{1,p}(\Omega,\partial\Omega)$ the weak solution of the boundary value problem (1.5) with the additional regularity $u \in W^{1,\infty}(\Omega)$, and W_ε defined by (2.15). Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u + W_\varepsilon u\|^p_{W^{1,p}(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \to 0. \tag{4.30}$$

Proof. Let us consider problems (2.5) and (4.19) and take as test functions $\psi = u - W_\varepsilon u$ and $\phi = u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon$, respectively, for $\mathcal{P}_\varepsilon u_\varepsilon$ arising in (2.12). Subtracting both expressions and using the definition of W_ε on $\overline{G}_\varepsilon$, we obtain

$$\begin{aligned}&\int\limits_{\Omega_\varepsilon} (|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon u - u_\varepsilon)\, dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x, u_\varepsilon)u_\varepsilon\, ds \\ &\leq -\mathcal{A}_{n,p}\int\limits_{\Omega} |u|^{p-2}u(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\, dx - \int\limits_{G_\varepsilon} f\mathcal{P}_\varepsilon u_\varepsilon\, dx - \int\limits_{G_\varepsilon} |\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\, dx.\end{aligned} \tag{4.31}$$

Besides, from (2.2), (2.18) and (4.31), we deduce

$$\begin{aligned}&K(\|\nabla(u - W_\varepsilon u - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)}) \\ &\leq \int\limits_{\Omega_\varepsilon} (|\nabla(u - W_\varepsilon u)|^{p-2}\nabla(u - W_\varepsilon u) - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon u - u_\varepsilon)\, dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x, u_\varepsilon)u_\varepsilon\, ds \leq \mathfrak{I}^1_\varepsilon + \mathfrak{I}^2_\varepsilon + \mathfrak{I}^3_\varepsilon\end{aligned} \tag{4.32}$$

where

$$\mathfrak{I}^1_\varepsilon \equiv \int\limits_{\Omega_\varepsilon} (|\nabla(u - W_\varepsilon u)|^{p-2}\nabla(u - W_\varepsilon u) - |\nabla u|^{p-2}\nabla u)\nabla(u - W_\varepsilon u - u_\varepsilon)\, dx,$$

$$\mathfrak{I}^2_\varepsilon \equiv -\mathcal{A}_{n,p}\int\limits_{\Omega} |u|^{p-2}u(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\, dx \quad \text{and} \quad \mathfrak{I}^3_\varepsilon \equiv -\int\limits_{G_\varepsilon} f\mathcal{P}_\varepsilon u_\varepsilon\, dx - \int\limits_{G_\varepsilon} |\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\, dx.$$

Let us show that $\mathfrak{I}^1_\varepsilon + \mathfrak{I}^2_\varepsilon + \mathfrak{I}^3_\varepsilon$ tends to zero as $\varepsilon \to 0$.

From (2.10) and (3.5), we apply Proposition 2.3 with $\eta_\varepsilon \equiv -W_\varepsilon u$ and $\varphi = \varphi_\varepsilon \equiv u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon$, and have that

$$\lim_{\varepsilon\to 0} \mathfrak{I}^1_\varepsilon = -\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla(W_\varepsilon u)|^{p-2}\nabla(W_\varepsilon u)\nabla(u - W_\varepsilon u - u_\varepsilon)\, dx.$$

Moreover, rewriting the computations (4.26)–(4.27) with minor modifications and using (3.12), we have

$$\lim_{\varepsilon\to 0} \mathfrak{I}^1_\varepsilon = \lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} \frac{\varepsilon\, 2^{2n-2}\mathcal{A}_{n,p}}{\omega_n(1-\alpha_\varepsilon)^{p-1}}|u|^{p-2}u(u - W_\varepsilon u - u_\varepsilon)\, ds.$$

Thus, using Lemma 2.8, (2.10) and (2.17), we obtain

$$\lim_{\varepsilon\to 0}(\mathfrak{I}^1_\varepsilon + \mathfrak{I}^2_\varepsilon) = 0. \tag{4.33}$$

Besides, since $\mathcal{P}_\varepsilon u_\varepsilon$ and $u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon$ are bounded in $W^{1,p}(\Omega)$ and $|G_\varepsilon| \to 0$ as $\varepsilon \to 0$, we derive

$$\lim_{\varepsilon\to 0} \mathfrak{I}^3_\varepsilon = 0. \tag{4.34}$$

Finally, gathering (4.32), (4.33) and (4.34), we obtain, as $\varepsilon \to 0$,

$$\|\nabla(u - W_\varepsilon u - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \to 0. \tag{4.35}$$

To get (4.30) from (4.35), we apply the Poincaré inequality for the extension $\mathcal{P}_\varepsilon(u - W_\varepsilon u - u_\varepsilon) \in W^{1,p}(\Omega, \partial\Omega)$ as in Theorem 3.2, and the theorem is proved. □

5. Critical relation for the adsorption: $p \in [2, n)$ and $\alpha \in (1, \frac{n}{n-p})$

In this section, we deal with sizes of cavities larger than the critical size. Because of the adsorption parameter, the constraints on the boundary of the cavities in (1.1) transform asymptotically into an obstacle problem with a nonlinear strange term $\mathcal{D}_n \sigma(x, u)$ that also contains information on the geometrical configuration of the original problem, namely, the area of the unit sphere and the scaling factor C_0^{n-1} (cf. (1.6)). We show the convergence of the extension of the solution of (1.1), as $\varepsilon \to 0$, towards that of (1.6) in the $W^{1,p}$-norm and compute bounds for discrepancies in the way stated by Theorem 5.2.

Theorem 5.1. *Let $\alpha \in (1, n/(n-p))$, $\gamma = \alpha(n-1) - n$ with $p \in [2, n)$, and let u_ε be the weak solution of* (1.1)*. Then, the limit function u of the extension of u_ε, defined by* (2.12)*, is the weak solution of problem* (1.6)*.*

Proof. First, we observe that the variational formulation of problem (1.6) is: find $u \in K_0$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla(v-u)\, dx + \mathcal{D}_n \int_\Omega \sigma(x,u)(v-u)\, dx \ge \int_\Omega f(v-u)\, dx, \qquad \forall v \in K_0, \tag{5.1}$$

where K_0 is defined by

$$K_0 = \{v \in W^{1,p}(\Omega, \partial\Omega) \,:\, v \ge 0 \text{ a.e. in } \Omega\}. \tag{5.2}$$

The existence and uniqueness of solution u of (5.1)–(5.2) follows from (2.2) and (2.18) (see the technique in Theorem 2.1). Besides, by Minty Lemma, problem (5.1) is equivalent to finding $u \in K_0$ such that

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla(v-u)\, dx + \mathcal{D}_n \int_\Omega \sigma(x,v)(v-u)\, dx \ge \int_\Omega f(v-u)\, dx, \qquad \forall v \in K_0. \tag{5.3}$$

Let us prove that the negative part of the limit function u, u^-, is equal to zero a.e. in Ω and, consequently, $u \in K_0$. Applying Lemma 2.6 and using that $u_\varepsilon^- = 0$ on S_ε and (2.10), we conclude

$$\|u_\varepsilon^-\|^p_{L^p(\Omega_\varepsilon)} \le K a_\varepsilon^{p-n} \varepsilon^n \|\nabla u_\varepsilon^-\|^p_{L^p(\Omega_\varepsilon)} \le K \varepsilon^{n-\alpha(n-p)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.4}$$

Thus, from (2.12) and the fact that $|G_\varepsilon| \to 0$, we have

$$\|u^-\|_{L^p(\Omega)} = \lim_{\varepsilon\to 0} \|u_\varepsilon^-\|_{L^p(\Omega_\varepsilon)} = 0. \tag{5.5}$$

In order to prove that the limit function u satisfies (5.3), we pass to the limit in (2.8) with $\psi = v \in K_0$. On account of (2.12) and the volume of G_ε, it follows that

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla(v-u_\varepsilon)\, dx = \int_\Omega |\nabla v|^{p-2} \nabla v \nabla(v-u)\, dx \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(v-u_\varepsilon)\, dx = \int_\Omega f(v-u)\, dx. \tag{5.6}$$

Let us show that, under the assumptions $\alpha \in (1, n/(n-p))$ and $\gamma = \alpha(n-1) - n$, the following equality holds:

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x,v)(v-u_\varepsilon)\, ds = \mathcal{D}_n \int_\Omega \sigma(x,v)(v-u)\, dx. \tag{5.7}$$

To do this, we introduce the function M_ε defined by $M_\varepsilon(x) \equiv M_\varepsilon^j(x)$, $x \in Y_\varepsilon^j \setminus \overline{G_\varepsilon^j}$, $j \in \Upsilon_\varepsilon$, where M_ε^j is a solution of the problem

$$\Delta_p M_\varepsilon^j = \mu_\varepsilon \text{ in } Y_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \qquad \partial_{\nu_p} M_\varepsilon^j = 1 \text{ on } \partial G_\varepsilon^j, \qquad \partial_{\nu_p} M_\varepsilon^j = 0 \text{ on } \partial Y_\varepsilon^j \setminus \partial G_\varepsilon^j, \tag{5.8}$$

and

$$\mu_\varepsilon = \frac{C_0^{n-1}\varepsilon^{\alpha(n-1)-n}\omega_n}{1-(a_\varepsilon\varepsilon^{-1})^n\omega_n}. \tag{5.9}$$

We assume that $\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}} M_\varepsilon^j dx = 0$. Taking as a test function M_ε^j in the integral identity for M_ε^j and applying the Hölder inequality, we obtain

$$\|\nabla M_\varepsilon^j\|^p_{L^p(Y_\varepsilon^j\setminus\overline{G_\varepsilon^j})} \le \Big|\int\limits_{\partial G_\varepsilon^j} M_\varepsilon^j ds\Big| \le |\partial G_\varepsilon^j|^{(p-1)/p}\|M_\varepsilon^j\|_{L^p(\partial G_\varepsilon^j)}.$$

Besides, using Lemma 2.5 and Lemma 2.4, we get

$$\begin{aligned}\|M_\varepsilon^j\|^p_{L^p(\partial G_\varepsilon^j)} &\le K\Big(a_\varepsilon^{n-1}\varepsilon^{-n}\|M_\varepsilon^j\|^p_{L^p(Y_\varepsilon^j\setminus\overline{G_\varepsilon^j})} + a_\varepsilon^{p-1}\|\nabla M_\varepsilon^j\|^p_{L^p(Y_\varepsilon^j\setminus\overline{G_\varepsilon^j})}\Big)\\ &\le K\Big(a_\varepsilon^{n-1}\varepsilon^{p-n} + a_\varepsilon^{p-1}\Big)\|\nabla M_\varepsilon^j\|^p_{L^p(Y_\varepsilon^j\setminus\overline{G_\varepsilon^j})} \le Ka_\varepsilon^{p-1}\|\nabla M_\varepsilon^j\|^p_{L^p(Y_\varepsilon^j\setminus\overline{G_\varepsilon^j})}.\end{aligned}$$

Hence, denoting by $\widehat{Y}_\varepsilon$ the set $\widehat{Y}_\varepsilon = \cup_{j\in\Upsilon_\varepsilon}(Y_\varepsilon^j \setminus \overline{G_\varepsilon^j})$,

$$\|\nabla M_\varepsilon^j\|_{L^p(Y_\varepsilon^j\setminus\overline{G_\varepsilon^j})} \le Ka_\varepsilon^{n/p} \text{ and } \|\nabla M_\varepsilon\|_{L^p(\widehat{Y}_\varepsilon)} \le K(a_\varepsilon\varepsilon^{-1})^{n/p}. \tag{5.10}$$

