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Admissibility and unifiability in contact logics

Philippe Balbiani[1] and Çiğdem Gencer[2]

[1]Institut de recherche en informatique de Toulouse
CNRS — Université de Toulouse
118 route de Narbonne, 31062 Toulouse Cedex 9, FRANCE
[2]Department of Mathematics and Computer Science
Istanbul Kültür University — Faculty of Science and Letters
Ataköy Campus 34156, Bakırköy-Istanbul, TURKEY

Abstract. Contact logics are logics for reasoning about the contact relations between regular subsets in a topological space. Admissible inference rules can be used to improve the performance of any algorithm that handles provability within the context of contact logics. The decision problem of unifiability can be seen as a special case of the decision problem of admissibility. In this paper, we examine the decidability of admissibility problems and unifiability problems in contact logics.

Keywords: Contact logics, admissibility, unifiability, decidability.

1 Introduction

The decision problem of unifiability in a logical system L can be formulated as follows: given a formula $\phi(X_1, \ldots, X_n)$, determine whether there exists formulas $\psi_1, \ldots, \psi_n$ such that $\phi(\psi_1, \ldots, \psi_n) \in L$. The research on unifiability was motivated by a more general decision problem, the admissibility problem: given an inference rule "from $\{\phi_1(X_1, \ldots, X_n), \ldots, \phi_m(X_1, \ldots, X_n)\}$, infer $\psi(X_1, \ldots, X_n)$", determine whether for all formulas $\chi_1, \ldots, \chi_n$, if $\{\phi_1(\chi_1, \ldots, \chi_n), \ldots, \phi_m(\chi_1, \ldots, \chi_n)\} \subseteq L$, then $\psi(\chi_1, \ldots, \chi_n) \in L$. In 1984, Rybakov [15] proved that there exists a decision procedure for determining whether a given inference rule is admissible in intuitionistic propositional logic. See also [16]. Later on, Ghilardi [11, 12] proved that intuitionistic propositional logic has a finitary unification type and extended this result to various extensions of $K4$. See also [9, 10] where decision procedures for unifiability in extensions of $K4$ are suggested.
Contact logics are logics for reasoning about the contact relations between regular subsets in a topological space [5, 17]. They are based on the primitive notion of regular regions and on the Boolean operations (empty region, complement of a region and union of two regions) that allow to obtain new regular regions from given ones. In contact logics, formulas are built from simple formulas of the form $C(a, b)$ and $a \equiv b$ — where a and b are terms in a Boolean language — using the Boolean constructs $\perp$, $\neg$ and $\vee$, the intuitive reading of $C(a, b)$ and $a \equiv b$ being "the regular regions denoted by a and b are in contact" and "the regular regions

denoted by a and b are equal". The main semantics of contact logics are the contact algebras of the regular subsets in a topological space [6–8]. But contact logics have also received a relational semantics that allow to use methods from modal logic for studying them [4].
In this setting, one important issue is the mechanization of reasoning in contact logics. Since admissible inference rules can be used to improve the performance of any algorithm that handles provability, it becomes natural to consider admissibility and unifiability within the context of contact logics. In this paper, we will examine variants of contact logics. The central result in this paper is the proof that the admissibility problem and the unifiability problem are decidable in these variants. In Section 2, we present the syntax and the semantics of these variants. Section 3 is about their axiomatization/completeness and their decidability/complexity. In Sections 4–6, we define the admissibility problem and we study its decidability. Section 7 is about the unifiability problem and its decidability. See [16] for details about admissibility and unifiability and [17] for details about contact logics.

2 Syntax and semantics of contact logics

In this section, we present the syntax and the semantics of contact logics. We adopt the standard rules for omission of the parentheses.

2.1 Syntax

To start with syntax, let us first consider a countable set AT of atomic terms (with typical members denoted x, y, etc) and a countable set AF of atomic formulas (with typical members denoted X, Y, etc). The terms (denoted a, b, etc) are inductively defined as follows:

– $a ::= x \mid 0 \mid -a \mid (a \sqcup b)$.

The other Boolean constructs for terms (1, $\sqcap$, etc) are defined as usual. We will use the following notations:

– a^0 for $-a$,
– a^1 for a.

Reading terms as regions, the constructs 0, $-$ and $\sqcup$ should be regarded as the empty region, the complement operation and the union operation. For all positive integers n and for all $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, formulas of the form $x_1^{\epsilon_1} \sqcap \ldots \sqcap x_n^{\epsilon_n}$ will be called monoms. In the sequel, we use $a(x_1, \ldots, x_n)$ to denote a term a whose atomic terms form a subset of $\{x_1, \ldots, x_n\}$. Considering $a(x_1, \ldots, x_n)$ as a formula in classical propositional logic, let $mon(a(x_1, \ldots, x_n))$ be the set of all monoms of the form $x_1^{\epsilon_1} \sqcap \ldots \sqcap x_n^{\epsilon_n}$ inconsistent with $-a(x_1, \ldots, x_n)$, that is to say $mon(a(x_1, \ldots, x_n)) = \{x_1^{\epsilon_1} \sqcap \ldots \sqcap x_n^{\epsilon_n} \colon (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$ and $a(x_1, \ldots, x_n)$ is a tautological consequence of $x_1^{\epsilon_1} \sqcap \ldots \sqcap x_n^{\epsilon_n}\}$. The formulas (denoted ϕ, ψ, etc) are inductively defined as follows:

- $\phi ::= X \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid C(a,b) \mid a \equiv b$.

The other Boolean constructs for formulas ($\top$, $\wedge$, etc) are defined as usual. We will use the following notations:

- $\bar{C}(a,b)$ for $\neg C(a,b)$,
- $a \not\equiv b$ for $\neg a \equiv b$,
- $a \leq b$ for $a \sqcap -b \equiv 0$.

Reading formulas as properties about regions, the constructs C and $\equiv$ should be regarded as the contact relation and the equality relation. Sets of formulas will be denoted Γ, Δ, etc. Formulas and sets of formulas are also called "expressions" (denoted α, β, etc). We shall say that an expression α is weak iff no atomic formula occurs in α. In the sequel, we use $\alpha(x_1,\ldots,x_n)$ to denote a weak expression α whose atomic terms form a subset of $\{x_1,\ldots,x_n\}$. A substitution is a function s assigning to each atomic term x a term $s(x)$ and to each atomic formula X a formula $s(X)$. As usual, s induces a homomorphism $s(\cdot)$ assigning to each term a a term $s(a)$ and to each expression α an expression $s(\alpha)$.

2.2 Semantics

Now, for the semantics. In [5, 17], the language of contact logics is interpreted either in relational structures, or in topological structures. In both cases, terms are interpreted by sets of points. The main difference between the two kinds of structures is the following: in relational structures, two regions are in contact when at least one point of the first region is related to at least one point of the second region whereas in topological structures, two regions are in contact when their topological closures have a nonempty intersection. The two semantics have been proved to be equivalent [5, 17]. In this paper, we only consider the relational semantics. A frame is a relational structure $\mathcal{F} = (W,R)$ where W is a non-empty set of points and R is a binary relation on W. A valuation based on $\mathcal{F}$ is a function V assigning to each atomic term x a subset $V(x)$ of W. V induces a function $(\cdot)^V$ assigning to each term a a subset $(a)^V$ of W such that

- $(x)^V = V(x)$,
- $(0)^V = \emptyset$,
- $(-a)^V = W \setminus (a)^V$,
- $(a \sqcup b)^V = (a)^V \cup (b)^V$.

