# $L^\infty$-NORM AND ENERGY QUANTIZATION FOR THE PLANAR LANE-EMDEN PROBLEM WITH LARGE EXPONENT

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ABSTRACT. For any smooth bounded domain $\Omega \subset \mathbb{R}^2$ we consider positive solutions to

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which satisfy the uniform energy bound

$$p\|\nabla u\|_\infty \le C$$

for $p > 1$. We prove convergence to $\sqrt{e}$ as $p \to +\infty$ of the $L^\infty$-norm of any solution. We further deduce quantization of the energy to multiples of $8\pi e$, thus completing the analysis performed in [5].

## 1. INTRODUCTION

This paper focuses on the asymptotic analysis, as $p \to +\infty$, of families of solutions to the Lane-Emden problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \qquad (\mathcal{P}_p)$$

where $\Omega$ is any smooth bounded planar domain.

This line of investigation started in [11, 12] for families $u_p$ of least energy solutions, for which a one-point concentration behavior in the interior of $\Omega$ is proved, as well as the $L^\infty$-*bounds*

$$\sqrt{e} \le \lim_{p\to+\infty} \|u_p\|_\infty \le C \qquad (1.1)$$

and the following estimate

$$\lim_{p\to+\infty} p\|\nabla u_p\|_2^2 = 8\pi e.$$

The bound in (1.1) was later improved in [1], where it was shown that for families of least energy solutions it holds:

$$\lim_{p\to+\infty} \|u_p\|_\infty = \sqrt{e}. \qquad (1.2)$$

Moreover in [1], [2] and [8] the Liouville equation in the whole plane

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U dx = 8\pi \end{cases} \qquad (1.3)$$

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was identified to be a *limit problem* for the Lane-Emden equation. Indeed in [1] it was proved that suitable rescalings around the maximum point of any least energy solution to ($\mathcal{P}_p$) converge, in $C^2_{loc}(\mathbb{R}^2)$, to the regular solution

$$U(x) = \log\left(\frac{1}{1+\frac{1}{8}|x|^2}\right)^2 \tag{1.4}$$

of (1.3). Hence least energy solutions exhibit only one concentration point and the local limit profile is given by (1.4). More general solutions having only one peak have been recently studied in [7], where their Morse index is computed and connections with the question of the uniqueness of positive solutions in convex domains are shown.

Observe that when $\Omega$ is a ball any solution to ($\mathcal{P}_p$) is radial by the result of Gidas, Ni and Nirenberg ([10]) and so the least energy is the unique solution for any $p > 1$.

In general in non-convex domains there may be families of solutions to ($\mathcal{P}_p$) other than the least energy ones. This is the case, for instance, of those found in [9] when the domain $\Omega$ is not simply connected, which have higher energy, precisely

$$\lim_{p\to+\infty} p\|\nabla u_p\|_2^2 = 8\pi e\cdot k,$$

for any fixed integer $k \geq 1$. These solutions exhibit a concentration phenomenon at $k$ distinct points in $\Omega$ as $p \to +\infty$ and their $L^\infty$-norm satisfies the same limit as in (1.2).

The question of characterizing the behavior of any family $u_p$ of positive solutions to ($\mathcal{P}_p$) naturally arises. This issue was studied in [5] in any general smooth bounded domain $\Omega$, under the uniform energy bound assumption

$$p\|\nabla u_p\|_2^2 \leq C \tag{1.5}$$

(see also [3], where the asymptotic analysis started for solutions of any sign, and the related papers [4, 6]). The results in [5] show that under the assumption (1.5) the solutions to ($\mathcal{P}_p$) are necessarily *spike-like*. More precisely in [5, Theorem 1.1] it is proved that, up to a subsequence, there exists an integer $k \geq 1$ and $k$ distinct points $x_i \in \Omega$, $i = 1,\dots,k$, such that, setting

$$\mathcal{S} = \{x_1,\dots,x_k\},$$

one has

$$\lim_{p\to+\infty} \sqrt{p}u_p = 0 \quad \text{in } C^2_{loc}(\bar{\Omega}\setminus\mathcal{S}) \tag{1.6}$$

and the energy satisfies

$$\lim_{p\to+\infty} p\|\nabla u_p\|_2^2 = 8\pi\sum_{i=1}^k m_i^2, \tag{1.7}$$

where $m_i$'s are positive constants given by

$$m_i = \lim_{\delta\to 0}\lim_{p\to+\infty}\max_{B_\delta(x_i)} u_p \tag{1.8}$$

which satisfy

$$m_i \geq \sqrt{e}. \tag{1.9}$$

Furthermore the location of the concentration points is shown to depend on the Dirichlet Green function $G$ of $-\Delta$ in $\Omega$ and on its regular part $H$

