STRUCTURES AND DYNAMICS

BY

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DISSERTATION

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Abstract

Our results are divided in three independent chapters.

In Chapter 2, we show that if g is a generic isometry of a generic subspace X of the Urysohn metric space $\mathbb{U}$ then g does not extend to a full isometry of $\mathbb{U}$. The same applies to the Urysohn sphere $\mathbb{S}$. Let $\boldsymbol{M}$ be a Fraïssé $\mathcal{L}$-structure, where $\mathcal{L}$ is a relational countable language and $\boldsymbol{M}$ has no algebraicity. We provide necessary and sufficient conditions for the following to hold: "For a generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$, every automorphism $f \in \mathrm{Aut}(\boldsymbol{A})$ extends to a full automorphism $\tilde{f} \in \mathrm{Aut}(\boldsymbol{M})$." From our analysis, a dichotomy arises and some structural results are derived that, in particular, apply to ω-stable Fraïssé structures without algebraicity. Results in Chapter 2 are separately published in [Pan15].

In Chapter 3, we develop a game-theoretic approach to anti-classification results for orbit equivalence relations and use this development to reorganize conceptually the proof of Hjorth's turbulence theorem. We also introduce a new dynamical criterion providing an obstruction to classification by orbits of Polish groups which admit a complete left invariant metric (CLI groups). We apply this criterion to the relation of equality of countable sets of reals and we show that the relations of unitary conjugacy of unitary and selfadjoint operators on the separable infinite-dimensional Hilbert space are not classifiable by CLI-group actions. Finally we show how one can adapt this approach to the context of Polish groupoids. Chapter 3 is joint work with Martino Lupini and can also be found in [LP16].

In Chapter 4, we develop a theory of projective Fraïssé limits in the spirit of Irwin-Solecki. The structures here will additionally support dual semantics as in [Sol10, Sol12]. Let Y be a compact metrizable space and let G be a closed subgroup of $\mathrm{Homeo}(Y)$. We show that there is always a projective Fraïssé limit $\boldsymbol{K}$ and a closed equivalence relation $\mathfrak{r}$ on its domain K that is definable in $\boldsymbol{K}$, so that the quotient of K under $\mathfrak{r}$ is homeomorphic to Y and the projection $K \to Y$ induces a continuous group embedding $\mathrm{Aut}(\boldsymbol{K}) \hookrightarrow G$ with dense image. The main results of Chapter 4 can also be found in [Pan16].

To Marianna, who never learned how to sing in tune.

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Chapter 1

Introduction

A *Polish space* is a separable topological space which admits a compatible complete metric. A *Polish group* is a topological group whose underlying topology is Polish. If G is a Polish group then a *Polish G-space* is a Polish space X, endowed with a continuous action $G \curvearrowright X$. We will focus on the following families of Polish groups.

- Let S_∞ be the group of all permutations of the natural numbers. This group inherits a Polish group topology from its action on the natural numbers, namely, the pointwise convergence topology. An important class of Polish groups is the class of all *non-Archimedean* Polish groups. These are groups which admit a neighborhood basis of the identity consisting of open subgroups. A Polish group is non-Archimedean if and only if it is isomorphic to a closed subgroup of S_∞; see [BK96, Theorem 1.5.1].

- Let (X, d) be a complete separable metric space and consider the group $\mathrm{Iso}(X, d)$ of all *isometries* of X. We view $\mathrm{Iso}(X, d)$ as a Polish group endowed, as above, with the pointwise convergence topology.

- Let K be a compact metrizable space and let $\mathrm{Homeo}(K)$ be the group of all *homeomorphisms* of K. We view $\mathrm{Homeo}(K)$ together with the compact-open topology, which renders it a Polish group.

