$\mathcal{L}_1$ ADAPTIVE CONTROL FOR NONLINEAR AND NON-SQUARE MULTIVARIABLE SYSTEMS

BY

HANMIN LEE

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Aerospace Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Doctoral Committee:

Professor Naira Hovakimyan, Chair
Professor Petros G. Voulgaris
Associate Professor Dusan M. Stipanovic
Associate Professor Srinivasa M. Salapaka

ABSTRACT

This research presents development of $\mathcal{L}_1$ adaptive output-feedback control theory for a class of uncertain, nonlinear, and non-square multivariable systems. The objective is to extend the $\mathcal{L}_1$ adaptive control framework to cover a wide class of underactuated systems with uniform performance and robustness guarantees.

This dissertation starts by investigating some structural properties of multivariable systems that are used in the development of $\mathcal{L}_1$ adaptive output feedback controllers. In particular, a state-decomposition is introduced for adaptive laws that only depends on the output signals. The existence of the decomposition is ensured by defining a virtual system for underactuated plants. Based on the mathematical findings, we propose a set of output feedback solutions for uncertain underactuated systems.

In adaptive control applications, a baseline control augmentation is often preferred, where the baseline controller defines the nominal system response. Adaptive controllers are incorporated into the control loop to improve the system response by recovering the nominal performance in the presence of uncertainties. This thesis provides a solution for $\mathcal{L}_1$ output feedback control augmentation. Stability and transient performance bounds are proven using Lyapunov analysis. To demonstrate the benefits of the $\mathcal{L}_1$ adaptive controllers we consider a missile system and an inverted pendulum, which are both underactuated systems.

Finally, we propose a filter design framework in the frequency domain. A new sufficient condition is presented to ensure stability of the closed loop and the reference systems, which is subsequently used in the optimal filter design. Existing $\mathcal{H}_\infty$ optimization techniques are leveraged to address the performance and robustness trade-off issues.

To my wife Jihye and our family with endless love and respect

ACKNOWLEDGMENTS

I am deeply grateful to my advisor Prof. Naira Hovakimyan for her guidance, inspiration, and support throughout my Ph.D studies. Her supervision has promoted my adherence to high academic standards and high quality work. Prof. Hovakimyan's indispensable advice and knowledge have equipped me with enhanced understanding of mathematical rigor, as well as practical insights on engineering problems. I would like to express my gratitude to the committee members Prof. Petros Voulgaris, Prof. Dusan Stipanovic, and Prof. Srinivasa Salapaka for their helpful comments to improve my dissertation.

My thanks also go to our research group members. The opportunities to work with Venanzio Cichella, Steven Snyder, and Hamidreza Jafarnejadsani allowed me to explore academic subjects with great pleasure, and thoughtful discussions have been a significant aid in developing the ideas of my dissertation. In addition, friendly support from all the other labmates is greatly acknowledged.

Special acknowledgment is given to the Korean Agency for Defense Development (ADD) for granting me the opportunity of doctoral study. I am utterly thankful to Prof. Youdan Kim, Prof. Chan Gook Park, Dr. Hang-Ju Cho, and my former supervisors VP. Ui-Jun Mun, VP. Hee Cherl Lee, and PR. Hamin Jeong, who motivated me to pursue doctoral studies and helped me secure a place at the University of Illinois at Urbana-Champaign. Additionally, recognition must be given to my considerate colleagues, who gave kind assistance to successfully complete my program of study.

Last but not least, heartfelt appreciation and thanks go toward my beloved wife Jihye and our dear family. I am indebted for her sacrifice, patience, and tolerance during the course of my graduate work. This could never have been completed without having her being besides me. She has been an exceptional friend and a lovely wife adding endless sparkle and love into my life. I am profoundly grateful to my parents, parents-in-law, and all family members for their persistent support and encouragement. The dissertation would not have been possible without their unconditional love.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\bar{\mathbb{C}}^+$	The set of complex numbers whose real parts are strictly positive or zero
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^-$	The set of complex numbers whose real parts are strictly negative
$\mathbb{I}_n$	The identity matrix with dimension n
$\bar{\mathbb{N}}$	The set of natural numbers including zero
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The set of n dimensional vectors whose entries are in $\mathbb{R}$
$\mathbb{R}^{n\times l}$	The set of $(n \times l)$ matrices whose entries are in $\mathbb{R}$
$\mathbb{R}^+$	The set of positive real numbers
$\bar{\mathbb{R}}^+$	The set of positive real numbers including zero
$\mathbb{R}[s]$	The ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$	The field of rational fractions associated with $\mathbb{R}[s]$
$\mathbb{R}^{p\times m}[s]$	The ring of $p \times m$ matrix polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}^{p\times m}(s)$	The set of $p \times m$ matrices whose entries are in $\mathbb{R}(s)$
$\mathbb{Z}$	The set of integers
$\mathbf{1}_m$	The vector $\mathbf{1}_m = [1, \cdots, 1] \in \mathbb{R}^m$
$\det A$	The determinant of a matrix A
$\mathrm{diag}\,(A_1, \ldots, A_n)$	The (block) diagonal matrix whose (i, i)th entry is A_i, $1 \le i \le n$
$\mathrm{eig}(A)$	The set of all eigenvalues of a matrix A
$\lambda_{\max}(P)$	The minimum eigenvalue of a positive (semi-) definite matrix P
$\lambda_{\min}(P)$	The minimum eigenvalue of a positive (semi-) definite matrix P
$\mathrm{normrank}(P(s))$	The normal rank of $P(s) \in \mathbb{R}^{p\times m}(s)$
$\mathrm{rank}(A)$	The rank of a matrix A

$\sqrt{A}$	The upper triangular matrix satisfying Cholesky decomposition, i.e., $A = \sqrt{A}^\top \sqrt{A}$
$\sup(\cdot)$	The supremum of a set or the essential supremum of a function
$A^\dagger$	The generalized inverse of a matrix A
$A^\top$	The transpose of a matrix A
$(x)_\tau, x_\tau$	The truncated signal of $x(t)$ such as $x_\tau(t) = 0$ for $t > \tau$, and $x_\tau(t) = x(t)$ for $t \leq \tau$
$\mathscr{L}(\cdot)$	The Laplace transform operator
$\|\cdot\|$	The matrix or vector ∞-norm
$\|\cdot\|_p$	The matrix or vector p-norm

CHAPTER. 1

Introduction

Stability of feedback systems and asymptotic tracking of the reference commands are the main goals in almost all control design problems. The main challenge that the control designers face stems from the presence of model uncertainties and external disturbances in the system. For example, flying vehicles are required to operate in large flight envelopes; hence, their control systems are inevitably affected by modeling uncertainties and unavoidable disturbances. Modern control techniques have been extensively investigated to ensure closed-loop stability and robust tracking performance. Adaptive control was motivated by the design of autopilots that could operate for large flight envelopes with sufficient robustness and desirable performance. Early developments in adaptive control were validated in experiments without in-depth robustness analysis, which led to the tragic flight test of the X-15, [1, 2]. The initial work in adaptive control was inspired by system identification and was focused on the paradigm of combining on-line parameter estimators and adjustable control laws [3–5]. The stability proofs of adaptive controllers were developed in [6–12]. As a result, adaptive control has become one of the most popular methods for dealing with system parametric and structural uncertainties in the last decade. With that said, most of the real-world applications have been based on full-state feedback measurements. While there has been a significant effort to develop output feedback extensions, most of the developments remained focused on square Multi-Input Multi-Output (MIMO) systems, subject to relative degree constraints manifested by strictly positive real transfer functions for desired system behavior. Such developments appear to be non-suitable for underactuated systems, where the number of inputs is less than the number of regulated output variables. In this thesis, we develop adaptive output-feedback solutions for nonlinear and underactuated systems using $\mathcal{L}_1$ adaptive control theory that has been recognized for its ability to deliver uniform performance with a priori robustness guarantees [2, 13–17].

1.1. Overview of $\mathcal{L}_1$ Adaptive Control

$\mathcal{L}_1$ adaptive control theory emerged in 2005 to address performance and robustness issues in adaptive control systems, [18, 19]. Robustness of model reference adaptive control (MRAC) systems has been significantly challenged in mid-eighties through the well-known Rohrs' example [20]. Following that seminal article, projection operator was introduced in [21] to ensure boundedness of the adaptive estimates of the parameters, leading to boundedness of the closed-loop signals in the presence of disturbances. Nevertheless, challenges remained with the prediction of the transient response and robustness margins of the closed-loop adaptive systems.

$\mathcal{L}_1$ adaptive control resolves these problems by introducing a new control architecture, with predictable transient response and robustness guarantees. The main elements of an $\mathcal{L}_1$ adaptive

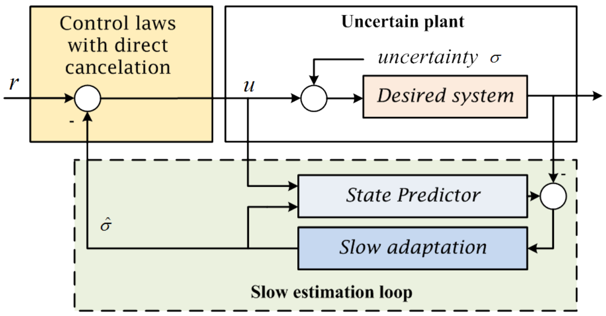

Figure 1.1: Model reference adaptive control

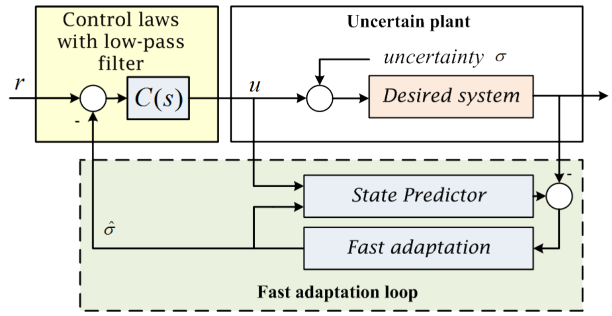

Figure 1.2: $\mathcal{L}_1$ adaptive control

controller are a low-pass filter, a state predictor and adaptation laws. In $\mathcal{L}_1$ architecture, the low pass filter plays a central role in preventing high frequency signals entering the control channel. This low-pass filter decouples the control loop from the estimation loop. Figures 1.1 and 1.2 illustrate architectural differences between standard Model Reference Adaptive Controllers (MRAC) and $\mathcal{L}_1$ adaptive controllers. Unlike MRAC, the $\mathcal{L}_1$ controller allows one to use high adaptation gains without losing robustness, which helps to improve the system performance in the presence of large and rapid variation of uncertainties. The benefits of $\mathcal{L}_1$ adaptive controllers have been extensively studied in [2,13,22], and it was shown that the $\mathcal{L}_1$ architecture leads to guaranteed robustness in the presence of fast adaptation. As a result, the adaptation gains in the architecture are limited only by the available hardware, as CPU and sensor sampling. In fact, the filtering structure provides a trade-off between robustness and performance. $\mathcal{L}_1$ adaptive controllers have been successfully employed in real-world aerospace applications involving performance recovery after challenging failure events [14–17, 23, 24].

The fundamental idea of $\mathcal{L}_1$ adaptive control is to compensate for uncertainties only within the bandwidth of the low-pass filter; this strategy seems to be less ambitious than full uncertainty compensation as MRAC aims to achieve. Similar ideas, in terms of control strategies, are found in Disturbance Observer Based (DOB) Control or Internal Model Control (IMC), which use disturbance estimation in feedback signals to compensate for uncertainties. Compared to $\mathcal{L}_1$ adaptive control, DOB/IMC require explicit system inversion to compute the estimate of the disturbance signals. This may limit the range of applications, since there are many physical plants for which it is difficult to obtain the inverse (e.g., non-square Multi-Input Multi-Output (MIMO) systems and nonlinear plants, to name a few). Moreover, the design process for a low-pass filter should include the procedure to obtain the inverse system [25–28].

The $\mathcal{L}_1$ adaptive architecture considers an auxiliary $\mathcal{L}_1$ reference system, which is the closed-loop system resulting from compensation of uncertainties within the low-pass filter bandwidth. The signals of the closed-loop $\mathcal{L}_1$ adaptive system approximate the signals of this closed-loop reference system in the presence of fast adaptation. The fast estimation loop achieves an implicit system inversion, similar to IMC/DOB controllers, yet without explicitly constructing an inverse. This architectural flexibility of $\mathcal{L}_1$ adaptive controller allows to explore a large class of uncertain systems, including underactuated systems that cannot be inverted. Since the design of the low-pass filter is decoupled from the estimation loop, one can account for system delays, control signal saturation and rate constraints, actuator and sensor dynamics in the estimation loop [29–31].

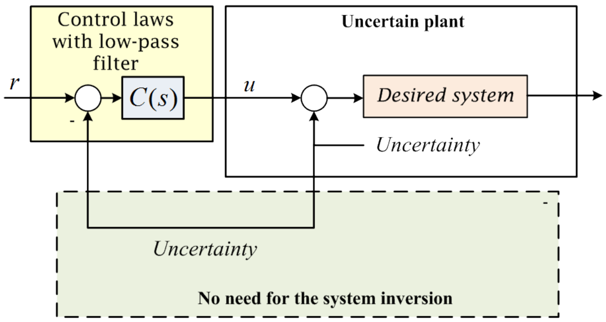

Figure 1.3: $\mathcal{L}_1$ reference system (not implementable)

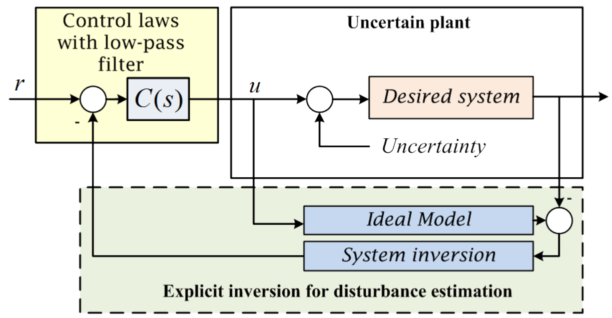

Figure 1.4: IMC

Taking advantage of the architectural flexibility of the $\mathcal{L}_1$ adaptive control structure, this dissertation outlines the design of $\mathcal{L}_1$ adaptive controllers for nonlinear and non-square MIMO systems, retaining all the benefits of existing $\mathcal{L}_1$ adaptive controllers.

1.2. Literature Review on Adaptive Output Feedback

A pioneering work in adaptive output feedback control was reported by Monopoli in 1974, which considered the use of auxiliary signals to design asymptotically stable model reference adaptive controllers for Linear Time-Invariant (LTI) Single-Input Single-Output (SISO) systems [8]. This work also led to further research on the topic of output-feedback control design [10, 12, 32]; however the output-feedback approaches generally are limited by structural assumptions.

Common approaches are based on passivity-type assumptions (e.g., Strictly Positive Realness (SPR) or its variations) [10, 33]. For example, the SPR condition allows one to apply the Kalman-Yakubovich-Popov (KYP) lemma, which presents an algebraic relationship between the internal states and output signals, thus admitting output-dependent adaptive laws. Notice that these assumptions limit the range of applications to systems with relative degree one. Several solutions are found in the literature to handle SISO plants with high relative degree. The authors of [9] borrowed the concept of auxiliary signals and extended the results to SISO systems with higher relative degree. Solutions based on adaptive back-stepping techniques have been proposed in [12, 34–36], where the plants were assumed to have a specific recursive structure and high relative degree.

In general, solutions to SISO systems can be straightforwardly extended to MIMO plants under similar structural requirements [10, 32, 33, 37]. The authors of [32, 37] extend the results from SISO SPR systems to MIMO systems. They introduce a modified interactor in order to relax the SPR assumptions, thus increasing the applicability of the result to include square MIMO systems with high relative degree; however, their applications are limited to Linear Time-Invariant (LTI)

MIMO systems. Similarly, the approaches in [38, 39] borrow concepts and tools from [12, 34–36] to address square MIMO systems with high relative degree. In these methods, the improvement in the transient performance is achieved by introducing a nonlinear damping term, which attenuates the effect of initial parameter errors. However, one silent drawback is the structural complexity of the controllers, which make it difficult to implement especially as the relative degree increases. Moreover, the approaches assume that the uncertainties are parametrized with unknown constants and output-dependent regressor functions. Their extension to time-varying uncertainties, which are dependent on the internal states, is not straightforward.

Other related papers in the literature are based on high-gain observers [40–42]. Since the state errors for the adaptation laws are estimated by the observer, the observer-based methods allow to use a state-feedback structure, which helps to deal with high relative degree. For example, the authors of [40] use the structure of the adaptive state-feedback control, and adopt a high-gain observer to provide error estimation for the adaptation laws. However, the saturation function for preventing the peaking phenomenon is based on the apriori knowledge of the initial conditions. When the stability domain is set to be larger, the saturation level needs to be increased, which can result in unacceptable transients since higher peaking signals are transmitted to the plant [43].

The majority of physical systems exhibits a non-square structure, or becomes underactuated when control augmentation is performed on a square system [44]. One way to deal with non-square MIMO plants is to employ solutions for square systems in combination with squaring (-down or -up) methods [45, 46] Squaring-down methods can be applied to overactuated systems [45] by reducing the excessive number of inputs; the use of existing square-based controllers in overactuated systems is not challenging. However, when dealing with underactuated systems, these methods ignore available measurements, thus limiting the use of output information. The disadvantage of squaring-down methods becomes even more evident when the system under consideration is non-minimum phase (e.g. missiles, inverted pendulums, etc.). In general, the extension of solutions for square systems to underactuated systems is not trivial [47].

Recent work on adaptive output feedback control of underactuated systems can be found in [48–50]. The authors of [48, 49] present a solution for square systems and its extension to non-square systems using square-up methods from [46]. They consider the systems in which the product between the input and output matrices is full rank; this assumption intrinsically implies that the system must have (vector) relative degree equal to one. The work in [48, 49] was extended in [51] to only deal with systems that have relative degree equal to two. In [50], the authors tackle underactuated systems by designing an adaptive controller with multi-rate inputs. The approach requires the lifted system to be Almost SPR (ASPR), and thus may not be applicable to systems with any relative degree.

In this thesis, we propose adaptive output-feedback solutions for a wider class of nonlinear and non-square systems. Our solutions are applicable to underactuated systems with arbitrary relative degree, and provide guaranteed transient and steady-state performance bounds. The solutions in this thesis are based on $\mathcal{L}_1$ adaptive control theory, which considers a filtering structure providing a trade-off between robustness and performance. With this architecture, the control loop is decoupled from the estimation loop, affording high adaptation gains. While $\mathcal{L}_1$ adaptive state-feedback controllers (e.g. [13, 52]) have been successfully employed in real-world applications [14–17, 24, 53], the literature directly concerned with output-feedback implementations is less extensive [30, 54–58]. $\mathcal{L}_1$ output-feedback solutions for Single-Input Single-Output (SISO) systems can be found in [55,57,58], and can be easily extended to square MIMO systems. For example, the approach of [55] can be employed for square nonlinear systems with high relative degree. $\mathcal{L}_1$ output feedback using input predictor [57] and $\mathcal{L}_1$ output feedback with model reference control [58] are applicable to square systems with a more easily verifiable stability condition. However, their extension to underactuated systems is challenging due to fundamental assumptions that hold only for square systems.

1.3. Main Contributions and Thesis Organization

In this thesis, we develop $\mathcal{L}_1$ adaptive output-feedback controllers for a class of nonlinear and underactuated systems. The main contributions are: (*i*) analysis of mathematical properties of multivariable systems, which can be used to develop $\mathcal{L}_1$ adaptive controllers for underactuated systems; (*ii*) control of nonlinear underactuated systems with arbitrary relative degree; (*iii*) analytical results on the performance bounds during the transient and steady-state; and (*iv*) a filter design framework, which is suitable for high order systems with frequency-domain specifications.

In Chapter 2, we introduce the relevant definitions and theoretical findings which are used in this thesis. This chapter includes topics from linear systems theory, definitions on system stability, and several approaches for uncertainty parametrization. In particular, we introduce a state decomposition with a verifiable condition for underactuated systems. The result is initially developed based on the assumption of relative degree one, and later is extended to systems with high relative degree.

Chapter 3 addresses control problems for underactuated nonlinear systems with relative degree one. In this chapter, we introduce $\mathcal{L}_1$ augmentation of state-feedback and dynamic output-feedback controllers. The $\mathcal{L}_1$ adaptive controllers are designed to recover the nominal performance. The theoretical results demonstrate that the transient response in the presence of non-zero initialization error is upper bounded by a strictly decreasing function, and that arbitrary small steady-state errors can be obtained by allowing high adaptation gains. The results are verified with illustrative examples.

In Chapter 4, we develop output-feedback solutions for underactuated systems with arbitrary relative degree. These approaches make the use of a virtual system and a right interactor that are presented in Chapter 2. The controller employs uncertainty estimation in the virtual system, which is an auxiliary structure for addressing systems with high relative degree. The performance of the controller is analyzed. A numerical example is provided to validate the theoretical findings.

We also derive piecewise constant adaptation laws for $\mathcal{L}_1$ adaptive control, which is more suitable for real-time applications. Analysis is presented to show that arbitrary performance bound can be assigned by selecting a sufficiently fast sampling rate.

Chapter 5 considers two different applications for underactuated systems. First, we consider missile-autopilot design applications. The proposed method developed in Chapter 3 is used to augment the baseline controller with $\mathcal{L}_1$ adaptive controller, resulting in the well-known three-loop autopilot. In this application, we show how the non-minimum phase zeros of the acceleration loop are addressed. Simulations show that the $\mathcal{L}_1$ adaptive controller improves the tracking performance per theoretical predictions. The second application is the inverted pendulum on a cart. The dynamics have high relative degree in this case. We demonstrate that the $\mathcal{L}_1$ controller developed in Chapter 4 can stabilize the system with arbitrarily small steady-state bounds. Simulation results are carried out to show the performance of the proposed controller.

Chapter 6 addresses a low-pass filter design problem for the $\mathcal{L}_1$ reference system. The low-pass filter is a key element in $\mathcal{L}_1$ adaptive control, which decides the trade-off between robustness and performance. In this chapter, a filter design framework to deal with frequency-domain specifications is proposed towards optimal trade-offs between robustness and performance. This approach avoids existing conservative designs that may occur in high order systems. We first present a new sufficient condition to guarantee stability of both the $\mathcal{L}_1$ reference system and the closed-loop $\mathcal{L}_1$ adaptive system. With this condition, a filter design framework is proposed with a suitable parametrization of the filter. The proposed method is more suitable especially for high order systems, in which frequency-domain specifications are easily incorporated. A design example illustrates the benefits of the proposed approach.

Finally, this thesis ends with concluding remarks addressing future research in Chapter 7.

CHAPTER. 2

Mathematical Preliminaries

In this chapter, we introduce a few structural properties of multivariable systems, and review stability theory. Mathematical results on underactuated systems[1] are presented towards obtaining $\mathcal{L}_1$ adaptive output-feedback solutions.

2.1. Linear Systems

Throughout this section, we consider the LTI system, denoted by $\mathcal{G}$:

$$\mathcal{G} : \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \quad x(0) = x_0, \tag{2.1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ with $\{A, B, C, D\}$ having appropriate dimensions. The transfer matrix of the system (2.1.1) is given by

$$G(s) = C(s\mathbb{I}_n - A)^{-1}B + D, \quad G(s) \in \mathbb{R}^{p\times m}(s). \tag{2.1.2}$$

2.1.1. *Zeros of Multivariable System*

This section starts by briefly reviewing mathematical definitions and tools that are used in the polynomial approach for multivariable system analysis. $\mathbb{R}^{p\times m}[s]$ is the ring of $p \times m$ matrix polynomials with coefficients in $\mathbb{R}$, and $\mathbb{R}^{p\times m}(s)$ is the set of $p \times m$ matrices whose entries are in the field of rational fractions.

Definition 2.1.1 (Degree of a polynomial matrix, [59])**.** *Let $P(s) \in \mathbb{R}^{p\times m}[s]$ be a $p \times m$ matrix polynomial. The degree of $P(s)$, denoted by* $\deg(P(s))$*, is defined as the largest integer k such that $B_k \neq 0$, where $P(s) = s^k B_k + s^{k-1}B_{k-1} + \cdots + sB_1 + B_0$ with $B_l \in \mathbb{R}^{p\times m}$,* $0 \le l \le k$ *and* $l \in \mathbb{Z}$*.*

Definition 2.1.2 (Infinite elementary divisor, [59])**.** *Let $P(s) \in \mathbb{R}^{p\times m}[s]$ and* $d = \deg(P(s))$*. The infinite elementary divisors of $P(s)$ are defined as the elementary divisors of the polynomial matrix $s^d P(1/s)$.*

Definition 2.1.3 (Unimodular polynomial matrix, [59])**.** *Let $U(s) \in \mathbb{R}^{n\times n}[s]$. $U(s)$ is called unimodular if $det(U(s)) = \alpha$ and $\alpha \neq 0$. Equivalently, $U(s)$ is unimodular if and only if there exists $U^{-1}(s) \in \mathbb{R}^{n\times n}[s]$ such that $U(s)U^{-1}(s) = U^{-1}(s)U(s) = \mathbb{I}_n$.*

Definition 2.1.4 (Normal rank of a polynomial matrix, [59])**.** *Let $P(s) \in \mathbb{R}^{p\times m}[s]$. The normal*

[1] Hereafter *underactuated system* denotes tall or square system.

rank of $P(s)$, denoted by normrank$(P(s))$*, is defined as*

$$\text{normrank}(P(s)) = \max\{\text{rank}(P(s)) : s \in \mathbb{C}\}.$$

Theorem 2.1.1 (Matrix polynomial decomposition, [60,61])**.** *Let $P(s) \in \mathbb{R}^{p\times m}[s]$ and $r =$ normrank$(P(s))$. Then, there exist unimodular matrices $U_1(s) \in \mathbb{R}^{p\times p}$ and $U_2(s) \in \mathbb{R}^{m\times m}[s]$ such that*

$$S_P(s) = \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} = U_1(s)P(s)U_2(s), \tag{2.1.3}$$

where $\Lambda(s) = \text{diag}\,(\lambda_1(s), ..., \lambda_r(s))$; $\lambda_i(s) \in \mathbb{R}[s]$ is a monic polynomial satisfying $\lambda_i(s)|\lambda_{i+1}(s)$ for $i = 1, ..., r-1$. $S_P(s) \in \mathbb{R}^{p\times m}[s]$ is called the Smith form of $P(s)$.

In Theorem 2.1.1, $\lambda_i(s)$ is uniquely determined by $P(s)$. Moreover, If $\Delta_i(s)$ is defined as the monic greatest common divisor of all the $i \times i$ non-zero minors of $P(s)$, then $\lambda_i(s) = \Delta_{i+1}(s)/\Delta_i(s)$ with $\Delta_1 = 1$, and $\{\Delta_i; 1 \leq i \leq r\}$ is the set of determinantal divisors of $P(s)$.

Definition 2.1.5 (Smith zero, [59])**.** *Let $P(s) \in \mathbb{R}^{p\times m}[s]$ and $r =$ normrank$(P(s))$. The $z_0 \in \mathbb{C}$ is called a Smith zero of $P(s)$ if z_0 is a root of $z_\Lambda = \prod_{i=1}^{r} \lambda_i(s)$, where $\lambda_i(s)$ is the i-th diagonal element of $\Lambda(s)$ given in* (2.1.3)*.*

Definition 2.1.6 (Smith-McMillan form, [60,61])**.** *Consider $G(s) \in \mathbb{R}^{p\times m}(s)$ with $r =$ normrank$(G(s))$. Let $d(s) \in \mathbb{R}[s]$ be the monic least common denominator of all non-zero entries of $G(s)$, and define $N(s) \in \mathbb{R}^{p\times m}[s]$ such that $G(s) = \frac{1}{d(s)}N(s)$; notice that $N(s)$ has the normal rank r. From Theorem 2.1.1, we can find unimodular matrices $U_1(s) \in \mathbb{R}^{p\times p}[s]$ and $U_2(s) \in \mathbb{R}^{m\times m}[s]$ such that $\Gamma(s) = U_1(s)N(s)U_2(s)$. Let $M_G(s) = U_1(s)G(s)U_2(s)$. Reducing the elements of $M_G(s)$ produces*

$$\frac{\lambda_i(s)}{d(s)} = \frac{\epsilon_i(s)}{\phi_i(s)}, \quad i = 1, ..., r, \tag{2.1.4}$$

where $\{\epsilon_i, \phi_i\}$ is coprime for each $i \in [1, \ldots, r]$. Then, $M_G(s)$ is called the Smith-McMillan form of $G(s)$, which is given by

$$M_G(s) = \begin{bmatrix} \Gamma_G(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1.5}$$

with $\Gamma_G(s) = \text{diag}\left(\frac{\epsilon_1(s)}{\phi_1(s)}, ..., \frac{\epsilon_r(s)}{\phi_r(s)}\right)$. Notice that $\epsilon_i(s)|\epsilon_{i+1}(s)$, $\phi_{i+1}(s)|\phi_i(s)$, $i \in [1, \ldots, r-1]$, and $d(s) = \phi_1(s)$.

Moreover, Let $\mathcal{A}_G$ be the set of zeros of $\epsilon_i(s)$ and $\phi_i(s)$ with $i \in [1, \ldots, r]$. Then, for each $\alpha \in \mathcal{A}_G$ the set of structural indices is defined as $\{\sigma_i; \sigma_i \leq \sigma_{i+1}, i = 1, ..., r-1\}_\alpha$ such that

$$\prod_{\alpha\in\mathcal{A}_G} M_\alpha(s) = \text{diag}\left(\frac{\epsilon_1(s)}{\phi_1(s)}, ..., \frac{\epsilon_r(s)}{\phi_r(s)}\right), \quad M_\alpha(s) = \text{diag}\left((s-\alpha)^{\sigma_1}, \ldots, (s-\alpha)^{\sigma_r}\right). \tag{2.1.6}$$

Definition 2.1.7 (Coprime polynomial matrices, [60])**.** $N_r(s) \in \mathbb{R}^{p\times m}[s]$, $D_r(s) \in \mathbb{R}^{m\times m}[s]$,

$N_l(s) \in \mathbb{R}^{p\times m}[s]$, and $D_l(s) \in \mathbb{R}^{p\times p}[s]$. Then

- *$N_r(s)$ and $D_r(s)$ are called right coprime, if there exist $X_r(s) \in \mathbb{R}^{m\times p}[s]$ and $Y_r(s) \in \mathbb{R}^{m\times p}[s]$ such that $X_r(s)N_r(s) + Y_r(s)D_r(s) = \mathbb{I}_m$.*
- *$N_l(s)$ and $D_l(s)$ are called left coprime, if there exist $X_r(s) \in \mathbb{R}^{p\times p}[s]$ and $Y_r(s) \in \mathbb{R}^{p\times p}[s]$ such that $N_l(s)X_l(s) + D_l(s)Y_l(s) = \mathbb{I}_p$.*

Now, let $G(s) \in \mathbb{R}^{p\times m}(s)$. A polynomial fraction description of $G(s)$ is defined as

$$G(s) = N_r(s)D_r^{-1}(s) = D_l^{-1}(s)N_l(s), \tag{2.1.7}$$

where $N_r(s) \in \mathbb{R}^{p\times m}[s]$, $D_r(s) \in \mathbb{R}^{m\times m}[s]$ are right coprime, and $N_l(s) \in \mathbb{R}^{p\times m}[s]$, $D_l(s) \in \mathbb{R}^{p\times p}[s]$ are left coprime.

Definition 2.1.8 (Matrix pencil, [59])**.** *Consider the system* (2.1.1)*. The matrix pencil $P_{\mathcal{G}}(s) \in \mathbb{R}^{p\times m}[s]$ of $\mathcal{G}$ is defined as*

$$P_{\mathcal{G}}(s) = -sP_E + P_A, \quad P_E = \begin{bmatrix} \mathbb{I}_n & 0 \\ 0 & 0_{p\times m} \end{bmatrix}, \quad P_A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.1.8}$$

Taking the Laplace transform on $\mathcal{G}$, one has the following relationship:

$$\begin{bmatrix} -x_0 \\ y(s) \end{bmatrix} = (-sP_E + P_A) \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} -s\mathbb{I}_n + A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix}. \tag{2.1.9}$$

The matrix pencil of the system $\mathcal{G}$ is also called a Rosenbrock's system matrix. The matrix polynomial $P_{\mathcal{G}}(s)$ exhibits the internal structure associated with the state-space model, while the transfer matrix $G(s)$ describes the structure of reachable and detectable states (input-output structure). The matrix $P_{\mathcal{G}}(s)$ illustrates that frequency-response and state-space methods are interrelated. [62].

Lemma 2.1.1 ([63])**.** *Consider the system $\mathcal{G}$ given in* (2.1.1)*, and the matrix pencil $P_{\mathcal{G}}(s)$ of $\mathcal{G}$. Then, there exist unitary matrices $S \in \mathbb{R}^{(n+p)\times(n+p)}$, $T \in \mathbb{R}^{(n+m)\times(n+m)}$ (i.e., $S^\top S = SS^\top = \mathbb{I}_{n+p}$ and $T^\top T = TT^\top = \mathbb{I}_{n+m}$) such that*

$$S^\top(-sP_E + P_A)T = \begin{bmatrix} -sE_f + A_f & 0 \\ * & -sE_\infty + A_\infty \end{bmatrix}, \tag{2.1.10}$$

where $-sE_f + A_f$, and $-sE_\infty + A_\infty$ contain all the finite and infinite elementary divisors of $-sP_E + P_A$, respectively.

Remark 2.1.1. *Lemma 2.1.1 is often called the Schur decomposition lemma.*

Lemma 2.1.2. *Consider the system $\mathcal{G}$ given in* (2.1.1)*, the matrix pencil $P_{\mathcal{G}}(s)$ of $\mathcal{G}$, and its transfer matrix $G(s)$ given in* (2.1.2)*. Let* $\text{eig}(A)$ *be the set of all eigenvalues of A. Suppose $z \notin \text{eig}(A)$. Then,* $\text{rank}(P_{\mathcal{G}}(z)) = n + \text{rank}(G(z))$*. Moreover,* $\text{normrank}(P_{\mathcal{G}}(s)) = n + \text{normrank}(G(s))$ *holds.*

Proof. Notice that for all $z \notin \text{eig}(A)$

$$P_{\mathcal{G}}(z) = (-zP_E + P_A) = \begin{bmatrix} I & 0 \\ -C(z\mathbb{I}_n - A)^{-1} & I \end{bmatrix} \begin{bmatrix} -z\mathbb{I}_n + A & B \\ 0 & G(z) \end{bmatrix}$$
$$= \begin{bmatrix} -z\mathbb{I}_n + A & 0 \\ C & G(z) \end{bmatrix} \begin{bmatrix} I & -(z\mathbb{I}_n - A)^{-1}B \\ 0 & I \end{bmatrix}.$$

By applying Sylvester's inequality, it follows that

$$\text{rank}(P_{\mathcal{G}}(z)) = n + \text{rank}(G(z)), \quad \forall z \notin \text{eig}(A).$$

Since $\text{normrank}(G(s)) = \max_{z\in\mathbb{C}}(\text{rank}(G(z)))$ and $\text{normrank}(P_{\mathcal{G}}(s)) = \max_{z\in\mathbb{C}}(\text{rank}(P_{\mathcal{G}}(z)))$, one concludes $\text{normrank}(P_{\mathcal{G}}(s)) = n + \text{normrank}(G(s))$. This completes the proof. □

Notice that Lemma 2.1.2 does not assume that (A, B, C) is a minimal realization of $G(s)$.

Definition 2.1.9 (Degeneracy)**.** *Consider the system $\mathcal{G}$ given in* (2.1.1)*. The system is called degenerate if*

$$\text{rank}(P_{\mathcal{G}}(s)) < n + \min(m, p), \quad \forall s \in \mathbb{C}. \tag{2.1.11}$$

Otherwise, it is called non-degenerate system.

Lemma 2.1.3. *The system $\mathcal{G}$ in* (2.1.1) *is degenerate, if and only if* $\text{normrank}(G(s)) < \min(m, p)$ *for all $s \in \mathbb{C}$. Therefore, the system is non-degenerate, if and only if* $\text{normrank}(G(s)) = \min(m, p)$*.*

Proof. Notice that the system is degenerate if and only if Equation (2.1.11) holds. Moreover, a necessary and sufficient condition for (2.1.11) is $\text{normrank}(P_{\mathcal{G}}(s)) < n + \min(m, p)$, which, together with Lemma 2.1.2, leads to $\text{normrank}(G(s)) < \min(m, p)$. □

Notice that degeneracy of the system is based on a rank condition of the matrix pencil $P_{\mathcal{G}}(s)$. However, the same rank condition can be applied to the transfer matrix $G(s)$, which implies that the minimal system can be degenerate.

Definition 2.1.10 (Markov parameters, [59])**.** *Consider the transfer function $G(s)$ given in* (2.1.2)*, let $M_0 = D$ and $M_k = CA^kB$ with $k \in \mathbb{N}$. Then, $M_k \in \mathbb{R}^{p\times m}$ is called a Markov parameter.*

Lemma 2.1.4 ([59]). *Consider the transfer function $G(s)$ given in* (2.1.2)*. Let* $\mathrm{eig}(A)$ *be the set of all eigenvalues of A. Then, if $|s| > \max(\mathrm{eig}(A))$, the following holds:*

$$G(s) = D + \sum_{k=1}^{\infty} s^{-k} C A^{k-1} B = \sum_{k=0}^{\infty} s^{-k} M_k, \tag{2.1.12}$$

where M_k is a Markov parameter of $G(s)$ with $k \in \overline{\mathbb{N}}$.

Proof. The proof of Lemma is found in [59]. □

Remark 2.1.2. *The right side of* (2.1.12) *is called a Laurent series expansion of $G(s)$. Notice that from Lemma 2.1.4,* $\lim_{s\to\infty} G(s) = D$ *holds.*

In SISO systems, the zeros are the roots of the numerator polynomial of the transfer function, and thus characterize blocking property of some signals through the systems. The extension of this definition to MIMO systems is not trivial, as in MIMO systems one has a matrix of transfer functions in the numerator. There have been different definitions introduced for zeros of MIMO systems [61,62]. The zeros of a MIMO system have played critical role in the decoupling structure [64], high gain control [65], and invariance of internal states [66,67], and model matching/factorization [68]. Zeros are important in adaptive systems as well, since many existing techniques are limited if the system has unstable zeros. In the literature, different definitions of zeros have been introduced to illustrate structural properties of the MIMO system [60,69,70]. The classical MIMO zeros are transmission zeros, decoupling zeros, system zeros, invairant zeros, and zeros at infinity.

Definition 2.1.11 (Output-zeroing direction)**.** *Consider the system in* (2.1.1)*. Let $z_i \in \mathbb{C}$ be a complex number. Then, the augmented vector $\begin{bmatrix} x_i^\top & u_i^\top \end{bmatrix}^\top \in \mathbb{R}^{n+m}$ is called an output-zeroing direction associated with z_i, if*

$$P_{\mathcal{G}}(z_i) \begin{bmatrix} x_i \\ u_i \end{bmatrix} = 0, \tag{2.1.13}$$

where $P_{\mathcal{G}}(z_i)$ is the matrix pencil of $\mathcal{G}$ evaluated at $s = z_i$.

Notice that the output-zeroing direction can represent transmission-blocking properties, since Equation (2.1.13) implies that $P_{\mathcal{G}}(s)$ loses its local rank at $s = z_i \notin \mathrm{eig}(A)$, and therefore from Lemma 2.1.2 it follows that $G(s)$ also has the rank deficiency at $s = z_i$. Moreover, if the system is fat (i.e., $m > p$), then the output-zeroing direction always exists for an infinite number of complex numbers (in the whole complex plane) regardless of the normal rank of $P_{\mathcal{G}}(s)$. However, when the system is square or tall (i.e., $m \leq p$) and is non-degenerate, it only has output-zero direction for a finite number of complex numbers (which will be called transmission zeros later). If the system is degenerate and tall or square, then it has zero directions associated with the whole complex plane.

Definition 2.1.12 (Poles and transmission zeros, [60])**.** *Consider a transfer matrix $G(s) \in \mathbb{R}^{p\times m}(s)$. Let $M_G(s)$ be the Smith-McMillan form of $G(s)$ given in* (2.1.5)*. Let $p_G(s) = \prod_{i=1}^{r} \phi_i(s)$ be the* *characteristic polynomial, and let* $z_G(s) = \prod_{i=1}^{r} \epsilon_i(s)$*. Then, the root of* $p_G(s)$ *is called a pole of* $G(s)$*, and the root of* $z_G(s)$ *is called a zero of* $G(s)$ *(i.e.,* $p_G(p_0) = 0$ *and* $z_G(z_0) = 0$*). The zeros of* $z_G(s)$ *are called* transmission zeros*.*

Remark 2.1.3. *Transmission zero are generalization of the classical definition of zeros in SISO systems, since they are associated with the reachable and controllable states of the MIMO systems [69]. Notice that transmission zeros are defined through the transfer matrix of the system.*

In MIMO systems, there are some elements which are both poles and zeros of a transfer matrix, while no such cases are found in SISO systems. In other words, a Smith zero of $G(s)$ (transmission zero) can also be a pole of $G(s)$. For example, consider

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{(s+3)}{(s+2)^2} \end{bmatrix}.$$

Then, the Smith-McMillan form of $G(s)$ is given by $M_G(s) = diag\left(\frac{1}{(s+1)^2(s+2)^2}, (s+2)\right)$, and therefore $s = -2$ is both a pole and a transmission zero of $G(s)$. Moreover, notice that the number of transmission zeros is always finite, which is less than or equal to $r = \text{normrank}(G(s))$.

Transmission zeros can be obtained from right/left coprime factorizations of $G(s)$.

Lemma 2.1.5. *Consider a transfer matrix* $G(s) \in \mathbb{R}^{p\times m}(s)$*. Suppose* $z_0 \in \mathbb{C}$ *is not a pole of* $G(s)$*. Then,* $z_0 \in \mathbb{C}$ *is a transmission zero if and only if* $\text{rank}(G(z_0)) < \text{normrank}(G(s))$*.*

Proof. The proof of Lemma 2.1.5 follows from Lemma 2.1.2. □

Lemma 2.1.6. *Consider a transfer matrix* $G(s) \in \mathbb{R}^{p\times m}(s)$*. Let* $r = \text{normrank}(G(s))$*, and let* $G(s)$ *have polynomial fraction representation such that*

$$G(s) = N_r(s)D_r^{-1}(s) = D_l^{-1}(s)N_l(s), \tag{2.1.14}$$

where $N_r(s) \in \mathbb{R}^{p\times m}[s]$*,* $D_r(s) \in \mathbb{R}^{m\times m}[s]$ *are right coprime, and* $N_l(s) \in \mathbb{R}^{p\times m}[s]$*,* $D_l(s) \in \mathbb{R}^{p\times p}[s]$ *are left coprime.*

(a) Let Z_{D_r}*,* Z_{D_l} *be the set of Smith zeros of* $D_r(s)$ *and* $D_l(s)$*, respectively. Then,* $p_0 \in \mathbb{C}$ *is a pole of* $G(s)$ $\Leftrightarrow$ $\det(D_r(p_0)) = 0$ $\Leftrightarrow$ $\det(D_l(p_0)) = 0$*. Therefore,* $Z_{D_r} = Z_{D_l}$*, which is equivalent to the set of poles of* $G(s)$*.*

(b) Let Z_{N_r}*,* Z_{N_l} *be the set of Smith zeros of* $N_r(s)$ *and* $N_l(s)$*, respectively. Then,* $Z_{N_r} = Z_{N_l}$*, which is equivalent to the set of transmission zeros of* $G(s)$*. Moreover,* $z_0 \in \mathbb{C}$ *is a transmission zero of* $G(s)$*, if and only if* $\text{rank}(N_l(z_0)) < \text{normrank}(N_l(s))$ *with* $r = \text{normrank}(N_l(s))$*. Equivalently,* $z_0 \in \mathbb{C}$ *is a transmission zero of* $G(s)$*, if and only if* $\text{rank}(N_r(z_0)) < \text{normrank}(N_r(s))$ *with* $r = \text{normrank}(N_l(s))$*.*

Proof. Let $M_G(s)$ be the Smith-McMillan form of $G(s)$. Then, $G(s) = U_1(s)M_G(s)U_2(s)$ for some unimodular matrices $U_1(s) \in \mathbb{R}^{p\times p}[s]$, and $U_2(s) \in \mathbb{R}^{m\times m}[s]$. Notice that $M_G(s) = E_G(s)R_d^{-1}(s) = L_d^{-1}(s)E_G(s)$, where

$$\begin{aligned}
E_G(s) =& \text{diag}\left(\epsilon_1(s), \cdots, \epsilon_r(s), 0_{(p-r)\times(m-r)}\right), \\
R_d(s) =& \text{diag}\left(\phi_1(s), \cdots, \phi_r(s), \mathbb{I}_{m-r}\right), \\
L_d(s) =& \text{diag}\left(\phi_1(s), \cdots, \phi_r(s), \mathbb{I}_{p-r}\right),
\end{aligned}$$

with $\{\epsilon_i(s), \phi_i(s); 1 \leq i \leq r\}$ being the set of diagonal elements of $M_G(s)$ in (2.1.5). Then, one has

$$G(s) = N_r(s)D_r^{-1}(s) = D_l^{-1}(s)N_l(s), \tag{2.1.15}$$

where $N_r(s) = U_1(s)E_G(s)$, $D_r(s) = U_2(s)R_d(s)$, $N_l(s) = E_G(s)U_2(s)$, and $D_l(s) = L_d(s)U_1(s)$ Since $N_r(s)$, $D_r(s)$ are right coprime, and $N_l(s)$, $D_l(s)$ are left comprime, Equation (2.1.15) is a coprime factorization of $G(s)$. From the fact that $U_1(s)$ and $U_2(s)$ are unimodular, it follows that

$$p_0 \in \mathbb{C} \text{ is a pole of } G(s) \quad \Leftrightarrow \quad \det(D_r(p_0)) = 0 \quad \Leftrightarrow \quad \det(D_l(p_0)) = 0.$$

Notice that $\text{normrank}(N_r(s)) = \text{normrank}(N_l(s)) = r$. Since the set of Smith zeros of $N_r(s)$ is $Z_{N_r} = \{z_0 \in \mathbb{C}; \phi_i(z_0) = 0, i \in [1, \ldots, r]\}$, Z_{N_r} is equivalent to the set of transmission zeros of $G(s)$. The same argument holds for Z_{N_l}. This completes the proof. □

Lemma 2.1.6 states that the transmission zeros can be obtained from the coprime factorization of the system. Then one can conclude that the MIMO transmission zeros generalize the SISO zeros, since $N_l(s)$ and $N_r(s)$ lose their local ranks at the transmission zeros. Notice that Lemma 2.1.6 cannot be used to determine the multiplicities of MIMO transmission zeros; a canonical form of the transfer matrix is required to decide the multiplicities [71].

Definition 2.1.13 (Decoupling Zeros, [69])**.** *Consider the system $\mathcal{G}$ given in* (2.1.1).

- *A complex number $z_0 \in \mathbb{C}$ is called input decoupling zero, if*

$$\text{rank}(D_I(z_0)) < n, \quad D_I(s) = \begin{bmatrix} -s\mathbb{I}_n + A & B \end{bmatrix}. \tag{2.1.16}$$

- *A complex number $z_0 \in \mathbb{C}$ is called output decoupling zero, if*

$$\text{rank}(D_O(z_0)) < n, \quad D_O(s) = \begin{bmatrix} -s\mathbb{I}_n + A \\ C \end{bmatrix}. \tag{2.1.17}$$

- *A complex number $z_0 \in \mathbb{C}$ is called input-output decoupling zero, if*

$$\text{rank}(D_I(z_0)) < n, \quad \text{rank}(D_O(z_0)) < n,$$

where $D_I(s)$, $D_O(s)$ are given in (2.1.16) *and* (2.1.17)*, respectively.*

Remark 2.1.4. *Decoupling zeros present the decoupling structure of the MIMO systems: input-decoupling, output-decoupling, and input-output decoupling. In state-space representation of the system, these zeros correspond to uncontrollable or unobservable modes of the system [60, 70].*

Lemma 2.1.7. *Consider the system $\mathcal{G}$ in* (2.1.1)*.*

- *The system has no input decoupling zeros if and only if the controllability matrix*

$$\mathcal{C} = [B, AB, \ldots, A^n B]$$

  has full rank n.

- *The system has no output decoupling zeros if and only if the observability matrix*

$$\mathcal{O} = [C^\top, (CA)^\top, \ldots, (CA^{n-1})^\top]^\top$$

  has full rank n.

Proof. The proof of Lemma 2.1.7 is given in [60]. □

The decoupling zeros are some eigenvalues of the system matrix A. More precisely, the input decoupling zeros (or output decoupling zeros) correspond to the uncontrollable (or unobservable) modes among the eigenvalues of A. Moreover, the intersection of the set of all decoupling zeros and that of all transmission zeros is the empty set, since the transmission zeros are defined on the controllable-observable subspace of the system. Rosenbrock introduced the set of system zeros which is the union of all transmission zeros and all decoupling zeros [72]. These system zeros can be calculated from specially formed minors of the matrix pencil.

Definition 2.1.14 (System zeros, [72])**.** *Consider the matrix pencil $P_{\mathcal{G}}(s)$ of the system* (2.1.1)*. Suppose the normal rank of $P_{\mathcal{G}}(s)$ is $r > 0$[2]. Let $m_{\mathcal{P},k}(s)$ be the r-th order non-zero minor of $P_{\mathcal{G}}(s)$, which is formed by taking the first n rows and n columns of $P_{\mathcal{G}}(s)$[3]. Let $z_{\mathcal{P}}(s)$ be the monic greatest common divisor of all these minors $m_{\mathcal{P},k}(s)$. Then, the roots of $z_{\mathcal{P}}(s)$ are called system zeros (i.e., $z_{\mathcal{P}}(z_0) = 0$).*

Remark 2.1.5. *System zeros exhibit the behavior of the system states. System zeros were firstly introduced in [73], and revised later in [72] to establish the exact set equality for transmission zeros and decoupling zeros.*

[2] In general, the normal rank of the matrix pencil $P_{\mathcal{G}}(s)$ is not the same as that of $G(s)$ (i.e., $n \leq r \leq \min(n+p, n+m)$); see also Lemma 2.1.2.

[3] Take all rows and columns of $(-s\mathbb{I}_n + A)$, and add appropriate $r-n$ rows (of $[C, D]$) and columns (of $[B^\top, D^\top]^\top$)

Theorem 2.1.2. *Consider the system* (2.1.1)*. Let* $\mathcal{Z}_S$ *be the set of system zeros,* $\mathcal{Z}_T$ *be the set of transmission zeros, and* $\mathcal{Z}_I$*,* $\mathcal{Z}_O$*,* $\mathcal{Z}_{IO}$ *be the sets of input-decoupling, output-decoupling, input-output-decoupling zeros, respectively. Then the following relationships hold:*

$$\mathcal{Z}_S = \mathcal{Z}_T \cup \mathcal{Z}_I \cup \mathcal{Z}_O, \quad |\mathcal{Z}_S| = |\mathcal{Z}_T| + |\mathcal{Z}_I| + |\mathcal{Z}_O| - |\mathcal{Z}_{IO}|.$$

Proof. The proof of Theorem 2.1.2 can be found in [72, 73]. □

Theorem 2.1.3. *Consider the system* (2.1.1)*. Let* $\mathcal{P}_S$ *be the set of eigenvalues of* A*,* $\mathcal{P}_T$ *be the set of poles in the transfer matrix of the system* (2.1.2)*, and* $\mathcal{Z}_I$*,* $\mathcal{Z}_O$*,* $\mathcal{Z}_{IO}$ *be the sets of input-decoupling, output-decoupling, input-output-decoupling zeros, respectively. Then, the following relationships hold:*

$$\mathcal{P}_S = \mathcal{P}_T \cup \mathcal{Z}_I \cup \mathcal{Z}_O, \quad |\mathcal{P}_S| = |\mathcal{P}_T| + |\mathcal{Z}_I| + |\mathcal{Z}_O| - |\mathcal{Z}_{IO}|.$$

Proof. The proof of Theorem 2.1.3 can be found in [62, 72, 73]. □

Definition 2.1.15 (Invariant zeros)**.** *Consider the matrix pencil* $P_{\mathcal{G}}(s)$ *of the system* (2.1.1)*. Let* $S_{\mathcal{P}}$ *be its Smith form and let* $r = \text{normrank}(P_{\mathcal{G}}(s))$*. The invariant zero of the system is defined as the root* $z_0 \in \mathbb{C}$ *such that* $z^I_{\mathcal{P}}(s) = \prod_{i=1}^{r} \lambda_i(s)$*, where* λ_i *is given in* (2.1.3)*.*

Remark 2.1.6. *Invariant zeros present the zero-output behaviors of MIMO systems, since they are associated with the state-space structure in which the system output is identically zero for non-zero inputs [62].*

Lemma 2.1.8. *Consider the matrix pencil* $P_{\mathcal{G}}(s)$ *of the system* (2.1.1)*. Then,* $z_0 \in \mathbb{C}$ *is an invariant zero if and only if* $\text{rank}(P_{\mathcal{G}}(z_0)) < \text{normrank}(P_{\mathcal{G}}(s))$*.*

Proof. The proof of Lemma 2.1.8 immediately follows from the Smith form of $P_{\mathcal{G}}(s)$. □

From Definitions 2.1.14 and 2.1.15 it follows that the set of invariant zeros is a subset of the set of system zeros. Moreover, from Lemma 2.1.5 and 2.1.8, the invariant zeros of the system $\mathcal{G}$ are analogous to the transmission zeros of its transfer matrix, since both are Smith zeros of $P_{\mathcal{G}}(s)$ and $G(s)$, respectively. Notice that the Smith zeros of $P_{\mathcal{G}}(s)$ can also exhibit the transmission-blocking properties in the matrix pencil in a similar way as the transmission zeros do in the transfer matrix, which results in output-zeroing problems.

The relationships among MIMO zeros are established according to the following lemma.

Lemma 2.1.9. *Consider the matrix pencil* $P_{\mathcal{G}}(s)$ *of the system* (2.1.1)*, and its transfer matrix given in* (2.1.2)*. Let* $\mathcal{Z}_S$*,* $\mathcal{Z}_V$*, and* $\mathcal{Z}_T$ *be the set of system zeros, invariant zeros, and transmission zeros, respectively. Then,*

(a) $\mathcal{Z}_T \subseteq \mathcal{Z}_V \subseteq \mathcal{Z}_S$*.*

(b) If (A, B, C) is an observable and controllable triple, then $\mathcal{Z}_T = \mathcal{Z}_V = \mathcal{Z}_S$.

(c) If the system is non-degenerate with $m = p$, then $\mathcal{Z}_V = \mathcal{Z}_S$.

Proof. The proof of Lemma 2.1.9(a) and 2.1.9(b) directly follow from definitions of each zero. The proof of Lemma 2.1.9(c) is found in [74]. □

Lemma 2.1.10. *Consider the matrix pencil $P_{\mathcal{G}}(s)$ of the system in* (2.1.1) *and the transfer matrix in* (2.1.2)*. Assume the system is non-degenerate. Let $\mathcal{Z}_V$, $\mathcal{Z}_I$, $\mathcal{Z}_O$, $\mathcal{Z}_{IO}$ be the set of invariant, input-decoupling, output-decoupling, input-output-decoupling zeros, respectively.*

(a) If the system is tall or square (i.e., $m \leq p$), then $\mathcal{Z}_O \subseteq \mathcal{Z}_V$, as well as $\mathcal{Z}_{IO} \subseteq \mathcal{Z}_V$.

(b) If the system is fat or square (i.e., $m \geq p$), then $\mathcal{Z}_I \subseteq \mathcal{Z}_V$, as well as $\mathcal{Z}_{IO} \subseteq \mathcal{Z}_V$.

Proof. Suppose $m \leq p$. Since the system is non-degenerate, $\text{normrank}(P_{\mathcal{G}}(s)) = n + m$, which implies that $P_{\mathcal{G}}(s)$ can lose its rank only at distinct complex numbers. Now, let $z_0 \in \mathbb{C}$ be an output-decoupling zero. Notice that the matrix $D_O(s)$ in (2.1.17) has the normal rank n, since it can only lose its local rank at $s \in \text{eig}(A)$. Let $v_1 \in \mathbb{R}^n$ be a vector of the null space of $D_O(z_0)$. Then $D_O(z_0)v_1 = 0$ holds. Now, let $v_t = [v_1^\top, 0]^\top \in \mathbb{R}^{n+m}$. Then, $P_{\mathcal{G}}(z_0)v_s = 0$. Therefore, $P_{\mathcal{G}}(z_0) < n + m$, and z_0 is the invariant zero following Lemma 2.1.8. The rest of the proof can be completed from a dual argument. □

Notice that Lemma 2.1.10 is not valid when the system is degenerate. However, the system zeros always include the decoupling zeros regardless of the degeneracy of the system as it follows from Theorem 2.1.2.

Lemma 2.1.11. *Consider the system given in* (2.1.1) *with $m \leq p$, and its transfer matrix in* (2.1.2)*. Let $\mathcal{Z}_V$, $\mathcal{Z}_T$,$\mathcal{Z}_I$, $\mathcal{Z}_O$, $\mathcal{Z}_{IO}$ be the set of invariant, transmission, input-decoupling, output-decoupling, input-output-decoupling zeros, respectively. Then, $\mathcal{Z}_V$ consists of all elements of $\mathcal{Z}_O$ and $\mathcal{Z}_T$, as well as some elements of $(\mathcal{Z}_I - \mathcal{Z}_{IO})$.*

Proof. The proof of Lemma 2.1.11 is given in [74]. □

Lemma 2.1.12. *Consider the non-degenerate systems $\mathcal{G}_1$, $\mathcal{G}_2$ with $m \leq p$ such that*

$$\begin{aligned} \mathcal{G}_1: \quad & \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t), \quad y_1(t) = C_1 x_1(t), \quad x_1(0) = 0, \\ \mathcal{G}_2: \quad & \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t), \quad y_2(t) = C_2 x_2(t), \quad x_2(0) = 0, \end{aligned} \tag{2.1.18}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u_1(t) \in \mathbb{R}^m$, $u_2(t) \in \mathbb{R}^m$, $y_1(t) \in \mathbb{R}^m$, and $y_2(t) \in \mathbb{R}^p$. Let $\mathcal{G}_2\mathcal{G}_1$ be the cascaded system of $\mathcal{G}_1$ and $\mathcal{G}_2$ with $u_2(t) = y_1(t)$. Then,

$$\mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1) = \mathcal{Z}_V(\mathcal{G}_1) \cup \mathcal{Z}_V(\mathcal{G}_1), \tag{2.1.19}$$

where $\mathcal{Z}_V(\cdot)$ denotes the set of invariant zeros of a system.

Proof. Notice that the matrix pencil of $\mathcal{G}_2\mathcal{G}_1$ satisfies

$$P_{\mathcal{G}_2\mathcal{G}_1}(s) = P_2(s)P_1(s), \quad \forall s \in \mathbb{C}, \tag{2.1.20}$$

where

$$P_{\mathcal{G}_2\mathcal{G}_1}(s) = \begin{bmatrix} -s\mathbb{I}_{n_1} + A_1 & 0 & B_1 \\ B_2C_1 & -s\mathbb{I}_{n_2} + A_2 & 0 \\ 0 & C_2 & 0 \end{bmatrix},$$

$$P_2(s) = \begin{bmatrix} \mathbb{I}_{n_1} & 0 & 0 \\ 0 & -s\mathbb{I}_{n_2} + A_2 & B_2 \\ 0 & C_2 & 0 \end{bmatrix}, \quad P_1(s) = \begin{bmatrix} -s\mathbb{I}_{n_1} + A_1 & 0 & B_1 \\ 0 & \mathbb{I}_{n_2} & 0 \\ C_1 & 0 & 0 \end{bmatrix}.$$

First, we show that the system $\mathcal{G}_2\mathcal{G}_1$ is non-degenerate by a contradiction argument. Suppose it is not true. Then, there exists a non-zero vector $\xi \in \mathbb{R}^{n_1+n_2+m}$ such $P_{\mathcal{G}_2\mathcal{G}_1}(s)\xi = 0$ for $\forall s \in \mathbb{C}$. Since $P_1(s)$ and $P_2(s)$ are full normal column rank, it is not possible that $P_1(s)P_2(s)\xi = 0$ for $\forall s \in \mathbb{C}$. Therefore, $P_{\mathcal{G}_2\mathcal{G}_1}(s)$ must be full normal column rank, which proves that it is non-degenerate system. i.e., $\text{normrank}(P_{\mathcal{G}_2\mathcal{G}_1}(s)) = n_1 + n_2 + m$.

Next, we prove Equation (2.1.19). Suppose $z_0 \in \mathcal{Z}_V(\mathcal{G}_1) \cup \mathcal{Z}_V(\mathcal{G}_2)$. Then, $P_{\mathcal{G}_1}(s = z_0) < n_1 + m$ or $P_{\mathcal{G}_2}(s = z_0) < n_2 + m$ holds, which further yields either $\text{rank}(P_1(z_0)) < n_1 + n_2 + m$ or $\text{rank}(P_2(z_0)) < n_1 + n_2 + m$ with $P_{\mathcal{G}_1}(s)$, $P_{\mathcal{G}_2}(s)$ being matrix pencils of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Therefore, from (2.1.20) it follows that $\text{rank}(P_{\mathcal{G}_2\mathcal{G}_1}(z_0)) < n_1 + n_2 + m$, which implies that $z_0 \in \mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1)$ and $\mathcal{Z}_V(\mathcal{G}_1) \cup \mathcal{Z}_V(\mathcal{G}_2) \subset \mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1)$. Now, let $z_1 \in \mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1)$. Since $\text{rank}(P_{\mathcal{G}_2\mathcal{G}_1}(z_1)) < n_1 + n + 2 + m$ holds, Sylvester's rank inequality in (2.1.20) leads to $\text{rank}(P_1(z_1)) + \text{rank}(P_2(z_2)) < 2(n_1 + n_2 + m)$. This implies that one of $P_1(z_1)$ and $P_2(z_1)$ must have a rank less than $(n_1 + n_2 + m)$. Therefore, either $\text{rank}(P_{\mathcal{G}_1}(z_1)) < n_1 + m$ or $\text{rank}(P_{\mathcal{G}_1}(z_1)) < n_2 + m$ holds. Since $\text{normrank}(P_{\mathcal{G}_1}(s)) = n_1 + m$ and $\text{normrank}(P_{\mathcal{G}_2}(s)) = n_2 + m$ hold by the hypothesis, it follows that $z_1 \in \mathcal{Z}_V(\mathcal{G}_1) \cup \mathcal{Z}_V(\mathcal{G}_2)$, which proves $\mathcal{Z}_V(\mathcal{G}_1) \cup \mathcal{Z}_V(\mathcal{G}_2) \supset \mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1)$. This completes the proof. □

Corollary 2.1.1. *Consider the non-degenerate systems $\mathcal{G}_1$, $\mathcal{G}_2$ given in (2.1.18) with $m \leq p$. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are controllable and observable systems. Then*

$$\mathcal{Z}_T(\mathcal{G}_2\mathcal{G}_1) \subset \mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1) = \mathcal{Z}_T(\mathcal{G}_1) \cup \mathcal{Z}_T(\mathcal{G}_1) \tag{2.1.21}$$

holds, where $\mathcal{Z}_T(\cdot)$, $\mathcal{Z}_V(\cdot)$ are the set of transmission zeros and that of invariant zeros, respectively.

Proof. The proof of Lemma 2.1.1 follows from Lemma 2.1.12, together with the fact that $\mathcal{Z}_V(\mathcal{G}_1) = \mathcal{Z}_T(\mathcal{G}_1)$, $\mathcal{Z}_V(\mathcal{G}_2) = \mathcal{Z}_T(\mathcal{G}_2)$, and $\mathcal{Z}_T(\mathcal{G}_2\mathcal{G}_1) \subset \mathcal{Z}_V(\mathcal{G}_2\mathcal{G}_1)$. □

Now, we introduce zeros at infinity which generalize the relative degree for MIMO systems.

Definition 2.1.16 (Zero at infinity (or infinite zero), [68])**.** *Let $G(s) \in \mathbb{R}^{p\times m}(s)$ be a transfer matrix. The zeros at infinity (or infinite zeros) of $G(s)$ are the zeros at $s = 0$ of $G(1/s)$.*

Remark 2.1.7. *Structural indices of the zeros at infinity (or infinite zeros) are the generalization of the relative degree in SISO systems. Therefore, the zeros at infinity are used to generalize the relative degree of SISO systems to MIMO systems [60]. For MIMO systems one considers the notion of* vector relative degree*, which will be defined later.*

The zero at infinity can be obtained from the Smith-McMillan form of $G(1/s)$ (see Definition (2.1.6)). Notice that the Smith-McMillan form of $G(1/s)$ provides the set of structural indices for $s = \alpha = 0$ (see (2.1.5)), which is often called infinite zero structure of $G(s)$. However, since the unimodular matrices can destroy the information concerning the infinite frequency structure of the system, the following lemma was developed to determine the zeros at infinity [70].

Lemma 2.1.13. *Let $G(s) \in \mathbb{R}^{p\times m}(s)$ be a transfer matrix. Let $M_G(s)$ be the Smith-McMillan form of $G(s)$, where*

$$M_G(s) = \begin{bmatrix} \Gamma_G(s) & 0 \\ 0 & 0 \end{bmatrix},$$

with $\Gamma_G(s) = \mathrm{diag}\,(\gamma_1(s), \cdot, \gamma_m(s))$; $\gamma_i(s) = \epsilon_i(s)/\phi_i(s) \in \mathbb{R}(s)$. Then, the infinite zero structure of $G(s)$ is the set of the relative degrees of $\gamma_i(s)$, $i \in 1, \ldots, r$.

Proof. See [70] and [75]. □

Remark 2.1.8. *Notice that from the Smith-McMillan form one has $G(s) = U_1(s)M_G(s)U_2(s)$, where $U_1(s)$, $U_2(s)$ are unimodular (thus biproper). Therefore, $U_1(s)$, $U_2(s)$ do not affect the infinite zero structure. Let $\mathcal{K}_G = \{k_i = \deg(\phi_i(s)) - \deg(\epsilon_i(s)); i \in 1, \ldots, r\}$. Then, $\mathcal{K}_G$ represents the infinite zero structure of $G(s)$.*

It is known that the order of infinite zero (i.e., $k_i \in \mathcal{K}_G$) is the number of inherent integrations between the input and output pairs [76]. Notice that order k_i of zero at infinity is equivalent to the relative degree in SISO systems. Therefore, infinite zeros generalize the relative degree of a SISO transfer function.

Up to this point we reviewed classical definitions and properties of transmission zeros, decoupling zeros, system zeros, invariant zeros, and zeros at infinity. Since the roots of polynomials are distinct in the complex domain, the number of those zeros is always finite in LTI systems. The interesting point is that the (classical) invariant zeros do not fully exhibit output-zeroing properties of the system; notice that the matrix pencil $P_{\mathcal{G}}(s)$ can lose its rank for the whole complex plane while the number of invariant zeros is finite by its definition. This leads to another direction of definitions (in a geometric point of view) for MIMO zeros along with different definitions for system's non-degeneracy [67].