Now, by means of M_ε, the integral on S_ε in (5.7) can be transformed into a volume integral. Thus, we can write

$$\begin{aligned}&\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,v)(v-u_\varepsilon)ds = \varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}} div(|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j\sigma(x,v)(v-u_\varepsilon))dx\\ =&\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}}|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j\nabla(\sigma(x,v)(v-u_\varepsilon))\,dx + \varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}}\Delta_p M_\varepsilon^j\sigma(x,v)(v-u_\varepsilon)\,dx\\ =&\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}}|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j\nabla(\sigma(x,v)(v-u_\varepsilon))\,dx + \varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}}\sigma(x,v)(v-u_\varepsilon)\,dx.\end{aligned} \tag{5.11}$$

From (5.10) and (2.10) we deduce

$$\varepsilon^{-\gamma}\int\limits_{\widehat{Y}_\varepsilon}|\nabla M_\varepsilon|^{p-1}|\nabla(\sigma(x,v)(v-u_\varepsilon))|\,dx \le K\varepsilon^{-\gamma}\Big(\int\limits_{\widehat{Y}_\varepsilon}|\nabla M_\varepsilon|^p\,dx\Big)^{(p-1)/p} \le K\varepsilon^{(n-\alpha(n-p))/p},$$

and, since $\alpha < n/(n-p)$,

$$\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}}|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j\nabla(\sigma(x,v)(v-u_\varepsilon))\,dx = 0. \tag{5.12}$$

In addition, by (5.9), (2.12) and the size of G_ε, we derive that

$$\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\mu_\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j\setminus\overline{G_\varepsilon^j}}\sigma(x,v)(v-u_\varepsilon)\,dx = C_0^{n-1}\omega_n\int\limits_\Omega\sigma(x,v)(v-u)\,dx. \tag{5.13}$$

Therefore, gathering (5.11), (5.12) and (5.13) yields (5.7), which concludes the proof. □

Theorem 5.2. *Let $\alpha \in (1, n/(n-p))$, $\gamma = \alpha(n-1) - n$ and $p \in [2, n)$. Let u_ε be the weak solution of* (1.1) *and $u \in W^{1,p}(\Omega, \partial\Omega)$ the weak solution of* (1.6) *with the additional regularity $u \in W^{1,\infty}(\Omega)$. Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u\|^p_{W^{1,p}(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon - u\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\beta, \tag{5.14}$$

where

$$\beta = \min\{(n - \alpha(n-p))(p-1)/p^2, \alpha - 1, (p-1)/p\}. \tag{5.15}$$

Proof. Let us introduce the following boundary value problem

$$\begin{cases} -\Delta_p v_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ \partial_{\nu_p} v_\varepsilon + \varepsilon^{-\gamma}\sigma(x, v_\varepsilon^+) + \varepsilon^{-\alpha(p-1)}\sigma(x, v_\varepsilon^-) = 0 & \text{for } x \in S_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.16}$$

Its variational formulation is: find $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ such that

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla\psi\, dx + \varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma(x, v_\varepsilon^+)\psi\, ds + \varepsilon^{-\alpha(p-1)}\int_{S_\varepsilon} \sigma(x, v_\varepsilon^-)\psi\, ds = \int_{\Omega_\varepsilon} f\psi\, dx, \qquad \forall\psi \in W^{1,p}(\Omega_\varepsilon, \partial\Omega). \tag{5.17}$$

The existence and uniqueness of solution v_ε of (5.17) follows from (2.2) and (2.18). In addition, taking $\psi = v_\varepsilon$ in (5.17) and considering $\mathcal{P}_\varepsilon v_\varepsilon$ the $W^{1,p}$– extension of v_ε to Ω (cf. Lemma 2.7), we apply the Poincaré inequality to obtain the estimates

$$\|v_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} \leq K, \quad \|v_\varepsilon^+\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\gamma, \quad \|v_\varepsilon^-\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^{\alpha(p-1)} \tag{5.18}$$

(see the proof of Theorem 2.1 for the technique where we have used the fact that $\sigma(x, v_\varepsilon^\pm)v_\varepsilon^\mp = 0$).

Now, applying Lemma 2.6 and estimates (5.18), we obtain

$$\|v_\varepsilon^-\|^p_{L^p(\Omega_\varepsilon)} \leq K\varepsilon^{n-\alpha(n-p)}. \tag{5.19}$$

Besides, setting $\psi = v_\varepsilon^-$ in (5.17) and taking into account the properties of $\sigma(x, u)$ and that

$$|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon = |\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ + |\nabla v_\varepsilon^-|^{p-2}\nabla v_\varepsilon^-, \tag{5.20}$$

we have

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon^-|^p\, dx + \varepsilon^{-\alpha(p-1)}\int_{S_\varepsilon} \sigma(x, v_\varepsilon^-)v_\varepsilon^-\, ds = \int_{\Omega_\varepsilon} f v_\varepsilon^-\, dx. \tag{5.21}$$

Hence, gathering (5.21) and (5.19), we conclude

$$\|\nabla v_\varepsilon^-\|^p_{L^p(\Omega_\varepsilon)} \leq K\varepsilon^{(n-\alpha(n-p))/p}. \tag{5.22}$$

Once we have shown

$$\|\nabla(v_\varepsilon^+ - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|v_\varepsilon^+ - u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^{(n-\alpha(n-p))/p}, \tag{5.23}$$

and

$$\|\nabla(v_\varepsilon^+ - u)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|v_\varepsilon^+ - u\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\beta, \tag{5.24}$$

where β is defined by (5.15), we get

$$\|\nabla(u_\varepsilon - u)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon - u\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\beta$$

and, rewriting the proof at the end of Theorem 3.2 with minor modifications, (5.14) holds, which proves the theorem.

Let us show (5.23) and (5.24). In order to prove (5.23), we consider problems (2.5) and (5.17) and take as test functions $\psi = v_\varepsilon^+ \in K_\varepsilon$ and $\psi = v_\varepsilon^+ - u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, respectively. Subtracting both expressions and using the properties of σ and the fact that $u_\varepsilon \geq 0$ in S_ε, we obtain

$$\begin{aligned}&\int_{\Omega_\varepsilon} \Big(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\Big)\nabla(v_\varepsilon^+ - u_\varepsilon)\,dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma(x,v_\varepsilon^+) - \sigma(x,u_\varepsilon))(v_\varepsilon^+ - u_\varepsilon)\,ds\\ &\leq -\,\varepsilon^{-\alpha(p-1)}\int_{S_\varepsilon}\sigma(x,v_\varepsilon^-)(v_\varepsilon^+ - u_\varepsilon)\,ds = \varepsilon^{-\alpha(p-1)}\int_{S_\varepsilon}\sigma(x,v_\varepsilon^-)u_\varepsilon\,ds \leq 0.\end{aligned}$$

Hence, on account of (5.20),

$$\begin{aligned}&\int_{\Omega_\varepsilon} \Big(|\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\Big)\nabla(v_\varepsilon^+ - u_\varepsilon)\,dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma(x,v_\varepsilon^+) - \sigma(x,u_\varepsilon))(v_\varepsilon^+ - u_\varepsilon)\,ds\\ &\leq -\int_{\Omega_\varepsilon}|\nabla v_\varepsilon^-|^{p-2}\nabla v_\varepsilon^-\nabla(v_\varepsilon^+ - u_\varepsilon)\,dx \leq \|\nabla v_\varepsilon^-\|^{p-1}_{L^p(\Omega_\varepsilon)}\|\nabla(v_\varepsilon^+ - u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}.\end{aligned} \tag{5.25}$$

Now, by (2.2), (2.18), (5.25) and (5.22), it follows

$$\begin{aligned}&K\Big(\|\nabla(v_\varepsilon^+ - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|v_\varepsilon^+ - u_\varepsilon\|^p_{L^p(S_\varepsilon)}\Big)\\ &\leq \int_{\Omega_\varepsilon}\Big(|\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\Big)\nabla(v_\varepsilon^+ - u_\varepsilon)\,dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma(x,v_\varepsilon^+) - \sigma(x,u_\varepsilon))(v_\varepsilon^+ - u_\varepsilon)\,ds\\ &\leq \widetilde{K}\varepsilon^{(n-\alpha(n-p))(p-1)/p^2}\|\nabla(v_\varepsilon^+ - u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}\end{aligned}$$

and, consequently, (5.23) holds.

In order to prove (5.24), we consider problems (5.1) and (5.17) and take as test functions $v = \mathcal{P}_\varepsilon {v_\varepsilon}^+ \in K_0$ and $\psi = v_\varepsilon^+ - u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, respectively. Subtracting both expressions and using (5.20) and the fact that $\sigma(x, v_\varepsilon^-)u \leq 0$ and $\sigma(x, v_\varepsilon^-)v_\varepsilon^+ = 0$ on S_ε, we obtain

$$\int_{\Omega_\varepsilon}\Big(|\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ - |\nabla u|^{p-2}\nabla u\Big)\nabla(v_\varepsilon^+ - u)\,dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma(x,v_\varepsilon^+) - \sigma(x,u))(v_\varepsilon^+ - u)\,ds \leq J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3 \tag{5.26}$$

where

$$J_\varepsilon^1 \equiv -\int_{\Omega_\varepsilon}|\nabla v_\varepsilon^-|^{p-2}\nabla v_\varepsilon^-\nabla(v_\varepsilon^+ - u)\,dx, \qquad J_\varepsilon^2 \equiv \int_{G_\varepsilon}|\nabla u|^{p-2}\nabla u\nabla(\mathcal{P}_\varepsilon {v_\varepsilon}^+ - u)\,dx - \int_{G_\varepsilon} f(\mathcal{P}_\varepsilon {v_\varepsilon}^+ - u)\,dx, \tag{5.27}$$

and

$$J_\varepsilon^3 \equiv \mathcal{D}_n\int_\Omega \sigma(x,u)(\mathcal{P}_\varepsilon {v_\varepsilon}^+ - u)\,dx - \varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(x,u)(v_\varepsilon^+ - u)\,ds.$$

Let us estimate each term J_ε^i for $i = 1, 2, 3$.

Taking into account (5.22), (5.18), (2.26), $u \in W^{1,\infty}(\Omega)$ and the embedding of the space $W^{1,p}(\Omega, \partial\Omega)$ into $L^{np/(n-p)}(\Omega)$ for $p < n$, we deduce

$$|J_\varepsilon^1| \leq \|\nabla v_\varepsilon^-\|_{L^p(\Omega_\varepsilon)}^{p-1} \|\nabla(v_\varepsilon^+ - u)\|_{L^p(\Omega_\varepsilon)} \leq K\varepsilon^{(n-\alpha(n-p))(p-1)/p^2} \tag{5.28}$$

and

$$\begin{aligned} |J_\varepsilon^2| &\leq K|G_\varepsilon|^{1/q} \|\nabla(\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u)\|_{L^p(\Omega)} + \|f\|_{L^{np/(np-n+p)}(G_\varepsilon)} \|\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u\|_{L^{np/(n-p)}(\Omega)} \\ &\leq K|G_\varepsilon|^{1/q} \|\nabla(\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u)\|_{L^p(\Omega)} + |G_\varepsilon|^{1/n} \|f\|_{L^q(\Omega)} \|\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u\|_{W^{1,p}(\Omega)} \\ &\leq K[\varepsilon^{(\alpha-1)n(p-1)/p} + \varepsilon^{\alpha-1}] \leq K\varepsilon^{\alpha-1}. \end{aligned} \tag{5.29}$$

To estimate J_ε^3 we use again the function $M_\varepsilon(x)$ defined by (5.8) to transform the integral on S_ε into a volume integral. Thus, see (5.11), we can write

$$\begin{aligned} J_\varepsilon^3 =& (\mathcal{D}_n - \varepsilon^{-\gamma}\mu_\varepsilon) \int_\Omega \sigma(x,u)(\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u)\,dx + \varepsilon^{-\gamma}\mu_\varepsilon \int_{\Omega\setminus\widehat{Y}_\varepsilon} \sigma(x,u)(\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u)\,dx \\ &- \varepsilon^{-\gamma} \int_{\widehat{Y}_\varepsilon} |\nabla M_\varepsilon|^{p-2} \nabla M_\varepsilon \nabla(\sigma(x,u)(\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u))\,dx. \end{aligned}$$