In the absence of local compactness it is often difficult to study these groups using classical methods. Nevertheless, various important results regarding large topological groups have been recently developed starting from a simple observation, namely, that the topology of the group in all examples above is canonically induced by its action on the associated underlying structure. A common theme behind these results is to study how various dynamical properties of the automorphism group of a structure translate to combinatorial properties of the structure itself and vice versa; see [Kec12] for a survey. Fraïssé theory is the core of this area of research, acting as a bookkeeping devise for the underlying combinatorics.

Our work in Chapter 2 is of this flavor. In particular, we examine various discrete and metric Fraïssé structures $\boldsymbol{M}$ and we study the combinatorics behind extending automorphisms of a generic substructure $\boldsymbol{N}$ of $\boldsymbol{M}$ to a global automorphism of $\boldsymbol{M}$. For the precise statements; see Section 1.1.

The results in Chapter 2 make use of an infinite game played with two Players, namely the Banach-Mazur game; see Section 2.1. In Chapter 3 we define two infinite games and use them to study obstructions in Borel reducibility of various orbit equivalence relations. The first game is a dynamical version of a *forth system* which we use to provide a new criterion for non-classifiability by CLI-group actions. Our second game is a "generous" version of a *back and forth system.* We use it to reorganize conceptually the proof of Hjorth's turbulence theorem. For the precise statements and further applications, see Section 1.2. For the proofs of these results; see Chapter 3.

The interplay between dynamics of non-Archimedean Polish groups and classical Fraïssé theory is established by the correspondence between closed subgroups of S_∞ and discrete Fraïssé structures $\boldsymbol{M}$. This correspondence was later extended to a correspondence between metric Fraïssé structures and closed subgroups of isometry groups $\mathrm{Iso}(X, d)$ of complete and separable metric spaces (X, d). Every Polish group G is a closed subgroup of some isometry group $\mathrm{Iso}(X, d)$ as above; see [GK03]. However, when G is represented as a closed subgroup of $\mathrm{Homeo}(K)$, for some compact metrizable space K, then it is more natural to study G through the automorphism group of a *projective Fraïssé limit.* Projective Fraïssé limits were introduced in [IS06]. Since then projective Fraïssé expressions have been found for various explicit examples of compact spaces. In Chapter 4 we show that any space K as above admits a projective Fraïssé expression. The precise statements and the relevant background are given in Section 1.3 below.

1.1 Extending automorphisms

The (separable) *Urysohn metric space* $(\mathbb{U}, \rho)$ was introduced in [Ury27] and it is the unique, up to isometry, Polish metric space that satisfies the following properties:

- (ultrahomogeneity) for every two finite isometric subspaces $A, B \subset \mathbb{U}$ and for every isometry $f : A \to B$, f extends to a full isometry $\tilde{f}$ of $\mathbb{U}$;
- (universality) every Polish metric space is isometric to a subspace of $\mathbb{U}$.

Huhunaišvili showed in [Huh55] that $\mathbb{U}$ satisfies a strengthening of the ultrahomogeneity property attained by replacing the adjective "finite" in A and B above with "compact."

There are spaces that enjoy a much stronger version of homogeneity. Consider for example the Euclidean metric space $\mathbb{R}^m$. Then, it is true that for every two, possibly infinite, metric subspaces A, B and every isometry $f : A \to B$, there is an isometry $\tilde{f}$ of the whole space $\mathbb{R}^m$ that extends f. In the case of $\mathbb{U}$, it was shown by Melleray in [Mel07] that Huhunaišvili's result cannot be extended further, i.e., if X is non-compact space then there are isometric copies A, B of X in $\mathbb{U}$ and an isometry $f : A \to B$ that does not extend to an isometry $\tilde{f}$ of $\mathbb{U}$. On the other hand, it is worth noting here that for any separable metric space X, adopting Uspenskij's use of Katětov's tower construction [Usp90, Kat86], we can find copies A, B of X in $\mathbb{U}$ so that any isometry $f : A \to B$ extends to a global isometry.

