Golden Ratio Based Partitions of the Integers

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Introduction

- Let $\mathbb{Z}^+ = \{1, 2, 3, 4, 5 \dots\}$.
- A **partition** of $\mathbb{Z}^+$ is a way of breaking $\mathbb{Z}^+$ into non-overlapping groups.
- The even and odd integers are a two set partition of $\mathbb{Z}^+$. $\{1, 2, 3, 4, 5, 6, \dots\} = \{1, 3, 5, 7, \dots\} \cup \{2, 4, 6, 8, \dots\}$.

Arithmetic Progressions

- A simple way of creating partitions is to take distinct **arithmetic progressions** in the integers.

Definition

Let $s, r \in \mathbb{Z}^+$. An arithmetic progression is a sequence of the form

$$f(k) = sk + r,$$

where $0 \leq r < s$.

- The even integers are given by $E(k) = 2k$ while the odd integers are $O(k) = 2k + 1$.
- More complex sets of progressions give more complex partitions. For instance, we can construct a 2 and 3 set case by dividing $\mathbb{Z}^+$ into the groups below

C_1	C_2	C_3
$7m+1$	$7m+2$	$7m+4$
$7m+3$	$7m+6$	$\vdots$
$7m+5$	$\vdots$	$\vdots$
$7m+7$	$\vdots$	$\vdots$

B_1	B_2
$3m+1$	$3m+2$
$3m+3$	$\vdots$
$\vdots$	$\vdots$
$\vdots$	$\vdots$

C_1	C_2	C_3
1	2	4
3	6	11
5	9	18
7	13	25
8	16	32
$\vdots$	$\vdots$	$\vdots$

B_1	B_2
1	2
3	5
4	8
6	11
7	14
$\vdots$	$\vdots$

- The first integers of the sets of the 3 part case are
  $C_1 = \{1, 3, 5, 7, 8, 10, 12, 14, 15, 17, 19, 21, \cdots\}$,
  $C_2 = \{2, 6, 9, 13, 16, 20, 23, 27, 30, 34, 37, 41, \cdots\}$,
  $C_3 = \{4, 11, 18, 25, 32, 39, 46, 53, 60, 67, 74, \cdots\}$,
  while the first in the sets of the 2 part case are
  $B_1 = \{1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, \dots\}$,
  $B_2 = \{2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, \dots\}$.
- If we classify the rows of the 3 part partition by whether the elements fall into the first or second column of the 2 part partition, we only get 5 out of 8 possibilities. Moreover, this classification has period of 12 together with reflection symmetry.
- Arithmetic progressions are easy to study because they are **periodic**. Their partitioning structure is simple. We study partitions composed of **semi-periodic** sequences.

Beatty Type Partitions

Beatty Sequences

- The floor function of a number a, denoted by $\lfloor a \rfloor$, is the integer part of a.
- $\phi = \frac{1+\sqrt{5}}{2} = 1.6810\dots$ is called the **Golden Ratio**. We have $\lfloor \phi \rfloor = \lfloor 1.6810 \dots \rfloor = 1$.

Theorem (Beatty's theorem)

Let α, β be two positive irrational numbers. Let A and B be two sequences such that $g(k) = \lfloor k\alpha \rfloor$ and $h(k) = \lfloor k\beta \rfloor$. Then $g(k)$ and $h(k)$ partition $\mathbb{Z}^+$ if and only if