Now, by (5.9), (5.18), (2.26) and (5.10), it follows that

$$\begin{aligned} |J_\varepsilon^3| \leq& K[\varepsilon^{(\alpha-1)n} \|\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u\|_{L^p(\Omega)} + \varepsilon^{-\gamma}\mu_\varepsilon |\Omega\setminus\widehat{Y}_\varepsilon|^{(p-1)/p} \|\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u\|_{L^p(\Omega)} \\ &+ \varepsilon^{-\gamma} \|\nabla M_\varepsilon\|_{L^p(\widehat{Y}_\varepsilon)}^{p-1} \|\nabla(\mathcal{P}_\varepsilon v_\varepsilon{}^+ - u)\|_{L^p(\Omega)}] \\ \leq& K[\varepsilon^{(\alpha-1)n} + \varepsilon^{(p-1)/p} + \varepsilon^{(\alpha-1)n(p-1)/p} + \varepsilon^{(n-\alpha(n-p))/p}]. \end{aligned} \tag{5.30}$$

Finally, gathering (2.2), (2.18), (5.26), (5.28), (5.29) and (5.30), (5.24) holds, which concludes the proof. □

6. Extreme cases for $p \in [2, n)$

We consider the rest of possible relations between the parameters α and γ which have not been considered in previous sections. Section 6.1 contains the results for the case of big cavities and small adsorption; the constraints on the boundary of the cavities in (1.1) transform asymptotically into an obstacle problem for the p-Laplacian in Ω, which ignores the adsorption parameter (which in fact can converge towards ∞); that is, as if Signorini conditions had been imposed (cf. [11] when $p = 2$). Section 6.3 contains the results for the case of small cavities; also the solution ignores asymptotically the adsorption parameter. In both cases, the convergence of the extension of the solution in the $W^{1,p}$-norm is proved along with bounds for discrepancies as stated in Theorems 6.2 and 6.5 respectively. Section 6.2 contains the case of large sizes of cavities and adsorption parameters; the solution of (1.1) vanishes asymptotically and we obtain estimates of the $W^{1,p}$-norm (cf. (6.8)).

6.1. The case $\alpha \in (1, \frac{n}{n-p})$ and $\gamma < \alpha(n-1) - n$

Theorem 6.1. *Let $\alpha \in (1, n/(n-p))$, $\gamma < \alpha(n-1) - n$ with $p \in [2, n)$, and let u_ε be the weak solution of* (1.1)*. Then, the limit function u of the extension of u_ε, defined by* (2.12)*, is the weak solution of problem* (1.7)*.*

Proof. We rewrite the proof of Theorem 5.1 with minor modifications; we briefly outline the main differences here.

The variational formulation of problem (1.7) is: find $u \in K_0$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u)\,dx \geq \int_\Omega f(v-u)\,dx, \quad \forall v \in K_0, \tag{6.1}$$

where K_0 is defined by (5.2). Besides, on account of the monotonicity of the function $|\lambda|^{p-2}\lambda$, problem (6.1) has a unique solution $u \in K_0$, which also satisfies

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v-u)\,dx \geq \int_\Omega f(v-u)\,dx, \quad \forall v \in K_0. \tag{6.2}$$

Let us note that (5.4) and (5.5) also hold in this case and, hence, the limit function u belongs to K_0. To prove that u satisfies (6.2), we pass to the limit in (2.8) with $\psi = v \in K_0$. It is easy to check that (5.6) holds. Moreover, under the assumptions $\alpha \in (1, n/(n-p))$ and $\gamma < \alpha(n-1)-n$, using (2.3) and (2.10), and computing $|S_\varepsilon|$, it follows that

$$\Big|\varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(x,v)(v-u_\varepsilon)ds\Big| \leq \varepsilon^{-\gamma}K[|S_\varepsilon| + |S_\varepsilon|^{(p-1)/p}\|u_\varepsilon\|_{L^p(S_\varepsilon)}] \leq K\varepsilon^{(\alpha(n-1)-n-\gamma)(p-1)/p} \to 0, \quad \text{as } \varepsilon \to 0,$$

which concludes the proof. □

Theorem 6.2. *Let $\alpha \in (1, n/(n-p))$, $\gamma < \alpha(n-1)-n$ and $p \in [2,n)$. Let u_ε be the weak solution of* (1.1) *and $u \in W^{1,p}(\Omega, \partial\Omega)$ the weak solution of* (1.7) *with the additional regularity $u \in W^{1,\infty}(\Omega)$. Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u\|^p_{W^{1,p}(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon - u\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\kappa, \tag{6.3}$$

where

$$\kappa = \min\{(n-\alpha(n-p))(p-1)/p^2, \alpha - 1, \alpha(n-1)-n-\gamma\}. \tag{6.4}$$

Proof. We use the technique in Theorem 5.2, that is, we consider v_ε the solution of problem (5.16), which satisfies estimates (5.18), (5.19) and (5.22). Besides, under the assumption $\alpha \in (1, n/(n-p))$, it is easy to check that (5.23) holds (see the proof of Theorem 5.2). Now, let us prove that, under the hypotheses of Theorem 6.2,

$$\|\nabla(v_\varepsilon^+ - u)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|v_\varepsilon^+ - u\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\kappa, \tag{6.5}$$

where u is the weak solution of (1.7) and κ is given by (6.4). Thus, gathering (5.23) and (6.5), we get

$$\|\nabla(u_\varepsilon - u)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon - u\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\kappa. \tag{6.6}$$

Moreover, since $|S_\varepsilon| \leq K\varepsilon^{\alpha(n-1)-n}$, we also have

$$\|\nabla(u_\varepsilon - u)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^\kappa.$$

To prove (6.5), we consider problems (6.1) and (5.17) and take as test functions $v = \mathcal{P}_\varepsilon {v_\varepsilon}^+ \in K_0$ and $\psi = v_\varepsilon^+ - u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, respectively, where u is the weak solution of (1.7). Subtracting both expressions and using (5.20) and the fact that $\sigma(x, v_\varepsilon^-)u \leq 0$ and $\sigma(x, v_\varepsilon^-)v_\varepsilon^+ = 0$ on S_ε, we obtain (5.26) where J_ε^1 and J_ε^2 are defined by (5.27) and J_ε^3 is

$$J_\varepsilon^3 \equiv -\varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(x,u)(v_\varepsilon^+ - u)\,ds.$$

Taking into account (5.22), (5.18), (2.26), $u \in W^{1,\infty}(\Omega)$ and the embedding of the space $W^{1,p}(\Omega, \partial\Omega)$ into $L^{np/(n-p)}(\Omega)$ for $p < n$, we deduce (5.28) and (5.29). Moreover, by Young inequality,

$$\Big|\int_{S_\varepsilon} \sigma(x,u)(v_\varepsilon^+ - u)\, ds\Big| \leq \delta \|v_\varepsilon^+ - u\|^p_{L^p(S_\varepsilon)} + K\delta^{-1/(p-1)}|S_\varepsilon| \leq \delta\|v_\varepsilon^+ - u\|^p_{L^p(S_\varepsilon)} + K\delta^{-1/(p-1)}\varepsilon^{\alpha(n-1)-n}, \quad (6.7)$$

with arbitrary $\delta > 0$. Therefore, from (2.2), (2.18), (5.26), (5.28), (5.29) and (6.7), it follows that

$$K\|\nabla(v_\varepsilon^+ - u)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}(K-\delta)\|v_\varepsilon^+ - u\|^p_{L^p(S_\varepsilon)} \leq \widetilde{K}(\delta^{-1/(p-1)}\varepsilon^{\alpha(n-1)-n-\gamma} + \varepsilon^{(n-\alpha(n-p))(p-1)/p^2} + \varepsilon^{\alpha-1}).$$

Now, choosing $\delta = K/2$ in the above expression yields (6.5).

To obtain (6.3) from (6.6), we apply the Poincaré inequality for the extension $\mathcal{P}_\varepsilon(u_\varepsilon - u) \in W^{1,p}(\Omega, \partial\Omega)$ as in Theorem 3.2, and the theorem is proved. □

6.2. The case $\alpha \in (1, \frac{n}{n-p})$ and $\gamma > \alpha(n-1) - n$

Theorem 6.3. *Let $\alpha \in (1, n/(n-p))$, $\gamma > \alpha(n-1) - n$ and $p \in [2, n)$. Then, the extension $\mathcal{P}_\varepsilon u_\varepsilon$ of the weak solution of* (1.1)*, defined by Theorem* 2.1*, verifies*

$$\|\mathcal{P}_\varepsilon u_\varepsilon\|^p_{W^{1,p}(\Omega)} \leq K[\varepsilon^{\gamma-\alpha(n-1)+n} + \varepsilon^{n-\alpha(n-p)}]^{1/p}, \quad (6.8)$$

and, consequently, $\mathcal{P}_\varepsilon u_\varepsilon$ converges to zero in $W^{1,p}(\Omega)$ when $\varepsilon \to 0$.

Proof. Applying Lemma 2.6 and estimate (2.10) yields

$$\|u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} \leq K[a_\varepsilon^{1-n}\varepsilon^n\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} + a_\varepsilon^{p-n}\varepsilon^n\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)}] \leq K[\varepsilon^{\gamma-\alpha(n-1)+n} + \varepsilon^{n-\alpha(n-p)}].$$

Besides, setting $\psi \equiv 0$ in the integral inequality (2.5) and using (2.2), we obtain

$$\|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq \|f\|_{L^q(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}.$$

Thus,

$$\|u_\varepsilon\|^p_{W^{1,p}(\Omega_\varepsilon)} \leq K[\varepsilon^{\gamma-\alpha(n-1)+n} + \varepsilon^{n-\alpha(n-p)}]^{1/p},$$

and, by (2.9), the theorem holds. □

6.3. The case $\alpha > \frac{n}{n-p}$ and $\gamma \in \mathbb{R}$

Theorem 6.4. *Let $\alpha > n/(n-p)$, $\gamma \in \mathbb{R}$ with $p \in [2, n)$, and let u_ε be the weak solution of* (1.1)*. Then, the limit function u of the extension of u_ε, defined by* (2.12)*, is the weak solution of the Dirichlet problem* (1.8)*.*

Proof. Let us take in (2.8) the test function $\psi = v - W_\varepsilon v \in K_\varepsilon$ where $v \in C_0^\infty(\Omega)$ and W_ε is the function defined by (2.15); since $W_\varepsilon = 1$ in $\overline{G}_\varepsilon$, we obtain

$$\int_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v)|^{p-2}\nabla(v - W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)dx \geq \int_{\Omega_\varepsilon} f(v - W_\varepsilon v - u_\varepsilon)dx$$

and we pass to the limit when $\varepsilon \to 0$. On account of (2.12) and (2.17), we deduce

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(v - W_\varepsilon v - u_\varepsilon)\, dx = \int_{\Omega} f(v-u)\, dx.$$

Besides, using (3.5), (2.12) and (2.17), we apply Proposition 2.3 with $\eta_\varepsilon \equiv -W_\varepsilon v$ and $\varphi = \varphi_\varepsilon \equiv v - W_\varepsilon v - \mathcal{P}_\varepsilon u_\varepsilon$ where $\mathcal{P}_\varepsilon u_\varepsilon$ is the $W^{1,p}$–extension defined in Theorem 2.1, and we obtain

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon v)|^{p-2}\nabla(v - W_\varepsilon v)\nabla(v - W_\varepsilon v - u_\varepsilon)\,dx = \int\limits_{\Omega} |\nabla v|^{p-2}\nabla v\nabla(v-u)\,dx.$$