Our focus here will be on a slightly different question. For every Polish metric space (X, d) there is canonical Polish topology for the hyperspace $\mathcal{F}(X)$ of all closed subsets of X, namely the Wijsman topology [Wij64, Bee91]. We say that for a *generic subspace* of X a certain property holds if the set of all closed subsets of X that have this property is a comeager subset of $\mathcal{F}(X)$ in the Wijsman topology. The question which motivates our investigation is whether for a generic subspace F of $\mathbb{U}$ every self-isometry of F extends to an isometry of $\mathbb{U}$. It turns out as a consequence of Lemma 2.5.5 that for a generic subspace F of $\mathbb{U}$ the space F is isometric to $\mathbb{U}$ itself. Since the generic subspace is of one isometry type, it therefore makes sense to ask whether for a *generic pair* F_1, F_2 of subspaces of $\mathbb{U}$ every isometry $f : F_1 \to F_2$ extends to a global isometry. Here, we identify pairs of closed subsets of $\mathbb{U}$ with points in $\mathcal{F}(\mathbb{U}) \times \mathcal{F}(\mathbb{U})$. Keeping in mind that $\mathbb{U}$ is just an instance of the general problem that we are going to deal with, consider the following definitions.

For every two isometric Polish metric spaces X, Y, we write $\mathrm{Iso}(X)$ for the space of all isometries of X and $\mathrm{Iso}(X, Y)$ for the space of all isometries from X onto Y. The spaces $\mathrm{Iso}(X)$ and $\mathrm{Iso}(X, Y)$ are Polish, equipped with the pointwise convergence topology; see [Kec95, Section 9B]. If A, B are isometric subsets of X we write $\mathcal{E}(A)$ to denote the set of all self-isometries of A that extend to a global isometry of X, and similarly by $\mathcal{E}(A, B)$ we denote the set of all isometries from A to B that extend to a global isometry of X.

Definition. Let A be a subspace of $\mathbb{U}$. We say that A is a *global subspace* if $\mathcal{E}(A) = \mathrm{Iso}(A)$, a *non-global subspace* if $\mathcal{E}(A) \subsetneq \mathrm{Iso}(A)$, or a *strongly non-global subspace* of $\mathbb{U}$ if $\mathcal{E}(A)$ is a meager subset of $\mathrm{Iso}(A)$.

Similarly, we say that a pair A, B of isometric subspaces of $\mathbb{U}$ is a *global pair* if $\mathcal{E}(A, B) = \mathrm{Iso}(A, B)$, a *non-global pair* if $\mathcal{E}(A, B) \subsetneq \mathrm{Iso}(A, B)$, or a *strongly non-global pair* in $\mathbb{U}$ if $\mathcal{E}(A, B)$ is a meager subset of $\mathrm{Iso}(A, B)$.

In Section 2.5, we give a "strongly negative" answer to both of our initial questions. The same results also follow for the Urysohn sphere $\mathbb{S}$ by adapting our methods in the bounded metric context.

Theorem 2.5.9. *Let $\mathbb{U}$ be the Urysohn space. Then, the generic subspace $F \in \mathcal{F}(\mathbb{U})$ as well as the generic pair A, B of subspaces of $\mathcal{F}(\mathbb{U})$ are strongly non-global.*

Before we develop the theory for the Urysohn space we undertake the task of answering the same questions in the countable setting, where instead of a metric space we consider ultrahomogeneous countable $\mathcal{L}$-structures $\boldsymbol{M}$ of some relational language $\mathcal{L}$. A structure $\boldsymbol{M}$ is called *ultrahomogeneous* if every isomorphism between finite substructures of $\boldsymbol{M}$ can be extended to a full automorphism of $\boldsymbol{M}$. The rationals with their natural ordering $(\mathbb{Q}, \leq)$ and the random graph $(\mathbb{G}, R)$ are two classical examples of countable ultrahomogeneous structures. Working in this new context we can ask the same questions if we first make the natural changes: we replace the word "subspace" with the word "substructure," the word "isometry" with the word "isomorphism," and we identify the space of all substructures of $\boldsymbol{M}$ with the Cantor space 2^M.