2.1.2. *Relative Degree and Interactor*

In this section, we introduce the vector relative degree of MIMO system which is a generalized relative degree (called a vector relative degree) in SISO system, and an interactor which cancels infinite zeros of a proper transfer matrix. This vector relative degree plays an important role in control system design (e.g., adaptive control [12, 32, 77], the factorization of transfer matrices with infinite zeros [75], and sliding mode control [78, 79]). In the literature, the vector relative degree is defined on square systems. In this thesis, we introduce a generalized definition for non-square systems.

Definition 2.1.17 (Vector relative degree). *Consider the transfer matrix $G(s) \in \mathbb{R}^{p\times m}(s)$. Let $r > 0$ be the normal rank of $G(s)$, and $\bar{k}_G \in \mathbb{R}^r$ be a vector whose entries are elements of $\mathcal{K}_G$, where $\mathcal{K}_G$ is a set representing the infinite zero structure of $G(s)$*[4]*. Then, k_G is called vector (or generalized) relative degree of $G(s)$. Moreover, $G(s)$ is said to have no infinite zeros, if $k_G = 0$.*

Hereafter, we focus on tall or square systems (i.e., underactuated systems), in which the number of inputs are less than or equal to the number of outputs.

Lemma 2.1.14. *Consider $P(s) \in \mathbb{R}^{p\times m}(s)$. Suppose $P(s)$ is a proper matrix with $m \leq p$ and* $\mathrm{normrank}(P(s)) = m$. *Then, $P(s)$ has no infinite zeros, if and only if $P(\infty) = \lim_{s\to\infty} P(s)$ has full rank.*

Proof. The proof of Lemma 2.1.14 is given in [75]. □

Lemma 2.1.15. *Consider the transfer matrix $G(s) \in \mathbb{R}^{p\times m}(s)$ given in (2.1.2) with $D = 0$. Suppose $G(s)$ is underactuated (i.e., $m \leq p$) with* $\mathrm{normrank}(G(s)) = m$. *Let $k_G \in \mathbb{R}^m$ be the vector relative degree such that $k_G = [k_1, \ldots, k_m]^\top$. Then*

$$M_{k_i} \neq 0, \quad i = [1, \ldots, m], \tag{2.1.22}$$

where $M_{k_i} = CA^{k_i-1}B$ is a Markov parameter of $G(s)$.

Proof. From the Smith-McMillan form of $G(s)$ we have $G(s) = U_1(s)M_G(s)U_2(s)$, where $U_1(s)$, $U_2(s)$ are unimodular matrices. Notice that from Lemma 2.1.4 it follows that for $|s| > \max(\mathrm{eig}(A))$ we have

$$s^{k_i} M_G(s) = \sum_{k=0}^{\infty} s^{k_i-k} U_1^{-1}(s) M_k U_2^{-1}(s). \tag{2.1.23}$$

Suppose $M_{k_i} = 0$. Then from (2.1.23) one has

$$\lim_{s\to\infty} s^{k_i} M_G(s) = 0,$$

[4] See Lemma 2.1.13 and Remark 2.1.8.

which contradicts to the fact that $M_G(s)$ has a diagonal element, which is a rational fraction of polynomials with the relative degree k_i. Therefore, $M_{k_i} \neq 0$. $\square$

Lemma 2.1.16. *Consider the transfer matrix $G(s)$ given in* (2.1.2) *with $D = 0$. Consider $G(s) \in \mathbb{R}^{p\times m}(s)$. Suppose $G(s)$ is underactuated (i.e., $m \leq p$) with* $\text{normrank}(G(s)) = m$*. Then, the vector relative degree is equal to $\mathbf{1}_m = [1, \ldots, 1] \in \mathbb{R}^m$, if and only if (CB) is full rank.*

Proof. Consider the Smith-McMillan form of $G(s)$ such that $G(s) = U_1(s)M_G(s)U_2(s)$, where $U_1(s)$, $U_2(s)$ are unimodular matrices. Notice that $sG(s) = U_1(s)(sM_G(s))U_2(s)$ holds. This implies that the vector relative degree of $sM_G(s)$ should be the zero vector (i.e. $sG(s)$ has no infinite zero structure), since $M_G(s)$ only has rational fractions of polynomials with relative degree $\mathbf{1}_m$. Moreover, from Lemma 2.1.4 it follows that $\lim_{s\to\infty} sG(s) = (CB)$. Finally, using Lemma 2.1.14 one concludes that (CB) is full rank, since $sG(s)$ has no infinite zero structure. The converse is straightforward. This completes the proof. $\square$

The infinite zeros of the system are defined on its transfer matrix. However, the following lemma links them to infinite zeros of its matrix pencil; this property can be used to obtain a right interactor (which will be defined later) by using a matrix pencil approach [80].

Lemma 2.1.17. *Consider the matrix pencil of the system $\mathcal{G}$ given in* (2.1.1)*, and let $G(s)$ be its transfer matrix given in* (2.1.2)*. Moreover, suppose $G(s)$ is underactuated (i.e., $m \leq p$) with* $\text{normrank}(G(s)) = m$*, and the realization of $\{A, B, C, D\}$ is detectable. Let $\mathcal{K}_G$ be the infinite zero structure of $G(s)$ with $l = |\mathcal{K}_G|$. Then, the matrix pencil $\mathcal{P}_G(s)$ has l infinite eigenvalues of respective orders k_i, $i \in [1, \ldots, l]$.*

Proof. The proof of Lemma 2.1.17 is given in [75]. $\square$

Now, we define an (right) interactor of $G(s)$ as follows.

Definition 2.1.18 (Interactor, [81])**.** *Consider the transfer matrix $G(s) \in \mathbb{R}^{p\times m}(s)$ given in* (2.1.2)*. The polynomial matrix $R^{-1}(s)$ is called an (right) interactor of $G(s)$, if*

$$\lim_{s\to\infty} G(s)R^{-1}(s) \tag{2.1.24}$$

is a full rank matrix.

The following lemma demonstrates that one can always find a right interactor for non-degenerate and underactuated systems.

Theorem 2.1.4 ([80])**.** *Consider the system $\mathcal{G}$ given in* (2.1.1) *and the transfer matrix $G(s)$ in* (2.1.2)*. Suppose $G(s)$ is an underactuated system with full normal column rank*[5]*. Then there exist*

[5] $\text{normrank}(G(s)) = m$ with $m \leq p$.

matrices $A_r \in \mathbb{R}^{n_r \times n_r}$*,* $B_r \in \mathbb{R}^{n_r \times m}$*,* $C_r \in \mathbb{R}^{m \times n_r}$ *and* $D_r \in \mathbb{R}^{m \times m}$*, such that*

$$P_{\mathcal{G}}(s) \begin{bmatrix} T_r & 0 \\ 0 & \mathbb{I}_m \end{bmatrix} = \begin{bmatrix} T_r & B_0 \\ 0 & D_0 \end{bmatrix} \begin{bmatrix} -s\mathbb{I}_{n_r} + A_r & B_r \\ C_r & D_r \end{bmatrix}, \tag{2.1.25}$$

and

$$\left| \begin{bmatrix} -s\mathbb{I}_{n_r} + A_r & B_r \\ C_r & D_r \end{bmatrix} \right| \neq 0, \quad \forall s \in \mathbb{C}, \tag{2.1.26}$$

where $P_{\mathcal{G}}(s)$ *is given in* (2.1.8)*, and* $T_r \in \mathbb{R}^{n \times n_r}$*,* $B_0 \in \mathbb{R}^{n \times m}$*,* $D_0 \in \mathbb{R}^{p \times m}$ *are some matrices; T and* D_0 *have full column rank. Moreover,*

(a) Let $K_r \in \mathbb{R}^{n \times m}$ *be any matrix, and let* $\bar{A}_r = A_r - K_r C_z$*,* $\bar{B}_r = B_r - K_r D_r$*, and* $\bar{B}_0 = B_0 + T_r K_r$*. Then, the following holds:*

$$\bar{G}(s) = G(s)R(s), \quad G(s) = \bar{G}(s)R^{-1}(s),$$

and

$$P_{\mathcal{G}}(s) \begin{bmatrix} T_r & 0 \\ 0 & \mathbb{I}_m \end{bmatrix} = \begin{bmatrix} T_r & \bar{B}_0 \\ 0 & D_0 \end{bmatrix} \begin{bmatrix} -s\mathbb{I}_{n_r} + \bar{A}_r & \bar{B}_r \\ C_r & D_r \end{bmatrix},$$

where $\bar{G}(s) = C(s\mathbb{I}_n - A)^{-1}\bar{B}_0 + D_0$*, and* $R(s) = C_r(s\mathbb{I}_{n_r} - \bar{A}_r)^{-1}\bar{B}_r + D_r$*. Therefore,* $R^{-1}(s)$ *is a right interactor of* $G(s)$*. Moreover, the set of* $\{\bar{A}_r, \bar{B}_r, C_r, D_r\}$ *also satisfies* (2.1.26)*; it is a minimal realization and has no finite invariant zeros.*

(b) The zeros of $R^{-1}(s)$ *are the eigenvalues of* $\bar{A}_r$*, and* K_r *can be chosen such that all zeros of* $R^{-1}(s)$ *are stable (i.e., the real parts of all eigenvalues of* $\bar{A}_r$ *are in the left-half complex plane).*

(c) If K_r *is chosen such that all eigenvalues of* $\bar{A}_r$ *are contained in* $\mathbb{C}^-$*, then the stabilizability of* (A, B) *guarantees that of* $(A, \bar{B})$*.*

(d) If K_r *is chosen such the* $\mathrm{eig}(\bar{A}_r) \cap \mathrm{eig}(A) = \emptyset$*, then the controllability of* (A, B) *guarantees that of* $(A, \bar{B})$*.*

Proof. The proof of Theorem 2.1.4 is given in [80]. □

Remark 2.1.9. *The authors of [80] used Lemma 2.1.17 in the proof of Theorem 2.1.4, and therefore one can find an interactor using the system matrix pencil. To obtain the interactor, one can take the following procedures:*

(a) Using a lower triangular Schur decomposition of the matrix pencil $G(s)$ *(see [63, Algorithm 4.1]), one obtains* $S \in \mathbb{R}^{(n+p)\times(n+p)}$*,* $T \in \mathbb{R}^{(n+m)\times(n+m)}$*,* $E_\infty \in \mathbb{R}^{n_\infty \times n_\infty}$*, and* $A_\infty \in \mathbb{R}^{n_\infty \times n_\infty}$ *which satisfy Lemma 2.1.1.*

(b) *Let* $n_r = n_\infty - m$, *and define a partitioned matrix* $E_1 \in \mathbb{R}^{n_\infty \times n_r}$ *such that* $E_\infty = [E_1, E_2]$.

(c) *From a singular value decomposition on* E_1, *one has* $U_1 \in \mathbb{R}^{n_\infty \times n_\infty}$, $V_1 \in \mathbb{R}^{n_r \times n_r}$ *such that* $U_1^\top E_1 V_1 = \Sigma_{E_1}$, $U_1^\top U_1 = \mathbb{I}_{n_\infty}$, *and* $V_1^\top V_1 = \mathbb{I}_{n_r}$.

(d) *Define* $U_\infty = \operatorname{diag}\left(V_1 \Sigma_r^{-1}, \mathbb{I}_m\right) U_1^\top \in \mathbb{R}^{n_\infty \times n_\infty}$, *where* $\Sigma_r \in \mathbb{R}^{n_r \times n_r}$ *is the diagonal matrix taking the first* n_r *elements (singular values) of* Σ_{E_1}.

(e) *Let* $S_m \in \mathbb{R}^{(n+p)\times(n+p-n_\infty)}$, $S_\infty \in \mathbb{R}^{(n+p)\times n_\infty}$, $T_m \in \mathbb{R}^{(n+m)\times(n-n_r)}$, $T_\infty \in \mathbb{R}^{(n+m)\times n_\infty}$ *be partitioned matrices such that* $S = [S_m, S_\infty]$, *and* $T = [T_m, T_\infty]$. *In addition, define* $S_u = S_\infty U_\infty^{-1}$ *and* $Z_u = U_\infty A_\infty$.

(f) *Then,* $T_z \in \mathbb{R}^{n \times n_r}$, $B_0 \in \mathbb{R}^{n \times m}$, *and* $D_0 \in \mathbb{R}^{p \times m}$ *are partitioned matrices satisfying*

$$S_u = \begin{bmatrix} T_r & B_0 \\ 0 & D_0 \end{bmatrix} \in \mathbb{R}^{(n+p)\times n_\infty}.$$

Moreover, $A_r \in \mathbb{R}^{n_r \times n_r}$, $B_r \in \mathbb{R}^{n_r \times m}$, $C_r \in \mathbb{R}^{m \times n_r}$, *and* $D_r \in \mathbb{R}^{m \times m}$ *are obtained by partitioning* Z_u *such as*

$$Z_u = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \in \mathbb{R}^{n_\infty \times n_\infty}.$$

The interactor can be used to find a virtual system which has vector relative degree $\mathbf{1}$.

2.1.3. *State Decomposition*

In this section, we consider a state-decomposition to decouple the system dynamics from the control input direction. The decomposition is valid if the system has vector relative degree $\mathbf{1}$. However, it will be shown that the decomposition is always possible in underactuated systems by introducing a virtual system. Consider the non-degenerate LTI system, denoted by $\mathcal{M}$:

$$\mathcal{M} : \dot{x}(t) = A_m x(t) + B_m u_x(t), \quad y(t) = C_m x(t), \quad x(0) = x_0, \tag{2.1.27}$$

where $x(t) \in \mathbb{R}^n$, $u_x(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ with $\{A_m, B_m, C_m\}$ being a stabilizable-detectable triple; A_m being Hurwitz, and B_m, C_m being full rank. Moreover, it is assumed that the system has full column norm rank with $m \le p$. The transfer matrix of the system (2.1.27) is given by

$$M(s) = C_m (s\mathbb{I}_n - A_m)^{-1} B_m, \quad M(s) \in \mathbb{R}^{p\times m}(s). \tag{2.1.28}$$

From Lemma 2.1.16 it follows that $(C_m B_m)$ has rank deficiency, if and only if $M(s)$ does not have the vector relative degree $\mathbf{1}_m = [1, \ldots, 1] \in \mathbb{R}^m$. However, using a right interactor of $sM(s)$, we can construct a virtual system which has the vector relative degree $\mathbf{1}_m$.

Corollary 2.1.2. *Consider the system* $\mathcal{M}$ *given in* (2.1.27), *and its transfer matrix* $M(s)$ *defined in* (2.1.28). *Suppose* $(C_m B_m)$ *is rank deficient. Then, there exist a stable transfer matrix* $Z(s)$ *and* *matrices,* $\bar{B} \in \mathbb{R}^{n\times m}$, $T_z \in \mathbb{R}^{n\times n_z}$, *such that*

$$\begin{aligned} Z(s) =& C_z(s\mathbb{I}_{n_z} - A_z)^{-1}B_z + D_z, \\ \bar{M}(s) =& C_m(s\mathbb{I}_n - A_m)^{-1}\bar{B} = M(s)Z^{-1}(s), \end{aligned} \tag{2.1.29}$$

and

$$\begin{aligned} A_mT_z =& T_zA_z + \bar{B}C_z, \quad C_mA_mT_z = C_m\bar{B}C_z, \\ B_m =& T_zB_z + \bar{B}D_z, \quad C_mB_m = C_m\bar{B}D_z, \end{aligned} \tag{2.1.30}$$

where $A_z \in \mathbb{R}^{n_z\times n_z}$, $B_z \in \mathbb{R}^{n_z\times m}$, $C_z \in \mathbb{R}^{n_z\times m}$, *and* $D_z \in \mathbb{R}^{n_z\times m}$ *satisfy*

$$\left| \begin{bmatrix} -s\mathbb{I}_{n_z} + A_z & B_z \\ C_z & D_z \end{bmatrix} \right| \neq 0, \quad \forall s \in \mathbb{C}, \tag{2.1.31}$$

and T_z *is full column rank. Moreover, the following hold:*

- $(A_m, \bar{B})$ *is stabilizable, and* $(C_m\bar{B})$ *is full rank.*
- *If the system* $\mathcal{M}$ *has no unstable invariant zeros, then the system of* $\{A_m, \bar{B}, C_m\}$ *does not possess unstable invariant zeros, and* $\bar{M}(s)$ *has no unstable transmission zeros.*

Proof. Notice that $(C_mB_m) = \lim_{s\to\infty}(sM(s))$. Let $G(s) = sM(s)$, and $G(s) = C(s\mathbb{I}_n - A)^{-1}B + D$ with $A = A_m$, $B = B_m$, $C = C_mA_m$, and $D = C_mB_m$. Since $\{A_m, B_m, C_m\}$ is stabilizable-detectable, and A_m is Hurwitz, the triple $\{A, B, C\}$ is also stabilizable and detectable. Therefore, from Theorem 2.1.4(a) it follows that there exists a right interactor $Z^{-1}(s)$ (having the stable $Z(s)$), which satisfies (2.1.31) and

$$\begin{bmatrix} -s\mathbb{I}_n + A_m & B_m \\ C_mA_m & C_mB_m \end{bmatrix} \begin{bmatrix} T_z & 0 \\ 0 & \mathbb{I}_m \end{bmatrix} = \begin{bmatrix} T_z & \bar{B} \\ 0 & \bar{D} \end{bmatrix} \begin{bmatrix} -s\mathbb{I}_{n_z} + A_z & B_z \\ C_z & D_z \end{bmatrix}, \tag{2.1.32}$$

with $T_z \in \mathbb{R}^{n\times n_z}$, $\bar{B} \in \mathbb{R}^{n\times m}$, and $\bar{D} \in \mathbb{R}^{p\times m}$; $(A_m, \bar{B})$ is stabilizable. Since Equation (2.1.32) holds, one has

$$(\bar{D} - C_m\bar{B})C_zA_z^{-1} = C_mT_z, \quad (\bar{D} - C_m\bar{B})D_z = C_mT_zB_z,$$

which further leads to

$$(\bar{D} - C_m\bar{B})(D_z - C_zA_z^{-1}B_z) = 0. \tag{2.1.33}$$

Notice that both $(D_z - C_zA_z^{-1}B_z)$ (see Equation (2.1.31)) and $\bar{D}$ are full rank (see Theorem 2.1.4). From (2.1.33) it follows that $\bar{D} = C_m\bar{B}$ holds. Therefore, $(C_m\bar{B})$ is full rank, and Equation (2.1.30) follows from (2.1.32).

Finally, Suppose that the system $\mathcal{M}$ has no unstable invariant zeros. Let $\bar{\mathcal{M}}$ be the system with $(A_m, \bar{B}, C_m)$. Notice that $(A_m, \bar{B}, C_m)$ is a stabilizable-detectable triple. Therefore, if one can show that $\bar{M}(s)$ has no unstable transmission zeros, then all invariant zeros of $\bar{\mathcal{M}}$ are in $\mathbb{C}^-$ (see Lemma 2.1.11). Since $M(s) = \bar{M}(s)Z(s)$ holds, the set of invariant zeros of $\mathcal{M}$ must contain transmission zeros of $\bar{M}(s)$ and $Z(s)$ (see Lemma 2.1.12). By the hypothesis, $\mathcal{M}$ has no unstable invariant zeros, which implies that $\bar{M}(s)$ has no unstable transmission zeros. This completes the proof. □

Remark 2.1.10. *Notice that if $(C_m B_m)$ is full rank, then $Z(s) = \mathbb{I}_m$. Moreover, if (A_m, B_m, C_m) is a controllable-observable triple, then $(A_m, \bar{B})$ can be controllable with an appropriate choice of $Z(s)$ (see Theorem 2.1.4).*

Notice that $M(s) = \bar{M}(s)Z(s)$ implies that MIMO systems with arbitrary relative degree can be represented with cascaded connections of $\bar{M}(s)$ and $Z(s)$, where $\bar{M}(s)$ is a virtual system with the vector relative degree $\mathbf{1}_m$, and $Z(s)$ is the stable inverse of the right interactor of $sM(s)$. Although this representation only exhibits an input-output connection, one can establish the relationship between the states of the original system and those of the virtual system, since Corollary 2.1.2 relies on the state-space structure (matrix pencil).

Corollary 2.1.3. *Consider the system $\mathcal{M}$ given in (2.1.27) with a non-zero initial condition $x_0 \in \mathbb{R}^n$. Let $x_v(t) \in \mathbb{R}^n$ and $x_z(t) \in \mathbb{R}^{n_z}$ be the states of the following cascaded system:*

$$\begin{aligned}
\dot{x}_z(t) =& A_z x_z(t) + B_z u_x(t), \quad u_v(t) = C_z x_z(t) + D_z u_x(t),\\
\dot{x}_v(t) =& A_m x_v(t) + \bar{B} u_v(t), \quad y_v(t) = C_m x_v(t),\\
x_v(0) =& x_0, \quad x_z(0) = 0,
\end{aligned} \tag{2.1.34}$$

where $y_v(t) \in \mathbb{R}^p$ is the output vector, and $A_z \in \mathbb{R}^{n_z \times n_z}, B_z \in \mathbb{R}^{n_z \times m}, C_z \in \mathbb{R}^{m \times n_z}, D_z \in \mathbb{R}^{m \times m}$, $\bar{B} \in \mathbb{R}^{n \times m}$ are defined in Corollary 2.1.2. Then, for all $t \geq 0$

$$x(t) = x_v(t) + T_z x_z(t), \quad y_v(t) = y(t), \tag{2.1.35}$$

where $T_z \in \mathbb{R}^{n \times n_z}$ is full column rank satisfying (2.1.30).

Proof. Notice that Equation (2.1.34) can be rewritten as

$$\begin{aligned}
\begin{bmatrix} \dot{x}_v(t) \\ \dot{x}_z(t) \end{bmatrix} &= \begin{bmatrix} A_m & \bar{B}C_z \\ 0 & A_z \end{bmatrix} \begin{bmatrix} x_v(t) \\ x_z(t) \end{bmatrix} + \begin{bmatrix} \bar{B}D_z \\ B_z \end{bmatrix} u_x(t),\\
y_v(t) &= \begin{bmatrix} C_m & 0 \end{bmatrix} \begin{bmatrix} x_v(t) \\ x_z(t) \end{bmatrix}.
\end{aligned} \tag{2.1.36}$$

Now, let $[x_t^\top(t), x_z^\top(t)]^\top = T_t[x_v^\top(t), x_z^\top(t)]^\top$ with $T_t = \begin{bmatrix} \mathbb{I}_n & T_z \\ 0 & \mathbb{I}_{n_z} \end{bmatrix}$. By applying a similarity transform with T_t, from (2.1.36) it follows, together with (2.1.30), that

$$\begin{aligned}\dot{x}_z(t) =& A_z x_t(t) + B_z u_x(t),\\ \dot{x}_t(t) =& A_m x_t(t) + B_m u_x(t), \quad y_v(t) = C_m x_t(t),\end{aligned}$$

with $x_t(0) = x_0$, and $x_z(0) = 0$. This implies that $x(t) = x_t(t)$ for all $t \geq 0$. Therefore, Equation (2.1.35) holds, which completes the proof. □

Corollary 2.1.3 provides a relationship between the states of the original system and those of its cascaded representation, i.e. $x_v(t)$ is the state vector of the virtual system and $x_z(t)$ is the state vector of the inverse of the right interactor. Up to this point, we have shown that any underactuated MIMO system can be rewritten as a cascaded representation of the virtual system and the inverse of an interactor. Moreover, the virtual system has relative degree one. Next we introduce important lemmas which are related to the relative degree.

Lemma 2.1.18. *Consider the system $\mathcal{M}$ in (2.1.27) with $m \leq p$. Then $(C_m B_m)$ is full rank, if and only if there is a matrix $H \in \mathbb{R}^{n\times p}$ such that $(\mathbb{I}_n - HC_m)B_m = 0$. Moreover, one such H can be found as follows*

$$H = B_m \left(C_m B_m\right)^{\dagger} ,$$

where $(C_m B_m)^{\dagger}$ is the generalized left inverse of $(C_m B_m)$.

Proof. Since $rank(C_m B_m) = rank(B_m)$, the proof of Lemma 2.1.18 follows from [82]. Moreover, it is easy to verify that $H = B_m(C_m B_m)^{\dagger}$ is a solution of $(\mathbb{I}_n - HC_m)B_m = 0$, since $(C_m B_m)^{\dagger}(C_m B_m) = \mathbb{I}_m$. □

Notice that $(\mathbb{I}_n - HC_m) \in \mathbb{R}^{n\times n}$ is a projection matrix, which projects $\mathbb{R}^n$ onto the subspace orthogonal to the range of B_m. The states $x(t)$ can be decomposed into $x(t) = v(t) + Hy(t)$, where $v(t) = (\mathbb{I}_n - HC_m)\, x(t)$.

Lemma 2.1.19. *Given the system $\mathcal{M}$ in (2.1.27) along with $m \leq p$, suppose $M(s)$ does not have unstable invariant zeros and $(C_m B_m)$ is full rank. Let $H = B_m \left(C_m B_m\right)^{\dagger}$. Then, the following relationships hold:*

- *If there is an unobservable mode of the pair $((\mathbb{I}_n - HC_m)A_m, C_m)$, it is an invariant zero of the system $M(s)$.*
- *Moreover, $((\mathbb{I}_n - HC_m)A_m, C_m)$ is detectable and there exists a gain $K_v \in \mathbb{R}^{n\times p}$ such that $A_v = ((\mathbb{I}_n - HC_m)A_m + K_v C_m)$ is Hurwitz.*

Proof. Suppose $z_i \in \mathbb{C}$ is an unobservable mode of $((\mathbb{I}_n - HC_m)A_m, C_m)$, where $H = B_m(C_m B_m)^{\dagger}$. We will show that z_i is a stable invariant zero of $M(s)$. By Popov-Belevitch-Hautus observability test [60, Chapter 3], there exists a non-zero vector $\xi_i \in R^n$ such that

$$(\mathbb{I}_n - HC_m)A_m\xi_i = z_i\xi_i, \quad C_m\xi_i = 0, \tag{2.1.37}$$

which yields

$$(z_i\mathbb{I}_n - A_m)\,\xi_i + HC_mA_m\xi_i = 0. \tag{2.1.38}$$

Now, let $\varsigma_i \in \mathbb{R}^m$ be $\varsigma_i = (C_mB_m)^\dagger C_mA_m\xi_i$. Then, it follows

$$B_m\varsigma_i = B_m(C_mB_m)^\dagger C_mA_m\xi_i = HC_mA_m\xi_i. \tag{2.1.39}$$

By combining (2.1.37), (2.1.38) and (2.1.39), it follows that

$$\mathcal{P}_M(z_i)\begin{bmatrix} -\xi_i \\ \varsigma_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathcal{P}_M(z_i) = \begin{bmatrix} -z_i\mathbb{I}_n + A_m & B_m \\ C_m & 0 \end{bmatrix}, \tag{2.1.40}$$

and $[-\xi_i^\top, \varsigma_i^\top]^\top \neq 0$. Notice that $rank(\mathcal{P}_M(z_i)) < n + m$ in (2.1.40). Therefore, applying Lemmas 2.1.3 and 2.1.5 implies that z_i must be an invariant zero of $M(s)$. Finally, since $M(s)$ does not have unstable invariant zeros, $z_i \in \mathbb{C}^-$ holds, which yields that $((\mathbb{I}_n - HC_m)A_m, C_m)$ is detectable. This completes the proof. □

The following lemma shows the invariant property of the rank condition on (C_mB_m).

Lemma 2.1.20. *Let (A_p, B_p, C_p) is a stabilizable-detectable triple of the non-degenerate MIMO system that represents an open-loop plant. Let $G_p(s) = \frac{y_p(s)}{u_p(s)} = C_p(s\mathbb{I} - A_p)^{-1}B_p$. Define $K_c(s)$ as its dynamic controller with $u_p(s) = K_c(s)y_p(s)$. Let (A_c, B_{c_1}, C_c, D_c) be a realization of $K_c(s)$ with appropriate dimensions such that (A_a, B_a, C_a) defined as*

$$A_a = \begin{bmatrix} A_p + B_pD_cC_p & B_pC_c \\ B_{c_1}C_p & A_c \end{bmatrix}, \quad B_a = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad C_a = \begin{bmatrix} C_p & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \tag{2.1.41}$$

is a stabilizable-detectable realization of the augmented input sensitivity function $M_a(s) = C_a(s\mathbb{I} - A_a)^{-1}B_a$. Further, assume (C_pB_p) is full rank. Then the following hold:

- *(C_aB_a) is full rank.*
- *If $z \in \mathbb{C}$ is an invariant zero of (A_a, B_a, C_a), it is the invariant zero of (A_p, B_p, C_p).*

Proof. By the definitions of B_a and C_a, it is obvious that (C_aB_a) is full rank. Since the system $M(s)$ is non-degenerate, the transmission zeros of the system can be obtained from the matrix pencil (Lemmas 2.1.3 and 2.1.5). Now, suppose $z \in \mathbb{C}$ is a transmission zero of $M(s)$. It follows from Lemmas 2.1.3 and 2.1.5 that there exists a non-zero vector $\xi^\top = \left[\xi_1{}^\top, \xi_2{}^\top, \xi_3{}^\top\right]$ satisfying

$$\begin{bmatrix} z\mathbb{I} - A_p - B_pD_cC_p & -B_pC_c & B_p \\ -B_{c_1}C_p & \lambda_z\mathbb{I} - A_c & 0 \\ -C_p & 0 & 0 \\ 0 & -\mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.1.42}$$

Notice that $\xi_2 = 0$ and $C_p\xi_1 = 0$. Therefore, Equation (2.1.42) can be reduced to

$$\mathcal{P}_{M_a}(z)\begin{bmatrix} -\xi_1 \\ \xi_3 \end{bmatrix} = 0, \quad \mathcal{P}_{M_a}(z) = \begin{bmatrix} -z\mathbb{I} + A_p & B_p \\ C_p & 0 \end{bmatrix}.$$

Since $[\xi_1^\top, \xi_3^\top]$ is a non-zero vector, applying Lemmas 2.1.3 and 2.1.5 concludes that z is the invariant zero of the system $G_p(s)$. This completes the proof. □

Remark 2.1.11. *Suppose that $\{A_p, B_p, C_p\}$ is a minimal realization of a given open-loop transfer matrix $G_p(s)$. It is well-known that any dynamic controller cannot move open-loop transmission zeros in the closed-loop system, but may add transmission zeros identical to the poles of the dynamic controller. However, Lemma 2.1.20 states that the input sensitivity function augmented with the controller has only open-loop transmission zeros.*

2.2. System Stability

In this section, we review stability of nonlinear systems. We begin by comparison functions which are denoted by $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ class functions.

Definition 2.2.1 ([83]). *Let $a > 0$ be a positive constant.*

(a) A function $\alpha : [0, a) \to \bar{\mathbb{R}}^+$ is called a class $\mathcal{K}$ function, if $\alpha(0) = 0$ and $\alpha(\cdot)$ is strictly increasing.

(b) A function $\alpha : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ is called a class $\mathcal{K}_\infty$-class function, if $\alpha(\cdot)$ is in class $\mathcal{K}$ and is radially unbounded (i.e., $\lim_{r\to\infty} \alpha(r) = \infty$).

(c) A function $\beta : \bar{\mathbb{R}}^+ \times [0, a) \to \bar{\mathbb{R}}^+$ is called a class $\mathcal{KL}$ function, if it verifies:

- *for each $t \in \bar{\mathbb{R}}^+$, $\beta(t, r)$ is in class $\mathcal{K}$ with respect to r;*
- *for each $r \in [0, a)$, $\beta(t, r)$ is decreasing with respect to t;*
- *for each $r \in [0, a)$, $\beta(t, r) \to 0$ as $t \to \infty$.*

(d) A function $\beta : \bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ is called a class $\mathcal{KL}_\infty$ function, if it satisfies:

- *for each $t \in \bar{\mathbb{R}}^+$, $\beta(t, r)$ is in class $\mathcal{KL}$ with respect to r;*

- *for each $r \in \bar{\mathbb{R}}^+$, $\beta(t,r)$ is decreasing with respect to t;*
- *for each $r \in \bar{\mathbb{R}}^+$, $\beta(t,r) \to 0$ as $t \to \infty$.*

2.2.1. $\mathcal{L}_p$ *stability*

$\mathcal{L}_p$ stability theory is essential to analyze the stability of MIMO systems. We consider the input-output map of the system given by $y = \mathcal{H}u$, where $\mathcal{H}$ is the mapping, and u, y are the input and output signals, respectively. We first define the spaces of signals

Definition 2.2.2 ([84, 85]). *The $\mathcal{L}^n_{(p,q)}$ space is defined as the set of measurable functions such that*

$$\mathcal{L}^n_{(p,q)} = \{f : \mathbb{R} \to \mathbb{R}^n; \|f\|_{\mathcal{L}_{(p,q)}} < \infty\},$$

where $\|\cdot\|_{\mathcal{L}_{(p,q)}}$ is given by

$$\|f\|_{\mathcal{L}_{(p,q)}} = \left(\int_{\mathbb{R}} \|f\|_q^p dt \right)^{1/p}, \quad 1 \le p \le \infty,$$
$$\|f\|_{\mathcal{L}_{(\infty,q)}} = \sup_{\mathbb{R}} \|f(t)\|_q .$$

Moreover, if $p = q$, then we simply use $\mathcal{L}^n_p$ to denote $\mathcal{L}^n_{(p,p)}$.

Remark 2.2.1. *In Definition 2.2.2, $\|\cdot\|_q$ is a spatial norm (i.e., for each $t \in \mathbb{R}$, $\|f(t)\|_q$ is interpreted as a vector q-norm in $\mathbb{R}^n$).*

The definition of $\mathcal{L}_p$ stability is given as follows.

Definition 2.2.3 ([83, 85]). *Consider the following input-output map $\mathcal{H} : \mathcal{L}^m_{(p,r)} \to \mathcal{L}^l_{(q,s)}$ with $y = \mathcal{H}u$, where $u(t) \in \mathcal{L}^m_{(p,r)}$ and $y(t) \in \mathcal{L}^l_{(q,s)}$; the map $\mathcal{H}$ is not necessarily linear. The map $\mathcal{H}$ is called $\mathcal{L}$-stable if there exist a class $\mathcal{K}$ function α, defined on $[0,\infty)$, and a nonnegative constant β such that*

$$\|(\mathcal{H}u)_\tau\|_{\mathcal{L}_{(r,s)}} \le \alpha(\|u_\tau\|_{\mathcal{L}_{(p,q)}}) + \beta, \quad \forall u \in \mathcal{L}^m_{(p,r)}, \quad \forall \tau \in [0,\infty).$$

Moreover, $\mathcal{H}$ is called finite-gain $\mathcal{L}$-stable if there exist nonnegative constants γ and β such that

$$\|(\mathcal{H}u)_\tau\|_{\mathcal{L}_{(r,s)}} \le \gamma\|u_\tau\|_{\mathcal{L}_{(p,q)}} + \beta, \quad \forall u \in \mathcal{L}^m_{(p,r)}, \quad \forall \tau \in [0,\infty). \tag{2.2.1}$$

Remark 2.2.2. *The terms L_2* stability *and $\mathcal{L}_\infty$* stability *will be used for $p = q = r = s = 2$, and $p = q = r = s = \infty$, respectively. The $\mathcal{L}_\infty$ stability is often referred to as Bounded-Input Bounded-Output (BIBO) stability.*

When $\mathcal{H}$ is a linear map, the $\mathcal{L}$ stability theory can be connected to the linear operator theory.

Definition 2.2.4 ([85]). *Let $\mathcal{O} : u \to y$ be a linear operator between $\mathcal{L}$ spaces. The induced norm (operator norm) of $\mathcal{O}$ is defined as*

$$\|\mathcal{O}\|_{(p,r)\to(q,s)} = \sup \frac{\|\mathcal{O}u\|_{\mathcal{L}_{(q,s)}}}{\|u\|_{\mathcal{L}_{(p,r)}}}.$$

Moreover, if $\|\mathcal{O}\|_{(p,r)\to(q,s)} < \infty$, then the operator $\mathcal{O}$ is called a bounded (or continuous) linear operator.

Now we focus on LTI MIMO systems. Let a stable and strictly proper LTI system (denoted by $\mathcal{G}_0$) be :

$$\mathcal{G}_0 : \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0, \tag{2.2.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$ with $\{A, B, C\}$ having appropriate dimensions. Notice that the LTI system of interest has zero initial condition in $\mathcal{L}_p$ stability definitions. The LTI system $\mathcal{G}_0$ can be regarded as a linear convolution operator $\mathcal{G}_0 : u(t) \to y(t)$ such that

$$y = \mathcal{G}_0 u = \int_0^\infty G_0(t-\tau)u(\tau)d\tau, \tag{2.2.3}$$

where $G_0(t)$ is the impulse matrix defined as

$$G_0(t) = \begin{cases} 0, & t < 0, \\ Ce^{At}B, & t \geq 0. \end{cases} \tag{2.2.4}$$

Remark 2.2.3. *Later we will see that a stable LTI system can be considered as a bounded linear operator. However, linear operators and LTI systems need to be distinguished in few different aspects: (1) the operators need to be defined between some function (signal) spaces, while LTI systems do not; (2) there are LTI systems which do not belong to the set of bounded operators (e.g. unstable systems); (3) special considerations are required when dealing with the case of LTI systems with $D \neq 0$ (proper LTI systems), since the impulse matrices of proper systems are not matrix functions in a conventional notion. Notice that the space of bounded linear operators is larger than the set of stable LTI systems.*

Theorem 2.2.1 ([86]). *Given the LTI system (as a linear convolution operator) $\mathcal{G}_0 : \mathcal{L}^m_{(p,r)} \to \mathcal{L}^l_{(q,s)}$ in (2.2.3), the following claims hold:*

(a) Let $p = r = 2$ and $q = s = 2$. Then, $\|\mathcal{G}_0\|_{(2,2)\to(2,2)} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G_0(j\omega))$, where $G_0(j\omega) = C(j\omega - A)^{-1}B$.

(b) Let $p = 1$, $r \in [1, \infty]$, and $q = s = 2$. Then $\|\mathcal{G}_0\|_{(1,r)\to(2,2)} = \|P^{1/2}\|_{r\to 2}$, where $P = B^\top W_o B$ with W_o being the observability Grammian (i.e., $A^\top W_o + W_o A + C^\top C = 0$); $\|\cdot\|_{r\to 2}$ is the matrix induced norm from r to 2.

(c) Let $p = r = 2$, $q = \infty$, and $s \in [1, \infty]$. Then, $\|\mathcal{G}_0\|_{(2,2)\to(\infty,s)} = \|Q^{1/2}\|_{\bar{s}\to 2}$, where $Q = C^\top W_c C$ with W_c being the controllability Grammian (i.e., $AW_c + W_c A^\top + BB^\top = 0$); $\|\cdot\|_{\bar{s}\to 2}$ is the matrix induced norm from $\bar{s}$ to 2, where $\bar{s}$ is the conjugate of s such that $1/s + 1/\bar{s} = 1$.

(d) Let $p = 1$, $r \in [1, \infty]$, $q = \infty$, and $s \in [1, \infty]$. Then, $\|\mathcal{G}_0\|_{(1,r)\to(\infty,s)} = \sup_{t\geq 0} \|G_0(t)\|_{r\to s}$, where $G_0(t)$ is the impulse matrix given in (2.2.4); $\|\cdot\|_{r\to s}$ is the matrix induced norm from r to s.

(e) Let $p = r \in [1, \infty]$, $q = s = \infty$. Then, $\|\mathcal{G}_0\|_{(r,r)\to(\infty,\infty)} = \max_{1\leq i\leq l} \|\mathrm{Row}_i(\hat{G}_0)\|_{\bar{r}}$, where $\mathrm{Row}_i(\cdot)$ takes the ith row of the matrix argument, $\hat{G}_0 = \|G_0(i,j)\|_{\mathcal{L}_{(\bar{p},1)}}$, and $\bar{p}$ is the conjugate of p such that $1/p + 1/\bar{p} = 1$; $G_0(i,j)$ is the (i,j) element of the impulse matrix given in (2.2.4)

Proof. The proof of Theorem 2.2.1 is found in [86]. □

Remark 2.2.4. *The interpretation of several induced norms follows:*

(a) The induced norm $\|\mathcal{G}_0\|_{(2,2)\to(2,2)}$ is the system gain between energy signals (i.e. $\mathcal{L}_2$-norm bounded signals), which is known as the $\mathcal{H}_\infty$ system norm. This induced norm can be used for the worst-case disturbance attenuation in terms of signal energy.

(b) The induced norm $\|\mathcal{G}_0\|_{(2,2)\to(\infty,\infty)}$ denotes the system gain for finite energy signals, which is known as the $\mathcal{H}_2$ system norm. This induced norm is often used to describe the worst-case peak amplitude due to finite energy signals.

(c) The induced norm $\|\mathcal{G}_0\|_{(\infty,\infty)\to(\infty,\infty)}$ indicates the system gain for bounded signals, which is known as the $\mathcal{L}_1$ system norm. This induced norm is often used to quantify the worst-case peak-to-peak gain.

Theorem 2.2.1 implies that a (strictly proper) LTI system can be regarded as an element in the subspace of (matrix) functions with bounded $\mathcal{L}_1$ norms. In other words, the operator space consisting of (strictly proper) LTI systems has one-to-one correspondence to the subspace of $\mathcal{L}_1$, which is composed of the corresponding impulse matrix functions.

Definition 2.2.5 ($\mathcal{L}_1$ norm, $\mathcal{H}_\infty$ norm)**.** *Let $\mathcal{G}$ be the system representing the map from inputs to outputs. Then, the $\mathcal{L}_1$-norm and $\mathcal{H}_\infty$-norm of the system $\mathcal{G}$ are defined as the following induced norms:*

$$\|\mathcal{G}\|_{\mathcal{L}_1} = \|\mathcal{G}\|_{(\infty,\infty)\to(\infty,\infty)}, \quad \|\mathcal{G}\|_{\mathcal{H}_\infty} = \|\mathcal{G}\|_{(2,2)\to(2,2)}.$$

Remark 2.2.5. *For a strictly proper LTI SISO system $\mathcal{G}_0$, $\|\mathcal{G}_0\|_{\mathcal{L}_1} = \|g_0\|_{\mathcal{L}_{(1,1)}}$ holds, where $g_0(t)$ is the impulse response of the system, i.e. the system's $\mathcal{L}_1$-norm is the $\mathcal{L}_1$-norm of its impulse response. However, since LTI MIMO systems have an impulse matrix, the $\mathcal{L}_1$ norm of the matrix function needs to be clarified. Let $G_0(t) = [g_{ij}(t)]_{1\leq i\leq p, 1\leq j\leq q}$ and define $\bar{g}_i : \mathbb{R} \to \mathbb{R}^m$, such as*

$\bar{g}_i(t) = [g_{i1}(t), \ldots, g_{im}(t)]^\top$ *with* $i \in [1, \ldots, p]$ *(i.e.,* $G_0(t) = [\bar{g}_1(t), \ldots, \bar{g}_p(t)]^\top$*). Then, the following are equivalent:*

$$\begin{aligned}
\|\mathcal{G}_0\|_{\mathcal{L}_1} &= \|\mathcal{G}_0\|_{(\infty,\infty)\to(\infty,\infty)} \\
&= \left\| \left[\|\bar{g}_1\|_{\mathcal{L}_{(1,1)}}, \|\bar{g}_2\|_{\mathcal{L}_{(1,1)}}, \ldots, \|\bar{g}_p\|_{\mathcal{L}_{(1,1)}} \right] \right\| = \max_{1\le i\le p} \|\bar{g}_i\|_{\mathcal{L}_{(1,1)}} \\
&= \max_{1\le i\le p} \int_0^\infty \left(\sum_{j=1}^m |g_{ij}(\tau)| \right) d\tau = \max_{1\le i\le p} \sum_{j=1}^m \|g_{ij}\|_{\mathcal{L}_{(1,1)}}
\end{aligned}$$

Notice that if the impulse matrix is a row vector signal (fat system), then $\|\mathcal{G}_0\|_{\mathcal{L}_1} = \|g\|_{\mathcal{L}_{(1,1)}}$ *holds; however, if it is a column vector (tall system), then* $\|\mathcal{G}_0\|_{\mathcal{L}_1} \neq \|g\|_{\mathcal{L}_{(1,1)}}$.

Remark 2.2.6. *The* $\mathcal{H}_\infty$*-norm of LTI systems can be calculated from the maximal singular value, [87]:*

$$\|G(s)\|_{\mathcal{H}_\infty} = \sup_\omega \sigma_{\max}(G(j\omega)).$$

Notice that if an LTI system has a non-zero D matrix, its impulse matrix is not well-defined. However the system norm (induced norm) still can be introduced from Definition 2.2.4 in a operator-theoretic way. This technical issue can be resolved by introducing distribution theory [88], where an impulse matrix of a proper LTI system is treated as a distribution; the $\mathcal{L}_p$ norm of the distribution is identical to the induced norm of the proper LTI system.

Definition 2.2.6 (Banach algebra, [89])**.** *Suppose* $g(t)$ *is a distribution with support in the interval* $[0, \infty)$ *of the form*

$$g(t) = \sum_{i=0}^{\infty} g_i \delta(t - t_i) + g_0(t), \tag{2.2.5}$$

where $0 \le t_0 < t_1 < \cdots$, $\delta(\cdot)$ *represents the unit impulse distribution,* $\{g_i; i \in \bar{\mathbb{N}}\}$ *is a sequence, and* $g_0(\cdot)$ *is a Lebesgue measurable function. The set* A *consists of all distributions* g *of the form* (2.2.5) *such that*

$$\|g\|_A = \int_0^\infty g_0(\tau) d\tau + \sum_{i=0}^{\infty} |g_i| < \infty.$$

The space A equipped with $\|\cdot\|_A$ in Definition 2.2.6 consists of systems with norm-bounded impulses (i.e. finite A-norm); in addition, $\|g\|_A = \sup \frac{\|\mathcal{G}u\|_{\mathcal{L}_\infty}}{\|u\|_{\mathcal{L}_\infty}}$ holds, where $\mathcal{G}$ is a convolution operator with a distribution $g(t)$. Therefore, $\mathcal{G}$ can represent proper and stable LTI systems. More precisely, for a convolution operator $\mathcal{G}$ the induced norm is equivalent to the A-norm of its impulse distribution. Since proper and stable LTI systems can be represented with impulse distributions, $\mathcal{G}$ can be used to represent proper and stable LTI systems; notice that the space A can include the set of proper and stable LTI systems, while the $\mathcal{L}_1$ space does not. The $\mathcal{L}_1$ norm of the LTI MIMO system $\mathcal{G}$ given in (2.1.1) can be computed as follows:

$$\|\mathcal{G}\|_{\mathcal{L}_1} = \|\mathcal{G}_0\|_{\mathcal{L}_1} + \|D\|_\infty,$$

where $\|\mathcal{G}_0\|_{\mathcal{L}_1}$ is the $\mathcal{L}_1$norm of the system, which is obtained from assuming $D = 0$. i.e.

$$\|\mathcal{G}_0\|_{\mathcal{L}_1} = \max_{1\le i\le p} \int_0^\infty \left(\sum_{j=1}^{m} |g_{ij}(\tau)| \right) d\tau,$$

where $g_{ij}(t)$ is the (i,j) element of $G_0(t) = Ce^{At}B$.

Remark 2.2.7. *A computation algorithm for the $\mathcal{L}_1$-norm with high precision can be found in [90].*

A set of bounded linear operators forms a Banach Algebra $\mathcal{A}$ equipped with an induced norm [84], i.e. for any $\mathcal{G}_1 \in \mathcal{A}$ and $\mathcal{G}_2 \in \mathcal{A}$ the following sub-multiplicative hold:

$$\|\mathcal{G}_1 \cdot \mathcal{G}_2\|_{op} \le \|\mathcal{G}_1\|_{op}\|\mathcal{G}_2\|_{op},$$

where $\|\cdot\|_{op}$ is an induced norm. The operator multiplication $\cdot$ is a continuous map $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$; in general, $\cdot$ is a composition of two operators. Therefore, one has

$$\|\mathcal{G}_1 \cdot \mathcal{G}_2\|_{\mathcal{L}_1} \le \|\mathcal{G}_1\|_{\mathcal{L}_1}\|\mathcal{G}_2\|_{\mathcal{L}_1}.$$

Notice that since the LTI systems having bounded $\mathcal{H}_\infty$ norm are always Bounded-Input/Bounded-Output stable (bounded $\mathcal{L}_1$-norm), there should be a relationship between these two norms.

Lemma 2.2.1 ([91], [92])**.** *Suppose $G(s)$ is a stable system, and let n be a McMillan degree*[6]*.*

1. *If $G(s) \in \mathbb{R}[s]$ is a strictly proper SISO system, then*

$$\|G(s)\|_{\mathcal{H}_\infty} \le \|G(s)\|_{\mathcal{L}_1} \le 2n\,\|G(s)\|_{\mathcal{H}_\infty}, \tag{2.2.6}$$

where n is a McMillan degree (the dimension of states in a minimal realization).

2. *If $G(s) \in \mathbb{R}[s]$ is a proper SISO system, then*

$$\|G(s)\|_{\mathcal{H}_\infty} \le \|G(s)\|_{\mathcal{L}_1} \le (2n+1)\,\|G(s)\|_{\mathcal{H}_\infty}. \tag{2.2.7}$$

3. *If $G(s) \in \mathbb{R}^{p\times m}[s]$ is a strictly proper MIMO system, then*

$$\|G(s)\|_{\mathcal{H}_\infty} \le \sqrt{p}\,\|G(s)\|_{\mathcal{L}_1}, \quad \|G(s)\|_{\mathcal{L}_1} \le 2n\sqrt{m}\,\|G(s)\|_{\mathcal{H}_\infty}. \tag{2.2.8}$$

[6] The dimension of states in a minimal realization.

4. If $G(s) \in \mathbb{R}^{p\times m}[s]$ is a proper MIMO system, then

$$\|G(s)\|_{\mathcal{H}_\infty} \leq \sqrt{p}\,\|G(s)\|_{\mathcal{L}_1}\,, \quad \|G(s)\|_{\mathcal{L}_1} \leq (2n+1)\sqrt{m}\,\|G(s)\|_{\mathcal{H}_\infty}\,. \tag{2.2.9}$$

Proof. The proof of Equations (2.2.6) and (2.2.7)(b) are given in [92, 93]. In Equation (2.2.8), the first inequalities are proven in [91]. Since Equation (2.2.9) follows straightforwardly from (2.2.8), we only prove the second inequality in (2.2.8) with the same idea as in [93]:

$$\int_0^\infty e^{A^\top\tau}C^\top Ce^{A\tau}d\tau = \int_0^\infty e^{A^\top\tau}BB^\top e^{A\tau}d\tau = \operatorname{diag}\left(\sigma_1,\dots,\sigma_n\right), \tag{2.2.10}$$

where σ_i is a Hankel singular value [92]. Let $C = [c_1^\top, c_2^\top, \dots, c_p^\top]^\top$ and $B = [b_1, b_2, \dots, b_m]$, where $c_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}^m$ are row and column vectors, respectively; $i \in [1,\dots,p]$ and $j \in [1,\dots,m]$. For each $k \in [1,\dots,n]$, $i \in [1,\dots,p]$, and $j \in [1,\dots,m]$ define

$$\phi_{jk}(t) = \frac{1}{\sqrt{\sigma_k}}u_k^\top e^{At}b_j \in \mathbb{R}, \tag{2.2.11}$$

and

$$\psi_{ik}(t) = \frac{1}{\sqrt{\sigma_k}}c_i e^{At}u_k \in \mathbb{R}, \tag{2.2.12}$$

where $u_k \in \mathbb{R}^n$ is a unit column vector with the ith element being 1. Notice that $\psi_{ik}(t)$ satisfies

$$\begin{aligned}\sum_{i=1}^p \psi_{ik}^2(t) =& \sum_{i=1}^p \psi_{ik}^\top(t)\psi_{ik}(t) = \frac{1}{\sigma_k}u_k^\top e^{A^\top t}\left(\sum_{i=1}^p c_i^\top c_i\right)e^{At}u_k\\ =&\frac{1}{\sigma_k}u_k^\top e^{A^\top t}C^\top Ce^{At}u_k,\end{aligned} \tag{2.2.13}$$

which, together with (2.2.10), leads to

$$\int_0^\infty \sum_{i=1}^p \psi_{ik}^2(\tau)d\tau = \frac{1}{\sigma_k}u_k^\top\left(\int_0^\infty e^{A^\top\tau}C^\top Ce^{A\tau}d\tau\right)u_k = 1, \quad k \in [1,\dots,n]. \tag{2.2.14}$$

This implies that $\sum_{i=1}^m \left(\|\psi_{ik}\|_{\mathcal{L}_2}\right)^2 = 1$ for $i \in [1,\dots,p]$ and $k \in [1,\dots,n]$. Similarly, since

$$\sum_{j=1}^m \phi_{jk}^2(t) = \frac{1}{\sigma_k}u_k^\top e^{At}BB^\top Ce^{At}u_k,$$

one has

$$\int_0^\infty \sum_{j=1}^m \phi_{jk}^2(\tau)d\tau = \frac{1}{\sigma_k}u_k^\top\left(\int_0^\infty e^{A\tau}BB^\top Ce^{A\tau}d\tau\right)u_k = 1, \quad k \in [1,\dots,n], \tag{2.2.15}$$

which leads to $\sum_{j=1}^m \left(\|\phi_{jk}\|_{\mathcal{L}_2}\right)^2 = 1$ for any $k \in [1,\dots,n]$.

Now, let $G(t)$ be the impulse matrix of $G(s)$, which is given by

$$G(t) = Ce^{At}B = [g_{ij}(t)]_{\substack{1\leq i\leq p\\ 1\leq j\leq m}}, \quad g_{ij}(t) = c_i e^{At} b_j. \tag{2.2.16}$$

Since $\sum_{k=1}^{n} u_k u_k^\top = \mathbb{I}_n$, it follows that

$$\sum_{k=1}^{n} \sigma_k \psi_{ik}\left(\frac{t}{2}\right)\phi_{jk}\left(\frac{t}{2}\right) = c_i e^{At} b_j = g_{ij}(t), \quad i \in [1,\ldots,p], \quad \in [1,\ldots,m]. \tag{2.2.17}$$

Moreover, for each $i \in [1,\ldots,p]$, we have

$$\begin{aligned}\sum_{j=1}^{m}\int_0^\infty |g_{ij}(\tau)|\, d\tau &= \sum_{j=1}^{m}\int_0^\infty \left|\sum_{k=1}^{n}\sigma_k \psi_{ik}\left(\frac{\tau}{2}\right)\phi_{jk}\left(\frac{\tau}{2}\right)\right| d\tau \\ &\leq 2\sum_{k=1}^{n}\sum_{j=1}^{m}\sigma_k \int_0^\infty |\eta_{ikj}(\tau)|\, d\tau \\ &\leq 2\sum_{k=1}^{n}\sum_{j=1}^{m}\sigma_k \left\|\eta_{ikj}\right\|_{\mathcal{L}_1},\end{aligned} \tag{2.2.18}$$

where $\eta_{ijk} = \eta_{ikj}(t) = \psi_{ik}(t)\phi_{jk}(t)$. Notice that from the definitions of $\psi_{ik}(\tau)$ and $\phi_{jk}(\tau)$, $\eta_{ikj} \in \mathcal{L}_1$ holds, and therefore, by applying Holder inequality [84], one has

$$\left\|\eta_{ijk}\right\|_{\mathcal{L}_1} \leq \left\|\psi_{ik}\right\|_{\mathcal{L}_2}\left\|\phi_{jk}\right\|_{\mathcal{L}_2}. \tag{2.2.19}$$

Moreover, Cauchy-Schwarz inequality, together with the fact that $\sum_{j=1}^{m}(\left\|\phi_{jk}\right\|_{\mathcal{L}_2})^2 = 1$, implies that

$$\sum_{j=1}^{m}\left\|\phi_{jk}\right\|_{\mathcal{L}_2} \leq \sqrt{m}\left(\sum_{j=1}^{m}(\left\|\phi_{jk}\right\|_{\mathcal{L}_2})^2\right)^{1/2} = \sqrt{m}. \tag{2.2.20}$$

Now, combining (2.2.18)-(2.2.20) yields

$$\sum_{j=1}^{m}\int_0^\infty |g_{ij}(\tau)|\, d\tau = 2\sqrt{m}\sum_{k=1}^{n}\sigma_k \left\|\psi_{ik}\right\|_{\mathcal{L}_2}, \quad i \in [1,\ldots,p], \tag{2.2.21}$$

which further leads to

$$\left\|G(s)\right\|_{\mathcal{L}_1} \leq 2\sqrt{m}\max_{1\leq i\leq p}\left(\sum_{k=1}^{n}\sigma_k \left\|\psi_{ik}\right\|_{\mathcal{L}_2}\right). \tag{2.2.22}$$

Since $\sum_{j=1}^{p}(\left\|\phi_{jk}\right\|_{\mathcal{L}_2})^2 = 1$ holds, it follow that for each $k \in [1,\ldots,n]$

$$\left\|\phi_{jk}\right\|_{\mathcal{L}_2}^2 \leq 1, \quad \forall j \in [1,\ldots,p], \tag{2.2.23}$$

which, together with (2.2.22), yields

$$\|G(s)\|_{\mathcal{L}_1} \le \sqrt{m} \sum_{k=1}^{n} \sigma_k. \tag{2.2.24}$$

Finally, since $\sigma_k \ge \|G(s)\|_{\mathcal{H}_\infty}$ (Hankel singular value theorem, [87]), $k \in [1, \ldots, n]$ holds, we obtain

$$\|G(s)\|_{\mathcal{L}_1} \le 2n\sqrt{m}\,\|G(s)\|_{\mathcal{H}_\infty}. \tag{2.2.25}$$

This completes the proof. □

Notice that the inequalities in Lemma 2.2.1 are not sharp. Moreover, one expects that the difference between $\mathcal{H}_\infty$-norm and $\mathcal{L}_\infty$-norm would be increasing, as the order of the system grows.

Finally, we introduce an important lemma from $\mathcal{H}_\infty$ theory.

Lemma 2.2.2 ([87]). *Let $G(s) \in \mathbb{R}^{p\times m}$ be a MIMO system with $G(s) = D + C(s\mathbb{I} - A)B$; $\{A, B, C, D\}$ is a realization of $G(s)$.*

(a) Suppose $A \in \mathbb{R}^{n\times n}$ is Hurwitz. Then, $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ if and only if

  (1) $R = \gamma^2\mathbb{I}_n - D^\top D \succ 0$,

  (2) and there exists $P = P^\top \succ 0$ such that $P(A + BR^{-1}D^\top C) + (A + BR^{-1}D^\top C)^\top P + PBR^{-1}B^\top P + C^\top(\mathbb{I}_n + DR^{-1}D^\top)C \prec 0$ with $(A + BR^{-1}(D^\top C + B^\top P))$ being Hurwitz.

(b) Suppose $A \in \mathbb{R}^{n\times n}$ is Hurwitz. Then, $\|G(s)\|_{\mathcal{H}_\infty} \le \gamma$ if and only if there exist real matrices $P \succcurlyeq 0$, L, and W such that

$$A^\top P + PA + C^\top C = -L^\top L, \quad D^\top C + B^\top P = -W^\top L, \quad \gamma^2\mathbb{I}_n - D^\top D = W^\top W.$$

(c) Suppose $A \in \mathbb{R}^{n\times n}$ has no imaginary axis eigenvalue. Then, $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ if and if only

  (a) $R = \gamma^2\mathbb{I}_n - D^\top D \succ 0$,

  (b) and H has no eigenvalue on the imaginary axis, where

$$H = \begin{bmatrix} A & 0 \\ -C^\top C & -A^\top \end{bmatrix} - \begin{bmatrix} -B \\ C^\top D \end{bmatrix} R^{-1} \begin{bmatrix} D^\top C & B^\top \end{bmatrix}.$$

Proof. The proof of Theorem 2.2.2 can be found in [87]. □

Lemma 2.2.2 is often called the Bounded-Real-Lemma.

2.2.2. *Lyapuvnov Stability and Its Extensions*

Consider the following system without any uncertainties

$$\dot{x} = f(x,t), \quad x(t_0) = x_0, \tag{2.2.26}$$

where $f : \mathbb{R}^n \times \overline{\mathbb{R}}^+ \to \mathbb{R}^n$ with $x_0 \in \mathbb{R}^n$ being an initial condition. We assume that the system (2.2.26) has a solution for any initial condition $x_0 \in \mathbb{R}^n$ and $t_0 \geq 0$.

Definition 2.2.7. *Let $x(t)$ be a solution of the system* (2.2.26)*. The point $x_e \in \mathbb{R}^n$ is called an equilibrium point of the system* (2.2.26)*, if $0 = f(x_e, t)$, $\forall t \in \overline{\mathbb{R}}^+$.*

Definition 2.2.8. *Let $x(t)$ be a solution of the system* (2.2.26)*. The point $x_a \in \mathbb{R}^n$ is called*

(a) locally attractive, if for each $t_0 \geq 0$ there exists $\delta(t_0) > 0$ such that $\|x_0 - x_a\| < \delta(t_0)$ implies $\lim_{t\to\infty} \|x(t; t_0, x_0) - x_a\| = 0$.

(b) locally equi-attractive (uniformly attractive) with respect to x_0, if for each $t_0 \geq 0$ and $\epsilon > 0$ there exist $\delta(t_0) > 0$ and $T(\epsilon, t_0) > 0$ such that if $\|x_0 - x_a\| < \delta(t_0)$ then $\|x(t; t_0, x_0) - x_a\| < \epsilon$ for all $t \geq t_0 + T$.

(c) locally uniformly attractive with respect to t_0 and x_0, if there exists $\delta > 0$ such that for each $\epsilon > 0$ there exists $T(\epsilon) > 0$ implying that for $\|x_0 - x_a\| < \delta$ one has $\|x(t; t_0, x_0) - x_a\| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$.

(d) globally attractive, if for each $t_0 > 0$, $x_0 \in \mathbb{R}^n$, and each $\epsilon > 0$ there exists $T(\epsilon, t_0, x_0) > 0$ such that $\|x(t; t_0, x_0) - x_a\| < \epsilon$ for all $t \geq t_0 + T(\epsilon, t_0, x_0)$ and all $x_0 \in \mathbb{R}^n$.

(e) globally uniformly attractive, if for each $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that $\|x(t; t_0, x_0) - x_a\| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$, all $t_0 > 0$, and all $x_0 \in \mathbb{R}^n$.

Notice that the definitions in 2.2.8 can be easily extended to a set instead of a point x_a. Moreover, Definition 2.2.8(a) should be read carefully; the limit depends on the choice of (t_0, x_0). An alternative definition is given as:

> *The point $x_a \in \mathbb{R}^n$ is locally attractive, if for each $t_0 \geq 0$ and $\epsilon > 0$ there exist $\delta(t_0) > 0$ and $T(\epsilon, t_0, x_0) > 0$ such that $\|x_0 - x_a\| < \delta(t_0)$ implies $\|x(t; t_0, x_0) - x_a\| < \epsilon$ for all $t \geq t_0 + T(\epsilon, t_0, x_0)$.*

The terms *uniformly* and *globally* on an attractive point need to be carefully taken into account. These terms apply to two independent variables (t_0, x_0), and two dependent variables, such as the ball of initial variables $\delta(t_0)$ and the rate of convergence $T(\epsilon, t_0, x_0)$. Technically, the uniformity in t_0 will remove all dependencies on t_0 in $\delta(t_0)$ and $T(\epsilon, t_0, x_0)$, and the uniformity in x_0 (called *equi-*) ignores all dependencies of x_0 on $T(\epsilon, t_0, x_0)$. However, when we say *uniformly attractive*, it means that the uniformity applies to both x_0 and t_0. Moreover, from the definitions in 2.2.8(d) and 2.2.8(e), global attraction does not necessarily imply uniform attraction with respect to (t_0, x_0), since the term *globally* indicates that the ball for initial conditions can be arbitrarily large. Therefore, we may allow a *globally (in* δ*) attractive* point, but not *uniformly (in* t_0*)* attractive one. While the condition on *global attraction* removes the dependency of t_0 on $\delta(t_0)$, the rate of convergence can still depend on the location of x_0 in $\mathbb{R}^n$. To summarize, the following claims hold.

- The definition in 2.2.8(b) implies that the rate of convergence is not dependent on the choice of x_0 in the ball.
- In Definition 2.2.8(c), the rate of convergence only depends on the size of $\|x(t; t_0, x_0) - x_a\|$, and the size of initial conditions is independent on t_0 as well. Therefore, it is automatically *locally equi-attractive.*
- In Definition 2.2.8(d), the term *globally* indicates that the ball of initial values can be arbitrarily large.
- In Definition 2.2.8(e), the uniformity is defined with respect to the choice of (t_0, x_0).

Next, we introduce several notions of point stability.

Definition 2.2.9. *Let* $x_e \in \mathbb{R}^n$ *be an equilibrium point of the system* (2.2.26)*, and let* $x_0 = x(t_0)$ *for some* $t_0 \in \overline{\mathbb{R}}^+$*. Then,* x_e *is called:*

(a) stable (in sense of Lyapunov), if for each $\epsilon > 0$ *and* $t_0 \geq 0$*, there exists* $\delta(\epsilon, t_0) > 0$ *such that if* $\|x_0 - x_e\| < \delta(\epsilon, t_0)$ *then* $\|x(t; t_0, x_0)\| < \epsilon$ *for all* $t \geq t_0 \geq 0$*.*

(b) Uniformly Stable (US), if there exists a class $\mathcal{K}$ *function* $\alpha(\cdot)$ *and* $c > 0$ *(independent of* t_0*) such that* $\|x(t)\| \leq \alpha(\|x_0\|)$*,* $\forall t \geq t_0 \geq 0$*,* $\forall \|x_0\| < c$*.*

(c) Asymptotically Stable (AS), if x_e *is stable and locally attractive.*

(d) Globally Asymptotically Stable (GAS), if x_e *is stable and globally attractive.*

(e) Uniformly Asymptotically Stable (UAS), if there exists a class $\mathcal{KL}$ *function* $\beta(\cdot, \cdot)$ *and* $c > 0$ *(independent of* t_0*) such that* $\|x(t)\| \leq \beta(\|x_0\|, t - t_0)$*,* $\forall t \geq t_0 \geq 0$*,* $\forall \|x_0\| < c$*.*

(f) Globally Uniformly Asymptotically Stable (GUAS), if there exists a class $\mathcal{KL}$ *function* $\beta(\cdot, \cdot)$ *such that* $\|x(t)\| \leq \beta(\|x_0\|, t - t_0)$*,* $\forall t \geq t_0 \geq 0$*,* $\forall x_0 \in \mathbb{R}^n$*.*

The term *globally* or *locally* in Definition 2.2.9 applies to the *attractive* property, not to the stability; therefore, the term *globally* must be used with *asymptotically.* In the literature, the term *globally stable* or *locally stable* are often used by omitting *asymptotically.* Notice that a point may be not (Lyapunov) stable, but could still be (globally) attractive (see Remark 2.2.8).