As a result,

- $(a^0)^V = W \setminus (a)^V$,
- $(a^1)^V = (a)^V$.

We shall say that V is balanced iff for all terms a, either $(a)^V = \emptyset$, or $(a)^V = W$, or $(a)^V$ is infinite and coinfinite. An interpretation is a subset I of AF. A model is a structure $\mathcal{M} = (W,R,V,I)$ where $\mathcal{F} = (W,R)$ is a frame, V is a valuation based on $\mathcal{F}$ and I is an interpretation. The satisfiability of a formula ϕ in $\mathcal{M}$, in symbols $\mathcal{M} \models \phi$, is defined as follows:

- $\mathcal{M} \models X$ iff $X \in I$,
- $\mathcal{M} \not\models \bot$,
- $\mathcal{M} \models \neg\phi$ iff $\mathcal{M} \not\models \phi$,
- $\mathcal{M} \models \phi \vee \psi$ iff either $\mathcal{M} \models \phi$, or $\mathcal{M} \models \psi$,
- $\mathcal{M} \models C(a, b)$ iff $((a)^V \times (b)^V) \cap R \neq \emptyset$,
- $\mathcal{M} \models a \equiv b$ iff $(a)^V = (b)^V$.

As a result,

- $\mathcal{M} \models \bar{C}(a, b)$ iff $((a)^V \times (b)^V) \cap R = \emptyset$,
- $\mathcal{M} \models a \not\equiv b$ iff $(a)^V \neq (b)^V$,
- $\mathcal{M} \models a \leq b$ iff $(a)^V \subseteq (b)^V$.

Let $\mathcal{F}$ be a frame. A formula ϕ is valid in $\mathcal{F}$, in symbols $\mathcal{F} \models \phi$, iff for all models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M} \models \phi$. A set Γ of formulas is valid in $\mathcal{F}$, in symbols $\mathcal{F} \models \Gamma$, iff for all formulas $\phi \in \Gamma$, $\mathcal{F} \models \phi$. Let $\mathcal{C}F$ be a class of frames. A formula ϕ is valid in $\mathcal{C}F$, in symbols $\mathcal{C}F \models \phi$, iff for all frames $\mathcal{F}$ in $\mathcal{C}F$, $\mathcal{F} \models \phi$. Let $\mathcal{C}F_0$ be the class of all frames. Obviously,

Proposition 1. *The following formulas are valid in $\mathcal{C}F_0$:*

- $C(x, y) \rightarrow x \not\equiv 0$,
- $C(x, y) \rightarrow y \not\equiv 0$,
- $C(x, y) \wedge x \leq z \rightarrow C(z, y)$,
- $C(x, y) \wedge y \leq z \rightarrow C(x, z)$,
- $C(x \sqcup y, z) \rightarrow C(x, z) \vee C(y, z)$,
- $C(x, y \sqcup z) \rightarrow C(x, y) \vee C(x, z)$.

In this paper, we will consider the following classes of frames:

- the class $\mathcal{C}F_r$ of all reflexive frames,
- the class $\mathcal{C}F_s$ of all symmetrical frames.

Obviously,

Proposition 2. *The following formula is valid in $\mathcal{C}F_r$:*

- $x \not\equiv 0 \rightarrow C(x, x)$.

The following formula is valid in $\mathcal{C}F_s$:

- $C(x, y) \rightarrow C(y, x)$.

3 Axiomatization and decidability of contact logics

In this section, we present the axiomatization and the decidability of contact logics. From now on, formulas will also be called "axioms" and pairs of the form (Γ, ϕ) where Γ is a finite set of formulas and ϕ is a formula will also be called "inference rules". When an axiom or an inference rule contains no occurrence of atomic formulas, it is qualified as "weak". An axiomatic system consists of a collection of axioms and a collection of inference rules. Let λ_0 be the axiomatic system consisting of

- a complete set of axioms for Classical Propositional Calculus (i.e. $X \rightarrow (Y \rightarrow X)$, $(X \rightarrow (Y \rightarrow Z)) \rightarrow ((X \rightarrow Y) \rightarrow (X \rightarrow Z))$, etc),
- a complete set of axioms for non-degenerate Boolean algebras (i.e. $x \sqcup (y \sqcup z) \equiv (x \sqcup y) \sqcup z$, $x \sqcup y \equiv y \sqcup x$, etc),
- the following axioms:
  - $C(x, y) \rightarrow x \not\equiv 0$,
  - $C(x, y) \rightarrow y \not\equiv 0$,
  - $C(x, y) \wedge x \leq z \rightarrow C(z, y)$,
  - $C(x, y) \wedge y \leq z \rightarrow C(x, z)$,
  - $C(x \sqcup y, z) \rightarrow C(x, z) \vee C(y, z)$,
  - $C(x, y \sqcup z) \rightarrow C(x, y) \vee C(x, z)$,
- the inference rule of modus ponens (i.e. $(\{X, X \rightarrow Y\}, Y)$).

We will consider extensions of λ_0 — denoted λ, μ, etc — by either adding new axioms, or adding new inference rules. The extension of λ_0 with a set A of axioms will be denoted $\lambda_0(A)$. The extension of λ_0 with a single axiom ϕ will be denoted $\lambda_0(\phi)$. In this paper, we will consider the following extensions of λ_0:

- $\lambda_r = \lambda_0(x \not\equiv 0 \rightarrow C(x, x))$,
- $\lambda_s = \lambda_0(C(x, y) \rightarrow C(y, x))$.

The extension of λ_0 with a single inference rule (Γ, ϕ) will be denoted $\lambda_0 + (\Gamma, \phi)$. A formula ϕ is said to be derivable in an extension λ of λ_0 from a finite set Γ of formulas, in symbols $\Gamma \vdash_\lambda \phi$, iff there exists a finite sequence $\phi_0, \ldots, \phi_m$ of formulas such that $\phi_m = \phi$ and for all nonnegative integers i, if $i \leq m$, then at least one of the following conditions holds:

- $\phi_i \in \Gamma$,
- there exists an axiom ψ in λ and there exists a substitution s such that $\phi_i = s(\psi)$,
- there exists an inference rule (Δ, ψ) in λ and there exists a substitution s such that $\phi_i = s(\psi)$ and $\{\phi_0, \ldots, \phi_{i-1}\} \supseteq s(\Delta)$.

The finite sequence $\phi_0, \ldots, \phi_m$ is called "derivation of ϕ in λ from Γ". The propositions below contain facts which can be found in most elementary logic texts.

Proposition 3. *Let Γ be a finite set of formulas and ϕ be a formula. If $\Gamma \vdash_\lambda \phi$, then for all substitutions s, $s(\Gamma) \vdash_\lambda s(\phi)$.*

Proposition 4. *Let Γ be a finite set of formulas and ϕ, ψ be formulas. The following conditions are equivalent:*

- $\Gamma \cup \{\phi\} \vdash_\lambda \psi$,
- $\Gamma \vdash_\lambda \phi \rightarrow \psi$.