If $\boldsymbol{M}$ is a $\mathcal{L}$-structure, $\mathrm{Age}(\boldsymbol{M})$ denotes the class of all finite $\mathcal{L}$-structures that can be embedded in $\boldsymbol{M}$. Countable ultrahomogeneous structures are also called *Fraïssé structures* because each such structure $\boldsymbol{M}$ can be attained as a limit (the so called *Fraïssé limit*) over $\mathrm{Age}(\boldsymbol{M})$. The class $\mathrm{Age}(\boldsymbol{M})$ is called a *Fraïssé class* if $\boldsymbol{M}$ is ultrahomogeneous. This approach, introduced by Fraïssé in [Fra54], allowed the systematic study of infinite ultrahomogeneous structures $\boldsymbol{M}$ through the study of the combinatorial properties of the finite objects lying in $\mathrm{Age}(\boldsymbol{M})$.

Here, we will limit our study to structures which have no algebraicity; see [Hod93, Cam90], or Section 2.2 for a definition. One of the known consequences that we also derive here from Lemma 2.2.9 is that $\boldsymbol{M}$ has no algebraicity if and only if for a generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$ the structure $\boldsymbol{A}$ is isomorphic to $\boldsymbol{M}$. In Section 2.2 we will see that for a Fraïssé structure $\boldsymbol{M}$ without algebraicity the generic substructure of $\boldsymbol{M}$ as well as the generic pair of substructures of $\boldsymbol{M}$ is either global or strongly non-global. Moreover, we will reflect the dividing line of this dichotomy to the following, central in this paper, property of Fraïssé classes.

Definition 2.2.4. Let $\mathcal{K}$ be a Fraïssé class and let $\boldsymbol{C} \in \mathcal{K}$. We say that $\boldsymbol{C}$ *splits* $\mathcal{K}$ if for every $\boldsymbol{D} \in \mathcal{K}$ and for every embedding $i : \boldsymbol{C} \to \boldsymbol{D}$ there are structures $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathcal{K}$, embeddings $j_1 : \boldsymbol{D} \to \boldsymbol{D}_1$ and $j_2 : \boldsymbol{D} \to \boldsymbol{D}_2$ and a bijection $f : D_1 \to D_2$, such that:

- $f \circ j_1 = j_2$;
- $f{\restriction}_{D_1 \setminus C}$ is an isomorphism between $\langle D_1 \setminus C \rangle_{\boldsymbol{D}_1}$ and $\langle D_2 \setminus C \rangle_{\boldsymbol{D}_2}$;
- f is not an isomorphism between $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$.

We say that $\mathcal{K}$ *splits* if there is a $\boldsymbol{C} \in \mathcal{K}$ that splits $\mathcal{K}$. In the language of graphs, a typical example of a Fraïssé class $\mathcal{K}$ that splits is the age of the random graph and a typical example of a Fraïssé class that does not split is the age of the countable complete graph. The main result of Section 2.2 will be the following theorem.

Proposition 2.2.19. *Let $\boldsymbol{M}$ be a Fraïssé structure that has no algebraicity and let $\mathcal{K}$ be the corresponding Fraïssé class.*

1. *If $\mathcal{K}$ splits then the generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$ is a strongly non-global substructure and the generic pair $\boldsymbol{A}, \boldsymbol{B}$ in $\boldsymbol{M}$ is a strongly non-global pair.*

2. *If $\mathcal{K}$ does not split then the generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$ is a global substructure and the generic pair $\boldsymbol{A}, \boldsymbol{B}$ in $\boldsymbol{M}$ is a global pair.*

Structures $\boldsymbol{M}$ whose corresponding age $\mathcal{K}$ does not split seem to be simpler than the ones having age that splits. In Section 2.3, we present some structural consequences for the structures $\boldsymbol{M}$ that have age which does not split. Theorem 2.3.1 states that ω-stable Fraïssé limits with no algebraicity have ages that do not split. We also provide an example showing that the converse is not true. Theorem 2.3.3 is a structural result regarding automorphism groups of Fraïssé limits which have no algebraicity and an age that does not split.