Thus, we get that u satisfies the following inequality

$$\int\limits_{\Omega} |\nabla v|^{p-2}\nabla v\nabla(v-u)\,dx \geq \int\limits_{\Omega} f(v-u)\,dx, \quad \forall v \in W^{1,p}(\Omega,\partial\Omega). \tag{6.9}$$

As usual, taking $v = u \pm \lambda\phi$ in (6.9) where $\phi \in W^{1,p}(\Omega,\partial\Omega)$ and passing to the limit as $\lambda \to +0$, we obtain that u satisfies the integral identity for problem (1.8), which concludes the proof. □

Theorem 6.5. *Let $\alpha > n/(n-p)$, $\gamma \in \mathbb{R}$ and $p \in [2,n)$. Let u_ε be the weak solution of* (1.1) *and $u \in W^{1,p}(\Omega,\partial\Omega)$ the weak solution of* (1.8) *with the additional regularity $u \in W^{1,\infty}(\Omega)$. Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u\|^p_{W^{1,p}(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^{\min((\alpha(n-p)-n)(p-1)/p,\alpha-1)}. \tag{6.10}$$

Proof. We consider the variational formulation of problem (1.8) and (2.5) and take as test functions $\phi = u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon$ and $\psi = u - W_\varepsilon u$, respectively, for W_ε the function defined by (2.15) and $\mathcal{P}_\varepsilon u_\varepsilon$ arising in (2.12). Subtracting both expressions and using that $W_\varepsilon = 1$ in $\overline{G}_\varepsilon$, we obtain

$$\begin{aligned}&\int\limits_{\Omega_\varepsilon}(|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon u - u_\varepsilon)\,dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,u_\varepsilon)u_\varepsilon\,ds\\ &\leq -\int\limits_{G_\varepsilon} f\mathcal{P}_\varepsilon u_\varepsilon\,dx - \int\limits_{G_\varepsilon}|\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\,dx.\end{aligned} \tag{6.11}$$

Besides, from (2.2), (2.18) and (6.11), we deduce

$$\begin{aligned}&K(\|\nabla(u - W_\varepsilon u - u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)})\\ &\leq \int\limits_{\Omega_\varepsilon}(|\nabla(u - W_\varepsilon u)|^{p-2}\nabla(u - W_\varepsilon u) - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)\nabla(u - W_\varepsilon u - u_\varepsilon)\,dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,u_\varepsilon)u_\varepsilon\,ds \leq Z^1_\varepsilon + Z^2_\varepsilon\end{aligned} \tag{6.12}$$

where

$$Z^1_\varepsilon \equiv \int\limits_{\Omega_\varepsilon}(|\nabla(u - W_\varepsilon u)|^{p-2}\nabla(u - W_\varepsilon u) - |\nabla u|^{p-2}\nabla u)\nabla(u - W_\varepsilon u - u_\varepsilon)\,dx,$$

and

$$Z^2_\varepsilon \equiv -\int\limits_{G_\varepsilon} f\mathcal{P}_\varepsilon u_\varepsilon\,dx - \int\limits_{G_\varepsilon}|\nabla u|^{p-2}\nabla u\nabla(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\,dx.$$

Let us estimate Z^1_ε and Z^2_ε.

From (2.23), (2.24), (2.16), (2.10) and the embedding of $L^r(\Omega)$ into $L^s(\Omega)$ for $s < r$, we have

$$|Z^\varepsilon_1| \leq K\|\nabla W_\varepsilon\|^{p-1}_{L^p(\Omega)}\|\nabla(u - W_\varepsilon u - \mathcal{P}_\varepsilon u_\varepsilon)\|_{L^p(\Omega)} \leq K\varepsilon^{(\alpha(n-p)-n)(p-1)/p}. \tag{6.13}$$

Moreover, by the embedding of the space $W^{1,p}(\Omega,\partial\Omega)$ into $L^{np/(n-p)}(\Omega)$ for $p<n$, we deduce

$$
\begin{aligned}
|Z_\varepsilon^2| \leq & \|f\|_{L^{np/(np-n+p)}(G_\varepsilon)}\|\mathcal{P}_\varepsilon u_\varepsilon\|_{L^{np/(n-p)}(\Omega)} + K|G_\varepsilon|^{1/q}\|\nabla(u-W_\varepsilon u-\mathcal{P}_\varepsilon u_\varepsilon)\|_{L^p(\Omega)} \\
\leq & |G_\varepsilon|^{1/n}\|f\|_{L^q(\Omega)}\|\mathcal{P}_\varepsilon u_\varepsilon\|_{W^{1,p}(\Omega)} + K|G_\varepsilon|^{1/q}\|\nabla(u-W_\varepsilon u-\mathcal{P}_\varepsilon u_\varepsilon)\|_{L^p(\Omega)} \\
\leq & K[\varepsilon^{\alpha-1}+\varepsilon^{(\alpha-1)n(p-1)/p}] \leq K\varepsilon^{\alpha-1}.
\end{aligned}
\tag{6.14}
$$

Finally, gathering (6.12), (6.13) and (6.14), we obtain

$$
\|\nabla(u-W_\varepsilon u-u_\varepsilon)\|^p_{L^p(\Omega_\varepsilon)}+\varepsilon^{-\gamma}\|u_\varepsilon\|^p_{L^p(S_\varepsilon)} \leq K\varepsilon^{\min((\alpha(n-p)-n)(p-1)/p),\alpha-1)}. \tag{6.15}
$$

To get (6.10) from (6.15), we apply the Poincaré inequality for the extension $\mathcal{P}_\varepsilon(u-W_\varepsilon u-u_\varepsilon)\in W^{1,p}(\Omega,\partial\Omega)$ as in Theorem 3.2, and the theorem is proved. □

7. The most critical relation when $p=n$

In this section, we consider the case where $p=n$, $n\geq 3$, and a more general geometry than that in Sections 3–6 (see Figure 1). For the sake of brevity, we only provide the homogenized problem of (1.1) and the corresponding corrector in the most critical situation, namely, what can be the analogous case to the big point in Figures 2-3. Further specifying, among all the possible relations between the parameters β_ε, ε and a_ε we consider the critical size of the perforations provided by the relation $\varepsilon^{n/(n-1)}\ln(a_\varepsilon^{-1})=O(1)$, and the critical relation for the adsorption parameter which is obtained when β_ε multiplied by the total area of the perforations is of order 1. Conditions (7.2) give the mentioned relations while (1.12) give particular choices or a_ε and β_ε satisfying (7.2).

Considering problem (1.1) in perforated domains Ω_ε, with isoperimetric perforations of arbitrary shape (cf. (7.1)), in Theorem 7.4, we prove the convergence of the solution towards that of the homogenized problem in (1.9), which is a boundary value problem in Ω with the strange term in the partial differential equation containing a double contribution on the boundary of the perforations, namely, the contribution due to the constraint $u_\varepsilon\geq 0$ and $\partial_{\nu_n}u_\varepsilon\geq-\beta_\varepsilon\sigma(x,u_\varepsilon)$. Due to the last constraint, the function $H(x,u)$ arising in the strange term is implicitly defined from a functional equation (cf. (1.10)) in which also the perimeter of the perforations l appears for any shape. We refer to Proposition 2.2 for the existence and uniqueness of the solution $H=H(x,u)$ of (1.10) and its properties, as well as Section 8 for examples of explicit solutions for certain data σ. The result on the corrector and improved convergence is in Theorem 7.5. We follow the scheme of proofs in Section 3.

Let us first introduce the geometrical configuration of the problem, the new test functions that we need to prove convergence and some preliminary results.

Let M be a finite subset of $\mathbb{Z}$ which we can identify with $\{1,2,\cdots,m_M\}$ for $m_M\in\mathbb{Z}$. Assume that we have the set $\mathcal{M}$ of domains D^m satisfying the following properties: for any $m\in M$, $\overline{D^m}\subset T_{1/4}\subset Y$, where $Y=(-1/2,1/2)^n$, $T_{1/4}=\{y\in\mathbb{R}^n:|y|<1/4\}$, D^m is diffeomorphic to a ball $m\in M$, and the area of D^m is equal to a given number $l>0$, i.e.

$$
|\partial D^m|=l,\ \forall m\in M. \tag{7.1}
$$

We define

$$
G_\varepsilon=\bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G^j+\varepsilon j)=\bigcup_{j\in\Upsilon_\varepsilon}G_\varepsilon^j,
$$

where G^j coincides with one of the domains D^m, $m\in M$, and $\Upsilon_\varepsilon=\{j\in\mathbb{Z}^n:\overline{G_\varepsilon^j}\subset Y_\varepsilon^j=\varepsilon Y+\varepsilon j, G_\varepsilon^j\cap\overline{\widetilde{\Omega}_\varepsilon}\neq\emptyset\}$ (see Figure 1). Obviously, we have $|\Upsilon_\varepsilon|\cong d\varepsilon^{-n}$, with some $d>0$, and

$$
\overline{G_\varepsilon^j}\subset T^j_{a_\varepsilon}\subset T^j_{\varepsilon/4}\subset Y^j_\varepsilon,
$$

where $T^j_{a_\varepsilon}$ and $T^j_{\varepsilon/4}$ are balls with radius a_ε and $\varepsilon/4$, respectively, and center P^j_ε, which coincides with the center of Y^j_ε. Now we can define

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ S_\varepsilon = \partial G_\varepsilon,\ \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Let us consider (1.1) when $p = n$, and the ε-depending parameters a_ε and β_ε satisfy

$$\beta_\varepsilon^{1/(n-1)}\, a_\varepsilon\, \varepsilon^{-n/(n-1)} \to \widetilde{C}_0^2 \quad \text{and} \quad \varepsilon^{n/(n-1)} \ln(4a_\varepsilon/\varepsilon) \to -\widetilde{\alpha}^2, \tag{7.2}$$

where $\widetilde{C}_0$ and $\widetilde{\alpha}$ are some constants different from zero. Recall that σ arising in (1.1) satisfies (2.1)–(2.3) with $\delta \in [n-1,\infty)$. Also, its variational formulation reads (2.5)–(2.6).

Using the monotonicity of the function $\sigma(x,u)$ with respect to $u \in \mathbb{R}$ (cf. (2.2)) and of the function $|\lambda|^{n-2}\lambda$ with respect $\lambda \in \mathbb{R}^n$, (2.3) and the continuous embedding of $W^{1,n}(\Omega_\varepsilon,\partial\Omega)$ into $L^r(S_\varepsilon)$ for $n \le r < \infty$, we have the following result for the solution of (1.1) (equivalently of (2.5)–(2.6)).

Theorem 7.1. *Let $\varepsilon > 0$, $f \in L^{n/(n-1)}(\Omega)$, and a_ε and β_ε given by* (7.2)*. Then, problem* (2.5)–(2.6) *has a unique solution $u_\varepsilon \in K_\varepsilon$ which also satisfies the inequality*

$$\int_{\Omega_\varepsilon} |\nabla\phi|^{n-2}\nabla\phi\nabla(\phi-u_\varepsilon)\,dx + \beta_\varepsilon \int_{S_\varepsilon} \sigma(x,\phi)(\phi-u_\varepsilon)\,ds \ge \int_{\Omega_\varepsilon} f(\phi-u_\varepsilon)\,dx, \qquad \forall\phi \in K_\varepsilon. \tag{7.3}$$

In addition, for u_ε the solution of (2.5)–(2.6)*, there exists an extension $\mathcal{P}_\varepsilon u_\varepsilon$ of u_ε to Ω, $\mathcal{P}_\varepsilon u_\varepsilon \in W^{1,n}(\Omega,\partial\Omega)$ with the following properties*

$$\|\mathcal{P}_\varepsilon u_\varepsilon\|_{W^{1,n}(\Omega)} \le K\|u_\varepsilon\|_{W^{1,n}(\Omega_\varepsilon)}, \qquad \|\nabla\mathcal{P}_\varepsilon u_\varepsilon\|_{L^n(\Omega)} \le K\|\nabla u_\varepsilon\|_{L^n(\Omega_\varepsilon)}$$

and

$$\|\mathcal{P}_\varepsilon u_\varepsilon\|^n_{W^{1,n}(\Omega)} + \beta_\varepsilon\|u_\varepsilon\|^n_{L^n(S_\varepsilon)} \le K\|f\|^{n/(n-1)}_{L^{n/(n-1)}(\Omega)}. \tag{7.4}$$

The proof of Theorem 7.1 holds by rewriting the proof of Theorem 2.1 with minor modifications. Considering (7.4), for each sequence of ε we can extract a subsequence (still denoted by ε) such that as $\varepsilon \to 0$

$$\mathcal{P}_\varepsilon u_\varepsilon \rightharpoonup u \text{ in } W^{1,n}(\Omega,\partial\Omega) \quad \text{and} \quad \mathcal{P}_\varepsilon u_\varepsilon \to u \text{ in } L^r(\Omega) \text{ for any } r \in [1,\infty), \tag{7.5}$$

for a certain function u which, once identified, provides the convergences (7.5) for the whole sequence of ε. The aim of the section is to obtain the homogenized problem satisfied by the function u in (7.5) (see Theorem 7.4).

