EXTRACTA MATHEMATICAE Vol. **31**, Núm. 1, 37 – 46 (2016)

Local Spectral Theory for Operators R and S Satisfying $RSR = R^2$

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Received February 5, 2016

Abstract: We study some local spectral properties for bounded operators R, S, RS and SR in the case that R and S satisfy the operator equation $RSR = R^2$. Among other results, we prove that S, R, SR and RS share Dunford's property (C) when $RSR = R^2$ and $SRS = S^2$.

Key words: Local spectral subspace, Dunford's property (C), operator equation.

AMS *Subject Class.* (2010): 47A10, 47A11, 47A53, 47A55.

1. Introduction and preliminaries

The equivalence of Dunford's property (C) for products RS and SR of operators $R \in L(Y, X)$ and $S \in L(X, Y)$, X and Y Banach spaces, has been studied in [2]. As noted in [13] the proof of Theorem 2.5 in [2] contains a gap which was filled up in [13, Theorem 2.7]. In [2] it was also studied property (C) for operators $R, S \in L(X)$ which satisfy the operator equations

$$RSR = R^2 \quad \text{and} \quad SRS = S^2. \tag{1}$$

A similar gap exists in the proof of Theorem 3.3 in [2], which states the equivalence of property (C) for R, S, RS and SR, when R, S satisfy (1).

In this paper we give a correct proof of this result and we prove further results concerning the local spectral theory of R, S, RS and SR, in particular we show several results concerning the quasi-nilpotent parts and the analytic cores of these operators. It should be noted that these results are established in a more general framework, assuming that only one of the operator equations in (1) holds.

Supported in part by MICINN (Spain), Grant MTM2013-45643.

We shall denote by X a complex infinite dimensional Banach space. Given a bounded linear operator $T \in L(X)$, the *local resolvent set* of T at a point $x \in X$ is defined as the union of all open subsets $\mathcal{U}$ of $\mathbb{C}$ such that there exists an analytic function $f : \mathcal{U} \to X$ satisfying

$$(\lambda I - T)f(\lambda) = x \quad \text{for all} \quad \lambda \in \mathcal{U} \ . \tag{2}$$

The local spectrum $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Obviously, $\sigma_T(x) \subseteq \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T.

The following result shows that $\sigma_T(Tx)$ and $\sigma_T(x)$ may differ only at 0. It was proved in [7] for operators satisfying the SVEP.

Lemma 1.1. *For every $T \in L(X)$ and $x \in X$ we have*

$$\sigma_T(Tx) \subseteq \sigma_T(x) \subseteq \sigma_T(Tx) \cup \{0\}. \tag{3}$$

Moreover, if T is injective then $\sigma_T(Tx) = \sigma_T(x)$ for all $x \in X$.

Proof. Take $S = T$ and $R = I$ in [6, Proposition 3.1]. ∎

For every subset $\mathcal{F}$ of $\mathbb{C}$, the *local spectral subspace* of T at $\mathcal{F}$ is the set

$$X_T(\mathcal{F}) := \{x \in X : \sigma_T(x) \subseteq \mathcal{F}\}.$$

It is easily seen from the definition that $X_T(\mathcal{F})$ is a linear subspace T-invariant of X. Furthermore, for every closed $\mathcal{F} \subseteq \mathbb{C}$ we have

$$(\lambda I - T)X_T(\mathcal{F}) = X_T(\mathcal{F}) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathcal{F}. \tag{4}$$

See [9, Proposition 1.2.16].

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_o \in \mathbb{C}$ (abbreviated SVEP at λ_o), if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f : \mathbf{D}_{\lambda_o} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{5}$$

is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. Clearly, the SVEP is inherited by the restrictions to invariant subspaces.

A variant of $X_T(\mathcal{F})$ which is more useful for operators without SVEP is the *glocal spectral subspace* $\mathcal{X}_T(\mathcal{F})$. For an operator $T \in L(X)$ and a closed subset $\mathcal{F}$ of $\mathbb{C}$, we define $\mathcal{X}_T(\mathcal{F})$ as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus \mathcal{F} \to X$ which satisfies

$$(\lambda I - T)f(\lambda) = x \quad \text{for all} \;\; \lambda \in \mathbb{C} \setminus \mathcal{F}.$$

Clearly $\mathcal{X}_T(\mathcal{F}) \subseteq X_T(\mathcal{F})$ for every closed $\mathcal{F} \subseteq \mathbb{C}$. Moreover T has SVEP if and only if

$$\mathcal{X}_T(\mathcal{F}) = X_T(\mathcal{F}) \quad \text{for all closed subsets} \quad \mathcal{F} \subseteq \mathbb{C}.$$

See [9, Proposition 3.3.2]. Note that $\mathcal{X}_T(\mathcal{F})$ and $X_T(\mathcal{F})$ are not closed in general.