Remark 2.2.8. *Consider* $\dot{x}(t) = -x(t) + e^{-t}cos(t)$*, for which* $x = 0$ *is not stable but globally attractive. To verify this, let* $x_e(t) = e^{-t}cos(t)$*, then* $\dot{x}_e(t) = f(x_e(t), t) = 0$ *holds from the differential equation. However, since* $\dot{x}_e(t) = \frac{d}{dt}(e^{-t}cos(t)) \neq 0$ *for some* $t > 0$*,* $x_e(t)$ *is not stationary with respect to* t*. Therefore, the system does not have an equilibrium point, since the equilibrium point should be stationary. Moreover, it is easy to show that* $x(t) \to 0$ *for any* $x_0 \in \mathbb{R}^n$*, so that* $x = 0$ *is globally attractive.*

In nonlinear system analysis, boundedness of the solutions is an important property, which differs from the notion of stability.

Definition 2.2.10. *Consider the system in* (2.2.26)*. The solution* $x(t)$ *of the system is said to be*

(a) uniformly bounded if there exists $c > 0$ *(independent of* t_0*) such that for each* $a \in (0, c)$ *there exists* $d = d(a) > 0$ *implying that for all* $\|x(t_0)\| \leq a$ *one has* $\|x(t)\| \leq d$*,* $\forall t \geq t_0$*.*

(b) globally uniformly bounded if for each $a \in \mathbb{R}^+$ *there exists* $d = d(a) > 0$ *such that* $\|x(t_0)\| \leq a$ *implies* $\|x(t)\| \leq d$*,* $\forall t \geq t_0$*.*

(c) uniformly ultimately bounded with ultimate bound b*, if there exists* b *(independent of* t_0*) and* c *(independent of* t_0*), and for each* $a \in (0, c)$ *there exists* $T(a, b)$*, such that* $\|x(t_0)\| \leq a$ *implies* $\|x(t)\| \leq b$*,* $\forall t \geq T(a, b) + t_0$*.*

(d) globally uniformly ultimately bounded with ultimate bound b*, if there exists* b *(independent of* t_0*) such that for each* $a \in \mathbb{R}^+$ *there exists* $T = T(a) > 0$*, implying that for* $\|x(t_0)\| \leq a$ *one has* $\|x(t)\| \leq b$*,* $\forall t + T(a) \geq t_0$*.*

In Definition 2.2.10, the term *uniformly* indicates that the bound d (or b) does not depend on t_0, and the term *ultimately* is used to show that boundedness of the solution holds after a certain time T. Moreover, the *ultimate boundedness* condition is stronger than the *boundedness* condition in the sense that a smaller bound can be used; notice that if b is chosen such that $b \geq d(a)$, the ultimate boundedness reduces to boundedness. Although not explicitly stated in the definition, ultimate boundedness is often used for the case that b is small, which indicates a better result than the boundedness.

Remark 2.2.9. *Lyapunov stability implies uniform boundedness; however, the converse is not true. An example is* $\dot{x}(t) = -x(t) + e^{-t}\cos(t)$*. As discussed in Remark 2.2.8, this system is not Lyapunov stable, since it does not have any equilibrium point. However, the attraction to the origin implies ultimate boundedness (and thus uniform boundedness).*

Up to this point we have investigated conventional definitions of stability. In practice, the conventional notions are not sufficient to describe the behavior of nonlinear systems when the systems under consideration are affected by undetermined signals such as control inputs and disturbances. Now, we consider advanced notions on stability. Consider the following system with an input signal:

$$\dot{x} = f(x, u, t), \quad x(t_0) = x_0, \tag{2.2.27}$$

where $x_0 \in \mathbb{R}^n$ is an initial condition, $f : \mathbb{R}^n \times \mathbb{R}^m \times \bar{\mathbb{R}}^+ \to \mathbb{R}^n$ is locally Lipschitz continuous with respect to (x, u) and piecewise continuous in t. Notice that $u(t)$ can be either a disturbance or control signal.

Definition 2.2.11 (0-GAS, [94])**.** *Consider the system* (2.2.27)*. Let the zero-input system of* (2.2.27) *be*

$$\dot{x} = \bar{f}(x, t), \quad x(t_0) = x_0, \tag{2.2.28}$$

where $\bar{f}(x, t) = f(x, 0, t)$. Then, the system (2.2.27) *is said to be* 0*-GAS, if the zero-input system* (2.2.28) *is globally asymptotically stable.*

Definition 2.2.12 (BIBS, CICS, [94,95])**.** *The system given in* (2.2.27) *is said to be Bounded-Input Bounded-State (BIBS) stable, if for some class $\mathcal{K}_\infty$ function α, the following holds:*

$$\|x(t)\| \le \max\left(\alpha(\|x_0\|), \alpha(\|u(t)\|)\right), \tag{2.2.29}$$

where $x(t) \in \mathbb{R}^n$ is a solution of the system. Moreover, the system is said to be Converging-Input Converging-State (CICS) stable, if

$$\lim_{t\to\infty} \|x(t)\| = 0, \tag{2.2.30}$$

for all $x_0 \in \mathbb{R}^n$ and all inputs $u(t)$ converging to 0*.*

Remark 2.2.10. *It is well-known that a linear system is* 0*-GAS, if and only if it is BIBS. However, this is not true for nonlinear systems. Consider the following nonlinear system*

$$\dot{x} = -x + (x^2 + 1)d, \quad x(0) = x_0, \tag{2.2.31}$$

where $x_0 \in \mathbb{R}^n$ is an initial condition. The system (2.2.31) *is* 0*-GAS, since it reduces to $\dot{x} = -x$ when $d \equiv 0$. However, the system is not BIBS. For example, let $d(t) = (2t + 2)^{-1/2}$, and $x_0 = \sqrt{2}$. Then, the solution is given by $x(t) = (2t + 2)^{1/2}$ which is unbounded. Even worse, the bounded disturbance $d(t) \equiv 1$ results in a finite-time explosion of the solution. Therefore,* 0*-GAS does not implies BIBS (i.e.,* 0*-GAS does not guarantee good behaviors with respect to the inputs). This motivates input-to-state stability [94].*

Definition 2.2.13 (ISS, [94, 96])**.** *The system given in* (2.2.27) *is said to be*

(a) locally (uniformly) Input-to-State Stable (ISS), if there exists a class $\mathcal{KL}$ function β, a class K function γ, $\delta_x > 0$ such that for each $x_0 \in \mathcal{D}_0$, each t_0, each $u \in \mathcal{L}_\infty$, the solution $x(t)$ satisfies

$$\|x(t)\| \le \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \le \tau \le t} \|u(\tau)\|), \quad \forall t \ge t_0,$$

where $\mathcal{D}_0 = \{x \in \mathbb{R}^n : \|x\| < \delta\}$.

(b) globally (uniformly) ISS if there exists a class $\mathcal{KL}$ *function* β*, a class* $\mathcal{K}$ *function* γ *such that for each* $x_0 \in \mathbb{R}^n$*, each* t_0*, each* $u \in \mathcal{L}_\infty$*, the solution* $x(t)$ *satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|), \quad \forall t \geq t_0.$$

Remark 2.2.11. *It can be shown that if the system* (2.2.27) *is ISS stable. Then, the system has the* 0*-GAS, BIBS, and CICS properties [96]. Moreover, the locally (respectively, globally) ISS implies the locally (respectively, globally) UAS, when* $u(t) \equiv 0$ *and* $x_e = 0$ *is an equilibrium point.*

Definitions 2.2.9 and 2.2.13 are used to describe system behavior around a point. In real-world applications, system uncertainties such as unmodeled dynamics and disturbances often prevent the states from going to zero. Therefore, practical stability is introduced to describe the behavior of solutions with respect to a neighborhood of the origin [97, 98].

Definition 2.2.14 (Practically UAS, [97])**.** *The system given in* (2.2.26) *is said to be (globally) practically UAS, if there exists a class* $\mathcal{KL}$ *function and a constant* $d > 0$ *such that for each* $x_0 \in \mathbb{R}^n$ *the following holds*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + d, \quad \forall t \geq t_0.$$

The definition of *local practical UAS* follows if the set of initial conditions is a local domain in $\mathbb{R}^n$. Practical input-to-state stability can be defined in a similar manner.

Definition 2.2.15 (Practically ISS, [98])**.** *The system given in* (2.2.27) *is said to be*

(a) locally practically ISS, if there exist a class $\mathcal{KL}$ *function* β*, a class* $\mathcal{K}$ *function* γ*, a constant* $d > 0$ *and* $\delta_x > 0$ *such that for each* x_0 *with* $\|x_0\| < \delta_x$*, each* t_0*, each* $u \in \mathcal{L}_\infty$*, the solution* $x(t)$ *satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|) + d, \quad \forall t \geq t_0.$$

(b) globally practically ISS, if there exist a class $\mathcal{KL}$ *function* β*, a class* $\mathcal{K}$ *function* γ*, and a constant* $d > 0$ *such that for each* $x_0 \in \mathbb{R}^n$*, each* t_0*, each* $u \in \mathcal{L}_\infty$*, the solution* $x(t)$ *satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|) + d, \quad \forall t \geq t_0.$$

In the literature, global stabilization is referred when a designer finds a single control parameter to stabilize nonlinear systems for all possible initial conditions [85, 99]. However, in general, global stabilization of nonlinear systems is not easily attained, if specific conditions are not imposed on the nonlinearities. Therefore, semi-global stabilization is often used to describe the stability of the systems. Roughly speaking, semi-global stabilization implies that one can always find a set of control parameters to stabilize the nonlinear system. The choice of the parameters is dependent on the bounds of initial conditions [99]. Consider the following nonlinear system:

$$\dot{x} = f(x, t, \theta), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{2.2.32}$$

where $x_0 \in \mathbb{R}^n$ is an initial condition, and $\theta \in \Theta$ with $\Theta \subseteq \mathbb{R}^l$ being a set of constant parameters (i.e., Θ indicates a set of all design parameters in general). In addition, suppose that for each $\theta \in \Theta$ $f_\theta(x, t)$ is locally Lipschitz continuous in x, and piecewise continuous in t, where $f_\theta(x, t) = f(x, t, \theta)$.

Definition 2.2.16 (UGPAS/USPAS, [97])**.** *The system given in* (2.2.32) *is said to be*

(a) Uniformly Globally Practically Asymptotically Stable (UGPAS), if for each $d > 0$ there exist $\theta^\star(d) \in \Theta$ and a class $\mathcal{KL}$ function β such that for all $x_0 \in \mathbb{R}^n$ and $t_0 \geq 0$

$$\|x(t; t_0, x_0, \theta^\star)\| \leq \beta(\|x_0\|, t - t_0) + d, \quad \forall t \geq t_0. \tag{2.2.33}$$

(b) Uniformly Semi-globally Practically Asymptotically Stable (USPAS), if for each $d > 0$ and $\delta > 0$ satisfying $\delta > d > 0$, there exist $\theta^\star(d, \delta) \in \Theta$ and a class $\mathcal{KL}$ function β, such that for all $x_0 \in \mathcal{D}_0$ and $t_0 \geq 0$

$$\|x(t; t_0, x_0, \theta^\star)\| \leq \beta(\|x_0\|, t - t_0) + d, \quad \forall t \geq t_0, \tag{2.2.34}$$

where $\mathcal{D}_0 = \{x \in \mathbb{R}^n : \|x\| < \delta\}$.

Notice that the function β in Definitions 2.2.16(a) and 2.2.16(b)) depends on d; d indicates the size of the invariant set where the state trajectories eventually remain. However, the semi-global stability (in Definition 2.2.16(b)) requires an additional dependency on the function β; β also relies on δ which indicates the size of the set of initial conditions. Moreover, the control parameter θ depends on both d and δ in the semi-global case, while it is independent of δ in Definition 2.2.16(a). This dependency indicates that the control parameters should be tuned according to initial conditions in the semi-global case. Similarly, the semi-global property can be applied to input-to-state stability. Consider the system

$$\dot{x} = f(x, u, t, \theta), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{2.2.35}$$

where $x_0 \in \mathbb{R}^n$ is an initial condition, and $\theta \in \Theta$ with $\Theta \subseteq \mathbb{R}^l$ being a set of constant parameters. In addition, suppose that for each $\theta \in \Theta$ $f_\theta(x, u, t)$ is locally Lipschitz continuous in (x, u), and piecewise continuous in t, where $f_\theta(x, u, t) = f(x, u, t, \theta)$.

Definition 2.2.17 (SPISS)**.** *The system given in* (2.2.35) *is said to be Semi-globally Practically ISS (SPISS), if for each $d > 0$, $\delta_x > 0$, and $\delta_r > 0$ satisfying $\delta_x > d$, there exist $\theta^\star(d, \delta_x, \delta_r) \in \Theta$*

and a class $\mathcal{KL}$ function β such that for all $x_0 \in \mathcal{D}_0$ and $t_0 \geq 0$

$$\|x(t; t_0, x_0, \theta^\star)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|) + d, \quad \|u\|_{\mathcal{L}_\infty} < \delta_r, \quad \forall t \geq t_0. \tag{2.2.36}$$

where $\mathcal{D}_0 = \{x \in \mathbb{R}^n : \|x\| < \delta_x\}$.

Remark 2.2.12. *Definitions 2.2.15(a) and 2.2.15(b) assume that the choice of design parameter θ is independent of initial conditions, reference commands, and disturbances. However, the parameter θ has dependencies on them in Definition 2.2.17.*

2.3. Uncertainty Parameterization

In adaptive control theory, parametrization of system uncertainties is a key technical issue. In the following, we introduce parametrization of an unknown LTI plant and nonlinear functions.

2.3.1. *LTI System Parametrization*

In this section, we introduce the bilinear parameterization of unknown LTI plants. Let $G_p(s)$ be an unknown LTI plant and $M(s)$ be a given (and known) desired model. The parametrization of $G_p(s)$ can be derived from solving a model matching problem, in which unknown coefficients of $G_p(s)$ will be represented as matched uncertainties of the known $M(s)$. The objective of model matching design is to find a control law to exactly follow the desired model, which establishes an algebraic relationship between $G_p(s)$ and $M(s)$; this is known as Model Reference Control (MRC) design. In fact, the method provides a solution to represent the unknown plant with the desired model, known (regressor) functions, and constant unknown parameters. For the sake of brevity, we consider SISO LTI plants (see [32] for MIMO plants). Let $G_p(s)$ and $M(s)$ be

$$G_p(s) = k_p \frac{Z_p(s)}{R_p(s)}, \qquad M(s) = k_m \frac{Z_m(s)}{R_m(s)}, \tag{2.3.1}$$

where $Z_p(s)$, $R_p(s)$, $Z_m(s)$, $R_m(s)$ are monic polynomials.

Assumption 2.3.1. *Assume that the unknown LTI plant $G(s)$ and the given desired model $M(s)$ given in* (2.3.1) *satisfy*

(a) k_p is an unknown positive constant, and $k_m > 0$ is known.

(b) The polynomials $Z_p(s)$ and $R_p(s)$ have no zeros in $\overline{\mathbb{C}}^+$ with unknown coefficients.

(c) The relative degree of $G_p(s)$ is known and identical to the relative degree of $M(s)$.

(d) The known polynomials $Z_m(s)$ and $R_m(s)$ have no zeros in $\overline{\mathbb{C}}^+$.

From MRC theory, it is known that there exist control gains and control laws to solve the model matching problem, such as

$$u(s) = \frac{\theta_1 \alpha(s)}{\Lambda(s)} u(s) + \frac{\theta_2 \alpha(s)}{\Lambda(s)} y(s) + \theta_3 y(s) + \theta_4 r(s)\,, \tag{2.3.2}$$

where $n \geq \deg R_m(s)$, $\theta_1\,, \theta_2 \in \mathbb{R}^{n-1}\,, \theta_3, \theta_4 \in \mathbb{R}$ are constant control gains, $\Lambda(s)$ is an arbitrary monic $n-1$ order polynomial containing $Z_m(s)$, and

$$\begin{aligned} &\alpha(s) = [s^{n-2}\,,\, s^{n-1}\,,\, \ldots\,,\, 1]\,, \quad \forall n \geq 2, \\ &\alpha(s) = 0\,, \quad n = 1\,. \end{aligned}$$

The control gains $\theta_i\,, \forall i \in \{1\,, \ldots\,, 4\}$ are selected to satisfy

$$\begin{aligned} &\theta_1^\top \alpha(s) R_p(s) + k_n(\theta_2^\top \alpha(s) + \theta_3 \Lambda(s)) Z_p(s) \\ =&\Lambda(s) R_p(s) - Z_p(s) \Lambda_0(s) R_m(s)\,, \quad \forall s \in \mathbb{C}\,, \end{aligned} \tag{2.3.3}$$

where $\Lambda_0(s)$ is the polynomial such that $\Lambda(s) = \Lambda_0(s) Z_m(s)$, and

$$\theta_4 = \frac{k_p}{k_m}\,. \tag{2.3.4}$$

Notice that the existence of $\{\theta_1\,, \theta_2\,, \theta_3\,, \theta_4\}$ follows from (2.3.3) under Assumption 2.3.1 [32]. Since $G_p(s)$ is unknown, θ_i, $i = 1, \ldots, 4$ is also unknown. However, $\Lambda(s)$ and $\alpha(s)$ can be defined from the minimal information about $G_p(s)$ and $M(s)$. Moreover, one can verify that the control law (2.3.2) gives a solution to the model matching such that

$$\begin{aligned} G_c(s) =& \frac{\theta_4 k_p Z_p(s) \Gamma^2(s)}{\Gamma(s)\left((\Gamma(s) - \theta_1^\top \alpha(s)) R_p(s) - k_p Z_p(s)(\theta_2^\top \alpha(s) + \theta_3 \Gamma(s))\right)}, \\ M(s) =& \frac{\theta_4 k_p Z_p(s) \Gamma^2(s)}{\Gamma(s)\left((\Gamma(s) - \theta_1^\top \alpha(s)) R_p(s) - k_p Z_p(s)(\theta_2^\top \alpha(s) + \theta_3 \Gamma(s))\right)}, \end{aligned}$$

where $G_c(s)$ is the closed-loop system with the control law (2.3.2), i.e. $y(s) = G_c(s) r(s) = M(s) r(s)$ holds for a given reference command $r(s)$.

Nest we investigate the state-space structure of this parametrization.

Theorem 2.3.1. *Suppose the unknown LTI system $G_p(s)$ and the given desired model $M(s)$ satisfy Assumption 2.3.1. Then, $G_p(s)$ can be represented as*

$$\dot{x}(t) = A_m x(t) + B_m(\omega u(t) + \theta^\top \phi(x)), \quad y(t) = C_m x(t), \quad x(0) = 0, \tag{2.3.5}$$

where $\{A_m, B_m, C_m\}$ is a realization of $M(s)$, and $\omega > 0$, θ, $\phi(x)$ are an unknown constant, unknown vector, and known regressor function with appropriate dimensions.

Proof. A realization of the (unknown) control law given in (2.3.2) can be rewritten in state space form as

$$\dot{w}_1(t) = F w_1(t) + g u(t), \quad \dot{w}_2(t) = F w_2(t) + g y_n(t), \quad w_1(0) = 0, \quad w_2(0) = 0, \tag{2.3.6}$$

$$u(t) = \theta_1^\top w_1(t) + \theta_2^\top w_2(t) + \theta_3 y(t) + \theta_4 r(t), \tag{2.3.7}$$

where $w_1(t), w_2(t) \in \mathbb{R}^{(n-1)}$ are auxiliary states, and $F \in \mathbb{R}^{(n-1)\times(n-1)}$ and $g \in \mathbb{R}^{n-1}$ are chosen such that $\det(sI - F) = \Lambda(s)$ and $(sI - F)^{-1} g = \frac{\alpha(s)}{\Lambda(s)}$. In other words, letting

$$\Lambda(s) = s^{n-1} + \lambda_{n-2} s^{n-2} + \ldots + \lambda_1 s^1 + \lambda_0 \,,$$

one has

$$F = \begin{bmatrix} -\lambda_{n-2} & -\lambda_{n-3} & -\lambda_{n-4} & \ldots & -\lambda_0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \quad g = [1\,0\,\ldots\,0]^\top .$$

By combining (2.3.6) and (2.3.7) with a state-space realization of $G_p(s)$, the closed-loop system $G_c(s)$ can be rewritten as

$$\dot{x}(t) = A_c x(t) + b_b \theta_4 r(t), \quad y(t) = c_b^\top x(t), \quad x_b(0) = 0, \tag{2.3.8}$$

where

$$A_c = \begin{bmatrix} A_p + b_p \theta_3 c_p^\top & b_p \bar{\theta}_1^\top & b_p \bar{\theta}_2^\top \\ g \theta_3 c_p^\top & F + g \theta_1^\top & g \theta_2^\top \\ g c_p^\top & 0 & F \end{bmatrix}, \quad b_b = \begin{bmatrix} b_n \\ g \\ 0 \end{bmatrix}, \quad c_b = \begin{bmatrix} c_p^\top \, 0\, 0 \end{bmatrix}^\top ,$$

with (A_p, b_p, c_p) being a realization of $G_p(s)$, and $x(t) = [x_p^\top(t), w_1^\top(t), w_2^\top(t)]$. Since $G_c(s) = M(s)$ holds, it follows that $M(s) = c_b^\top (sI - A_c)^{-1} b_b \bar{\theta}_4$, and therefore ($A_m$, b_m, and c_m) is the realization of the desired model $M(s)$, with $A_m = A_c$, $b_m = b_b \theta_4$ and $c_m = c_b$. Finally, let $\omega = \theta_4^{-1}$, $\theta = [-\theta_4^{-1}\theta_1^\top, -\theta_4^{-1}\theta_2, -\theta_4^{-1}\theta_3^\top]^\top$, and $\phi(x) = [w_1^\top(t), w_2^\top(t), y(t)]^\top$; notice that $\phi(x)$ has a state-dependency, but it is a known and computable function without the knowledge of the internal states of the unknown plant. Finally, from (2.3.7) one has

$$r(t) = \omega u(t) + \theta^\top \phi(x),$$

which, together with (2.3.8), leads to Equation (2.3.5). This completes the proof. □

2.3.2. *Nonlinear Function Parametrization*

Consider the nonlinear function $f : \mathbb{R}^{n_m} \times \mathbb{R} \to \mathbb{R}^{n_p}$ subject to the following Assumption.

Assumption 2.3.2. *There exists $b_0 > 0$ such that*

$$\|f(0,t)\| < b_0, \quad \forall t \geq 0,$$

where b_0 is a known constant. Moreover, for any $\delta > 0$ there exist $d_\delta > 0$, and $b_\delta > 0$ such that

$$\left\|\frac{\partial f(X,t)}{\partial X}\right\| \leq d_\delta, \quad \left\|\frac{\partial f(X,t)}{\partial t}\right\| \leq b_\delta, \quad \forall \|X\| < \delta,$$

where d_δ and b_δ are known constants.

Lemma 2.3.1. *Let $\tau > 0$, and let $X(t) = [X_1^\top(t), X_2^\top(t)]^\top$ be a continuous and (piecewise) differentiable function, where $X_1(t) \in \mathbb{R}^{n_1}$, $X_2(t) \in \mathbb{R}^{n_2}$. Suppose that $\|\dot{X}(t)\|$ is finite for all $0 \leq t \leq \tau$. Consider a nonlinear function $f(X,t)$ satisfying Assumption 2.3.2 and*

$$\|f(X,t)\| < \bar{d}_X \|X_1(t)\| + \bar{b}_X, \quad \|X_\tau\|_{\mathcal{L}_\infty} \leq \rho_X, \quad 0 \leq t \leq \tau,$$

for some $\rho_X > 0$, $\bar{d}_X > 0$ and $\bar{b}_X > 0$, Then, there exist continuous and (piecewise) differentiable $\theta(t)$ and $\sigma(t)$, such that

$$f(X,t) = \theta(t)\|X_1(t)\| + \sigma(t)\,, \quad \forall t \in [0,\tau],$$

and

$$\|\theta(t)\| \leq \bar{d}_X, \quad \|\dot{\theta}(t)\| \leq \bar{l}_\theta, \quad \|\sigma(t)\| \leq \bar{b}_X, \quad \|\dot{\sigma}(t)\| \leq \bar{l}_\sigma,$$

where $\bar{l}_\theta$, $\bar{l}_\sigma$ are computable finite bounds.

Proof. See [2, Lemma A.9, Lemma A.10]. □

CHAPTER. 3

$\mathcal{L}_1$ Adaptive Output-Feedback for MIMO Systems with Vector Relative Degree One

This chapter presents $\mathcal{L}_1$ adaptive output feedback controllers for underactuated systems with vector relative degree $\mathbf{1}_m$. We present two design approaches: (i) $\mathcal{L}_1$ adaptive controller as the main controller of the system, and (ii) $\mathcal{L}_1$ adaptive augmentation of a dynamic baseline controller. With the state decomposition introduced in Chapter 2, the proposed methods directly tackle underactuated systems without additional squaring-up process, which makes the design procedures simpler. Throughout this chapter, we consider the following class of unknown input gain and nonlinear functions:

Assumption 3.0.1. *The unknown constant input gain Ω is assumed to be a (nonsingular) strictly row-diagonally dominant matrix with $sgn(\Omega_{ii})$ known. Moreover, the input gain satisfies $\Omega \in \mathcal{C}_\Omega$, where $\mathcal{C}_\Omega \subseteq \mathbb{R}^{m\times m}$ is a known convex compact set.*

Assumption 3.0.2. *There exists $b_0 > 0$ such that*

$$\|f(0,t)\| \leq b_0, \quad \forall t \geq 0,$$

where b_0 is a known constant. Moreover, for any $\delta > 0$ there exist $c_\delta > 0$, and $d_\delta > 0$ such that

$$\left\|\frac{\partial f(x,t)}{\partial t}\right\| \leq c_\delta, \quad \left\|\frac{\partial f(x,t)}{\partial x}\right\| \leq d_\delta, \quad \forall \|x\| < \delta,$$

where c_δ and d_δ are known constants.

3.1. $\mathcal{L}_1$ Adaptive Controller

In this section, we introduce the $\mathcal{L}_1$ adaptive output feedback controller for underactuated MIMO system. Consider the following MIMO system:

$$\begin{aligned}\dot{x}(t) =& A_p x(t) + B_m(\Omega u(t) + f(x,t)),\\ y(t) =& C_m x(t), \quad x(0) = x_0,\end{aligned} \tag{3.1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input, and measurable output vectors, respectively, with $m \leq p$, $x_0 \in \mathbb{R}^n$ is an unknown initial value, $\Omega \in \mathbb{R}^{m\times m}$ is an unknown constant input gain, and $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$ is an unknown function representing matched uncertainties; Ω, and $f(x,t)$ satisfy Assumptions 3.0.1 and 3.0.2, respectively. $A_p \in \mathbb{R}^{n\times n}$ is a known (nominal) matrix, and $B_m \in \mathbb{R}^{n\times m}$ and $C_m \in \mathbb{R}^{p\times n}$ are known full rank matrices; (A_p, B_m, C_m) is a stabilizable and detectable triple which represents the nominal system. To proceed, the following assumptions are made:

Assumption 3.1.1. *The nominal system $G_p(s) = C_m(s\mathbb{I}_n - A_p)^{-1}B_m$ has no unstable transmission zeros, and (C_mB_m) is full rank.*

Assumption 3.1.2. *$K_m \in \mathbb{R}^{m\times n}$ is chosen so that $A_m = A_0 - B_mK_m$ is Hurwitz, and*

$$M(s) = C_m(s\mathbb{I}_n - A_m)^{-1}B_m$$

represents desired responses.

Remark 3.1.1. *Notice that Assumption 3.1.1 leads to the fact that $M(s)$ has no unstable zeros, since the state-feedback gain cannot change zeros of the system.*

The formal problem statement at hand is given as follows:

Problem 3.1.1. *Let the system described by Equation (3.1.1) satisfy Assumptions 3.0.1, 3.0.2, 3.1.1, and 3.1.2. The control objective is to design an adaptive output feedback control law for $u(t)$ so that $y(t)$ tracks $y_m(t)$ governed by $y_m(s) = M(s)K_gr(s)$, where $K_g \in \mathbb{R}^{m\times m_r}$ is a known gain and $r(s)$ is the Laplace transform of a given reference command $r(t) \in \mathbb{R}^{m_r}$.*

3.1.1. *Design of L1 adaptive controller*

In this section, we propose an $\mathcal{L}_1$ adaptive output feedback controller to solve Problem 3.1.1. Firstly, we introduce a few variables of interest. Define

$$c_\Omega = \max_{\Omega\in\mathcal{C}_\Omega} \|\Omega K_m\|, \quad c_{\bar{\Omega}} = \max_{\Omega\in\mathcal{C}_\Omega} \|(\mathbb{I}_m - \Omega)K_m\|. \tag{3.1.2}$$

Let $\rho_0 > 0$ be a given upper bound satisfying $\|x_0\| \le \rho_0$. For a given $\delta > 0$, define

$$L_\delta = \frac{\bar{\delta}(\delta)}{\delta}\left(d_{\bar{\delta}(\delta)} + c_{\bar{\Omega}}\right), \quad \bar{\delta}(\delta) = \delta + \bar{\gamma}. \tag{3.1.3}$$

where $d_{\bar{\delta}(\delta)}$ is introduced in Assumptions 3.0.2. Define

$$\kappa_m = \sup_{t\ge 0}\|e^{A_vt}\|, \quad \kappa_v = \sqrt{n\frac{\lambda_{\max}(\bar{P}_v)}{\lambda_{\min}(P_v)}}, \quad \kappa_y = \sqrt{n\frac{\lambda_{\max}(\bar{P}_v)}{\lambda_{\min}(P_y)}}, \tag{3.1.4}$$

where $P_y \in \mathbb{R}^{p\times p}$ be a given positive definite matrix, and

$$\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m), \quad H = B_m(C_mB_m)^\dagger. \tag{3.1.5}$$

Moreover,

$$A_v = (\mathbb{I}_n - HC_m)A_m + K_vC_m \tag{3.1.6}$$

is assumed to be Hurwitz with a given $K_v \in \mathbb{R}^{n\times p}$ (such K_v always exists, see Remark 3.1.1 and Lemma 2.1.19), and $P_v \in \mathbb{R}^{n\times n}$ is the positive definite matrix which solves $A_v^\top P_v + P_vA_v = -Q$ for

$Q \succ \epsilon_q \mathbb{I}_n$ with a small positive number ϵ_q.

Let $D(s)$ be a $m \times m$ transfer function matrix chosen so that for all $\Omega \in \mathcal{C}_\Omega$

$$C(s) = \Omega D(s) \left(\mathbb{I}_m + \Omega D(s)\right)^{-1} \tag{3.1.7}$$

is stable and strictly proper, and $C(0) = \mathbb{I}_m$. Moreover, the choice of $D(s)$ ensures that for all $\Omega \in \mathcal{C}_\Omega$ there exists $\rho_r > 0$ such that

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_r - \rho_{ext} - \rho_{int}}{L_{\rho_r}\rho_r}, \tag{3.1.8}$$

where

$$\begin{aligned} \rho_{ext} &= \|H_0(s)C(s)\|_{\mathcal{L}_1} \|K_g r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} b_0, \quad \rho_{int} = (\kappa_m + \kappa_x)\rho_0, \\ \kappa_x &= \|H_1(s)\|_{\mathcal{L}_1} \kappa_y + (\|H_2(s)\|_{\mathcal{L}_1} + \left\|\bar{H}_2(s)\right\|_{\mathcal{L}_1})\kappa_v, \end{aligned} \tag{3.1.9}$$

with

$$\begin{aligned} H_0(s) =& (\mathbb{I}_n s - A_m)^{-1} B_m, \quad G(s) = H_0(s)(\mathbb{I}_m - C(s)), \\ H_1(s) =& H_0(s)C_1(s), \quad H_2(s) = H_0(s)C_2(s), \quad \bar{H}_2(s) = H_0(s)\bar{C}(s)K_m, \end{aligned} \tag{3.1.10}$$

and

$$C_1(s) = (s+\mu)C(s)B^\dagger H, \quad C_2(s) = C(s)B^\dagger H C_m A_m, \quad \bar{C}(s) = (\mathbb{I}_m - C(s))\Omega. \tag{3.1.11}$$

Notice that L_{ρ_r} satisfies (3.1.3) with d_{ρ_x} (given in Assumption 3.0.2) and

$$\rho_x = \rho_r + \bar{\gamma}. \tag{3.1.12}$$

Moreover, $\mu > 0$ is chosen to satisfy $\mu > \alpha$, where $\alpha = \frac{\alpha_1^* + \alpha_2^*}{2}$ solves

$$\begin{aligned} &(\alpha_1^*, \alpha_2^*) = \arg\min(\alpha_1 + \alpha_2), \\ &\quad \text{subject to} \quad \alpha_1 > 0, \ \alpha_2 > 0, \ \left(\frac{\epsilon_1}{\alpha_1} + \frac{\epsilon_2}{\alpha_2}\right) \le \epsilon_q, \end{aligned} \tag{3.1.13}$$

with $\epsilon_1 = \left\|\sqrt{P_y} C_m B_m K_m\right\|_2^2$ and $\epsilon_2 = \left\|\sqrt{P_y} C_m B_m\right\|_2^2 m L_{\rho_r}^2$.

Now, the adaptive control input $u(t) \in \mathbb{R}^m$ is defined as

$$u(s) = D(s)(K_g r(s) - \hat{\eta}_t(s) - K_m \hat{x}(s)) - K_m \hat{x}(s), \tag{3.1.14}$$

where $r(s)$ is the Laplace transform of a reference command $r(t) \in \mathbb{R}^m$, $K_g \in \mathbb{R}^{m\times m}$ is a known constant matrix, and $\hat{x}(t) = \hat{v}(t) + Hy(t)$; $\hat{v}(t) \in \mathbb{R}^n$ is provided by the state-output predictor, and

$\hat{\eta}_t(s)$ is the Laplace transform of

$$\hat{\eta}_t(t) = \hat{\Omega}(t)u(t) + \hat{\theta}(t)\|\hat{x}(t)\| + \hat{\sigma}(t), \tag{3.1.15}$$

where $\hat{\Omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ are given by the adaptive laws. The following state-output predictor is considered:

$$\begin{aligned}\dot{\hat{v}}(t) =& A_v\hat{x}_v - P_v^{-1}A_m^\top C_m^\top P_y\tilde{y}(t) - K_v y(t), \quad \hat{v}(0) = 0,\\ \dot{\hat{y}}(t) =& -\mu\tilde{y}(t) + C_m B_m(\hat{\eta}_t(t) + K_m\hat{x}(t)) + C_m A_m\hat{x}(t), \quad \hat{y}(0) = y_0\end{aligned} \tag{3.1.16}$$

where $y_0 = C_m x_0$ is assumed to be known, and $\tilde{y}(t) = \hat{y}(t) - y(t)$. The adaptive laws are defined using the projection operator:

$$\begin{aligned}\dot{\hat{\Omega}}(t) =& \Gamma_\Omega \text{Proj}(\hat{\Omega}(t), -e(t)u^\top(t)), \quad \hat{\Omega}(0) = \mathbb{I}_m,\\ \dot{\hat{\theta}}(t) =& \Gamma_\theta \text{Proj}(\hat{\theta}(t), -e(t)\|\hat{x}(t)\|), \quad \hat{\theta}(0) = 0,\\ \dot{\hat{\sigma}}(t) =& \Gamma_\sigma \text{Proj}(\hat{\sigma}(t), -e(t)), \quad \hat{\sigma}(0) = 0,\end{aligned} \tag{3.1.17}$$

where $e(t) = B_m^\top C_m^\top P_y\tilde{y}(t)$, $\text{Proj}(\cdot,\cdot)$ denotes the projection operator [21], and $\Gamma_\Omega > 0$, $\Gamma_\theta > 0$, $\Gamma_\sigma > 0$ are real constant adaptation gains. Figure 3.1 depicts the structure of the proposed controller.

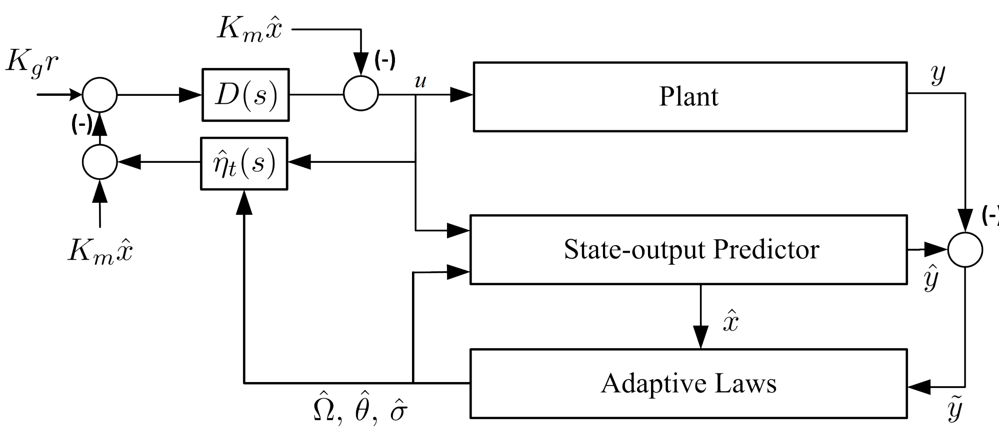

Figure 3.1: $\mathcal{L}_1$ output feedback control with state-feedback gain

Notice that $\bar{\gamma}$ can be chosen sufficiently small, so that $\rho_r \approx \rho_x$. Later, it will be shown that ρ_x characterize a positively invariant set of the closed-loop system.

3.1.2. *Stability and performance analysis*

In this section, the stability analysis of the proposed $\mathcal{L}_1$ adaptive output feedback controller is presented. First, we analyze the $\mathcal{L}_1$ reference system stability. Consider the following closed-loop reference system

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + B_m\Omega\left(u_{ref}(t) + K_m x_{ref}(t)\right) \tag{3.1.18}$$

$$+ B_m((\mathbb{I}_m - \Omega)K_m x_{ref}(t) + f(x_{ref}, t)), \tag{3.1.19}$$

$$y_{ref}(t) = C_m x_{ref}(t), \quad x_{ref}(0) = 0,$$

with

$$u_{ref}(s) = C_0(s)\left(K_g r(s) - \eta_{ref}(s) - \sigma(s)\right) - K_m x_{ref}(s), \tag{3.1.20}$$

where $x_{ref}(t) \in \mathbb{R}^n$, $y_{ref}(t) \in \mathbb{R}^p$ are the reference system state and output vectors, respectively, and

$$C_0(s) = D(s)(\mathbb{I}_m + \Omega D(s))^{-1}. \tag{3.1.21}$$

Moreover, $\eta_{ref}(s)$ and $\sigma(s)$ are the Laplace transform of

$$\eta_{ref}(t) = (\mathbb{I}_m - \Omega)K_m x_{ref}(t) + f(x_{ref}, t) - f(0, t), \quad \sigma(t) = f(0, t), \tag{3.1.22}$$

respectively. Notice that the system (3.1.18) and (3.1.20) compensates uncertainties within a filter bandwidth, defining the reference system of the proposed $\mathcal{L}_1$ control laws. Since the system is not implementable due to the unknowns signals, we use it only for analysis purposes. To prove the closed-loop stability, we first introduce a condition for the semi-global stability of the ideal reference system. Then, it will be shown that the difference between the closed-loop system and the ideal reference system is semi-globally attractive with arbitrarily small bounds, which guarantees the closed-loop stability.

Lemma 3.1.1. *Consider the closed-loop reference system given in* (3.1.18) *and* (3.1.20) *subject to design constraints given in* (3.1.2) - (3.1.13)*. Then, for each* $\Omega \in \mathcal{C}_\Omega$ *and each* $\tau > 0$ *one has*

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{rx}, \tag{3.1.23}$$

where

$$\rho_{rx} = \rho_r - \gamma_{x0}\rho_0, \quad \gamma_{x0} = \frac{\kappa_x + \kappa_m}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \tag{3.1.24}$$

with κ_m*,* κ_x*, and* $G(s)$ *are given in* (3.1.4), (3.1.9), *and* (3.1.10), *respectively. Moreover, the following holds: for each* $\Omega \in \mathcal{C}_\Omega$ *and each* $\tau > 0$

$$\left\|u_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{ru}, \tag{3.1.25}$$

where

$$\rho_{ru} = \|C_0(s)\|_{\mathcal{L}_1}\left(\|K_g r\|_{\mathcal{L}_\infty} + L_{\rho_r}\rho_{rx} + b_0\right) + \|K_m\|\rho_{rx}. \tag{3.1.26}$$

Proof. Substituting the control law given in (3.1.20) into (3.1.18) yields

$$\begin{aligned} x_{ref}(s) =& H_0(s)C(s)K_g r(s) + G(s)(\eta_{ref}(s) + \sigma(s)), \\ y_{ref}(s) =& C_m x_{ref}(s)\,, \end{aligned} \tag{3.1.27}$$

where $H_0(s)$ and $G(s)$ are given in (3.1.10); $\eta_{ref}(s)$, $\sigma(s)$ are the Laplace transform of the signals, $\eta_{ref}(t)$ and $\sigma(t)$ given in (3.1.22), respectively. Notice that Equation (3.1.8) implies that $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$, which in turn yields $\rho_{rx} > 0$.

Now, we prove Equation (3.1.23) using a contradiction argument. Suppose it is not true. Since $\|x_{ref}(0)\| = 0 < \rho_{rx}$, it follows from the system in (3.1.27) that there exists $\tau > 0$ such that $\|x_{ref}(\tau)\| = \rho_{rx}$, while $\|x_{ref}(t)\| < \rho_{rx}$ for $0 \leq t < \tau$. Therefore, the following must hold:

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} = \rho_{rx}. \tag{3.1.28}$$

Since $\rho_{rx} \leq \rho_r \leq \rho_x$, Assumption 3.0.2, along with (3.1.3) and (3.1.22), leads to

$$\left\|\eta_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq L_{\rho_r} \left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty}. \tag{3.1.29}$$

Therefore, combining (3.1.27) and (3.1.29) yields

$$\rho_{rx} = \left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_r - \gamma_{x_0}\rho_0, \tag{3.1.30}$$

where γ_{x_0} is given in (3.1.24). Since $\rho_{rx} = \rho_r - \gamma_{x_0}\rho_0$ holds in (3.1.24), Equation (3.1.30) contradicts (3.1.28), thus proving (3.1.23). Finally, combining (3.1.20), (3.1.23), and (3.1.29) concludes Equation (3.1.25). This completes the proof. □

Notice that Lemma 3.1.1 present semi-global stability of the $\mathcal{L}_1$ reference system, since the condition in (3.1.8) is always verified as the bandwidth of a low pass filter increases. Moreover, from the Laplace transform of the closed-loop reference system in (3.1.18) and (3.1.20), it follows that

$$(y_m - y_{ref})(s) = G_m(s)(K_g r(s) - \eta_{ref}(s) - \sigma(s)), \tag{3.1.31}$$

where $\eta_{ref}(s)$, $\sigma(s)$ are the Laplace transform of bounded signals (given in (3.1.22)), and $G_m(s) = M(s)\left(\mathbb{I}_m - C(s)\right)$ with $G_m(0) = 0$. Equation (3.1.31) implies that $y_{ref}(t)$ can closely approximate $y_m(t)$ with $C(s) \approx \mathbb{I}_m$. However, the high-bandwidth filter may result in loss of robustness to time delay [2]. Therefore, the choice of a filter gives a trade-off between performance and robustness.

Now, we analyze the performance of the proposed controller. To proceed, we introduce several variables of interests. Let

$$\begin{aligned}\gamma_{u_0} =&(\|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r} + \|K_m\|)\gamma_{x_0} + \|C_3(s)\|_{\mathcal{L}_1} \kappa_y + (\|C_4(s)\|_{\mathcal{L}_1} + \left\|\bar{C}_4(s)\right\|_{\mathcal{L}_1})\kappa_v),\\ \gamma_u =&(\|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r} + \|K_m\|)\gamma_x + \frac{\|C_3(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_y)}} + \frac{\|C_4(s)\|_{\mathcal{L}_1} + \left\|\bar{C}_4(s)\right\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_v)}},\\ \gamma_x =&\frac{\lambda_{\min}(P_y)^{-\frac{1}{2}} \|H_1(s)\|_{\mathcal{L}_1} + \lambda_{\min}(P_v)^{-\frac{1}{2}}(\|H_2(s)\|_{\mathcal{L}_1} + \left\|\bar{H}_2(s)\right\|_{\mathcal{L}_1})}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}},\end{aligned} \tag{3.1.32}$$

where $G(s)$, $H_1(s)$, $H_2(s)$, $\bar{H}_2(s)$ are given in (3.1.10), and

$$C_3(s) = (s+\mu)C_0(s)B_m^{\dagger}H, \quad C_4(s) = C_0(s)B_m^{\dagger}HC_mA_m, \quad \bar{C}_4(s) = (\mathbb{I}_m - D(s)\Omega)^{-1}K_m. \tag{3.1.33}$$

Let $\epsilon_r > 0$ satisfy

$$\gamma_x\epsilon_r < \bar{\gamma}, \quad \gamma_x\epsilon_r < \bar{\gamma}, \quad \forall\Omega \in \mathcal{C}_\Omega. \tag{3.1.34}$$

Finally, define

$$\rho_u = \rho_{ru} + \rho_{du}, \quad \rho_{dx} = \gamma_{x0}\rho_0 + \bar{\gamma}, \quad \rho_{du} = \gamma_{u0}\rho_0 + \bar{\gamma}, \tag{3.1.35}$$

where ρ_{ru} is given in (3.1.26).

Lemma 3.1.2. *Let $\tau > 0$. Suppose $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u$. Then, the nonlinear function $f(x,t)$ in* (3.1.1) *can be rewritten as*

$$f(x,t) = \theta(t)\|x(t)\| + \sigma(t), \tag{3.1.36}$$

where $\theta(t) \in \mathbb{R}^m$ and $\sigma(t) \in \mathbb{R}^m$ satisfy

$$\|\theta(t)\| \leq d_{\rho_x}, \quad \|\dot{\theta}(t)\| \leq l_\theta, \quad \|\sigma(t)\| \leq b_0, \quad \|\dot{\sigma}(t)\| \leq l_\sigma, \tag{3.1.37}$$

for all $0 \leq t \leq \tau$; l_θ and l_σ are computable bounds.

Proof. Since $\|x_\tau\|_{\mathcal{L}_\infty} < \rho_x$ and $\|u_\tau\|_{\mathcal{L}_\infty} < \rho_u$ holds from the hypothesis, Equation (3.1.1) implies that $\|\dot{x}_\tau\|_{\mathcal{L}_\infty}$ is finite. Therefore, the proof of Lemma 3.1.2 follows from Lemma 2.3.1. □

Lemma 3.1.3. *Let $v(t) = (\mathbb{I}_n - HC_m)\,x(t)$. Given the system* (3.1.1)*, the control laws defined in* (3.1.16) - (3.1.17) *and with the design constraints, the estimation errors $\tilde{v}(t) = \hat{v}(t) - v(t)$ and $\tilde{y}(t) = \hat{y}(t) - y(t)$ have the following bounds for all $t \geq 0$:*

$$\|\tilde{v}(t)\| \leq \frac{\rho(t)}{\sqrt{\lambda_{\min}(P_v)}}, \quad \|\tilde{y}(t)\| \leq \frac{\rho(t)}{\sqrt{\lambda_{\min}(P_y)}}, \tag{3.1.38}$$

where

$$\rho(t) = \sqrt{\left(x_0^\top \bar{P}_v x_0 - \frac{\theta_1 - \theta_0}{\Gamma}\right) e^{-\lambda_1 t} + \frac{\theta_1}{\Gamma}}\,, \tag{3.1.39}$$

with $\Gamma = \min\left(\Gamma_\Omega, \Gamma_\theta, \Gamma_\sigma\right)$*, and*

$$\begin{aligned}
&\theta_1 = \theta_0 + \frac{4m}{\lambda_1}\left(d_{\rho_x} l_\theta + b_0 l_\sigma\right),\\
&\theta_0 = 4 \max_{\Omega\in\mathcal{C}_\Omega, t\geq 0}\left(tr\left(\Omega^\top\Omega\right) + m d_{\rho_x}^2 + m b_0^2\right),\\
&\lambda_1 = \left(\max\left(\frac{\lambda_{\max}(P_v)}{\lambda_{\min}(Q_v)}, \frac{\lambda_{\max}(P_y)}{2(\mu-\alpha)\lambda_{\min}(P_y)}\right)\right)^{-1}, \quad Q_v = Q - \epsilon_q \mathbb{I}_n,
\end{aligned} \tag{3.1.40}$$

Proof. Pre-multiplying both sides of (3.1.1) by $(\mathbb{I}_n - HC_m)$ leads to

$$\begin{aligned}
\dot{v}(t) =& A_H v(t) + A_H H y(t),\\
\dot{y}(t) =& C_m A_m v(t) + C A_m H y(t) + C_m B_m K_m x(t) + C_m B_m\left(\Omega u(t) + f(x,t)\right),\\
x(t) =& v(t) + H y(t), \quad v(0) = v_0, \quad y(0) = y_0,
\end{aligned} \tag{3.1.41}$$

where $A_H = (\mathbb{I}_n - HC_m)A_m$, $y_0 = C_m x_0$, and v_0 is given in (3.1.40). Let τ be a positive constant. Since $C_m\hat{v}(t) - (\mathbb{I}_p - C_m H)y(t) = C_m\tilde{v}(t)$ holds from $C_m v(t) = C_m(\mathbb{I}_n - HC_m)x(t)$, subtracting (3.1.41) from (3.1.16) yields the following error dynamics:

$$\begin{aligned}
\dot{\tilde{v}}(t) =& A_v\tilde{v}(t) - P_v^{-1}A_m^\top C_m^\top P_y\tilde{y}(t), \quad \tilde{v}(0) = -v_0,\\
\dot{\tilde{y}}(t) =& -\mu\tilde{y}(t) + C_m B_m\tilde{\eta}_\sigma(t) + C_m A_m\tilde{v}(t), \quad \tilde{y}(0) = 0,
\end{aligned} \tag{3.1.42}$$

where $A_v = A_H + K_v C_m$ is Hurwitz, and

$$\tilde{\eta}_\sigma(t) = \tilde{\Omega}(t)u(t) + \hat{\theta}(t)\|\hat{x}(t)\| + \hat{\sigma}(t) - \left(f(x,t) - f(0,t)\right) - \sigma(t), \tag{3.1.43}$$

with $\tilde{\Omega}(t) = \hat{\Omega}(t) - \Omega$ and $\sigma(t) = f(0,t)$. Since $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u$ holds from the hypothesis, Equation (3.1.43), along with (3.1.36), can be rewritten as $\tilde{\eta}_\sigma(t) = \tilde{\eta}_t(t) + \phi(t)$, where

$$\tilde{\eta}_t(t) = \tilde{\Omega}(t)u(t) + \tilde{\theta}(t)\|\hat{x}(t)\| + \tilde{\sigma}(t), \tag{3.1.44}$$

and $\phi(t) = \phi_1(t) + \phi_2(t)$ with $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$, $\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t)$, and

$$\phi_1(t) = K_m\left(\hat{x}(t) - x(t)\right) = K_m\tilde{v}(t), \quad \phi_2(t) = \theta(t)\left(\|\hat{x}(t)\| - \|x(t)\|\right). \tag{3.1.45}$$

Now, consider the Lyapunov function

$$V(t) =\tilde{v}^\top(t)P_v\tilde{v}(t) + \tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{tr(\tilde{\Omega}^\top(t)\tilde{\Omega}(t))}{\Gamma_\Omega} + \frac{\tilde{\theta}^\top(t)\tilde{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\tilde{\sigma}(t)}{\Gamma_\sigma}, \tag{3.1.46}$$

where $P_y \in \mathbb{R}^{p\times p}$ is chosen to be positive definite and $P_v \succ 0$ is the solution of $A_v^\top P_v + P_v A_v = -Q$ for a given $Q \in \mathbb{R}^{n\times n}$; $Q \succ \epsilon_q \mathbb{I}_n$. By combining (3.1.17) and (3.1.42), the derivative of (3.1.46) is given by

$$\dot{V}(t) \leq -\tilde{v}^\top(t)Q\tilde{v}(t) - 2\mu\tilde{y}^\top(t)P_y\tilde{y}(t) - \frac{2\tilde{\theta}^\top\dot{\theta}}{\Gamma_\theta} - \frac{2\tilde{\sigma}^\top\dot{\sigma}}{\Gamma_\sigma} + 2\tilde{y}^\top(t)P_yC_mB_m\phi(t). \tag{3.1.47}$$

Notice that the completion of squares yields

$$\begin{aligned} 2\tilde{y}^\top(t)P_yC_mB_m\phi_1(t) &\leq \alpha_1\tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{1}{\alpha_1}\left\|\sqrt{P_y}C_mB_mK_m\right\|_2^2\|\tilde{v}(t)\|_2^2, \\ 2\tilde{y}^\top(t)P_yC_mB_m\phi_2(t) &\leq \alpha_2\tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{1}{\alpha_2}\left\|\sqrt{P_y}C_mB_m\right\|_2^2\|\phi_2(t)\|_2^2, \end{aligned} \tag{3.1.48}$$

where α_1, α_2 are arbitrary positive constants. By using the fact that $d_{\rho_x} < L_{\rho_r}$, from (3.1.48) and (3.1.37) it follows that

$$\|\phi_2(t)\|_2 \leq \sqrt{m}L_{\rho_r}\|\tilde{v}(t)\|_2, \quad \forall t \geq 0. \tag{3.1.49}$$

Choose $\alpha = \alpha_1^* + \alpha_2^*$ for the given (α_1^*, α_2^*) in (3.1.13). By combining (3.1.47) - (3.1.49), the following holds

$$\dot{V}(t) \leq -\tilde{v}^\top(t)Q_v\tilde{v}(t) - 2(\mu-\alpha)\tilde{y}^\top(t)P_y\tilde{y}(t) - \frac{2\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} - \frac{2\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma}, \tag{3.1.50}$$

where $Q_v = Q - \epsilon_q\mathbb{I}_n \succ 0$. Notice that μ was chosen to ensure $\mu - \alpha > 0$. Furthermore, the bounds given in (3.1.37), together with (3.1.2), lead to

$$\frac{\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma} \leq \frac{\theta_1 - \theta_0}{2\Gamma}\lambda_1, \tag{3.1.51}$$

where $\Gamma = \min(\Gamma_\Omega, \Gamma_\Theta, \Gamma_\sigma)$. Combining (3.1.50) and (3.1.51) gives

$$\dot{V}(t) \leq -\tilde{v}^\top(t)Q_v\tilde{v}(t) - 2(\mu-\alpha)\tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{\theta_1 - \theta_0}{\Gamma}\lambda_1, \tag{3.1.52}$$

where θ_0, θ_1, and λ_1 are give in (3.1.40). The projection operator in (3.1.17) ensures

$$\max_{\Omega\in\mathcal{C}_\Omega}\left(\frac{tr(\tilde{\Omega}^\top(t)\tilde{\Omega}(t))}{\Gamma_\Omega} + \frac{\tilde{\theta}^\top(t)\tilde{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\tilde{\sigma}(t)}{\Gamma_\sigma}\right) \leq \frac{\theta_0}{\Gamma}, \quad 0 \leq t \leq \tau, \tag{3.1.53}$$

which leads to

$$V(t) \leq \tilde{v}^\top(t) P_v \tilde{v}(t) + \tilde{y}^\top(t) P_y \tilde{y}(t) + \frac{\theta_1}{\Gamma} - \frac{1}{\lambda_1} \frac{\theta_1 - \theta_0}{\Gamma} \lambda_1, \quad \forall t \geq 0. \tag{3.1.54}$$

Notice that

$$\tilde{v}^\top(t) P_v \tilde{v}(t) + \tilde{y}^\top(t) P_y \tilde{y}(t) \leq \frac{1}{\lambda_1} \left(\tilde{v}^\top(t) Q_v \tilde{v}(t) + 2(\mu - \alpha) \tilde{y}^\top(t) P_y \tilde{y}(t) \right), \quad 0 \leq t \leq \tau. \tag{3.1.55}$$

From (3.1.54) and (3.1.55) it follows that

$$-\tilde{v}^\top(t) Q_v \tilde{v}(t) - 2(\mu - \alpha) \tilde{y}^\top(t) P_y \tilde{y}(t) \leq -\lambda_1 \left(V(t) - \frac{\theta_1}{\Gamma} \right) - \frac{\theta_1 - \theta_0}{\Gamma} \lambda_1, \tag{3.1.56}$$

which, along with (3.1.52), yields

$$\dot{V}(t) \leq -\lambda_1 \left(V(t) - \frac{\theta_1}{\Gamma} \right), \quad \forall t \geq 0. \tag{3.1.57}$$

Gronwall-Bellman inequality further leads to

$$\sqrt{V(t)} \leq \rho_v(t, t_0), \quad 0 \leq t_0 \leq t \leq \tau, \tag{3.1.58}$$

where

$$\rho_v(t, t_0) = \sqrt{\left(V(t_0) - \frac{\theta_1}{\Gamma} \right) e^{-\lambda_1 (t - t_0)} + \frac{\theta_1}{\Gamma}}. \tag{3.1.59}$$

Finally, using $V(0) \leq v_0^\top P_v v_0 + \frac{\theta_0}{\Gamma}$, $\|\tilde{y}(t)\| \leq \frac{\rho_v(t,0)}{\sqrt{\lambda_{\min}(P_y)}}$, and $\|\tilde{v}(t)\| \leq \frac{\rho_v(t,0)}{\sqrt{\lambda_{\min}(P_v)}}$ concludes the upper bounds in (3.1.38), which completes the proof. □

The upper bounds in (3.1.38) depend on the initial condition through $v_0 = (\mathbb{I}_n - HC_m)x_0$, regardless of y_0. The steady-state bounds can be made arbitrary small by high adaptation gains. Notice that the effect due to the non-zero initial condition depends on the choice of K_v and μ (with Q_v and P_y). Now, we present the results on the transient and steady-state performance.

Theorem 3.1.1. *Consider the closed-loop system with $\mathcal{L}_1$ adaptive output feedback controller defined via* (3.1.14) – (3.1.17)*, subject to the design constraints in* (3.1.2) - (3.1.13)*. Suppose the adaptation gains are chosen sufficiently high to satisfy*

$$\Gamma > \frac{\theta_1}{\epsilon_\gamma}, \tag{3.1.60}$$

where $\epsilon_\gamma > 0$ satisfies (3.1.34)*, $\Gamma = \min(\Gamma_\Omega, \Gamma_\theta, \Gamma_\sigma)$, and θ_1 is given in* (3.1.40)*. Then, the following upper bounds holds:*

$$\|\tilde{x}_{ref}\|_{\mathcal{L}_\infty} < \rho_{dx}, \quad \|\tilde{u}_{ref}\|_{\mathcal{L}_\infty} < \rho_{du}, \tag{3.1.61}$$

and

$$\|x\|_{\mathcal{L}_\infty} \leq \rho_x, \quad \|u\|_{\mathcal{L}_\infty} \leq \rho_u. \tag{3.1.62}$$

Moreover, for each $\Omega \in \mathcal{C}_\Omega$, there exist constants $\gamma_{dx} > 0$, $\gamma_{dy} > 0$, and strictly decreasing functions $\upsilon_{dx}(t)$, $\upsilon_{dy}(t)$ such that for all $t \geq 0$

$$\|x_{ref}(t) - x(t)\| \leq \upsilon_{dx}(t)\|v_0\| + \frac{\gamma_{dx}}{\sqrt{\Gamma}}, \quad \|y_{ref}(t) - y(t)\| \leq \upsilon_{dy}(t)\|v_0\| + \frac{\gamma_{dy}}{\sqrt{\Gamma}}, \tag{3.1.63}$$

Proof. Define

$$\eta(t) = (\mathbb{I}_m - \Omega)K_m x(t) + f(x,t) - f(0,t), \tag{3.1.64}$$

and $\tilde{\eta}_{ref}(t) = \eta_{ref}(t) - \eta(t)$, where $\eta_{ref}(t)$ is given in (3.1.22). Let $\tilde{x}_{ref}(t) = x_{ref}(t) - x(t)$, $\tilde{u}_{ref}(t) = u_{ref}(t) - u(t)$, and $\tilde{y}_{ref}(t) = y_{ref}(t) - y(t)$. First, we prove (3.1.61) using a contradiction argument. Suppose it is not true. From (3.1.8) it follows that $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$. Using the definition of ρ_{dx} in (3.1.35) one can obtain $\rho_{dx} > \rho_0$, which leads to $\|\tilde{x}_{ref}(0)\| < \rho_{dx}$. Moreover, $\|\tilde{u}_{ref}(0)\| = 0 < \rho_0$ holds. From continuity, there exists $\tau > 0$ such that

$$\|\tilde{x}_{ref}(\tau)\| = \rho_{dx}, \quad \text{or} \quad \|\tilde{u}_{ref}(\tau)\| = \rho_{du},$$

while

$$\|\tilde{x}_{ref}(t)\| < \rho_{dx}, \qquad \|\tilde{u}_{ref}(t)\| < \rho_{du}, \quad 0 \leq t \leq \tau,$$

where ρ_{du} is given in (3.1.35). This leads to

$$\left\|\tilde{x}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{dx}, \quad \left\|\tilde{u}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{du}. \tag{3.1.65}$$

From (3.1.35) it follows that $\rho_{dx} = \rho_x - \rho_{rx}$ and $\rho_{du} = \rho_u - \rho_{ru}$. Since Lemma 3.1.1 holds, Equation (3.1.65) yields

$$\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x, \quad \|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u. \tag{3.1.66}$$

Since $(d_{\rho_x} + c_{\bar{\Omega}}) < L_{\rho_r}$, Assumption 3.0.2 along with (3.1.66) yields

$$\|\tilde{\eta}_{ref}(t)\| \leq L_{\rho_r}\|\tilde{x}_{ref}(t)\|, \quad 0 \leq t \leq \tau. \tag{3.1.67}$$

Moreover, from Lemma 3.1.2 and Equation (3.1.15) it follows that

$$\hat{\eta}_t(t) + K_m\hat{x}(t) = \Omega(u(t) + K_m x(t)) + \tilde{\eta}_t(t) + \phi(t) + \eta(t) + \sigma(t), \tag{3.1.68}$$

where $\tilde{\eta}_t(t)$, $\eta(t)$ are given in (3.1.44) and (3.1.64), respectively, $\sigma(t) = f(0,t)$, and $\phi(t) = \phi_1(t) + \phi_2(t)$; $\phi_1(t)$ and $\phi_2(t)$ are defined in (3.1.45). By substituting (3.1.68) into the control law (3.1.14), it follows that

$$u(s) = C_0(s)\left(K_g r(s) - \eta(s) - \sigma(s) - \tilde{\eta}_t(s) - \phi(s)\right) - \bar{C}_0(s)\phi_1(s) - K_m x(s), \tag{3.1.69}$$

where $\bar{C}_0(s) = (\mathbb{I}_m + D(s)\Omega)^{-1}$. Further, by substituting (3.1.69) into (3.1.1), the Laplace transform of the closed-loop system is written as

$$\begin{aligned} x(s) =& H_0(s) r_f(s) + G(s)(\eta(s) + \sigma(s)) + x_{in}(s) \\ & - H_0(s)C(s)(\tilde{\eta}_t(s) + \phi(s)) - H_0(s)\bar{C}(s)\phi_1(s), \\ y(s) =& C_m x(s)\,, \end{aligned} \tag{3.1.70}$$

where $x_{in}(s) = (s\mathbb{I}_n - A_m)^{-1}x_0$, $r_f(s) = C(s)K_g r(s)$, and $\bar{C}(s) = \Omega\bar{C}_0(s)$; $C(s)$ is given in (3.1.7), and $G(s)$, $H_0(s)$ are defined in (3.1.10). Notice that $(\mathbb{I}_m - C(s)) = (I + \Omega D(s))^{-1}$ leads to

$$\bar{C}(s) = (\mathbb{I}_m - C(s))\Omega. \tag{3.1.71}$$

Let $\tilde{x}_{ref}(t) = x_{ref}(t) - x(t)$, $\tilde{\eta}_{ref}(t) = \eta_{ref}(t) - \eta(t)$, and $\tilde{y}_{ref}(t) = y_{ref}(t) - y(t)$. Now, by subtracting (3.1.70) from (3.1.27), together with (3.1.71), it follows that

$$\begin{aligned} \tilde{x}_{ref}(s) =& G(s)\left(\tilde{\eta}_{ref}(s) + \Omega\phi_1(s)\right) + H_0(s)\phi_f(s) - x_{in}(s), \\ \tilde{y}_{ref}(s) =& C_m \tilde{x}_{ref}(s), \end{aligned} \tag{3.1.72}$$

where $\phi_f(s) = C(s)\left(\tilde{\eta}_t(s) + \phi(s)\right)$. Moreover, from (3.1.20) and (3.1.69) one has

$$\tilde{u}_{ref}(s) = -C_0(s)\tilde{\eta}_{ref}(s) - K_m\tilde{x}_{ref}(s) + \bar{C}_0(s)\phi_1(s) + \Omega^{-1}\phi_f(s). \tag{3.1.73}$$

Notice that Equation (3.1.3), together with Assumption 3.0.2, yields

$$\|\tilde{\eta}_{ref}(t)\| \leq \|(\mathbb{I}_m - \Omega)K_m\|\|\tilde{x}_{ref}(t)\| + d_{\rho_x}\|\tilde{x}_{ref}(t)\| < L_{\rho_r}\|\tilde{x}_{ref}(t)\|, \tag{3.1.74}$$

for $0 \leq t \leq \tau$. The Laplace transform of (3.1.42) implies

$$\phi_f(s) = C(s)\left(\tilde{\eta}_t(s) + \phi(s)\right) = C_1(s)\tilde{y}(s) - C_2(s)\tilde{v}(s), \tag{3.1.75}$$

where $C_1(s)$ and $C_2(s)$ are given in (3.1.11). Now, combining (3.1.72) - (3.1.75), together with Lemma 3.1.3, leads to

$$\left\|\tilde{u}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \gamma_{u_0}\rho_0 + \gamma_u\sqrt{\frac{\theta_1}{\Gamma}}, \quad \left\|\tilde{x}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \gamma_{x_0}\rho_0 + \gamma_x\sqrt{\frac{\theta_1}{\Gamma}}, \tag{3.1.76}$$

where γ_{x_0}, $\{\gamma_{u_0}, \gamma_u, \gamma_x\}$ are given in (3.1.24), and (3.1.32), respectively. From the hypothesis in (3.1.60), Equation (3.1.76) further yields

$$\left\|\tilde{u}_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{dx}, \quad \left\|\tilde{x}_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{du}, \tag{3.1.77}$$

which shows the clear contradiction to (3.1.65), thus proving (3.1.61). Moreover, by applying the triangular inequality on $\|\tilde{x}_{ref}\|$ and $\|\tilde{u}_{ref}\|$, Equation (3.1.62) follows

Next, we prove Equation (3.1.63). Notice that $\phi_f(t)$ is a bounded signal from the fact that $C_1(s)$, $C_1(s)$ are stable and proper. Since $\Omega\phi_1(t) = \Omega K_m \tilde{v}(t)$ holds in (3.1.45), $\Omega\phi_1(t)$ is also a bounded signal. Let $C(s) = C_f(s\mathbb{I}_m - A_f)^{-1}B_f$ in (3.1.27), where $\{A_f \in \mathbb{R}^{n_f\times n_f}, B_f \in \mathbb{R}^{n_f\times m}, C_f \in \mathbb{R}^{m\times m}\}$ is a minimal realization of $C(s)$. Then, a state-space realization of (3.1.72) takes the form:

$$\begin{aligned}\dot{\tilde{x}}_c(t) &= A_c\tilde{x}_c(t) + B_c(\tilde{\eta}_{ref}(t) + \Omega\phi_1(t)) + B_r\phi_f(t),\\ \tilde{x}_{ref}(t) &= C_c\tilde{x}_c(t), \quad \tilde{x}_c(0) = [-x_0^\top, 0]^\top,\end{aligned} \tag{3.1.78}$$

with

$$A_c = \begin{bmatrix} A_m & B_mC_f \\ 0 & A_f \end{bmatrix}, \quad B_c = \begin{bmatrix} B_m \\ -B_f \end{bmatrix}, \quad B_r = \begin{bmatrix} B_m \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} \mathbb{I}_n & 0 \end{bmatrix}, \tag{3.1.79}$$

where $\tilde{x}_c(t) = [\tilde{x}_{ref}^\top(t), x_f^\top(t)]^\top \in \mathbb{R}^{n_c\times n_c}; n_c = n + n_f$, and $x_f(t) \in \mathbb{R}^{n_f}$ is some internal state vector in (3.1.72) Notice that $G(s) = C_c(s\mathbb{I}_{n_c} - A_c)^{-1}B_c = H_0(s)(\mathbb{I}_m - C(s))$ holds. In addition, $\{A_c, B_c, C_c\}$ is detectable and stabilizable, since pole-zero cancellations of $G(s)$ happen on $\mathbb{C}^-$, if any. Let $t_m \geq 0$. For a given initial condition $x_c(t_m)$ and $t \geq t_0$, the solution of (3.1.78) is given by

$$\begin{aligned}\tilde{x}_c(t) =& e^{A_c(t-t_m)}\tilde{x}_c(t_m) + \int_{t_m}^t e^{A_c(t-\tau)}B_r\phi_f(\tau)d\tau \\ &+ \int_{t_m}^t e^{A_c(t-\tau)}B_c(\tilde{\eta}_{ref}(\tau) + \Omega_1\phi(t))d\tau, \quad t \geq t_m \geq 0,\end{aligned} \tag{3.1.80}$$

Now, using the continuity of the $\mathcal{L}_1$-norm, one may take a sufficiently small $\lambda_0 > 0$ such that $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1} < 1/L$. Define $A_{\lambda_0} = A_c + \lambda_0\mathbb{I}_{n_c}$, $\bar{x}_c(t) = e^{\lambda_0(t-t_m)}\tilde{x}_c(t)$, $\bar{\phi}_f(t) = e^{\lambda_0(t-t_m)}\phi_f(t)$, $\bar{\eta}_{ref}(t) = e^{\lambda_0(t-t_m)}\tilde{\eta}_{ref}(t)$, and $\bar{\phi}_1(t) = e^{\lambda_0(t-t_m)}\Omega\phi_1(t)$. Multiplying both sides of (3.1.80) by $e^{\lambda_0(t-t_0)}C_c$ yields

$$\begin{aligned}\bar{x}_{ref}(t) =& C_ce^{A_{\lambda_0}(t-t_m)}\tilde{x}_c(t_m) + \int_{t_m}^t C_ce^{A_{\lambda_0}(t-\tau)}B_r\bar{\phi}_f(\tau)d\tau \\ &+ \int_{t_m}^t C_ce^{A_{\lambda_0}(t-\tau)}B_c(\bar{\eta}_{ref}(\tau) + \bar{\phi}_1(\tau))d\tau,\end{aligned} \tag{3.1.81}$$

where $\bar{x}_{ref}(t) = e^{\lambda_0(t-t_m)}\tilde{x}_{ref}(t)$. Notice that A_{λ_0} is Hurwitz by $\|G(s-\lambda_0)\|_{\mathcal{L}_1} < \infty$. Since Equation (3.1.74) holds for all $t \geq 0$, by using the fact that $C_c = [\mathbb{I}_{n_c}, 0]$, from (3.1.81) and (3.1.74) it follows that for all $t \geq t_0$

$$\|\bar{x}_{ref}\|_{\mathcal{L}_\infty[t_0,t]} \leq \frac{\beta_0}{1-\beta_1L}\|\tilde{x}_c(t_0)\| + \frac{\beta_1}{1-\beta_1L}\|\bar{\phi}_1\|_{\mathcal{L}_\infty[t_0,t]} + \frac{\beta_2}{1-\beta_1L}\|\bar{\phi}_f\|_{\mathcal{L}_\infty[t_0,t]}, \tag{3.1.82}$$

where $\beta_0 = \sup_{0\leq\tau}\|e^{A_{\lambda_0}\tau}\|$, $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1}$, and $\beta_2 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}B_r\right\|_{\mathcal{L}_1}$. Now, multiplying both sides of (3.1.80) by $e^{\lambda_0(t-t_0)}$, and combining (3.1.74) and (3.1.82), one can obtain

$$\|\bar{x}_c(t)\| \leq \kappa_0\|\tilde{x}_c(t_m)\| + \kappa_1\left\|\bar{\phi}_f\right\|_{\mathcal{L}_\infty[t_m,t]} + \kappa_2\left\|\bar{\phi}_1\right\|_{\mathcal{L}_\infty[t_m,t]}, \tag{3.1.83}$$

where

$$\begin{aligned}\kappa_0 =&\beta_0(1+\frac{L\beta_3}{1-\beta_1 L}), \quad \kappa_1 = \beta_2(1+\frac{L\beta_3}{1-\beta_1 L}),\\ \kappa_2 =&\frac{\beta_3}{1-\beta_1 L}, \quad \beta_3 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}B_c\right\|_{\mathcal{L}_1}.\end{aligned} \tag{3.1.84}$$

Since $\|\bar{x}_c(t)\| = e^{\lambda_0(t-t_m)}\|\tilde{x}_c(t)\|$ holds, Equation (3.1.83) can be rewritten by

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)}\|\tilde{x}_c(t_0)\| + \kappa_1\|\phi_f\|_{\mathcal{L}_\infty[t_m,t]} + \kappa_2\|\Omega\phi_1\|_{\mathcal{L}_\infty[t_m,t]}, \quad \forall t \geq t_m \geq 0. \tag{3.1.85}$$

Since $\|\tilde{y}(t)\| \leq \frac{\rho_v(t,t_m)}{\sqrt{\lambda_{\min}(P_y)}}$, and $\|\tilde{v}(t)\| \leq \frac{\rho_v(t,t_m)}{\sqrt{\lambda_{\min}(P_v)}}$, substituting (3.1.45) and (3.1.75) into (3.1.85) yields

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)}\|\tilde{x}_c(t_m)\| + \gamma_1 \sup_{t_m\leq\tau\leq t}\rho_v(\tau,t_m), \tag{3.1.86}$$

where $\rho_v(\tau,t_m)$ is given in (3.1.59), and

$$\gamma_1 = \frac{\kappa_1\left\|C_1(s)\right\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_y)}} + \frac{\kappa_1\left\|C_2(s)\right\|_{\mathcal{L}_1} + \kappa_2\|\Omega K_m\|}{\sqrt{\lambda_{\min}(P_v)}}.$$

Moreover, notice that

$$\sup_{t_m\leq\tau\leq t}\rho_v(\tau,t_m) \leq \sqrt{V(t_m)} + \sqrt{\frac{\theta_1}{\Gamma}}, \tag{3.1.87}$$

where $V(t)$ is given in (3.1.46), and θ_1, Γ are defined in (3.1.40). Let $t_m = \frac{t}{2}$. combining (3.1.58), (3.1.86) and (3.1.87) leads to

$$\|\tilde{x}_c(t)\| \leq \upsilon_x(t)\|v_0\| + \frac{\gamma_x}{\sqrt{\Gamma}}, \tag{3.1.88}$$

where

$$\begin{aligned}\upsilon_x(t) &= \gamma_1\sqrt{n\lambda_{\max}(P_v)}(\kappa_0 e^{-\frac{\lambda_0}{2}t} + e^{-\frac{\lambda_1}{4}t}) + \kappa_0^2 e^{-\lambda_0 t},\\ \gamma_x &= \kappa_0\gamma_1\sqrt{\theta_0} + 3\sqrt{\theta_1},\end{aligned}$$

with θ_0 given in (3.1.40). Since $\|\tilde{x}_{ref}(t)\| \leq \|\tilde{x}_c(t)\|$, letting $\upsilon_y(t) = \|C_m\|\rho_x(t)$ and $\gamma_y = \|C_m\|\gamma_x$ concludes (3.1.63). This completes the proof. □

Notice that high adaptation gains produce arbitrarily small steady-state bounds in (3.1.63), and the transient bounds to unknown initial conditions are guaranteed by strictly decreasing functions. Since the transient bounds are independent of adaptation gains, adaptation gains can be made arbitrarily high without undesirable transient behaviors. Moreover, combining Lemma 3.1.1 and Theorem 3.1.1 proves the semi-global stability of the closed-loop system.