To do this, we introduce the functions Q_ε and $\mathsf{W}_\varepsilon \in W^{1,n}(\Omega,\partial\Omega)$ as follows: For $j \in \Upsilon_\varepsilon$, let $q^j_\varepsilon(x)$ be the solution of the problem

$$\begin{cases} \Delta_n q^j_\varepsilon = 0 & \text{in } T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ q^j_\varepsilon = 1 & \text{on } \partial G^j_\varepsilon, \\ q^j_\varepsilon = 0 & \text{on } \partial T^j_{\varepsilon/4}, \end{cases} \tag{7.6}$$

and we introduce the function $Q_\varepsilon \in W^{1,n}(\Omega,\partial\Omega)$ by setting

$$Q_\varepsilon(x) = q^j_\varepsilon(x), \quad x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \tag{7.7}$$

extended by 1 inside G^j_ε, $j \in \Upsilon_\varepsilon$, and by 0 in $\mathbb{R}^n \setminus \bigcup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}$.

Similarly, for $j \in \Upsilon_\varepsilon$, let $\mathsf{w}^j_\varepsilon(x)$ be the solution of the problem

$$\begin{cases} \Delta_n \mathsf{w}^j_\varepsilon = 0 & \text{in } T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}}, \\ \mathsf{w}^j_\varepsilon = 1 & \text{on } \partial T^j_{a_\varepsilon}, \\ \mathsf{w}^j_\varepsilon = 0 & \text{on } \partial T^j_{\varepsilon/4}. \end{cases} \tag{7.8}$$

It can be easily verified that

$$\mathsf{w}_\varepsilon^j = \Big(\ln\Big(\frac{4a_\varepsilon}{\varepsilon}\Big)\Big)^{-1}\ln\Big(\frac{4|x-P_\varepsilon^j|}{\varepsilon}\Big) \tag{7.9}$$

We define the function $\mathsf{W}_\varepsilon \in W^{1,n}(\Omega,\partial\Omega)$ by setting

$$\mathsf{W}_\varepsilon(x) = \mathsf{w}_\varepsilon^j(x), \quad x \in T^j_{\varepsilon/4}\setminus \overline{T^j_{a_\varepsilon}},\ j\in\Upsilon_\varepsilon, \tag{7.10}$$

extended by 1 inside $T^j_{a_\varepsilon}$, $j\in\Upsilon_\varepsilon$, and by 0 in $\mathbb{R}^n\setminus\bigcup_{j\in\Upsilon_\varepsilon}T^j_{\varepsilon/4}$. Thus, we compute

$$\begin{aligned}
\|\nabla\mathsf{W}_\varepsilon\|^m_{L^m(\Omega)} &\le K|\varepsilon\ln(4a_\varepsilon/\varepsilon)|^{-m} \qquad \text{if } 1\le m<n,\\
\|\nabla\mathsf{W}_\varepsilon\|^n_{L^n(\Omega)} &\le K|\varepsilon^{n/(n-1)}\ln(4a_\varepsilon/\varepsilon)|^{1-n}
\end{aligned}$$

and, since $\varepsilon^{n/(n-1)}\ln(4a_\varepsilon/\varepsilon)\to-\widetilde{\alpha}^2$ as $\varepsilon\to0$, we have

$$\|\nabla\mathsf{W}_\varepsilon\|^m_{L^m(\Omega)} \le K\varepsilon^{m/(n-1)} \quad \text{if } 1\le m<n, \qquad \|\nabla\mathsf{W}_\varepsilon\|^n_{L^n(\Omega)}\le K, \tag{7.11}$$

and

$$\mathsf{W}_\varepsilon \rightharpoonup 0 \ \text{ in } W^{1,n}(\Omega) \text{ as } \varepsilon\to0. \tag{7.12}$$

It should be noted that because of the geometry of the G^j, in general, the function q^j_ε, defined by (7.6), cannot be explicitly constructed. Lemma 7.2 provides us some properties for Q_ε by means of comparison with W_ε in Ω (see Lemma 2 in [39] for the proof).

Lemma 7.2. *Let us assume that* $\varepsilon^{n/(n-1)}\ln(4a_\varepsilon/\varepsilon)\to-\widetilde{\alpha}^2$ *as* $\varepsilon\to0$. *Let* Q_ε *and* W_ε *be defined by* (7.7) *and* (7.10) *respectively. Then, we have*

$$\|\mathsf{W}_\varepsilon - Q_\varepsilon\|_{W^{1,n}(\Omega)} \le K\varepsilon^{\frac{1}{n-1}}. \tag{7.13}$$

Also for the sake of completeness, we introduce the following result.

Lemma 7.3. *Let* $\widetilde{Y}_\varepsilon=\varepsilon(-1/2,1/2)^n\setminus a_\varepsilon\overline{G}_0$ *where* G_0 *is a domain of* $\mathbb{R}^n$ *diffeomorphic to a ball, and* $0<a_\varepsilon<\varepsilon/4$ *such that* $a_\varepsilon\overline{G}_0\subset\varepsilon(-1/2,1/2)^n$. *If* $\varphi\in W^{1,n}(\widetilde{Y}_\varepsilon)$, *then*

$$\|\varphi\|^n_{L^n(a_\varepsilon\partial G_0)} \le K[a_\varepsilon^{n-1}\varepsilon^{-n}\|\varphi\|^n_{L^n(\widetilde{Y}_\varepsilon)} + a_\varepsilon^{n-1}|\ln(\varepsilon/a_\varepsilon)|^{n-1}\|\nabla\varphi\|^n_{L^n(\widetilde{Y}_\varepsilon)}].$$

The proof of Lemma 7.3 holds applying the technique in Lemma 2 in [33] for $p=2$ (cf. Lemma 2.5).

Theorem 7.4. *Let* a_ε *and* β_ε *satisfy* (7.2) *and let* u_ε *be the weak solution of problem* (1.1) *with* $p=n$. *Then, the limit function* u *of the extension of* u_ε, *defined by* (7.5), *is the weak solution of the problem* (1.9)–(1.10).

Proof. First, let us note that on account that Proposition 2.2, equation (1.10) has a unique solution $H\equiv H(x,u)$, which is a continuously differentiable function in $\overline{\Omega}\times(\mathbb{R}\setminus\{0\})$ and continuous in $\overline{\Omega}\times\mathbb{R}$, and satisfies $H(x,0)=0$, (2.20) and (2.21) with $p=n$. Also, we observe that the weak solution of problem (1.9) is the solution in $W^{1,n}(\Omega,\partial\Omega)$ of the integral equation

$$\int_\Omega |\nabla u|^{n-2}\nabla u\nabla\phi\,dx + \widetilde{\mathcal{A}_n}\int_\Omega\Big(|H(x,v^+)|^{n-2}H(x,v^+)+|v^-|^{n-2}v^-\Big)\phi\,dx = \int_\Omega f\phi\,dx, \qquad \forall\phi\in W^{1,n}(\Omega,\partial\Omega). \tag{7.14}$$

From the monotonicity of the function $|\lambda|^{n-2}\lambda$ and (2.20) with $p=n$, the existence and uniqueness of solution of (7.14) holds (cf. [30]).

Let us consider the function

$$\phi = v - Q_\varepsilon(H(x,v^+)+v^-), \tag{7.15}$$

where $v \in C_0^\infty(\Omega)$, Q_ε is the function defined by (7.7) and $H(x,\tau)$ is the solution of the functional equation (1.10). Because of (7.7) and (2.21), we can check that $\phi \geq 0$ on S_ε and, hence, it belongs to K_ε. We now take ϕ as a test function in (7.3); by definition of Q_ε, we get

$$\begin{aligned}&\int_{\Omega_\varepsilon} |\nabla(v - Q_\varepsilon(H(x,v^+) + v^-))|^{n-2}\nabla(v - Q_\varepsilon(H(x,v^+) + v^-))\nabla(v - Q_\varepsilon(H(x,v^+) + v^-) - u_\varepsilon)\,dx \\ &+ \beta_\varepsilon \int_{S_\varepsilon} \sigma(x, v^+ - H(x,v^+))(v^+ - H(x,v^+) - u_\varepsilon)\,ds \geq \int_{\Omega_\varepsilon} f(v - Q_\varepsilon(H(x,v^+) + v^-) - u_\varepsilon)\,dx\end{aligned} \tag{7.16}$$

and we pass to the limit in (7.16) when $\varepsilon \to 0$.

We denote by $\mathcal{T}_\varepsilon$ the first integral on the left hand side of (7.16) and by $\widetilde{H}$ the function $\widetilde{H} \equiv H(x,v^+) + v^-$. Thus, we have

$$\begin{aligned}\mathcal{T}_\varepsilon =& \int_{\Omega_\varepsilon} \Big(|\nabla(v - \mathsf{W}_\varepsilon\widetilde{H} + (\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})|^{n-2} - |\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\Big)\nabla(v - Q_\varepsilon\widetilde{H})\nabla(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)\,dx \\ &+ \int_{\Omega_\varepsilon} |\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla(v - Q_\varepsilon\widetilde{H})\nabla(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)\,dx = \mathcal{T}_\varepsilon^a + \mathcal{T}_\varepsilon^b + \mathcal{T}_\varepsilon^c + \mathcal{T}_\varepsilon^d,\end{aligned}$$

where

$$\mathcal{T}_\varepsilon^a \equiv \int_{\Omega_\varepsilon} \Big(|\nabla(v - \mathsf{W}_\varepsilon\widetilde{H} + (\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})|^{n-2} - |\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\Big)\nabla(v - Q_\varepsilon\widetilde{H})\nabla(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)\,dx,$$

$$\mathcal{T}_\varepsilon^b \equiv \int_{\Omega_\varepsilon} |\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})\nabla(v - \mathsf{W}_\varepsilon\widetilde{H} - u_\varepsilon)\,dx,$$

$$\mathcal{T}_\varepsilon^c \equiv \int_{\Omega_\varepsilon} |\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla((\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})\nabla(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)\,dx$$

and

$$\mathcal{T}_\varepsilon^d \equiv \int_{\Omega_\varepsilon} |\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})\nabla((\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})\,dx.$$