A formula ϕ is said to be provable in λ, in symbols $\vdash_\lambda \phi$, iff $\emptyset \vdash_\lambda \phi$. In this case, every derivation of ϕ in λ from $\emptyset$ is called "proof of ϕ in λ". The provable formulas of λ will be called "theorems of λ". We will denote by $Th(\lambda)$ the set of all theorems of λ. We shall say that λ is consistent iff $\bot \not\in Th(\lambda)$. We will denote by $CF(\lambda)$ the class of all frames $\mathcal{F}$ such that $\mathcal{F} \models Th(\lambda)$ and we will denote by $CF_{fin}(\lambda)$ the class of all finite frames $\mathcal{F}$ such that $\mathcal{F} \models Th(\lambda)$. We shall say that λ is balanced iff for all formulas ϕ, the following conditions are equivalent:

- $\phi \not\in Th(\lambda)$,
- there exists a countable frame $\mathcal{F} \in CF(\lambda)$, there exists a balanced valuation V on $\mathcal{F}$ and there exists an interpretation I such that $(\mathcal{F}, V, I) \not\models \phi$.

λ_0 itself is balanced, but also most extensions of λ_0 considered in [5, 17] like λ_r and λ_s are balanced. In [5, 17], one can also find the facts contained in the following

Proposition 5. *Let ϕ be a formula. The following conditions are equivalent:*

- $\phi \in Th(\lambda_0)$,
- $\mathcal{C}F_0 \models \phi$.

Proposition 6. *Let ϕ be a formula. The following conditions are equivalent:*

- $\phi \in Th(\lambda_r)$,
- $\mathcal{C}F_r \models \phi$.

The following conditions are equivalent:

- $\phi \in Th(\lambda_s)$,
- $\mathcal{C}F_s \models \phi$.

More generally,

Proposition 7. *Let ϕ be a formula. If there exists a finite set A of axioms such that $\lambda = \lambda_0(A)$, then the following conditions are equivalent:*

- $\phi \in Th(\lambda)$,
- $CF(\lambda) \models \phi$,
- $CF_{fin}(\lambda) \models \phi$.

A consequence of Proposition 7 is the following

Proposition 8. *If there exists a finite set A of axioms such that $\lambda = \lambda_0(A)$, then $Th(\lambda)$ is decidable.*

Later on, we will use Propositions 3–8 without explicit reference.

4 Admissibility: definitions

Let λ be an extension of λ_0. An inference rule (Γ, ϕ) is said to be admissible in λ iff for all substitutions s, if $s(\Gamma) \subseteq Th(\lambda)$, then $s(\phi) \in Th(\lambda)$. The next proposition indicates that inference rules admissible in λ do not increase $Th(\lambda)$ when added to λ.

Proposition 9. *Let (Γ, ϕ) be an inference rule. If (Γ, ϕ) is admissible in λ, then $Th(\lambda + (\Gamma, \phi)) = Th(\lambda)$.*

Proof. Suppose (Γ, ϕ) is admissible in λ. If $Th(\lambda + (\Gamma, \phi)) \neq Th(\lambda)$, then obviously, there exists a formula ψ such that $\psi \in Th(\lambda + (\Gamma, \phi))$ and $\psi \notin Th(\lambda)$. Hence, there exists a proof $\psi_0, \ldots, \psi_m$ of ψ in $\lambda+(\Gamma, \phi)$. Since (Γ, ϕ) is admissible in λ, each use of (Γ, ϕ) in $\psi_0, \ldots, \psi_m$ can be replaced by a corresponding proof in λ. Thus, there exists a proof of ψ in λ. Therefore, $\psi \in Th(\lambda)$: a contradiction.

Inference rules that are admissible in λ can be used to improve the performance of any algorithm that handles λ-provability. In this respect, the following decision problem, called "admissibility problem in λ", in symbols $ADM(\lambda)$, is of the utmost importance:

- input: an inference rule (Γ, ϕ),
- output: determine whether (Γ, ϕ) is admissible in λ.

Applicability of inference rules that are admissible in λ to ameliorate algorithms for λ-provability incites us to study the decidability of $ADM(\lambda)$. To start this study, let us first define the notion of derivability in λ. We shall say that an inference rule (Γ, ϕ) is derivable in λ iff $\Gamma \vdash_\lambda \phi$. It happens that derivability is a special case of admissibility.

Proposition 10. *Let (Γ, ϕ) be an inference rule. If (Γ, ϕ) is derivable in λ, then (Γ, ϕ) is admissible in λ.*

Proof. Suppose (Γ, ϕ) is derivable in λ. If (Γ, ϕ) is not admissible in λ, then there exists a substitution s such that $s(\Gamma) \subseteq Th(\lambda)$ and $s(\phi) \notin Th(\lambda)$. Since (Γ, ϕ) is derivable in λ, $\Gamma \vdash_\lambda \phi$. Hence, $s(\Gamma) \vdash_\lambda s(\phi)$. Thus, there exists a derivation $\phi_0, \ldots, \phi_m$ of $s(\phi)$ in λ from $s(\Gamma)$. Since $s(\Gamma) \subseteq Th(\lambda)$, each use of $s(\Gamma)$ in $\phi_0, \ldots, \phi_m$ can be replaced by a corresponding proof in λ. Therefore, there exists a proof of $s(\phi)$ in λ. Consequently, $s(\phi) \in Th(\lambda)$: a contradiction.

Nevertheless, in the general case, it may happen that derivability and admissibility in such-or-such contact logic do not coincide. It suffices, for instance, to consider the inference rule $(\{C(x, y)\}, C(y, x))$. Since for all substitutions s, $s(C(x, y)) \notin Th(\lambda_0)$, $(\{C(x, y)\}, C(y, x))$ is admissible in λ_0. Since $C(x, y) \to C(y, x) \notin Th(\lambda_0)$, $(\{C(x, y)\}, C(y, x))$ is not derivable in λ_0. As a result, the following decision problem, called "derivability problem in λ", in symbols $DER(\lambda)$, has its importance:

- input: an inference rule (Γ, ϕ),
- output: determine whether (Γ, ϕ) is derivable in λ.

Obviously,

Proposition 11. *If there exists a finite set A of axioms such that $\lambda = \lambda_0(A)$, then $DER(\lambda)$ is decidable.*

We shall say that λ is structurally complete iff for all inference rules (Γ, ϕ), if (Γ, ϕ) is admissible in λ, (Γ, ϕ) is derivable in λ. By Propositions 10 and 11,

Proposition 12. *If λ is structurally complete and there exists a finite set A of axioms such that $\lambda = \lambda_0(A)$, then $ADM(\lambda)$ is decidable.*

Now, we intend to extend Proposition 12 to structurally incomplete extensions of λ_0. However, in this paper, we will only be able to study the decidability of the following decision problem, called "weak admissibility problem in λ", in symbols $wADM(\lambda)$:

- input: a weak inference rule (Γ, ϕ),
- output: determine whether (Γ, ϕ) is admissible in λ.