The proofs of the main theorems of Section 2.2 and Section 2.5 use infinite games. In Section 2.1, we define the Banach Mazur game and we state the main result regarding this game that we are going to use. A short note on the Wijsman hyperspace topology is given in Chapter 2.4.

1.2 Complexity of orbit equivalence relations

Classification problems arise naturally in many areas of mathematics. Consider for example the problem of classifying all bounded selfadjoint operators $T \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ on a Hilbert space $\mathcal{H}$ up to unitary equivalence. If $\mathcal{H}$ is of finite dimension n, then we can assign to every selfadjoint operator T the tuple $f(T) = (\lambda_0, \ldots, \lambda_{n-1})$ of its increasingly ordered eigenvalues, counting repetition. The map $f\colon \mathcal{B}(\mathcal{H})_{\mathrm{sa}} \to \mathbb{R}^n$ has the property that T, S are unitarily equivalent if and only if $f(T) = f(S)$. Moreover, since f is computed by an explicit formula, one can recover the relation of unitary conjugacy of selfadjoint operators from the relation of equality on $\mathbb{R}^n$. The same is far from being true when $\mathcal{H}$ is infinite dimensional. In this case, there is no Borel map $f\colon \mathcal{B}(\mathcal{H})_{\mathrm{sa}} \to Y$, where Y is a Polish space, so that T, S are unitarily conjugate if and only if $f(T) = f(S)$; see [CN90]. The situation is actually much worse, as we discuss below.

Let E, F be two equivalence relations on a Polish spaces X, Y, respectively. A (E, F)-*homomorphism* is a map $f\colon X \to Y$ with the property $xEy \Rightarrow f(x)Ef(y)$. A (E, F)-homomorphism f is *a reduction* from E to F, if $xEy \Leftrightarrow f(x)Ef(y)$. Given E, F as above we are interested in the problem of whether there is a reduction f from E to F that is moreover Borel as a map from X to Y. In this case, E is said to be *Borel reducible* to F. Recall that E is *concretely classifiable* if E is Borel reducible to equality on some Polish space Y. One can associate to every countable language $\mathcal{L}$ the Polish space $\mathrm{Mod}(\mathcal{L})$ of all $\mathcal{L}$-structures with domain $\mathbb{N}$. For $x, y \in \mathrm{Mod}(\mathcal{L})$ we write $x \cong_{\mathcal{L}} y$ if x, y are isomorphic $\mathcal{L}$-structures. An equivalence relation E on a Polish space X is *classifiable by countable structures* if it is Borel reducible to $\cong_{\mathcal{L}}$ for some countable language $\mathcal{L}$. As we noted above the relation of unitary conjugacy of bounded selfadjoint operators on an infinite-dimensional Hilbert space is not concretely classifiable. This result was then strengthened by Kechris and Sofronidis [KS01], who proved that such a relation is not classifiable by countable structures. The same conclusions hold for unitary operators.

Particularly interesting is the case when the equivalence relation E on X is obtained as the *orbit equivalence relation* E_G^X of a continuous action of a Polish group G on X. In particular, all examples mentioned so far are of this form. For instance, unitary equivalence of selfadjoint operators is induced by the action of the unitary group $\mathcal{U}(\mathcal{H})$ on $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ by conjugation. Similarly, the relation $\cong_{\mathcal{L}}$ of isomorphism of $\mathcal{L}$-structures is induced by the canonical logic action of S_∞ on $\mathrm{Mod}(\mathcal{L})$. Finally, the equality relation on a Polish space is induced by the action of the trivial group. Both aforementioned anti-classification results stem from the careful study of which restrictions the topology on G puts on a G-space X. To be precise, consider the following general problem.

