FINITE ELEMENT METHODS: 1970's AND BEYOND
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STABILIZED FINITE ELEMENT METHODS

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Abstract. We give a brief overview of stabilized finite element methods and illustrate the developments applied to the advection-diffusion equation.

Key words: Stabilized methods

1 INTRODUCTION

On the introduction and generalization of stabilized finite elements lays one of the major contributions of Tom Hughes.

Stabilized finite element methods are formed by adding to the standard Galerkin method terms that are mesh-dependent, consistent and numerically stabilizing. Starting with the Streamline Upwind Petrov-Galerkin method (SUPG - see [10, 37]), a generalization was proposed for the Stokes problem (see [40]) that circumvents the need to satisfy the Babuška-Brezzi condition [1, 6], which showed the potential of extending the idea to various applications. Designing the perturbation term as in the least-squares formulation, the Galerkin/least-squares method was applied to various structural and fluid problems (see [18, 38, 39, 41] and references therein).

Related to this effort and at the same time, a number of other developments were taking place. Under the leadership of Claes Johnson a series of articles presented analysis of the scalar advective diffusive equation [46], the incompressible Navier-Stokes [47] and a generalization of the discontinuity capturing term [48] introduced earlier in [45].

In the 80's at Stanford newer developments were being made and Tom led an extraordinary effort that opened new doors in many areas. Many are described in more detail elsewhere in this book. Let us for example mention acoustics [26, 27], where stabilized methods added at almost no cost a term that improved the accuracy of the Galerkin method. This was an area where finite elements had been dismissed as expensive, and all of a sudden, with a more accurate method, it became competitive.

The 80's were inspiring and led to many other developments in the 90's and today. Let us mention some of the contributions by application (a complete list is not feasible for this article - other references can be found in this book). For advection-diffusion, reaction-diffusion and advection-diffusion-reaction scalar equations see [11, 12, 14, 17, 15, 23, 24, 28, 32, 30, 61]. For Stokes and incompressible Navier-Stokes equations see [3, 8, 13, 16, 19, 22, 25, 49, 58, 33, 34]. For compressible Navier-Stokes see [33, 34, 42, 43, 44]. For moving boundaries and interfaces see [53, 59, 60]. For shallow water flows see [5, 29, 31]. Analysis and more references can be found in the book [55].

In this note we revisit stabilized finite element methods for the advective-diffusive equation. The SUPG, the Galerkin/least-squares and the Unusual stabilized methods are displayed and their analysis, recalled.

2 THE ADVECTIVE-DIFFUSIVE MODEL

The advective-diffusive model consists of finding the scalar-valued function $u(\boldsymbol{x})$ such that

$$-\kappa\Delta u + \boldsymbol{a}\cdot\nabla u = f\,, \tag{1}$$

where κ is the positive constant viscosity coefficient (or diffusivity), $\boldsymbol{a}$ is the velocity field and f is a source function. To simplify notation, in what follows we assume $\boldsymbol{a}$ to be constant, although the subsequent analysis and considerations equally apply to velocities which are piecewise constant in the partition of Ω. Ω is the domain assumed to be smooth. In addition, for simplicity, consider a homogeneous Dirichlet boundary condition:

$$u = 0 \qquad \text{on}\,\Gamma = \partial\Omega\,. \tag{2}$$

The variational formulation corresponding to (1)-(2) is: find $u \in H_0^1(\Omega)$ such that

$$\kappa(\nabla u, \nabla v) + (\boldsymbol{a}\cdot\nabla u, v) = (f, v) \qquad \forall v \in H_0^1(\Omega), \tag{3}$$

where $(\,\cdot\,,\,\cdot\,)$ indicates integration over Ω and $H_0^1(\Omega)$ is the Sobolev space with square integrable value and derivative, and zero value on the boundary. We denote by $||\cdot||_1$ the norm associated with this space and by $||\cdot||$ the $L_2(\Omega)$ norm.