We end this section with the following

Proposition 13. *Let $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ be a weak inference rule. The following conditions are equivalent:*

- *$(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ is not admissible in λ,*
- *there exists terms $a_1, \ldots, a_n$ such that $\Gamma(a_1, \ldots, a_n) \subseteq Th(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin Th(\lambda)$.*

Proof. $(\Rightarrow)$ Suppose $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ is not admissible in λ. Hence, there exists a substitution s such that $s(\Gamma(x_1, \ldots, x_n)) \subseteq Th(\lambda)$ and $s(\phi(x_1, \ldots, x_n)) \notin Th(\lambda)$. Let $a_1, \ldots, a_n$ be terms such that for all positive integers i, if $i \leq n$, then $a_i = s(x_i)$. Since $s(\Gamma(x_1, \ldots, x_n)) \subseteq Th(\lambda)$ and $s(\phi(x_1, \ldots, x_n)) \notin Th(\lambda)$, $\Gamma(a_1, \ldots, a_n) \subseteq Th(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin Th(\lambda)$.
$(\Leftarrow)$ Suppose there exists terms $a_1, \ldots, a_n$ such that $\Gamma(a_1, \ldots, a_n) \subseteq Th(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin Th(\lambda)$. Let s be a substitution such that for all positive integers i, if $i \leq n$, then $s(x_i) = a_i$. Since $\Gamma(a_1, \ldots, a_n) \subseteq Th(\lambda)$ and $\phi(a_1, \ldots, a_n) \notin Th(\lambda)$, $s(\Gamma(x_1, \ldots, x_n)) \subseteq Th(\lambda)$ and $s(\phi(x_1, \ldots, x_n)) \notin Th(\lambda)$. Hence, $(\Gamma(x_1, \ldots, x_n), \phi(x_1, \ldots, x_n))$ is not admissible in λ.

5 Admissibility: useful lemmas

Let λ be an extension of λ_0. The decidability of $wADM(\lambda)$ is difficult to establish and we defer proving it till next section. In the meantime, we present useful lemmas. Let n be a nonnegative integer. Let Φ_n be the set of all weak formulas with atomic terms in $x_1, \ldots, x_n$. We define on Φ_n the equivalence relation $\equiv^n_\lambda$ as follows:

- $\phi(x_1, \ldots, x_n) \equiv^n_\lambda \psi(x_1, \ldots, x_n)$ iff $\phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n) \in Th(\lambda)$.

Obviously, considered as formulas in classical propositional logic, the terms $a(x_1, \ldots, x_n)$ and $b(x_1, \ldots, x_n)$ are equivalent iff $mon(a(x_1, \ldots, x_n)) = mon(b(x_1, \ldots, x_n))$. Hence, there exists exactly 2^{2^n} pairwise non-equivalent terms in $x_1, \ldots, x_n$. Since each weak formula $\phi(x_1, \ldots, x_n)$ in Φ_n is a Boolean combination of elementary formulas of the form $C(a(x_1, \ldots, x_n), b(x_1, \ldots, x_n))$ or of the form $a(x_1, \ldots, x_n) \equiv b(x_1, \ldots, x_n)$,

Lemma 1. $\equiv_\lambda^n$ *has finitely many equivalence classes on* Φ_n.

Let A_n be the set of all n-tuples of terms. Note that n-tuples of terms in A_n may contain occurrences of atomic terms distinct from x_1, …, x_n. Given $(a_1, \ldots, a_n) \in A_n$, a frame $\mathcal{F} \in CF(\lambda)$ and a valuation V on $\mathcal{F}$, let

– $\Phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}$ be the set of all C-free weak formulas $\phi(x_1, \ldots, x_n)$ such that $(\mathcal{F}, V) \models \phi(a_1, \ldots, a_n)$.

Consider a complete list $\phi_1(x_1, \ldots, x_n), \ldots, \phi_k(x_1, \ldots, x_n)$ in $\Phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}$ of representatives for each equivalence class on $\Phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}$ modulo $\equiv_\lambda^n$ and define

– $\phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) = \phi_1(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_k(x_1, \ldots, x_n)$.

Obviously,

Lemma 2. $(\mathcal{F}, V) \models \phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}(a_1, \ldots, a_n)$.

Hence, $\phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n)$ is in $\Phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}$. Let

– $\Phi_{a_1,\ldots,a_n} = \{\phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n)$: $\mathcal{F} \in CF(\lambda)$ and V is a valuation on $\mathcal{F}\}$.

Consider a complete list $\psi_1(x_1, \ldots, x_n), \ldots, \psi_l(x_1, \ldots, x_n)$ in $\Phi_{(a_1,\ldots,a_n)}$ of representatives for each equivalence class on $\Phi_{(a_1,\ldots,a_n)}$ modulo $\equiv_\lambda^n$ and define

– $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) = \psi_1(x_1, \ldots, x_n) \vee \ldots \vee \psi_l(x_1, \ldots, x_n)$.

We have the

Lemma 3. $\psi_{(a_1,\ldots,a_n)}(a_1, \ldots, a_n) \in Th(\lambda)$.

Proof. Suppose $\psi_{(a_1,\ldots,a_n)}(a_1, \ldots, a_n) \notin Th(\lambda)$. Thus, there exists a frame $\mathcal{F} \in CF(\lambda)$ and there exists a valuation V on $\mathcal{F}$ such that $(\mathcal{F}, V) \not\models \psi_{(a_1,\ldots,a_n)}(a_1, \ldots, a_n)$. Let i be a positive integer such that $1 \leq i \leq l$ and $\phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n)$ is equivalent to $\psi_i(x_1, \ldots, x_n)$ modulo $\equiv_\lambda^n$. Since $\mathcal{F} \models Th(\lambda)$ and, by Lemma 2, $(\mathcal{F}, V) \models \phi^{\mathcal{F},V}_{(a_1,\ldots,a_n)}(a_1, \ldots, a_n)$, $(\mathcal{F}, V) \models \psi_i(a_1, \ldots, a_n)$. Therefore, $(\mathcal{F}, V) \models \psi_{(a_1,\ldots,a_n)}(a_1, \ldots, a_n)$: a contradiction.

Moreover,

Lemma 4. *For all C-free weak formulas $\phi(x_1, \ldots, x_n)$, the following conditions are equivalent:*

– $\phi(a_1, \ldots, a_n) \in Th(\lambda)$,
– $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n) \in Th(\lambda)$.

Proof. ($\Rightarrow$) Suppose $\phi(a_1,\ldots,a_n) \in Th(\lambda)$. If $\psi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) \rightarrow \phi(x_1,\ldots,x_n) \notin Th(\lambda)$, then there exists a frame $\mathcal{F} \in CF(\lambda)$ and there exists a valuation V on $\mathcal{F}$ such that $(\mathcal{F},V) \not\models \psi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) \rightarrow \phi(x_1,\ldots,x_n)$. Hence, $(\mathcal{F},V) \models \psi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n)$ and $(\mathcal{F},V) \not\models \phi(x_1,\ldots,x_n)$. Thus, there exists a positive integer i such that $i \leq l$ and $(\mathcal{F},V) \models \psi_i(x_1,\ldots,x_n)$. Let $\mathcal{F}' \in CF(\lambda)$ be a frame and V' be a valuation on $\mathcal{F}'$ such that $\phi^{\mathcal{F}',V'}_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n)$ is equivalent to $\psi_i(x_1,\ldots,x_n)$ modulo $\equiv^n_\lambda$. Since $(\mathcal{F},V) \models Th(\lambda)$ and $(\mathcal{F},V) \models \psi_i(x_1,\ldots,x_n)$, $(\mathcal{F},V) \models \phi^{\mathcal{F}',V'}_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n)$. Therefore, for all C-free weak formulas $\theta(x_1,\ldots,x_n)$, if $(\mathcal{F}',V') \models \theta(a_1,\ldots,a_n)$, then $(\mathcal{F},V) \models \theta(x_1,\ldots,x_n)$. Since $\mathcal{F}' \models Th(\lambda)$ and $\phi(a_1,\ldots,a_n) \in Th(\lambda)$, $(\mathcal{F}',V') \models \phi(a_1,\ldots,a_n)$. Since for all C-free weak formulas $\theta(x_1,\ldots,x_n)$, if $(\mathcal{F}',V') \models \theta(a_1,\ldots,a_n)$, then $(\mathcal{F},V) \models \theta(x_1,\ldots,x_n)$, $(\mathcal{F},V) \models \phi(x_1,\ldots,x_n)$: a contradiction.
($\Leftarrow$) Suppose $\psi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) \rightarrow \phi(x_1,\ldots,x_n) \in Th(\lambda)$. Consequently, $\psi_{(a_1,\ldots,a_n)}(a_1,\ldots,a_n) \rightarrow \phi(a_1,\ldots,a_n) \in Th(\lambda)$. By Lemma 3, $\psi_{(a_1,\ldots,a_n)}(a_1,\ldots,a_n) \in Th(\lambda)$. Since $\psi_{(a_1,\ldots,a_n)}(a_1,\ldots,a_n) \rightarrow \phi(a_1,\ldots,a_n) \in Th(\lambda)$, $\phi(a_1,\ldots,a_n) \in Th(\lambda)$.