Given a closed subspace Z of X and $T \in L(X)$, we denote by $T|Z$ the restriction of T to Z.

Lemma 1.2. [2, Lemmas 2.3 and 2.4] *Let $\mathcal{F}$ be a closed subset of $\mathbb{C}$ and $T \in L(X)$.*

(1) *If $0 \in \mathcal{F}$ and $Tx \in X_T(\mathcal{F})$ then $x \in X_T(\mathcal{F})$.*

(2) *Suppose T has SVEP, $Z := X_T(\mathcal{F})$ is closed, and $A := T|X_T(\mathcal{F})$. Then $X_T(\mathcal{K}) = Z_A(\mathcal{K})$ for all closed $\mathcal{K} \subseteq \mathcal{F}$.*

Lemma 1.3. *Suppose that T has SVEP and $\mathcal{F}$ is a closed subset of $\mathbb{C}$ such that $0 \notin \mathcal{F}$. If $X_T(F \cup \{0\})$ is closed then $X_T(\mathcal{F})$ is closed.*

Proof. Set $Z := X_T(\mathcal{F} \cup \{0\})$ and $S := T|Z$. By [9, Proposition 1.2.20] we have $\sigma(S) \subseteq \mathcal{F} \cup \{0\}$. We suppose first that $0 \notin \sigma(S)$. Then $\sigma(S) \subseteq \mathcal{F}$, hence $Z = Z_S(\mathcal{F})$. By Lemma 1.2 we have $Z_S(\mathcal{F}) = X_T(\mathcal{F})$, so $X_T(\mathcal{F})$ is closed. For the case $0 \in \sigma(S)$, we set $\mathcal{F}_0 := \sigma(S) \cap \mathcal{F}$. Then $\sigma(S) = \mathcal{F}_0 \cup \{0\}$. Since $0 \in \sigma(S)$, by Lemma 1.2 we have $Z = Z_S(\mathcal{F}_0) \oplus Z_S(\{0\})$ and

$$Z_S(\mathcal{F}_0) = Z_S(\sigma(S) \cap \mathcal{F}) = Z_S(\mathcal{F}) = X_T(\mathcal{F}),$$

hence $X_T(\mathcal{F})$ is closed. ∎

2. Operator equation $RSR = R^2$

Operators $S, R \in L(X)$ satisfying the operator equations $RSR = R^2$ and $SRS = S^2$ were studied first in [12], and more recently in [10], [11], [8], and other papers. An easy example of operators for which these equations hold is given in the case that $R = PQ$ and $S = QP$, where $P, Q \in L(X)$ are idempotents. A remarkable result of Vidav [12, Theorem 2] shows that if R, S are self-ajoint operators on a Hilbert space then the equations (1) hold if and only if there exists an (uniquely determined) idempotent P such that $R = PP^*$ and $S = P^*P$, where P^* is the adjoint of P.

The operators R, S, SR and RS for which the equations (1) hold share many spectral properties ([10], [11]), and local spectral properties as decomposability, property (β) and SVEP ([8]). In this section we consider the permanence of property (C), property (Q) in this context.

It is easily seen that if $0 \notin \sigma(R) \cap \sigma(S)$ then $R = S = I$, so this case is trivial. Thus we shall assume that $0 \in \sigma(R) \cap \sigma(S)$. Evidently, the operator equation $RSR = R^2$ implies

$$(SR)^2 = SR^2 \quad \text{and} \quad (RS)^2 = R^2 S.$$

LEMMA 2.1. *Suppose that $R, S \in L(X)$ satisfy $RSR = R^2$. Then for every $x \in X$ we have*

$$\sigma_R(Rx) \subseteq \sigma_{SR}(x) \quad \text{and} \quad \sigma_{SR}(SRx) \subseteq \sigma_R(x). \tag{6}$$

Proof. For the first inclusion, suppose that $\lambda_0 \notin \sigma_{SR}(x)$. Then there exists an open neighborhood $\mathcal{U}_0$ of λ_0 and an analytic function $f : \mathcal{U}_0 \to X$ such that

$$(\lambda I - SR) f(\lambda) = x \quad \text{for all} \;\; \lambda \in \mathcal{U}_0.$$

From this it follows that

$$\begin{aligned} Rx &= R(\lambda I - SR) f(\lambda) = (\lambda R - RSR) f(\lambda) \\ &= (\lambda R - R^2) f(\lambda) = (\lambda I - R)(Rf)(\lambda), \end{aligned}$$

for all $\lambda \in \mathcal{U}_0$. Since $Rf : \mathcal{U}_0 \to X$ is analytic we get $\lambda_0 \notin \sigma_R(Rx)$.