If we set $v = u$ in (3), use Poincaré-Friedrichs to bound from below the left-hand-side and Cauchy-Schwarz to bound from above the right-hand-side, we have:

$$\kappa C||u||_1^2 \leq \kappa||\nabla u||^2 = (f, u) \leq ||f||||u|| \leq ||f||||u||_1$$

Dividing the left and right terms by $\kappa C||u||_1$ we get:

$$||u||_1 \leq \frac{1}{\kappa C}||f|| \tag{4}$$

Remark 1 *This estimate indicates the intrinsic problem underlying this model equation. Namely, for small values of κ, small variations on the data f can lead to large variations on the solution u.*

3 STABILIZED FINITE ELEMENT METHODS

The standard Galerkin method is constructed based on the variational formulation (3) by taking a subspace of $H_0^1(\Omega)$ spanned by continuous piecewise polynomials. In two dimensions the support of these functions is a mesh partition of Ω into triangles and/or quadrilaterals that do not overlap. We denote by V_h such space of functions and write the Galerkin method as: find $u_h \in V_h$ such that

$$\kappa(\nabla u_h, \nabla v) + (\boldsymbol{a} \cdot \nabla u_h, v) = (f, v) \qquad \forall v \in V_h\,. \tag{5}$$

This method inherits the stability of the continuous problem (4) and it yields to spurious oscillations when the convective coefficient is larger than the diffusive coefficient (more precisely, in terms of the mesh Peclet number: when $|\boldsymbol{a}|h/\kappa >> 1$). The stabilized finite element methods for this model can be written as: find $u_h \in V_h$ such that

$$B(u_h, v) = F(v) \qquad \forall v \in V_h \tag{6}$$

where

$$\begin{aligned} B(u,v) &= \kappa(\nabla u, \nabla v) + (\boldsymbol{a} \cdot \nabla u, v) + S(u,v) &(7)\\ F(v) &= (f, v). &(8) \end{aligned}$$

$S(u,v)$ indicates the additional terms added to the standard variational formulation. These are added such that consistency is preserved and numerical stability, enhanced. There are three different terms that are usually considered for this model, namely:

$$\begin{aligned} S_1(u,v) &= \sum_K \tau_K(-\kappa\Delta u + \boldsymbol{a}\cdot\nabla u - f, \boldsymbol{a}\cdot\nabla v)_K &(9)\\ S_2(u,v) &= \sum_K \tau_K(-\kappa\Delta u + \boldsymbol{a}\cdot\nabla u - f, \boldsymbol{a}\cdot\nabla v - \kappa\Delta v)_K &(10)\\ S_3(u,v) &= \sum_K \tau_K(-\kappa\Delta u + \boldsymbol{a}\cdot\nabla u - f, \boldsymbol{a}\cdot\nabla v + \kappa\Delta v)_K &(11) \end{aligned}$$

where K denotes an arbitrary element of our partition, τ_K is a stability coefficient to be described below and $(\,\cdot\,,\,\cdot\,)_K$ denotes integration over K.

Remark 2 *The first stabilization S_1 corresponds to the additional term in the SUPG formulation [10].*

Remark 3 *The second stabilization S_2 is a least-squares type modification [41]. The method in this case is denoted by Galerkin/least-squares (or GLS for short).*

Remark 4 *The third stabilization S_3 gives rise to an unusual stabilized finite element method (USFEM) and it was first proposed in [17]. The method is also suggested by static condensation of bubbles added to the finite element space V_h (see [2, 7, 15]).*

Remark 5 *In [36] it was shown that stabilized methods and the USFEM method stem from incorporating analytically the unresolved scales (subgrid scales) into the finite element solution (resolved scales).*

The original stability parameters suggested in [10] were derived by comparing to finite difference stencils and were limited to linear interpolations. The stability parameter τ_K we will be using accommodates usage of higher order interpolations and it can be understood *a posteriori* based on *a priori* error analysis. The formulae are as follows (see [17]):

$$\begin{aligned}
\tau_K &= \frac{h_K}{2|\boldsymbol{a}|_p}\xi(\mathrm{Pe}_K) && (12)\\
\mathrm{Pe}_K &= \frac{m_k|\boldsymbol{a}|_p h_K}{2\kappa} && (13)\\
\xi(\mathrm{Pe}_K) &= \begin{cases} \mathrm{Pe}_K & 0 \le \mathrm{Pe}_K < 1 \\ 1 & \mathrm{Pe}_K \ge 1 \end{cases} && (14)\\
|\boldsymbol{a}|_p &= \left(\sum_{i=1}^{N} |a_i|^p\right)^{1/p} \qquad 1 \le p < \infty && (15)\\
m_k &= \min\left\{\frac{1}{3}, 2C_k\right\} && (16)\\
C_k \sum_K h_K^2 ||\Delta v||_{0,K}^2 &\le ||\nabla v||_0^2 \qquad v \in V_h && (17)
\end{aligned}$$

Remark 6 *An alternative definition of τ obviates the definition of the mesh parameter h_K and the inverse estimate constant C_k [20].*

4 HIGHLIGHTS OF AN ERROR ANALYSIS

In this section we sketch an error analysis for the unusual stabilized method, as introduced in [17] - see S_3 above. We also refer to this work for a complete analysis.