We define on A_n the equivalence relation $\cong^n_\lambda$ as follows:

- $(a_1,\ldots,a_n) \cong^n_\lambda (b_1,\ldots,b_n)$ iff for all weak formulas $\phi(x_1,\ldots,x_n)$ in Φ_n, $\phi(a_1,\ldots,a_n) \in Th(\lambda)$ iff $\phi(b_1,\ldots,b_n) \in Th(\lambda)$.

By Lemma 1,

Lemma 5. $\cong^n_\lambda$ *has finitely many equivalence classes on* A_n.

It is of interest to consider the equivalence relation $\cong^n_\lambda$, seeing that, according to our definitions,

Lemma 6. *If* $(a_1,\ldots,a_n) \cong^n_\lambda (b_1,\ldots,b_n)$*, then for all weak inference rules* $(\Gamma(x_1,\ldots,x_n),\phi(x_1,\ldots,x_n))$*, the following conditions are equivalent:*

- $\Gamma(a_1,\ldots,a_n) \subseteq Th(\lambda)$ *and* $\phi(a_1,\ldots,a_n) \notin Th(\lambda)$,
- $\Gamma(b_1,\ldots,b_n) \subseteq Th(\lambda)$ *and* $\phi(b_1,\ldots,b_n) \notin Th(\lambda)$.

Now, we define on A_n the equivalence relation $\simeq^n_\lambda$ as follows:

- $(a_1,\ldots,a_n) \simeq^n_\lambda (b_1,\ldots,b_n)$ iff for all C-free weak formulas $\phi(x_1,\ldots,x_n)$ in Φ_n, $\phi(a_1,\ldots,a_n) \in Th(\lambda)$ iff $\phi(b_1,\ldots,b_n) \in Th(\lambda)$.

Obviously,

Lemma 7. *If* $(a_1,\ldots,a_n) \cong^n_\lambda (b_1,\ldots,b_n)$*, then* $(a_1,\ldots,a_n) \simeq^n_\lambda (b_1,\ldots,b_n)$.

Moreover, by Lemma 1,

Lemma 8. $\simeq^n_\lambda$ *has finitely many equivalence classes on* A_n.

The key things to note about the equivalence relations $\cong^n_\lambda$ and $\simeq^n_\lambda$ are contained in the following lemmas.

Lemma 9. *The following conditions are equivalent:*

- $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$,
- $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \leftrightarrow \psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \in Th(\lambda)$.

Proof. ($\Rightarrow$) Suppose $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$. If $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \leftrightarrow \psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \notin Th(\lambda)$, then either $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \rightarrow \psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \notin Th(\lambda)$, or $\psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \rightarrow \psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \notin Th(\lambda)$. Without loss of generality, let us assume that $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \rightarrow \psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \notin Th(\lambda)$. By Lemma 4, $\psi_{(b_1,\ldots,b_n)}(a_1, \ldots, a_n) \notin Th(\lambda)$. Since $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$, $\psi_{(b_1,\ldots,b_n)}(b_1, \ldots, b_n) \notin Th(\lambda)$. By Lemma 3, $\psi_{(b_1,\ldots,b_n)}(b_1, \ldots, b_n) \in Th(\lambda)$: a contradiction.
($\Leftarrow$) Suppose $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \leftrightarrow \psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \in Th(\lambda)$. Hence, for all C-free weak formulas $\phi(x_1, \ldots, x_n)$, $\psi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) \rightarrow \phi(x_1, \ldots, x_n) \in Th(\lambda)$ iff $\psi_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \rightarrow \phi(x_1, \ldots, x_n) \in Th(\lambda)$. By Lemma 4, for all C-free weak formulas $\phi(x_1, \ldots, x_n)$, $\phi(a_1, \ldots, a_n) \in Th(\lambda)$ iff $\phi(b_1, \ldots, b_n) \in Th(\lambda)$. Thus, $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$.

Lemma 10. *If λ is balanced and $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$, then $(a_1, \ldots, a_n) \cong^n_\lambda (b_1, \ldots, b_n)$.*