For the second inclusion, let $\lambda_0 \notin \sigma_R(x)$. Then there exists an open neighborhood $\mathcal{U}_0$ of λ_0 and an analytic function $f : \mathcal{U}_0 \to X$ such that

$$(\lambda I - R) f(\lambda) = x \quad \text{for all} \;\; \lambda \in \mathcal{U}_0.$$

Consequently,

$$\begin{aligned} SRx &= SR(\lambda I - R) f(\lambda) = (\lambda SR - SR^2) f(\lambda) \\ &= (\lambda SR - (SR)^2) f(\lambda) = (\lambda I - SR)(SRf)(\lambda), \end{aligned}$$

for all $\lambda \in \mathcal{U}_0$, and since $(SR)f$ is analytic we obtain $\lambda_0 \notin \sigma_{SR}(SRx)$. ∎

THEOREM 2.2. *Let $S, R \in L(X)$ satisfy $RSR = R^2$, and let $\mathcal{F}$ be a closed subset of $\mathbb{C}$ with $0 \in \mathcal{F}$. Then $X_R(\mathcal{F})$ is closed if and only if so is $X_{SR}(\mathcal{F})$.*

Proof. Suppose that $X_R(\mathcal{F})$ is closed and let (x_n) be a sequence of $X_{SR}(\mathcal{F})$ which converges to $x \in X$. We need to show that $x \in X_{SR}(\mathcal{F})$. For every $n \in \mathbb{N}$ we have $\sigma_{SR}(x_n) \subseteq \mathcal{F}$ and hence, by Lemma 2.1, we have $\sigma_R(Rx_n) \subseteq \mathcal{F}$, i.e. $Rx_n \in X_R(\mathcal{F})$. Since $0 \in \mathcal{F}$, by Lemma 1.2 we have $x_n \in X_R(\mathcal{F})$, and since $X_R(\mathcal{F})$ is closed, $x \in X_R(\mathcal{F})$, i.e. $\sigma_R(x) \subseteq \mathcal{F}$. Now from Lemma 2.1 we derive $\sigma_{SR}(SRx) \subseteq \mathcal{F}$, and this implies $SRx \in X_{SR}(\mathcal{F})$. Again by Lemma 1.2, we obtain $x \in X_{SR}(\mathcal{F})$, thus $X_{SR}(\mathcal{F})$ is closed.

Conversely, suppose that $X_{SR}(\mathcal{F})$ is closed and let (x_n) be a sequence of $X_R(\mathcal{F})$ which converges to $x \in X$. Then $\sigma_R(x_n) \subseteq \mathcal{F}$ for every $n \in \mathbb{N}$, hence $\sigma_{SR}(SRx_n) \subseteq \mathcal{F}$, i.e. $SRx_n \in X_{SR}(\mathcal{F})$ by Lemma 2.1. But $0 \in \mathcal{F}$, so, by Lemma 1.2, $x_n \in X_{SR}(\mathcal{F})$. Since $X_{SR}(\mathcal{F})$ is closed, $x \in X_{SR}(\mathcal{F})$, hence $\sigma_{SR}(x) \subseteq \mathcal{F}$. Now from Lemma 2.1 we obtain $\sigma_R(Rx) \subseteq \mathcal{F}$, i.e. $Rx \in X_R(\mathcal{F})$, and the condition $0 \in \mathcal{F}$ implies $x \in X_R(\mathcal{F})$. ∎

The following result is inspired by [8, Theorem 2.1].

LEMMA 2.3. *Let $S, R \in L(X)$ be such that $RSR = R^2$ and one of the operators R, SR, RS has SVEP. Then all of them have SVEP. Additionally, if $SRS = S^2$ and one of R, S, SR, RS has SVEP then all of them have SVEP.*

Proof. By [6, Proposition 2.1], SR has SVEP if and only if RS has SVEP. So it is enough to prove that R has SVEP at λ_0 if an only if so has RS.