$$H(x,y) = -G(x,y) + \frac{1}{2\pi}\log\frac{1}{|x-y|}. \tag{1.10}$$

according to the following system

$$-m_i \nabla_x H(x_i,x_i) + \sum_{\ell\neq i} m_\ell \nabla_x G(x_i,x_\ell) = 0,$$

and moreover

$$\lim_{p\to+\infty} pu_p = 8\pi\sum_{i=1}^{k} m_i G(\cdot,x_i) \ \text{ in } C^2_{loc}(\bar\Omega\setminus\mathcal{S}).$$

In [5, Lemma 4.1] it is also proved that a suitable rescaling of $u_p$ around each concentration point converges to the regular solution $U$ in (1.4).
The convergence (1.6) and the inequality (1.9) immediately imply the following bound on the $L^\infty$-norm:

$$\sqrt{e} \leq \lim_{p\to+\infty} \|u_p\|_\infty \leq C. \tag{1.11}$$

In [5] it was conjectured that for all solutions to $(\mathcal{P}_p)$, under the assumption (1.5), *one should have the equality in* (1.9) *and hence also the equality in the left hand side of* (1.11).

As far as we know the only case where the equality has been proved is when $u_p$ are least-energy solutions (see [1]), as recalled before. The proof of [1] strongly uses the minimality property of the solutions and does not apply to general families of solutions.

Here we answer this question by computing the exact value of the $m_i$'s for any solution. Our result is the following:

**Theorem 1.1.** *Let $u_p$ be a family of solutions to $(\mathcal{P}_p)$ satisfying (1.5). Then*

(i) $m_i = \sqrt{e}, \quad \forall i = 1,\dots,k$
(ii) $\lim_{p\to+\infty}\|u_p\|_\infty = \sqrt{e}$.

This theorem shows that, for $p$ large, all peaks of the solutions have essentially the same height, namely $\sqrt{e}$. Moreover Theorem 1.1 allows to improve the analysis of the asymptotic behavior of the solutions to $(\mathcal{P}_p)$ performed in [5], in particular implying, by (1.7), a *quantization of the energy* to integer multiples of $8\pi e$ as $p$ goes to infinity:

**Corollary 1.2.** *Let $u_p$ be a family of solutions to $(\mathcal{P}_p)$ and assume that (1.5) holds. Then there exist a number $k\in\mathbb{N}$ and a sequence $p_n\to+\infty$ as $n\to+\infty$ such that one has*

$$\lim_{n\to\infty} p_n\int_\Omega |\nabla u_{p_n}(x)|^2\,dx = 8\pi e\cdot k. \tag{1.12}$$

This shows that each concentration point carries the same mass and implies that solutions to $(\mathcal{P}_p)$ can exist, for $p$ large, only at the level of energies given by multiples of $8\pi e$. Hence this number plays in the two dimensional case the same role as the best Sobolev constant $S$ in dimension $N\geq 3$.

## 2. Proof of Theorem 1.1

Let $k \geq 1$ and $x_i \in \Omega$, $i = 1, \dots, k$, be as in the introduction and let us keep the notation $u_p$ to denote the corresponding subsequence of the family $u_p$ for which the results in [5] hold true.

In particular (see [5, Theorem 1.1 & Lemma 4.1]) for $r > 0$ such that $B_{3r}(x_j) \subset \Omega$, for any $j = 1, \dots, k$, and $B_{3r}(x_j) \cap B_{3r}(x_i) = \emptyset$, for any $i, j = 1, \dots, k$, $j \neq i$, letting $y_{i,p} \in \Omega$ be a sequence defined as

$$u_p(y_{i,p}) := \max_{B_{2r}(x_i)} u_p \tag{2.13}$$

it follows that

$$\lim_{p \to +\infty} y_{i,p} = x_i, \tag{2.14}$$

$$\lim_{p \to +\infty} u_p(y_{i,p}) = m_i, \tag{2.15}$$

$$\lim_{p \to +\infty} \varepsilon_{i,p} \left( := \left[ p u_p(y_{i,p})^{p-1} \right]^{-1/2} \right) = 0 \tag{2.16}$$

and setting

$$w_{i,p}(y) := \frac{p}{u_p(y_{i,p})} (u_p(y_{i,p} + \varepsilon_{i,p} y) - u_p(y_{i,p})), \quad y \in \Omega_{i,p} := \frac{\Omega - y_{i,p}}{\varepsilon_{i,p}}, \tag{2.17}$$

then

$$\lim_{p \to +\infty} w_{i,p} = U \ \text{ in } \ C^2_{loc}(\mathbb{R}^2), \tag{2.18}$$

where $U$ is as in (1.4).