Proof. Suppose λ is balanced and $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$. If $(a_1, \ldots, a_n) \ncong^n_\lambda (b_1, \ldots, b_n)$, then there exists a weak formula $\phi(x_1, \ldots, x_n)$ in Φ_n such that $\phi(a_1, \ldots, a_n) \in Th(\lambda)$ not-iff $\phi(b_1, \ldots, b_n) \in Th(\lambda)$. Without loss of generality, let us assume that $\phi(a_1, \ldots, a_n) \in Th(\lambda)$ and $\phi(b_1, \ldots, b_n) \notin Th(\lambda)$. Since λ is balanced, there exists a countable frame $\mathcal{F} \in CF(\lambda)$ and there exists a balanced valuation V on $\mathcal{F}$ such that $(\mathcal{F}, V) \not\models \phi(b_1, \ldots, b_n)$. By Lemma 2, $(\mathcal{F}, V) \models \phi^{\mathcal{F},V}_{(b_1,\ldots,b_n)}(b_1, \ldots, b_n)$. Since $\mathcal{F} \models Th(\lambda)$, $\neg\phi^{\mathcal{F},V}_{(b_1,\ldots,b_n)}(b_1, \ldots, b_n) \notin Th(\lambda)$. Since $(a_1, \ldots, a_n) \simeq^n_\lambda (b_1, \ldots, b_n)$, $\neg\phi^{\mathcal{F},V}_{(b_1,\ldots,b_n)}(a_1, \ldots, a_n) \notin Th(\lambda)$. Since λ is balanced, there exists a countable frame $\mathcal{F}' \in CF(\lambda)$ and there exists a balanced valuation V' on $\mathcal{F}'$ such that $(\mathcal{F}', V') \models \phi^{\mathcal{F},V}_{(b_1,\ldots,b_n)}(a_1, \ldots, a_n)$. Suppose $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$. Now, consider $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$. If $(b_1^{\epsilon_1} \sqcap \ldots \sqcap b_n^{\epsilon_n})^V = \emptyset$, then $(\mathcal{F}, V) \models b_1^{\epsilon_1} \sqcap \ldots \sqcap b_n^{\epsilon_n} \equiv 0$. Hence, $\phi^{\mathcal{F},V}_{(b_1,\ldots,b_n)}(x_1, \ldots, x_n) \rightarrow x_1^{\epsilon_1} \sqcap \ldots \sqcap x_n^{\epsilon_n} \equiv 0 \in Th(\lambda)$. Since $\mathcal{F}' \models Th(\lambda)$ and $(\mathcal{F}', V') \models \phi^{\mathcal{F},V}_{(b_1,\ldots,b_n)}(a_1, \ldots, a_n)$, $(\mathcal{F}', V') \models a_1^{\epsilon_1} \sqcap \ldots \sqcap a_n^{\epsilon_n} \equiv 0$. Thus, $(a_1^{\epsilon_1} \sqcap \ldots \sqcap a_n^{\epsilon_n})^{V'} = \emptyset$. Similarly, the reader may easily verify that if $(b_1^{\epsilon_1} \sqcap \ldots \sqcap b_n^{\epsilon_n})^V = W$, then $(a_1^{\epsilon_1} \sqcap \ldots \sqcap a_n^{\epsilon_n})^{V'} = W'$ and if $(b_1^{\epsilon_1} \sqcap \ldots \sqcap b_n^{\epsilon_n})^V$ is infinite and coinfinite, then $(a_1^{\epsilon_1} \sqcap \ldots \sqcap a_n^{\epsilon_n})^{V'}$ is infinite and coinfinite. In all cases, there exists a bijection $f_{(\epsilon_1,\ldots,\epsilon_n)}$ from $(b_1^{\epsilon_1} \sqcap \ldots \sqcap b_n^{\epsilon_n})^V$ to $(a_1^{\epsilon_1} \sqcap \ldots \sqcap a_n^{\epsilon_n})^{V'}$. Let f be the union of all $f_{(\epsilon_1,\ldots,\epsilon_n)}$ when $(\epsilon_1, \ldots, \epsilon_n)$ describes $\{0, 1\}^n$. The reader may easily verify that f is a bijection from W to W' such that for all $u \in W$ and for all $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$, $u \in (b_1^{\epsilon_1} \sqcap \ldots \sqcap b_n^{\epsilon_n})^V$ iff $f(u) \in (a_1^{\epsilon_1} \sqcap \ldots \sqcap a_n^{\epsilon_n})^{V'}$. Let R'_f be the binary relation on W' defined by $u' R'_f v'$ iff $f^{-1}(u') R f^{-1}(v')$. We define $\mathcal{F}'_f = (W', R'_f)$. Obviously, f is an isomorphism from $\mathcal{F}$ to $\mathcal{F}'_f$. Since $\mathcal{F} \models Th(\lambda)$, $\mathcal{F}'_f \models Th(\lambda)$. Since $\phi(a_1, \ldots, a_n) \in Th(\lambda)$, $(\mathcal{F}'_f, V') \models \phi(a_1, \ldots, a_n)$. Therefore, $(\mathcal{F}, V) \models \phi(b_1, \ldots, b_n)$: a contradiction.

6 Admissibility: decidability

Let λ be an extension of λ_0. By Proposition 13 and Lemmas 5–8 and 10, $wADM(\lambda)$ would be decidable if λ is balanced, $Th(\lambda)$ is decidable and a complete set of representatives for each class on A_n modulo $\simeq_\lambda^n$ could be computed. Let k be a nonnegative integer. Given $(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in A_n$, we define on $\{0,1\}^k$ the equivalence relation $\sim^k_{(a_1,\ldots,a_n)}$ as follows:

- $(\epsilon_1, \ldots, \epsilon_k) \sim^k_{(a_1,\ldots,a_n)} (\epsilon'_1, \ldots, \epsilon'_k)$ iff for all positive integers i, if $i \leq n$, then $z_1^{\epsilon_1} \sqcap \ldots \sqcap z_k^{\epsilon_k} \in mon(a_i(z_1, \ldots, z_k))$ iff $z_1^{\epsilon'_1} \sqcap \ldots \sqcap z_k^{\epsilon'_k} \in mon(a_i(z_1, \ldots, z_k))$.

Obviously,

Lemma 11. $\sim^k_{(a_1,\ldots,a_n)}$ *has at most* 2^n *equivalence classes on* $\{0,1\}^k$.

Hence, there exists a one-to-one function f assigning to each equivalence class $\mid (\epsilon_1, \ldots, \epsilon_k) \mid_{\sim^k_{(a_1,\ldots,a_n)}}$ an n-tuple $f(\mid (\epsilon_1, \ldots, \epsilon_k) \mid_{\sim^k_{(a_1,\ldots,a_n)}}) \in \{0,1\}^n$. By means of the one-to-one function f, for all positive integers i, if $i \leq n$, then we define the term $b_i(x_1, \ldots, x_n)$ as follows:

- $b_i(x_1, \ldots, x_n) = \bigsqcup\{x_1^{\epsilon'_1} \sqcap \ldots \sqcap x_n^{\epsilon'_n} \colon z_1^{\epsilon_1} \sqcap \ldots \sqcap z_k^{\epsilon_k} \in mon(a_i(z_1, \ldots, z_k))$ and $f(\mid (\epsilon_1, \ldots, \epsilon_k) \mid_{\sim^k_{(a_1,\ldots,a_n)}}) = (\epsilon'_1, \ldots, \epsilon'_n)\}$.

Given a nonempty set W, the reader may easily verify the following

Lemma 12.
- *for all valuations* V *on* W*, there exists a valuation* V' *on* W *such that for all positive integers* i*, if* $i \leq n$*, then* $(a_i(z_1, \ldots, z_k))^V = (b_i(x_1, \ldots, x_n))^{V'}$*,*
- *for all valuations* V *on* W*, there exists a valuation* V' *on* W *such that for all positive integers* i*, if* $i \leq n$*, then* $(b_i(x_1, \ldots, x_n))^V = (a_i(z_1, \ldots, z_k))^{V'}$*.*

The key thing to note about the terms $b_i(x_1, \ldots, x_n), \ldots, b_i(x_1, \ldots, x_n)$ is contained in the following

Lemma 13. $(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \simeq_\lambda^n (b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$.

Proof. Suppose $(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \not\simeq_\lambda^n (b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$. Hence, there exists a C-free weak formula $\phi(y_1, \ldots, y_n)$ in Φ_n such that $\phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in Th(\lambda)$ not-iff $\phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n)) \in Th(\lambda)$. Thus, we have to consider the following two cases.
Case $\phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k)) \in Th(\lambda)$ **and** $\phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n)) \notin Th(\lambda)$**.** Hence, there exists a frame $\mathcal{F} \in CF(\lambda)$ and there exists a valuation V on $\mathcal{F}$ such that $(\mathcal{F}, V) \not\models \phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$. By Lemma 12, there exists a valuation V' on $\mathcal{F}$ such that for all positive integers i, if $i \leq n$, then $(b_i(x_1, \ldots, x_n))^V = (a_i(z_1, \ldots, z_k))^{V'}$. Since $(\mathcal{F}, V) \not\models \phi(b_1(x_1, \ldots, x_n), \ldots, b_n(x_1, \ldots, x_n))$, $(\mathcal{F}, V') \not\models \phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k))$. Thus, $\mathcal{F} \not\models \phi(a_1(z_1, \ldots, z_k), \ldots, a_n(z_1, \ldots, z_k))$. Since $\mathcal{F} \in CF(\lambda)$,

$\phi(a_1(z_1,\dots,z_k),\dots,a_n(z_1,\dots,z_k)) \not\in Th(\lambda)$: a contradiction.
Case $\phi(a_1(z_1,\dots,z_k),\dots,a_n(z_1,\dots,z_k)) \not\in Th(\lambda)$ and $\phi(b_1(x_1,\dots,x_n),\dots,b_n(x_1,\dots,x_n)) \in Th(\lambda)$. Similar to the case $\phi(a_1(z_1,\dots,z_k),\dots,a_n(z_1,\dots,z_k)) \in Th(\lambda)$ and $\phi(b_1(x_1,\dots,x_n),\dots,b_n(x_1,\dots,x_n)) \not\in Th(\lambda)$.