3.1.3. *Application to Reference Tracking Control*

In this section, we introduce adaptive integral control for a reference tracking problem using the proposed method. Consider the open-loop system

$$\begin{aligned} \dot{x}_p(t) &= A_0 x_p(t) + B_p\left(\Omega u + f_p(x_p, t)\right), \\ y_p(t) &= C_p x_p(t), \quad x_p(0) = 0, \end{aligned} \tag{3.1.89}$$

where $x_p(t) \in \mathbb{R}^{n_p}$, $u(t) \in \mathbb{R}^m$, $y_p(t) \in \mathbb{R}^{p_p}$ are the state, input and measurable output vectors, respectively, with $p_p \geq m$. Moreover, $A_0 \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, $C_p \in \mathbb{R}^{p_p \times n_p}$ are known matrices; (A_0, B_p) is a controllable pair, (A_0, C_p) is an observable pair, and $\text{rank}(C_p B_p) = m$. $\Omega \in \mathbb{R}^{m \times m}$ is an unknown constant input gain, and $f_p : \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}^m$ represents matched uncertainties. Let $z(t) = T_{z_p} y_p(t)$ be a performance output, where $T_{z_p} \in \mathbb{R}^{m \times n_p}$ is a given matrix.

For the purposes of zero-tracking error, let us define the tracking error as $e_z(t) = r_{cmd}(t) - z(t)$. Then, the augmented plant with the tracking error can be written as

$$\begin{aligned} \dot{x}(t) =& A_0 x(t) + B_m\left(\Omega u + f(x,t)\right) + B_z r_{cmd}(t), \\ y(t) =& C_m x(t), \\ z(t) =& T_z C_m x(t), \quad x(0) = x_0, \end{aligned} \tag{3.1.90}$$

where $f(x,t) = f_p([\mathbb{I}_{n_p}, 0]x, t)$, $x(t) = [x_p^\top(t), e_z^\top(t)]^\top$ is the augmented state, $y(t) = [y_p^\top(t), e_z^\top(t)]^\top$ is the augmented output, and $x_0 = [0, e_z^\top(0)]^\top$; $A_0 \in \mathbb{R}^{n \times n}$, $B_m \in \mathbb{R}^{n \times m}$, $B_z \in \mathbb{R}^{n \times m}$, $C_m \in \mathbb{R}^{p \times m}$, $T_z \in \mathbb{R}^{m \times p}$ are defined as

$$\begin{aligned} A_0 &= \begin{bmatrix} A_p & 0 \\ -T_{z_p} C_p & 0 \end{bmatrix}, \quad B_m = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} 0 \\ \mathbb{I}_m \end{bmatrix}, \\ C_m &= \begin{bmatrix} C_p & 0 \\ 0 & \mathbb{I}_m \end{bmatrix}, \quad T_z = \left[T_{z_p}, 0\right], \end{aligned} \tag{3.1.91}$$

with $n = n_p + m$, and $p = p_p + m$. To proceed, we assume the system in (3.1.90) satisfies Assumptions 3.0.1, 3.0.2, 3.1.1, and 3.1.2 with $M(s) = C_m(s\mathbb{I}_n - A_m)^{-1} B_m$.

Remark 3.1.2. *Notice that from the definitions of B_m and C_m in* (3.1.91) $\text{rank}(C_p B_p) = m$ *implies* $\text{rank}(C_m B_m) = m$.

The control objective is to construct adaptive control laws such that the performance output tracks the given reference command $r_{cmd}(t) \in \mathbb{R}^m$. The control law is given in (3.1.14) with $K_g = 0$, and the adaptive law is defined in (3.1.17). Since the system (3.1.90) has $B_z r_{cmd}(t)$, the modified state-output predictor is introduced as follows:

$$\begin{aligned}\dot{\hat{v}}(t) =& A_v\hat{x}(t) - P_v^{-1}A_m^\top C_m^\top P_y\tilde{y}(t) - K_v y(t) + B_H r_{cmd}(t), \quad \hat{v}(0) = 0, \\ \dot{\hat{y}}(t) =& -\mu\tilde{y}(t) + C_m B_m(\hat{\eta}_t(t) + K_m\hat{x}_v(t)) + C_m B_z r_{cmd}(t) + C_m A_m\hat{x}(t), \quad \hat{y}(0) = y_0,\end{aligned} \tag{3.1.92}$$

where $y_0 = C_m x_0$ is known, $\hat{x}(t) = \hat{v}(t) + Hy(t)$, and $B_H = (\mathbb{I}_n - HC_m)B_z$. Design constraints are similar with those in Section 3.1.1.

The modified closed-loop reference system can be written as

$$\begin{aligned}\dot{x}_{ref}(t) =& A_m x_{ref}(t) + B_z r_{cmd}(t) + B_m\left(\Omega\left(u_{ref}(t) + K_m x_{ref}(t)\right) + f(x_{ref}, t)\right), \\ y_{ref}(t) =& C_m x_{ref}(t), \\ z_{ref}(t) =& T_z C_m x_{ref}(t), \quad x_{ref}(0) = x_0,\end{aligned} \tag{3.1.93}$$

with

$$u_{ref}(s) = -C_0(s)(\eta_{ref}(s) + \sigma(s)) - K_m x_{ref}(s), \tag{3.1.94}$$

where $x_{ref}(t) \in \mathbb{R}^n$, $y_{ref}(t) \in \mathbb{R}^p$, $z_{ref}(t) \in \mathbb{R}^m$ are the reference system state, measured output, and performance output vectors, respectively, and $C_0(s)$ is given in (3.1.21); $\eta_{ref}(s)$, and $\sigma(s)$ are the Laplace transform of the signals $\eta_{ref}(t)$, and $\sigma(t)$ defined in (3.1.22).

Corollary 3.1.1. *Given the system in* (3.1.90)*, the closed-loop system via the $\mathcal{L}_1$ adaptive controller defined in* (3.1.14)*,* (3.1.17)*, and* (3.1.92)*, subject to the $\mathcal{L}_1$–norm condition in* (3.1.8)*, has the following upper bound:*

$$\|z_{ref} - z\|_{\mathcal{L}_\infty} \leq \|T_z\|\frac{\gamma_y}{\sqrt{\Gamma}},$$

for some $\gamma_y > 0$, where $\Gamma = \min(\Gamma_\Omega, \Gamma_\Theta, \Gamma_\sigma)$.

Proof. Notice that the modified state-output predictor in (3.1.92) and the augmented plant in (3.1.90) yield the error dynamics in (3.1.42), so that Lemma 3.1.3 holds. Therefore, the proof of Corollary 3.1.1 follows from Theorem 3.1.1, and thus is omitted. □

3.1.4. *Illustrative example*

In this section, we provide a numerical example to demonstrate the performance of the proposed method. Consider the uncertain plant given in (3.1.89) with the nominal matrices:

$$A_p = \begin{bmatrix} -1.0190 & 0.9051 & 0.0022 \\ 0.8223 & -1.0770 & 0.1756 \\ 0 & 0 & -20.2 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & 57.3 & 0 \\ 16.25 & 0.9788 & 0.0485 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ -20.2 \end{bmatrix}.$$

To verify the proposed controller, the following uncertainties are taken into account:

$$\Omega = 0.8, \quad f(x,t) = 0.5x_1^3 - 0.15x_2^2 + \sin(t),$$

where x_i is the ith element of $x \in \mathbb{R}^3$. The goal is to design an adaptive controller so that $y_1(t)$ tracks a given reference command $r(t) \in \mathbb{R}$, where $y_1(t)$ is the first element of $y(t) \in \mathbb{R}^2$. For the purposes of command tracking, we define tracking error $e_z(t) = r(t) - y_1(t)$, and apply the proposed method in Section 3.1.3. The desired model is obtained via designing the state-feedback gain

$$K_m = [5.5210, 37.4448, 0.2854, -2.2361],$$

where we used standard LQR techniques for the augmented system in (3.1.90) with the weighting matrices $Q_m = \text{diag}(0, 1, 0, 5)$, and $R_m = 1$. The set of parameters in the $\mathcal{L}_1$ adaptive controller are given by

$$Q = 10\mathbb{I}_4, \quad P_y = 0.2\mathbb{I}_3, \quad \mu = 20, \quad D(s) = 30/s, \quad \Gamma = 500, \quad K_v = \begin{bmatrix} -0.16 & -0.64 & 0 \\ -0.70 & -0.06 & 0.01 \\ -2.83 & -11.85 & -0.53 \\ 0.82 & 0.05 & -1.21 \end{bmatrix}.$$

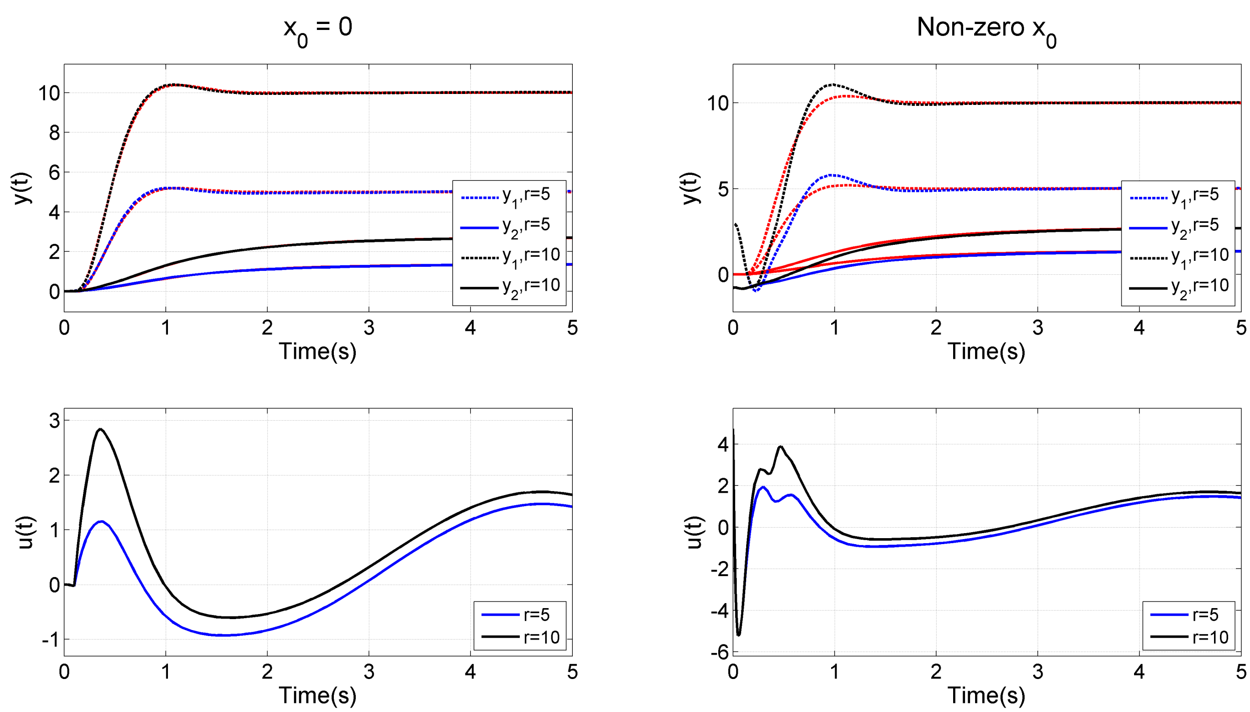

Figure 3.2: System responses and control inputs with initial conditions

Figure 3.2 shows the system response and control inputs for $r(t) = 5$ and $r(t) = 10$, where two different initial conditions $x_0 = 0$ and $x_0 = \frac{\pi}{180}[-3, 3, -5]^\top$ are used. In Figure 3.2, red-dotted lines are the desired responses for $y_1(t)$, and red-solid lines represent the desired reference signals for $y_2(t)$. The results clearly indicate that the transient responses due to the non-zero initial conditions decay over time, and that the small tracking errors are achieved by nullifying the effects of system uncertainties.

3.2. $\mathcal{L}_1$ Adaptive Augmentation of a Baseline Controller

In this section, we introduce $\mathcal{L}_1$ adaptive augmentation of a baseline controller. In output-feedback systems, the baseline augmented system is often a non-square and underactuated system [44]. In this chapter, we develop the $\mathcal{L}_1$ adaptive output feedback controller for this class of systems.

Consider the system with matched uncertainties as

$$\begin{aligned} \dot{x}_p(t) =& A_p x_p(t) + B_p(\Omega u_p(t) + f_p(x_p, t)), \\ y_p(t) =& C_p x(t), \quad x_p(0) = x_{p_0}, \end{aligned} \tag{3.2.1}$$

where $x_p(t) \in \mathbb{R}^{n_p}$, $u_p(t) \in \mathbb{R}^m$, $y_p(t) \in \mathbb{R}^{p_p}$ are the state, input, and measurable output vectors with $m \leq p_p$, respectively, and $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times m}$, $C_p \in \mathbb{R}^{p_p \times n_p}$ are known matrices, with (A_p, B_p, C_p) being controllable and observable; $x_{p0} \in \mathbb{R}^{n_p}$ is an initial condition, $\Omega \in \mathbb{R}^{m \times m}$ represents the unknown constant input gain satisfying Assumption 3.0.1, and $f_p(x_p, t)$ is an unknown function describing matched uncertainties. The baseline control structure is assumed to be

$$\begin{aligned} \dot{x}_c(t) =& A_c x_c(t) + B_{c_1} y_p(t) + B_{c_2} r(t), \\ u_b(t) =& C_c x_c(t) + D_c y_p(t), \quad x_c(0) = 0, \end{aligned} \tag{3.2.2}$$

where $x_c(t) \in \mathbb{R}^{n_c}$, $r(t) \in \mathbb{R}^{p_c}$, $u_b(t) \in \mathbb{R}^m$, are the controller state, reference command, and baseline controller input vectors, respectively. $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_{c_1} \in \mathbb{R}^{n_c \times p_p}$, $B_{c_2} \in \mathbb{R}^{n_c \times p_c}$, $C_c \in \mathbb{R}^{m \times n_c}$, and $D_c \in \mathbb{R}^{m \times p_p}$ are known matrices. Let $u_p(t) = u_b(t) + u(t)$, where $u(t) \in \mathbb{R}^m$ is the adaptive controller input, which will be defined later. By combining (3.2.1) and (3.2.2), the augmented plant is rewritten as

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B_m \left(\Omega u(t) + f(x, t) \right) + B_z r(t) \,, \\ y(t) &= C_m x(t) \,, \\ z(t) &= C_z x(t) \,, \quad x(0) = x_0 \,, \end{aligned} \tag{3.2.3}$$

where $x(t) = \left[x_p^\top(t), x_c^\top(t) \right]^\top \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ is the augmented measurable output, $z(t) \in \mathbb{R}^{p_c}$ is the performance output such that $z(t) = T_z y_p(t)$ for some matrix $T_z \in \mathbb{R}^{p_c \times p_p}$; $x_0 = [x_{p_0}^\top, 0]^\top$ is the initial condition, and $A_m \in \mathbb{R}^{n \times n}$, $B_m \in \mathbb{R}^{n \times m}$, $B_r \in \mathbb{R}^{n \times m}$, $C_m \in \mathbb{R}^{p \times n}$, $C_z \in \mathbb{R}^{p_c \times n}$ are known matrices (with $n = n_p + n_c$, $p = p_p + n_c$, $p < n$, and $m < n$) defined as

$$\begin{aligned} A_m &= \begin{bmatrix} A_p + B_p D_c C_p & B_p C_c \\ B_{c_1} C_p & A_c \end{bmatrix}, \quad B_m = \begin{bmatrix} B_p \\ 0 \end{bmatrix}, \\ B_z &= \begin{bmatrix} 0 \\ B_{c_2} \end{bmatrix}, C_m = \begin{bmatrix} C_p & 0 \\ 0 & \mathbb{I}_{n_c} \end{bmatrix}, C_z = \begin{bmatrix} T_z C_p & 0 \end{bmatrix}. \end{aligned} \tag{3.2.4}$$

Finally, $f(x,t)$ is the unknown function satisfying Assumption 3.0.2 and

$$f(x,t) = (\Omega - \mathbb{I}_m)\left(C_c x_c(t) + D_c C_p x_p(t)\right) + f_p(x_p,t).$$

To proceed, consider the following assumptions:

Assumption 3.2.1. *The baseline controller* (3.2.2) *is designed so that the performance output* $z(t) \in \mathbb{R}^{p_c}$ *tracks the desired response* $z_m(t) \in \mathbb{R}^{p_c}$ *for the nominal system* ($\Omega = \mathbb{I}_m,\ f(x_p,t) \equiv 0$)*. In other words,*

$$z(s) = z_m(s) = M_z(s) r(s), \tag{3.2.5}$$

where $M_z(s) = C_z(s\mathbb{I}_n - A_m)^{-1}B_z$*, and* $r(s)$ *is the Laplace transform of the reference command* $r(t) \in \mathbb{R}^{p_c}$*.*

Assumption 3.2.2. *Let the augmented input sensitivity function* $M(s)$ *be*

$$M(s) = C_m \left(s\mathbb{I}_n - A_m\right)^{-1} B_m\,, \tag{3.2.6}$$

where A_m *is Hurwitz,* B_m*,* C_m *are full rank, and* (A_m, B_m)*,* (A_m, C_m) *are controllable and observable pairs, respectively. Moreover, the following hold:*

- *The open-loop system* (A_p, B_p, C_p) *has no unstable transmission zeros; i.e. if* $z_i \in \mathbb{C}$ *is a transmission zero of the system, then* z_i *has a strictly negative real part.*
- $(C_p B_p)$ *is full rank.*

Remark 3.2.1. *It follows from Lemma 2.1.20 that* $M(s)$ *does not have unstable transmission zeros and*

$$\mathrm{rank}(C_m B_m) = m. \tag{3.2.7}$$

The problem is formally stated as follows:

Problem 3.2.1. *Let the augmented system described by Equation* (3.2.3) *satisfy Assumptions 3.0.1, 3.0.2, 3.2.1, and 3.2.2. The control objective is to design an output feedback control law for* $u(t)$ *that compensates for the uncertainties and ensures that* $z(t)$ *tracks the desired response* $z_m(t)$ *with uniform bounds both in transient and steady-state.*

3.2.1. *Design of* $\mathcal{L}_1$ *adaptive controller*

In this section, we introduce the $\mathcal{L}_1$ adaptive output feedback controller for the MIMO system (3.2.3). Before stating the main results, we introduce several design variables. Let $\rho_0 > 0$ be a given constant satisfying $\|x_0\| \leq \rho_0$ with $x_0 \in \mathbb{R}^n$ being the initial condition, and let $\bar{\gamma} > 0$ be an arbitrarily small constant. For a given $\delta > 0$ define

$$L_\delta = \frac{\bar{\delta}(\delta)}{\delta} d_{\bar{\delta}(\delta)}, \quad \bar{\delta}(\delta) = \delta + \bar{\gamma}\,, \tag{3.2.8}$$

where $d_{\bar{\delta}(\delta)}$ is introduced in Assumption 3.0.2. Let $K_v \in \mathbb{R}^{n\times p}$ be the gain chosen so that

$$A_v = (\mathbb{I}_n - HC_m)A_m + K_v C_m \tag{3.2.9}$$

is Hurwitz (from Remark 3.2.1 and Lemma 2.1.19 such K_v exists) with $H = B_m(C_m B_m)^\dagger$. Let $\epsilon_q > 0$ be any positive number, $P_y \in \mathbb{R}^{p\times p}$ be a given positive definite matrix, and $P_v \in \mathbb{R}^{n\times n}$ be the positive definite matrix which solves

$$A_v^\top P_v + P_v A_v = -Q \tag{3.2.10}$$

for a positive definite $Q \in \mathbb{R}^{n\times n}$ with $\epsilon_q < \lambda_{\min}(Q)$. Define

$$\kappa_m = \sup_{t\geq 0} \|e^{A_m t}\|, \quad \kappa_y = \sqrt{n\frac{\lambda_{\max}(\bar{P}_v)}{\lambda_{\min}(P_v)}}, \quad \kappa_v = \sqrt{n\frac{\lambda_{\max}(\bar{P}_v)}{\lambda_{\min}(P_y)}}, \tag{3.2.11}$$

where $\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m)$. Let $D(s)$ be a $m \times m$ transfer matrix such that for all $\Omega \in \mathcal{C}_\Omega$

$$C(s) = \Omega C_0(s), \tag{3.2.12}$$

is a stable and strictly proper transfer matrix with $C(0) = \mathbb{I}_m$, where

$$C_0(s) = D(s)(\mathbb{I}_m + \Omega D(s))^{-1}. \tag{3.2.13}$$

Moreover, the choice of $D(s)$ must ensure that for all $\Omega \in \mathcal{C}_\Omega$, there exists $\rho_r > 0$ such that

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_r - \rho_{ext} - \rho_{int}}{L_{\rho_r}\rho_r}, \tag{3.2.14}$$

where

$$\begin{aligned} \rho_{ext} =& \|H_z(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} b_0, \\ \rho_{int} =& (\kappa_m + \kappa_x)\rho_0, \\ \kappa_x =& \|H_1(s)\|_{\mathcal{L}_1} \kappa_y + \|H_2(s)\|_{\mathcal{L}_1} \kappa_v, \end{aligned} \tag{3.2.15}$$

with κ_m, κ_y,κ_v being given in (3.2.11),

$$\begin{aligned} H_z(s) =& (s\mathbb{I}_n - A_m)^{-1}B_z, \quad H_0(s) = (s\mathbb{I}_n - A_m)^{-1}B_m, \\ H_1(s) =& H_0(s)C_1(s), \quad H_2(s) = H_0 C_2(s), \\ G(s) =& H_0(s)\left(\mathbb{I}_m - C(s)\right), \end{aligned} \tag{3.2.16}$$

and

$$C_1(s) = (s+\mu)C(s)B_m^{\dagger}H, \quad C_2(s) = C(s)B_m^{\dagger}HC_mA_m. \tag{3.2.17}$$

Notice that L_{ρ_r} satisfies (3.2.8) with d_{ρ_x} and

$$\rho_x = \rho_r + \bar{\gamma}. \tag{3.2.18}$$

Finally, let $\mu \in \mathbb{R}$ be chosen to satisfy

$$\mu > \alpha^{\star}, \quad \alpha^{\star} = \frac{mL_{\rho_r}^2}{\epsilon_q}\left\|\sqrt{P_y}C_mB_m\right\|_2^2. \tag{3.2.19}$$

We consider the control input $u(t) \in \mathbb{R}^m$ according to the following law:

$$u(s) = -D(s)\hat{\eta}_t(s)\,, \tag{3.2.20}$$

where $D(s) \in \mathbb{C}^{m\times m}$ is chosen to satisfy the design constraints via (3.2.8) - (3.2.19), and $\hat{\eta}_t(s)$ is the Laplace transform of

$$\hat{\eta}_t(t) = \hat{\Omega}(t)u(t) + \hat{\theta}(t)\|\hat{x}(t)\| + \hat{\sigma}(t), \tag{3.2.21}$$

where $\hat{x}(t) = \hat{v}(t) + Hy(t)$; $\hat{v}(t) \in \mathbb{R}^n$ is provided by the state-output predictor, and $\hat{\Omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ are given in the adaptive laws. The following state-output predictor is used:

$$\begin{aligned}
\dot{\hat{v}}(t) =& A_v\hat{x}(t) - K_vy(t) - P_v^{-1}A_m^{\top}C_m^{\top}P_y\tilde{y}(t) + (\mathbb{I}_n - HC_m)\,B_zr(t),\\
\dot{\hat{y}}(t) =& C_mA_m\hat{x}(t) - \mu\tilde{y}(t) + C_mB_m\hat{\eta}_t(t) + C_mB_zr(t),\\
\hat{x}(t) =& \hat{v}(t) + Hy(t), \quad \hat{v}(0) = 0, \quad \hat{y}(0) = y_0,
\end{aligned} \tag{3.2.22}$$

where $\tilde{y}(t) = \hat{y}(t) - y(t)$, and $y_0 \in \mathbb{R}^p$ is a known initial output vector with $y_0 = C_mx_0$. The adaptive laws are defined using the projection operator:

$$\begin{aligned}
\dot{\hat{\Omega}}(t) &= \Gamma_{\Omega}\text{Proj}(\hat{\Omega}(t), -e_y(t)u^{\top}(t)), \quad \hat{\Omega}(0) = \mathbb{I}_m,\\
\dot{\hat{\theta}}(t) &= \Gamma_{\theta}\text{Proj}(\hat{\theta}(t), -e_y(t)\|(\hat{x}(t))\|), \quad \hat{\Theta}(0) = 0,\\
\dot{\hat{\sigma}}(t) &= \Gamma_{\sigma}\text{Proj}(\hat{\sigma}(t), -e_y(t)), \quad \hat{\sigma}(0) = 0,
\end{aligned} \tag{3.2.23}$$

where $e_y(t) = B_m^{\top}C_m^{\top}P_y\tilde{y}(t)$, $\Gamma_{\Omega} > 0$, $\Gamma_{\Theta} > 0$, $\Gamma_{\sigma} > 0$ are real constant adaptation gains and $\text{Proj}(\cdot,\cdot)$ denotes the projection operator [21]. Figure 3.3 depicts the proposed control architecture.

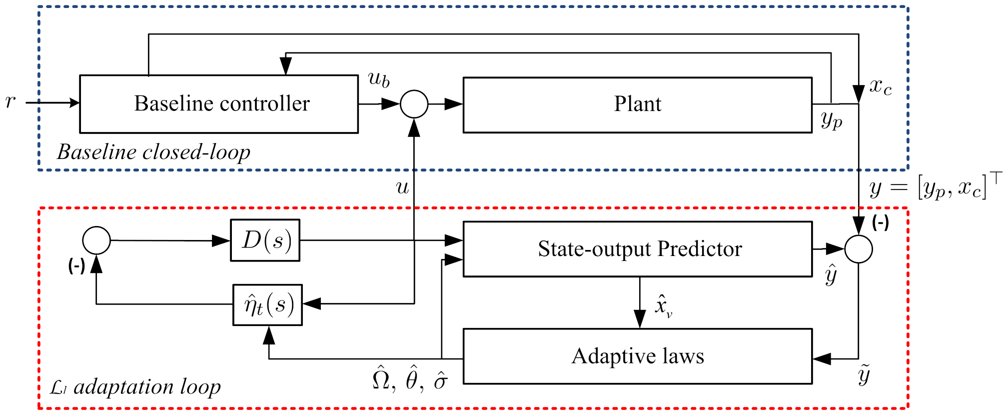

Figure 3.3: $\mathcal{L}_1$ output feedback control architecture

3.2.2. *Stability and Performance Analysis*

In this section, the reference system of $\mathcal{L}_1$ adaptive controller is introduced, and its stability analysis is presented. Consider the following closed-loop reference system

$$\begin{aligned} \dot{x}_{ref}(t) =& A_m x_{ref}(t) + B_m \left(\Omega u_{ref}(t) + f(x_{ref}, t)\right) + B_z r(t), \quad x_{ref}(0) = 0, \\ y_{ref}(t) =& C_m x_{ref}(t), \quad z_{ref}(t) = C_z x_{ref}(t), \end{aligned} \tag{3.2.24}$$

with

$$u_{ref}(s) = -C_0(s)\left(\eta_{ref}(s) + \sigma(s)\right), \tag{3.2.25}$$

where $x_{ref}(t) \in \mathbb{R}^n$, $y_{ref}(t) \in \mathbb{R}^p$, $z_{ref}(t) \in \mathbb{R}^{p_c}$ are the reference system state, output, and performance output vectors, respectively. Moreover, $\eta_{ref}(s)$, and $\sigma(s)$ are the Laplace transforms of $\eta_{ref}(t)$, and $\sigma(t)$ given by

$$\sigma(t) = f(0, t), \quad \eta_{ref}(t) = f(x_{ref}, t) - f(0, t). \tag{3.2.26}$$

The closed-loop reference system in (3.2.24) and (3.2.25) defines *the best achievable performance* of the $\mathcal{L}_1$ adaptive architecture [2]. Notice that the system is not implementable as it depends on the unknowns; it is used only for analysis purposes.

Lemma 3.2.1. *Consider the closed-loop reference system given in* (3.2.24) *and* (3.2.25) *and design constraints defined via* (3.2.8) - (3.2.19)*. Then, for each* $\Omega \in \mathcal{C}_\Omega$ *and each* $\tau > 0$ *the following holds*

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{rx}, \tag{3.2.27}$$

where

$$\rho_{rx} = \rho_r - \gamma_{x_0}\rho_0, \quad \gamma_{x_0} = \frac{\kappa_x + \kappa_m}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \tag{3.2.28}$$

with κ_m*,* κ_x*, and* $G(s)$ *being given in* (3.2.11)*,* (3.2.15)*, and* (3.2.16)*, respectively. Moreover,* $\left\|u_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{ru}$ *holds, where*

$$\rho_{ru} = \|C_0(s)\|_{\mathcal{L}_1} \left(L_{\rho_r}\rho_{rx} + b_0\right), \tag{3.2.29}$$

with $C_0(s)$ *defined in* (3.2.13)*.*

Proof. Notice that the definition of ρ_{rx} in (3.2.28), together with (3.2.14), yields

$$\rho_{rx} = \rho_r - \gamma_{x_0}\rho_0 > \frac{\rho_{ext}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \tag{3.2.30}$$

which leads to $\rho_{rx} > 0$. Now, we prove Equation (3.2.27) by a contradiction argument. Suppose it is not true. Since $\|x_{ref}(0)\| = 0 < \rho_{xr}$, from the continuity it follows that there exists $\tau > 0$ such that $\|x_{ref}(\tau)\| = \rho_{xr}$, while $\|x_{ref}(t)\| < \rho_{xr}$ for $0 \leq t < \tau$. Therefore, the following must hold:

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} = \rho_{xr}. \tag{3.2.31}$$

By substituting (3.2.25) into (3.2.24), the Laplace transform of the closed-loop reference system is given by

$$\begin{aligned} x_{ref}(s) =& H_z(s)r(s) + G(s)(\eta_{ref}(s) + \sigma(s)), \\ u_{ref}(s) =& -C_0(s)(\eta_{ref}(s) + \sigma(s)), \\ y_{ref}(s) =& C_m x_{ref}(s), \quad z_{ref}(s) = C_z x_{ref}(s), \end{aligned} \tag{3.2.32}$$

where $\eta_{ref}(s)$, $\sigma(s)$ are the Laplace transforms of $\eta_{ref}(t)$, and $\sigma(t)$ given in (3.2.26), and $C_0(s)$, $\{H_z(s), G(s)\}$ are defined in (3.2.13), and (3.2.16), respectively. Since $\rho_{rx} \leq \rho_r \leq \rho_x$ and $d_{\rho_x} \leq L_{\rho_r}$ hold, Assumption 3.0.2, along with (3.2.8), leads to

$$\left\|\eta_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq L_{\rho_r} \left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty}. \tag{3.2.33}$$

From (3.2.32) and (3.2.33) it follows

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{ext} + \|G(s)\|_{\mathcal{L}_1} L_{\rho_r} \left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty}. \tag{3.2.34}$$

Since $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$ is derived from (3.2.14), Equation (3.2.34) can be rewritten as

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \frac{\rho_{ext}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \tag{3.2.35}$$

which, together with (3.2.30), yields

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{rx}. \tag{3.2.36}$$

This contradicts (3.2.31), thus proving (3.2.27). Finally, combining (3.2.27), (3.2.32), and (3.2.33) yields

$$\left\|u_{ref_{\tau}}\right\|_{\mathcal{L}_\infty} \leq \rho_{ru},$$

where ρ_{ur} is given in (3.2.29). This completes the proof. □

By choosing $D(s)$ such that $C(s) \approx \mathbb{I}_m$, Equation (3.2.14) can be always verified, and thus implies the semi-global stability of the ideal $\mathcal{L}_1$ reference system in (3.2.24) - (3.2.25). Moreover, the input sensitivity function of the reference system is given by

$$(z_{ref} - z_m)(s) = G_z(s)(\eta_{ref}(s) + \sigma(s)),$$

where $G_z(s) = C_z(s\mathbb{I} - A_m)^{-1}B_m(\mathbb{I}_m - C(s))$, and $(\eta_{ref}(s) + \sigma(s))$ is the Laplace transform of $f(x_{ref}, t)$ which is bounded by Lemma 3.2.1. Therefore, while $C(s) \approx \mathbb{I}_m$ makes the bound $\|z_{ref} - z_m\|_{\mathcal{L}_\infty}$ arbitrarily small, it also reduces the input sensitivity to zero. However, choosing $C(s)$ with high bandwidth (i.e. $C(s) \approx \mathbb{I}_m$) can result in loss of robustness [2].

In what follows, the closed-loop system stability is analyzed and the transient performance bounds are presented. Let

$$\begin{aligned}
\gamma_{u_0} &= \|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r}\gamma_{x_0} + (\|C_3(s)\|_{\mathcal{L}_1}\kappa_y + \|C_4(s)\|_{\mathcal{L}_1}\kappa_v), \\
\gamma_u &= \|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r}\gamma_x + \frac{\|C_3(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_y)}} + \frac{\|C_4(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_v)}}, \\
\gamma_x &= \frac{\lambda_{\min}(P_y)^{-\frac{1}{2}}\|H_1(s)\|_{\mathcal{L}_1} + \lambda_{\min}(P_v)^{-\frac{1}{2}}\|H_2(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}},
\end{aligned} \tag{3.2.37}$$

where $\{\kappa_y, \kappa_v\}$, γ_{x_0} are given in (3.2.11) and (3.2.28), respectively, and

$$C_3(s) = (s + \mu)C_0(s)B_m^{\dagger}H, \quad C_4(s) = C_0(s)B_m^{\dagger}HC_mA_m. \tag{3.2.38}$$

Next, $\epsilon_\gamma > 0$ is chosen to satisfy

$$\gamma_x\epsilon_\gamma < \bar{\gamma}, \quad \gamma_u\epsilon_\gamma < \bar{\gamma}, \quad \forall\Omega \in \mathcal{C}_\Omega. \tag{3.2.39}$$

Finally, let ρ_u, ρ_{dx}, and ρ_{du} be

$$\begin{aligned}
\rho_u &= \rho_{ru} + \rho_{du}, \\
\rho_{dx} = \gamma_{x_0}\rho_0 + \bar{\gamma}, &\quad \rho_{du} = \gamma_{u_0}\rho_0 + \bar{\gamma},
\end{aligned} \tag{3.2.40}$$

respectively, where ρ_{ru} is defined in (3.2.29).

Notice that from Lemma 3.1.2 the nonlinear function can be parameterized as follows:

$$f(x,t) = \theta(t)\|x(t)\| + \sigma(t), \tag{3.2.41}$$

where $\theta(t) \in \mathbb{R}^m$ and $\sigma(t) \in \mathbb{R}^m$ satisfy

$$\begin{aligned} \|\theta(t)\| \leq d_{\rho_x}, \quad \|\dot{\theta}(t)\| \leq l_\theta, \quad 0 \leq t \leq \tau, \\ \|\sigma(t)\| \leq b_0, \quad \|\dot{\sigma}(t)\| \leq l_\sigma, \quad 0 \leq t \leq \tau, \end{aligned} \tag{3.2.42}$$

with $l_\theta = l_\theta(\rho_x, \rho_u)$ and $l_\sigma = l_\sigma(\rho_x, \rho_u)$ being bounded.

Lemma 3.2.2. *Consider the system given in* (3.2.3) *with control laws and design constraints defined via* (3.2.8) - (3.2.23)*. Let $\tilde{v}(t) = \hat{v}(t) - v(t)$ and $\tilde{y}(t) = \hat{y}(t) - y(t)$, where $v(t) = (\mathbb{I}_n - HC_m)\,x(t)$. Let $\tau > 0$ be a positive constant. Then, if $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u$, the following bounds hold for all $t \geq 0$*

$$\begin{aligned} \|\tilde{v}(t)\| \leq& \kappa_v e^{-\frac{\lambda_1^\star}{2}t}\|x_0\| + \sqrt{\frac{\theta_1}{\lambda_{\min}(P_v)}}\frac{1}{\sqrt{\Gamma}}, \\ \|\tilde{y}(t)\| \leq& \kappa_y e^{-\frac{\lambda_1^\star}{2}t}\|x_0\| + \sqrt{\frac{\theta_1}{\lambda_{\min}(P_y)}}\frac{1}{\sqrt{\Gamma}}, \end{aligned} \tag{3.2.43}$$

where κ_y, κ_v are defined in (3.2.11)*, and*

$$\begin{aligned} \lambda_1^\star =& \left(\max\left(\frac{\lambda_{\max}(P_v)}{\lambda_{\min}(Q_v)}, \frac{\lambda_{\max}(P_y)}{(\mu-\alpha^\star)\lambda_{\min}(P_y)}\right)\right)^{-1}, \\ \theta_1 =& \theta_0 + \frac{4m}{\lambda_1^\star}\left(d_{\rho_x}l_\theta + b_0 l_\sigma\right), \\ \theta_0 =& 4\sup_{\Omega\in\mathcal{C}_\Omega}\left(tr\left(\Omega^\top\Omega\right) + n(d_{\rho_x}^2 + b_0^2)\right), \\ \Gamma =& \min\left(\Gamma_\Omega, \Gamma_\Theta, \Gamma_\sigma\right), \quad Q_v = Q - \epsilon_q\mathbb{I}_n > 0, \end{aligned} \tag{3.2.44}$$

with $l_\theta = l_\theta(\rho_x, \rho_u)$ and $l_\sigma = l_\sigma(\rho_x, \rho_u)$ satisfying (3.2.42).

Proof. Let $\tau > 0$ satisfy $\|x_\tau\|_{\mathcal{L}_\infty} < \rho_x$ and $\|u_\tau\|_{\mathcal{L}_\infty} < \rho_u$. Pre-multiplying both sides of (3.2.3) by $(\mathbb{I}_n - HC_m)$ and taking the derivative of $y(t)$ yield

$$\begin{aligned} \dot{v}(t) =& A_H v(t) + A_H H y(t) + (\mathbb{I}_n - HC_m)B_z r(t), \quad v(0) = v_0 \\ \dot{y}(t) =& C_m A_m\left(v(t) + Hy(t)\right) + C_m B_m\left(\Omega u(t) + f(x,t)\right) + C_m B_z r(t), \quad y(0) = y_0 \\ x(t) =& v(t) + Hy(t), \end{aligned} \tag{3.2.45}$$

where $A_H = (\mathbb{I}_n - HC_m)A_m$, $v_0 = (\mathbb{I} - HC_m)x_0$, and $y_0 = C_m x_0$. By using the fact that $C_m v(t) = C_m(\mathbb{I}_n - HC_m)x(t)$, and subtracting (3.2.22) from (3.2.45), it follows, together with Lemma 3.1.2, that

$$\begin{aligned}\dot{\tilde{v}}(t) =& A_v\tilde{v}(t) - P_v^{-1}A_m^\top C_m^\top P_y\tilde{y}(t), \quad \tilde{v}(0) = -v_0 \\ \dot{\tilde{y}}(t) =& C_m A_m \tilde{v}(t) + C_m B_m(\tilde{\eta}_t(t) + \phi(t)) - \mu\tilde{y}(t), \quad \tilde{y}(0) = 0,\end{aligned} \tag{3.2.46}$$

where A_v is given in (3.2.9), and

$$\tilde{\eta}_t(t) = \tilde{\Omega}(t)u(t) + \tilde{\theta}(t)\|\hat{x}(t)\| + \tilde{\sigma}(t), \tag{3.2.47}$$

with $\tilde{\Omega}(t) = \hat{\Omega}(t) - \Omega$, $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$, $\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t)$, $\sigma(t) = f(0,t)$, and

$$\phi(t) = \theta(t)\left(\|\hat{x}(t)\| - \|x(t)\|\right). \tag{3.2.48}$$

Now, consider the Lyapunov function

$$V(t) = \tilde{v}^\top(t)P_v\tilde{v}(t) + \tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{tr(\tilde{\Omega}^\top(t)\tilde{\Omega}(t))}{\Gamma_\Omega} + \frac{\tilde{\theta}^\top(t)\tilde{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\tilde{\sigma}(t)}{\Gamma_\sigma}, \tag{3.2.49}$$

where $P_y \in \mathbb{R}^{p\times p}$ is chosen to be positive definite and $P_v > 0$ is the solution of $A_v^\top P_v + P_v A_v = -Q$ for a given $Q \in \mathbb{R}^{n\times n}$; notice that Q is chosen to satisfy $Q > \epsilon_q \mathbb{I}_n$. By combining (3.2.23) and (3.2.46), the derivative of (3.2.49) satisfies

$$\begin{aligned}\dot{V}(t) \leq& -\tilde{v}^\top(t)Q\tilde{v}(t) - 2\mu\tilde{y}^\top(t)P_y\tilde{y}(t) - \frac{2\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} \\ &- \frac{2\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma} + 2\tilde{y}^\top(t)P_y C_m B_m \phi(t).\end{aligned} \tag{3.2.50}$$

Using the completion of squares, one has

$$2\tilde{y}^\top(t)P_yC_mB_m\phi(t) \leq \alpha\tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{1}{\alpha}\|\sqrt{P}_yC_mB_m\|_2^2\|\phi(t)\|_2^2, \tag{3.2.51}$$

where α is any positive constant. Moreover, combining (3.2.42) and (3.2.48), along with the fact that $d_{\rho_x} \leq L_{\rho_r}$, leads to

$$\|\phi(t)\|_2 \leq \sqrt{m}L_{\rho_r}\|\tilde{v}(t)\|. \tag{3.2.52}$$

Choose $\alpha^\star = \frac{\alpha}{2}$, with $\alpha^\star$ being given in (3.2.19). By combining (3.2.50) - (3.2.52), it follows that

$$\dot{V}(t) \leq -\tilde{v}^\top(t)Q_v\tilde{v}(t) - 2(\mu - \alpha^\star)\tilde{y}^\top(t)P_y\tilde{y}(t) - \frac{2\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} - \frac{2\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma}, \tag{3.2.53}$$

where $Q_v = Q - \epsilon_q\mathbb{I}_n > 0$. Notice that μ is chosen to satisfy $\mu > \alpha^\star$ (see (3.2.19)). Furthermore, Equation (3.2.42) implies

$$\frac{\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma} \leq \frac{2m}{\Gamma}(d_{\rho_x} l_\theta + b_0 l_\sigma), \tag{3.2.54}$$

where $l_\theta = l_\theta(\rho_x, \rho_u)$ and $l_\sigma = l_\sigma(\rho_x, \rho_u)$ are given in (3.2.42), and $\Gamma = \min(\Gamma_\Omega, \Gamma_\theta, \Gamma_\sigma)$. The projection operator in (3.2.23) ensures that

$$\sup_{t\geq 0}\left(\frac{tr(\tilde{\Omega}^\top\tilde{\Omega})}{\Gamma_\Omega} + \frac{\tilde{\theta}^\top\tilde{\theta}}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top\tilde{\sigma}}{\Gamma_\sigma}\right) \leq \frac{\theta_0}{\Gamma}, \tag{3.2.55}$$

where θ_0 is defined in (3.2.44). From the definition of $\lambda_1^\star$ given in (3.2.44), it follows that for all $t \geq 0$

$$\tilde{v}^\top(t)P_v\tilde{v}(t) + \tilde{y}^\top(t)P_y\tilde{y}(t) \leq \frac{1}{\lambda_1^\star}(\tilde{v}^\top(t)Q_v\tilde{v}(t) + 2(\mu - \alpha^\star)\tilde{y}^\top(t)P_y\tilde{y}(t)). \tag{3.2.56}$$

Next, from (3.2.52) - (3.2.54) one has

$$\dot{V}(t) \leq -\tilde{v}^\top(t)Q_v\tilde{v}(t) - 2(\mu - \alpha^\star)\tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{4m}{\Gamma}(d_{\rho_x} l_\theta + b_0 l_\sigma), \tag{3.2.57}$$

which, along with (3.2.49), (3.2.55), and (3.2.56), leads to

$$\dot{V}(t) \leq -\lambda_1^\star\left(V(t) - \frac{\theta_1}{\Gamma}\right), \quad t \geq 0,$$

where θ_1 is given in (3.2.44), and $l_\theta = l_\theta(\rho_x, \rho_u)$, $l_\sigma = l_\sigma(\rho_x, \rho_u)$ satisfy (3.2.42). Gronwell-Bellman inequality further yields

$$\sqrt{V(t)} \leq \upsilon_v(t, t_0), \quad 0 \leq t_0 \leq t, \tag{3.2.58}$$

which leads to

$$\|\tilde{y}(t)\| \leq \frac{\upsilon_v(t,t_0)}{\sqrt{\lambda_{\min}(P_y)}}, \quad \|\tilde{v}(t)\| \leq \frac{\upsilon_v(t,t_0)}{\sqrt{\lambda_{\min}(P_v)}}, \tag{3.2.59}$$

where

$$\upsilon_v(t, t_0) = \sqrt{\left(V(t_0) - \frac{\theta_1}{\Gamma}\right)e^{-\lambda_1^\star(t-t_0)} + \frac{\theta_1}{\Gamma}}. \tag{3.2.60}$$

Finally, since $V(0) \leq x_0^\top\bar{P}_v x_0 + \frac{\theta_0}{\Gamma}$ with $\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m)$, by letting $t_0 = 0$, Equation (3.2.59) reduces to (3.2.43), which completes the proof. □

Theorem 3.2.1. *Consider the closed-loop system with $\mathcal{L}_1$ adaptive output feedback controller defined via* (3.2.20) – (3.2.23)*, subject to the $\mathcal{L}_1$–norm condition in* (3.2.14)*. Suppose the adaptation gain is chosen sufficiently high to satisfy*

$$\Gamma > \frac{\theta_1}{{\epsilon_\gamma}^2}, \tag{3.2.61}$$

where Γ, θ_1 *are defined in* (3.2.44)*, and* ϵ_γ *satisfies* (3.2.39)*. Then, the following upper bounds hold:*

$$\|x_{ref} - x\|_{\mathcal{L}_\infty} \leq \rho_{dx}, \quad \|u_{ref} - u\|_{\mathcal{L}_\infty} \leq \rho_{du}, \tag{3.2.62}$$

and

$$\begin{aligned} \|y_{ref} - y\|_{\mathcal{L}_\infty} \leq & \|C_m\|\rho_{dx}, \quad \|z_{ref} - z\|_{\mathcal{L}_\infty} \leq \|C_z\|\rho_{dx}, \\ \|x\|_{\mathcal{L}_\infty} \leq & \rho_x, \quad \|u\|_{\mathcal{L}_\infty} \leq \rho_u. \end{aligned} \tag{3.2.63}$$

Moreover, for each $\Omega \in \mathcal{C}_\Omega$ *there exist positive constants* γ_{dx}*,* γ_{dy}*, and* γ_{dz} *depending on* ρ_x *and* ρ_u*, and strictly decreasing functions* $\upsilon_{dx}(t)$*,* $\upsilon_{dy}(t)$*, and* $\upsilon_{dz}(t)$*, such that for all* $t \geq 0$

$$\begin{aligned} \|x_{ref}(t) - x(t)\| \leq & \upsilon_{dx}(t)\|x_0\| + \frac{\gamma_{dx}}{\sqrt{\Gamma}}, \\ \|y_{ref}(t) - y(t)\| \leq & \upsilon_{dy}(t)\|x_0\| + \frac{\gamma_{dy}}{\sqrt{\Gamma}}, \\ \|z_{ref}(t) - z(t)\| \leq & \upsilon_{dz}(t)\|x_0\| + \frac{\gamma_{dz}}{\sqrt{\Gamma}}. \end{aligned} \tag{3.2.64}$$

Proof. Let $\tilde{x}_{ref}(t) = x_{ref}(t) - x(t)$, $\tilde{u}_{ref}(t) = u_{ref}(t) - u(t)$, $\tilde{y}_{ref}(t) = y_{ref}(t) - y(t)$, $\tilde{z}_{ref}(t) = z_{ref}(t) - z(t)$, and $\tilde{\eta}_{ref}(t) = f(x_{ref}, t) - f(x, t)$. First, we prove Equation (3.2.62) by a contradiction argument. Suppose it is not true. Since $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$ holds from (3.2.14), Equation (3.2.40), together with the fact that $\kappa_m \geq 1$ in (3.2.11), leads to

$$\rho_{dx} > (\kappa_x + \kappa_m)\rho_0 + \bar{\gamma} > \rho_0, \tag{3.2.65}$$

which implies that $\|\tilde{x}_{ref}(0)\| < \rho_{dx}$, and $\|\tilde{u}_{ref}(0)\| = 0 < \rho_{du}$. From continuity, there exists $\tau > 0$ such that

$$\|\tilde{x}_{ref}(\tau)\| = \rho_{dx} \quad \text{or} \quad \|\tilde{u}_{ref}(\tau)\| = \rho_{du}, \tag{3.2.66}$$

while

$$\|\tilde{x}_{ref}(t)\| < \rho_{dx}, \quad \|\tilde{u}_{ref}(t)\| < \rho_{du}, \quad 0 \leq t < \tau, \tag{3.2.67}$$

where ρ_{du} is given in (3.2.40). Moreover, from (3.2.66) and (3.2.67) the following must hold:

$$\left\|\tilde{x}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{dx}, \quad \left\|\tilde{u}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{du}. \tag{3.2.68}$$

Notice that combining (3.2.18), (3.2.28) and (3.2.40) leads to

$$\rho_{dx} = \rho_x - \rho_{rx}, \quad \rho_{du} = \rho_u - \rho_{ru},$$

which, together with Lemma 3.2.1 and the triangular inequalities on $\|\tilde{x}_{ref}\|$ and $\|\tilde{u}_{ref}\|$, yields

$$\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x, \quad \|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u. \tag{3.2.69}$$

Since $d_{\rho_x} \leq L_{\rho_r}$ holds, Assumption 3.0.2 along with (3.2.69) yields

$$\|\tilde{\eta}_{ref}(t)\| \leq L_{\rho_r} \|\tilde{x}_{ref}(t)\|, \quad 0 \leq t \leq \tau. \tag{3.2.70}$$

Moreover, using Lemma 3.1.2, from (3.2.21) one has

$$\hat{\eta}_t(t) = \Omega u(t) + f(x,t) + \tilde{\eta}_t(t) + \phi(t), \tag{3.2.71}$$

where $\tilde{\eta}_t(t)$, and $\phi(t)$ are defined in (3.2.47), and (3.2.48), respectively. Substituting (3.2.71) into (3.2.20) leads to

$$u(s) = -C_0(s)\left(\eta(s) + \sigma(s) + \tilde{\eta}_t(s) + \phi(s)\right), \tag{3.2.72}$$

where $\eta(s)$, $\sigma(s)$, $\tilde{\eta}_t(s)$, and $\phi(s)$ are the Laplace transforms of $(f(x,t)-\sigma(t))$, $\sigma(t)$, $\tilde{\eta}_t(t)$, and $\phi(t)$, respectively; $C_0(s)$ is given in (3.2.13). Next, combining (3.2.3) and (3.2.72) yields

$$\begin{aligned} x(s) =& H_z(s)r(s) + G(s)(\eta(s)+\sigma(s)) - H_0(s)\phi_f(s) + (s\mathbb{I}_n - A_m)^{-1}x_0, \\ y(s) =& C_m x(s), \quad z(s) = C_z x(s), \end{aligned} \tag{3.2.73}$$

where $H_z(s)$, $H_0(s)$, $G(s)$ are given in (3.2.16), and

$$\phi_f(s) = C(s)(\tilde{\eta}_t(s) + \phi(s)). \tag{3.2.74}$$

By subtracting (3.2.73) from (3.2.32), it follows that

$$\begin{aligned} \tilde{x}_{ref}(s) =& G(s)\tilde{\eta}_{ref}(s) + H_0(s)\phi_f(s) - (s\mathbb{I}_n - A_m)^{-1}x_0, \\ \tilde{y}_{ref}(s) =& C_m \tilde{x}_{ref}(s), \quad \tilde{z}_{ref}(s) = C_z \tilde{x}_{ref}(s), \end{aligned} \tag{3.2.75}$$

and

$$\tilde{u}_{ref}(s) = C_0(s)\left(\tilde{\eta}_t(s) + \phi(s) - \tilde{\eta}_{ref}(s)\right). \tag{3.2.76}$$

Since $(C_m B_m)^{\dagger}(C_m B_m) = \mathbb{I}_m$, from (3.2.46) one has

$$C_0(s)(\tilde{\eta}_t(s) + \phi(s)) = C_3(s)\tilde{y}(s) - C_4(s)\tilde{v}(s), \tag{3.2.77}$$

and

$$\phi_f(s) = C(s)(\tilde{\eta}_t(s) + \phi(s)) = C_1(s)\tilde{y}(s) - C_2(s)\tilde{v}(s), \tag{3.2.78}$$

where $\{C_1(s), C_2(s)\}$, and $\{C_3(s), C_4(s)\}$ are defined in (3.2.17) and (3.2.38), respectively; $C_1(s)$, $C_2(s)$, $C_3(s)$, and $C_4(s)$ are all stable and proper transfer function matrices. By combining (3.2.14), (3.2.70), and (3.2.75) - (3.2.78), it follows, together with Lemma 3.2.2, that

$$\left\|\tilde{u}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \gamma_{u_0}\rho_0 + \gamma_u \sqrt{\frac{\theta_1}{\Gamma}}, \quad \left\|\tilde{x}_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \gamma_{x_0}\rho_0 + \gamma_x \sqrt{\frac{\theta_1}{\Gamma}}, \tag{3.2.79}$$

where θ_1 is given in (3.2.44). Since the choice of adaptation gains ensures that $\gamma_x\sqrt{\frac{\theta_1}{\Gamma}} < \bar{\gamma}$ and $\gamma_u\sqrt{\frac{\theta_1}{\Gamma}} < \bar{\gamma}$, Equation (3.2.79), along with (3.2.40), implies

$$\left\|\tilde{u}_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{dx}, \quad \left\|\tilde{x}_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{du}, \tag{3.2.80}$$

which shows the clear contradiction to (3.2.68), thus proving (3.2.62). Moreover, by applying the triangular inequality on $\|\tilde{x}_{ref}\|$ and $\|\tilde{u}_{ref}\|$, Equation (3.2.63) follows.

Next, we prove Equation (3.2.64). Notice that $\phi_f(t)$ of Equation (3.2.78) is bounded, since $C_1(s)$, $C_2(s)$ are stable transfer matrices, and Lemma 3.2.2 holds. Let $C(s) = C_f(s\mathbb{I}_m - A_f)^{-1}B_f$, where $A_f \in \mathbb{R}^{n_f\times n_f}$, $B_f \in \mathbb{R}^{n_f\times m}$, $C_f \in \mathbb{R}^{m\times m}$ present a minimal realization. Then, the state-space realization of the system given in (3.2.75) and (3.2.76) can take the form of

$$\begin{aligned} \dot{\tilde{x}}_c(t) &= A_c\tilde{x}_c(t) + B_c\tilde{\eta}_{ref}(t) + B_f\phi_f(t), \\ \tilde{x}_{ref}(t) &= C_c\tilde{x}_c(t), \quad \tilde{x}_c(0) = [-x_0^\top, 0]^\top, \end{aligned} \tag{3.2.81}$$

with

$$A_c = \begin{bmatrix} A_m & B_mC_f \\ 0 & A_f \end{bmatrix}, \quad B_c = \begin{bmatrix} B_m \\ -B_f \end{bmatrix}, \quad B_f = \begin{bmatrix} B_m \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} \mathbb{I}_n & 0 \end{bmatrix}, \tag{3.2.82}$$

where $\tilde{x}_c(t) = [\tilde{x}_{ref}^\top(t), \tilde{x}_f^\top(t)]^\top \in \mathbb{R}^{n_c\times n_c}$ is the state of the system (3.2.81); $n_c = n + n_f$, and $\tilde{x}(t) \in \mathbb{R}^{n_f}$ is an internal state vector. Let $t_m \geq 0$. For a given $\tilde{x}_c(t_m)$ and $t \geq t_m$, the solution of (3.2.81) is given by

$$\tilde{x}_c(t) = e^{A_c(t-t_m)}\tilde{x}_c(t_m) + \int_{t_m}^t e^{A_c(t-\tau)}B_f\phi_f(\tau)d\tau + \int_{t_m}^t e^{A_c(t-\tau)}B_c\tilde{\eta}_{ref}(\tau)d\tau. \tag{3.2.83}$$

Notice that $G(s) = C_c(s\mathbb{I}_{n_c} - A_c)^{-1}B_c = H_0(s)(\mathbb{I}_m - C(s))$. Since Equation (3.2.14) implies $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$, the continuity of the $\mathcal{L}_1$-norm allows to take a sufficiently small $\lambda_0 > 0$ such that $\beta_1 = \|G(s - \lambda_0)\|_{\mathcal{L}_1} < 1/L_{\rho_r}$. Define $A_{\lambda_0} = A_c + \lambda_0\mathbb{I}_{n_c}$, $\bar{x}_c(t) = e^{\lambda_0(t-t_m)}\tilde{x}_c(t)$, $\bar{\phi}_f(t) = e^{\lambda_0(t-t_m)}\phi_f(t)$, and $\bar{\eta}_{ref}(t) = e^{\lambda_0(t-t_m)}\tilde{\eta}_{ref}(t)$. Multiplying both sides of (3.2.83) by $e^{\lambda_0(t-t_m)}C_c$ yields

$$\begin{aligned} \bar{x}_{ref}(t) =& C_ce^{A_{\lambda_0}(t-t_m)}\tilde{x}_c(t_m) + \int_{t_m}^t C_ce^{A_{\lambda_0}(t-\tau)}B_f\bar{\phi}_f(\tau)d\tau \\ &+ \int_{t_m}^t C_ce^{A_{\lambda_0}(t-\tau)}B_c\bar{\eta}_{ref}(\tau)d\tau, \end{aligned} \tag{3.2.84}$$

where $\bar{x}_{ref}(t) = e^{\lambda_0(t-t_m)}\tilde{x}_{ref}(t)$. Notice that A_{λ_0} is Hurwitz by $\|G(s - \lambda_0)\|_{\mathcal{L}_1} < \infty$. From (3.2.70), one has

$$\|\bar{\eta}_{ref}(t)\| \leq L_{\rho_r}\|\bar{x}_{ref}(t)\|, \quad \forall t \geq 0. \tag{3.2.85}$$

By combining (3.2.84) and (3.2.85) and using the fact that $C_c = [\mathbb{I}_{n_c}, 0]$, it follows that for all $t \geq t_m$

$$\|\bar{x}_{ref}\|_{\mathcal{L}_\infty[t_m,t]} \leq \frac{\beta_0}{1-\beta_1 L_{\rho_r}} \|\tilde{x}_c(t_m)\| + \frac{\beta_2}{1-\beta_1 L_{\rho_r}} \left\|\bar{\phi}_f\right\|_{\mathcal{L}_\infty[t_m,t]}, \tag{3.2.86}$$

where $\beta_0 = \sup_{0\leq\tau} \|e^{A_{\lambda_0}\tau}\|$, $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1}$, and $\beta_2 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}B_f\right\|_{\mathcal{L}_1}$. Now, multiplying both sides of (3.2.83) by $e^{\lambda_0(t-t_m)}$, and combining (3.2.85) - (3.2.86), one can obtain

$$\|\bar{x}_c(t)\| \leq \kappa_0 \|\tilde{x}_c(t_m)\| + \kappa_1 \left\|\bar{\phi}_f\right\|_{\mathcal{L}_\infty[t_m,t]},$$

which further yields

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)} \|\tilde{x}_c(t_m)\| + \kappa_1 \|\phi_f\|_{\mathcal{L}_\infty[t_m,t]}, \tag{3.2.87}$$

where

$$\kappa_0 = \beta_0\left(1 + \frac{L_{\rho_r}\beta_3}{1-\beta_1 L_{\rho_r}}\right), \quad \kappa_1 = \beta_2\left(1 + \frac{L_{\rho_r}\beta_3}{1-\beta_1 L_{\rho_r}}\right), \tag{3.2.88}$$

with $\beta_3 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}B_c\right\|_{\mathcal{L}_1}$. Substituting (3.2.59), (3.2.60) and (3.2.78) into (3.2.87) leads to

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)} \|\tilde{x}_c(t_m)\| + \gamma_1 \left(\upsilon_v(t_m, 0) + \sqrt{\frac{\theta_1}{\Gamma}}\right), \tag{3.2.89}$$

where $\upsilon_v(\cdot,\cdot)$ is defined in (3.2.60), and

$$\gamma_1 = \kappa_1 \left(\frac{\|C_1(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_y)}} + \frac{\|C_2(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_v)}}\right).$$

Since Equation (3.2.87) holds for any $0 \leq t_m \leq t$, using Lemma 3.2.2, and Equations (3.2.78) and (3.2.87) results in

$$\|\tilde{x}_c(t_m)\| \leq \kappa_0 e^{-\lambda_0 t_m} \|x_0\| + \gamma_1 \sqrt{n\lambda_{\max}(\bar{P}_v)} \|x_0\| + \gamma_1 \sqrt{\frac{\theta_1}{\Gamma}}.$$

Since $V(0) \leq n\lambda_{\max}(\bar{P}_v)\|x_0\|^2 + \frac{\theta_0}{\Gamma}$ with $\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m)$, setting $t_m = t/2$, and substituting (3.2.60) and (3.2.90) into (3.2.89) yield

$$\|\tilde{x}_c(t)\| \leq \upsilon_{dx}(t)\|x_0\| + \frac{\gamma_{dx}}{\sqrt{\Gamma}}, \quad t \geq 0, \tag{3.2.90}$$

where

$$\begin{aligned}
\upsilon_{dx}(t) =& \kappa_0^2 e^{-\lambda_0 t} + \kappa_0\gamma_1\sqrt{n\lambda_{\max}(\bar{P}_v)} e^{-\frac{\lambda_0}{2}t} + \gamma_1\sqrt{n\lambda_{\max}(\bar{P}_v)} e^{-\frac{\lambda_1^\star}{4}t}, \\
\gamma_{dx} =& (\kappa_0 + 2)\gamma_1\sqrt{\theta_1}.
\end{aligned}$$

Since $\|\tilde{x}_{ref}(t)\| \leq \|\tilde{x}_c(t)\|$, letting $\gamma_{dy} = \|C_m\|\gamma_{dx}$, $\upsilon_{dy}(t) = \|C_m\|\upsilon_{dx}(t)$, $\gamma_{dz} = \|C_z\|\gamma_{dx}$, and $\upsilon_{dz}(t) = \|C_z\|\upsilon_{dx}(t)$ reduces to (3.2.64). This completes the proof. $\square$

The semi-global result for the closed-loop system stability directly follows from Lemma 3.2.1 and Theorem 3.2.1. Notice that the transient performance due to the non-zero initial conditions is upper-bounded by strictly decreasing functions that are not dependent on the adaptation gain. Moreover, high adaptation gains lead to arbitrarily small steady-state performance bounds.