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

- The special property of ϕ is that

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1,$$

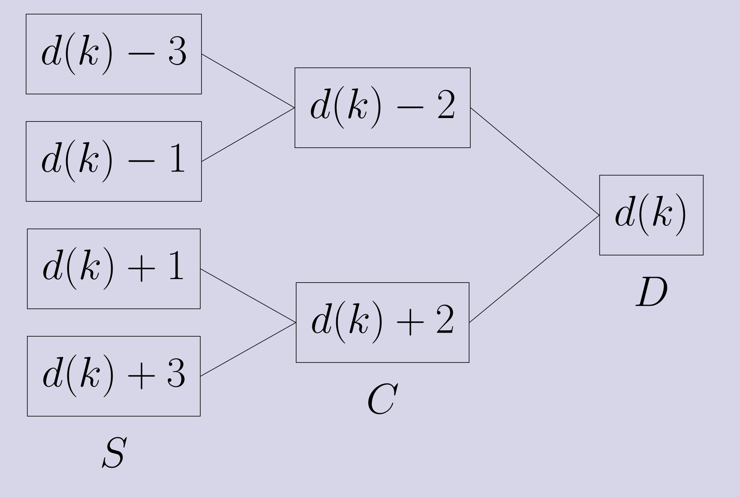
so, $a(k) = \lfloor k\phi \rfloor$ and $b(k) = \lfloor k\phi^2 \rfloor$ partition $\mathbb{Z}^+$.

Almost Beatty Partition

Theorem (Uspensky's Theorem)

*Beatty's Theorem does not hold for three (or more) sequences. That is, if α, β and γ are arbitrary positive numbers, then $\lfloor k\alpha \rfloor$, $\lfloor k\beta \rfloor$ and $\lfloor k\gamma \rfloor$ do **not** partition the positive integers.*

Our work concerns constructions we have created which extend the A, B partition. The following construction is in 3 parts.



$$d(k) = 3\lfloor k\phi \rfloor + k,$$
$$c(k) = \lfloor k\phi \rfloor + 2k - 1,$$
$$s(k) = \lceil c(k)/2 \rceil.$$

2-Column ϕ Partition

- Define the sets A and B as

$$A = \{a(k)\}_{k=1}^{\infty},$$
$$B = \{b(k)\}_{k=1}^{\infty}.$$

- The sequences $a(k)$ and $b(k)$ give a partition of $\mathbb{Z}^+$ with semi-periodic structure.

A	B
1	2
3	5
4	7
6	10
8	13
9	15
$\vdots$	$\vdots$

3-Column ϕ Partition

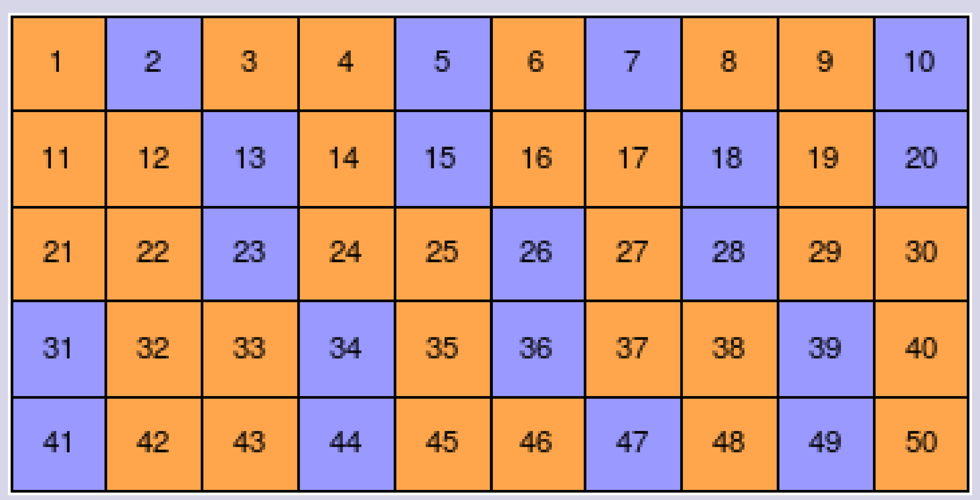
- Define the sets S, C, D as

$$S = \{s(k)\}_{k=1}^{\infty}, \; C = \{c(k)\}_{k=1}^{\infty},$$
$$D = \{d(k)\}_{k=1}^{\infty}.$$

- The sequences $d(k)$, $c(k)$, and $s(k)$ give a partition of $\mathbb{Z}^+$ with similar structure to the A, B partition.