First, it follows from definition:

$$\begin{aligned}
m_k &\le 2C_k, && (18)\\
\xi(\mathrm{Pe}_K) &\le \mathrm{Pe}_K. && (19)
\end{aligned}$$

Then stability can be established as follows:

$$(\boldsymbol{a} \cdot \nabla v, v) = 0 \quad \forall v \in V_h \tag{20}$$

and

$$B(v,v) = 0 + \kappa\|\nabla v\|^2 + \|\tau^{1/2}\boldsymbol{a} \cdot \nabla v\|^2 - \sum_K \|\tau^{1/2}\kappa\Delta v\|_K^2 \tag{21}$$

On the other hand,

$$\begin{aligned}
\tau_K &= \frac{h_K}{2|\boldsymbol{a}|_p}\xi(\mathrm{Pe}_K) && (22)\\
&= \frac{m_k h_K^2}{4\kappa}\frac{\xi(\mathrm{Pe}_K)}{\mathrm{Pe}_K} && (23)\\
&\le \frac{m_k h_K^2}{4\kappa} && (24)
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_K \|\tau^{1/2}\kappa\Delta v\|_K^2 &\leq \frac{m_k\kappa}{4}\sum_K h_K^2\|\Delta v\|_K^2 &(25)\\ &\leq \frac{m_k}{4C_k}\kappa\|\nabla v\|^2 &(26)\\ &\leq \frac{\kappa}{2}\|\nabla v\|^2 &(27)\end{aligned}$$

Thus, the stability estimate follows:

$$B(v,v) \geq \frac{1}{2}\left(\kappa\|\nabla v\|^2 + \|\tau^{1/2}\boldsymbol{a}\cdot\nabla v\|^2\right) \tag{28}$$

Since the method is consistent, i.e., for the solution of the differential equation u

$$B(u,v) = L(v) \qquad \forall v \in V_h \tag{29}$$

we have that stability and consistency yield convergence:

Theorem 1 *The solution of the stabilized method u_h converges to the exact solution of the advective-difusive model u as follows:*

$$\begin{aligned}\kappa\|\nabla(u_h-u)\|^2 \quad + \quad & \|\tau^{1/2}\boldsymbol{a}\cdot\nabla(u_h-u)\|^2 \leq &(30)\\ & C\sum_K h^{2k}|u|_{k+1,K}^2\left(H(\mathrm{Pe}_K-1)h|\boldsymbol{a}|_p + H(1-\mathrm{Pe}_K)\kappa\right) &(31)\end{aligned}$$

with $H(\cdot)$ the Heaviside function given by:

$$H(x-y) = \begin{cases} 0, & x<y \\ 1, & x>y \end{cases} \tag{32}$$

For a proof we refer to [17].

Remark 7 *This convergence result accommodates zones of advection dominated regimes* $(\mathrm{Pe}_K > 1)$ *and zones of diffusion dominated regimes* $(\mathrm{Pe}_K < 1)$*, as typically found in practice.*

5 NUMERICAL EXAMPLES

In this section the performance of the SUPG method will be illustrated with two one-dimensional numerical examples.

5.1 Boundary layer

The first example consists of an unresolved boundary layer in $\Omega = (0,1)$, $a = 1$, $\kappa = 0.005$, $f(x) = 0$, with boundary conditions $u(0) = 0$ and $u(1) = 1$. The element Peclet number is 10, which renders the Galerkin method unstable.

Figure 1 shows the exact solution along the Galerkin and SUPG solutions. The Galerkin solution has poor stability, whereas the SUPG solution is stable and highly nodal-wise accurate.

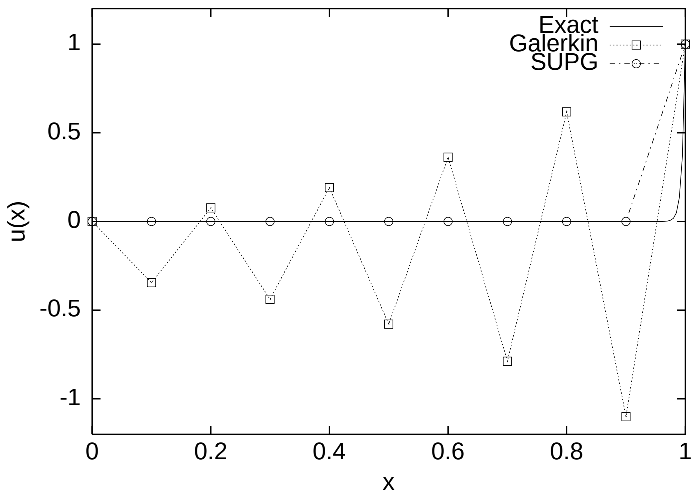

Figure 1: Boundary layer problem. Exact and numerical solutions.