CHAPTER. 4

$\mathcal{L}_1$ adaptive controller for MIMO Systems with Arbitrary Vector Relative Degree

In this chapter, we relax the vector relative degree condition that was made in the previous chapter. Although the relative degree constraint is crucial for the use of the state-decomposition, it may not be satisfied in many physical systems, which in turn limits the range of applications. This chapter shows that the problem can be resolved by introducing a virtual system and a right interactor. Figures 4.1 and 4.2 illustrate the uncertain open-loop system and the cascaded connection of the virtual system and the inverse system of a right interactor, where $M(s)$ is the desired system, and $Z(s)$ denotes the inverse of a right interactor. The main idea is to design the estimation loop based on the virtual system, in which state-decomposition is allowed. Since the cascaded connection fully describes the open-loop system, this method is effective to tackle the systems with high relative degree.

This chapter starts by introducing an $\mathcal{L}_1$ adaptive controller for nonlinear underactuated systems. Next, we present the piecewise constant adaptation laws for the $\mathcal{L}_1$ adaptive controller, which are more suitable for real-time applications.

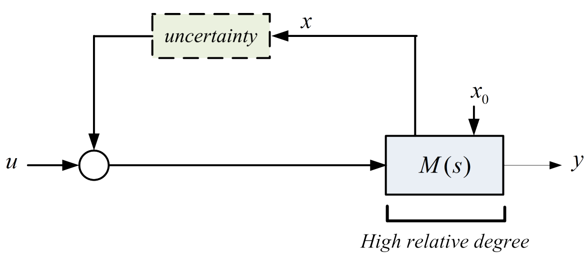

Figure 4.1: Uncertain system

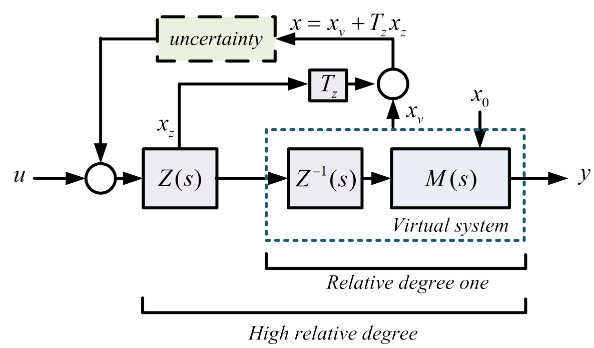

Figure 4.2: Cascaded representation

4.1. $\mathcal{L}_1$ Adaptive Control for Nonlinear Systems

In this section, we introduce the $\mathcal{L}_1$ adaptive output-feedback controller for the systems with arbitrary relative degree. Consider the following MIMO system

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B_m(\omega u(t) + f(x,t))\,, \\ y(t) &= C_m x(t)\,, \quad x(0) = x_0\,, \end{aligned} \tag{4.1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and measurable output vectors, respectively, with $p \geq m$, and $x_0 \in \mathbb{R}^n$ is the initial state. Moreover, $A_m \in \mathbb{R}^{n\times n}$ is a known Hurwitz matrix, $B_m \in \mathbb{R}^{n\times m}$ and $C_m \in \mathbb{R}^{p\times n}$ are known matrices. Let (A_m, B_m, C_m) be the minimal realization of $M(s) = C_m (s\mathbb{I}_n - A_m)^{-1} B_m$, which describes the desired dynamics of the closed-loop system; suppose $M(s)$ has full column rank m. Finally, $\omega > 0$ is an unknown constant input gain, and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function representing system uncertainties.

Assumption 4.1.1. *$M(s)$ does not have unstable transmission zeros.*

Remark 4.1.1. *Notice that we do not require $(C_m B_m)$ be full rank.*

Assumption 4.1.2. *The unknown constant input gain satisfies $\omega \in \mathcal{C}_\omega$, where $\mathcal{C}_\omega = [\omega_l, \omega_u]$ is a known compact set with $0 < \omega_l < \omega_u$.*

Assumption 4.1.3. *There exists $b_0 > 0$ such that*

$$\|f(0,t)\| < b_0, \quad \forall t \geq 0,$$

where b_0 is a known constant. Moreover, for any $\delta > 0$ there exist $d_\delta > 0$, and $b_\delta > 0$ such that

$$\left\| \frac{\partial f(x,t)}{\partial x} \right\| \leq d_\delta, \quad \left\| \frac{\partial f(x,t)}{\partial t} \right\| \leq b_\delta, \quad \forall \|x\| < \delta,$$

where d_δ and b_δ are known constants.

Then, the problem at hand is stated as follows:

Problem 4.1.1. *Consider the system described by Equation (4.1.1) satisfying Assumptions 4.1.1-4.1.3. Design an adaptive control law for $u(t)$ such that $y(t)$ tracks the desired response $y_m(t)$ both in transient and steady state, where $y_m(t)$ is the signal with the Laplace transform of $y_m(s) = M(s)K_g r(s)$ with $K_g \in \mathbb{R}^{m \times m_r}$ being a feed-forward gain, and $r(t) \in \mathbb{R}^{m_r}$ being a reference signal.*

4.1.1. *Uncertainty parameterization*

Let $\{A_z, B_z, C_z, D_z\}$ be the set of system matrices of $Z(s)$ defined for $M(s)$, and $T_z \in \mathbb{R}^{n \times n_z}$, $\bar{B} \in \mathbb{R}^{n \times m}$ be matrices satisfying (2.1.30). Consider the following systems:

$$\begin{aligned} \dot{x}_u(t) =& A_z x_u(t) + B_z u(t), \\ u_z(t) =& C_z x_u(t) + D_z u(t), \quad x_u(0) = 0, \end{aligned} \tag{4.1.2}$$

and

$$\dot{x}_f(t) = A_z x_f(t) + B_z f(T_g x_g + T_z x_f, t), \quad x_f(0) = 0, \tag{4.1.3}$$

where $x_g(t) = [x_v^\top(t), x_u^\top(t)]^\top$, $T_g = [\mathbb{I}_n, \omega T_z]$, and $f(\cdot, t)$ satisfy Assumption 4.1.3. The state $x_v(t) \in \mathbb{R}^n$ is governed by the following *virtual* system:

$$\begin{aligned} \dot{x}_v(t) =& A_m x_v(t) + \bar{B}(\omega u_z(t) + \bar{f}(X,t)), \\ y_v(t) =& C_m x_v(t), \quad x_v(0) = x_0, \end{aligned} \tag{4.1.4}$$

where

$$\bar{f}(X,t) = C_z x_f(t) + D_z f(T_g x_g + T_z x_f, t), \tag{4.1.5}$$

with $X = [x_g^\top(t), x_f^\top(t)]^\top$. By letting $x_z(t) = x_f(t) + \omega x_u(t)$, from Corollary 2.1.3 and Equations (4.1.2) - (4.1.5) we have $x(t) = T_g x_g(t) + T_z x_f(t)$ and $y_v(t) = y(t)$ for any $t \geq 0$, where $x(t)$, $y(t)$ are solutions of (4.1.1).

Lemma 4.1.1. *Consider the systems in* (4.1.2) - (4.1.5). *Let* $\tau > 0$, $\rho_x > 0$, *and* $\rho_u > 0$. *Suppose* $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x$, *and* $\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u$, *where* $x(t) = T_g x_g(t) + T_z x_f(t)$. *The function* $\bar{f}(X,t)$ *in* (4.1.5) *can be represented as:*

$$\bar{f}(X,t) = \theta(t)\|x_g(t)\| + \sigma(t), \quad 0 \leq t \leq \tau,$$

where

$$\begin{aligned} \|\theta(t)\| \leq \bar{d}_{\rho_x} \quad \|\dot{\theta}(t)\| \leq \bar{l}_\theta, \\ \|\sigma(t)\| \leq \bar{b}_{\rho_x}, \quad \|\dot{\sigma}(t)\| \leq \bar{l}_\sigma, \end{aligned} \tag{4.1.6}$$

with $\bar{l}_\theta$, $\bar{l}_\sigma$ *being computable finite bounds, and* $\bar{d}_{\rho_x}$, $\bar{b}_{\rho_x}$ *being given by*

$$\begin{aligned} \bar{d}_{\rho_x} =& \max_{\omega \in \mathcal{C}_\omega} (\|C_z T_z^\dagger T_g\| + \|D_z\|\|T_g\| d_{\rho_x}), \\ \bar{b}_{\rho_x} =& \|C_z T_z^\dagger\|\rho_x + \|D_z\| \, \|T(s)\|_{\mathcal{L}_1} \, d_{\rho_x}^2 \rho_x \\ &+ \|D_z\|(\|T(s)\|_{\mathcal{L}_1} \, d_{\rho_x} + 1) b_0. \end{aligned} \tag{4.1.7}$$

with $T(s) = T_z(s\mathbb{I}_{n_z} - A_z)^{-1} B_z$, *and* b_0, d_{ρ_x} *being given in Assumption 4.1.3.*

Proof. Since $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x$, from (4.1.3) it follows that

$$\|T_z x_z\|_{\mathcal{L}_\infty[0,\tau]} \leq \|T(s)\|_{\mathcal{L}_1} \, d_{\rho_x} \rho_x + \|T(s)\|_{\mathcal{L}_1} \, b_0, \tag{4.1.8}$$

where d_{ρ_x}, b_0 are given in Assumption 4.1.3, and

$$T(s) = T_z(s\mathbb{I}_{n_z} - A_z)^{-1} B_z.$$

Moreover, notice that $x_z(t) = T_z^\dagger T_g x_g(t) + T_z^\dagger x(t)$, where $T_z^\dagger$ is the generalized inverse of T_z. From (4.1.5), one has

$$\bar{f}(X,t) =C_z T_z^\dagger T_g x_g(t) + C_z T_z^\dagger x(t) + D_z f(T_g x_g + T_z x_f, t), \tag{4.1.9}$$

Using Assumption 4.1.3 on $f(x,t)$, it can be shown that the partial derivatives of $\bar{f}(X,t)$ are (semi-globally) bounded. Since $\|T_g x_g + T_z x_f\|_{\mathcal{L}_\infty[0,\tau]} \leq \rho_x$ holds, from (4.1.8) and (4.1.9) it follows that

$$\|\bar{f}(X,t)\| < \bar{d}_{\rho_x}\|x_g(t)\| + \bar{b}_{\rho_x}, \quad 0 \le t \le \tau,$$

where $\bar{d}_{\rho_x}$, $\bar{b}_{\rho_x}$ are given in (4.1.7). Finally, since $\|x_g\|_{\mathcal{L}_\infty[0,\tau]}$ and $\|\dot{x}_g\|_{\mathcal{L}_\infty[0,\tau]}$ are finite, from Lemma 2.3.1 the conclusion follows. This completes the proof. □

Remark 4.1.2. *The conservative bounds on $\theta(t)$ and $\sigma(t)$ are dependent on the choice of $Z(s)$.*

4.1.2. *Design of $\mathcal{L}_1$ adaptive controller*

In this section, the $\mathcal{L}_1$ adaptive output feedback controller that solves Problem 4.1.1 is presented. Before stating the main result, we introduce several design variables. Let $\rho_0 > 0$ be a given constant satisfying $\|x_0\| \le \rho_0$ with $x_0 \in \mathbb{R}^n$ being an initial condition, and choose $\bar{\gamma} > 0$ to be an arbitrarily small constant. For a given $\delta > 0$ define

$$L_\delta = \frac{\bar{\delta}(\delta)}{\delta} d_{\bar{\delta}(\delta)}, \quad \bar{\delta}(\delta) = \delta + \bar{\gamma}\,, \tag{4.1.10}$$

where $d_{\bar{\delta}(\delta)}$ is introduced in Assumption 4.1.3. Choose $Z^{-1}(s)$ to be a right interactor of $sM(s)$ such that

$$Z(s) = C_z(s\mathbb{I}_{n_z} - A_z)^{-1}B_z + D_z,$$

where A_z is Hurwitz, and $\{A_z \in \mathbb{R}^{n_z\times n_z}, B_z \in \mathbb{R}^{n_z\times m}, C_z \in \mathbb{R}^{m\times n_z}\}$ is a minimal realization of $Z(s)$. Notice that the existence of $Z(s)$ is guaranteed by Theorem 2.1.4. Now, let $T_z \in \mathbb{R}^{n\times n_z}$ and $\bar{B} \in \mathbb{R}^{n\times m}$ be matrices that satisfy Corollary 2.1.2. Choose $K_v \in \mathbb{R}^{n\times p}$ as a stabilizing gain so that

$$A_v = A_H + K_v C_m \tag{4.1.11}$$

is Hurwitz (from Lemma 2.1.19 such K_v exists), where

$$A_H = (\mathbb{I}_n - HC_m)A_m, \quad H = \bar{B}(C_m\bar{B})^\dagger, \tag{4.1.12}$$

with $(C_m\bar{B})^\dagger$ being the generalized inverse of $(C_m\bar{B})$. Let $P_y \in \mathbb{R}^{p\times p}$ be a given positive definite matrix, and $P_v \in \mathbb{R}^{n\times n}$ be the positive definite matrix which solves

$$A_v^\top P_v + P_v A_v = -Q \tag{4.1.13}$$

for a positive definite $Q \in \mathbb{R}^{n\times n}$ with $\epsilon_q < \lambda_{\min}(Q)$. Define

$$\begin{gathered}\kappa_m = \sup_{t\ge 0}\|e^{A_m t}\|,\\ \kappa_y = \sqrt{n\frac{\lambda_{\max}(\bar{P}_v)}{\lambda_{\min}(P_v)}}, \quad \kappa_v = \sqrt{n\frac{\lambda_{\max}(\bar{P}_v)}{\lambda_{\min}(P_y)}},\end{gathered} \tag{4.1.14}$$

where $\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m)$. Let $D(s)$ be a $m \times m$ transfer matrix such that, for all $\omega \in \mathcal{C}_\omega$

$$C(s) = \omega C_0(s)$$

is stable with $C(0) = \mathbb{I}_m$, and $C(s)Z^{-1}(s)$ is strictly proper, where

$$C_0(s) = D(s)(\mathbb{I}_m + \omega D(s))^{-1}. \tag{4.1.15}$$

Moreover, it is assumed that $D(s)$ ensures that there exists $\rho_r > 0$ such that

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_r - \rho_{ext} - \rho_{int}}{L_{\rho_r}\rho_r}, \quad \omega \in \mathcal{C}_\omega, \tag{4.1.16}$$

where

$$\begin{aligned}
\rho_{ext} =& \|H_0(s)C(s)K_g(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} b_0, \\
\rho_{int} =& (\kappa_m + \kappa_x)\rho_0, \\
\kappa_x =& \|H_1(s)\|_{\mathcal{L}_1} \kappa_y + \|H_2(s)\|_{\mathcal{L}_1} \kappa_v,
\end{aligned} \tag{4.1.17}$$

with κ_m, κ_y, and κ_v being given in (4.1.14). Moreover,

$$\begin{aligned}
H_0(s) =& (s\mathbb{I}_n - A_m)^{-1}B_m, \\
H_1(s) =& \omega H_0(s)C_1(s), \quad H_2(s) = \omega H_0(s)C_2(s), \\
G(s) =& H_0(s)\left(\mathbb{I}_m - C(s)\right),
\end{aligned} \tag{4.1.18}$$

and

$$\begin{aligned}
C_1(s) =& (s + \alpha)C_0(s)Z^{-1}(s)(C_m\bar{B})^\dagger, \\
C_2(s) =& C_0(s)Z^{-1}(s)(C_m\bar{B})^\dagger C_m A_m,
\end{aligned} \tag{4.1.19}$$

where $\alpha > 0$ will be defined later. Notice that L_{ρ_r} satisfies (4.1.10) with d_{ρ_x} and

$$\rho_x = \rho_r + \bar{\gamma}. \tag{4.1.20}$$

Finally, let $\alpha > 0$ be chosen to satisfy

$$\alpha_y = 2\alpha - \alpha_\phi > 0, \quad \alpha_\phi = \frac{m\bar{d}_{\rho_x}^2}{\epsilon_q}\left\|\sqrt{P_y}C_m\bar{B}\right\|_2^2, \tag{4.1.21}$$

where $\bar{d}_{\rho_x}$ is given in (4.1.7), and $\sqrt{P_y} \in \mathbb{R}^{p\times p}$ is the upper triangular matrix satisfying the Cholesky decomposition; $P_y = \sqrt{P_y}^\top \sqrt{P_y}$.

Remark 4.1.3. *Clearly, for small $\bar{\gamma} > 0$, we have $\rho_x \approx \rho_r$; ρ_r is used to characterize the conservative bounds on the positive invariant set of the closed-loop system.*

Next, consider the following control law

$$u(s) = D(s)K_g r(s) - D(s)Z^{-1}(s)\hat{\eta}_t(s)\,, \tag{4.1.22}$$

where $\hat{\eta}_t(s)$ is the Laplace transform of

$$\hat{\eta}_t(t) = \hat{\omega}(t)u_z(t) + \hat{\theta}(t)\|\hat{x}_g(t)\| + \hat{\sigma}(t), \tag{4.1.23}$$

and $\hat{\omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ are defined later in the adaptive laws, $u_z(t)$ is given in (4.1.2), and $x_g(t) = [\hat{x}_v^\top(t), x_u^\top(t)]^\top$; $x_u(t)$ is defined in (4.1.2), and $\hat{x}_v(t) = \hat{v}(t) + Hy(t)$ with $\hat{v}(t)$ being given by the following predictor:

$$\begin{aligned}\dot{\hat{v}}(t) =& A_v\hat{x}_v(t) - K_v y(t) - P_v^{-1}A_m^\top C_m^\top P_y\tilde{y}(t),\\ \dot{\hat{y}}(t) =& -\alpha\tilde{y}(t) + C_m A_m\hat{x}_v(t) + C_m\bar{B}\hat{\eta}_t(t),\\ \hat{v}(0) =& 0, \quad \hat{y}(t) = y_0,\end{aligned} \tag{4.1.24}$$

where $y_0 = C_m x_0$ is assumed to be known, $\tilde{y}(t) = \hat{y}(t) - y(t)$, and A_v is given in (4.1.11). Finally, the following adaptive laws are used:

$$\begin{aligned}\dot{\hat{\omega}}(t) =& \Gamma_\omega \mathrm{Proj}(\hat{\omega}(t), -u_z^\top(t)e_y(t)), \quad \hat{\omega}(0) = 1,\\ \dot{\hat{\theta}}(t) =& \Gamma_\theta \mathrm{Proj}(\hat{\theta}(t), -\|\hat{x}_g(t)\|e_y(t)), \quad \hat{\theta}(0) = 0,\\ \dot{\hat{\sigma}}(t) =& \Gamma_\sigma \mathrm{Proj}(\hat{\sigma}(t), -e_y(t)), \quad \hat{\sigma}(0) = 0,\end{aligned} \tag{4.1.25}$$

where Γ_ω, Γ_θ, Γ_σ are positive adaptation gains, $e_y(t) = \bar{B}^\top C_m^\top P_y\tilde{y}(t)$, and $\mathrm{Proj}(\cdot,\cdot)$ denotes the projection operator [21].

4.1.3. *Stability and Performance Analysis*

Consider the following closed-loop reference system

$$\begin{aligned}\dot{x}_{ref}(t) =& A_m x_{ref}(t) + B_m\left(\omega u_{ref}(t) + f(x_{ref}, t)\right),\\ y_{ref}(t) =& C_m x_{ref}(t), \quad x_{ref}(0) = 0,\end{aligned} \tag{4.1.26}$$

with

$$u_{ref}(s) = C_0(s)\left(K_g r(s) - \eta_{ref}(s) - \sigma(s)\right), \tag{4.1.27}$$

where $x_{ref}(t) \in \mathbb{R}^n$, $y_{ref}(t) \in \mathbb{R}^p$ are the reference system states and outputs, respectively, $r(s)$ is the Laplace transform of the reference command $r(t) \in \mathbb{R}^{m_r}$, $K_g \in \mathbb{R}^{m\times m_r}$ is a feed-forward gain, and $C_0(s)$ is given in (4.1.15). Moreover, $\eta_{ref}(s)$ and $\sigma(s)$ are the Laplace transforms of

$$.\eta_{ref}(t) = f(x_{ref}, t) - f(0, t), \quad \sigma(t) = f(0, t), \tag{4.1.28}$$

respectively.

The reference system given in (4.1.26) and (4.1.27) assumes compensation of uncertainties within the filter bandwidth.

Lemma 4.1.2. *Consider the closed-loop reference system given in* (4.1.26) *and* (4.1.27) *and design constraints defined in* (4.1.10) - (4.1.20)*. Then, for each* $\omega \in \mathcal{C}_\omega$ *and* $\tau > 0$ *the following bound holds*

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{rx}, \tag{4.1.29}$$

where

$$\rho_{rx} = \rho_r - \frac{\rho_{int}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}} > 0, \tag{4.1.30}$$

with ρ_{int}*,* $G(s)$ *given in* (4.1.17) *and* (4.1.18)*, respectively. Moreover,*

$$\left\|u_{ref_\tau}\right\|_{\mathcal{L}_\infty} \leq \rho_{ru}, \tag{4.1.31}$$

where

$$\rho_{ru} = \|C_0(s)\|_{\mathcal{L}_1} (L_{\rho_r}\rho_{rx} + b_0) + \|C_0(s)K_g\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}, \tag{4.1.32}$$

with $C_0(s)$ *defined in* (4.1.15)*.*

Proof. Notice that from (4.1.16) and (4.1.30) one has

$$\rho_{rx} > \rho_{ext} \geq 0\,, \tag{4.1.33}$$

where ρ_{ext} is defined in (4.1.17). Next, it will be shown that Equation (4.1.29) holds by a contradiction argument. Suppose it is not true for some $\tau' > 0$. Then, since $x_{ref}(t)$ is continuous and $x_{ref}(0) = 0$, there exists $\tau' \in [0, \tau]$ such that

$$\|x_{ref}(\tau')\| = \rho_{rx}\,, \quad \|x_{ref}(t)\| < \rho_{rx}\,, \quad \forall t \in [0, \tau')\,,$$

which yields

$$\left\|x_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} = \rho_{rx}\,. \tag{4.1.34}$$

By substituting the control law in (4.1.27) into (4.1.26), the Laplace transform of the closed-loop reference system is written as

$$\begin{aligned} x_{ref}(s) &= H_0 C(s) K_g r(s) + G(s)\eta_{ref}(s) + G(s)\sigma(s)\,, \\ y_{ref}(s) &= C_m x_{ref}(s)\,, \end{aligned} \tag{4.1.35}$$

which further leads to

$$\left\|x_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq \|H_0(s)C(s)K_g\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} \left\|\eta_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} \|\sigma\|_{\mathcal{L}_\infty}, \tag{4.1.36}$$

where $C(s)$, $\{H_0(s), G(s)\}$ are given in (4.1.2) and (4.1.18), respectively, and $\eta_{ref}(s)$, $\sigma(s)$ are the Laplace transform signals defined in (4.1.28). Since $\left\|x_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq \rho_{rx} < \rho_x$ and $d_{\rho_x} < L_{\rho_r}$ hold, from Assumption 4.1.3 it follows that

$$\left\|\eta_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq L_{\rho_r} \left\|x_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty}, \quad \|\sigma\|_{\mathcal{L}_\infty} \leq b_0. \tag{4.1.37}$$

Since Equation (4.1.16) implies $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$, combining (4.1.36) and (4.1.37) yields

$$\left\|x_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq \frac{\|H_0(s)C(s)K_g\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} b_0}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}},$$

which, together with (4.1.16), leads to

$$\rho_{xr} = \left\|x_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq \frac{\rho_{ext} + \|G(s)\|_{\mathcal{L}_1} b_0}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}} < \rho_r - \frac{\rho_{int}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}} = \rho_{xr} \tag{4.1.38}$$

Therefore, Equation (4.1.38) shows a contradiction to (4.1.34), thus proving (4.1.29). Finally, Equation (4.1.31) follows from combining (4.1.27) and (4.1.37). This completes the proof. □

Notice that the condition given in (4.1.16) depends on the upper bound of the partial derivative of $f(x,t)$, which, in turn, depends upon the unknown initial condition. Thus, the stability result in Lemma 4.1.2 is semi-global. However, in the case, when the uncertain function $f(x,t)$ has globally bounded partial derivatives (e.g. $d_\delta \equiv L$ for some constant $L > 0$), it is straightforward to verify that Equation (4.1.16) provides a uniform condition for stabilization (i.e., $\|G(s)\|_{\mathcal{L}_1} L < 1$).

Remark 4.1.4. *Notice that the present approach requires a minimum order filter (i.e., $C(s)Z^{-1}(s)$ is proper), when the system has high relative degree. Such condition is typical for output-feedback approaches. For example, the methods of [55, 57] require choosing a low-pass filter dependent upon the systems relative degree.*

To demonstrate the stability of the closed-loop system with the proposed $\mathcal{L}_1$ adaptive control laws (4.1.22)-(4.1.25), it will be shown that the difference between the closed-loop system and the ideal reference system is semi-globally bounded with arbitrarily small steady-state bounds. Before stating the main results, we introduce a few variables of interest. Let

$$
\begin{aligned}
\gamma_{u_0} =& \left\|C_0(s)\right\|_{\mathcal{L}_1} L_{\rho_r}\gamma_{x_0} + (\left\|C_1(s)\right\|_{\mathcal{L}_1}\kappa_y + \left\|C_2(s)\right\|_{\mathcal{L}_1}\kappa_v),\\
\gamma_{x_0} =& \frac{\kappa_x+\kappa_m}{1-\left\|G(s)\right\|_{\mathcal{L}_1} L_{\rho_r}},\\
\gamma_u =& \left\|C_0(s)\right\|_{\mathcal{L}_1} L_{\rho_r}\gamma_x + \frac{\left\|C_1(s)\right\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_y)}} + \frac{\left\|C_2(s)\right\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_v)}},\\
\gamma_x =& \frac{\lambda_{\min}(P_y)^{-\frac{1}{2}}\left\|H_1(s)\right\|_{\mathcal{L}_1} + \lambda_{\min}(P_v)^{-\frac{1}{2}}\left\|H_2(s)\right\|_{\mathcal{L}_1}}{1-\left\|G(s)\right\|_{\mathcal{L}_1} L_{\rho_r}},
\end{aligned} \tag{4.1.39}
$$

where $\{\kappa_m, \kappa_y, \kappa_v\}$, κ_x, $\{H_1(s), H_2(s)\}$, and $\{C_1(s), C_2(s)\}$ are given in (4.1.14), (4.1.17), (4.1.18), and (4.1.19), respectively. Next, $\epsilon_\gamma > 0$ is chosen to satisfy

$$
\gamma_x\epsilon_\gamma < \bar{\gamma}, \quad \gamma_u\epsilon_\gamma < \bar{\gamma}, \quad \forall\omega\in\mathcal{C}_\omega. \tag{4.1.40}
$$

Finally, let ρ_u, ρ_{dx}, and ρ_{du} be

$$
\begin{aligned}
\rho_u &= \rho_{ru} + \rho_{du},\\
\rho_{dx} = \gamma_{x_0}\rho_0 + \bar{\gamma}, &\quad \rho_{du} = \gamma_{u_0}\rho_0 + \bar{\gamma},
\end{aligned} \tag{4.1.41}
$$

respectively, where ρ_{ru} is defined in (4.1.32).

Lemma 4.1.3. *Consider the system given by Equation* (4.1.1) *with control law defined in* (4.1.22)-(4.1.25)*. Let $\tau > 0$ be a positive constant. If $\|x_\tau\|_{\mathcal{L}_\infty} \le \rho_x$ and $\|u_\tau\|_{\mathcal{L}_\infty} \le \rho_u$, then the output-estimation error for all $t \in [0,\tau]$ is bounded as follows:*

$$
\|\tilde{y}(t)\| \le \kappa_y e^{-\frac{\lambda_1}{2}t}\|x_0\| + \sqrt{\frac{\theta_1}{\lambda_{\min}(P_y)}}\frac{1}{\sqrt{\Gamma}}, \tag{4.1.42}
$$

where κ_y is defined in (4.1.14)*, and*

$$
\begin{aligned}
\lambda_1 &= \min\left(\frac{\lambda_{\min}(Q_v)}{\lambda_{\max}(P_v)}, \alpha_y\right),\\
\theta_1 &= \theta_0 + 4m\frac{\bar{d}_{\rho_x}\bar{l}_\theta + \bar{b}_{\rho_x}\bar{l}_\sigma}{\lambda_1},\\
\theta_0 &= 4\left(\omega_u^2 + m\bar{d}_{\rho_x}^2 + m\bar{b}_{\rho_x}^2\right),\\
\Gamma &= \min\left(\Gamma_\omega, \Gamma_\theta, \Gamma_\sigma\right),
\end{aligned} \tag{4.1.43}
$$

with $Q_v = Q - \epsilon_q\mathbb{I}_n \succ 0$, $\alpha_y > 0$ is given in (4.1.21)*, and $\bar{d}_{\rho_x}$, $\bar{l}_\theta$, $\bar{b}_{\rho_x}$, $\bar{l}_\sigma$ are bounds satisfying* (4.1.6)*.*

Proof. Notice that Lemma 4.1.1 holds from the hypothesis. From (4.1.2) - (4.1.4) it follows that

$$\begin{aligned}\dot{x}_v(t) =& A_m x_v(t) + \bar{B}(\omega u_z(t) + \theta(t)\|x_g(t)\| + \sigma(t)),\\ y(t) =& C_m x_v(t), \quad x_v(0) = x_0,\end{aligned} \tag{4.1.44}$$

where $x_g(t) = [x_v^\top(t), x_u^\top(t)]$, and $x_u(t)$, $u_z(t)$ are given in (4.1.2). Notice that $x(t) = x_v(t) + T_z(\omega x_u(t) + x_f(t))$ holds, where $x_f(t)$ is defined in (4.1.3), and $T_z \in \mathbb{R}^{n\times n_f}$ satisfies (2.1.30). Next, define $v(t) = (\mathbb{I}_n - HC_m)x_v(t)$. By pre-multiplying both sides of (4.1.44) by $(\mathbb{I}_n - HC_m)$ and taking the derivative of $y(t)$, it follows that

$$\begin{aligned}\dot{v}(t) =& A_H v(t) + A_H H y(t), \quad v(0) = v_0,\\ \dot{y}(t) =& C_m A_m v(t) + C_m A_m H y(t) + C_m\bar{B}(\omega u_z(t) + \theta(t)\|x_g(t)\| + \sigma(t)), \quad y(0) = y_0,\end{aligned} \tag{4.1.45}$$

where $v_0 = (\mathbb{I}_n - HC_m)x_0$, and $\{A_H, H\}$ is given in (4.1.12). Let

$$\tilde{\eta}_t(t) = \tilde{\omega}(t)u_z(t) + \tilde{\theta}(t)\|\hat{x}_g(t)\| + \tilde{\sigma}(t), \tag{4.1.46}$$

and

$$\phi(t) = \theta(t)(\|\hat{x}_g(t)\| - \|x_g(t)\|), \tag{4.1.47}$$

where $\tilde{\omega}(t) = \hat{\omega}(t) - \omega$, $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$, and $\tilde{\sigma}(t) = \hat{\sigma}(t) - \sigma(t)$. Define

$$\tilde{v}(t) = \hat{v}(t) - v(t), \quad \tilde{y}(t) = \hat{y}(t) - y(t). \tag{4.1.48}$$

Then, subtracting (4.1.45) from (4.1.24) yields

$$\begin{aligned}\dot{\tilde{v}}(t) =& A_v\tilde{v}(t) - P_v^{-1}A_m^\top C_m^\top P_y\tilde{y}(t),\\ \dot{\tilde{y}}(t) =& -\alpha\tilde{y}(t) + C_m A_m\tilde{v}(t) + C_m\bar{B}(\tilde{\eta}_t(t) + \phi(t)),\\ \tilde{v}(0) =& -v_0, \quad \tilde{y}(0) = 0,\end{aligned} \tag{4.1.49}$$

where A_v is Hurwitz (see (4.1.11)), and $\tilde{\eta}_t(t)$, $\phi(t)$ are given in (4.1.46), and (4.1.47), respectively. Now, consider the Lyapunov function:

$$V(t) = \tilde{v}^\top(t)P_v\tilde{v}(t) + \tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{\tilde{\omega}^2(t)}{\Gamma_\omega} + \frac{\tilde{\theta}^\top(t)\tilde{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\tilde{\sigma}(t)}{\Gamma_\sigma}. \tag{4.1.50}$$

Taking the derivative of (4.1.50), and substituting (4.1.25) and (4.1.49), one has

$$\begin{aligned}\dot{V}(t) \leq& -\tilde{v}^\top(t)Q\tilde{v}(t) - 2\alpha\tilde{y}^\top(t)P_y\tilde{y}(t) - \frac{2\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta}\\ &- \frac{2\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma} + 2\tilde{y}^\top(t)P_yC_m\bar{B}\phi(t),\end{aligned} \tag{4.1.51}$$

where $Q \succ \epsilon_q\mathbb{I}_n$ is positive definite satisfying (4.1.13). Notice that $\|\|\hat{x}_g(t)\| - \|x_g(t)\|\| \leq \|\tilde{v}(t)\|$

holds. Then from (4.1.47) and (4.1.6) it follows that

$$2\tilde{y}^\top(t)P_yC_m\bar{B}\phi(t) \leq \alpha_\phi\tilde{y}^\top(t)P_y\tilde{y}(t) + \frac{m\bar{d}_{\rho_x}^2\|\sqrt{P_y}C_m\bar{B}\|_2^2}{\alpha_\phi}\|\tilde{v}(t)\|_2^2, \tag{4.1.52}$$

where $\bar{d}_{\rho_x}$, α_ϕ are given in (4.1.7) and (4.1.21), respectively. Further, from (4.1.51) and (4.1.52) one has

$$\dot{V}(t) \leq -\tilde{v}^\top(t)Q_v\tilde{v}(t) - \alpha_y\tilde{y}^\top(t)P_y\tilde{y}(t) - \frac{2\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} - \frac{2\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma}, \tag{4.1.53}$$

where $Q_v = Q - \epsilon_q\mathbb{I}_n \succ 0$ and $\alpha_y > 0$ (see (4.1.21)). Notice that from Lemma 4.1.1 it follows that for $0 \leq t \leq \tau$

$$\frac{2\tilde{\theta}^\top(t)\dot{\theta}(t)}{\Gamma_\theta} + \frac{2\tilde{\sigma}^\top(t)\dot{\sigma}(t)}{\Gamma_\sigma} \leq \frac{\theta_1 - \theta_0}{\Gamma}\lambda_1,$$

and the projection operator in (4.1.25) ensures

$$\frac{\tilde{\omega}^2(t)}{\Gamma_\omega} + \frac{\tilde{\theta}^\top(t)\tilde{\theta}(t)}{\Gamma_\theta} + \frac{\tilde{\sigma}^\top(t)\tilde{\sigma}(t)}{\Gamma_\sigma} \leq \frac{\theta_0}{\Gamma}, \tag{4.1.54}$$

where Γ, θ_0, θ_1, λ_1 are given in (4.1.43). Since

$$-\tilde{v}^\top(t)Q_v\tilde{v}(t) - \alpha_y\tilde{y}^\top(t)P_y\tilde{y}(t) \leq -\lambda_1(\tilde{v}^\top(t)P_v\tilde{v}(t) + \tilde{y}^\top(t)P_y\tilde{y}(t)),$$

combining (4.1.53) - (4.1.54), along with (4.1.50), leads to

$$\dot{V}(t) \leq -\lambda_1\left(V(t) - \frac{\theta_1}{\Gamma}\right).$$

Choose $t_0 \in \mathbb{R}$ to be $0 \leq t_0 \leq t \leq \tau$. Then, Gronwell-Bellman inequality yields

$$\sqrt{V(t)} \leq \upsilon_v(t, t_0), \quad 0 \leq t_0 \leq t \leq \tau, \tag{4.1.55}$$

which gives

$$\|\tilde{v}(t)\| \leq \frac{\upsilon_v(t, t_0)}{\sqrt{\lambda_{\min}(P_v)}}, \quad \|\tilde{y}(t)\| \leq \frac{\upsilon_v(t, t_0)}{\sqrt{\lambda_{\min}(P_y)}}, \tag{4.1.56}$$

where

$$\upsilon_v(t, t_0) = \sqrt{\left(V(t_0) - \frac{\theta_1}{\Gamma}\right)e^{-\lambda_1(t-t_0)} + \frac{\theta_1}{\Gamma}}. \tag{4.1.57}$$

Finally, since $V(0) \leq x_0^\top\bar{P}_vx_0 + \frac{\theta_0}{\Gamma}$ with $\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m)$, from letting $t_0 = 0$ it follows that

$$\begin{aligned}
\|\tilde{y}(t)\| \leq & \kappa_y e^{-\frac{\lambda_1}{2}t}\|x_0\| + \sqrt{\frac{\theta_1}{\lambda_{\min}(P_y)}}\frac{1}{\sqrt{\Gamma}}, \\
\|\tilde{v}(t)\| \leq & \kappa_v e^{-\frac{\lambda_1}{2}t}\|x_0\| + \sqrt{\frac{\theta_1}{\lambda_{\min}(P_v)}}\frac{1}{\sqrt{\Gamma}},
\end{aligned} \tag{4.1.58}$$

where κ_v, κ_y are given in (4.1.14). This completes the proof. □

Notice that high adaptation gains make the estimation errors arbitrarily small.

Theorem 4.1.1. *Consider the closed-loop system with $\mathcal{L}_1$ adaptive output feedback controller defined via* (4.1.22)-(4.1.25)*, subject to the design constraints in* (4.1.10)-(4.1.21)*. Suppose the adaptation gains are chosen sufficiently high to satisfy*

$$\Gamma > \frac{\theta_1}{{\epsilon_\gamma}^2},$$

where Γ, θ_1 are defined in (4.1.43)*, and ϵ_γ satisfies* (4.1.40)*. Then, the following upper bounds hold:*

$$\|x_{ref} - x\|_{\mathcal{L}_\infty} \leq \rho_{dx}, \quad \|u_{ref} - u\|_{\mathcal{L}_\infty} \leq \rho_{du}, \tag{4.1.59}$$

and

$$\begin{aligned}
\|y_{ref} - y\|_{\mathcal{L}_\infty} &\leq \|C_m\|\rho_{dx}, \\
\|x\|_{\mathcal{L}_\infty} \leq \rho_x, & \quad \|u\|_{\mathcal{L}_\infty} \leq \rho_u.
\end{aligned} \tag{4.1.60}$$

Moreover, for each $\omega \in \mathcal{C}_\omega$ there exist positive constants γ_{dx}, γ_{dy}, and γ_{dz}, and strictly decreasing functions of $\upsilon_{dx}(t)$, $\upsilon_{dy}(t)$, and $\upsilon_{dz}(t)$, such that for all $t \geq 0$

$$\begin{aligned}
\|x_{ref}(t) - x(t)\| \leq & \upsilon_{dx}(t)\|x_0\| + \frac{\gamma_{dx}}{\sqrt{\Gamma}}, \\
\|y_{ref}(t) - y(t)\| \leq & \upsilon_{dy}(t)\|x_0\| + \frac{\gamma_{dy}}{\sqrt{\Gamma}}.
\end{aligned} \tag{4.1.61}$$

Proof. Define $\tilde{x}_{ref}(t) = x_{ref}(t) - x(t)$, $\tilde{u}_{ref}(t) = u_{ref}(t) - u(t)$, $\tilde{y}_{ref}(t) = y_{ref}(t) - y(t)$, and $\tilde{\eta}_{ref}(t) = f(x_{ref}, t) - f(x, t)$. First, it will be shown that Equation (4.1.59) holds by a contradiction argument. Suppose it is not true. Notice that since $\kappa_m \geq 1$ in (4.1.14), it follows that $\gamma_{x_0} > 1$, which leads to $\rho_{dx} > \rho_0$, and $\|\tilde{x}_{ref}(0)\| = \rho_0 < \rho_{dx}$, where γ_{x_0}, ρ_{dx} are given in (4.1.39), and (4.1.41), respectively. Moreover, since $\|\tilde{u}_{ref}(0)\| = 0 < \rho_{du}$ with ρ_{du} being given in (4.1.41), from the continuity of the solutions it follows that there exists $\tau' > 0$ such that

$$\|\tilde{x}_{ref}(\tau')\| = \rho_{dx} \quad \text{or} \quad \|\tilde{u}_{ref}(\tau')\| = \rho_{du},$$

while

$$\|\tilde{x}_{ref}(t)\| < \rho_{dx}, \quad \|\tilde{u}_{ref}(t)\| < \rho_{du}, \quad 0 \le t < \tau',$$

which implies that the following must hold:

$$\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \le \rho_{dx}, \quad \left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \le \rho_{du}. \tag{4.1.62}$$

Notice that from (4.1.20), (4.1.30), and (4.1.41) it follows that

$$\rho_{dx} = \rho_x - \rho_{rx}, \quad \rho_{du} = \rho_u - \rho_{ru}.$$

Then, the triangular inequalities on (4.1.62), together with (4.1.29) and (4.1.31), yield

$$\|x_{\tau'}\|_{\mathcal{L}_\infty} \le \rho_x, \quad \|u_{\tau'}\|_{\mathcal{L}_\infty} \le \rho_u, \tag{4.1.63}$$

which, together with Assumption 4.1.3 and the fact that $d_{\rho_x} \le L_{\rho_r}$, leads to

$$\|\tilde{\eta}_{ref}(t)\| \le L_{\rho_r}\|\tilde{x}_{ref}(t)\|, \quad 0 \le t \le \tau'. \tag{4.1.64}$$

Since Equation (4.1.63) holds, from Lemma 4.1.1, Equation (4.1.23) can be rewritten as

$$\hat{\eta}_t(t) = \omega u_z(t) + \bar{f}(X,t) + \tilde{\eta}_t(t) + \phi(t), \tag{4.1.65}$$

where $u_z(t)$, $\bar{f}(X,t)$, $\tilde{\eta}_t(t)$, $\phi(t)$ are given in (4.1.2), (4.1.5), (4.1.46), and (4.1.47), respectively. Notice that $x(t) = T_g x_g(t) + T_z x_f(t)$, and therefore from (4.1.2) and (4.1.3) it follows that

$$\omega u_z(t) + \bar{\eta}(s) = Z(s)(\omega u(s) + \eta(s)), \tag{4.1.66}$$

where $\bar{\eta}(s)$, $\eta(s)$ are the Laplace transforms of $\bar{f}(X,t)$ and $f(x,t)$, respectively. Substituting (4.1.65) and (4.1.66) into (4.1.22) leads to

$$\begin{aligned} u(s) =& C_0(s)(K_g r(s) - \eta(s)) - \phi_c(s), \\ \phi_c(s) =& C_0(s)Z^{-1}(s)(\tilde{\eta}_t(s) + \phi(s)), \end{aligned} \tag{4.1.67}$$

where $C_0(s)$ is given in (4.1.15); $C_0(s)Z^{-1}(s)$ is a stable and strictly proper transfer matrix. Combining the Laplace transform of (4.1.1) with (4.1.67) yields

$$\begin{aligned} x(s) =& H_r(s)r(s) + G(s)\eta(s) - \omega H_0(s)\phi_c(s) + x_{in}(s), \\ y(s) =& C_m x(s), \end{aligned} \tag{4.1.68}$$

where $H_r(s)$, $H_0(s)$, $G(s)$ are given in (4.1.18), and $x_{in}(s) = (s\mathbb{I}_n - A_m)^{-1}x_0$. By subtracting

(4.1.67) and (4.1.68) from (4.1.35), it follows that

$$\begin{aligned}\tilde{x}_{ref}(s) =& G(s)\tilde{\eta}_{ref}(s) + \omega H_0(s)\phi_c(s) - x_{in}(s),\\ \tilde{y}_{ref}(s) =& C_m\tilde{x}_{ref}(s),\end{aligned} \tag{4.1.69}$$

and

$$\tilde{u}_{ref}(s) = -C_0(s)\tilde{\eta}_{ref}(s) + \phi_c(s). \tag{4.1.70}$$

Since $(C_mB_m)^{\dagger}(C_mB_m) = \mathbb{I}_m$, from (4.1.49) one has

$$\phi_c(s) = C_0(s)Z^{-1}(s)(\tilde{\eta}_t(s) + \phi(s)) = C_1(s)\tilde{y}(s) - C_2(s)\tilde{v}(s), \tag{4.1.71}$$

where $\{C_1(s), C_2(s)\}$, and $\{\tilde{y}(t), \tilde{v}(t)\}$ are defined in (4.1.19), and (4.1.48), respectively; $C_1(s)$, $C_2(s)$ are all stable and proper transfer function matrices. From (4.1.16) it can be shown that $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$. Therefore, combining (4.1.64), and (4.1.69)-(4.1.71) yields

$$\begin{aligned}\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq& \frac{\|H_1(s)\|_{\mathcal{L}_1}\|\tilde{y}_{\tau'}\|_{\mathcal{L}_\infty} + \|H_2(s)\|_{\mathcal{L}_1}\|\tilde{v}_{\tau'}\|_{\mathcal{L}_\infty}}{1-\|G(s)\|_{\mathcal{L}_1} L_{\rho_r}} + \frac{\kappa_m\rho_0}{1-\|G(s)\|_{\mathcal{L}_1} L_{\rho_r}},\\ \left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq& \|C_1(s)\|_{\mathcal{L}_1}\|\tilde{y}_{\tau'}\|_{\mathcal{L}_\infty} + \|C_2(s)\|_{\mathcal{L}_1}\|\tilde{v}_{\tau'}\|_{\mathcal{L}_\infty} + \|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r}\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty},\end{aligned}$$

where κ_m, $\{H_1(s), H_2(s)\}$, $\{C_1(s), C_2(s)\}$ are given in (4.1.14), (4.1.18), and (4.1.19), respectively. Since Equation (4.1.58) holds for $0 \leq t \leq \tau'$, one has

$$\begin{aligned}\left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq& \gamma_{u0}\rho_0 + \gamma_u\sqrt{\frac{\theta_1}{\Gamma}},\\ \left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq& \gamma_{x0}\rho_0 + \gamma_x\sqrt{\frac{\theta_1}{\Gamma}},\end{aligned} \tag{4.1.72}$$

where γ_{u_0}, γ_{x_0}, γ_u, γ_x are given in (4.1.39), and θ_1 is defined in (4.1.43). Since $\Gamma > 0$ is chosen so that $\gamma_x\sqrt{\frac{\theta_1}{\Gamma}} < \bar{\gamma}$ and $\gamma_u\sqrt{\frac{\theta_1}{\Gamma}} < \bar{\gamma}$, from (4.1.72) it follows that

$$\left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} < \rho_{dx}, \quad \left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} < \rho_{du},$$

which contradict to (4.1.62), thus proving (4.1.59). Moreover, Equation (4.1.60) is obtained from applying the triangular inequality on $\|\tilde{x}_{ref}\|$ and $\|\tilde{u}_{ref}\|$.

Next, we prove Equation (4.1.61). Let $A_b \in \mathbb{R}^{n_b\times n_b}$, $B_b \in \mathbb{R}^{n_b\times m}$, and $C_b \in \mathbb{R}^{m\times m}$ be a minimal realization of $C(s)$ with the appropriate dimension n_b. Then, the system given in (4.1.69) and (4.1.70) can be represented as

$$\begin{aligned}\dot{\tilde{x}}_c(t) =& A_c\tilde{x}_c(t) + B_c\tilde{\eta}_{ref}(t) + \bar{B}_c\omega\phi_c(t),\\ \tilde{x}_{ref}(t) =& C_c\tilde{x}_c(t), \quad \tilde{x}_c(0) = [-x_0^\top, 0]^\top,\end{aligned} \tag{4.1.73}$$

with

$$A_c = \begin{bmatrix} A_m & B_m C_b \\ 0 & A_b \end{bmatrix}, \quad B_c = \begin{bmatrix} B_m \\ -B_b \end{bmatrix}, \quad \bar{B}_c = \begin{bmatrix} B_m \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} \mathbb{I}_n & 0 \end{bmatrix},$$

where $\tilde{x}_c(t) = [\tilde{x}_{ref}^\top(t), \tilde{x}_b^\top(t)]^\top \in \mathbb{R}^{n_c \times n_c}$ is the state vector with $n_c = n + n_b$. Let $t_m \geq 0$. Then, from (4.1.73) it follows that for $t \geq t_m$

$$\tilde{x}_c(t) = e^{A_c(t-t_m)}\tilde{x}_c(t_m) + \int_{t_m}^{t} e^{A_c(t-\tau)}\bar{B}_c\omega\phi_c(\tau)d\tau + \int_{t_m}^{t} e^{A_c(t-\tau)}B_c\tilde{\eta}_{ref}(\tau)d\tau. \tag{4.1.74}$$

Notice that it can be shown that $G(s) = C_c(s\mathbb{I}_{n_c} - A_c)^{-1}B_c = H_0(s)(\mathbb{I}_m - C(s))$. Since from (4.1.16) $\|G(s)\|_{\mathcal{L}_1} L_{\rho_r} < 1$ holds, from the continuity of the $\mathcal{L}_1$-norm it follows that one may take a sufficiently small $\lambda_0 > 0$ such that $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1} < 1/L_{\rho_r}$. Next, let $A_{\lambda_0} = A_c + \lambda_0\mathbb{I}_{n_c}$, and define $\bar{x}_c(t) = e^{\lambda_0(t-t_m)}\tilde{x}_c(t)$, $\bar{\phi}_c(t) = \omega e^{\lambda_0(t-t_m)}\phi_c(t)$, $\bar{x}_{ref}(t) = e^{\lambda_0(t-t_m)}\tilde{x}_{ref}(t)$, and $\bar{\eta}_{ref}(t) = e^{\lambda_0(t-t_m)}\tilde{\eta}_{ref}(t)$. Since Assumption 4.1.3 implies that

$$\|\bar{\eta}_{ref}(t)\| \leq L_{\rho_r}\|\bar{x}_{ref}(t)\|, \tag{4.1.75}$$

multiplying both sides of (4.1.74) by $e^{\lambda_0(t-t_m)}C_c$ leads to

$$\|\bar{x}_{ref}\|_{\mathcal{L}_\infty[t_m,t]} \leq \frac{\beta_0}{1-\beta_1 L_{\rho_r}}\|\tilde{x}_c(t_m)\| + \frac{\beta_2}{1-\beta_1 L_{\rho_r}}\left\|\bar{\phi}_c\right\|_{\mathcal{L}_\infty[t_m,t]}, \tag{4.1.76}$$

where $\beta_0 = \sup_{0\leq\tau}\|e^{A_{\lambda_0}\tau}\|$, $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1}$, and $\beta_2 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}\bar{B}_c\right\|_{\mathcal{L}_1}$. By combining (4.1.74) - (4.1.76), it can be shown that

$$\|\bar{x}_c(t)\| \leq \kappa_0\|\tilde{x}_c(t_m)\| + \kappa_1\left\|\bar{\phi}_c\right\|_{\mathcal{L}_\infty[t_m,t]},$$

which further gives

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)}\|\tilde{x}_c(t_m)\| + \omega_u\kappa_1\|\phi_c\|_{\mathcal{L}_\infty[t_m,t]}, \tag{4.1.77}$$

where $\omega_u > 0$ is the upper bound of ω, and

$$\kappa_0 = \beta_0(1 + \frac{L_{\rho_r}\beta_3}{1-\beta_1 L_{\rho_r}}), \quad \kappa_1 = \beta_2(1 + \frac{L_{\rho_r}\beta_3}{1-\beta_1 L_{\rho_r}}), \tag{4.1.78}$$

with $\beta_3 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}B_c\right\|_{\mathcal{L}_1}$. Substituting (4.1.71) together with (4.1.55) - (4.1.57) into (4.1.77) leads to

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)}\|\tilde{x}_c(t_m)\| + \gamma_1\left(\upsilon_v(t_m,0) + \sqrt{\frac{\theta_1}{\Gamma}}\right), \tag{4.1.79}$$

where $\upsilon_v(\cdot,\cdot)$, θ_1 are defined in (4.1.57) and (4.1.43), respectively, and

$$\gamma_1 = \omega_u \kappa_1 \left(\frac{\|C_1(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_y)}} + \frac{\|C_2(s)\|_{\mathcal{L}_1}}{\sqrt{\lambda_{\min}(P_v)}} \right).$$

Notice that from (4.1.43) and (4.1.57) it follows that

$$\upsilon(t_m, 0) \le \sqrt{n\lambda_{\max}(\bar{P}_v)}\|x_0\| + \sqrt{\frac{\theta_1}{\Gamma}},$$

which, together with (4.1.79), results in

$$\|\tilde{x}_c(t_m)\| \le \kappa_0 e^{-\lambda_0 t_m}\|x_0\| + \gamma_1\sqrt{n\lambda_{\max}(\bar{P}_v)}\|x_0\| + \gamma_1\sqrt{\frac{\theta_1}{\Gamma}}, \tag{4.1.80}$$

where $\bar{P}_v = (\mathbb{I}_n - HC_m)^\top P_v(\mathbb{I}_n - HC_m)$. Set $t_m = t/2$. Substituting (4.1.80) into (4.1.79), and using (4.1.57), one has

$$\|\tilde{x}_{ref}(t)\| \le \|\tilde{x}_c(t)\| \le \upsilon_{dx}(t)\|x_0\| + \frac{\gamma_{dx}}{\sqrt{\Gamma}}, \quad t \ge 0,$$

where

$$\begin{aligned}\upsilon_{dx}(t) =& \kappa_0^2 e^{-\lambda_0 t} + \kappa_0\gamma_1\sqrt{n\lambda_{\max}(\bar{P}_v)}e^{-\frac{\lambda_0}{2}t} + \gamma_1\sqrt{n\lambda_{\max}(\bar{P}_v)}e^{-\frac{\lambda_1}{4}t},\\ \gamma_{dx} =& (\kappa_0 + 2)\gamma_1\sqrt{\theta_1},\end{aligned}$$

with λ_1 being given in (4.1.43). Finally, letting $\gamma_{dy} = \|C_m\|\gamma_{dx}$ and $\upsilon_{dy}(t) = \|C_m\|\upsilon_{dx}(t)$ yields (4.1.61). This completes the proof. □

Notice that the steady-state bounds can be tuned to arbitrarily small values by increasing the adaptation gains. This does not affect the upper bound functions υ_{dx} and υ_{dy}. Moreover, the closed-loop system is semi-globally stabilized, which follows from Lemma 4.1.2 and Theorem 4.1.1. Notice that the stability result becomes global, when the uncertain function $f(x,t)$ has globally bounded partial derivatives.

4.1.4. *Illustrative example*

To illustrate the performance of the proposed $\mathcal{L}_1$ adaptive controller, we present simulation results on a numerical example. Consider the uncertain plant given in (4.1.1) with

$$A_m = \begin{bmatrix} -1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1.5 \\ 0 & -1 & -1.5 & 1.5 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad C_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \tag{4.1.81}$$

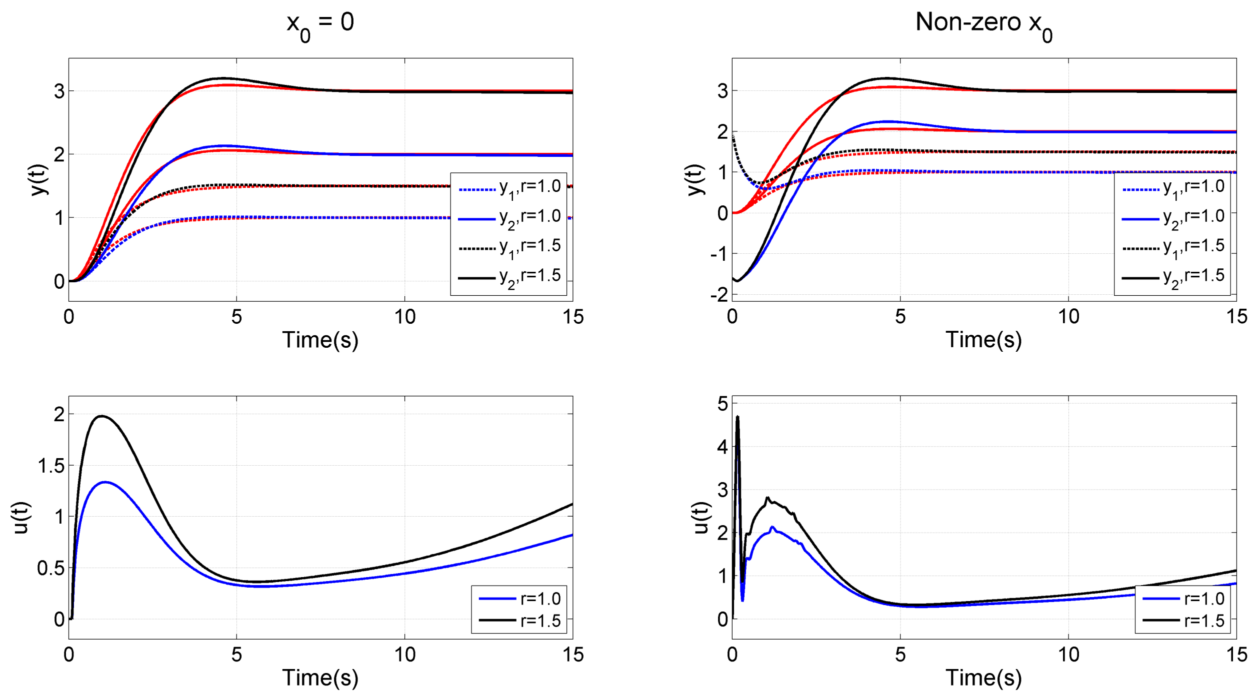

Figure 4.3: System responses and control inputs

$\Omega = 0.8$, and

$$f(x,t) = 0.017\|x\|_2^2 + 0.05\tanh(0.5x_1)x_1 - 0.08x_3^2 - 0.5x_3\cos(0.1t) + 0.5x_4\sin(0.15t) - 0.05(1 - e^{-0.7t}).$$

Since $(C_m B_m)$ is not full rank, the design procedure of the $\mathcal{L}_1$ controller starts by choosing the interactor $Z(s) = \frac{1.3}{s+1.5}$ (see Remark 2.1.9). Let $K_g = 1$ and $D(s) = \frac{2}{s}\frac{4s^2+434s+4}{s^2+101s+0.8}$. The design parameters for the proposed method are given by

$$Q = 4\mathbb{I}_3, \quad P_y = 0.1\mathbb{I}_2, \quad \mu = 20, \quad \Gamma = 200\,, \; K_v = \begin{bmatrix} -3.87 & 0.98 \\ 0.27 & -5.8 \\ 0.7 & -0.74 \\ 1.25 & -1.15 \end{bmatrix}.$$

Figure 4.3 shows the system response and control inputs for $r(t) = 1$ and $r(t) = 1.5$; the left plots illustrates the results for the system with $x_0 = 0$, and the left plots present simulation results for the system initialized with $x_0 = [2, -2, 2, -2]^\top$. Notice that red-dotted lines represent the desired responses of $y_1(t)$, and red-solid lines are used for the desired responses of $y_2(t)$. As shown in Figure 4.3, the proposed controller shows that the effects of non-zero x_0 vanish over time, and the steady-state errors remain small. This validates the theoretical results.

4.2. Piecewise Constant Adaptation Laws for $\mathcal{L}_1$ Adaptive Control

In this section, we present piecewise constant adaptation laws for $\mathcal{L}_1$ adaptive output-feedback control. The proposed scheme extends the existing $\mathcal{L}_1$ adaptive control with piecewise constant adaptation laws to underactuated systems. In the existing architecture, the procedure to obtain the inverse of the desired model is required, which is not trivial for underactuated systems. Using the state-decomposition technique, we demonstrate that the piecewise constant adaptation laws can be extended to underactuated systems.

Consider the system with nonlinear uncertainties as

$$\begin{aligned} \dot{x}(t) =& A_m x(t) + B_m(\Omega u(t) + f(x,t)), \\ y(t) =& C_m x(t), \quad x(0) = x_0, \end{aligned} \tag{4.2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, the input and the measurable output vectors with $m \leq p$, respectively, $A_m \in \mathbb{R}^{n\times n}$, $B_m \in \mathbb{R}^{n\times m}$, $C_m \in \mathbb{R}^{p\times n}$ are known matrices with A_m being Hurwitz, and (A_m, B_m, C_m) is a controllable-observable pair; notice that $M(s) = C_m(s\mathbb{I}_n - A_m)^{-1}B_m$ presents the desired response. Finally, $\Omega \in \mathbb{R}^{m\times m}$ is an unknown input gain, $f(x,t) \in \mathbb{R}^m$ is an unknown function representing system uncertainties, and $x_0 \in \mathbb{R}^n$ is an initial condition.