Using Hölder inequality, (7.13), (7.11) and (7.4), it follows

$$|\mathcal{T}_\varepsilon^d| \leq \|\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})\|_{L^n(\Omega_\varepsilon)}^{n-1}\|\nabla((\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})\|_{L^n(\Omega_\varepsilon)}$$

and

$$|\mathcal{T}_\varepsilon^c| \leq \|\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})\|_{L^n(\Omega_\varepsilon)}^{n-2}\|\nabla((\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})\|_{L^n(\Omega_\varepsilon)}\|\nabla(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)\|_{L^n(\Omega_\varepsilon)},$$

which converge towards zero as $\varepsilon \to 0$. Moreover, taking into account the inequalities (9.3) and (9.4) with $p = n$ (see, for the technique, the estimate $|R_\varepsilon^a|$ for $p > 3$ in the proof of Proposition 2.3), we have

$$\begin{aligned}|\mathcal{T}_\varepsilon^a| \leq& K\int_{\Omega_\varepsilon} |\nabla((\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})|(|\nabla(v - \mathsf{W}_\varepsilon\widetilde{H})| + |\nabla((\mathsf{W}_\varepsilon - Q_\varepsilon)\widetilde{H})|)^{n-3} \\ &\times |\nabla(v - Q_\varepsilon\widetilde{H})||\nabla(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)|\,dx \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla(v-\mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla(v-\mathsf{W}_\varepsilon\widetilde{H})\nabla(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon)\,dx.$$

Now, we write the last integral, $\mathcal{T}_\varepsilon^b$, as

$$\int_{\Omega_\varepsilon}|\nabla(v-\mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla(v-\mathsf{W}_\varepsilon\widetilde{H})\nabla(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon)\,dx=\mathcal{Z}_\varepsilon^a+\mathcal{Z}_\varepsilon^b+\mathcal{Z}_\varepsilon^c+\mathcal{Z}_\varepsilon^d,$$

where

$$\mathcal{Z}_\varepsilon^a\equiv\int_{\Omega_\varepsilon}(|\nabla(v-\mathsf{W}_\varepsilon\widetilde{H})|^{n-2}-|\nabla v|^{n-2})\nabla v\nabla(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon)\,dx,\quad \mathcal{Z}_\varepsilon^b\equiv\int_{\Omega_\varepsilon}|\nabla v|^{n-2}\nabla v\nabla(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon)\,dx,$$

$$\mathcal{Z}_\varepsilon^c\equiv-\int_{\Omega_\varepsilon}(|\nabla(v-\mathsf{W}_\varepsilon\widetilde{H})|^{n-2}-|\nabla(\mathsf{W}_\varepsilon\widetilde{H})|^{n-2})\nabla(\mathsf{W}_\varepsilon\widetilde{H})\nabla(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon)\,dx$$

and

$$\mathcal{Z}_\varepsilon^d\equiv-\int_{\Omega_\varepsilon}|\nabla(\mathsf{W}_\varepsilon\widetilde{H})|^{n-2}\nabla(\mathsf{W}_\varepsilon\widetilde{H})\nabla(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon)\,dx.$$

From Proposition 2.3, (7.11) and (7.4), we obtain $|\mathcal{Z}_\varepsilon^a|\to 0$ and $|\mathcal{Z}_\varepsilon^c|\to 0$ as $\varepsilon\to 0$. Besides, on account of (7.12), (7.5) and the size of G_ε, we deduce

$$\lim_{\varepsilon\to 0}\mathcal{Z}_\varepsilon^b=\int_\Omega|\nabla v|^{n-2}\nabla v\nabla(v-u)\,dx.$$

Finally, by (7.11), we get

$$\lim_{\varepsilon\to 0}\mathcal{Z}_\varepsilon^d=-\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla\mathsf{W}_\varepsilon|^{n-2}\nabla\mathsf{W}_\varepsilon\nabla(|\widetilde{H}|^{n-2}\widetilde{H}(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon))\,dx.$$

Thus, gathering the above convergences, we obtain

$$\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon=\int_\Omega|\nabla v|^{n-2}\nabla v\nabla(v-u)\,dx-\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla\mathsf{W}_\varepsilon|^{n-2}\nabla\mathsf{W}_\varepsilon\nabla(|\widetilde{H}|^{n-2}\widetilde{H}(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon))\,dx. \tag{7.17}$$

Now, let us consider the second term on the the right hand side of (7.16) and let us prove that

$$\lim_{\varepsilon\to 0}\Big(\beta_\varepsilon\int_{S_\varepsilon}\sigma(x,v^+-H(x,v^+))(v^+-H(x,v^+)-u_\varepsilon)\,ds-\int_{\Omega_\varepsilon}|\nabla\mathsf{W}_\varepsilon|^{n-2}\nabla\mathsf{W}_\varepsilon\nabla(|\widetilde{H}|^{n-2}\widetilde{H}(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon))\,dx\Big)$$
$$\leq\widetilde{\mathcal{A}_n}\int_\Omega\Big(|H(x,v^+)|^{n-2}H(x,v^+)+|v^-|^{n-2}v^-\Big)(v-u)\,dx. \tag{7.18}$$

By the definition of W_ε and the Green formula, we have

$$\begin{aligned}&\int\limits_{\Omega_\varepsilon} |\nabla \mathsf{W}_\varepsilon|^{n-2}\nabla \mathsf{W}_\varepsilon \nabla(|\widetilde{H}|^{n-2}\widetilde{H}(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon))\,dx \\ &=\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} |\nabla \mathsf{w}^j_\varepsilon|^{n-2}\partial_{\nu_n}\mathsf{w}^j_\varepsilon|\widetilde{H}|^{n-2}\widetilde{H}(v-u_\varepsilon)\,ds+\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}} |\nabla \mathsf{w}^j_\varepsilon|^{n-2}\partial_{\nu_n}\mathsf{w}^j_\varepsilon|\widetilde{H}|^{n-2}\widetilde{H}(v-\widetilde{H}-u_\varepsilon)\,ds.\end{aligned}\tag{7.19}$$

Moreover, using the properties of $H(x,u)$, we get $H(x,v^+)v^-=0$ and, hence,

$$|\widetilde{H}|^{n-2}\widetilde{H}=|H(x,v^+)|^{n-2}H(x,v^+)+|v^-|^{n-2}v^-.\tag{7.20}$$

Thus, combining (7.19), (7.20) and (7.9) yields

$$\int\limits_{\Omega_\varepsilon} |\nabla \mathsf{W}_\varepsilon|^{n-2}\nabla \mathsf{W}_\varepsilon \nabla(|\widetilde{H}|^{n-2}\widetilde{H}(v-\mathsf{W}_\varepsilon\widetilde{H}-u_\varepsilon))\,dx=\mathcal{H}^a_\varepsilon+\mathcal{H}^b_\varepsilon\tag{7.21}$$

where

$$\mathcal{H}^a_\varepsilon\equiv\Big|\frac{4}{\varepsilon\ln(\frac{4a_\varepsilon}{\varepsilon})}\Big|^{n-2}\frac{4}{\varepsilon\ln(\frac{4a_\varepsilon}{\varepsilon})}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}}(|H(x,v^+)|^{n-2}H(x,v^+)+|v^-|^{n-2}v^-)(v-u_\varepsilon)\,ds$$

and

$$\mathcal{H}^b_\varepsilon\equiv-\Big|\frac{1}{a_\varepsilon\ln(\frac{4a_\varepsilon}{\varepsilon})}\Big|^{n-2}\frac{1}{a_\varepsilon\ln(\frac{4a_\varepsilon}{\varepsilon})}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{a_\varepsilon}}(|H(x,v^+)|^{n-2}H(x,v^+)+|v^-|^{n-2}v^-)(v^+-H(x,v^+)-u_\varepsilon)\,ds.$$

On account of (7.5) and (7.2), we apply Lemma 2.8 and have

$$-\lim_{\varepsilon\to0}\mathcal{H}^a_\varepsilon=\widetilde{\mathcal{A}_n}\int\limits_\Omega\Big(|H(x,v^+)|^{n-2}H(x,v^+)+|v^-|^{n-2}v^-\Big)(v-u)\,dx.\tag{7.22}$$

Therefore, the proof of (7.18) is completed by showing

$$\lim_{\varepsilon\to0}\Big(\beta_\varepsilon\int\limits_{S_\varepsilon}\sigma(x,v^+-H(x,v^+))(v^+-H(x,v^+)-u_\varepsilon)\,ds-\mathcal{H}^b_\varepsilon\Big)\le0.\tag{7.23}$$

To prove (7.23), we have to introduce a set of functions $\{m^j_\varepsilon\}_{j\in\Upsilon_\varepsilon}$: for each $j\in\Upsilon_\varepsilon$, we consider the problem

$$\Delta_n m^j=0 \text{ in } T_1\setminus\overline{G^j},\qquad \partial_{\nu_n}m^j=\frac{l}{\omega_n}\text{ on }\partial T_1,\qquad \partial_{\nu_n}m^j=-1\text{ on }\partial G^j,\tag{7.24}$$

which has a unique solution defined up to an additive constant. We note that because for $j\in\Upsilon_\varepsilon$, $G^j\in\mathcal{M}$, we are dealing with a finite number of different functions (7.24). For $j\in\Upsilon_\varepsilon$, we set

$$m^j_\varepsilon(x)=\varepsilon^{\frac{n}{n-1}}m^j\Big(\frac{x-P^j_\varepsilon}{a_\varepsilon}\Big)\quad\text{for } x\in T^j_{a_\varepsilon}\setminus\overline{G^j_\varepsilon}.\tag{7.25}$$

It is easy to see that $m^j_\varepsilon(x)$ is a solution of the following problem

$$\Delta_n m^j_\varepsilon = 0 \text{ in } T^j_{a_\varepsilon} \setminus \overline{G^j_\varepsilon}, \qquad \partial_{\nu_n} m^j_\varepsilon = a_\varepsilon^{1-n}\varepsilon^n \frac{l}{\omega_n} \text{ on } \partial T^j_{a_\varepsilon}, \qquad \partial_{\nu_n} m^j_\varepsilon = -a_\varepsilon^{1-n}\varepsilon^n \text{ on } \partial G^j_\varepsilon. \tag{7.26}$$

We take $h_\varepsilon \equiv (|H(x,v^+)|^{n-2}H(x,v^+) + |v^-|^{n-2}v^-)(v^+ - H(x,v^+) - u_\varepsilon)$ as a test function in the integral identity for problem (7.26) and we get

$$\Big|\sum_{j\in\Upsilon_\varepsilon} \int_{T^j_{a_\varepsilon}\setminus\overline{G^j_\varepsilon}} |\nabla m^j_\varepsilon|^{n-2}\nabla m^j_\varepsilon \nabla h_\varepsilon\, dx\Big| = \varepsilon^n a_\varepsilon^{1-n}\Big|\int_{S_\varepsilon} h_\varepsilon ds - \frac{l}{\omega_n}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{a_\varepsilon}} h_\varepsilon\, ds\Big| \tag{7.27}$$

Now, by (7.25), it follows

$$\int_{T^j_{a_\varepsilon}\setminus\overline{G^j_\varepsilon}} |\nabla_x m^j_\varepsilon|^n dx = a_\varepsilon^{-n} a_\varepsilon^n \varepsilon^{n^2/(n-1)} \int_{T_1\setminus\overline{G^j}} |\nabla_y m^j|^n\, dy \le K\varepsilon^{n^2/(n-1)}$$

and, hence,

$$\sum_{j\in\Upsilon_\varepsilon} \int_{T^j_{a_\varepsilon}\setminus\overline{G^j_\varepsilon}} |\nabla m^j_\varepsilon|^n\, dx \le K\varepsilon^{n^2/(n-1)}\varepsilon^{-n} = K\varepsilon^{n/(n-1)}. \tag{7.28}$$

Thus, from (7.27), (7.28) and (7.4), we derive

$$\varepsilon^n a_\varepsilon^{1-n}\Big|\int_{S_\varepsilon} h_\varepsilon\, ds - \frac{l}{\omega_n}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{a_\varepsilon}} h_\varepsilon ds\Big| \le K\varepsilon. \tag{7.29}$$