Furthermore by the result in [5, Proposition 4.3 & Lemma 4.4] we have that for any $\gamma \in (0, 4)$ there exists $R_\gamma > 1$ such that

$$w_{i,p}(z) \leq (4 - \gamma) \log \frac{1}{|z|} + \widetilde{C}_\gamma, \qquad \forall\ i = 1, \dots, k \tag{2.19}$$

for some $\widetilde{C}_\gamma > 0$, provided $R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{i,p}}$ and $p$ is sufficiently large.

The pointwise estimate (2.19) implies the following uniform bound, which will be the key to use the dominated convergence theorem in the proof of Theorem 1.1:

**Lemma 2.1.**

$$0 \leq \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \leq \begin{cases} 1 & \text{for } |z| \leq R_\gamma \\ C_\gamma \frac{1}{|z|^{4-\gamma}} & \text{for } R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{j,p}} \end{cases}. \tag{2.20}$$

*Proof.* Observe that by (2.14)

$$B_r(y_{i,p}) \subset B_{2r}(x_i), \text{ for } p \text{ sufficiently large},$$

as a consequence

$$w_{i,p} \leq 0, \quad \text{in } B_{\frac{r}{\epsilon_{i,p}}}(0) \ (\subset \Omega_{i,p}), \text{ for } p \text{ large}, \tag{2.21}$$

which implies the first bound in (2.20).
For $p$ sufficiently large, by (2.21) and (2.19), we also get the second bound in (2.20):

$$0 \leq \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p = e^{p \log\left(1 + \frac{w_{j,p}(z)}{p}\right)} \leq e^{w_{j,p}(z)} \leq C_\gamma \frac{1}{|z|^{4-\gamma}}$$

for $R_\gamma \le |z| \le \frac{r}{\varepsilon_{j,p}}$. □

*Proof of Theorem 1.1.* Observe that by the assumption (1.5) and Hölder inequality

$$\begin{aligned}(0\le)\ p\int_\Omega u_p^p(x)dx &\le p^{\frac{1}{p+1}}|\Omega|^{\frac{1}{p+1}}\left[p\int_\Omega|\nabla u_p|^2dx\right]^{\frac{p}{p+1}}\\ &= p\int_\Omega|\nabla u_p|^2dx(1+o_p(1))\\ &\overset{(1.5)}{\le} C+o_p(1),\end{aligned}$$

so that, by the properties of the Green function $G$,

$$\begin{aligned}\int_{\Omega\setminus B_{2r}(x_j)}G(y_{j,p},x)u_p^p(x)dx &\le C_r\int_{\Omega\setminus B_{2r}(x_j)}u_p^p(x)dx\\ &\le C_r\int_\Omega u_p^p(x)dx = O\left(\frac{1}{p}\right)\end{aligned} \tag{2.22}$$

and similarly, observing that for $p$ large enough the points $y_{j,p}\in B_{\frac{r}{2}}(x_j)$ by (2.14) and $B_{\frac{r}{2}}(x_j)\subset B_r(y_{j,p})\subset B_{2r}(x_j)$, also

$$\begin{aligned}\int_{B_{2r}(x_j)\setminus B_r(y_{j,p})}G(y_{j,p},x)u_p^p(x)dx &\le \int_{\{\frac{r}{2}<|x-x_j|<2r\}}G(y_{j,p},x)u_p^p(x)dx\\ &\le C_{\frac{r}{2}}\int_\Omega u_p^p(x)dx = O\left(\frac{1}{p}\right).\end{aligned} \tag{2.23}$$