Assumption 4.2.1. *$M(s)$ has no unstable transmission zeros.*

Assumption 4.2.2. *The unknown constant input gain $\Omega \in \mathbb{R}^{m\times m}$ is assumed to be an unknown (nonsingular) strictly row-diagonally dominant matrix with sign(Ω_{ii}) known. Moreover, it is assumed that there exists a known compact convex set $\mathcal{C}_\Omega$ such that $\Omega \in \mathcal{C}_\Omega$, and that a nominal system input gain $\Omega_0 \in \mathbb{R}^{m\times m}$ is known.*

Assumption 4.2.3. *Define $\mathcal{B}_\delta = \{x \in \mathbb{R}^n; \|x\| < \delta\}$ with $\delta > 0$. The unknown function $f(x,t)$: $(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^m$ is assumed to have the following properties:*

- *There exists $b_0 > 0$ such that $\|f(0,t)\| < b_0$ for all $t \geq 0$.*
- *For each $\delta > 0$, there exists $d_\delta > 0$ such that $\|f(x,t) - f(y,t)\| \leq d_\delta \|x - y\|$ for all $x, y \in \mathcal{B}_\delta$, uniformly in $t \in \mathbb{R}$.*

With the above setup, the problem is formally stated as follows:

Problem 4.2.1. *Let the system described by Equation* (4.2.1) *satisfy Assumptions 4.2.2 – 4.2.3. The objective is to design an adaptive output feedback control law for $u(t)$ so that the output $y(t)$ tracks the desired response $y_m(t)$ both in transient and steady state, where $y_m(t) \in \mathbb{R}^p$ is the signal with the Laplace transform of $y_m(s) = M(s)K_g r(s)$; $r(t) \in \mathbb{R}^{m_r}$ is a reference command, and $K_g \in \mathbb{R}^{m\times m_r}$ is a known feed-forward gain.*

4.2.1. *Design of $\mathcal{L}_1$ adaptive controller*

In this section, $\mathcal{L}_1$ adaptive output feedback controller is developed to solve Problem 4.2.1. Let $\rho_0 > 0$ be a given constant satisfying $\|x_0\| \le \rho_0$ with $x_0 \in \mathbb{R}^n$ being an initial condition, and choose $\bar{\gamma} > 0$ to be an arbitrarily small constant. For a given $\delta > 0$ define

$$L_\delta = \frac{\bar{\delta}(\delta)}{\delta} d_{\bar{\delta}(\delta)}, \quad \bar{\delta}(\delta) = \delta + \bar{\gamma}\,, \tag{4.2.2}$$

where $d_{\bar{\delta}(\delta)}$ is introduced in Assumption 4.2.3. Let $Z(s)$ and $\bar{M}(s)$ be given as

$$Z(s) = C_z(s\mathbb{I}_{n_z} - A_z)^{-1}B_z + D_z, \quad \bar{M}(s) = C_m(s\mathbb{I}_n - A_m)^{-1}\bar{B}, \tag{4.2.3}$$

which satisfy Corollary 2.1.2. Notice that the existence of $Z(s)$ and $\bar{M}(s)$ is guaranteed. Choose $K_v \in \mathbb{R}^{n\times p}$, such that $A_v = A_H + K_v C_m$ is Hurwitz (see Lemma 2.1.18), where

$$A_H = (\mathbb{I}_n - HC_m)A_m, \quad H = \bar{B}(C_m\bar{B})^\dagger, \tag{4.2.4}$$

with $(C_m\bar{B})^\dagger$ being the generalized inverse of $(C_m\bar{B})$. Now, let $\lambda_v > 0$ be the decay rate of $e^{A_v t}$, such that

$$\|e^{A_v t}\| \le \kappa_{v0} e^{-\lambda_v t}, \quad \kappa_{v0} = \sup_{t\ge 0}\|e^{A_v t}\|, \quad \forall t \ge 0. \tag{4.2.5}$$

Choose $\mu > \lambda_v$ and define

$$\kappa_m = \sup_{t\ge 0}\|e^{A_m t}\|, \quad \kappa_y = \frac{2\kappa_v\|C_m A_m\|}{\mu - \lambda_v}, \quad \kappa_v = \|\mathbb{I}_n - HC_m\|\kappa_{v0}\,. \tag{4.2.6}$$

Next, let $D(s)$ be a $m \times m$ transfer matrix such that

$$C(s) = \Omega D(s)(\mathbb{I}_m + \Omega D(s))^{-1} \tag{4.2.7}$$

is stable with $C(0) = \mathbb{I}_m$, and $D(s)Z^{-1}(s)$ is a proper transfer matrix. In addition, the choice of $D(s)$ ensures that there exists $\rho_r > 0$ such that for all $\Omega \in \mathcal{C}_\Omega$

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_r - \rho_{ext} - \rho_{int}}{L_{\rho_r}\rho_r}\,, \tag{4.2.8}$$

where

$$G(s) = H_0(s)\left(\mathbb{I}_m - C(s)\right), \quad H_0(s) = (s\mathbb{I}_n - A_m)^{-1}B_m, \tag{4.2.9}$$

and

$$\begin{aligned} \rho_{ext} &= \|H_0(s)C(s)K_g\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} + \|G(s)\|_{\mathcal{L}_1} b_0, \\ \rho_{int} &= (\kappa_m + \kappa_x)\rho_0\,, \\ \kappa_x &= \|H_1(s)\|_{\mathcal{L}_1}\kappa_v + (\|H_2(s)\|_{\mathcal{L}_1} + \|H_3(s)\|_{\mathcal{L}_1})\kappa_y, \end{aligned} \tag{4.2.10}$$

with $K_g \in \mathbb{R}^{m\times m_r}$ being a feed-forward gain, $r(t) \in \mathbb{R}^{m_r}$ being a reference command, and κ_m, κ_v being given in (4.2.6); $H_1(s)$, $H_2(s)$, and $H_3(s)$ are defined as

$$\begin{aligned} H_1(s) =& H_0(s)C(s)Z^{-1}(s)(C_m\bar{B})^{\dagger}\bar{C}_m\bar{A}_m, \\ H_2(s) =& H_0(s)C(s)Z^{-1}(s)(C_m\bar{B})^{\dagger}(s+\mu), \\ H_3(s) =& \mu H_0(s)C(s)Z^{-1}(s)(C_m\bar{B})^{\dagger}, \end{aligned} \tag{4.2.11}$$

and L_{ρ_r} is given in (4.2.2) with

$$\rho_x = \rho_r + \bar{\gamma}. \tag{4.2.12}$$

Let $T_s > 0$ be the sampling rate of CPU. The adaptive control input $u(t) \in \mathbb{R}^m$ is governed by

$$\begin{aligned} u(s) &= D(s)K_g r(s) - D(s)Z^{-1}(s)\hat{\eta}_t(s), \\ \hat{\eta}_t(s) &= \hat{u}_v(s) + \hat{\eta}_v(s), \end{aligned} \tag{4.2.13}$$

where

$$\hat{u}_v(s) = Z(s)\Omega_0 u(s), \quad \hat{\eta}_v(s) = (C_m\bar{B})^{\dagger}e^{\mu\mathbb{I}_p T_s}\hat{\eta}_H(s), \tag{4.2.14}$$

with Ω_0 being a (known) nominal input gain; $\hat{\eta}_H(s)$ will be shortly defined in the adaptation laws. The following state-output predictor is used:

$$\begin{aligned} \dot{\hat{v}}(t) =& A_H\hat{v}(t) + A_H H y(t) + K_v\left(C_m\hat{v}(t) - (\mathbb{I}_p - C_m H)y(t)\right), \quad \hat{v}(0) = 0, \\ \dot{\hat{y}}(t) =& -\mu(\hat{y}(t) - y(t)) + C_m A_m(\hat{v}(t) + Hy(t)) + C_m\bar{B}\hat{u}_v(t) + \hat{\eta}_H(t), \quad \hat{y}(0) = y_0, \end{aligned} \tag{4.2.15}$$

where A_H, H are given in (4.2.4), and $\hat{u}_v(t)$ is defined in (4.2.14). The adaptive laws for $\hat{\eta}_H(t)$ are defined as

$$\hat{\eta}_H(t) = -\Phi^{-1}(T_s)e^{-\mu T_s}\tilde{y}_H(t)\,, \tag{4.2.16}$$

where

$$\tilde{y}_H(t) = \hat{y}(kT_s) - y(kT_s)\,, \quad \forall t \in [kT_s, (k+1)T_s)\,, \quad \forall k \in \overline{\mathbb{N}}\,, \tag{4.2.17}$$

and

$$\Phi(T_s) = \mu^{-1}(1 - e^{-\mu T_s})\,. \tag{4.2.18}$$

Notice that $\mu > 0$ can be a small number, since a small variable $\lambda_v > 0$ always satisfies (4.2.5). Notice that when the system does not have vector relative degree $\mathbf{1}_m$, the proposed method requires a minimum order for the filter such that $C(s)Z^{-1}(s)$ is proper (see also Remark 4.1.4).

4.2.2. *Stability and performance analysis*

Consider the following closed-loop reference system

$$\begin{aligned}\dot{x}_{ref}(t) =& A_m x_{ref}(t) + B_m \left(\Omega u_{ref}(t) + f(x_{ref}, t)\right), \\ y_{ref}(t) =& C_m x_{ref}(t), \quad x_{ref}(0) = 0,\end{aligned} \tag{4.2.19}$$

with

$$u_{ref}(s) = C_0(s)(K_g r(s) - \eta_{ref}(s) - \sigma(s)), \tag{4.2.20}$$

where $x_{ref}(t) \in \mathbb{R}^n$, $y_{ref}(t) \in \mathbb{R}^p$ are the state and output vectors, respectively. Moreover,

$$C_0(s) = D(s) \left(\mathbb{I}_m + \Omega D(s)\right)^{-1}, \tag{4.2.21}$$

and $\eta_{ref}(s)$, $\sigma(s)$ are the Laplace transforms of the following signals:

$$\eta_{ref}(t) = f(x_{ref}, t) - f(0, t), \quad \sigma(t) = f(0, t). \tag{4.2.22}$$

The closed-loop reference system in (4.2.19) and (4.2.20) is identical to the one in Section 4.1.3.

Lemma 4.2.1. *Consider the closed-loop reference system in* (4.2.19) *and* (4.2.20) *subject to* (4.2.8). *Then, for each* $\tau > 0$ *the following bounds hold:*

$$\left\|x_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{rx}, \tag{4.2.23}$$

$$\left\|u_{ref_\tau}\right\|_{\mathcal{L}_\infty} < \rho_{ru}, \tag{4.2.24}$$

where ρ_{rx} *and* ρ_{ru} *are defined as*

$$\rho_{rx} = \rho_r - \frac{\rho_{int}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}} > 0, \quad \rho_{ru} = \|C_0(s)K_g\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} + \|C_0(s)\|_{\mathcal{L}_1} \left(L_{\rho_r}\rho_{rx} + b_0\right). \tag{4.2.25}$$

Moreover, ρ_{int}*,* $G(s)$*,* $C_0(s)$ *are defined in* (4.2.10), (4.2.9), *and* (4.2.21), *respectively.*

Proof. Notice that substituting (4.2.20) into (4.2.19) yields

$$\begin{aligned}x_{ref}(s) &= H_0 C(s) K_g r(s) + G(s)\eta_{ref}(s) + G(s)\sigma(s), \\ y_{ref}(s) &= C_m x_{ref}(s),\end{aligned} \tag{4.2.26}$$

where $C(s)$, $\{H_0(s), G(s)\}$ are given in (4.2.7) and (4.2.9), respectively, and $\eta_{ref}(s)$, $\sigma(s)$ are the Laplace transform signals defined in (4.2.22). The system (4.2.26) is identical with the reference system in Section 4.1. Therefore, the proof follows from the previous analysis (see the proof of Lemma 4.1.2). □

To proceed, we introduce a few definitions and variables of interest. The following variables are used in the analysis of transient and steady-steady bounds:

$$\rho_u = \rho_{ru} + \rho_{du}, \quad \rho_{dx} = \gamma_{x_0}\rho_0 + \bar{\gamma}, \quad \rho_{du} = \gamma_{u_0}\rho_0 + \bar{\gamma}, \tag{4.2.27}$$

where ρ_{ru} is given in (4.2.25), and

$$\begin{aligned} \gamma_{x_0} =& \frac{\kappa_x + \kappa_m}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \\ \gamma_{u_0} =& \|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r}\gamma_{x_0} + \|C_1(s)\|_{\mathcal{L}_1}\kappa_v + (\|C_2(s)\|_{\mathcal{L}_1} + \|C_3(s)\|_{\mathcal{L}_1})\kappa_y, \\ \gamma_x =& \frac{\|H_1(s)\|_{\mathcal{L}_1} + \|H_2(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \\ \gamma_u =& \|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r}\gamma_x + \|C_2(s)\|_{\mathcal{L}_1} + \|C_3(s)\|_{\mathcal{L}_1} \end{aligned} \tag{4.2.28}$$

with κ_m, κ_x being given in (4.2.6) and (4.2.10); $C_i(s)$, $i = 1, 2, 3$ are defined as

$$\begin{aligned} C_1(s) =& C_0(s)Z^{-1}(s)(\bar{C}_m\bar{B}_m)^{\dagger}\bar{C}_m\bar{A}_m, \\ C_2(s) =& C_0(s)Z^{-1}(s)(\bar{C}_m\bar{B}_m)^{\dagger}(s + \mu), \\ C_3(s) =& \mu C_0(s)Z^{-1}(s)(\bar{C}_m\bar{B}_m)^{\dagger}. \end{aligned} \tag{4.2.29}$$

Next, define

$$\alpha_1(T_s) = \left(1 - e^{-(\mu-\lambda_v)T_s}\right), \quad \alpha_2(T_s) = \|C_m\bar{B}\|\Phi(T_s), \tag{4.2.30}$$

and

$$\Delta_u = \max_{\Omega\in\mathcal{C}_\Omega} \|\Omega_0 - \Omega\| \, \|Z(s)\|_{\mathcal{L}_1} \rho_u, \quad \Delta_s = \|Z(s)\|_{\mathcal{L}_1} \left(L_{\rho_r}\rho_r + b_0\right), \tag{4.2.31}$$

where $Z(s)$, λ_v, κ_y and $\Phi(T_s)$ are given in (4.2.3), (4.2.5), (4.2.6) and (4.2.18). Moreover, let

$$\gamma(T_s) = 2\alpha_2(T_s)(\Delta_u + \Delta_f)\,. \tag{4.2.32}$$

Lemma 4.2.2. *Given the definitions of variables in* (4.2.27) - (4.2.32)*, the following holds*

$$\lim_{T_s\to 0} \gamma(T_s) = 0. \tag{4.2.33}$$

Moreover, there exists $\bar{T}_s > 0$*, such that for all* $T_s \in [0, \bar{T}_s]$

$$\gamma_x\gamma(T_s) < \bar{\gamma}, \quad \gamma_u\gamma(T_s) < \bar{\gamma}, \quad \forall\Omega \in \mathcal{C}_\Omega, \tag{4.2.34}$$

where γ_x *and* γ_u *are given in* (4.2.28).

Proof. The proof of Lemma 4.2.2 is straightforward from the definition of $\gamma(T_s)$. □

Lemma 4.2.3. *Let $\tau > 0$, and assume $T_s > 0$ satisfies* (4.2.34)*. Then*

$$\|\tilde{y}(t)\| \leq \kappa_y \alpha_1(T_s) e^{-\lambda_v t}\|x_0\| + \gamma(T_s), \quad 0 \leq t \leq \tau, \tag{4.2.35}$$

where κ_y, $\alpha_1(T_s)$ and $\gamma(T_s)$ are given in (4.2.6), (4.2.30) *and* (4.2.32)*, respectively.*

Proof. Let $\tilde{u}_v(t)$ and $\eta_v(t)$ be the signals of the Laplace transform of

$$\tilde{u}_v(s) = Z(s)(\Omega_0 - \Omega)u(s), \quad \eta_v(s) = Z(s)(\eta(s) + \sigma(s)), \tag{4.2.36}$$

where $\eta(s)$ is the Laplace transform of $\eta(t) := f(x,t) - f(0,t)$, and $Z(s)$ is defined in (4.2.3); notice that $\sigma(s)$ is the Laplace transform of $\sigma(t)$ given in (4.2.22). Since $\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x$, from Assumption 4.2.3 it follows that for all $0 \leq t \leq \tau$

$$\|f(x,t)\| \leq d_{\rho_x}\rho_x + b_0 \leq L_{\rho_r}\rho_r + b_0, \tag{4.2.37}$$

which further leads to

$$\|\eta_v(t)\| \leq \|Z(s)\|_{\mathcal{L}_1}\left(L_{\rho_r}\rho_r + b_0\right). \tag{4.2.38}$$

Similarly, from $\|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u$ one has

$$\|\tilde{u}_v(t)\| \leq \max_{\Omega \in \mathcal{C}_\Omega} \|\Omega_0 - \Omega\|_\infty \|Z(s)\|_{\mathcal{L}_1}\rho_u, \quad 0 \leq t \leq \tau. \tag{4.2.39}$$

Notice that from Corollary 2.1.3 the system (4.2.1) can be represented by

$$\begin{aligned} \dot{x}_v(t) =& A_m x_v(t) + \bar{B}(\hat{u}_v(t) - \tilde{u}_v(t) + \eta_v(t)), \\ y(t) =& C_m x_v(t), \quad x_v(0) = x_0, \end{aligned} \tag{4.2.40}$$

where $\hat{u}_v(t)$ is given in (4.2.14), and $\tilde{u}_v(t)$, $\eta_v(t)$ are defined in (4.2.36). Now, let $x_v(t) = v(t) + Hy(t)$ with

$$v(t) = (\mathbb{I}_n - HC_m)x_v(t), \tag{4.2.41}$$

where H is given by (4.2.4). By pre-multiplying both sides of (4.2.40) by $(\mathbb{I}_n - HC_m)$ and taking the derivative of $y(t)$, Equation (4.2.40) can be rewritten by

$$\begin{aligned} &\dot{v}(t) = A_H v(t) + A_H H y(t), \quad v(0) = v_0, \\ &\dot{y}(t) = C_m A_m v(t) + C_m A_m H y(t) + C_m \bar{B}\left(\hat{u}_v(t) - \tilde{u}_v(t) + \eta_v(t)\right), \quad y(0) = y_0, \end{aligned} \tag{4.2.42}$$

where A_H, H are given in (4.2.4), $v_0 = (\mathbb{I}_n - HC_m)x_0$ and $y_0 = C_m x_0$. Then, subtracting (4.2.42) from (4.2.15), together with the fact that $C_m\hat{v}(t) - (\mathbb{I}_p - C_m H)y(t) = C_m\tilde{v}(t)$, yields

$$\begin{aligned}
&\dot{\tilde{v}}(t) = A_v\tilde{v}(t), \quad v(0) = -v_0\,,\\
&\dot{\tilde{y}}(t) = -\mu\tilde{y}(t) + C_m A_m\tilde{v}(t) + \hat{\eta}_H(t) + C_m\bar{B}\left(\tilde{u}_v(t) - \eta_v(t)\right), \quad y(0) = 0\,,
\end{aligned} \tag{4.2.43}$$

where $\tilde{v}(t) = \hat{v}(t) - v(t)$ and $\tilde{y}(t) = \hat{y}(t) - y(t)$. Next, let $t_0 \geq 0$ and $t_k = kT_s + t_0$, where $k \in \bar{\mathbb{N}}$. By substituting (4.2.16) into (4.2.43), and integrating (4.2.43), one has

$$\begin{aligned}
\tilde{y}(\tau' + t_k) =& \alpha_0(\tau', T_s)e^{-\mu\tau'}\tilde{y}(t_k) + \int_0^{\tau'} e^{-\mu\mathbb{I}_p(\tau'-\xi)}C_m A_m\tilde{v}(t_k+\xi)d\xi\\
&+ \int_0^{\tau'} e^{-\mu\mathbb{I}_p(\tau'-\xi)}C_m\bar{B}(\tilde{u}_v(t_k+\xi) - \eta_v(t_k+\xi))d\xi, \quad \tau' \in [0, T_s],
\end{aligned} \tag{4.2.44}$$

where $\alpha_0(\tau', T_s)$ is defined as

$$\alpha_0(\tau', T_s) = 1 - \Phi(\tau')\Phi^{-1}(T_s)e^{-\mu(T_s-\tau')}\,. \tag{4.2.45}$$

Notice that from (4.2.5) it follows that

$$\|\int_0^{\tau'} e^{-\mu\mathbb{I}_p(\tau'-\xi)}C_m A_m\tilde{v}(t_k+\xi)d\xi\| \leq \alpha_1(\tau')e^{-\lambda_v(t_k+\tau'-t_0)}\|\tilde{v}(t_0)\|, \tag{4.2.46}$$

where

$$\bar{\alpha}_1(\tau') = k_{v_0}\frac{\|C_m A_m\|}{\mu - \lambda_v}\alpha_1(\tau'), \quad \forall\tau' \in [0, T_s], \tag{4.2.47}$$

and κ_{v_0}, λ_v are given in (4.2.5); $\alpha_1(\cdot)$ is defined in (4.2.30). Moreover, since

$$\|\tilde{u}_v(t)\| \leq \Delta_u, \quad \|\eta_v(t)\| \leq \Delta_f, \quad 0 \leq t \leq \tau, \tag{4.2.48}$$

it can be shown that

$$\|\int_0^{\tau'} e^{-\mu\mathbb{I}_p(\tau'-\xi)}C_m\bar{B}(\tilde{u}_v(t_k+\xi) + \eta_v(t_k+\xi))d\xi\| \leq \alpha_2(\tau')(\Delta_u + \Delta_f), \quad \forall\tau' \in [0, T_s], \tag{4.2.49}$$

where $\alpha_2(\cdot)$ is given in (4.2.30), with $\Phi(\tau')$ being given by (4.2.18). Since

$$\|\tilde{y}(t_k)\| \leq \bar{\alpha}_1(T_s)e^{-\lambda_v(t_k-t_0)}\|\tilde{v}(t_0)\| + \alpha_2(T_s)(\Delta_u + \Delta_f), \quad \forall k \in \mathbb{N}, \tag{4.2.50}$$

from (4.2.44) it follows that

$$\begin{aligned}
\|\tilde{y}(\tau' + t_k)\| \leq& \left(e^{-(\mu-\lambda_v)\tau'}\alpha_0(\tau', T_s)\bar{\alpha}_1(T_s) + \bar{\alpha}_1(\tau')\right)e^{-\lambda_v(\tau'+t_k-t_0)}\|\tilde{v}(t_0)\|\\
&\left(e^{-\mu\tau'}\alpha_0(\tau', T_s)\alpha_2(T_s) + \alpha_2(\tau')\right)(\Delta_u + \Delta_f), \quad \tau' \in [0, T_s].
\end{aligned} \tag{4.2.51}$$

Notice that since $\bar{\alpha}_1(\tau') \leq \bar{\alpha}_1(T_s)$, $\alpha_2(\tau') \leq \alpha_2(T_s)$, and $0 \leq \Phi(\tau')\Phi^{-1}(T_s) \leq 1$ hold for $\tau' \in [0, T_s]$, one has

$$\begin{aligned} e^{-(\mu-\lambda_v)\tau'}\alpha_0(\tau', T_s)\bar{\alpha}_1(T_s) + \bar{\alpha}_1(\tau') &\le 2\alpha_1(T_s), \\ e^{-\mu\tau'}\alpha_0(\tau', T_s)\alpha_2(T_s) + \alpha_2(\tau') &\le 2\alpha_2(T_s), \end{aligned} \tag{4.2.52}$$

for all $\tau' \in [0, T_s]$. Therefore, from (4.2.50), (4.2.51) and (4.2.52) it can be shown that

$$\|\tilde{y}(t)\| \le 2\bar{\alpha}_1(T_s)e^{-\lambda_v(t-t_0)}\|\tilde{v}(t_0)\| + \gamma(T_s), \quad 0 \le t_0 \le t \le \tau, \tag{4.2.53}$$

where $\gamma(T_s)$ is given in (4.2.32). Notice that from (4.2.47) it follows that

$$2\|\mathbb{I}_n - HC_m\|\bar{\alpha}_1(T_s) = \kappa_y\alpha_1(T_s), \tag{4.2.54}$$

where κ_y, $\alpha_1(T_s)$ are defined in (4.2.6) and (4.2.30), respectively. Finally, since $v_0 = (\mathbb{I}_n - HC_m)x_0$, letting $t_0 = 0$ yields

$$\|\tilde{y}(t)\| \le \kappa_y\alpha_1(T_s)e^{-\lambda_v t}\|x_0\| + \gamma(T_s), \quad 0 \le t \le \tau. \tag{4.2.55}$$

This completes the proof. □

Remark 4.2.1. *From Lemma 4.2.2, the adaptation sampling time $T_s > 0$ can be chosen such that $\gamma(T_s)$ becomes arbitrarily small.*

Remark 4.2.2. *Since $\alpha_1(T_s)$ and $\gamma(T_s)$ converge to zero as $T_s \to 0$, the transient bounds and steady-state bounds for $\tilde{y}(t)$ can be small enough.*

Theorem 4.2.1. *Suppose $T_s > 0$ is chosen such that*

$$\gamma_x\gamma(T_s) < \bar{\gamma}, \quad \gamma_u\gamma(T_s) < \bar{\gamma}, \quad \forall\Omega \in \mathcal{C}_\Omega, \tag{4.2.56}$$

where γ_x and γ_u are given in (4.2.28)*. Then, the $\mathcal{L}_1$ adaptive output feedback controller defined via* (4.2.13) – (4.2.18)*, subject to the $\mathcal{L}_1$–norm condition* (4.2.8)*, provides the following upper bounds:*

$$\|u_{ref} - u\|_{\mathcal{L}_\infty} \le \rho_{du}, \quad \|x_{ref} - x\|_{\mathcal{L}_\infty} \le \rho_{dx}, \tag{4.2.57}$$

and

$$\begin{aligned} \|y_{ref} - x\|_{\mathcal{L}_\infty} &\le \|C_m\|\rho_{dx}, \\ \|x\|_{\mathcal{L}_\infty} \le \rho_x, &\quad \|u\|_{\mathcal{L}_\infty} \le \rho_u\,. \end{aligned} \tag{4.2.58}$$

Moreover, for each $\Omega \in \mathcal{C}_\Omega$ there exist strictly decreasing functions $\upsilon_x(t)$, $\upsilon_y(t)$, and positive constants γ_{dx}, γ_{dy}, such that

$$\|x_{ref} - x\|_{\mathcal{L}_\infty} \le \upsilon_x(t)\|x_0\| + \gamma_{dx}\bar{\gamma}, \quad \|y_{ref} - y\|_{\mathcal{L}_\infty} \le \upsilon_y(t)\|x_0\| + \gamma_{dy}\bar{\gamma}\,. \tag{4.2.59}$$

Proof. First, define

$$\tilde{\eta}_v(s) = \hat{\eta}_v(s) + \tilde{u}_v(s) - \eta_v(s)\,, \tag{4.2.60}$$

where $\hat{\eta}_v(s)$, $\{\tilde{u}_v(s), \eta_v(s)\}$ are given in (4.2.14) and (4.2.36), respectively. Notice that the control laws in (4.2.13) can be rewritten by

$$u(s) = D(s)Z^{-1}(s)\left(Z(s)K_g r(s) - Z(s)\Omega u(s) - \eta_v(s) - \tilde{\eta}_v(s)\right),$$

which further leads to

$$u(s) = C_0(s)\left(K_g r(s) - \eta(s) - \sigma(s)\right) - C_0(s)Z^{-1}(s)\tilde{\eta}_v(s), \tag{4.2.61}$$

where $C_0(s)$ is given in (4.2.21); recall that $\eta(s)$, $\sigma(s)$ are the Laplace transforms of $\eta(t) = (f(x,t) - f(0,t))$ and $\sigma(t) = f(0,t)$, respectively. From (4.2.61) and (4.2.1), the closed-loop system is given by

$$\begin{aligned} x(s) &= H_0(s)C(s)K_g r(s) + G(s)(\eta(s) + \sigma(s)) - H_0(s)C(s)Z^{-1}(s)\tilde{\eta}_v(s) + x_{in}(s), \\ y(s) &= C_m x(s), \end{aligned} \tag{4.2.62}$$

where $\{H_0(s), G(s)\}$, $C(s)$ are given in (4.2.9) and (4.2.7), respectively, and $x_{in}(s) = (s\mathbb{I}_n - A_m)^{-1}x_0$. Now, let $\tilde{x}_{ref}(t) = x_{ref}(t) - x(t)$, $\tilde{u}_{ref}(t) = u_{ref}(t) - u(t)$, and $\tilde{\eta}_{ref}(t) = \eta_{ref}(t) - \eta(t)$. Then, subtracting (4.2.62) from (4.2.26) yields

$$\begin{aligned} \tilde{x}_{ref}(s) &= G(s)\tilde{\eta}_{ref}(s) + H_0(s)C(s)Z^{-1}(s)\tilde{\eta}_v(s) - x_{in}(s), \\ \tilde{y}_{ref}(s) &= C_m \tilde{x}_{ref}(s). \end{aligned} \tag{4.2.63}$$

Similarly, from (4.2.61) and (4.2.20) it follows that

$$\tilde{u}_{ref}(s) = -C_0(s)\tilde{\eta}_{ref}(s) + C_0(s)Z^{-1}(s)\tilde{\eta}_v(s). \tag{4.2.64}$$

Next, we prove (4.2.57) by a contradiction argument. Suppose that it is not true. Notice that $\rho_{dx} > \rho_0$. Since $\|\tilde{x}_{ref}(0)\| \le \rho_0 < \rho_{dx}$ and $\|\tilde{u}_{ref}(0)\| = 0 < \rho_{du}$ hold, the continuity of a solution in (4.2.63) implies that there exists $\tau' > 0$, such that the following must hold:

$$\begin{gathered} \left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} = \rho_{dx}, \quad \left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \le \rho_{du}, \\ \text{or} \\ \left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \le \rho_{dx}, \quad \left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} = \rho_{du}. \end{gathered} \tag{4.2.65}$$

Notice that $\rho_{rx} + \rho_{dx} = \rho_x$ holds, and the triangular inequality, together with (4.2.23), (4.2.24) and (4.2.65), yields

$$\|x_{\tau'}\|_{\mathcal{L}_\infty} \le \rho_{rx} + \rho_{dx} = \rho_x, \quad \|u_{\tau'}\|_{\mathcal{L}_\infty} \le \rho_{ru} + \rho_{du} = \rho_u, \tag{4.2.66}$$

where ρ_{rx}, ρ_{ru} are given in (4.2.25), and ρ_{dx}, ρ_u, ρ_{du} are defined in (4.2.27). Since $\rho_{dx} < \rho_x$ and $d_{\rho_x} < L_{\rho_r}$, Assumption 4.2.3 leads to

$$\|\tilde{\eta}_{ref}(t)\| \leq L_{\rho_r}\|\tilde{x}_{ref}(t)\|, \quad \forall t \in [0,\tau']. \tag{4.2.67}$$

Next, by taking the Laplace transform of (4.2.43) and using the fact that $\tilde{y}(0) = 0$, it follows, in conjunction with (4.2.60), that

$$\tilde{\eta}_v(s) + (1 - e^{\mu T_s})(C_m\bar{B})^\dagger\hat{\eta}_H(s) = (C_m\bar{B})^\dagger(s+\mu)\tilde{y}(s) - (C_m\bar{B})^\dagger\bar{C}_m\bar{A}_m\tilde{v}(s)\,. \tag{4.2.68}$$

Moreover, the adaptation laws in (4.2.16), together with (4.2.18), implies

$$(1 - e^{\mu T_s})\hat{\eta}_H(s) = \mu\tilde{y}_H(s)\,, \tag{4.2.69}$$

where $\tilde{y}_H(s)$ is the Laplace transform of $\tilde{y}_H(t)$ given in (4.2.17). Therefore, combining (4.2.68) and (4.2.69) yields

$$C_0(s)Z^{-1}(s)\tilde{\eta}_v(s) = C_1(s)\tilde{v}(s) + C_2(s)\tilde{y}(s) - C_3(s)\tilde{y}_H(s), \tag{4.2.70}$$

where $C_1(s)$, $C_2(s)$, and $C_3(s)$ are given in (4.2.29).

From (4.2.63), (4.2.67) and (4.2.70) it follows that

$$\begin{aligned}\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq & \frac{\|H_1(s)\|_{\mathcal{L}_1}\|\tilde{v}\|_{\mathcal{L}_\infty} + \|H_2(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty} + \|H_3(s)\|_{\mathcal{L}_1}\|\tilde{y}_H\|_{\mathcal{L}_\infty}}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}} \\ & + \frac{\kappa_m\rho_0}{1 - \|G(s)\|_{\mathcal{L}_1} L_{\rho_r}}, \\ \left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq & \|C_0(s)\|_{\mathcal{L}_1} L_{\rho_r}\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} + \|C_1(s)\|_{\mathcal{L}_1}\|\tilde{v}\|_{\mathcal{L}_\infty} \\ & + \|C_2(s)\|_{\mathcal{L}_1}\|\tilde{y}\|_{\mathcal{L}_\infty} + \|C_3(s)\|_{\mathcal{L}_1}\|\tilde{y}_H\|_{\mathcal{L}_\infty}\,,\end{aligned} \tag{4.2.71}$$

where κ_m and $\{H_1(s), H_2(s), H_3(s)\}$ are given in (4.2.6) and (4.2.11), respectively. Since $\|x_{\tau'}\|_{\mathcal{L}_\infty} \leq \rho_x$ and $\|u_{\tau'}\|_{\mathcal{L}_\infty} \leq \rho_u$, from Lemma 4.2.3 and (4.2.5) it follows that for all $t \in [0,\tau']$

$$\|\tilde{y}_H(t)\| \leq \kappa_y\rho_0 + \gamma(T_s), \quad \|\tilde{y}_H(t)\| \leq \kappa_y\rho_0 + \gamma(T_s), \quad \|\tilde{v}(t)\| \leq \kappa_v\rho_0, \tag{4.2.72}$$

which, along with (4.2.71), leads to

$$\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq \gamma_{x_0}\rho_0 + \gamma_x\gamma(T_s), \quad \left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} \leq \gamma_{u_0}\rho_0 + \gamma_u\gamma(T_s). \tag{4.2.73}$$

Since $\gamma_x\gamma(T_s) < \bar{\gamma}$ and $\gamma_u\gamma(T_s) < \bar{\gamma}$ (see (4.2.34)), it follows that $\left\|\tilde{x}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} < \rho_{dx}$ and $\left\|\tilde{u}_{ref_{\tau'}}\right\|_{\mathcal{L}_\infty} < \rho_{du}$, which contradict to (4.2.65). Therefore, Equation (4.2.57) is proven. It can be shown that Equation (4.2.58) holds by applying the triangular inequality.

Now, we prove (4.2.59). Let $\phi_c(t)$ is the signal with the Laplace transform

$$\phi_c(s) = \Omega C_0(s) Z^{-1}(s) \tilde{\eta}_v(s), \tag{4.2.74}$$

with $\tilde{\eta}_v(s)$ being given in (4.2.60). Notice that $\phi_c(t)$ is a bounded signal, because from (4.2.70) it follows that

$$\phi_c(s) = \Omega C_1(s)\tilde{v}(s) + \Omega C_2(s)\tilde{y}(s) - \Omega C_3(s)\tilde{y}_H(s), \tag{4.2.75}$$

and $\tilde{v}(t)$, $\tilde{y}(t)$, $\tilde{y}_H(t)$ are all bounded for $t \geq 0$ (see Lemma 4.2.3), where $C_1(s)$, $C_2(s)$, $C_3(s)$ are stable and proper transfer matrices given in (4.2.29).

Applying a similar method from the proof of Theorem 4.2.1, one has

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)} \|\tilde{x}_c(t_m)\| + \kappa_1 \|\phi_c\|_{\mathcal{L}_\infty[t_m,t]}, \quad 0 \leq t_m \leq t, \tag{4.2.76}$$

where $\tilde{x}_c(t) = [\tilde{x}_{ref}^\top(t), \tilde{x}_b^\top(t)]^\top \in \mathbb{R}^{n_c \times n_c}$ is the internal state vector of the system (4.2.63) with $n_c > n$; κ_0, λ_0 and κ_1 are some positive constants (see the proof of Theorem 4.1.1). Since (4.2.75) implies that

$$\|\phi_c\|_{\mathcal{L}_\infty[t_m,t]} \leq \|\Omega C_1(s)\|_{\mathcal{L}_1} \|\tilde{v}\|_{\mathcal{L}_\infty[t_m,t]} + \|\Omega C_2(s)\|_{\mathcal{L}_1} \|\tilde{y}\|_{\mathcal{L}_\infty[t_m,t]} + \|\Omega C_3(s)\|_{\mathcal{L}_1} \|\tilde{y}_H\|_{\mathcal{L}_\infty[t_m,t]}, \tag{4.2.77}$$

by combining (4.2.5), (4.2.53) and (4.2.77), and (4.2.76), it follows that

$$\|\tilde{x}_c(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)} \|\tilde{x}_c(t_m)\| + \kappa_1\kappa_2 \|\tilde{v}(t_m)\| + \kappa_1\kappa_3 \gamma(T_s), \tag{4.2.78}$$

where

$$\begin{aligned}
\kappa_2 &= \|\Omega C_1(s)\|_{\mathcal{L}_1} \kappa_{v_0} + 2\bar{\alpha}_1(T_s)(\|\Omega C_2(s)\|_{\mathcal{L}_1} + \|\Omega C_3(s)\|_{\mathcal{L}_1}), \\
\kappa_3 &= \|\Omega C_2(s)\|_{\mathcal{L}_1} + \|\Omega C_3(s)\|_{\mathcal{L}_1}
\end{aligned}$$

with κ_{v_0}, $\bar{\alpha}_1(T_s)$ being given in (4.2.5) and (4.2.47), respectively. Moreover, by letting $t_m = t/2$ and using $\|x_c(0)\| = \|x_0\|$ and $\|v_0\| \leq \|\mathbb{I} - HC_m\|\|x_0\|$, from (4.2.78) it can be shown that

$$\|\tilde{x}_c(t)\| \leq \upsilon_{dx}(t)\|x_0\| + \gamma_{dx}\gamma(T_s),$$

where

$$\begin{aligned}
\upsilon_{dx}(t) =& \kappa_0^2 e^{-\lambda_0 t} + \|\mathbb{I} - HC_m\|\kappa_1\kappa_2(\kappa_0 e^{-\frac{\lambda_0}{2}t} + e^{-\frac{\lambda_v}{2}t}), \\
\gamma_{dx} =& (\kappa_0 + 1)\kappa_1\kappa_3.
\end{aligned} \tag{4.2.79}$$

This proves (4.2.58). Finally, defining $\gamma_{dy} = \|C_m\|\gamma_{dx}$ and $\upsilon_{dy}(t) = \|C_m\|\upsilon_{dx}(t)$ yields Equation (4.2.59). This completes the proof. □

Assume that the system (4.2.1) has zero initial condition. From Equation 4.2.44 in the proof of Lemma 4.2.3, one can show that for each $t_k = kT_s$, $k \in \mathbb{N}$,

$$\int_0^{T_s} e^{\mu \mathbb{I}_p \xi} \hat{\eta}_v(t_k) d\xi = \int_0^{T_s} e^{\mu \mathbb{I}_p - \xi} (\tilde{u}_v(t_{k-1} + \xi) - \eta_v(t_{k-1} + \xi)) d\xi, \tag{4.2.80}$$

where $\hat{\eta}_v(t_k)$ is given in (4.2.16). Notice that $\hat{\eta}_v(t_k)$ is an exponentially weighted average of uncertainties over the interval ($e^{\mu\xi}$ is a weighting function, and $\tilde{u}_v$, η_v represent the input gain mismatch and nonlinear uncertainties after passing through $Z(s)$). Therefore, the proposed adaptive laws provide a piecewise constant estimation of system uncertainties; the estimate can be improved by using small enough $T_s > 0$. Moreover, the steady-state bound can be made arbitrarily small, since $\gamma(T_s)$ given in (4.2.32) gets close to zero with $T_s \to 0$ (see (4.2.18)).

4.2.3. *Illustrative example*

In this section, we verify the proposed controller on an academic example. Consider the following nonlinear system:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 1 & -5 & 2 \\ 1 & 0 & -5.5 \end{bmatrix}}_{A_m} x(t) + \underbrace{\begin{bmatrix} 2 \\ 2.5 \\ -3 \end{bmatrix}}_{B_m} (\Omega u(t) + f_\Delta(x,t)),$$

$$y(t) = \underbrace{\begin{bmatrix} -5 & 10 & 5 \\ 2.5 & -2 & 0 \end{bmatrix}}_{C_m} x(t),$$

where $\Omega \in [0.5, 1.5]$ is an unknown input gain, and the unknown function is given by

$$f_\Delta(x,t) = B_m^\dagger \begin{bmatrix} 1.65\|x\| + 5\tanh(0.5x_1)x_1 \\ -0.3x_3^2 - 0.2(1 - e^{-0.3t}) \\ -x_3\cos(1.256t) \end{bmatrix}.$$

The design parameters for $\mathcal{L}_1$ adaptive controllers are chosen such that

$$D(s) = \frac{15}{s(s/30+1)}, \quad Z(s) = \frac{4.09}{s+3}, \quad \mu = 1.0, \quad K_v = \begin{bmatrix} 0.20 & -0.42 \\ -1.30 & 0.40 \\ 0.50 & -0.60 \end{bmatrix}. \tag{4.2.81}$$

Figures 4.4 - 4.5 illustrate the performance of the proposed controller according to different choices of sampling rates ($T_s = 2Hz$ and $T_s = 20Hz$). In Figure 4.4, it is observed that the lower sampling rate ($T_s = 20Hz$) makes the system responses close to the reference system. Figure 4.5

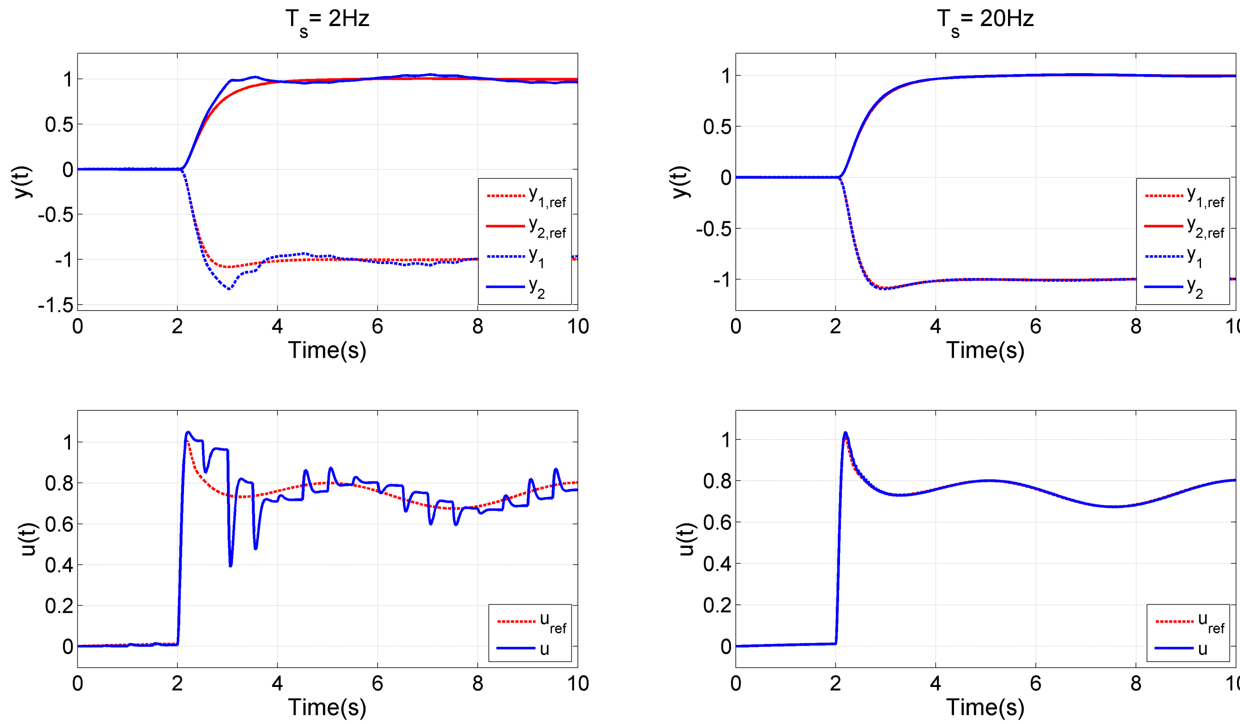

Figure 4.4: Effects of the choice of sampling rates

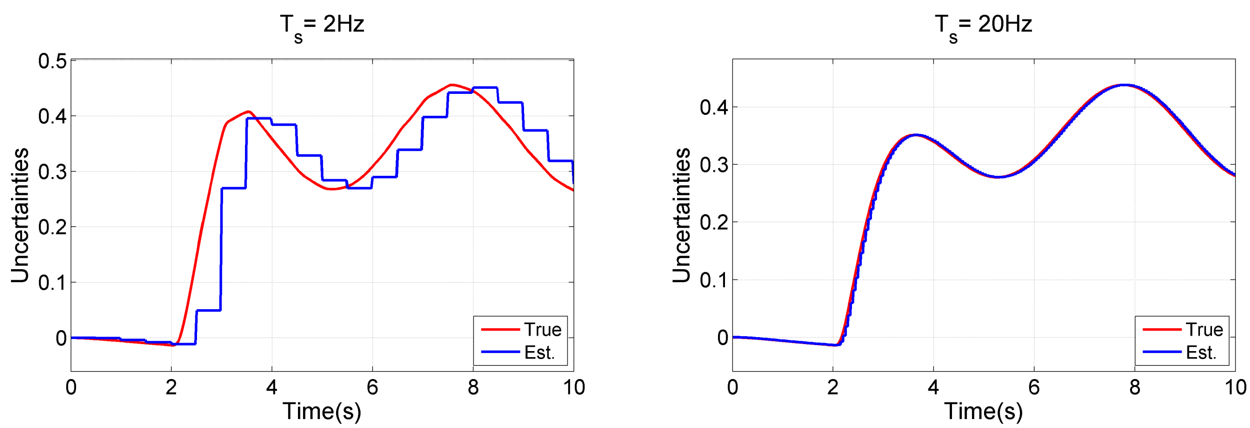

Figure 4.5: Piecewise estimation of uncertainties

shows the piecewise constant estimation of a disturbance signal; the estimate is improved with $T_s = 20Hz$. In Figure 4.6, we present the simulation results for the unit step command, and the sinusoidal command $r(t) = sin(t)$, where the system is initialized with non-zero initial condition $x_0 = [0.4, -0.4, 0.4]$. As predicted in the analysis, the effects of the non-zero initial condition are decreasing over time, and the steady-state errors remain small; notice that the choice of a small sampling rate T_s reduces the errors.

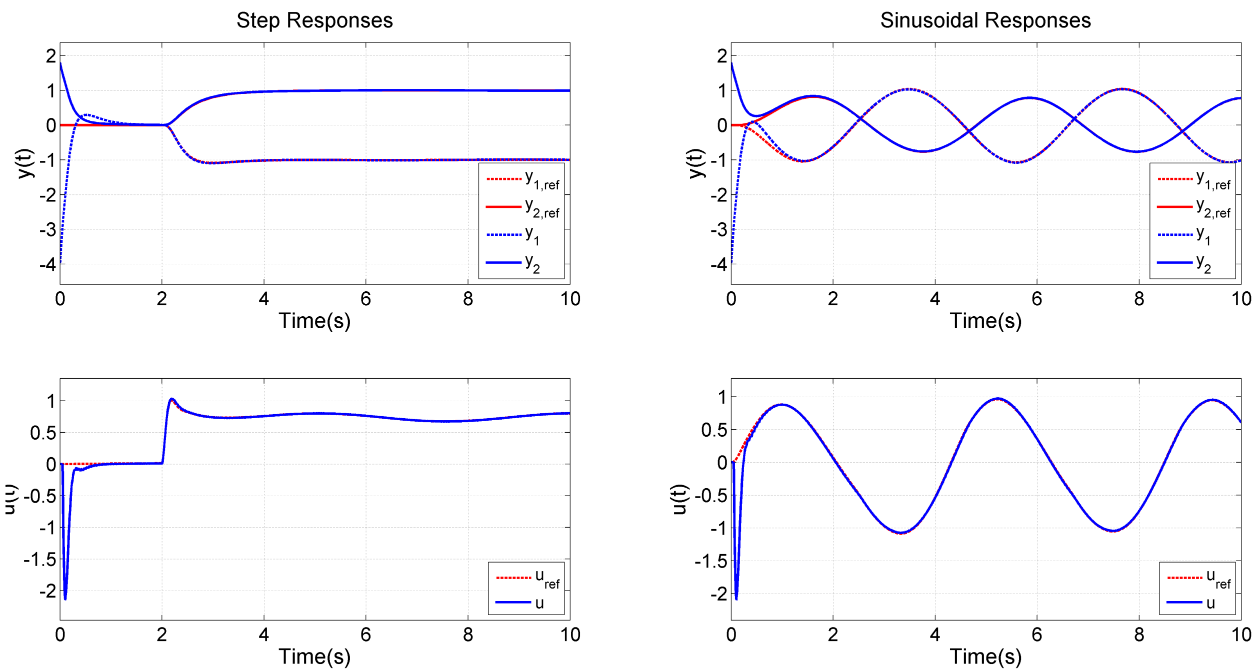

Figure 4.6: System responses for $x_0 = [0.4, -0.4, 0.4]$

CHAPTER. 5

Applications

This chapter considers longitudinal autopilot of missile and inverted pendulum. For both applications, we validate the $\mathcal{L}_1$ adaptive controllers and provide simulation results.

5.1. Design of Missile Longitudinal Autopilot

A classic approach for missile control systems is the three-loop autopilot (dubbed Raytheon three-loop autopilot), which uses only the acceleration and angular rate signals and has been successfully employed in real applications (e.g., Sparrow, Patriot Anti-Cruise Missile, Standard Missile Block IV, and Enhanced Fiber Optic Guided Missile, to name a few) [100–102]. The three-loop autopilot is characterized by a rate loop to improve overall damping properties, a synthetic stabilization loop to tolerate some instability (i.e., $M_\alpha > 0$), and an acceleration loop for command tracking [102, 103]. The aerodynamic characteristics of missile systems related to the three-loop autopilot design are well studied in [104], and the conncetion with modern control methodologies is found in [105]. In [106], the authors show that the three-loop autopilot has guaranteed robustness properties of the optimal control around the trim condition. To extend these robustness properties along the large flight envelope with performance guarantees we consider $\mathcal{L}_1$ adaptive augmentation of the three-loop autopilot.

Several approaches for adaptive missile autopilot can be found in the literature. For example, the authors of [107, 108] use dynamic inversion for missile autopilot design. An approach with a neural network and output redefinition was introduced in [109, 110]. Notice that these methods are based on full state information, so they may require additional observers for state estimation and corresponding analysis for the closed-loop stability.

Adaptive augmentation of the three-loop autopilot is not straightforward. There are a few issues: 1) adaptive output feedback approaches are mainly derived under the assumption of minimum-phase and square systems, 2) the transfer function from the fin command to the measured acceleration shows typically non-minimum phase features in missile systems, and 3) the missile dynamics present a non-square system when the set of all available signals is considered.

In this section, an $\mathcal{L}_1$ adaptive output feedback controller is presented for a missile longitudinal autopilot using both acceleration and pitch rate signals, as discussed in Chapter 3.

5.1.1. *Linearized model for missile longitudinal dynamics*

The transfer function of the linearized longitudinal missile dynamics with a first-order actuator is given by [102, Chapter 21]:

$$\begin{bmatrix} \frac{n_z(s)}{\delta_{cmd}(s)} \\ \frac{q(s)}{\delta_{cmd}(s)} \end{bmatrix} = -\frac{(M_\alpha Z_\delta - Z_\alpha M_\delta)}{M_\alpha} \begin{bmatrix} \frac{V_m}{1845}\left(1 - \frac{Z_\delta s^2}{M_\alpha Z_\delta - Z_\alpha M_\delta}\right)\left(1 + \frac{Z_\alpha}{M_\alpha}s - \frac{s^2}{M_\alpha}\right)^{-1}\left(\frac{1}{\tau_a s+1}\right) \\ \left(1 + \frac{M_\delta s}{M_\alpha Z_\delta - Z_\alpha M_\delta}\right)\left(1 + \frac{Z_\alpha}{M_\alpha}s - \frac{s^2}{M_\alpha}\right)^{-1}\left(\frac{1}{\tau_a s+1}\right) \end{bmatrix}, \tag{5.1.1}$$

where $\delta_{cmd}(s)$, $n_z(s)$ and $q(s)$ are the Laplace transforms of the actuator command, body acceleration, and pitch rate, respectively, τ_a is the time constant of the actuator, V_m is the velocity of the missile, and Z_α, Z_δ, M_α, M_δ are aerodynamic derivatives.

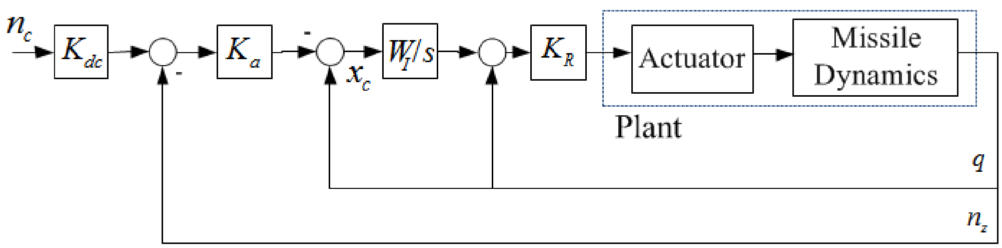

Figure 5.1: Three-loop autopilot

The three-loop autopilot structure is given in Figure 5.1. Along the lines of the representation given in Section 3.2, the controller dynamics of the three-loop autopilot are written as

$$\begin{aligned} \dot{x}_c &= B_{c_1} y_p + B_{c_2} n_c, \\ u_b &= C_c x_c + D_c y_p, \end{aligned} \tag{5.1.2}$$

where n_c is an acceleration command, and

$$B_{c_1} = \begin{bmatrix} K_a & 1 \end{bmatrix}, \quad B_{c_2} = -K_{dc}K_a, \quad C_c = K_R W_I, \quad D_c = \begin{bmatrix} 0 & K_R \end{bmatrix},$$

with $y_p = [n_z, q]^\top$ and K_{dc}, K_a, W_I, K_R being the control gains. In tail-controlled missile systems, it is observed that the transfer function of $\frac{n_z(s)}{\delta_{cmd}(s)}$ has an unstable zero

$$w_z = \sqrt{\frac{M_\alpha Z_\delta - M_\delta Z_\alpha}{Z_\delta}},$$

while the transfer function of $\frac{q(s)}{\delta_{cmd}(s)}$ has a stable zero at $s = -1/T_a$, where T_a is the turning rate time constant defined as

$$T_a = \frac{M_\delta}{M_\alpha Z_\delta - M_\delta Z_\alpha}.$$

Notice that the transmission zeros of the open-loop system in (5.1.1) are the common zeros of $\frac{n_z(s)}{\delta_{cmd}(s)}$, and $\frac{q(s)}{\delta_{cmd}(s)}$, so that the open-loop transfer function matrix cannot have unstable transmission zeros.

Moreover, it is easy to verify that (C_pB_p) is full rank, where (A_p, B_p, C_p) is the minimal realization of (5.1.1). Therefore, from Lemma 2.1.20, it follows that the augmented input sensitivity function $M(s)$ has no unstable transmission zeros, which implies that the augmented system together with (5.1.1) and (5.1.2) satisfies Assumption 3.1.2. Figure 5.2 illustrates the proposed control scheme.

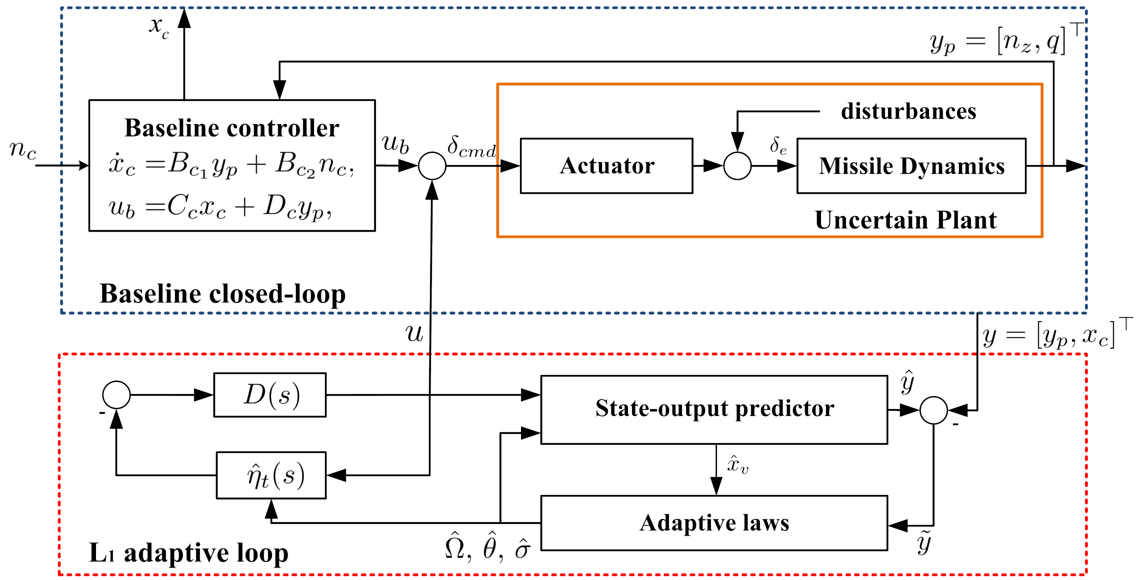

Figure 5.2: Adaptive control extension of three-loop autopilot

5.1.2. *Linear Model Simulation*

To verify the performance of the proposed $\mathcal{L}_1$ controller, simulation results are illustrated in this section. The nominal aerodynamics coefficients and three-loop autopilot gains are obtained from [102, Chapter 23]:

$$
\begin{aligned}
&V_m = 3000\, ft/s, \quad Z_\alpha = -2.94\, s^{-1}, \quad Z_\delta = -0.65\, s^{-1}\,, \\
&M_\alpha = -642\, s^{-2}, \quad M_\delta = -555\, s^{-2}, \quad \tau_a = 0.0106\, s\,, \\
&K_{dc} = 1.5348 \quad K_a = 1.15\,, \quad W_I = 12.9\,, \quad K_R = 0.0928\,.
\end{aligned}
$$

The autopilot design of the nominal system shows the desired time constant $0.3\,s$, the cross-over frequency $55\, rad/s$, and the delay margin $15\, ms$ with the given 1^{st} order actuator. It is known that the selection of $C(s)$ defines the trade-off between the performance to command tracking and the robustness to a time delay. Figure 5.3 illustrates the numerically determined time delay margin and maximum singular values of the input sensitivity function to the performance output according to different choices of the bandwidth of the filter $C(s)$ (we assume that $\Omega = \mathbb{I}_m$ and $D(s) = \frac{k}{s}\mathbb{I}_m$ in this analysis).

In the design, the filter bandwidth is chosen to be $15\, Hz$ for $\Omega = \mathbb{I}_m$ (i.e., $D(s) = 94.25/s$), which gives $10\, ms$ of the time delay margin in the closed-loop system. The design parameters for

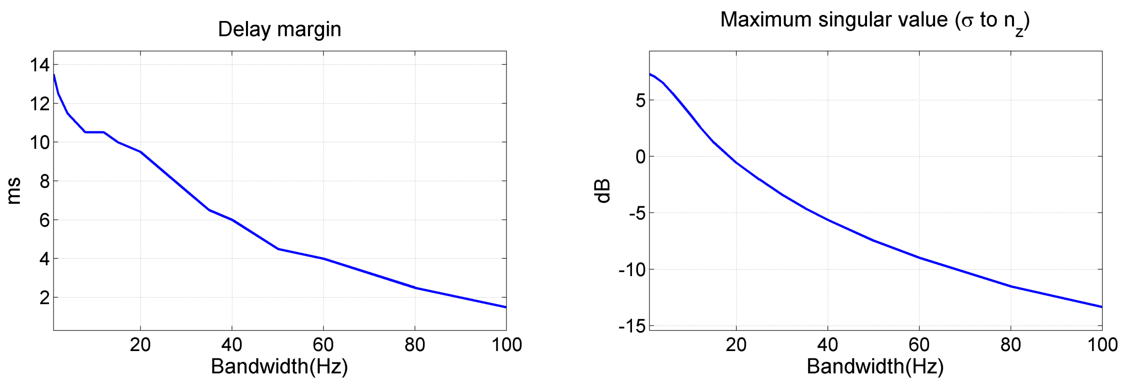

Figure 5.3: Time delay margin and input sensitivity

the proposed controller are selected as

$$K_v = \begin{bmatrix} -13.59 & 4.06 & 0.00 \\ 10.40 & 3.04 & 0.00 \\ -10.77 & 7.83 & 0.00 \\ -1.15 & -1.00 & -35.00 \end{bmatrix}, \quad \mu = 30, \quad Q_v = 10^3\mathbb{I}_4, \quad P_y = \mathbb{I}_3,$$

$$\Gamma_\Omega = \Gamma_\Theta = \Gamma_\sigma = 100, \quad D(s) = 94.25/s.$$

Figure 5.4 depicts the system response and control history of the nominal system for a 10 g step command. Notice that for the nominal system, the controller with and without augmentation produces the same response. This is the expected and correct behavior; no additional action is required by the augmenting controller.

Three kinds of matched disturbances are taken into account during simulations: an actuator bias $\sigma_1(t) = 2$, a sinusoidal input bias $\sigma_2(t) = 2\sin(4\pi t)$, and nonlinear matched uncertainties $\eta(t) = 0.5\sin(2t)\|y(t)\|_2 + \sin(3t)$. Simulation results for a 10 g step command are illustrated in Figures 5.5 – 5.7. When there is a constant bias (Figure 5.5), the three-loop autopilot is eventually able to compensate for the disturbance, but the response time is slower relative to the nominal system, and a larger actuator deflection is commanded. In Figure 5.6, the sinusoidal disturbance has a more pronounced presence in the output, when the three-loop system is used alone. The $\mathcal{L}_1$ augmentation is able to greatly reduce the effect of the disturbance on the output, improving the tracking performance. When nonlinear uncertainties are considered, the three-loop controller is not able to stabilize the system. This can be seen in Figure 5.7. However, the $\mathcal{L}_1$ augmentation is able to reject the disturbance and achieve tracking performance similar to the nominal system.