Suppose that R has SVEP at λ_0 and let $f : \mathcal{U}_0 \to X$ be an analytic function on an open neighborhood $\mathcal{U}_0$ of λ_0 for which $(\lambda I - RS)f(\lambda) \equiv 0$ on U_0. Then $RSf(\lambda) = \lambda f(\lambda)$ and

$$0 = RS(\lambda I - RS)f(\lambda) = (\lambda RS - (RS)^2)f(\lambda) = (\lambda RS - (R^2 S)f(\lambda) \\ = (\lambda I - R)RSf(\lambda).$$

The SVEP of R at λ_0 implies that

$$RSf(\lambda) = \lambda f(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{U}_0.$$

Hence $f \equiv 0$ on U_0, and we conclude that RS has SVEP at λ_0.

Conversely, suppose that RS has SVEP at λ_0 and let $f : \mathcal{U}_0 \to X$ be an analytic function on an open neighborhood $\mathcal{U}_0$ of λ_0 such that $(\lambda I - R)f(\lambda) \equiv 0$ on U_0. Then $R^2 f(\lambda) = \lambda R f(\lambda) = \lambda^2 f(\lambda)$ for all $\lambda \in \mathcal{U}_0$. Moreover,

$$\begin{aligned} 0 &= RS(\lambda I - R)f(\lambda) = \lambda RSf(\lambda) - R^2 f(\lambda) = \lambda RSf(\lambda) - \lambda^2 f(\lambda) \\ &= (\lambda I - RS)(-\lambda f(\lambda)), \end{aligned}$$

and since RS has SVEP at λ_0 we have $\lambda f(\lambda) \equiv 0$, hence $f(\lambda) \equiv 0$, so R has SVEP at λ_0.

The second assertion is clear, if $SRS = S^2$, just interchanging R and S in the argument above, the SVEP fo S holds if and only if SR, or equivalently RS, has SVEP. ∎

We now consider the result of Theorem 2.2 when $0 \notin \mathcal{F}$.

THEOREM 2.4. *Let $\mathcal{F}$ be a closed subset of $\mathbb{C}$ such that $0 \notin \mathcal{F}$. Suppose that $R, S \in L(X)$ satisfy $RSR = R^2$ and R has SVEP. Then we have*

(1) *If $X_R(\mathcal{F} \cup \{0\})$ is closed then $X_{SR}(\mathcal{F})$ is closed.*

(2) *If $X_{SR}(\mathcal{F} \cup \{0\})$ is closed then $X_R(\mathcal{F})$ is closed.*

Proof. (1) Let us denote $\mathcal{F}_1 := \mathcal{F} \cup \{0\}$. The set $\mathcal{F}_1$ is closed, and by assumption $X_R(\mathcal{F}_1)$ is closed. Since $0 \in F_1$ then $X_{SR}(\mathcal{F}_1)$ is closed, by Theorem 2.2. Moreover, the SVEP for R is equivalent to the SVEP for SR by Lemma 2.3. Then $X_{SR}(\mathcal{F})$ is closed by Lemma 1.3.

(2) The argument is similar: if $X_{SR}(\mathcal{F} \cup \{0\})$ is closed then $X_R(\mathcal{F} \cup \{0\})$ by Theorem 2.2, and since R has SVEP, $X_R(\mathcal{F})$ is closed by Lemma 1.3. ∎

DEFINITION 2.5. An operator $T \in L(X)$ is said to have *Dunford's property* (C) (abbreviated *property* (C)) if $\mathcal{X}_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$.

It should be noted that Dunford property (C) implies SVEP.

THEOREM 2.6. *Suppose that $S, R \in L(X)$ satisfy $RSR = R^2$, and any one of the operators R, SR, RS, has property (C).Then all of them have property (C). If, additionally, $SRS = S^2$ and one of R, S, RS, SR has property (C), then all of them have property (C).*

Proof. Since property (C) implies SVEP, all the operators have SVEP by Lemma 2.3. Moreover the equivalence of property (C) for SR and RS has been proved in [2] (see also [13]). So it is enough to prove that R has property (C) if an only if so has RS.

Suppose that R has property (C) and let $\mathcal{F}$ be a closed set. If $0 \in \mathcal{F}$ then $X_{SR}(\mathcal{F})$ is closed, by Theorem 2.2, while in the case where $0 \notin \mathcal{F}$ we have that $X_R(\mathcal{F} \cup \{0\})$ is closed, and hence, by part (1) of Theorem 2.4, the SVEP for R ensures that also in this case $X_{SR}(\mathcal{F})$ is closed. Therefore, SR has property (C).