By means of the lemmas presented above, let us prove the following

Proposition 14. *A complete set of representatives for each class on A_n modulo $\simeq^n_\lambda$ can be computed.*

Proof. By Lemma 13, the set of all n-tuples of terms on $\{x_1,\dots,x_n\}$ constitutes a complete set of representatives for each class on A_n modulo $\simeq^n_\lambda$. Since there exists exactly 2^{2^n} pairwise non-equivalent terms of the form $b(x_1,\dots,x_n)$, a complete set of representatives for each class on A_n modulo $\simeq^n_\lambda$ can be computed.

As a result,

Proposition 15. *If λ is balanced and there exists a finite set A of axioms such that $\lambda = \lambda_0(A)$, then $wADM(\lambda)$ is decidable.*

Proof. Suppose λ is balanced and there exists a finite set A of axioms such that $\lambda = \lambda_0(A)$. We define an algorithm taking as input a weak inference rule $(\Gamma(x_1,\dots,x_n),\phi(x_1,\dots,x_n))$ and returning the value *true* iff $(\Gamma(x_1,\dots,x_n),\phi(x_1,\dots,x_n))$ is admissible in λ as follows:

- compute a complete set $\{(a^1_1,\dots,a^1_n),\dots,(a^N_1,\dots,a^N_n)\}$ of representatives for each class on A_n modulo $\simeq^n_\lambda$;
- if there exists a positive integer k such that $k \leq N$, $\Gamma(a^k_1,\dots,a^k_n) \subseteq Th(\lambda)$ and $\phi(a^k_1,\dots,a^k_n) \not\in Th(\lambda)$ then return $false$ else return *true*.

By Propositions 13 and 14 and Lemmas 5–8 and 10, this algorithm is sound and complete with respect to $wADM(\lambda)$ and can be executed.

However, the exact complexity of $wADM(\lambda)$ is not known.

7 Unifiability

Let λ be an extension of λ_0. A formula ϕ is said to be unifiable in λ iff there exists a substitution s such that $s(\phi) \in Th(\lambda)$. It happens that if λ is consistent, then unifiability is a special case of admissibility.

Proposition 16. *Let ϕ be a formula. If λ is consistent, then the following conditions are equivalent:*

- *ϕ is unifiable in λ,*
- *$(\{\phi\},\bot)$ is not admissible in λ.*

Proof. Suppose λ is consistent.
($\Rightarrow$) Suppose ϕ is unifiable in λ. If $(\{\phi\}, \perp)$ is admissible in λ, then for all substitutions s, if $s(\phi) \in Th(\lambda)$, then $\perp \in Th(\lambda)$. Since λ is consistent, $\perp \notin Th(\lambda)$. Since for all substitutions s, if $s(\phi) \in Th(\lambda)$, then $\perp \in Th(\lambda)$, for all substitutions s, $s(\phi) \notin Th(\lambda)$. Hence, ϕ is not unifiable in λ: a contradiction.
($\Leftarrow$) Suppose $(\{\phi\}, \perp)$ is not admissible in λ. Hence, there exists a substitution s such that $s(\phi) \in Th(\lambda)$. Thus, ϕ is unifiable in λ.

Now, let us consider the following decision problem, called "weak unifiability problem in λ", in symbols $wUNI(\lambda)$:

- input: a weak formula ϕ,
- output: determine whether ϕ is unifiable in λ.

Lemma 14. *For all weak formulas $\phi(x_1, \ldots, x_n)$, the following conditions are equivalent:*

- *ϕ is unifiable in λ,*
- *there exists $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$ such that $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$.*

Proof. ($\Rightarrow$) Suppose $\phi(x_1, \ldots, x_n)$ is unifiable in λ. Hence, there exists a substitution s such that $s(\phi(x_1, \ldots, x_n)) \in Th(\lambda)$. Let t be a ground substitution. Since $s(\phi(x_1, \ldots, x_n)) \in Th(\lambda)$, $t(s(\phi(x_1, \ldots, x_n))) \in Th(\lambda)$. Let $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$ be obtained from $(t(s(x_1)), \ldots, t(s(x_n)))$ by applying ordinary reasoning in non-degenerate Boolean algebras. Since $t(s(\phi(x_1, \ldots, x_n))) \in Th(\lambda)$, $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$.
($\Leftarrow$) Suppose there exists $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$ such that $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$. Let s be a substitution such that for all positive integers i, if $i \leq n$, then $s(x_i) = \epsilon_i$. Since $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$, $s(\phi(x_1, \ldots, x_n)) \in Th(\lambda)$. Thus, ϕ is unifiable in λ.

Hence, it is easy to check that when $Th(\lambda)$ is decidable, $wUNI(\lambda)$ is decidable. Now, remark that for all weak formulas $\phi(x_1, \ldots, x_n)$ and for all $(\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n$, $\phi(\epsilon_1, \ldots, \epsilon_n)$ is equivalent modulo $\equiv_\lambda^n$ to one of the following elementary formulas: $\perp$, $\top$, $C(1, 1)$, $\bar{C}(1, 1)$. Moreover, even when $Th(\lambda)$ is undecidable, the elementary formula in $\{\perp, \top, C(1, 1), \bar{C}(1, 1)\}$ that is equivalent modulo $\equiv_\lambda^n$ to $\phi(\epsilon_1, \ldots, \epsilon_n)$ in λ can be computed. As a result, in all cases, i.e. whatever is the decidability status of $Th(\lambda)$,

Proposition 17. *$wUNI(\lambda)$ is decidable.*

Remark that the elementary formula in $\{\perp, \top, C(1, 1), \bar{C}(1, 1)\}$ that is equivalent modulo $\equiv_\lambda^n$ to $\phi(\epsilon_1, \ldots, \epsilon_n)$ in λ can be computed in linear time. As a result,

Proposition 18. *$wUNI(\lambda)$ is in NP.*

It happens that if λ is consistent, then the satisfiability problem in Boolean Logic is reducible to $wUNI(\lambda)$.