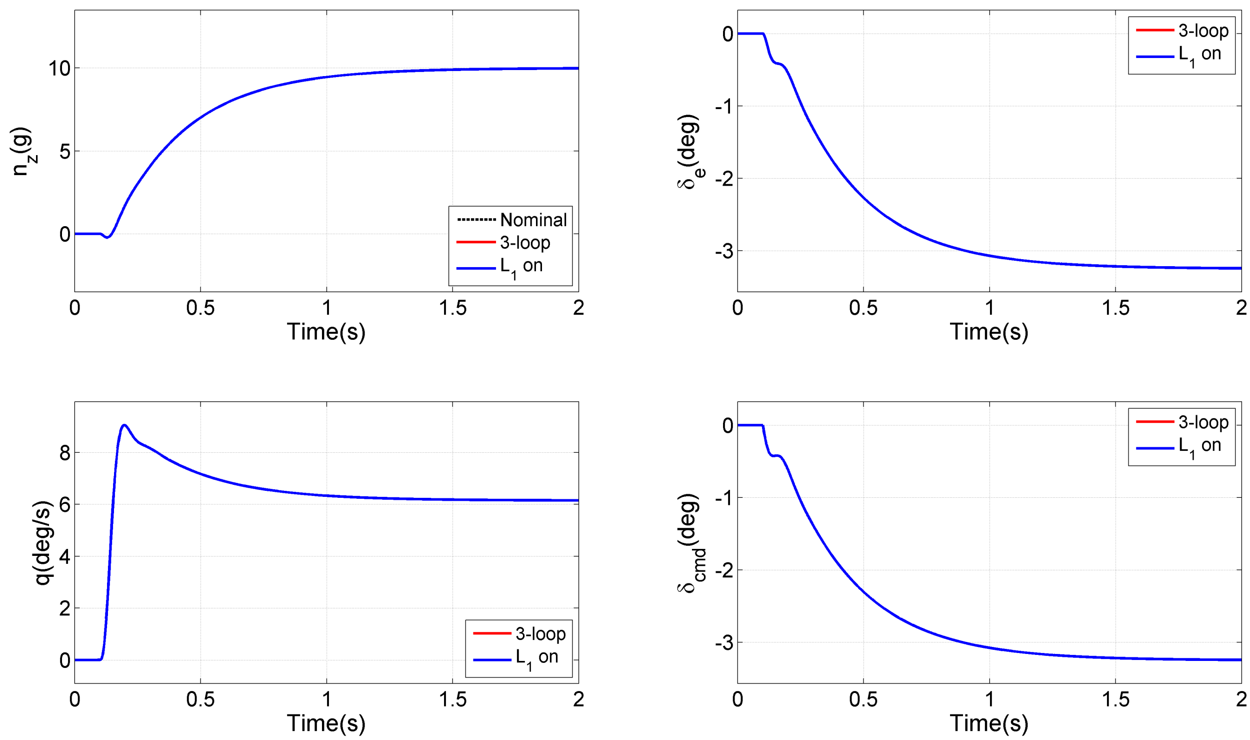

Figure 5.4: Response of the nominal system

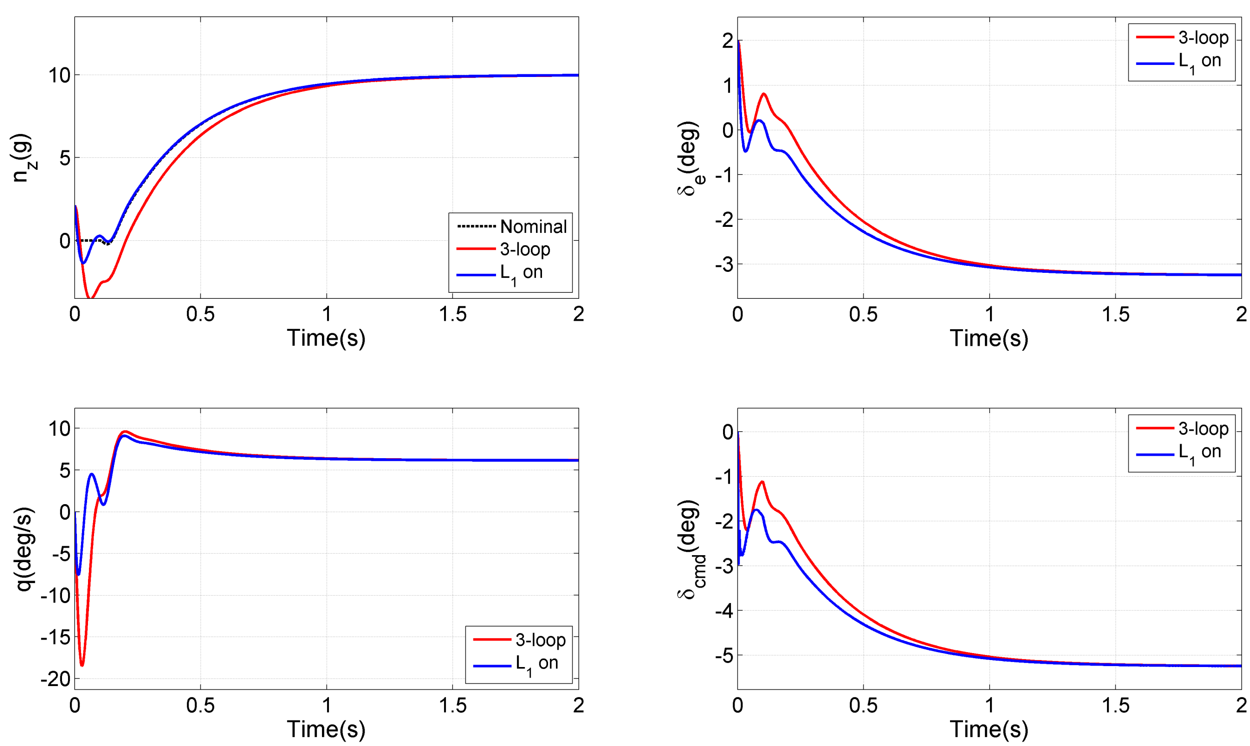

Figure 5.5: Response of the system for $\sigma(t) = \sigma_1(t)$

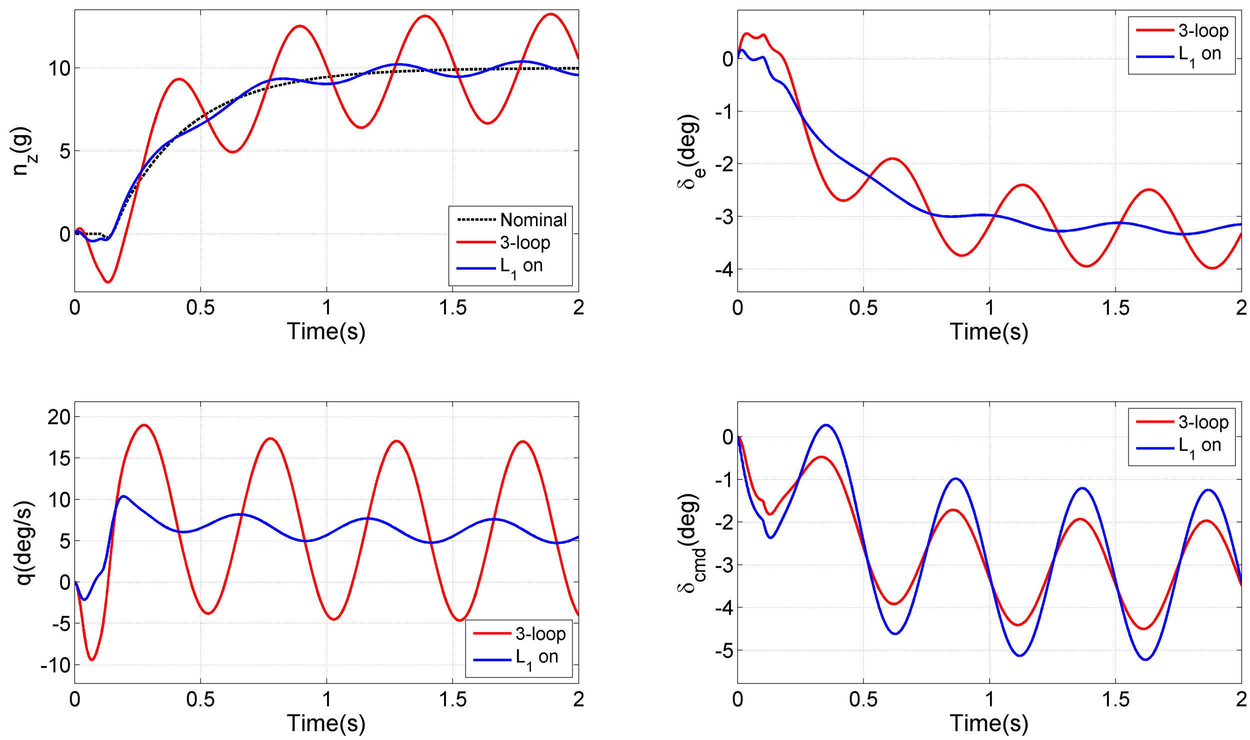

Figure 5.6: Response of the system for $\sigma(t) = \sigma_2(t)$

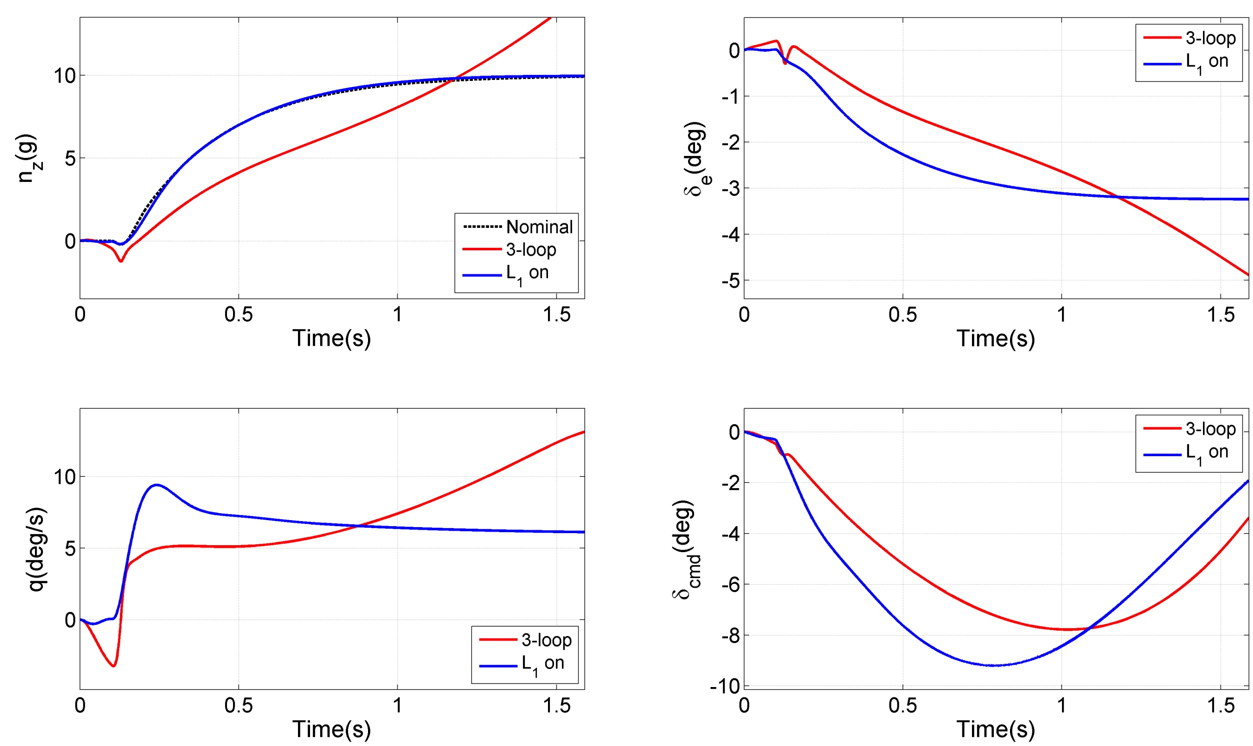

Figure 5.7: Response of the system for the disturbance $\eta(t)$

5.1.3. *Nonlinear Model Simulation*

In this section, we demonstrate the proposed method with a nonlinear missile model. The nonlinear longitudinal model for a missile system is given in [111]:

$$
\begin{aligned}
\dot{M}(t) &= \frac{1}{v_s}\left(-|n_z(t)|\sin(|\alpha(t)|) + A_x M^2(t)\cos(\alpha(t))\right),\\
\dot{\alpha}(t) &= K_\alpha M(t) C_n(\alpha(t), \delta_e(t), M(t))\cos(\alpha(t)) + q(t),\\
\dot{q}(t) &= K_q M^2(t) C_m(\alpha(t), \delta_e(t), M(t)),\\
n_z(t) &= K_z M^2(t) C_n(\alpha(t), \delta_e(t), M(t)),
\end{aligned}
\tag{5.1.3}
$$

where A_x, K_α, K_q, and K_z are given constants, v_s is the speed of sound, $M(t)$ is the Mach number, $\alpha(t)$ is the angle of attack, $q(t)$ is the pitch rate, $n(t)$ is the acceleration, $\delta_e(t)$ is the fin deflection, and $C_n(\alpha(t), \delta_e(t), M(t))$ and $C_m(\alpha(t), \delta_e(t), M(t))$ are the aerodynamic coefficients written as nonlinear functions of $M(t)$, $\alpha(t)$, and $q(t)$ (See details in [111]). The actuator dynamics are given by

$$\dot{\delta}_e(t) = \tau(\delta_c(t) - \delta_e(t)),$$

where $\tau = 188.5\ rad/s$, and $\delta_c(t)$ is the fin command. The goal is to design an adaptive controller to track a given acceleration command $n_c(t)$ with measurable outputs of $n_z(t)$ and $q(t)$. The nominal model is obtained from a linearization at $M = 3.0$ and $\alpha = 0$. The baseline controller gains are chosen to have an infinite gain margin, and a phase margin of 75° with a phase cross-over frequency 52 rad/s; the gains of three-loop autopilot in (5.1.2) are:

$$K_{dc} = 1.1873, \quad K_a = -0.0553, \quad W_I = 17.5144, \quad K_R = 0.4667.$$

For the $\mathcal{L}_1$ adaptive controller, the low-pass filter and the adaptation gains are chosen as $D(s) = 188.5/s$, and $\Gamma_\Omega = \Gamma_\theta = \Gamma_\sigma = 200$, respectively. The predictor gain K_v is obtained from pole assignment such that A_v has the poles $[-50, -52.5, -55, -57.5]$; $\mu = 30$, $Q = diag(50, 5, 5, 50)$, and $P_y = 0.1\mathbb{I}_3$.

For the purposes of demonstrating robustness to uncertainties, we perform nonlinear simulations with perturbed moment coefficients defined as

$$C_m = C_m((1 + \Delta_a)\alpha(t), (1 + \Delta_e)\delta_e(t), M(t)),$$

where Δ_a and Δ_e represent perturbations in the aerodynamic derivatives of C_{m_α} and $C_{m_{\delta_e}}$, respectively. The following scenarios of perturbations $\Delta_i = (\Delta_a, \Delta_e)$ are chosen as:

$$\Delta_0 = (0, 0), \quad \Delta_1 = (-0.5, -0.5), \quad \Delta_2 = (-2.0, -0.25), \quad \Delta_3 = (-2.5, 0).$$

Figures 5.8 - 5.15 illustrate system responses and control histories of the nonlinear missile system for a given reference command. In this simulation, the Mach number deviates from 3.0 to 2.3 and the angle-of-attack varies from approximately $-15°$ to $15°$ as shown in Figure 5.8 - 5.11. Since the $\mathcal{L}_1$ controller is designed at the point $(M, \alpha_0) = (3, 0)$, the uncertainties in the simulation are the nonlinearities of the model and the variations of aerodynamic coefficients. In Figures 5.8 5.9, the $\mathcal{L}_1$ adaptive controller provides similar response to the baseline three-loop controller, when slight variations of the aerodynamic derivatives occur (i.e., Δ_0 and Δ_1). However, for the cases of large uncertainties (i.e., Δ_2, Δ_3), the response of the three-loop controller is shown through the dotted black lines and can be compared directly to the solid blue lines in Figures 5.10 5.11. The adaptive output-feedback controller shows a tangible improvement in performance, almost nullifying the oscillations that were present with the three-loop controller.

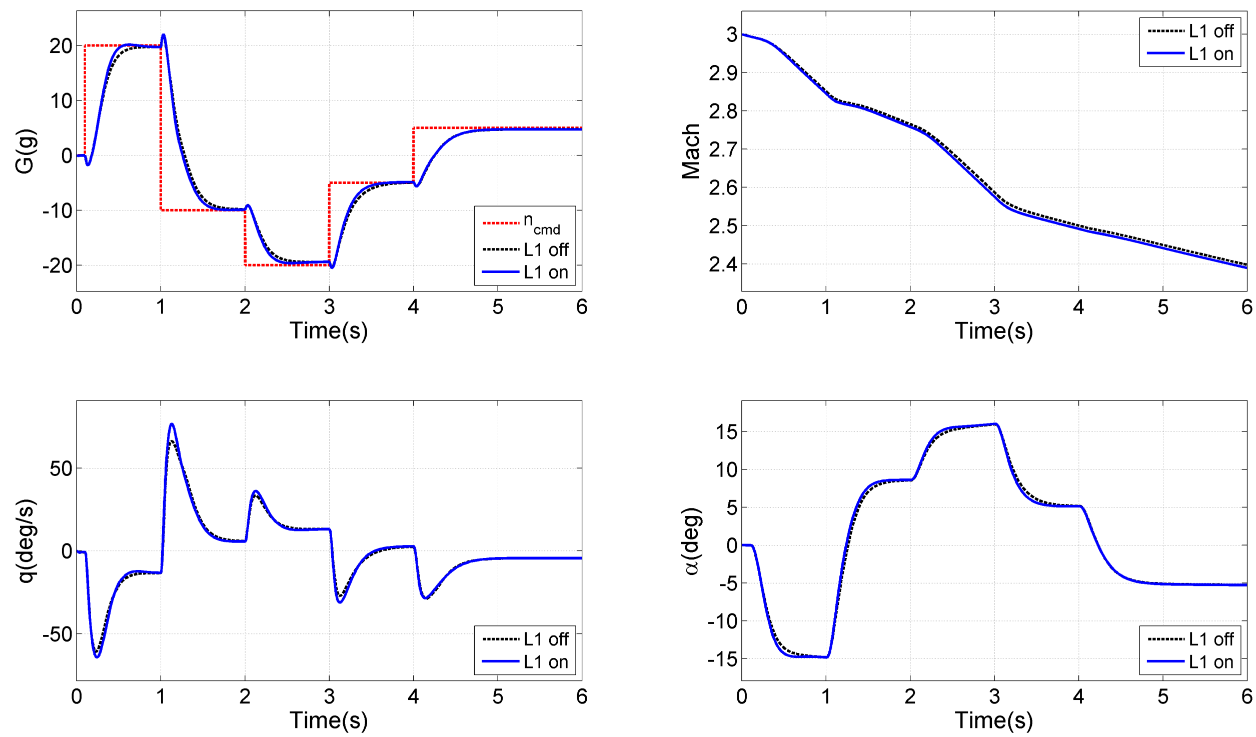

Figure 5.8: System responses for Δ_0

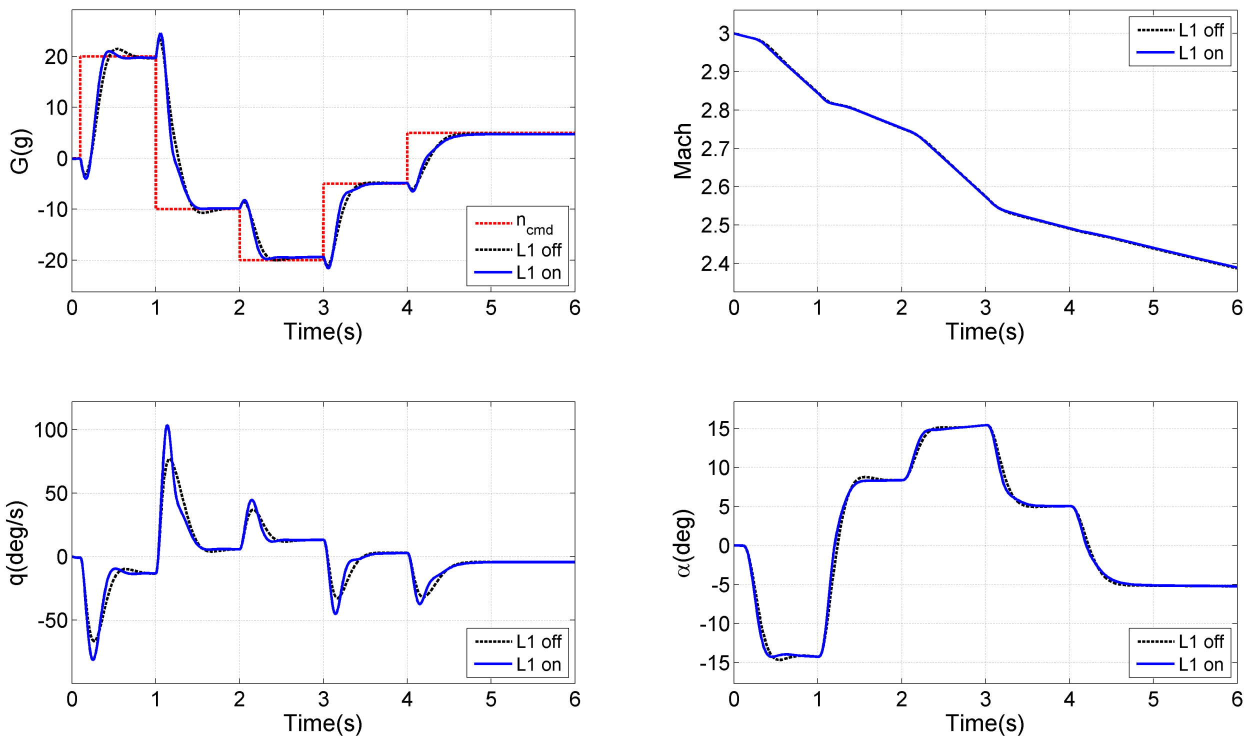

Figure 5.9: System responses for Δ_1

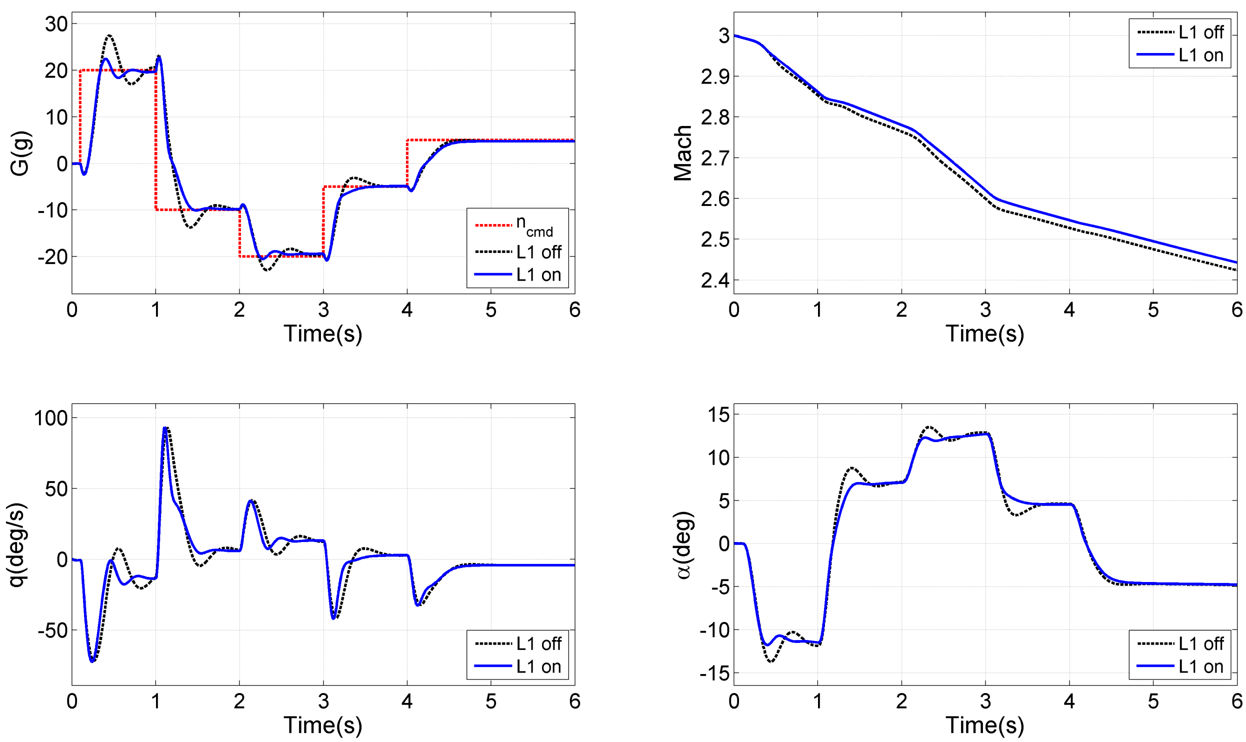

Figure 5.10: System responses for Δ_2

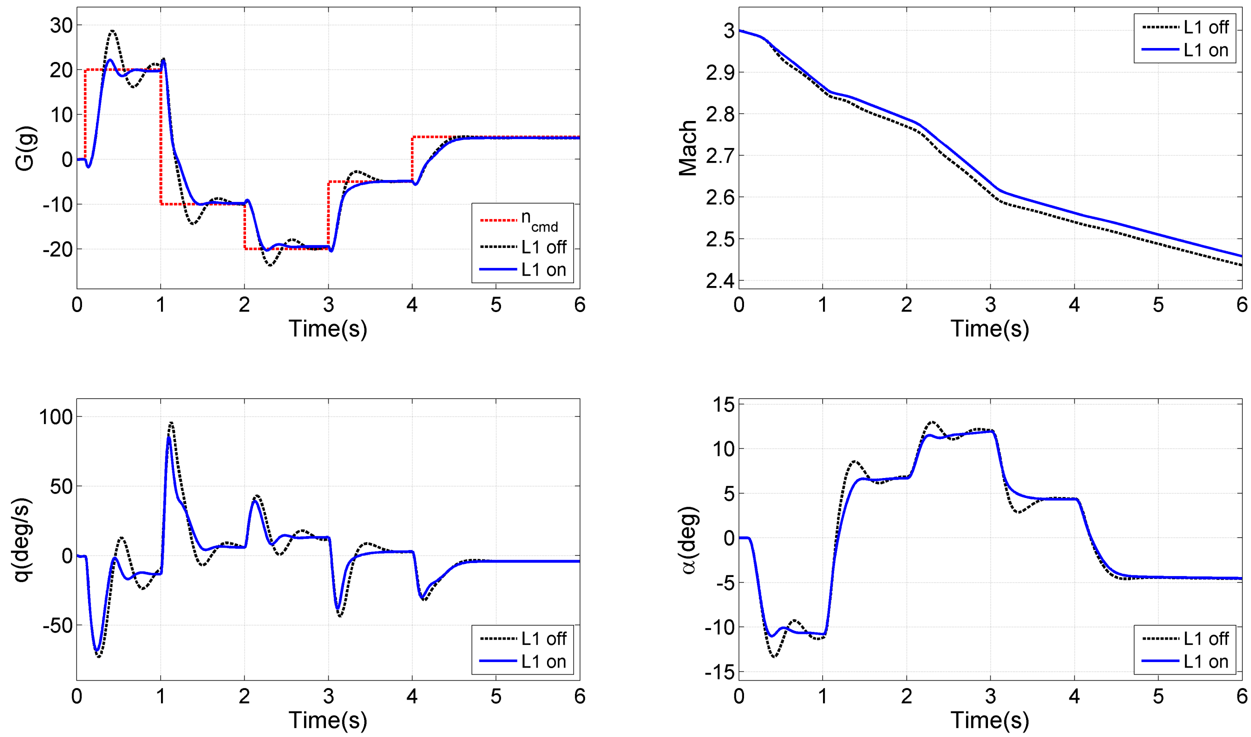

Figure 5.11: System responses for Δ_3

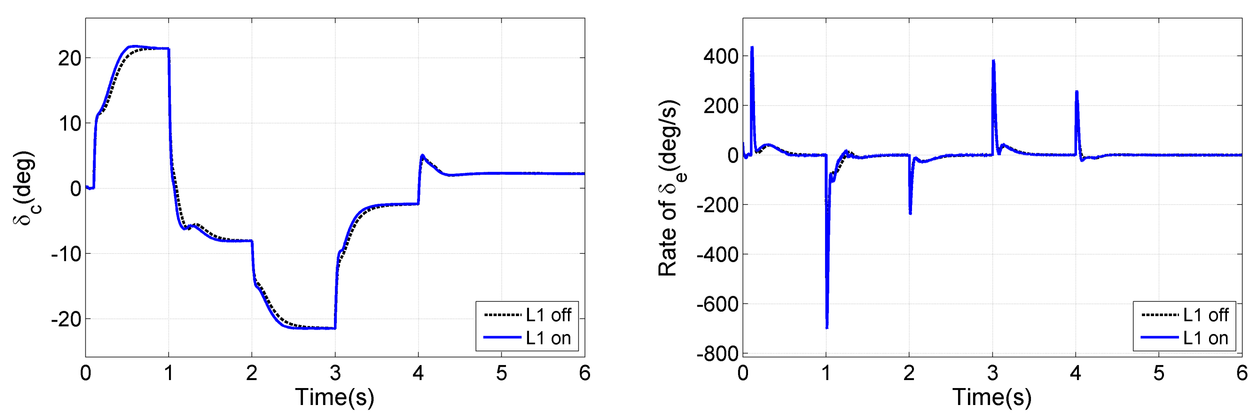

Figure 5.12: Control histories for Δ_0

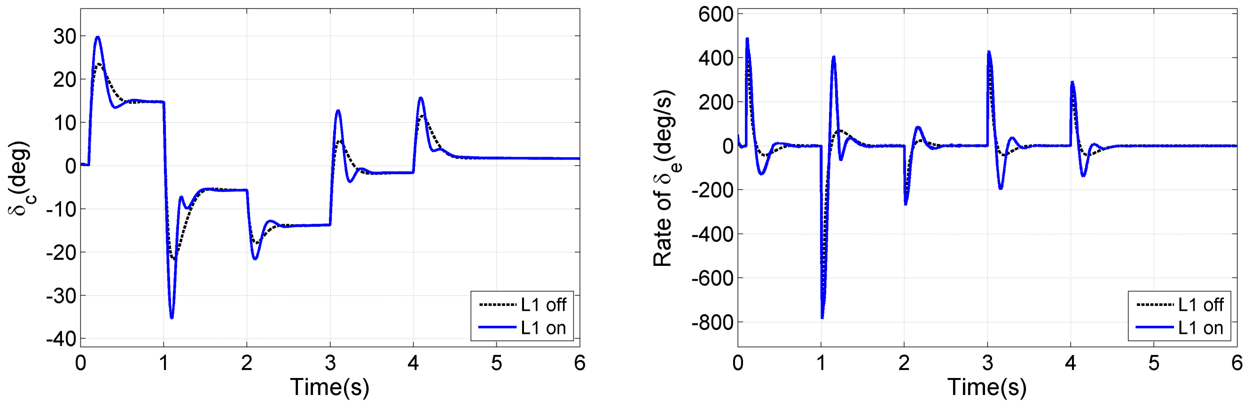

Figure 5.13: Control histories for Δ_1

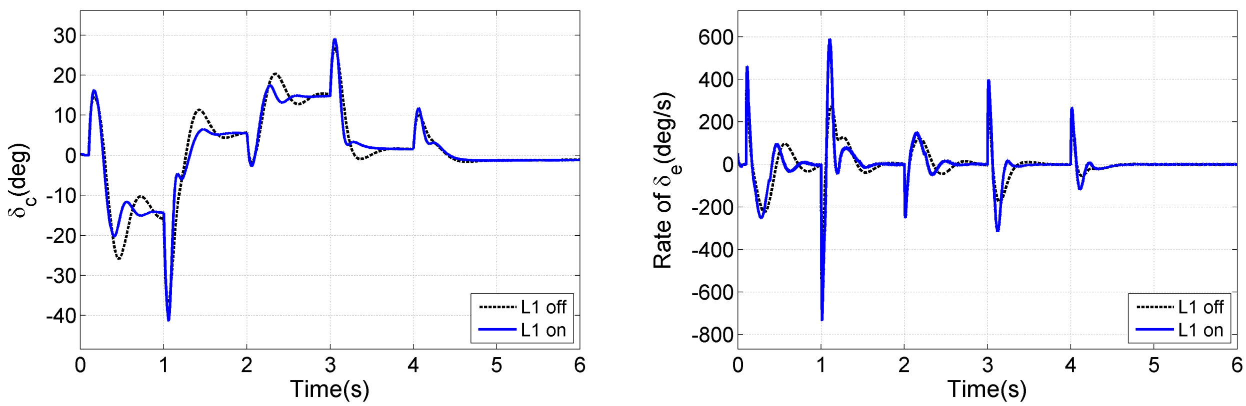

Figure 5.14: Control histories for Δ_2

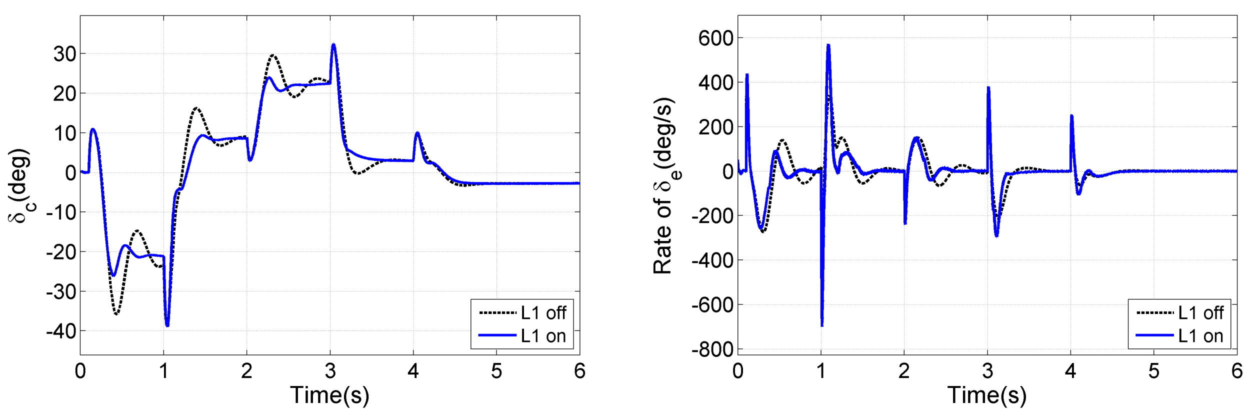

Figure 5.15: Control histories for Δ_3

5.2. Inverted Pendulum on a Cart

In this section, we consider the control problem of the inverted pendulum which is an underactuated system with relative degree 2. Notice that the linearized dynamics of the system has rank deficiency for the product of its input and output matrices. Therefore, we will consider the controller from Section 4.2. As shown in Figure 5.16, an input force $F(t)$ acts on the cart for

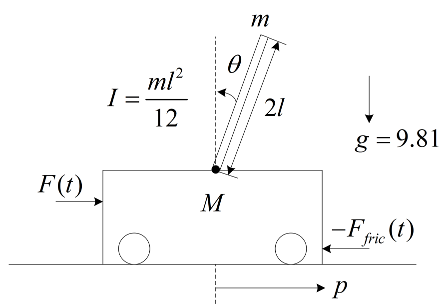

Figure 5.16: Inverted pendulum on a cart

the purpose of tracking a reference position p while maintaining the inverted pendulum balanced upright. The nonlinear model is given by

$$\begin{aligned} \frac{F(t)+F_{fric}(t)+d(t)}{M+m} &= \ddot{p}(t) + \frac{ml\cos\theta(t)\ddot{\theta}(t) - ml\sin\theta(t)\dot{\theta}^2(t)}{M+m}, \\ 0 &= ml\cos\theta\ddot{p}(t) - mgl\sin\theta(t) + (I+ml^2)\ddot{\theta}(t), \\ F(t) &= \omega u(t) - \nu\dot{p}(t), \end{aligned} \tag{5.2.1}$$

where $p(t) \in \mathbb{R}$, $\theta(t) \in \mathbb{R}$ are the cart position and pendulum angle (measurable outputs), respectively, $u(t)$ is the voltage input, and $F_{fric}(t)$ represents the nonlinear dynamic friction given by [112]

$$\begin{aligned} F_{fric}(t) =& -73\dot{p}(t) - 121z(t)\left(1 - 70\frac{\|\dot{p}(t)\|}{h(\dot{p}(t))}\right), \\ \dot{z}(t) =& \dot{p}(t) - 121\frac{\|\dot{p}(t)\|}{h(\dot{p}(t))}z(t), \end{aligned} \tag{5.2.2}$$

with $h(\dot{p}(t)) = -(0.04287 + 0.0432e^{-(\frac{\dot{p}(t)}{0.105})^2})(m+M)g$. Moreover, the definitions of system parameters are given in (5.1). The nominal system parameters are selected as follows [112]:

$$M_0 = 0.815, \quad m_0 = 0.210, \quad l_0 = 0.305, \quad \omega_0 = 1.719, \quad \nu_0 = 7.682, \tag{5.2.3}$$

and two sets of parameter variations are taken into account:

$$\begin{aligned} \mathcal{S}_1 =& \{M = 1.2M_0, m = 0.7m_0, l = 0.7l_0, \omega = 0.8\omega_0, \nu = 0.7\nu_0\}, \\ \mathcal{S}_2 =& \{M = 1.2M_0, m = 0.8m_0, l = 0.8l_0, \omega = 0.8\omega_0, \nu = 0.5\nu_0\}. \end{aligned} \tag{5.2.4}$$

Table 5.1: Definitions of parameters

Parameter	Description	Unit
M	mass of the cart	kg
m	mass of the pendulum	kg
l	length of the pendulum	m
ω	voltage to force conversion factor	N/V
ν	electrical resistance factor	Ns/m
I	moment of inertia of the pendulum	kgm^2

The control objective is to make the closed-loop system track a given position command by using both $p(t)$ and $\theta(t)$. For the purposes of comparison, we first consider a standard LQR controller [60, 112]. The gain can be computed from the linearizion of the nonlinear model (5.2.1) at $(p_e, \theta_e) = (0, 0)$, together with $\cos\theta(t) \approx 1$:

$$K_{lqr} = [-7.0711, -13.5752, -42.5823, -7.6058]$$

with the weighting matrics $Q_{lqr} = \operatorname{diag}(50, 10, 150, 5)$ and $R_{lqr} = 1$. As for the proposed $\mathcal{L}_1$ controller given in Section 4.2, the desired model is chosen identical to the nominal (linearized) closed-loop system obtained by the LQR controller:

$$A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 14.62 & 20.64 & 88.23 & 15.87 \\ 0 & 0 & 0 & 1 \\ -44.26 & -62.47 & -237.34 & -48.04 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 \\ 2.07 \\ 0 \\ -6.26 \end{bmatrix},$$

$$C_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

with the state vector $x(t) = [p(t), \dot{p}(t), \theta(t), \dot{\theta}(t)]^\top$, and the reference position command $r(t)$.

Remark 5.2.1. *The nominal (linearized) open-loop transfer function from $u(t)$ to $p(t)$ has an unstable zero. However, the transfer matrix from u to $[p, \theta]^\top$ does not possess unstable transmission zeros, which guarantees that $M(s)$ has no unstable transmission zeros.*

Since the desired model is obtained from the linearizion, the uncertain function $f(x, t)$ in (4.2.1) includes the linearizion errors, parameter variations, nonlinear friction $F_{fric}(t)$, and disturbance signal $d(t)$. The set of parameters for the $\mathcal{L}_1$ adaptive controller is given by

$$K_g = -7.0711, \quad Z(s) = \frac{-9.323}{s + 17}, \quad D(s) = \frac{30}{s(s/70 + 1)(s/100 + 1)}, \quad \mu = 1, \quad T_s = 0.005,$$

and

$$K_v = \begin{bmatrix} -5.16 & -30.92 & -1.42 & 62.34 \\ -1.42 & -36.03 & -3.79 & 76.71 \end{bmatrix}^\top .$$

In simulation, we considers two cases: (1) the nominal nonlinear dynamics with system parameters in (5.2.3), $F_{fric}(t) \equiv 0$, $d(t) \equiv 0$, and zero initialization errors; (2) the nonlinear dynamics with parametric variations in (5.2.4), the nonlinear friction given by (5.2.2), the input disturbance $d(t) = 3sin(t)$, and non-zero initial condition $x_0 = [-0.5, -1, 0.1745, 0]^\top$. Figures 5.17 and 5.18 illustrates the simulation results for the first case. From the plots it can be noted that there is no significant difference in the performance of the solutions; this is not surprising, since the only uncertainties that affect the performance of the controllers are the linearization errors. Figures 5.19 and 5.20 present the system responses and control inputs for the second case. As shown in Figure 5.19, the $\mathcal{L}_1$ controller ensures close tracking of the position, and boundedness of the angle within a neighborhood of zero, in spite of the uncertainties and non-zero initial error.

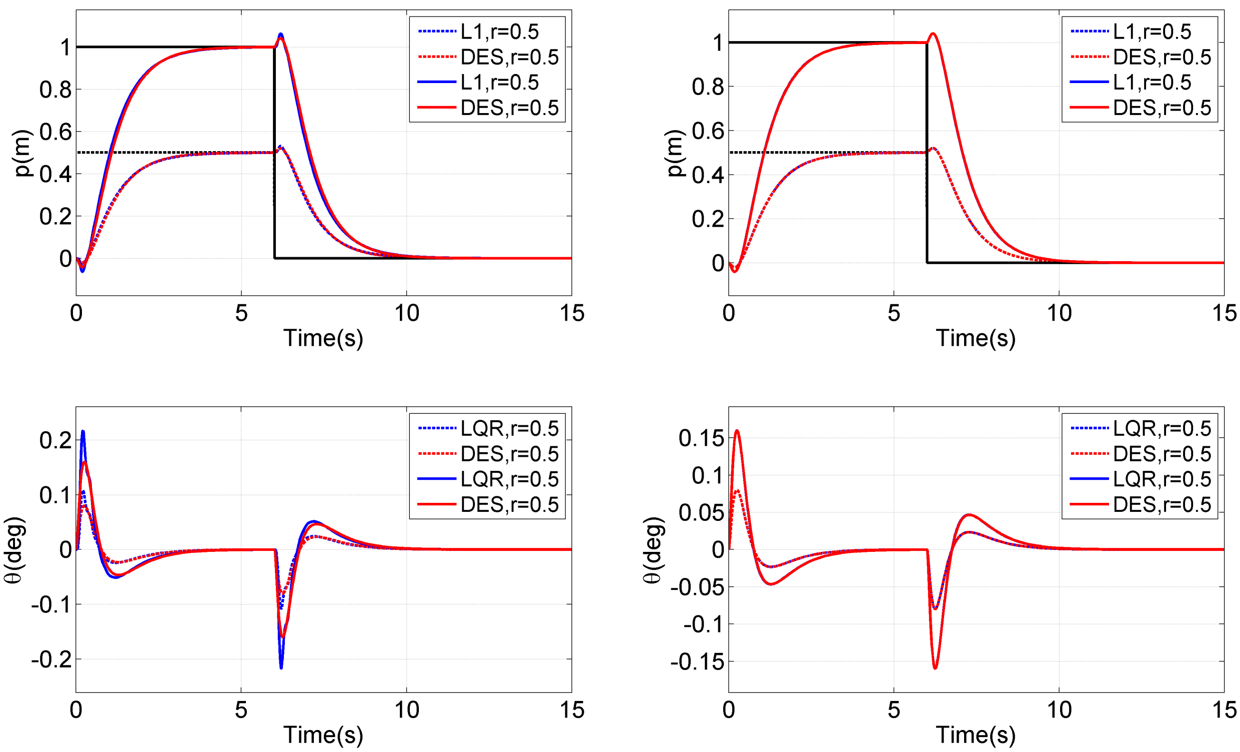

Figure 5.17: Inverted pendulum: position, and angle for case 1

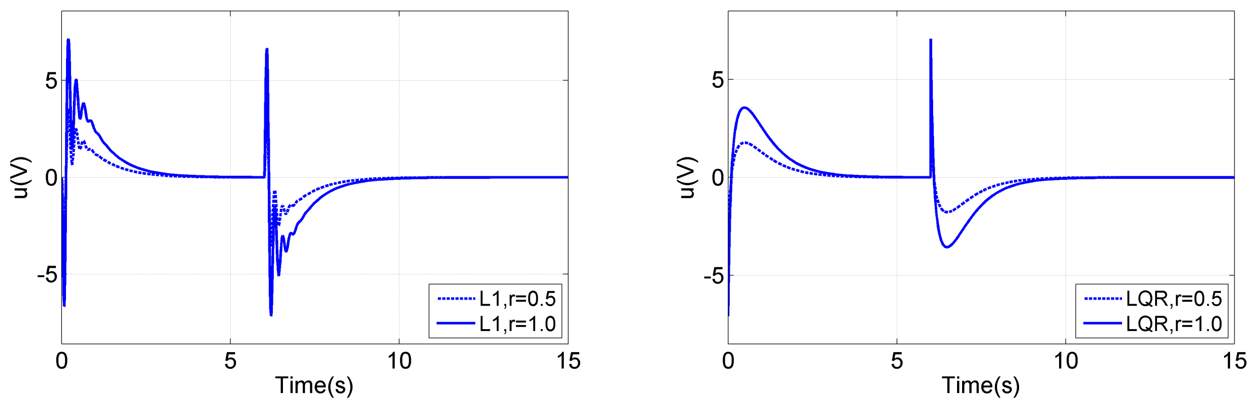

Figure 5.18: Inverted pendulum: control input for case 1

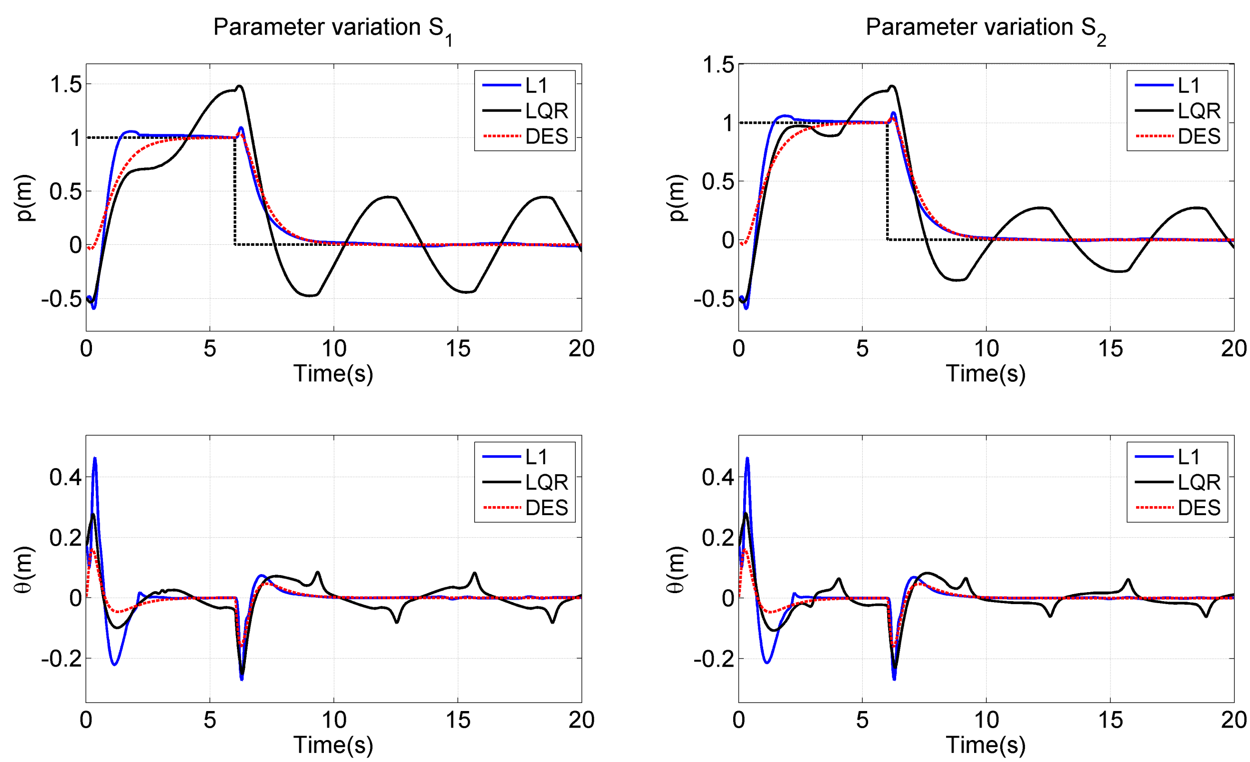

Figure 5.19: Inverted pendulum: position, and angle for case 2

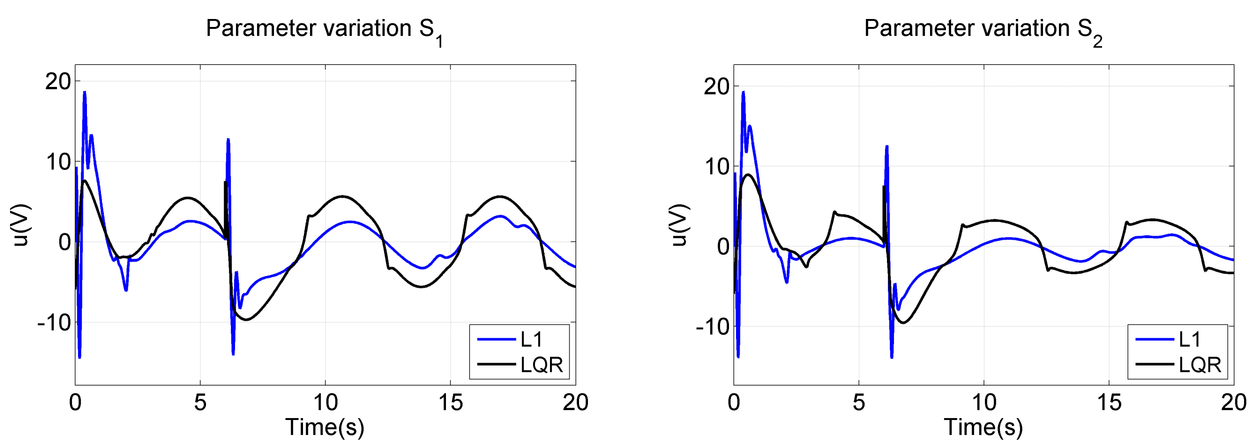

Figure 5.20: Inverted pendulum: control input for case 3

CHAPTER. 6

Design of the Lowpass filter for $\mathcal{L}_1$ Reference System Performance Optimization

In the previous chapters, we demonstrated that the $\mathcal{L}_1$ reference system is $\mathcal{L}_\infty$ stable with respect to external signals, if a low- pass filter is designed to satisfy the $\mathcal{L}_1$-norm stability condition. We also proved that the closed-loop $\mathcal{L}_1$ adaptive system converges to the reference system in the presence of fast estimation rates. For optimization of the reference system performance, one needs a systematic design procedure for the low-pass filter towards achieving satisfactory trade-off between robustness and performance of the closed-loop system. The filter design problem for MIMO systems is overly challenging as compared to SISO systems.

Since the stability condition is formulated with an $\mathcal{L}_1$ norm constraint, the problem is reduced to $\mathcal{L}_1$-norm optimization problem. In robust control literature, the continuous-time $\mathcal{L}_1$ optimal controllers (minimizing a worst-case norm) are known to be in the form of irrational controllers. The authors of [113] proposed optimizing a star-norm (which is an upper bound of the $\mathcal{L}_1$ norm) to solve the filter design problem. In [114], a D-K iteration procedure for $\mathcal{L}_1$-norm optimization was suggested. However, these methods seem to be rather conservative for an optimal solution; the approach in [113] ignores a performance measure and only takes a fixed time-delay into account in the design process; the D-K iteration method may not be a suitable approach for continuous-time $\mathcal{L}_1$-norm minimizing problems, since it produces an irrational filter [91]. Recent progress in this direction is based on discretization of the continuous-time system. The optimal solution for the filter is obtained in the discrete-time domain, and then the discrete optimal filter is converted into its continuous-time version [115]. This method is more tractable since it gives a rational approximation of the optimal solution; the authors of [115] borrowed the idea of Euler approximation [116] which is a suitable approximation, guaranteeing the closeness to the continuous-time optimal solution with a small sampling time. In this approach, both robustness and performance of the reference system are taken into account. However, the use of small time-steps in the discretization inevitably results in undesirable high-order filters. As a result, an additional order reduction step is required; notice that the reduction should be performed with an $\mathcal{L}_1$ optimization setup. This is known to be a challenging problem because the optimal reduction may result in an irrational filter.

In real-world applications, performance specifications are often given in the frequency domain, which necessarily require obtaining an optimal solution within an $\mathcal{H}_\infty$ optimization framework. Frequency-domain approaches are well established in control engineering, allowing the designers to utilize commercial off-the-shelf optimization tools. Many efficient numerical solvers are found for the $\mathcal{H}_\infty$ optimization with successful applications, providing optimal (or suboptimal) solutions to the problems that include structured uncertainties, uncertain time delays, and requirements of the control structure.

The key challenge in dealing with the frequency-domain specifications for the $\mathcal{L}_1$ reference system is a conservative design result especially with high order systems. This stems from converting the $\mathcal{L}_1$ norm condition to a corresponding constraint in the frequency domain. For example, the authors of [117] use the fact that the $\mathcal{L}_1$-norm is upper and lower bounded by the $\mathcal{H}_\infty$-norm (see Lemma 2.2.1):

$$\frac{1}{\sqrt{p}}\left\|G(s)\right\|_{\mathcal{H}_\infty} \leq \left\|G(s)\right\|_{\mathcal{L}_1} \leq 2n\sqrt{m}\left\|G(s)\right\|_{\mathcal{H}_\infty}, \tag{6.0.1}$$

where $G(s) \in \mathbb{R}^{p\times m}(s)$ and n is the dimension of the state vector of $G(s)$. In [117], the authors solve

$$\sqrt{m}L\left\|G(s)\right\|_{\mathcal{H}_\infty} < \frac{1}{2n} \tag{6.0.2}$$

to ensure $\left\|G(s)\right\|_{\mathcal{L}_1} L < 1$, where $L > 0$ is the Lipschitz constant of the unknown nonlinearity. However, as the system order increases, the solution becomes more conservative, since the norm inequality between the $\mathcal{L}_1$-norm and the $\mathcal{H}_\infty$-norm depends on the order of $G(s)$.

In this chapter, we avoid the conservative conversions given in (6.0.1) and (6.0.2) by proposing a filter design method with a new stability condition. The condition is formulated in the frequency domain for stability of both the $\mathcal{L}_1$ reference system and the closed-loop $\mathcal{L}_1$ adaptive system. A suitable parameterization of the low-pass filter makes the design problem solvable in a standard $\mathcal{H}_\infty$ optimization framework. Moreover, frequency-domain specifications are easily taken into the framework.

6.1. Stability Condition for $\mathcal{L}_1$ Adaptive Systems

In this section, we first develop a new sufficient condition for the stability of the $\mathcal{L}_1$ reference system, and then show that this condition also guarantees the closed-loop stability for $\mathcal{L}_1$ adaptive systems. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}(t) =& A_m x(t) + B_m(\Omega u(t) + f(x,t)),\\ y(t) =& C_m x(t), \quad x(0) = x_0,\end{aligned} \tag{6.1.1}$$

where $\{A_m \in \mathbb{R}^{n\times n}, B_m \in \mathbb{R}^{n\times m}, C_m \in \mathbb{R}^{p\times n}\}$ is a controllable-observable triple representing the desired model $M(s) = C_m(s\mathbb{I}_n - A_m)^{-1}B_m$, $\Omega \in \mathbb{R}^{m\times m}$ is an unknown input gain, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ represents system uncertainties. Now, suppose the unknown input gain and the nonlinear function $f(x,t)$ satisfy the following assumptions:

Assumption 6.1.1. *There exist $L > 0$ and $B_0 > 0$ such that*

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\|, \quad \|f(0,t)\| < b_0, \quad x, y \in \mathbb{R}^n, \quad \forall t \geq 0. \tag{6.1.2}$$

Assumption 6.1.2. *The unknown constant input gain Ω is assumed to be a (nonsingular) strictly row-diagonally dominant matrix with $sgn(\Omega_{ii})$ known. Moreover, the input gain satisfies $\Omega \in \mathcal{C}_\Omega$, where $\mathcal{C}_\Omega \subseteq \mathbb{R}^{m\times m}$ is a known convex compact set.*

Remark 6.1.1. *For the sake of simplicity, we assume a globally Lipschitz condition in Assumption 6.1.1. For locally Lipschitz continuous functions, $\mathbb{R}^n$ can be replaced with $\mathcal{D}_x = \{x \in \mathbb{R}^n : \|x\| < \rho_x\}$, where ρ_x denotes the size of a positive invariant set.*

We start by analyzing the $\mathcal{L}_1$ reference system stability. Let $Z^{-1}(s)$ be a right interactor of $sM(s)$ and $D(s) \in \mathbb{R}^{m\times m}[s]$ be a proper transfer matrix such that

$$C(s) = \Omega D(s)(\mathbb{I}_m + \Omega D(s))^{-1} \tag{6.1.3}$$

is a low pass filer with $C(0) = \mathbb{I}_m$. Moreover, it is assumed that $D(s)Z^{-1}(s)$ is a proper transfer matrix. Now, consider the following reference system:

$$\begin{aligned} \dot{x}_{ref}(t) =& A_m x_{ref}(t) + B_m(\Omega u_{ref}(t) + f(x_{ref}, t)), \\ y_{ref}(t) =& C_m x_{ref}(t), \quad x_{ref}(0) = 0, \end{aligned} \tag{6.1.4}$$

and

$$u_{ref}(s) =\Omega^{-1} C(s) \left(K_g r(s) - \eta_{ref}(s) - \sigma(s)\right), \tag{6.1.5}$$

where $C(s)$ is given in (6.1.3), $\eta_{ref}(s) = \mathscr{L}\left(\eta_{ref}\right)$, and $\sigma(s) = \mathscr{L}\left(\sigma\right)$ with

$$\eta_{ref}(t) = f(x_{ref}, t) - f(0, t), \quad \sigma(t) = f(0, t). \tag{6.1.6}$$

Lemma 6.1.1. *Consider the closed-loop reference system given in* (6.1.4) *and* (6.1.5)*. Suppose that the lowpass filter satisfies*

$$\|G(s)\|_{\mathcal{L}_1} L < 1, \tag{6.1.7}$$

where L is given in Assumption 6.1.1, $G(s) = H_0(s)(\mathbb{I}_m - C(s))$, and $H_0(s) = (s\mathbb{I}_n - A_m)^{-1}B_m$. Then, the closed-loop reference system is practically ISS with respect to the reference command $r(t)$[7]. Moreover, there exists $\lambda_0 > 0$, $\kappa_0 > 0$, $\kappa_1 > 0$, and $\kappa_2 > 0$ such that for all $t \geq t_0 \geq 0$.

$$\|x_{ref}(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_0)}\|x_{ref}(t_0)\| + \kappa_1\|r_f\|_{\mathcal{L}_\infty[t_0,t]} + \kappa_2\|\sigma\|_{\mathcal{L}_\infty[t_0,t]}, \tag{6.1.8}$$

with $r_f(s) = C(s)K_g r(s)$.

Proof. Consider the closed-loop reference system defined in (6.1.4) and (6.1.5). Let $r_f(s) = C(s)K_g r(s)$. Substituting the control law given in (6.1.5) into (6.1.4) yields

$$\begin{aligned} x_{ref}(s) =& H_0(s) r_f(s) + G(s)\left(\eta_{ref}(s) + \sigma(s)\right) + x_{in}(s), \\ y_{ref}(s) =& C_m x_{ref}(s), \end{aligned}$$

[7] See Definition 2.2.15.

where $H_0(s) = (s\mathbb{I}_n - A_m)^{-1}B_m$, $G(s) = H_0(s)(\mathbb{I}_m - C(s))$, and $x_{in}(s) = (s\mathbb{I}_n - A_m)^{-1}x_{ref}(t_0)$. Define $C(s) = C_f(s\mathbb{I}_m - A_f)^{-1}B_f$, where $\{A_f \in \mathbb{R}^{n_f\times n_f}, B_f \in \mathbb{R}^{n_f\times m}, C_f \in \mathbb{R}^{m\times m}\}$ is a minimal realization of $C(s)$. Then, a state-space realization of (6.1.9) is given by

$$\begin{aligned}\dot{x}_c(t) &= A_c x_c(t) + B_c(\eta_{ref}(t) + \sigma(t)) + B_r r_f(t),\\ x_{ref}(t) &= C_c x_c(t), \quad x_c(t_0) = [x_{ref}(t_0)^\top, 0]^\top,\end{aligned} \tag{6.1.9}$$

with

$$A_c = \begin{bmatrix} A_m & B_m C_f \\ 0 & A_f \end{bmatrix}, \quad B_c = \begin{bmatrix} B_m \\ -B_f \end{bmatrix}, \quad B_r = \begin{bmatrix} B_m \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} \mathbb{I}_n & 0 \end{bmatrix}, \tag{6.1.10}$$

where $x_c(t) = [x_{ref}^\top(t), x_f^\top(t)]^\top \in \mathbb{R}^{n_c\times n_c}$;$n_c = n + n_f$, and $x_f(t) \in \mathbb{R}^{n_f}$ is some internal state vector in (6.1.9). Therefore, the following solution for a given initial condition $x_c(t_0)$ can be obtained:

$$x_c(t) = e^{A_c(t-t_0)}x_c(t_0) + \int_{t_0}^{t} e^{A_c(t-\tau)}B_r r_f(\tau)d\tau + \int_{t_0}^{t} e^{A_c(t-\tau)}B_c(\eta_{ref}(\tau) + \sigma(\tau))d\tau. \tag{6.1.11}$$

Using the continuity of the $\mathcal{L}_1$-norm, one may take a sufficiently small $\lambda_0 > 0$ such that $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1} < 1/L$. Define $A_{\lambda_0} = A_c + \lambda_0\mathbb{I}_{n_c}$, $\bar{x}_c(t) = e^{\lambda_0(t-t_0)}x_c(t)$, $\bar{r}_f(t) = e^{\lambda_0(t-t_0)}r_f(t)$, $\bar{\eta}_{ref}(t) = e^{\lambda_0(t-t_0)}\eta_{ref}(t)$, and $\bar{\sigma}(t) = e^{\lambda_0(t-t_0)}\sigma(t)$. Multiplying both sides of (6.1.11) by $e^{\lambda_0(t-t_0)}C_c$, yields

$$\begin{aligned}\bar{x}_{ref}(t) =& C_c e^{A_{\lambda_0}(t-t_0)}x_c(t_0) + \int_{t_0}^{t} C_c e^{A_{\lambda_0}(t-\tau)}B_r\bar{r}_f(\tau)d\tau\\ &+ \int_{t_0}^{t} C_c e^{A_{\lambda_0}(t-\tau)}B_c(\bar{\eta}_{ref}(\tau) + \bar{\sigma}(\tau))d\tau,\end{aligned} \tag{6.1.12}$$

where $\bar{x}_{ref}(t) = e^{\lambda_0(t-t_0)}x_{ref}(t)$. Notice that A_{λ_0} is Hurwitz by $\|G(s-\lambda_0)\|_{\mathcal{L}_1} < \infty$. Moreover, Assumption 6.1.1, along with (6.1.7) and (6.1.6), implies

$$\|\bar{\eta}_{ref}(t)\| \leq L\|\bar{x}_{ref}(t)\|, \quad \forall t \geq t_0. \tag{6.1.13}$$

By combining (6.1.12) and (6.1.13) and using the fact that $C_c = [\mathbb{I}_{n_c}, 0]$, it follows that for all $t \geq t_0$

$$\|\bar{x}_{ref}\|_{\mathcal{L}_\infty[t_0,t]} \leq \frac{\beta_0}{1-\beta_1 L}\|x_c(t_0)\| + \frac{\beta_1}{1-\beta_1 L}\|\bar{\sigma}\|_{\mathcal{L}_\infty[t_0,t]} + \frac{\beta_2}{1-\beta_1 L}\|\bar{r}_f\|_{\mathcal{L}_\infty[t_0,t]}, \tag{6.1.14}$$

where $\beta_0 = \sup_{0\leq\tau}\|e^{A_{\lambda_0}\tau}\|$, $\beta_1 = \|G(s-\lambda_0)\|_{\mathcal{L}_1}$, and $\beta_2 = \left\|(s\mathbb{I}_{n_c} - A_{\lambda_0})^{-1}B_r\right\|_{\mathcal{L}_1}$. Now, multiplying both sides of (6.1.11) by $e^{\lambda_0(t-t_0)}$, and combining (6.1.13) and (6.1.14), one can obtain

$$\|\bar{x}_c(t)\| \leq \kappa_0\|x_c(t_0)\| + \kappa_1\|\bar{r}_f\|_{\mathcal{L}_\infty[t_0,t]} + \kappa_2\|\bar{\sigma}\|_{\mathcal{L}_\infty[t_0,t]}, \tag{6.1.15}$$

where

$$\begin{aligned}\kappa_0 =&\beta_0(1+\frac{L\beta_3}{1-\beta_1 L}), \quad \kappa_1 = \beta_2(1+\frac{L\beta_3}{1-\beta_1 L}),\\ \kappa_2 =&\frac{\beta_3}{1-\beta_1 L}, \quad \beta_3 = \left\|(s\mathbb{I}_{n_c}-A_{\lambda_0})^{-1}B_c\right\|_{\mathcal{L}_1}.\end{aligned} \tag{6.1.16}$$

Since

$$e^{-\epsilon_0(t-t_0)}\|\bar{\sigma}\|_{\mathcal{L}_\infty[t_0,t]} \le \|\sigma\|_{\mathcal{L}_\infty[t_0,t]}, \quad e^{-\epsilon_0(t-t_0)}\|\bar{r}_f\|_{\mathcal{L}_\infty[t_0,t]} \le \|r_f\|_{\mathcal{L}_\infty[t_0,t]},$$

and $\|\bar{x}_c(t)\| = e^{\lambda_0(t-t_0)}\|x_c(t)\|$ hold, Equation (6.1.15) can be rewritten by

$$\|x_c(t)\| \le \kappa_0 e^{-\lambda_0(t-t_0)}\|x_c(t_0)\| + \kappa_1\|r_f\|_{\mathcal{L}_\infty[t_0,t]} + \kappa_2\|\sigma\|_{\mathcal{L}_\infty[t_0,t]}, \quad \forall t \ge t_0. \tag{6.1.17}$$

Since $r_f(s) = C(s)K_g r(s)$ holds, and $\sigma(t)$ is bounded, Equation (6.1.17) implies that the closed-loop reference system is practically ISS. Finally, the fact that $\|x_c(t)\| \ge \|x_{ref}(t)\|$ concludes (6.1.8), which completes the proof. □

Lemma 6.1.1 provides a further result on the $\mathcal{L}_1$ reference system stability; notice that Lemma 6.1.1 indicates the ISS property of the reference system, while the $\mathcal{L}_1$ literature mainly demonstrates the BIBO stability [2].

Remark 6.1.2. *It can be shown that if the nonlinear function f is locally Lipschitz continuous, then Equation* (6.1.8) *still holds depending on the upper bound of initial conditions. Notice that $C(s)$ can always be chosen to satisfy* (6.1.7) *by increasing the filter bandwidth. Therefore, the closed-loop reference system becomes semi-globally practically ISS with respect to $r(t)$ in this case.*

Next, we introduce a new sufficient condition for the ISS $\mathcal{L}_1$ reference system.

Lemma 6.1.2. *Consider the closed-loop reference system given in* (6.1.4) *and 6.1.5. Suppose that a low-pass filter $C(s)$ is chosen to satisfy*

$$\|G(s)\|_{\mathcal{H}_\infty}\sqrt{m}L < 1, \tag{6.1.18}$$

where L is given in Assumption 6.1.1, $G(s) = H_0(s)(\mathbb{I}_m - C(s))$ and $H_0(s) = (s\mathbb{I}_n - A_m)^{-1}B_m$. Then, the closed-loop system is practically ISS with respect to $r(t)$. Moreover, there exists $\bar{\lambda}_0 > 0$, $\bar{\kappa}_0 > 0$, $\bar{\kappa}_1 > 0$, and $\bar{\kappa}_2 > 0$ such that for all $t \ge t_0 \ge 0$

$$\|x_{ref}(t)\| \le \bar{\kappa}_0 e^{-\bar{\lambda}_0(t-t_0)}\|x_{ref}(t_0)\| + \bar{\kappa}_1\|r_f\|_{\mathcal{L}_\infty[t_0,t]} + \bar{\kappa}_2\|\sigma\|_{\mathcal{L}_\infty[t_0,t]}, \tag{6.1.19}$$

with $r_f(s) = C(s)K_g r(s)$.

Proof. Consider the reference system given in (6.1.4) and (6.1.5). Let $r_f(s) = C(s)K_g r(s)$. By applying a state-space realization, it follows that for $t \geq t_0$ the internal state of the system $x_c(t)$ satisfies

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c(\eta_{ref}(t) + \sigma(t)) + B_r r_f(t), \\ x_{ref}(t) &= C_c x_c(t), \quad x_c(t_0) = [x_{ref}(t_0)^\top, 0]^\top, \end{aligned} \tag{6.1.20}$$

where A_c, B_c, and C_c are given in (6.1.10). Since $G(s) = H_0(s)(\mathbb{I}_m - C(s)) = C_c(s\mathbb{I}_n - A_c)B_c$ holds with the observable and controllable $\{A_c, B_c, C_c\}$, A_c should be Hurwitz.

Suppose $\|G(s)\|_{\mathcal{H}_\infty} < \gamma$ with $\gamma = \frac{1}{\sqrt{m}L}$. From Lemma 2.2.2(a), there exists $P = P^\top \succ 0$ such that

$$A_c^\top P + PA_c + \gamma^2 PB_cB_c^\top P + C_c^\top C_c = -Q, \quad Q \succ 0. \tag{6.1.21}$$

Choose a small enough $\epsilon > 0$ such that $Q - \epsilon\mathbb{I}_{n_c} \succ 0$, and consider a Lyapunov function $V(t) = x_c(t)^\top P x_c(t)$. Taking the derivative of $V(t)$ and substituting (6.1.20) into the derivative, one has

$$\dot{V}(t) = x_c^\top(t)(A_c^\top P + PA_c)x_c(t) + 2x_c^\top PB_c\eta_{ref}(t) + 2x_c^\top PB_c\sigma_{ref}(t) + 2x_c^\top PB_r r_f(t). \tag{6.1.22}$$

Notice that from Assumption 6.1.1 one obtains

$$\eta_{ref}^\top(t)\eta_{ref}(t) \leq mL^2 x_{ref}^\top(t)x_{ref}(t), \quad t \geq t_0. \tag{6.1.23}$$

Since

$$2x_c^\top PB_c\eta_{ref}(t) \leq \gamma^2 x_c^\top(t)PB_cB_c^\top P^\top x_c(t) + \frac{1}{\gamma^2}\eta_{ref}^\top(t)\eta_{ref}(t),$$

from (6.1.22) it follows, along with (6.1.23), that

$$2x_c^\top PB_c\eta_{ref}(t) \leq \gamma^2 x_c^\top(t)PB_cB_c^\top Px_c(t) + x_c^\top(t)C_c^\top C_c x_c(t). \tag{6.1.24}$$

Now, combining (6.1.21), (6.1.22) and (6.1.24) brings

$$\dot{V}(t) \leq -x_c^\top(t)Qx_c(t) + 2x_c^\top PB_c\sigma_{ref}(t) + 2x_c^\top PB_r r_f(t). \tag{6.1.25}$$

Let $\epsilon_\sigma > 0$ and $\epsilon_f > 0$ satisfy $\epsilon_\sigma + \epsilon_f < \epsilon$, and define $Q_\epsilon = Q - (\epsilon_\sigma + \epsilon_f)\mathbb{I}_{n_c} \succ 0$. Then, by applying square completions in (6.1.25) it can be shown that

$$\dot{V}(t) \leq -x_c^\top(t)Q_\epsilon x_c(t) + \gamma_\sigma\|\sigma_{ref}(t)\|^2 + \gamma_f\|r_f(t)\|^2, \tag{6.1.26}$$

where $\gamma_\sigma = \frac{n}{\epsilon_\sigma}\|PB_c\|_2^2 > 0$ and $\gamma_f = \frac{n}{\epsilon_f}\|PB_r\|_2^2 > 0$. Moreover, since $x_c^\top(t)Q_\epsilon x_c(t) \geq \frac{\lambda_{\min}(Q_\epsilon)}{\lambda_{\max}(P)}V(t)$, Equation (6.1.26) can be rewritten as

$$\dot{V}(t) \leq -\lambda_0 V(t) + \gamma_\sigma\|\sigma_{ref}(t)\|^2 + \gamma_f\|r_f(t)\|^2, \tag{6.1.27}$$

where $\lambda_0 = \frac{\lambda_{\min}(Q_\epsilon)}{\lambda_{\max}(P)}$. The Comparison Lemma from (6.1.27) implies that

$$V(t) \leq e^{-\lambda_0(t-t_0)}V(t_0) + \int_{t_0}^{t} e^{-\lambda_0(t-\tau)}(\gamma_\sigma\|\sigma_{ref}(\tau)\|^2 + \gamma_f\|r_f(\tau)\|^2)d\tau,$$

which further yields

$$V(t) \leq e^{-\lambda_0(t-t_0)}V(t_0) + \gamma'_\sigma \sup_{\tau\in[t_0,t]} \|\sigma_{ref}(\tau)\|^2 + \gamma'_f \sup_{\tau\in[t_0,t]} \|r_f(\tau)\|^2, \tag{6.1.28}$$

where $\gamma'_\sigma = \frac{\gamma_\sigma}{\lambda_0}$ and $\gamma'_f = \frac{\gamma_f}{\lambda_0}$. Notice that

$$\sup_{\tau\in[t_0,t]} \|\sigma_{ref}(\tau)\|^2 \leq \left(\sup_{\tau\in[t_0,t]} \|\sigma_{ref}(\tau)\|\right)^2, \quad \sup_{\tau\in[t_0,t]} \|r_f(\tau)\|^2 \leq \left(\sup_{\tau\in[t_0,t]} \|r_f(\tau)\|\right)^2.$$

Therefore, from (6.1.28) it follows that

$$\|x_c(t)\| \leq \bar{\kappa}_0 e^{-\bar{\lambda}_0(t-t_0)}\|x_c(t_0)\| + \bar{\kappa}_1 \sup_{\tau\in[t_0,t]} \|\sigma_{ref}(\tau)\| + \bar{\kappa}_2 \sup_{\tau\in[t_0,t]} \|r_f(\tau)\|, \tag{6.1.29}$$

where $\bar{\lambda}_0 = \lambda_0/2$, $\bar{\kappa}_0 = \sqrt{n\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$, $\bar{\kappa}_1 = \sqrt{\frac{\gamma'_\sigma}{\lambda_{\min}(P)}}$, and $\bar{\kappa}_2 = \sqrt{\frac{\gamma'_f}{\lambda_{\min}(P)}}$ Finally, since $\|x_{ref}(t)\| \leq \|x_c(t)\|$ and $x_c(t_0) = [x_{ref}^\top(t_0), 0]^\top$ holds, from (6.1.29) one concludes (6.1.19). This completes the proof. □

Remark 6.1.3. *In Lemma 6.1.2, the condition* (6.1.18) *is formulated in the frequency domain. Notice that the stability condition is less conservative than the condition in* (6.0.2). *Moreover, the inequality in* (6.1.18) *does not depend on system order*[8].