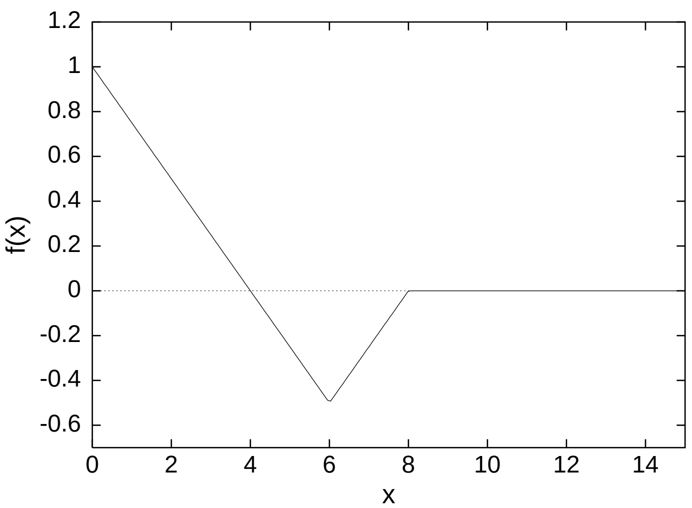

Figure 2: Convection with source term problem. Source term.

5.2 Convection with source term

Contrary to classical upwind methods, stabilized methods are consistent. The advantage of this key feature is shown in this example, where convection is combined with the independent source term $f(x)$ given by the Figure 2 [50].

The problem to be solved in $\Omega = (0, 15)$ has parameters $a = 1$, $\kappa = 0$, and boundary condition $u(0) = 0$. Figure 3 shows the Galerkin and SUPG solutions against the exact solution. Note that both are very good. However, plain upwind methods and inconsistent methods can yield to very inaccurate solutions.

6 QUO VADIS?

Trying to anticipate future developments due to the introduction of stabilized methods is an exercise that is certainly doomed to fail. The interest on stabilized methods continues to grow, judging by the number of papers citing these works.

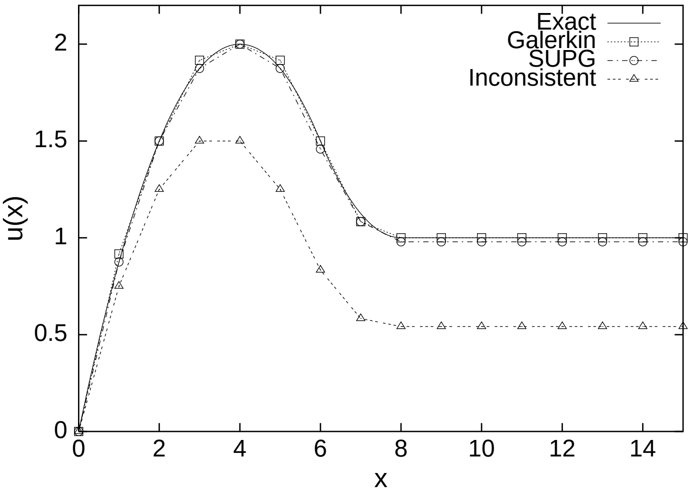

Figure 3: Convection with source term problem. Exact and numerical solutions.

To mention some of the advances published in the past two years alone, we find applications in: viscoelastic flows [4], human lung respiratory system [57], magnetohydrodynamics [56], flow in porous media [54], viscoplastic flows [52, 51], crystal growth [62], stochastic models for flows with chemical reactions [35]. Stabilized methods offer a framework that is appealing for generalizations and extensions to other applications.

In the following sections in this book we will see that the methodology can be also understood from constructing richer subspaces (other than the usual ones spanned by polynomials) in the Galerkin method. The appearance of the relationship between these methods [7, 2] evolved to the introduction of residual-free bubbles [9, 21] and multiscale formulations [36].

7 ACKNOWLEDGMENTS

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