Let us prove (7.23). To do it, we write

$$\beta_\varepsilon \int_{S_\varepsilon} \sigma(x, v^+ - H(x,v^+))(v^+ - H(x,v^+) - u_\varepsilon)\, ds - \mathcal{H}^b_\varepsilon = \mathcal{X}^a_\varepsilon + \mathcal{X}^b_\varepsilon + \mathcal{X}^c_\varepsilon + \mathcal{X}^d_\varepsilon \tag{7.30}$$

where

$$\mathcal{X}^a_\varepsilon \equiv \Big(\Big|\frac{1}{a_\varepsilon \ln(4a_\varepsilon/\varepsilon)}\Big|^{n-2}\frac{1}{a_\varepsilon \ln(4a_\varepsilon/\varepsilon)} + \frac{\varepsilon^n a_\varepsilon^{1-n}}{\widetilde{\alpha}^{2(n-1)}}\Big)\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{a_\varepsilon}} h_\varepsilon\, ds,$$

$$\mathcal{X}^b_\varepsilon \equiv \frac{\varepsilon^n a_\varepsilon^{1-n}}{\widetilde{\alpha}^{2(n-1)}}\Big(\frac{\omega_n}{l}\int_{S_\varepsilon} h_\varepsilon\, ds - \sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{a_\varepsilon}} h_\varepsilon ds\Big), \quad \mathcal{X}^c_\varepsilon \equiv \frac{\omega_n \varepsilon^n a_\varepsilon^{1-n}(\beta_\varepsilon \varepsilon^{-n} a_\varepsilon^{n-1} - \widetilde{C}_0^{2(n-1)})}{l\widetilde{\alpha}^{2(n-1)}\widetilde{C}_0^{2(n-1)}}\int_{S_\varepsilon} h_\varepsilon\, ds$$

and

$$\mathcal{X}^d_\varepsilon \equiv \beta_\varepsilon\Big(\int_{S_\varepsilon} \sigma(x, v^+ - H(x,v^+))(v^+ - H(x,v^+) - u_\varepsilon)\, ds - \frac{\omega_n}{l\widetilde{\alpha}^{2(n-1)}\widetilde{C}_0^{2(n-1)}}\int_{S_\varepsilon} h_\varepsilon\, ds\Big).$$

From (7.29), it is clear that

$$\lim_{\varepsilon\to 0} \mathcal{X}^b_\varepsilon = 0. \tag{7.31}$$

Besides, applying Lemma 7.3, we obtain

$$\|w\|^n_{L^n(S_\varepsilon)} \le K(a_\varepsilon^{n-1}\varepsilon^{-n}\|w\|^n_{L^n(\Omega_\varepsilon)} + a_\varepsilon^{n-1}|\ln(\varepsilon/a_\varepsilon)|^{n-1}\|\nabla w\|^n_{L^n(\Omega_\varepsilon)})$$

for all $w \in W^{1,n}(\Omega_\varepsilon)$ and, thus,

$$\Big|\int\limits_{S_\varepsilon} h_\varepsilon\, ds\Big| \leq K a_\varepsilon^{(n-1)^2/n}\varepsilon^{1-n}(a_\varepsilon^{n-1}\varepsilon^{-n} + a_\varepsilon^{n-1}|\ln(\varepsilon/a_\varepsilon)|^{n-1})^{1/n}\|h_\varepsilon\|_{W^{1,n}(\Omega_\varepsilon)}.$$

Then, using (7.2) and (7.4), we deduce

$$\lim_{\varepsilon\to 0}\mathcal{X}_\varepsilon^c = 0. \tag{7.32}$$

In a similar way, we have

$$\lim_{\varepsilon\to 0}\mathcal{X}_\varepsilon^a = 0. \tag{7.33}$$

Now, taking into account that H is the solution of the functional equation (1.10) and that $v^-(v^+ - H(x,v^+)) = 0$ and $u_\varepsilon \geq 0$ on S_ε, we conclude that

$$\mathcal{X}_\varepsilon^d = -\beta_\varepsilon\widetilde{\mathcal{B}_n}\int\limits_{S_\varepsilon}|v^-|^{n-2}v^-(v^+ - H(x,v^+) - u_\varepsilon)\, ds \leq 0. \tag{7.34}$$

Then, gathering (7.30), (7.33), (7.31), (7.32) and (7.34) yields (7.23), and, from (7.21), (7.22) and (7.23), (7.18) holds.

Using (7.17) and (7.18), we obtain that the limit of the left hand side of (7.16) is bounded from above by the following expression

$$\int\limits_\Omega |\nabla v|^{n-2}\nabla v\nabla(v-u)\, dx + \widetilde{\mathcal{A}_n}\int\limits_\Omega\Big(|H(x,v^+)|^{n-2}H(x,v^+) + |v^-|^{n-2}v^-\Big)(v-u)\, dx.$$

In addition, (7.5), (7.12), (7.13) and the fact that $|G_\varepsilon| \to 0$, as $\varepsilon \to 0$, give

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} f(v - Q_\varepsilon\widetilde{H} - u_\varepsilon)dx = \int\limits_\Omega f(v-u)\, dx.$$

Therefore, taking limits in (7.16), we obtain that u satisfies the following inequality

$$\int\limits_\Omega |\nabla v|^{n-2}\nabla v\nabla(v-u)\, dx + \widetilde{\mathcal{A}_n}\int\limits_\Omega\Big(|H(x,v^+)|^{n-2}H(x,v^+) + |v^-|^{n-2}v^-\Big)(v-u)\, dx \geq \int\limits_\Omega f(v-u)\, dx, \tag{7.35}$$

for all $v \in W^{1,n}(\Omega,\partial\Omega)$. As usual, taking $v = u \pm \lambda\phi$ in (7.35) where $\phi \in W^{1,n}(\Omega,\partial\Omega)$ and passing to the limit as $\lambda \to +0$, we get (7.14), which concludes the proof. □

Finally we state the corrector result whose proof is performed by re-writing the proof of Theorem 3.2 with the suitable modifications introduced by the value of p, the definition (7.7) and the Theorem 7.4.

Theorem 7.5. *Let a_ε and β_ε satisfy* (7.2) *and let u_ε be the weak solution of problem* (1.1) *with $p = n$. Let $u \in W^{1,n}(\Omega,\partial\Omega)$ the weak solution of the boundary value problem* (1.9) *with the additional regularity $u \in W^{1,\infty}(\Omega)$, and Q_ε defined by* (7.7)*. Then, as $\varepsilon \to 0$, we have*

$$\|u_\varepsilon - u + Q_\varepsilon(H(x,u^+) + u^-)\|^n_{W^{1,n}(\Omega_\varepsilon)} + \beta_\varepsilon\|u_\varepsilon - u^+ + H(x,u^+)\|^n_{L^n(S_\varepsilon)} \to 0.$$

8. Final comments

Here, we gather some comments and remarks about the extensions of the results throughout the paper.

As regards the most critical situation (point I in the table of Section 1), let us note that in the case where $\sigma(x,u) = b(x)|u|^{p-2}u$, with $b(x)$ a strictly positive continuously differentiable function in $\overline{\Omega}$, we can solve explicitly the functional equation (2.19) for $p \in [2,n]$; namely, we can define the solution H of (2.19) explicitly in terms of $b(x)$ and u. As a matter of fact, we obtain

$$H(x,u) = \frac{(\varrho b(x))^{1/(p-1)}}{1 + (\varrho b(x))^{1/(p-1)}}\, u, \tag{8.1}$$

where ϱ contains information on the averaged constant of the problem (cf. also (2.21) and (2.22)).

Since all the results of the paper apply to this case, $\sigma(x,u) = b(x)|u|^{p-2}u$, we observe that the dependence of the nonlinear strange term on $b(x)$ ranges from linear to nonlinear or no dependence (cf. the table in Section 1); the nonlinear dependence appearing for the most critical case (cf. Sections 3 and 7).

Also, an important point to underline is that in the case where $b(x) \equiv b$ is a positive constant, even for the most critical situation, arbitrary shapes of the cavities (periodically placed) can be considered and some kind of capacity constant will likely appear in the homogenized problem. The latter can be easily shown for $p = 2$ suitably modifying proofs in Section 3, although, to our knowledge, the result for variational inequalities is not found in the literature.

As regards the geometrical configuration of the problem, we observe that, for $p \in [2,n)$, the limit behavior of the solution of (1.1) remains to be obtained in the cases where the cavities G_ε are not balls or there is not periodicity of the structure. For the case where $p = 2$, different shapes of the domains have been considered in [24, 25] for boundary value problems outside the most critical situation (namely, outside the big point in Figures 2 and 3). As a matter of fact, the local problem obtained from the microstructure of the original problem, strongly depends on the center of the cavities and makes it difficult to guess the homogenized problem. This fact has been observed in very different homogenization problems in perforated media, with linear partial differential equations and with different boundary conditions or constraints on the boundary condition, and always related with critical sizes of the cavities. Sometimes the difficulty can be overcome by considering periodicity of the coefficients arising in the partial differential equations or more restrictive parameters and functions arising on the Robin boundary conditions (cf., e.g., [11, 35] and references therein). Also let us note that other techniques avoiding local problems could allow less restrictive geometrical configuration to be considered: see [7] and [35] for two second order elliptic operators with oscillating coefficients and two types of cavities, in each periodicity cell, with Signorini condition or nonlinear Robin condition on the boundary of one of these cavities. However, we highlight that, in the existing literature, the most critical case (cf. Point I on the table of Section 1.1) for arbitrary shapes of perforations has not been considered even when $p = 2$: cf. [24] for arbitrary shapes when $\sigma = bu$ and $p = 2$; see the above paragraph in this connection.

For nonlinear Robin boundary conditions, even for the Laplace operator, in the most critical situation, the problem has been unsolved for a long time. [24] considers perforations that are not necessarily balls but the problem for the most critical relation remained as an open problem for any geometry of the perforations until [21]. [21] appears as the first paper in the literature where an implicitly defined homogenized problem is outlined, the perforations being for balls. In fact [21, 24, 25, 44] consider the Laplace operator in perforated media over the whole domain and their results complement each other. However [21, 44] consider only spherical cavities while the cavities can be of different shapes in [24, 25] but for relations between parameters outside the big point. See [18, 20] for a long list of references on related problems.

In the most critical situation, for nonlinear Robin boundary conditions, the Laplacian and $n = 2$, namely, $p = n = 2$, we refer to [37] for general geometries of the cavities and to [18] when $p = 2$, $n = 3$ and the domain is perforated by tubes. An extension to $p = n \geq 3$ can be found in [39]. Here, in Section 7, we consider a different problem (cf. (1.1)), with unilateral constraints. Also, a more restrictive σ is considered in [18, 37, 39]. However, it should be emphasized that, in any case, the perimeter of the perforations arises in the strange term instead of the shape of the perforations as one might expect for $n \geq 3$. In these cases, the difference to broach the problem for the different values of p, namely $2 \leq p < n$ and $p = n$, recalls the difference when $n \geq 3$ and $n = 2$ for the Laplacian in [9], or the Stokes equations in [2], both with Dirichlet conditions on the boundary of the perforations. When considering the adsorption, the perimeter of the perforations also arises in the homogenized problem (cf. [37] for further details on the differences when $p = n = 2$.)

Related with the nonlinear data of our problem, we note that the hypotheses (2.1)–(2.3) on the function $\sigma(x, u)$ in this paper seem to be optimal and allow us to provide a general framework for results and proofs. However, many of the results hold true under weaker hypotheses for σ. Actually, the strong monotonicity outlined in (2.2) can be changed by the weaker hypothesis of strict monotonicity or only monotonicity depending on the relations for parameters or the appropriate improved convergence. This can be seen in a simple way when verifying proofs.

However, as noticed in [22], the adsorption isotherms used mostly in the literature are of the form $\sigma(x, u) = g(u)$ with g a positive strictly increasing function in $[0, \infty)$. In this connection, we also note that certain proofs can be adapted for functions σ both with less smoothness or increasing requirements. We refer to [5] for explicit definitions of σ arising in models from ecology, hydrogeology or chemical reactions, for comments on possible extensions when $u \leq 0$, and for further references.