Conversely, suppose that SR has property (C). For every closed subset $\mathcal{F}$ containing 0, $X_R(\mathcal{F})$ is closed by Theorem 2.2. If $0 \notin \mathcal{F}$ then $X_{SR}(\mathcal{F} \cup \{0\})$ is closed, hence $X_R(\mathcal{F})$ is closed by part (2) of Theorem 2.4 and we conclude that R has property (C).

If additionally, $SRS = S^2$ then, by interchanging S with R, the same argument above proves the second assertion, so the proof is complete. ∎

Next we consider the case when $\mathcal{F}$ is a singleton set, say $\mathcal{F} := \{\lambda\}$. The glocal spectral subspace $\mathcal{X}_T(\{\lambda\})$ coincides with the *quasi-nilpotent part* $H_0(\lambda I - T)$ of $\lambda I - T$ defined by

$$H_0(\lambda I - T) := \{x \in X : \limsup_{n\to\infty} \|(\lambda I - T)^n x\|^{1/n} = 0\}.$$

See [1, Theorem 2.20]. In general $H_0(\lambda I - T)$ is not closed, but it coincides with the kernel of a power of $\lambda I - T$ in some cases [3, Theorem 2.2].

DEFINITION 2.7. An operator $T \in L(X)$ is said to have the *property* (Q) if $H_0(\lambda I - T)$ is closed for every $\lambda \in \mathbb{C}$.

It is known that if $H_0(\lambda I - T)$ is closed then T has SVEP at λ ([4]), thus,

$$\text{property } (C) \Rightarrow \text{property } (Q) \Rightarrow \text{SVEP}.$$

Therefore, for operators T having property (Q) we have $H_0(\lambda I - T) = X_T(\{\lambda\})$.

In [13, Corollary 3.8] it was observed that if $R \in L(Y, X)$ and $S \in L(X, Y)$ are both injective then RS has property (Q) precisely when SR has property (Q).

Recall that $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $T(X)$ is closed and the kernel ker T is finite-dimensional, and T is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$, if the range $T(X)$ has finite codimension.

THEOREM 2.8. *Let $R, S \in L(X)$ satisfying $RSR = R^2$, and $R, S \in \Phi_+(X)$ or $R, S \in \Phi_-(X)$. Then R has property (Q) if and only if so has SR.*

Proof. Suppose that $R, S \in \Phi_+(X)$ and R has property (Q). Then R has SVEP and, by Lemma 2.3, also SR has SVEP. Consequently, the local and glocal spectral subspaces relative to the a closed set coincide for R and SR. By assumption $H_0(\lambda I - R) = X_R(\{\lambda\})$ is closed for every $\lambda \in \mathbb{C}$, and $H_0(SR) = X_{SR}(\{0\})$ is closed by Theorem 2.2. Let $0 \neq \lambda \in \mathbb{C}$. By [9, Proposition 3.3.1, part (f)]

$$X_R(\{\lambda\} \cup \{0\}) = X_R(\{\lambda\}) + X_R(\{0\}) = H_0(\lambda I - R) + H_0(R).$$

Since $R \in \Phi_+(X)$ the SVEP at 0 implies that $H_0(R)$ is finite-dimensional, see [1, Theorem 3.18], so $X_R(\{\lambda\} \cup \{0\})$ is closed. Then part (1) of Theorem 2.4 implies that $H_0(\lambda I - SR) = X_{SR}(\{\lambda\})$ is closed, hence SR has property (Q).

Conversely, suppose that SR has property (Q). If $\lambda = 0$ then $H_0(SR) = X_{RS}(\{0\})$ is closed by assumption, and $H_0(R) = X_R(\{0\})$ is closed by Theorem 2.2. In the case $\lambda \neq 0$ we have

$$X_{SR}(\{\lambda\} \cup \{0\}) = X_{SR}(\{\lambda\}) + X_{SR}(\{0\}) = H_0(\lambda I - SR) + H_0(SR).$$

Since SR has SVEP and $SR \in \Phi_+(X)$, $H_0(SR)$ is finite dimensional by [1, Theorem 3.18]. So $X_{SR}(\{\lambda\} \cup \{0\})$ is closed. By part (2) of Theorem 2.4, $X_R(\{\lambda\}) = H_0(\lambda I - R)$ is closed. Therefore R has property (Q).