Proposition 19. *Let $a(x_1, \ldots, x_n)$ be a term. If λ is consistent, then the following conditions are equivalent:*

– $a(x_1, \dots, x_n)$ *is satisfiable in Boolean Logic,*
– $a(x_1, \dots, x_n) \equiv 1$ *is unifiable in* λ*.*

Proof. Suppose λ is consistent.
($\Rightarrow$) Suppose $a(x_1, \dots, x_n)$ is satisfiable in Boolean Logic. Hence, there exists $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ such that $a(\epsilon_1, \dots, \epsilon_n)$ is equivalent to 1 in Boolean Logic. Thus, $a(\epsilon_1, \dots, \epsilon_n) \equiv 1 \in Th(\lambda)$. Let s be a substitution such that for all positive integers i, if $i \leq n$, then $s(x_i) = \epsilon_i$. Since $a(\epsilon_1, \dots, \epsilon_n) \equiv 1 \in Th(\lambda)$, $s(a(x_1, \dots, x_n) \equiv 1) \in Th(\lambda)$. Therefore, $a(x_1, \dots, x_n) \equiv 1$ is unifiable in λ.
($\Leftarrow$) Suppose $a(x_1, \dots, x_n) \equiv 1$ is unifiable in λ. Hence, there exists a substitution s such that $s(a(x_1, \dots, x_n) \equiv 1) \in Th(\lambda)$. Let t be a ground substitution. Since $s(a(x_1, \dots, x_n) \equiv 1) \in Th(\lambda)$, $t(s(a(x_1, \dots, x_n) \equiv 1)) \in Th(\lambda)$. Let $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ be obtained from $(t(s(x_1)), \dots, t(s(x_n)))$ by applying ordinary reasoning in Boolean Logic. If $a(x_1, \dots, x_n)$ is not satisfiable in Boolean Logic, then $a(\epsilon_1, \dots, \epsilon_n)$ is equivalent to 0 in Boolean Logic. Since λ is consistent, $\perp \notin Th(\lambda)$. Thus, there exists a frame $\mathcal{F} \in CF(\lambda)$. Let V be a valuation on $\mathcal{F}$ such that for all positive integers i, if $i \leq n$, then

– if $t(s(x_i))$ is equivalent to 0 in Boolean Logic, then $V(x_i) = \emptyset$,
– if $t(s(x_i))$ is equivalent to 1 in Boolean Logic, then $V(x_i) = W$.

Therefore, for all positive integers i, if $i \leq n$, then $V(x_i) = (t(s(x_i)))^V$. Since $a(\epsilon_1, \dots, \epsilon_n)$ is equivalent to 0 in Boolean Logic, $(\mathcal{F}, V) \not\models t(s(a(x_1, \dots, x_n) \equiv 1))$. Since $\mathcal{F} \in CF(\lambda)$, $t(s(a(x_1, \dots, x_n) \equiv 1)) \notin Th(\lambda)$ a contradiction.

As a result,

Proposition 20. *If* λ *is consistent, then* $wUNI(\lambda)$ *is* NP*-hard.*

Now, we give a syntactic result for unifiability and non-unifiability of a weak formula in λ.

Proposition 21. *Let* $\phi(x_1, \dots, x_n)$ *be a weak formula. If* λ *is consistent and* $C(1, 1) \in Th(\lambda)$*, then the following conditions are equivalent:*

– $\phi(x_1, \dots, x_n)$ *is not unifiable in* λ*,*
– $\phi(x_1, \dots, x_n) \to \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$*.*

Proof. Suppose λ is consistent and $C(1, 1) \in Th(\lambda)$.
($\Rightarrow$) Suppose $\phi(x_1, \dots, x_n)$ is not unifiable in λ. If $\phi(x_1, \dots, x_n) \to \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1\colon 1 \leq i \leq n\} \notin Th(\lambda)$, then there exists a frame $\mathcal{F} \in CF(\lambda)$ and there exists a valuation V on $\mathcal{F}$ such that $(\mathcal{F}, V) \not\models \phi(x_1, \dots, x_n) \to \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1\colon 1 \leq i \leq n\}$. Hence, $(\mathcal{F}, V) \models \phi(x_1, \dots, x_n)$ and $(\mathcal{F}, V) \not\models \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1\colon 1 \leq i \leq n\}$. Suppose $\mathcal{F} = (W, R)$. Thus, for all positive integers i, if $i \leq n$, then either $V(x_i) = \emptyset$, or $V(x_i) = W$. Let s be a substitution such that for all positive integers i, if $i \leq n$, then

– if $V(x_i) = \emptyset$, then $s(x_i) = 0$,
– if $V(x_i) = W$, then $s(x_i) = 1$.

Therefore, for all positive integers i, if $i \leq n$, then $V(x_i) = (s(x_i))^V$ Since $(\mathcal{F}, V) \models \phi(x_1, \ldots, x_n)$, $(\mathcal{F}, V) \models \phi(s(x_1), \ldots, s(x_n))$. Since for all positive integers i, if $i \leq n$, then $s(x_i)$ is either equal to 0, or equal to 1, $\phi(s(x_1), \ldots, s(x_n))$ is equivalent modulo $\equiv_\lambda^n$ to an elementary formula in $\{\perp, \top, C(1,1), \bar{C}(1,1)\}$. Since $\mathcal{F} \models Th(\lambda)$, $C(1,1) \in Th(\lambda)$ and $(\mathcal{F}, V) \models \phi(s(x_1), \ldots, s(x_n))$, $\phi(s(x_1), \ldots, s(x_n))$ is equivalent modulo $\equiv_\lambda^n$ to $\top$. Consequently, $\phi(x_1, \ldots, x_n)$ is unifiable in λ: a contradiction.
($\Leftarrow$) Suppose $\phi(x_1, \ldots, x_n) \rightarrow \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$. If $\phi(x_1, \ldots, x_n)$ is unifiable in λ, then by Lemma 14, there exists $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$ such that $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$. Let s be a substitution such that for all positive integers i, if $i \leq n$, then $s(x_i) = \epsilon_i$. Since $\phi(x_1, \ldots, x_n) \rightarrow \bigvee\{x_i \not\equiv 0 \wedge x_i \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$, $s(\phi(x_1, \ldots, x_n)) \rightarrow \bigvee\{s(x_i) \not\equiv 0 \wedge s(x_i) \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$. Since for all positive integers i, if $i \leq n$, then $s(x_i) = \epsilon_i$, $\phi(\epsilon_1, \ldots, \epsilon_n) \rightarrow \bigvee\{\epsilon_i \not\equiv 0 \wedge \epsilon_i \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$. Since $\phi(\epsilon_1, \ldots, \epsilon_n) \in Th(\lambda)$, $\bigvee\{\epsilon_i \not\equiv 0 \wedge \epsilon_i \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$. Since $(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, $\bigwedge\{\epsilon_i \equiv 0 \vee \epsilon_i \equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$. Since $\bigvee\{\epsilon_i \not\equiv 0 \wedge \epsilon_i \not\equiv 1\colon 1 \leq i \leq n\} \in Th(\lambda)$, λ is not consistent: a contradiction.

8 Conclusion

Admissibility problems and unifiability problems are decidable in many modal logics [1–3, 13, 14], but modal logics for which they become undecidable are known [18]. Nevertheless, very little is known about these problems in some of the most important modal logics considered in Computer Science and Artificial Intelligence. For example, the decidability and the complexity of the unification problem for the following modal logics remains open: modal logic K, multi-modal variants of K, sub-Boolean modal logics.
In this paper, we have examined variants of contact logics. The central result in this paper is the proof that the weak admissibility problem and the weak unifiability problem are decidable in these variants.
Much remains to be done. For example, λ being a consistent extension of λ_0, Propositions 16 and 20 imply that $wADM(\lambda)$ is $coNP$-hard, but the exact complexity of $wADM(\lambda)$ is not known. One may also consider the admissibility problem $ADM(\lambda)$ defined in Section 4 and the following unifiability problem: given a formula ϕ, determine whether ϕ is unifiable in λ. Finally, there is also the related question of the unification type of λ. Our conjecture is that the unification type of most extensions of λ_0 considered in [5, 17] is finitary.

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