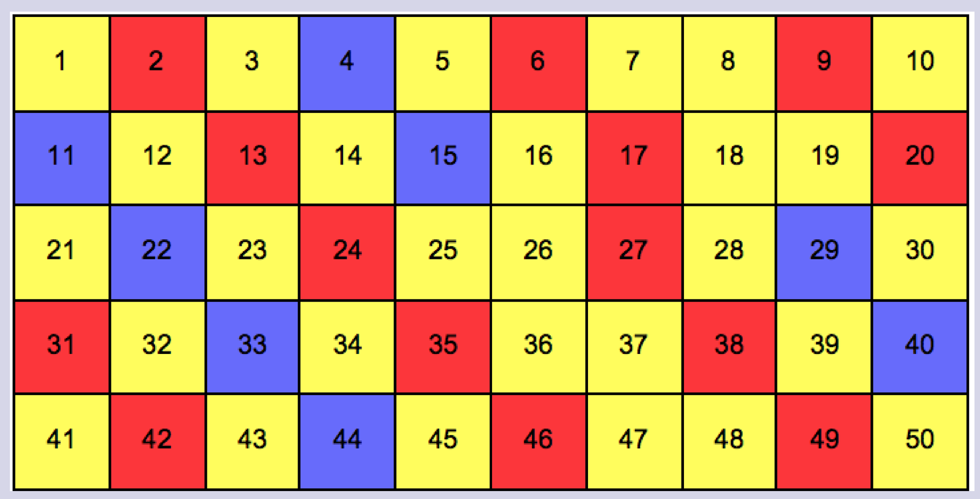
S	C	D
1	2	4
3	6	11
5	9	15
7	13	22
8	17	29
10	20	33
$\vdots$	$\vdots$	$\vdots$

Grid of 2-Column Partition



ϕ-Partition of Integers in 2 Columns

Grid of 3-Column Partition



ϕ-Partition of Integers in 3 Columns

Results: Properties of the 3-Column ϕ Partition

Let $\{x\}$ denote the fractional part of x.

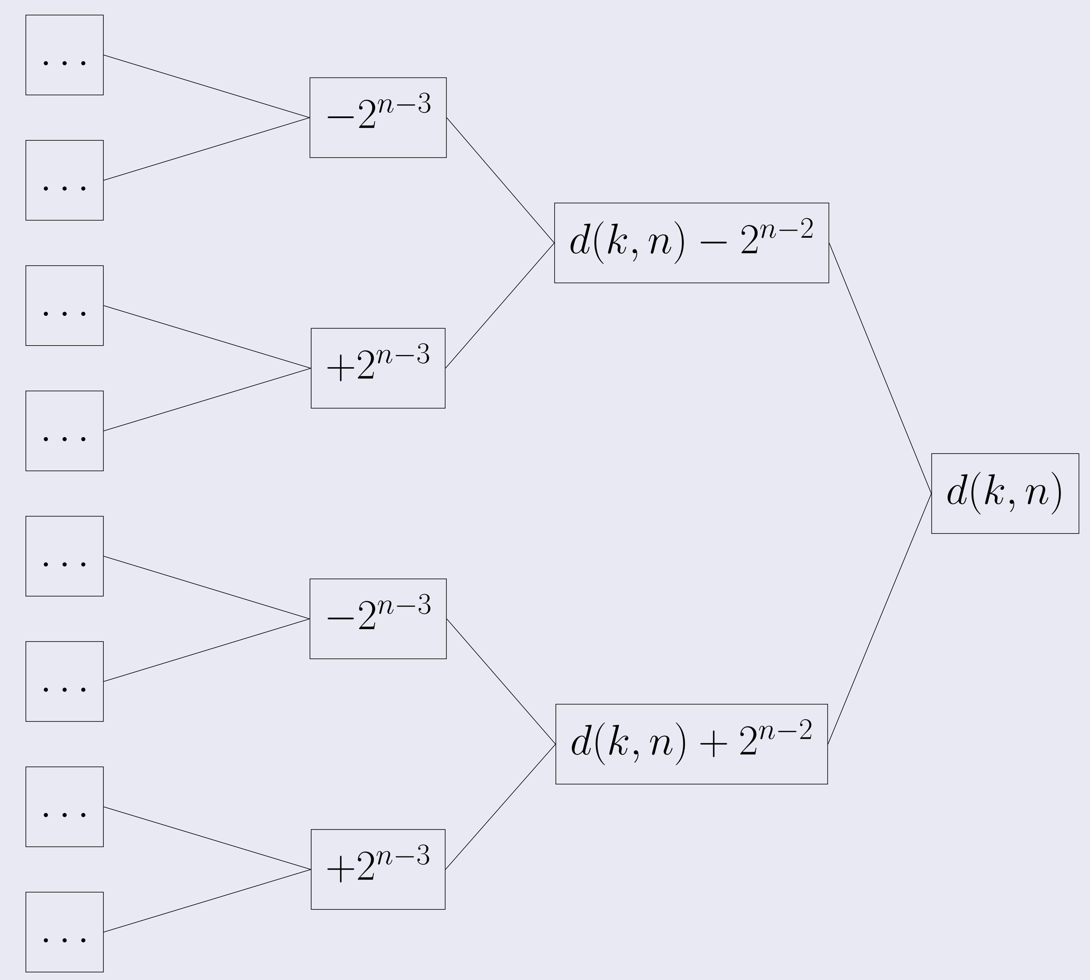
- Let $a(k) = \lfloor k\phi \rfloor$ and $b(k) = \lfloor k\phi^2 \rfloor$. Then $\{a(k)\phi\} + \phi\{b(k)\phi\} = 1$.
- Let $d(k) = 3\lfloor k\phi \rfloor + k$ and $c(k) = \lfloor k\phi \rfloor + 2k - 1$ as above. Then