By the Green representation formula, using the previous estimates, we then get

$$\begin{aligned}u_p(y_{j,p}) &= \int_\Omega G(y_{j,p},x)u_p^p(x)dx\\ &= \int_{B_{2r}(x_j)}G(y_{j,p},x)u_p^p(x)dx+\int_{\Omega\setminus B_{2r}(x_j)}G(y_{j,p},x)u_p^p(x)dx\\ &\overset{(2.22),(2.23)}{=} \int_{B_r(y_{j,p})}G(y_{j,p},x)u_p^p(x)dx+o_p(1)\\ &\overset{(2.17)}{=} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)}G(y_{j,p},y_{j,p}+\varepsilon_{j,p}z)\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz+o_p(1)\\ &\overset{(1.10)}{=} -\frac{u_p(y_{j,p})}{p}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)}H(y_{j,p},y_{j,p}+\varepsilon_{j,p}z)\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz\\ &\quad -\frac{u_p(y_{j,p})}{2\pi p}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)}\log|z|\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz\\ &\quad -\frac{u_p(y_{j,p})\log\varepsilon_{j,p}}{2\pi p}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)}\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz+o_p(1)\\ &= A_p+B_p+C_p+o_p(1).\end{aligned} \tag{2.24}$$

Since $H$ is smooth and $x_j \notin \partial\Omega$, by (2.14) and (2.16) we get

$$\lim_{p\to+\infty} H(y_{j,p}, y_{j,p}+\varepsilon_{j,p}z) = H(x_j,x_j), \text{ for any } z\in\mathbb{R}^2,$$

so by (2.15), the convergence (2.18) and the uniform bounds in (2.20) we can apply the dominated convergence theorem, and since the function $z\mapsto \frac{1}{|z|^{4-\gamma}}$ is integrable in $\{|z|>R_\gamma\}$ choosing $\gamma\in(0,2)$ we deduce

$$\lim_{p\to+\infty} u_p(y_{j,p})\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} H(y_{j,p},y_{j,p}+\varepsilon_{j,p}z)\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz$$
$$= m_j H(x_j,x_j)\int_{\mathbb{R}^2} e^U \overset{(1.3)}{=} 8\pi\, m_j H(x_j,x_j),$$

from which

$$A_p := -\frac{u_p(y_{j,p})}{p}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} H(y_{j,p},y_{j,p}+\varepsilon_{j,p}z)\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz = o_p(1). \tag{2.25}$$

For the second term in (2.24) we apply again the dominated convergence theorem, using (2.20) and observing now that the function $z\mapsto\frac{\log|z|}{|z|^{4-\gamma}}$ is integrable in $\{|z|>R_\gamma\}$ and that $z\mapsto\log|z|$ is integrable in $\{|z|\le R_\gamma\}$. Hence we get

$$\lim_{p\to+\infty} u_p(y_{j,p})\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \log|z|\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz = m_j\int_{\mathbb{R}^2}\log|z|e^{U(z)}dz<+\infty$$

and this implies that

$$B_p \;:=\; -\frac{u_p(y_{j,p})}{2\pi p}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \log|z|\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz = o_p(1). \tag{2.26}$$

Finally for the last term in (2.24) let us observe that by the definition of $\varepsilon_{j,p}$ in (2.16)

$$\log\varepsilon_{j,p} = -\frac{(p-1)}{2}\log u_p(y_{j,p}) - \frac{1}{2}\log p, \tag{2.27}$$

again by the dominated convergence theorem

$$\lim_{p\to+\infty}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)}\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz = \int_{\mathbb{R}^2} e^U \overset{(1.3)}{=} 8\pi, \tag{2.28}$$

and it follows

$$\begin{aligned} C_p \;&:=\; -\frac{u_p(y_{j,p})\log\varepsilon_{j,p}}{2\pi p}\int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)}\left(1+\frac{w_{j,p}(z)}{p}\right)^p dz \\ &\overset{(2.28)}{=}\; -\frac{u_p(y_{j,p})\log\varepsilon_{j,p}}{2\pi p}\,(8\pi+o_p(1)) \\ &\overset{(2.27)}{=}\; u_p(y_{j,p})\left[\frac{(p-1)}{p}\log u_p(y_{j,p})+\frac{\log p}{p}\right](2+o_p(1)). \end{aligned} \tag{2.29}$$

Substituting (2.25), (2.26) and (2.29) into (2.24) we get

$$u_p(y_{j,p}) = u(y_{j,p})\left[\frac{(p-1)}{p}\log u_p(y_{j,p})+\frac{\log p}{p}\right](2+o_p(1))+o_p(1),$$

passing to the limit as $p \to +\infty$ and using (2.15) we conclude that

$$\log m_j = \frac{1}{2}.$$

From this we immediately deduce statement (i). Then, due to the property (1.6), also (ii) follows. □

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