Remark 8.1. Note that, in this paper, we give results for the p-Laplacian on perforated domains, by tiny cavities, with constraints for solutions and their normal derivatives on the boundary of the cavities, which are completly new in the literature. Dealing with unilateral constraints for the p-Laplacian and the homogenization of perforated media, we mention very different problems and results in [36] for Signorini conditions (when $\alpha = 1$) and [26, 41] for obstacle problems. For different constraints and sizes of perforations, we provide a map of all possible homogenized problems and construct the corresponding correctors (see Figures 2 and 3). In particular, we obtain seven different limits when $p \in [2, n)$, most than ever found for the p-Laplace operator in perforated media (see [28, 40] to compare). In this connection, [40] considers a ε-dependent boundary value problem with generalized Robin condition (no constraints for solutions), without any corrector result and with a more restrictive σ. Except one, the strange terms in [40] are different since they cannot get the double influence coming from the constraints on the solutions and on their normal derivatives (cf. Section 1); among all the homogenized problems here obtained, only (1.5) and (1.8) coincide with some homogenized problems in [40] for some σ. In addition, to improve the weak convergence obtained in [40], it suffices to re-write the proofs for correctors in Sections 2–6 with the suitable modifications. Similar comments apply to the results in [37] and [39] when $p = n = 2$ and $p = n \geq 3$, respectively, and to our new strange term in (1.9) and corrector in Theorem 7.5. Dealing with the results in [16, 19, 23], see the end of Section 1.

9. Appendix

To avoid introducing technical details in Section 2.1, we provide here the proof of Propositions 2.2 and 2.3 that we have not found in the literature.

Proof of Proposition 2.2. Let us consider $z = \tau - H$ and rewrite the equation as $|z-\tau|^{p-2}(z-\tau) + \varrho\,\sigma(x, z) = 0$. Considering the continuously differentiable function $F(x, \tau, z) = |z - \tau|^{p-2}(z - \tau) + \varrho\,\sigma(x, z)$ defined from a domain of $\mathbb{R}^{n+2}$ into $\mathbb{R}$ and taking into account that only the points of the form $(x, \tau, z) = (x, 0, 0)$, $x \in \overline{\Omega}$, can verify $F(x, \tau, z) = 0$ and $\partial_z F(x, \tau, z) = 0$, the implicit function theorem provides the continuously differentiable function in $\overline{\Omega} \times (\mathbb{R} \setminus \{0\})$, $z = U(x, \tau)$ such that

$$|U(x, \tau) - \tau|^{p-2}(U(x, \tau) - \tau) + \varrho\,\sigma(x, U(x, \tau)) = 0.$$

Then, writing $H(x, \tau) = \tau - U(x, \tau)$ for $(x, \tau) \in \overline{\Omega} \times (\mathbb{R} \setminus \{0\})$ and $H(x, 0) = 0$ for $x \in \overline{\Omega}$ (cf. (2.4)), we verify that H is the unique continuous function in $\overline{\Omega} \times \mathbb{R}$ which satisfies (2.19).

Moreover, on account of (2.19) and the monotonicity of the functions $|\lambda|^{p-2}\lambda$ and $\sigma(x,z)$ (cf. (2.18) and (2.2), respectively), we have

$$\begin{aligned}&(|H(x,u)|^{p-2}H(x,u)-|H(x,v)|^{p-2}H(x,v))(u-v)\\ &=(|H(x,u)|^{p-2}H(x,u)-|H(x,v)|^{p-2}H(x,v))(H(x,u)-H(x,v))\\ &\quad+\varrho\,(\sigma(x,u-H(x,u))-\sigma(x,v-H(x,v)))(u-H(x,u)-v+H(x,v))\\ &\geq k_3|H(x,u)-H(x,v)|^p+\varrho\,k_1|u-H(x,u)-v+H(x,v)|^p\geq \widetilde{k}_1|u-v|^p,\end{aligned}$$

for all $x\in\overline{\Omega}$, $u,v\in\mathbb{R}$, and certain constant $\widetilde{k}_1>0$, and (2.20) holds. Condition (2.20) and the fact that $H(x,0)=0$ imply (2.22). Finally, using (2.19), (2.22) and (2.4) we can prove (2.21), which concludes the proof of the proposition. □

Proof of Proposition 2.3. First, we write the term on the left hand side of (2.23) as

$$\int_{\Omega_\varepsilon}\Big(|\nabla(v+\eta_\varepsilon)|^{p-2}\nabla(v+\eta_\varepsilon)-|\nabla v|^{p-2}\nabla v\Big)\nabla\varphi\,dx=R^a_\varepsilon+R^b_\varepsilon+R^c_\varepsilon \tag{9.1}$$

where

$$R^a_\varepsilon\equiv\int_{\Omega_\varepsilon}\Big(|\nabla(v+\eta_\varepsilon)|^{p-2}-|\nabla v|^{p-2}\Big)\nabla v\nabla\varphi\,dx,\qquad R^b_\varepsilon\equiv\int_{\Omega_\varepsilon}\Big(|\nabla(v+\eta_\varepsilon)|^{p-2}-|\nabla\eta_\varepsilon|^{p-2}\Big)\nabla\eta_\varepsilon\nabla\varphi\,dx$$

and

$$R^c_\varepsilon\equiv\int_{\Omega_\varepsilon}|\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\nabla\varphi\,dx.$$

Then, we apply the inequality

$$||\mathsf{a}+\mathsf{b}|^{p-2}-|\mathsf{b}|^{p-2}|\leq|\mathsf{a}|^{p-2},\qquad\forall\mathsf{a},\mathsf{b}\in\mathbb{R}^n, \tag{9.2}$$

for $p\in(2,3]$, obtained from the Minkowskii and the triangle inequalities, and the inequalities

$$(a+b)^{p-2}-b^{p-2}\leq a(p-2)(a+b)^{p-3},\qquad\forall a,b\geq 0, \tag{9.3}$$

and

$$(a+b)^{p-3}\leq K(a^{p-3}+b^{p-3}),\qquad\forall a,b\geq 0, \tag{9.4}$$

for $p>3$, to estimate the term $R_\varepsilon\equiv R^a_\varepsilon+R^b_\varepsilon$ arising in (2.23).

Let us assume $p\in(2,3]$. Using (9.2) and the Hölder inequality we obtain

$$|R^a_\varepsilon|\leq\int_{\Omega_\varepsilon}||\nabla(v+\eta_\varepsilon)|^{p-2}-|\nabla v|^{p-2}||\nabla v||\nabla\varphi|dx\leq\int_{\Omega_\varepsilon}|\nabla\eta_\varepsilon|^{p-2}|\nabla v||\nabla\varphi|dx\leq K\|\nabla\eta_\varepsilon\|^{p-2}_{L^{(p-2)p/(p-1)}(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)}$$

and

$$|R^b_\varepsilon|\leq\int_{\Omega_\varepsilon}||\nabla(v+\eta_\varepsilon)^{p-2}-|\nabla\eta_\varepsilon|^{p-2}||\nabla\eta_\varepsilon||\nabla\varphi|\,dx\leq\int_{\Omega_\varepsilon}|\nabla v|^{p-2}|\nabla\eta_\varepsilon||\nabla\varphi|\,dx\leq K\|\nabla\eta_\varepsilon\|_{L^{p/(p-1)}(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)},$$

and, hence, (2.24) holds for $p\in(2,3]$.

Let us assume $p>3$. We can write

$$\begin{aligned}|R^a_\varepsilon|&\leq\int_{\Omega_\varepsilon}||\nabla(v+\eta_\varepsilon)|^{p-2}-|\nabla v|^{p-2}||\nabla v||\nabla\varphi|\,dx\\ &=\int_{\Omega_\varepsilon\cap D^+_\varepsilon}(|\nabla(v+\eta_\varepsilon)|^{p-2}-|\nabla v|^{p-2})|\nabla v||\nabla\varphi|\,dx+\int_{\Omega_\varepsilon\cap D^-_\varepsilon}(|\nabla v|^{p-2}-|\nabla(v+\eta_\varepsilon)|^{p-2})|\nabla v||\nabla\varphi|\,dx,\end{aligned}$$

where $D_\varepsilon^+ = \{x : |\nabla(v+\eta_\varepsilon)(x)|^{p-2} - |\nabla v(x)|^{p-2} > 0\}$ and $D_\varepsilon^- = \{x : |\nabla(v+\eta_\varepsilon)(x)|^{p-2} - |\nabla v(x)|^{p-2} < 0\}$. Now, using (9.3), (9.4) and the Hölder inequality we obtain

$$\begin{aligned}&\int_{\Omega_\varepsilon\cap D_\varepsilon^+}(|\nabla(v+\eta_\varepsilon)|^{p-2} - |\nabla v|^{p-2})|\nabla v||\nabla\varphi|\,dx \le \int_{\Omega_\varepsilon\cap D_\varepsilon^+}((|\nabla v| + |\nabla\eta_\varepsilon|)^{p-2} - |\nabla v|^{p-2})|\nabla v||\nabla\varphi|\,dx\\ \le&(p-2)\int_{\Omega_\varepsilon}|\nabla\eta_\varepsilon|(|\nabla v| + |\nabla\eta_\varepsilon|)^{p-3}|\nabla v||\nabla\varphi|\,dx \le K\int_{\Omega_\varepsilon}(|\nabla\eta_\varepsilon||\nabla v|^{p-2} + |\nabla\eta_\varepsilon|^{p-2}|\nabla v|)|\nabla\varphi|\,dx\\ \le&K(\|\nabla\eta_\varepsilon\|_{L^{p/(p-1)}(\Omega)} + \|\nabla\eta_\varepsilon\|^{p-2}_{L^{(p-2)p/(p-1)}(\Omega)})\|\nabla\varphi\|_{L^p(\Omega)}.\end{aligned}$$

Similarly, we can estimate

$$\begin{aligned}&\int_{\Omega_\varepsilon\cap D_\varepsilon^-}(|\nabla v|^{p-2} - |\nabla(v+\eta_\varepsilon)|^{p-2})|\nabla v||\nabla\varphi|\,dx\\ \le&\int_{\Omega_\varepsilon\cap D_\varepsilon^-}((|\nabla(v+\eta_\varepsilon)| + |\nabla\eta_\varepsilon|)^{p-2} - |\nabla(v+\eta_\varepsilon)|^{p-2})|\nabla v||\nabla\varphi|\,dx\\ \le&(p-2)\int_{\Omega_\varepsilon}|\nabla\eta_\varepsilon|(|\nabla(v+\eta_\varepsilon)| + |\nabla\eta_\varepsilon|)^{p-3}|\nabla v||\nabla\varphi|\,dx \le K\int_{\Omega_\varepsilon}(|\nabla\eta_\varepsilon||\nabla v|^{p-2} + |\nabla\eta_\varepsilon|^{p-2}|\nabla v|)|\nabla\varphi|\,dx\\ \le&K(\|\nabla\eta_\varepsilon\|_{L^{p/(p-1)}(\Omega)} + \|\nabla\eta_\varepsilon\|^{p-2}_{L^{(p-2)p/(p-1)}(\Omega)})\|\nabla\varphi\|_{L^p(\Omega)}.\end{aligned}$$

Consequently,

$$|R_\varepsilon^a| \le K[\|\nabla\eta_\varepsilon\|_{L^{p/(p-1)}(\Omega)} + \|\nabla\eta_\varepsilon\|^{p-2}_{L^{(p-2)p/(p-1)}(\Omega)}]\|\nabla\varphi\|_{L^p(\Omega)}.$$

Similar arguments allow us to obtain the same estimate for $|R_\varepsilon^b|$ and, thus, (2.24) also holds for $p > 3$.

Now, (2.25) follows from (9.1) and the fact that R_ε^c also converges to zero as $\varepsilon \to 0$ under the assumption $\|\nabla\eta_\varepsilon\|_{L^p(\Omega)} \to 0$, which concludes the proof. □

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