$$\{c(k)\phi\} + \phi\{d(k)\phi\} = \begin{cases} 1, & \text{if } \{k\phi\} > \frac{1}{\sqrt{5}}, \\ 2, & \text{if } \{k\phi\} < \frac{1}{\sqrt{5}}. \end{cases}$$

- $\{s(k)\phi\} = \frac{\sqrt{5}}{2\phi}\{k\phi\} + b$, where b takes on one of 8 values:

$$\{-1/2, 0, 1/2, \frac{1-\sqrt{5}}{4}, \frac{3-\sqrt{5}}{4}, \frac{5-\sqrt{5}}{4}, 1 - \frac{\sqrt{5}}{2}, 2 - \frac{\sqrt{5}}{2}\}.$$

n-Column ϕ Partition



- This construction partitions the integers into n groups with the rightmost two sequences having a closed form in terms of $a(k)$.
- $d(k, n) = (2^{n-1} - 1)a(k) + k$ and $c(k, n) = a(k) + (2^{n-1} - 2)k - (2^{n-2} - 1)$.
- The column densities in $\mathbb{Z}^+$ give an interpolation of the identities $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$ with convergence to an arithmetic progression partition as $n \to \infty$.

Column Densities

How do the 2 and 3 column ϕ partitions overlap with one another?

A	B
1	2
3	5
4	7
6	10
8	13
9	15
$\vdots$	$\vdots$

S	C	D
1	2	4
3	6	11
5	9	15
7	13	22
8	17	29
10	20	33
$\vdots$	$\vdots$	$\vdots$

- For a given integer $a(k)$ or $b(k)$, can we figure out whether it lies in D, C, or S?
- If we mark the rows of D, C, S with A and B dependent on whether the integers in that row lie in A or B, only 5 of 8 possibilities occur. What are the frequencies and why?
- We can instead mark the A, B integer pairs by how they appear in D, C, and S. Numerical data suggests the following density values:

Pair	SC	CS	DS	CD	SS	DC	SS	CC	DD
Density	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{\phi-1}{5}$	$\frac{3-\phi}{5}$	0	0	0	0

References

- Beatty, Samuel (1926). *Problem 3173.* American Mathematical Monthly. 33 (3): 159. doi:10.2307/2300153
- Uspensky, J. V. (1927). *On a problem arising out of the theory of a certain game.* Amer. Math. Monthly 34 (1927), pp. 516–521.