Up to this point the reference system stability has been discussed. Now, we analyze the behavior of the closed-loop $\mathcal{L}_1$ adaptive system. Given the nonlinear system in (6.1.1), consider the $\mathcal{L}_1$ adaptive control laws:

$$u(s) = D(s)K_g r(s) - D(s)Z^{-1}(s)\hat{\eta}_t(s), \tag{6.1.30}$$

where $r(s)$ is the Laplace transform of a reference command $r(t) \in \mathbb{R}^{m_r}$, and $K_g \in \mathbb{R}^{m\times m_r}$ is a known feed-forward gain, and $\hat{\eta}_t(s)$ is the Laplace transform of $\hat{\eta}_t(t) \in \mathbb{R}^m$, which represents the signals generated by the predictor and the adaptive laws in the $\mathcal{L}_1$-adaptive control structure.

Remark 6.1.4. *Notice that the control law in* (6.1.30) *does not assume a specific structure for the predictor and the adaptive laws, and therefore Equation* (6.1.30) *can represent the control laws that we discussed in the previous chapter (see* (4.1.22)*, and* (4.2.13)*).*

Lemma 6.1.3. *Consider the nonlinear system* (6.1.5)*, subject to the following assumptions:*

[8]McMillan degree, the dimension of states in a minimal realization of the system.

(a) $C(s)$ *is chosen such that the closed-loop reference system satisfies* (6.1.8)*;*

(b) *the control structure* (6.1.30) *guarantees that there exist a class* $\mathcal{KL}_\infty$ *function* β_ϕ *and a constant* $\gamma > 0$*, such that*

$$\|\phi_t(t)\| \leq \beta_\phi(\|x(t_0)\|, t - t_0) + \gamma, \quad 0 \leq t_0 \leq t, \tag{6.1.31}$$

where $\phi_t(t)$ *is the signal with the Laplace transform of*

$$\phi_t(s) = C(s)Z^{-1}(s)(\hat{\eta}_t(s) - \eta_t(s)), \tag{6.1.32}$$

with $\eta_t(s) = Z(s)(\Omega u + \mathscr{L}(f))$.

Then, there exist a class $\mathcal{KL}_\infty$ *function* β_{dx} *and a class* $\mathcal{K}_\infty$ *class function* γ_{dx} *such that*

$$\|x_{ref}(t) - x(t)\| \leq \beta_{dx}(\|x_0\|, t - t_0) + \gamma_{dx}(\gamma). \tag{6.1.33}$$

Proof. Let $t_0 \geq 0$, and

$$\tilde{\eta}_t(t) = \hat{\eta}_t(t) - \eta_t(s). \tag{6.1.34}$$

Combining (6.1.34) and (6.1.30) yields

$$u(s) = D(s)K_g r(s) + D(s)Z^{-1}(s)(\eta_t(s) + \tilde{\eta}_t(t)),$$

which, together with the fact that $\mathscr{L}(f) = \mathscr{L}(\eta) + \mathscr{L}(\sigma)$, leads to

$$u(s) = \Omega^{-1}C(s)\left(K_g r(s) - \eta(s) - \sigma(s)\right) - \Omega^{-1}C(s)Z^{-1}(s)\tilde{\eta}_t(s), \tag{6.1.35}$$

where $\eta(s) = \mathscr{L}(\eta)$, and $\sigma(s) = \mathscr{L}(\sigma)$ with

$$\eta(t) = f(x, t) - f(0, t), \quad \sigma(t) = f(0, t),$$

Substituting (6.1.35) into (6.1.1) yields the following closed-loop system:

$$\begin{aligned} x(s) =& H_0(s)r_f(s) + G(s)(\eta(s) + \sigma(s)) + x_{in}(s) - H_0(s)C(s)\tilde{\eta}_t(s) \\ y(s) =& C_m x(s)\,, \end{aligned} \tag{6.1.36}$$

where $x_{in}(s) = (s\mathbb{I}_n - A_m)^{-1}x_0$, $r_f(s) = C(s)K_g r(s)$, $G(s) = H_0(s)(\mathbb{I}_m - C(s))$, and $H_0(s) = (s\mathbb{I}_n - A_m)^{-1}B_m$. Let $\tilde{x}_{ref}(t) = x_{ref}(t) - x(t)$, $\tilde{\eta}_{ref}(t) = \eta_{ref}(t) - \eta(t)$, and $\tilde{y}_{ref}(t) = y_{ref}(t) - y(t)$. Now, by subtracting (6.1.36) from (6.1.9), it follows that

$$\begin{aligned} \tilde{x}_{ref}(s) =& G(s)\tilde{\eta}_{ref}(s) + H_0(s)\phi_t(s) + x_{in}(s), \\ \tilde{y}_{ref}(s) =& C_m \tilde{x}_{ref}(s), \end{aligned} \tag{6.1.37}$$

and

$$\|\tilde{\eta}_{ref}(t)\| \leq L\|x_{ref}(t)\|, \quad t \geq 0, \tag{6.1.38}$$

where $\phi_t(s) = C(s)Z^{-1}(s)\tilde{\eta}_t(s)$, and $x_{in}(s) = (s\mathbb{I}_n - A_m)^{-1}\tilde{x}_{ref}(0)$ with $\tilde{x}_{ref}(0) = -x_0$. Notice that the system (6.1.37) with (6.1.38) has the same structure as in (6.1.9), and therefore from (6.1.8) it follows that

$$\|\tilde{x}_{ref}(t)\| \leq \kappa_0 e^{-\lambda_0(t-t_m)}\|\tilde{x}_{ref}(t_m)\| + \kappa_1\|\phi_t\|_{\mathcal{L}_\infty[t_m,t]}, \quad t_m \leq t, \tag{6.1.39}$$

for some $\kappa_0 > 0$, $\lambda_0 > 0$, and $\kappa_1 > 0$. Suppose $t_m \geq t_0$. From (6.1.31) it follows that

$$\begin{aligned}\|\phi_t\|_{\mathcal{L}_\infty[t_0,t_m]} &= \sup_{t_0\leq\tau\leq t_m}\|\phi_t(\tau)\| \\ &\leq \sup_{t_0\leq\tau\leq t_m}(\beta_\phi(\|\tilde{x}_{ref}(t_0)\|, \tau - t_0)) + \gamma \leq \beta_\phi(\|\tilde{x}_{ref}(t_0)\|, 0) + \gamma,\end{aligned} \tag{6.1.40}$$

and

$$\|\phi_t\|_{\mathcal{L}_\infty[t_m,t]} \leq \beta_\phi(\|\tilde{x}_{ref}(t_m)\|, 0) + \gamma. \tag{6.1.41}$$

Next, combining (6.1.39) and (6.1.40) yields

$$\begin{aligned}\|\tilde{x}_{ref}(t)\| \leq &\kappa_0 e^{-\lambda_0(t-t_0)}\|\tilde{x}_{ref}(t_0)\| + \kappa_0\kappa_1 e^{-\lambda_0(t-t_m)}\beta_\phi(\|\tilde{x}_{ref}(t_0)\|, 0) + \kappa_0\kappa_1\gamma \\ &+ \kappa_1\|\phi_t\|_{\mathcal{L}_\infty[t_m,t]},\end{aligned} \tag{6.1.42}$$

which, together with (6.1.41), leads to

$$\begin{aligned}\|\tilde{x}_{ref}(t)\| \leq &\kappa_0 e^{-\lambda_0(t-t_0)}\|\tilde{x}_{ref}(t_0)\| + \kappa_0\kappa_1 e^{-\lambda_0(t-t_m)}\beta_\phi(\|\tilde{x}_{ref}(t_0)\|, 0) \\ &+ (\kappa_0 + 1)\kappa_1\gamma + \kappa_1\beta_\phi(\|\tilde{x}_{ref}(t_m)\|, 0).\end{aligned} \tag{6.1.43}$$

Since

$$\beta_\phi(\|\tilde{x}_{ref}(t_m)\|, 0) \leq \beta_\phi(\|\kappa_0 e^{-\lambda_0(t_m-t_0)}\|\tilde{x}_{ref}(t_0)\| + \kappa_1\beta_\phi(\|\tilde{x}_{ref}(t_0)\|, 0) + \kappa_1\gamma\|, 0),$$

by letting $t_m = (t+t_0)/2$, one has a class $\mathcal{KL}_\infty$ function β_m and a class $\mathcal{K}_\infty$ function γ_m, such that

$$\beta_\phi(\|\tilde{x}_{ref}(t_m)\|, 0) \leq \beta_m(\|\tilde{x}_{ref}(t_0)\|, t - t_0) + \gamma_m(\gamma). \tag{6.1.44}$$

Finally, combining (6.1.43) and (6.1.44), it can be shown that

$$\|\tilde{x}_{ref}(t)\| \leq \beta_{dx}(\|\tilde{x}_{ref}(t_0)\|, t - t_0) + \gamma_{dx}(\gamma), \quad 0 \leq t_0 \leq t,$$

where β_{dx} and γ_{dx} are some class $\mathcal{KL}_\infty$ and $\mathcal{K}_\infty$ functions, respectively. Therefore, letting $\tilde{x}_{ref}(t_0) = -x_0$ concludes (6.1.33), which completes the proof. □

Remark 6.1.5. *Equation* (6.1.33) *implies that the transient due to non-zero initialization is quantified with a strictly decreasing function, and that the steady-state errors remain bounded; notice* *that the errors are bounded by $\gamma_{dx}(\cdot)$ that converges to zero as $\gamma \to 0$. The small γ is achieved with high adaptation gains.*

The hypothesis (a) of Lemma 6.1.3 holds if a low-pass filter $C(s)$ is chosen to satisfy either (6.1.7) or (6.1.18) (see Lemmas 6.1.1 and 6.1.2). Moreover, it can be easily shown that the condition (b) of Lemma 6.1.3 is always guaranteed by the bounded estimation errors of the $\mathcal{L}_1$ controllers of this thesis. These observations allow us to state the following theorem.

Theorem 6.1.1. *Consider the nonlinear system given in* (6.1.1)*, satisfying Assumptions 6.1.1 and 6.1.2. Suppose*

(a) the low pass filter $C(s)$ is designed to ensure either $\|G(s)\|_{\mathcal{L}_1} L < 1$ or $\|G(s)\|_{\mathcal{H}_\infty} \sqrt{m}L < 1$, where $G(s) = (s\mathbb{I}_n - A_m)^{-1}B_m(\mathbb{I}_m - C(s))$;

(b) The controller in (6.1.30) *is implemented with the estimation laws that satisfy* (6.1.31)*.*

Then, the closed-loop system is practically ISS with respect to the reference command $r(t)$.

Proof. Notice that

$$\|x(t)\| \leq \|x_{ref}(t)\| + \|x(t) - x_{ref}(t)\|, \quad \forall t \geq 0. \tag{6.1.45}$$

Therefore, the proof of Theorem 6.1.1 directly follows from Lemmas 6.1.1 - 6.1.3. □

Remark 6.1.6. *It is easy to show that the closed-loop $\mathcal{L}_1$ adaptive system becomes semi-globally practically ISS with respect to $r(t)$[9] if the nonlinear function $f(x,t)$ in* (6.1.1) *is assumed to be locally Lipschitz continuous (see also Remark 6.1.2).*

6.2. Filter Design with $\mathcal{H}_\infty$ Optimization Theory

Now, we introduce the a filter design method for frequency-domain specifications, where the condition in (6.1.18) is used as the stability condition. The optimal filter design problem is formally stated as follows:

Problem 6.2.1.

$$\min_{C(s)\ \textit{stabilizing}} \|\mathcal{T}_\Omega\|_{\mathcal{H}_\infty}$$

$$\textit{subject to}$$

$$\|H_0(s)(\mathbb{I}_m - C(s))\|_{\mathcal{H}_\infty} < \sqrt{m}L, \quad C(0) = \mathbb{I}_m, \quad \Omega \in \mathcal{C}_\Omega,$$

where $\mathcal{T}_\Omega$ is a map from the external input w to the performance output z.

[9] See Definition 2.2.17.

Now, the objective is to reformulate Problem 6.2.1 into a standard robust performance problem such that

$$\min_{\text{stabilizing } Q(s)} \|\mathcal{F}_l(P_\Delta(s), Q(s))\|_{\mathcal{H}_\infty},$$
$$\text{subject to } \|\Delta\|_{\mathcal{H}_\infty} < \gamma,$$

where $\mathcal{F}_l$ is the Lower Linear Fractional Transform (LLFT) of the generalized plants consisting of an uncertain plant model $P_\Delta(s)$ and a controller $Q(s)$; Δ is a norm-bounded uncertain block, and $\gamma > 0$ is a given constant. Notice that Problem 6.2.1 includes the algebraic constraint $C(0) = \mathbb{I}_m$, which is not easy to handle within standard $\mathcal{H}_\infty$ frameworks. To tackle this issue, we first develop a feedback structure of $\mathcal{L}_1$ reference system, which satisfies the low-pass filter constraint.

Consider the closed-loop reference system given in (6.1.4) and (6.1.5). From (6.1.3) and (6.1.5) it follows that

$$u_{ref}(s) = D(s)(K_g r(s) - \Omega u_{ref}(s) - \eta_{ref}(s) - \sigma(s)),$$

where $\eta_{ref}(s)$ and $\sigma(s)$ are given in (6.1.6). Figure 6.1 illustrates the feedback structure of the $\mathcal{L}_1$ reference system.

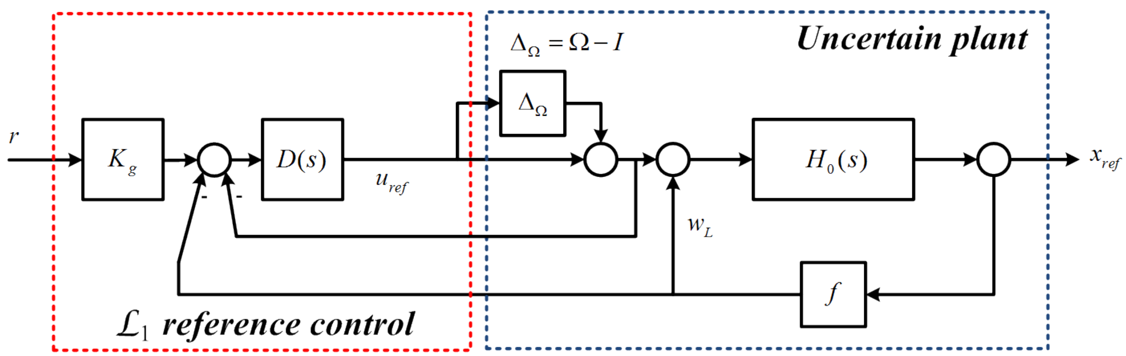

Figure 6.1: Feedback structure of $\mathcal{L}_1$ reference system

By letting

$$D(s) = \frac{1}{s}Q(s), \tag{6.2.1}$$

we notice that $C(0) = \mathbb{I}_m$ is always satisfied, where $Q(s) \in \mathbb{R}^{m\times m}(s)$ is any stable and proper matrix; this implies that $C(s)$ can be parameterized with $Q(s)$, and therefore an optimal filter can be obtained by finding an optimal $Q(s)$. Moreover, from (6.1.5) and (6.1.3) it follows that

$$\eta_{ref}(s) + \sigma(s) = K_g r(s) - (\mathbb{I}_m + \Omega D(s))D^{-1}(s)u_{ref}(s). \tag{6.2.2}$$

Notice that from (6.1.4) one has

$$\Omega u_{ref}(s) + \eta_{ref}(s) + \sigma(s) = B_m^\dagger(s\mathbb{I}_n - A_m)x_{ref}(s), \tag{6.2.3}$$

which, in conjunctions with (6.2.2), leads to

$$\begin{aligned} u_{ref}(s) &= D(s)(K_g r(s) - B_m^\dagger(s\mathbb{I}_n - A_m)x_{ref}(s)) \\ &= Q(s)(F_1(s)r(s) - F_2(s)x_{ref}(s)), \end{aligned} \tag{6.2.4}$$

where $F_1(s) = K_g\frac{1}{s}$ and $F_2(s) = \frac{1}{s}B_m^\dagger(s\mathbb{I}_n - A_m)$. Then, combining (6.2.4) and (6.1.4) yields the feedback structure given in Figure 6.2, where Δ_Ω is the uncertain block that satisfies $\|\Delta_\Omega\|_{\mathcal{H}_\infty} \leq \max_{\Omega\in\mathcal{C}_\Omega} \|\Omega - \mathbb{I}_m\|_2$.

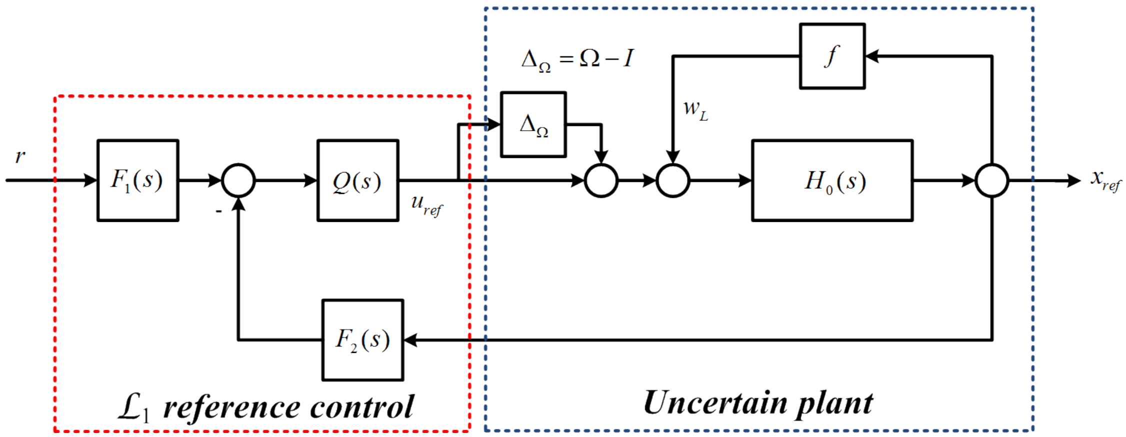

Figure 6.2: Feedback structure with Q-parametrization

Remark 6.2.1. *We notice that the reference control law in* (6.2.4) *is not implementable since it depends on the unmeasurable states. We use it only for the purposes of filter design. As it was stated in [118], the $\mathcal{L}_1$ reference controller can be implemented for some classes of systems. For example, in [118] it is shown that an implementable reference controller for square output-feedback systems can be formulated with the use of the system inverse. In cases, when the system inverse is not straightforward to obtain, the $\mathcal{L}_1$ controller shows clear benefits over other robust control methods. Underactuated systems belong to that class of problems, where the inverse is not defined.*

From the $\mathcal{H}_\infty$ optimization theory, the condition in (6.1.18) can be represented with a norm-bounded uncertain block Δ_L such that $\|\Delta_L\|_{\mathcal{H}_\infty} < \sqrt{m}L$. Therefore, the stability problem for the $\mathcal{L}_1$ reference system can be tackled by solving the standard robust stability problem:

Problem 6.2.2. *Given the feedback system in Figure 6.3, find $Q(s)$ that stabilizes the system, subject to $\|\Delta_L\|_{\mathcal{H}_\infty} < \sqrt{m}L$ and $\|\Delta_\Omega\|_{\mathcal{H}_\infty} < \max_{\Omega\in\mathcal{C}_\Omega} \|\Omega - \mathbb{I}_m\|_2$.*

Notice that the algebraic constraint $C(0) = \mathbb{I}_m$ is always satisfied in the feedback structure of Figure 6.3.

Remark 6.2.2. *As discussed in Chapter 4, the system with high vector relative degree requires a minimum order of the filter (i.e., $C(s)Z^{-1}(s)$ is proper, where $Z(s) \in \mathbb{R}^{m\times m}(s)$ is a (given) stable*

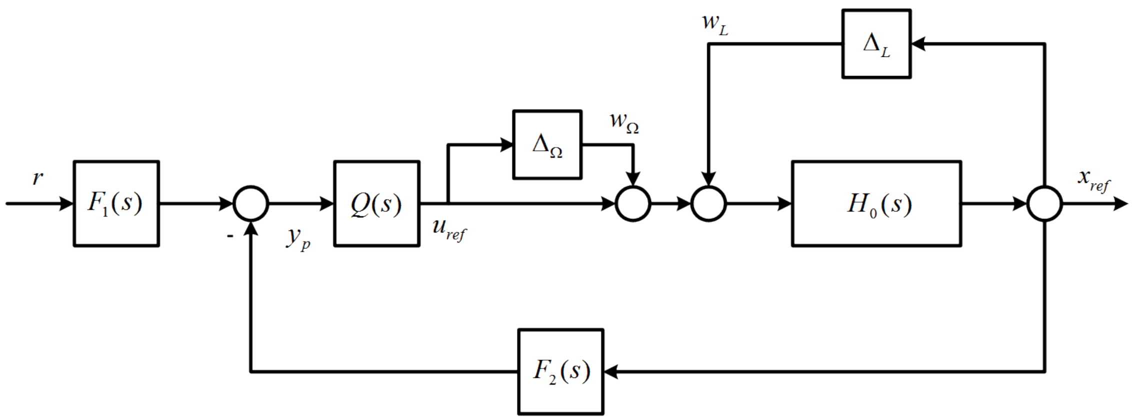

Figure 6.3: Feedback structure for the $\mathcal{L}_1$ reference system stability problem

and proper transfer matrix). This issue can be addressed by using block modifications in Figure 6.3: $F_1(s)$ and $F_2(s)$ can be replaced with $F_1'(s) = F_1(s)Z(s)$ and $F_2'(s) = F_2(s)Z(s)$, respectively. Then, from an optimal solution for $Q(s)$, the low-pass filter can be obtained by letting $D(s) = \frac{1}{s}Q(s)Z(s)$.

Next, we introduce a $\mathcal{H}_\infty$ optimization framework for the design of an optimal filter with frequency-domain specifications. Since the stability condition for the $\mathcal{L}_1$ reference system is reformulated into Problem 6.2.2, the optimal design with the specifications simply follows from conventional $\mathcal{H}_\infty$ design procedures; additional weighting functions are taken into Problem 6.2.2 to realize the frequency-domain specifications. Therefore, Problem 6.2.1 can be re-stated as the robust performance problem:

Problem 6.2.3. *Consider the feedback structure given in Figure 6.4, where $W_I(s)$, $W_U(s)$, and $W_E(s)$ are weighting functions. We solve*

$$\min_{Q(s)\ \textit{stabilizing}} \|\mathcal{F}_l(P_\Delta, Q(s))\|_{\mathcal{H}_\infty}$$

$$\textit{subject to}$$

$$\|\Delta_L\|_{\mathcal{H}_\infty} < \sqrt{m}L, \quad \|\Delta_\Omega\|_{\mathcal{H}_\infty} < \max_{\Omega \in \mathcal{C}_\Omega} \|\Omega - \mathbb{I}_m\|_2, \quad \|\Delta_I\|_{\mathcal{H}_\infty} < 1,$$

where $\mathcal{F}_l$ is the LLFT that represents a map from r to $[e_u, e_y]^\top$, and P_Δ is the uncertain plant that includes the uncertain block $\Delta = \mathrm{diag}\,(\Delta_L, \Delta_\Omega, \Delta_I)$.

Remark 6.2.3. *Since the constraints in Problem 6.2.3 are uniform over system order, Problem 6.2.3 is more suitable to deal with high order systems (see also Remark 6.1.3). Moreover, any algebraic constraints are not observed in Problem 6.2.3, and therefore it is solved by using efficient $\mathcal{H}_\infty$ numerical solvers (e.g., μ synthesis, and non-smooth optimization techniques [87, 92, 119]).*

Different types of weighting blocks can be introduced to obtain desired robustness and performance. For the criteria of weighting function selection, one can refer to [119–121]:

- Weight $W_I(s)$ for time-delay margins
  $W_I(s)$ is chosen for a time-delay consideration. $W_I(s)$ is typically set to be

$$W_I(s) = \epsilon \frac{(\theta/\epsilon) + 1}{(\theta/p)s + 1} \tag{6.2.5}$$

  for the input delay $e^{-\theta s}$ (Pade Approximation, [119]), where p is a large number and $\epsilon \ll 2$ denotes the allowed steady-state relative gain error, while θ is the admissible time delay error. This approximation implies that the uncertainty amounts to 100% at the frequency $1/\theta$.

- Weight $W_U(s)$ for control efforts
  $W_U(s)$ penalizes the control input. A typical setup is $W_U(s) = s$, which is used to to avoid fast changes in the inputs. Notice that the use of $W_U(s)$ to penalize the inputs at the low frequency range makes little sense, since the controller needs a certain magnitude for the input to be effective [120].

- Weight $W_E(s)$ for performance measures
  $W_E(s)$ indicates the allowed magnitude for the tracking errors at each frequency. One example is to consider: (i) steady-state offset less than A; (ii) closed-loop bandwidth higher than ω_B, and (iii) amplification of high-frequency noise less than an factor M. These specifications can be realized as the weighting function that has a stair-like asymptote [121]:

$$W_E(s) = \frac{1}{M}\frac{\tau_P s + 1}{\tau_P s + A/M}, \quad \tau_P = \frac{1}{M\omega_B}, \tag{6.2.6}$$

  Notice that ω_B is the frequency at which the magnitude is almost $0dB$, and $1/M$, $1/A$ represent the minimum and the maximum magnitude of the asymptote, respectively. Then, M is the magnitude of the allowed errors for $\omega > \omega_B$, and A denotes the allowed magnitude error at a low frequency range (less than ω_B) [120,121].

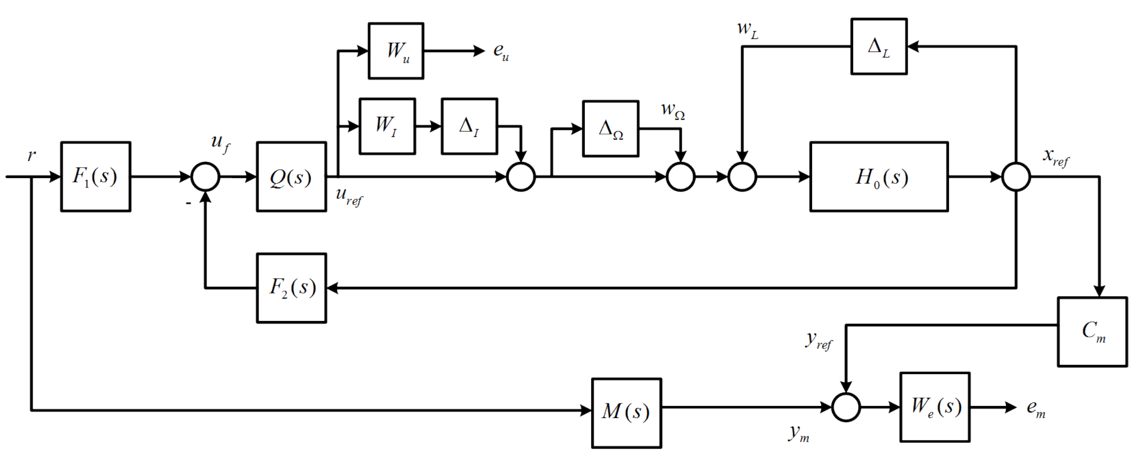

Figure 6.4: Uncertain plants for μ synthesis with weighting functions $W_u(s)$ and $W_e(s)$

Remark 6.2.4. *The benefits of the method in this section are summarized as follows: (i) the standard $\mathcal{H}_\infty$ design procedures for frequency-domain specifications are allowed; (ii) the design procedure is systematically implemented for an optimal trade-off between performance of robustness; and (iii) the solution is less conservative especially for high order systems.*

6.3. Design Example

In this section, we design a lowpass filter with $\mathcal{H}_\infty$ optimization method. Consider the nonlinear dynamics:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -2 & 0 & 1 \\ 1 & -5 & 2 \\ 1 & 0 & 5.5 \end{bmatrix}}_{A_m} x(t) + \underbrace{\begin{bmatrix} 1 \\ 2.5 \\ -3 \end{bmatrix}}_{B_m} (u(t) + f_\Delta(x,t)),$$

$$y(t) = \underbrace{\begin{bmatrix} -5 & 10 & 5 \\ 2.5 & -2 & 0 \end{bmatrix}}_{C_m} x(t),$$

where $f_\Delta(x,t)$ satisfies

$$\|f(x,t) - f(y,t)\|_2 \leq L \|x-y\|_2, \quad \|f(0,t)\|_2 \leq b_0, \qquad x, y \in \mathbb{R}^n, t \geq 0, \tag{6.3.1}$$

for some $L > 0$ and $b_0 > 0$. The objective is to design a low-pass filter $C(s)$ (as in (6.1.3)) to guarantee the stability in the $\mathcal{L}_1$ reference system (6.1.4). To deal with frequency-domain specifications, we use the $\mathcal{H}_\infty$ framework proposed in Section 6.2 (see Problem 6.2.3). From the

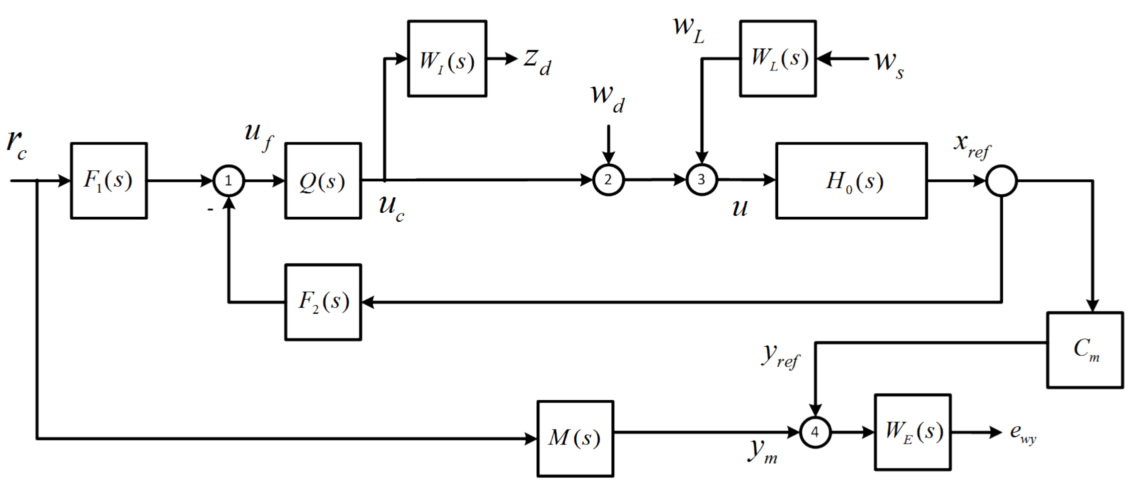

Figure 6.5: Block diagram for the filter design

$\mathcal{H}_\infty$ theory, the robust performance problem in 6.2.3 can be reformulated into the $\mathcal{H}_\infty$ synthesis problem:

Problem 6.3.1. *Let $W_L(s) = L$. Given the feedback structure $\mathcal{T}$ in Figure 6.5, we solve*

$$\min_{\text{stabilizing } Q(s)} \{\|\mathcal{T} : w_d \to z_d\|_{\mathcal{H}_\infty}, \|\mathcal{T} : r_c \to e_{wy}\|_{\mathcal{H}_\infty}\}$$
$$\text{subject to} \quad \|\mathcal{T} : w_s \to x_{ref}\|_{\mathcal{H}_\infty} < 1.$$

Notice that $\mathcal{T} : w_L \to x_{ref}$ in Figure 6.5 corresponds to the transfer matrix $G(s)$ given in (6.1.18).

For the filter design, we use $W_I(s)$ in (6.2.5) for a delay margin around $0.15s$, and W_E in (6.2.6) for the low-frequency tracking performance; $W_I(s)$ is set with $\theta = 0.15$, $\epsilon = 0.2$, and $p = 10$, and $W_E(s)$ is computed with $M = 1$, $A = 3.16 \times 10^{-5}$, and $\tau_B = 18.85$. Figure 6.6 illustrates the weighting functions that are used in this optimization. Notice that $(C_m B_m)$ is not full rank, and

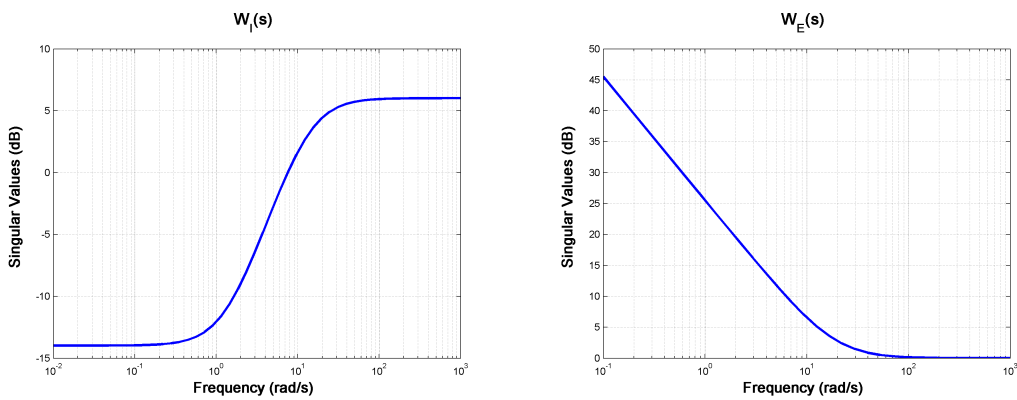

Figure 6.6: Weighting functions

therefore we obtain an right interactor $Z(s)$. The right interactor is computed as $Z(s) = 4/(s+3)$ by following the procedure in Remark 2.1.9. Since Problem 6.3.1 is structured and non-convex, we use a MATLAB solver (Hifoo, [122]), and obtain

$$C(s) = \frac{3.672}{s^2 + 3s + 3.672}, \quad D(s) = \frac{3.672}{s^2 + 3s}$$

with

$$\|G(s)\|_{\mathcal{H}_\infty} = 0.9687, \quad \|G(s)\|_{\mathcal{L}_1} = 0.9248.$$

To demonstrate the performance of the $\mathcal{L}_1$ adaptive system, we consider a numerical simulation with the $\mathcal{L}_1$ adaptive controller of Section 4.2 with the design parameters given in Section 4.2.3; notice that the filter has been changed to the one that we obtained in this section. In the simulation, arbitrary uncertainties with Lipschitz constant $L = 1.0324$, and any input time delays within $[0, 0.12]$ are taken. Figure 6.7 illustrate the $\mathcal{L}_1$ reference system and the closed-loop system responses, and control histories of both systems are given in Figure 6.8. Although the effect of the time delay is not incorporated in the closed-loop analysis, the result shows that the $\mathcal{L}_1$ adaptive controller renders the system response close to the $\mathcal{L}_1$ reference system for uncertain time delays within the specified margin.

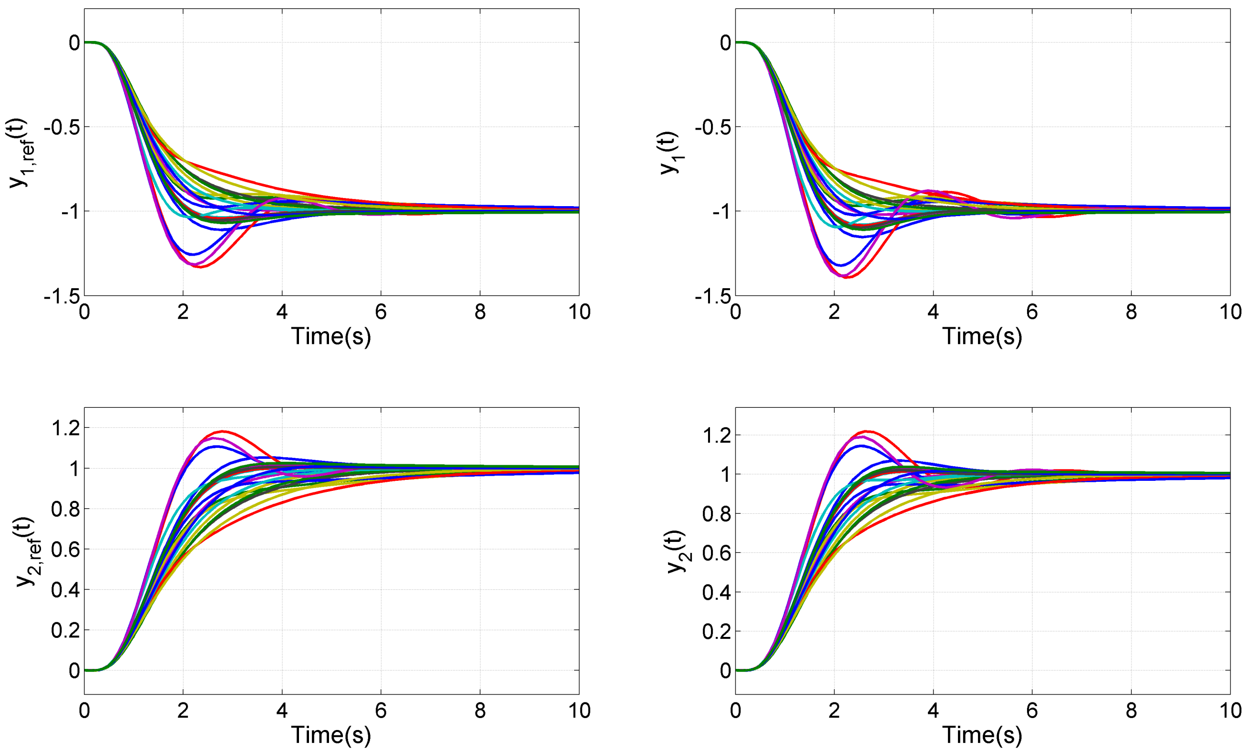

Figure 6.7: System responses of the $\mathcal{L}_1$ reference and the closed-loop system

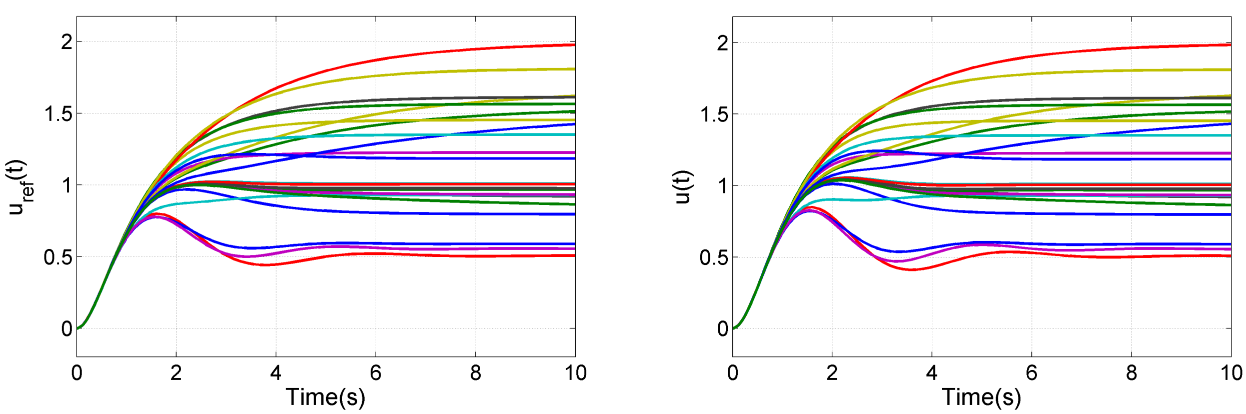

Figure 6.8: Control histories of the $\mathcal{L}_1$ reference and the closed-loop system

CHAPTER. 7

Conclusion and Future research

7.1. Conclusion

This dissertation develops $\mathcal{L}_1$ adaptive output feedback controllers for uncertain, nonlinear, and underactuated systems. Prior results were limited to square MIMO systems, [55, 57, 58].

In Chapter 2, we introduced the required theoretical background. It was shown that a state decomposition approach can be effectively used along with a right interactor to create a virtual system.

In Chapter 3, we consider underactuated nonlinear systems with vector relative degree one. With mild assumptions on the uncertainties, the proposed controller guarantees semi-global stabilization of the closed-loop system. In Chapter 4, approaches to deal with the underactuated systems with high relative degree are addressed. By using a right interactor, a virtual system is introduced which allows for state-decomposition for the $\mathcal{L}_1$ adaptive closed-loop system. Chapter 5 verifies the performance of the proposed controllers with practical examples. In Chapter 6, we develop a low-pass filter design method in the frequency domain. A frequency-domain condition to guarantee the closed-loop stability is introduced. With this condition, the design method suitably deals with frequency-domain specifications. A trade-off between robustness and performance can be optimally performed in the frequency domain by utilizing existing $\mathcal{H}_\infty$ optimization techniques.

7.2. Future research

Future research will focus on extensions of $\mathcal{L}_1$ adaptive control to cover wider classes of MIMO systems. First, the approach will be extended to handle unmatched uncertainties in underactuated systems. The compensation for unmatched uncertainties is a challenging issue in control system design. From geometric control theory, it is known that unmatched uncertainties of underactuated systems may not be perfectly canceled out through the control input, even if they are estimated with high precision [123]. However, it is possible that some class of unmatched nonlinearities can be dealt with by using $\mathcal{L}_1$ adaptive controllers. For example, if the systems are square and all states are measurable, the unmatched uncertainties can be compensated within a filter bandwidth [2].

Second, we envision to develop $\mathcal{L}_1$ adaptive control for switched systems, which can facilitate a multi-model $\mathcal{L}_1$ adaptive control for complex systems that may require different reference behaviors around different trim conditions. This approach cna cpature systems with much larger class of uncertainties.

Finally, the sampled-data applications for $\mathcal{L}_1$ adaptive control will be addressed. Since most controllers (if not all) are implemented in digital computers, the overall control system needs to be tackled in the sampled-data framework. For example, many important cyber-physical systems such as power grids, transportation and financial systems are sampled-data systems. The development of control strategies to handle vulnerability to cyber attacks has become an active research area of interest [124]. In the literature, it has been shown that the $\mathcal{L}_1$ adaptive control with piecewise-constant adaptation can be analyzed in the sampled-data framework [125] by providing an upper bound for sampling rates for closed-loop stability. The approach is based on square-system analysis. For underactuated systems, the $\mathcal{L}_1$ adaptive controller presented in 4.2 can be extended to sampled-data systems. This will enable the use of $\mathcal{L}_1$ adaptive controller in a much broader range of applications.

References

[1] L. W. Taylor, Jr. and E. J. Adkins, "Adaptive control and the X-15," in *Princeton University Conference on Aircraft Flying Qualities*, (Princeton, NJ), June 1965.

[2] N. Hovakimyan and C. Cao, $\mathcal{L}_1$ *Adaptive Control Theory.* Philadelphia, PA: Society for Industrial and Applied Mathematics, 2010.

[3] R. Kalman, "Design of self-optimizing control systems," *ASME Transactions*, vol. 80, pp. 468–478, 1958.

[4] K. J. Åström and B. Wittenmark, "On self-tuning regulators," *Automatica*, vol. 9, pp. 185–199, March 1973.

[5] L. Ljung, *System Identification—Theory for the User.* Englewood Cliffs, New Jersey: Prentice Hall, January 1987.

[6] R. E. Bellman, *Adaptive Control Processes—A Guided Tour.* Princeton, NJ: Princeton University Press, 1961.

[7] Y. D. Landau, *Adaptive Control: the Model Reference Approach.* Control & Systems Theory, New York, NY: Marcel Dekker, Inc., 1979.

[8] R. V. Monopoli, "Model reference adaptive control with an augmented error signal," *IEEE Transactions on Automatic Control*, vol. 19, pp. 474–484, October 1974.

[9] A. Feuer and A. S. Morse, "Adaptive control of single-input, single-output linear systems," *IEEE Transactions on Automatic Control*, vol. 23, pp. 557–569, August 1978.

[10] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems.* Information and System Sciences, Englewood Cliffs, NJ: Prentice Hall, 1989.

[11] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control.* Englewood Cliffs, NJ: Prentice Hall, 1991.

[12] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design.* New York, NY: John Wiley & Sons, 1995.

[13] C. Cao and N. Hovakimyan, "Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture with guaranteed transient performance," *IEEE Transactions on Automatic Control*, vol. 53, pp. 586–591, March 2008.

[14] I. M. Gregory, C. Cao, E. Xargay, N. Hovakimyan, and X. Zou, "$\mathcal{L}_1$ adaptive control design for nasa airstar flight test vehicle," in *AIAA Guidance, Navigation and Control Conference*, (Chicago, IL), August 2009.

[15] N. Hovakimyan, C. Cao, E. Kharisov, E. Xargay, and I. M. Gregory, "$\mathcal{L}_1$ adaptive control for safety-critical systems," *IEEE Control Systems Magazine*, vol. 31, pp. 54–104, October 2011.

[16] A. Pettersson, K. J. Åström, A. Robertsson, and R. Johansson, "Augmenting l1 adaptive control of piecewise constant type to a fighter aircraft. performance and robustness evaluation for rapid maneuvering," in *AIAA Guidance, Navigation and Control Conference*, (Minneapolis, MN), American Institute of Aeronautics and Astronautics, December 2012.

[17] K. Ackerman, E. Xargay, R. Choe, N. Hovakimyan, M. C. Cotting, R. B. Jeffrey, M. P. Blackstun, T. P. Fulkerson, T. R. Lau, and S. S. Stephens, "$\mathcal{L}_1$ stability augmentation system for calspan's variable-stability learjet," in *AIAA Guidance, Navigation and Control Conference*, (San Diego, CA), American Institute of Aeronautics and Astronautics, January 2016.

[18] C. Cao and N. Hovakimyan, "Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture, Part I: Control signal and asymptotic stability," in *American Control Conference*, (Minneapolis, MN), pp. 3397–3402, June 2006.

[19] C. Cao and N. Hovakimyan, "Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture, Part II: Guaranteed transient performance," in *American Control Conference*, (Minneapolis, MN), pp. 3403–3408, June 2006.

[20] C. E. Rohrs, L. S. Valavani, M. Athans, and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics," *IEEE Transactions on Automatic Control*, vol. 30, pp. 881–889, September 1985.

[21] J.-B. Pomet and L. Praly, "Adaptive nonlinear regulation: Estimation from the Lyapunov equation," *IEEE Transactions on Automatic Control*, vol. 37, pp. 729–740, June 1992.

[22] C. Cao, V. Patel, K. Reddy, N. Hovakimyan, E. Lavretsky, and K. Wise, "Are phase and time-delay margins always adversely affected by high-gain?," in *AIAA Guidance, Navigation and Control Conference*, American Institute of Aeronautics and Astronautics, August 2006.

[23] B. Michini and J. How, "$\mathcal{L}_1$ adaptive control for indoor autonomous vehicles: Design process and flight testing," in *AIAA Guidance, Navigation and Control Conference*, American Institute of Aeronautics and Astronautics, August 2009.

[24] H. Lee, S. Snyder, and N. Hovakimyan, "L1 adaptive control within a flight envelope protection system," *Journal of Guidance, Control and Dynamics*, vol. 40, pp. 1013–1026, April 2017.

[25] K. Ohnishi, "A new servo method in mechatronics," *Transaction on Japanese Society of Electrical Engineering*, vol. 107, no. D, pp. 83–86, 1987.

[26] W.-H. Chen, J. Yang, L. Guo, and S. Li, "Disturbance-observer-based control and related methodsan overview," *IEEE Transactions on Industrial Electronics*, vol. 63, no. 2, pp. 1083–1095, 2016.

[27] C. E. Garcia and M. Morari, "Internal model control. a unifying review and some new results," *Industrial & Engineering Chemistry Process Design and Development*, vol. 21, pp. 308–323, April 1982.

[28] M. Campi, W. S. Lee, and B. Anderson, "New filters for internal model control design," *International Journal of Robust and Nonlinear Control*, vol. 4, no. 6, pp. 757–775, 1994.

[29] E. Kharisov, K. K. Kim, N. Hovakimyan, and X. Wang, "Limiting behavior of $\mathcal{L}_1$ adaptive controllers," in *AIAA Guidance, Navigation and Control Conference*, American Institute of Aeronautics and Astronautics, August 2011.

[30] E. Kharisov, *$\mathcal{L}_1$ Adaptive Output-feedback Control Architectures.* PhD thesis, University of Illinois at Urnbana–Champaign, January 2014.

[31] X. Wang and N. Hovakimyan, "$\mathcal{L}_1$ adaptive controller for nonlinear time-varying reference systems," *Systems & Control Letters*, vol. 61, pp. 455–463, 2012.

[32] P. A. Ioannou and J. Sun, *Robust Adaptive Control.* Upper Saddle River, NJ: Prentice Hall, 1996.

[33] I. Bar-Kana and H. Kaufman, "Simple adaptive control of uncertain systems," *International Journal of Adaptive Control and Signal Processing*, vol. 2, pp. 133–143, June 1988.

[34] P. Kokotovic, M. Krstic, and I. Kanellakopoulos, "Backstepping to passivity: recursive design of adaptive systems," in *IEEE Conference on Decision and Control*, Institute of Electrical and Electronics Engineers, December 1992.

[35] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, "Adaptive nonlinear output-feedback control with an observer-based identifier," in *American Control Conference*, pp. 2821–2825, IEEE, 1993.

[36] M. Jankovic, "Adaptive nonlinear output feedback tracking with a partial high-gain observer and backstepping," *IEEE Transactions on Automatic Control*, vol. 42, no. 1, pp. 106–113, 1997.

[37] G. Tao and P. A. Ioannou, "Robust model reference adaptive control for multivariable plants," *International Journal of Adaptive Control and Signal Processing*, vol. 2, pp. 217–248, Semtempber 1988.

[38] R. R. Costa, L. Hsu, A. K. Imai, and P. Kokotović, "Lyapunov-based adaptive control of MIMO systems," *Automatica*, vol. 39, pp. 1251–1257, July 2003.

[39] Y. Ling and G. Tao, "Adaptive backstepping control design for linear multivariable plants," in *IEEE Conference on Decision and Control*, Institute of Electrical and Electronics Engineers, December 1996.

[40] H. Khalil, "Adaptive output feedback control of nonlinear systems represented by input-output models," *Transactions on Automatic Control*, vol. 41, no. 2, pp. 177–188, 1996.

[41] X. Zhang and Y. Lin, "Adaptive output feedback tracking for a class of nonlinear systems," *Automatica*, vol. 48, pp. 2372–2376, September 2012.

[42] J. Du, X. Hu, H. Liu, and C. L. P. Chen, "Adaptive robust output feedback control for a marine dynamic positioning system based on a high-gain observer," *Transactions on Neural Networks and Learning Systems*, vol. 26, pp. 2775–2786, November 2015.

[43] F. Esfandiari and H. K. Khalil, "Output feedback stabilization of fully linearizable systems," *International Journal of Control*, vol. 56, pp. 1007–1037, November 1992.

[44] E. Lavretsky and K. A. Wise, *Robust and Adaptive Control With Aerospace Applications*. Springer London, 2013.

[45] P. Sannuti and A. Saberi, "Special coordinate basis for multivariable linear systems—finite and infinite zero structure, squaring down and decoupling," *International Journal of Control*, vol. 45, pp. 1655–1704, May 1987.

[46] P. Misra, "Numerical algorithms for squaring-up non-square systems, part ii: General case," in *1998 American Control Conference*, (San Francisco, CA), June 1998.

[47] G. Tao, "Multivariable adaptive control: A survey," *Automatica*, vol. 50, pp. 2737–2764, November 2014.

[48] E. Lavretsky, "Adaptive output feedback design using asymptotic properties of LQG/LTR controllers," *IEEE Transactions on Automatic Control*, vol. 57, pp. 1587–1591, June 2012.

[49] T. E. Gibson, Z. Qu, A. M. Annaswamy, and E. Lavretsky, "Adaptive output feedback based on closed-loop reference models," *Transactions on Automatic Control*, vol. 60, no. 10, pp. 2728–2733, 2015.

[50] I. Mizumoto, T. Chen, S. Ohdaira, M. Kumon, and Z. Iwai, "Adaptive output feedback control of general MIMO systems using multirate sampling and its application to a cart–crane system," *Automatica*, vol. 43, pp. 2077–2085, December 2007.

[51] Z. Qu, A. M. Annaswamy, and E. Lavretsky, "Adaptive output-feedback control for relative degree two systems based on closed-loop reference models," in *IEEE Conference on Decision and Control*, Institute of Electrical and Electronics Engineers, December 2015.

[52] E. Xargay, N. Hovakimyan, and C. Cao, "$\mathcal{L}_1$ adaptive controller for multi-input multi-output systems in the presence of nonlinear unmatched uncertainties," in *American Control Conference*, Institute of Electrical and Electronics Engineers, June 2010.

[53] J. Wang, V. V. Patel, C. Cao, N. Hovakimyan, and E. Lavretsky, "Novel $\mathcal{L}_1$ adaptive control methodology for aerial refueling with guaranteed transient performance," *Journal of Guidance, Control and Dynamics*, vol. 31, pp. 182–193, January 2008.

[54] C. Cao and N. Hovakimyan, "$\mathcal{L}_1$ adaptive output feedback controller for systems of unknown dimension," *IEEE Transactions on Automatic Control*, vol. 53, pp. 815–821, April 2008.

[55] C. Cao and N. Hovakimyan, "$\mathcal{L}_1$ adaptive output-feedback controller for non-stricly-positive-real reference systems: Missile longitudinal autopilot design," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 32, pp. 717–726, May-June 2009.

[56] M. Bichlmeier, "An $\mathcal{L}_1$ adaptive output feedback controller using modified piecewise constant adaptation law," in *AIAA Guidance, Navigation and Control Conference*, American Institute of Aeronautics and Astronautics, January 2016.

[57] E. Kharisov and N. Hovakimyan, "$\mathcal{L}_1$ adaptive output feedback controller for minimum phase systems," in *American Control Conference*, (San Francisco, CA), June–July 2011.

[58] H. Lee, V. Cichella, and N. Hovakimyan, "$\mathcal{L}_1$ adaptive output feedback augmentation of model reference control," in *American Control Conference*, (Portland, OR), June 2014.

[59] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulars*. Princeton University Press, 2009.

[60] P. J. Antsaklis and A. N. Michel, *Linear systems*. Englewood Cliffs, NJ: McGraw-Hill, 1997.

[61] C. B. Schrader and M. K. Sain, "Research on system zeros: a survey," *International Journal of Control*, vol. 50, pp. 1407–1433, October 1989.

[62] A. G. J. MacFarlane and N. Karcanias, "Poles and zeros of linear multivariable systems : a survey of the algebraic, geometric and complex-variable theory," *International Journal of Control*, vol. 24, pp. 33–74, July 1976.

[63] P. V. Dooren, "The computation of kronecker's canonical form of a singular pencil," *Linear Algebra and its Applications*, vol. 27, pp. 103–140, October 1979.

[64] N. Karcanias and B. Kouvaritakis, "The output zeroing problem and its relationship to the invariant zero structure : a matrix pencil approach," *International Journal of Control*, vol. 30, pp. 395–415, September 1979.

[65] U. Shaked and B. Kouvaritakis, "The zeros of linear optimal control systems and their role in high feedback gain stability design," *IEEE Transactions on Automatic Control*, vol. 22, pp. 597–599, August 1977.

[66] G. Verghese and T. Kailath, "Eigenvector chains for finite and infinite zeros of rational matrices," in *1979 18th IEEE Conference on Decision and Control including the Symposium on Adaptive Processes*, Institute of Electrical and Electronics Engineers, December 1979.

[67] J. Tokarzewski, "A note on dynamical interpretation of invariant zeros in mimo lti systems and algebraic criterions of degeneracy.," in *Proc. Int. Symp. Mathematical Theory of Networks and Systems*, June 2000.

[68] P. Antsaklis, "On finite and infinite zeros in the model matching problem," in *IEEE Conference on Decision and Control*, Institute of Electrical and Electronics Engineers, December 1986.

[69] H. Rosenbrock, *State-Space and Multivariable Theory.* New York, NY: John Wiley & Sons, 1970.

[70] T. Kailath, *Linear Systems.* Upper Saddle River, NJ: Prentice Hall, 1980.

[71] W. Wolovich, "On the numerators and zeros and rational transfer matrices," *IEEE Transactions on Automatic Control*, vol. 18, pp. 544–546, October 1973.

[72] H. Rosenbrock, "Correspondence : Correction to 'the zeros of a system'," *International Journal of Control*, vol. 20, pp. 525–527, September 1974.

[73] H. H. Rosenbrock, "The zeros of a system†," *International Journal of Control*, vol. 18, pp. 297–299, July 1973.

[74] H. H. Rosenbrock, "Comments on poles and zeros of linear multivariable systems: A survey of the algebraic, geometric and complex-variable theory," *International Journal of Control*, vol. 26, pp. 157–161, July 1977.

[75] B. R. Copeland and M. G. Safonov, "Zero cancelling compensators for singular control problems and their application to the inner–outer factorization problem," *International Journal of Robust and Nonlinear Control*, vol. 2, pp. 139–164, August 1992.

[76] P. Sannuti and A. Saberi, "Special coordinate basis for multivariable linear systems—finite and infinite zero structure, squaring down and decoupling," *International Journal of Control*, vol. 45, pp. 1655–1704, May 1987.

[77] Y. Mutoh and R. Ortega, "Interactor structure estimation for adaptive control of discrete-time multivariable nondecouplable systems," *Automatica*, vol. 29, pp. 635–647, May 1993.

[78] G. C. Verghese, B. Fernandez, and J. K. Hedrick, "Stable, robust tracking by sliding mode control," *Systems & control letters*, vol. 10, no. 1, pp. 27–34, 1988.

[79] T. R. Oliveira, A. J. Peixoto, E. V. L. Nunes, and L. Hsu, "Control of uncertain nonlinear systems with arbitrary relative degree and unknown control direction using sliding modes," *International Journal of Adaptive Control and Signal Processing*, vol. 21, no. 8-9, pp. 692–707, 2007.

[80] X. Xin and T. Mita, "A simple state-space design of an interactor for a non-square system via system matrix pencil approach," *Linear Algebra and its Applications*, vol. 351-352, pp. 809–823, August 2002.

[81] W. A. Wolovich and P. L. Falb, "Invariants and canonical forms under dynamic compensation," *SIAM Journal on Control and Optimization*, vol. 14, pp. 996–1008, November 1976.

[82] J. Chen, R. J. Patton, and H.-Y. Zhang, "Design of unknown input observers and robust fault detection filters," *International Journal of Control*, vol. 63, pp. 85–105, January 1996.

[83] H. Khalil, *Nonlinear systems.* Prentice Hall Upper Saddle River, NJ, 2002.

[84] J. B. Conway, *A Course in Functional Analysis.* Springer New York, 1985.

[85] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach.* Princeton University Press, 2011.

[86] V. Chellaboina, W. M. Haddad, D. S. Bernstein, and D. A. Wilson, "Induced convolution operator norms of linear dynamical systems," *Mathematics of Control, Signals, and Systems*, vol. 13, pp. 216–239, September 2000.

[87] M. Green and D. J. Limebeer, *Linear Robust Control.* Courier Corporation, 2012.

[88] G. Grubb, *Distributions and operators*, vol. 252. Springer Science & Business Media, 2008.

[89] M. Vidyasagar, "Optimal rejection of persistent bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 31, pp. 527–534, June 1986.

[90] N. K. Rutland and P. G. Lane, "Computing the 1-norm of the impulse response of linear time-invariant systems," *Systems & Control Letters*, vol. 26, pp. 211–221, October 1995.

[91] M. Dahleh and I. Diaz-Bobillo, *Control of uncertain systems: a linear programming approach.* Prentice Hall, 1995.

[92] S. P. Boyd and C. H. Barratt, *Linear controller design: limits of performance.* Prentice Hall, 1990.

[93] S. P. Boyd, *Robust Control of Multivariable and Large Scale Systems.* Minneapolis, MN: Honeywell Systems and Research Center, 1985.

[94] E. D. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and optimal control theory*, pp. 163–220, Springer, 2008.

[95] I. Karafyllis, "A system-theoretic framework for a wide class of systems i: Applications to numerical analysis," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 876–899, 2007.

[96] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351–359, 1995.

[97] A. Chaillet and A. Lora, "Uniform global practical asymptotic stability for time-varying cascaded systems," *European journal of control*, vol. 12, no. 6, pp. 595–605, 2006.

[98] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for iss systems and applications," *Mathematics of Control, Signals, and Systems*, vol. 7, no. 2, pp. 95–120, 1994.

[99] A. Isidori, *Nonlinear Control Systems II.* London: Springer-Verlag, 1999.

[100] M. W. Fossier, "The development of radar homing missiles," *Journal of Guidance, Control and Dynamics*, vol. 7, pp. 641–651, November 1984.

[101] F. S. Kramer, "Multivariable autopilot design and implementation for tactical missiles," tech. rep., DTIC Document, 1998.

[102] P. Zarchan, *Tactical and Strategic Missile Guidance.* Reston, VA: American Institute of Aeronautics and Astronautics, Inc., 2012.

[103] D. Stallard, "An approach to autopilot design for homing interceptor missiles," in *Navigation and Control Conference*, American Institute of Aeronautics and Astronautics, August 1991.

[104] F. Nesline and M. L. Nesline, "How autopilot requirements constrain the aerodynamic design of homing missiles," in *American Control Conference*, (San Diego, CA, USA), Institute of Electrical & Electronics Engineers, June 1984.

[105] C. P. Mracek and D. B. Ridgely, "Missile longitudinal autopilots: Connections between optimal control and classical topologies," in *AIAA Guidance, Navigation and Control Conference*, (San Francisco, CA), American Institute of Aeronautics and Astronautics, August 2005.

[106] C. P. Mracek and D. B. Ridgely, "Missile longitudinal autopilots: Comparison of multiple three loop topologies," in *AIAA Guidance, Navigation and Control Conference*, American Institute of Aeronautics and Astronautics, August 2005.

[107] R. Hindman and W. Shell, "Missile autopilot design using adaptive nonlinear dymanic inversion," in *2005 American Control Conference*, (Portland, OR, USA), Institute of Electrical & Electronics Engineers (IEEE), June 2005.

[108] F. Peter, F. Holzapfel, E. Xargay, and N. Hovakimyan, "$\mathcal{L}_1$ adaptive augmentation of a missile autopilot," in *AIAA Guidance, Navigation and Control Conference*, (Minneapolis, MN), August 2012.

[109] A. J. Calise, M. Sharma, and J. E. Corban, "Adaptive autopilot design for guided munitions," *Journal of Guidance, Control, and Dynamics*, vol. 23, pp. 837–843, September 2000.

[110] M. McFarland and A. Calise, "Adaptive nonlinear control of agile antiair missiles using neural networks," *IEEE Transactions on Control Systems Technology*, vol. 8, no. 5, pp. 749–756, 2000.

[111] R. Nichols, R. Reichert, and W. Rugh, "Gain scheduling for h-infinity controllers: a flight control example," *Transactions on Control Systems Technology*, vol. 1, pp. 69–79, June 1993.

[112] S. A. Campbell, S. Crawford, and K. Morris, "Friction and the inverted pendulum stabilization problem," *Journal of Dynamic Systems, Measurement, and Control*, vol. 130, no. 5, p. 054502, 2008.

[113] D. Li, N. Hovakimyan, C. Cao, and K. Wise, "Filter design for feedback-loop trade-off of $\mathcal{L}_1$ adaptive controller: A linear matrix inequality approach," in *AIAA Guidance, Navigation and Control Conference*, (Honolulu, HI), p. 6280, August 2008.

[114] M. Naghnaeian, P. G. Voulgaris, and N. Hovakimyan, "On robustness of $\mathcal{L}_1$ adaptive control with time varying perturbations and filter design," in *American Control Conference*, (Montreal, Canada), pp. 1937–1942, June 2012.

[115] H. Jafarnejadsani, D. Sun, H. Lee, and N. Hovakimyan, "Optimized $\mathcal{L}_1$ adaptive controller for trajectory tracking of an indoor quadrotor," *Journal of Guidance, Control and Dynamics*, pp. 1–13, February 2017.

[116] F. Blanchini and M. Sznaier, "Rational l1 suboptimal compensators for continuous-time systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 7, pp. 1487–1492, 1994.

[117] K. K. K. Kim, E. Kharisov, and N. Hovakimyan, "Filter design for $\mathcal{L}_1$ adaptive output-feedback controller," in *IEEE Conference on Decision and Control*, pp. 5653–5658, IEEE, December 2011.

[118] E. Kharisov, K. K. K. Kim, X. Wang, and N. Hovakimyan, "Limiting behavior of $\mathcal{L}_1$ adaptive controllers," in *AIAA Guidance, Navigation and Control Conference*, 2011.

[119] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design.* JOHN WILEY & SONS, 2001.

[120] P. Lundstrom, S. Skogstad, and Z. C. Wang, "Weight selection for h-infinity and mu-control methods: insights and examples from process control," in *In Symposium on Robust Control System Design Using H-infinity and Related Methods*, 1991.

[121] P. Lundstrom, S. Skogestad, and Z.-Q. Wang, "Uncertainty weight selection for h-infinity and mu-control methods," in *IEEE Conference on Decision and Control*, pp. 1537–1542, IEEE, 1991.

[122] J. V. Burke, D. Henrion, A. S. Lewis, and M. L. Overton, "Hifoo-a matlab package for fixed-order controller design and h-infinity optimization," *IFAC Proceedings Volumes*, vol. 39, no. 9, pp. 339–344, 2006.

[123] G. Marro, F. Morbidi, L. Ntogramatzidis, and D. Prattichizzo, "Geometric control theory for linear systems: a tutorial," in *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems–MTNS*, vol. 5, 2010.

[124] M. Naghnaeian, N. Hirzallah, and P. G. Voulgaris, "Dual rate control for security in cyber-physical systems," in *2015 54th IEEE Conference on Decision and Control*, pp. 1415–1420, December 2015.

[125] H. L. Hamidreza Jafarnejadsani and N. Hovakimyan, "An $\mathcal{L}_1$ adaptive control design for output-feedback sampled-data systems," in *American Control Conference*, (Seattle, WA), May 2017.