The proof in the case where $R, S \in \Phi_-(X)$ is analogous. ∎

Corollary 2.9. *Let $S, R \in L(X)$ satisfy the operator equations* (1)*. If one of the operators R, S, RS and SR is bounded below and has property (Q), then all of them have property (Q).*

Proof. Note that all the operators R, S, RS, and SR are injective when one of them is injective [8, Lemma 2.3], and the same is true for being upper semi-Fredholm [8, Theorem 2.5]. Hence, if one of the operators is bounded below, then all of them are bounded below.

By Theorem 2.8 property (Q) for R and for SR are equivalent. So the same is true for S and RS, and also for RS and SR since R and S are injective. ∎

The *analytical core* $K(T)$ of $T \in L(X)$ is defined [1, Definition 1.20] as the set of all $\lambda \in \mathbb{C}$ for which there exists a constant $\delta > 0$ and a sequence (u_n) in X such that $x = u_0$, and $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n \|x\|$ for each $n \in \mathbf{N}$. The following characterization can be found in [1, Theorem 2.18]:

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X : 0 \notin \sigma_T(x)\}.$$

The analytical core of T is an invariant subspace and, in general, is not closed.

THEOREM 2.10. *Suppose that $R, S \in L(X)$ satisfy $RSR = R^2$.*

(1) *If $0 \neq \lambda \in \mathbb{C}$, then $K(\lambda I - R)$ is closed if and only $K(\lambda I - SR)$ is closed, or equivalently $K(\lambda I - RS)$ is closed.*

(2) *If R is injective, then $K(R)$ is closed if and only $K(SR)$ is closed, or equivalently $K(RS)$ is closed.*

Proof. (1) Suppose $\lambda \neq 0$ and $K(\lambda I - R)$ closed. Let (x_n) be a sequence of $K(\lambda I - SR)$ which converges to $x \in X$. Then $\lambda \notin \sigma_{SR}(x_n)$ and hence, by Lemma 2.1, $\lambda \notin \sigma_R(Rx_n)$, thus $Rx_n \in K(\lambda I - R)$. Since $Rx_n \to Rx$ and $K(\lambda I - R)$ is closed, it then follows that $Rx \in K(\lambda I - R)$, i.e., $\lambda \notin \sigma_R(Rx)$. Since $\lambda \neq 0$, by Lemma 1.1 we have $\lambda \notin \sigma_R(x)$, hence $\lambda \notin \sigma_{SR}(SRx)$ again by Lemma 2.1. By Lemma 1.1 this implies $\lambda \notin \sigma_{SR}(x)$. Therefore $x \in K(\lambda I - SR)$, and consequently, $K(\lambda I - SR)$ is closed.

Conversely, suppose that $\lambda \neq 0$ and $K(\lambda I - SR)$ is closed. Let (x_n) be a sequence of $K(\lambda I - R)$ which converges to $x \in X$. Then $\lambda \notin \sigma_R(x_n)$ and, by Lemma 2.1, we have $\lambda \notin \sigma_{SR}(SRx_n)$. By Lemma 1.1 then we have $\lambda \notin \sigma_{SR}(x_n)$, so $x_n \in K(\lambda I - SR)$, and hence $x \in K(\lambda I - SR)$, since the last set is closed. This implies that $\lambda \notin \sigma_{SR}(x)$, and hence $\lambda \notin \sigma_R(Rx)$, again by Lemma 2.1. By Lemma 1.1 we have $\lambda \notin \sigma_R(x)$, so $x \in K(\lambda I - R)$. Therefore, $K(\lambda I - R)$ is closed. The equivalence $K(\lambda I - SR)$ is closed if and only if $K(\lambda I - RS)$ is closed was proved in [13, Corollary 3.3].

(2) The proof is analogous to that of part (1) applying Lemma 1.1. ∎

COROLLARY 2.11. *Suppose $RSR = R^2$, $SRS = S^2$ and $\lambda \neq 0$. Then the following statements are equivalent:*

(1) *$K(\lambda I - R)$ is closed;*

(2) *$K(\lambda I - SR)$ is closed;*

(3) *$K(\lambda I - RS)$ is closed;*

(4) *$K(\lambda I - S)$ is closed.*

When R is injective, the equivalence also holds for $\lambda = 0$.

Proof. The equivalence of (3) and (4) follows from Theorem 2.10, interchanging R and S. Since, as noted in the proof of Corollary 2.9, the injectivity of R is equivalent to the injectivity of S, the equivalence of (1) and (4) also holds for $\lambda = 0$. ∎

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