Problem 1.2.1. *Given a class of Polish groups $\mathcal{C}$, which dynamical conditions on a Polish G-space X ensure that the corresponding orbit equivalence relation is not Borel reducible to E_H^Y for some Borel action of a Polish group H in $\mathcal{C}$ on a Polish space Y?*

By [BK96, Corollary 5.1.6], one can assume in Problem 1.2.1 that the action of H on Y is continuous. The property of having meager orbits and a dense orbit provides such a criterion for the class $\mathcal{C}$ of *compact Polish groups* [Gao09, Proposition 6.1.10]. This is used in [CN90] to prove that bounded selfadjoint operators on an infinite-dimensional Hilbert space are not concretely classifiable up to unitary conjugacy. Hjorth's turbulence theory, initially developed by Hjorth in [Hjo00], addresses this problem in the case when $\mathcal{C}$ is the class of *non-Archimedean Polish groups.* Turbulence is used in [KS01] to prove that bounded selfadjoint operators on an infinite-dimensional Hilbert space are not classifiable by countable structures up to unitary conjugacy. Turbulence has played a key role in Borel complexity theory in the last two decades and it is to this day essentially the only known method to prove unclassifiability by countable structures; see [Hjo97, FTT14, FW04, Lup14, KLP15, KLP10, AM15a, AM15b, ST09a, ST09b, ST10, HL, TL12, IKT09, KTD12, Kec10, Gao09, Hjo00, Kec02, Far12]. There has been so far little progress into obtaining similar criteria for other interesting classes of Polish groups.

The purpose of Chapter 3 is two-fold. Our first goal is to introduce a game-theoretic approach to Problem 1.2.1. This approach consists in endowing the space X/G of orbits of a Polish G-space X with different graph structures, and then showing that a Baire measurable (E_G^X, E_H^Y)-homomorphism $f\colon X \to Y$ induces a graph homomorphism $X/G \to Y/G$ after restricting to an invariant dense G_δ set. This perspective allows us to give a short conceptual proof of Hjorth's turbulence theorem, avoiding the substantial amount of bookkeeping of

Hjorth's original argument [Hjo00]; see also [Gao09, Chapter 10].

Our second goal is to use the aforementioned game-theoretic approach to address Problem 1.2.1 for the class of *CLI groups*. Recall that a CLI group is a Polish group that admits a *compatible, complete, left-invariant metric*. It is easy to see that a Polish group is CLI, if and only if it admits compatible, complete, right-invariant metric. Every locally compact group, as well as every solvable Polish group—in particular, every abelian Polish group—is CLI [HS99, Corollary 3.7]. Such a class of groups has been considered in several papers so far. For instance, [Bec98, Corollary 5.C.6] settled the topological Vaught conjecture for CLI groups. It is also proved in [Bec98, Theorem 5.B.2] that CLI groups satisfy an analog of the Glimm-Effros dichotomy. In [Gao98, Theorem 1.1] it is shown that the non-Archimedean CLI groups are precisely the automorphism groups of countable structures whose Scott sentence does not have an uncountable model. The class of CLI groups has been further studied in [Mal11], where it is shown that if forms a coanalytic non-Borel subset of the class of Polish groups.

A fundamental tool in the study of dichotomies for orbit equivalence relations from [Bec98] is the notion of ι -embeddability for points in a Polish G-space. If X is a G-space then we say that ι-embeds in y, if there is a left-Cauchy sequence (g_n) in G so that $g_n x$ converges to y. Here we will work with right-embeddings—or what we call *Becker-embeddings*—instead: we say that x *Becker-embeds* in y, if there is right-Cauchy sequence (g_n) in G so that $g_n y$ converges to x. We prove that a Baire-measurable homomorphism between orbit equivalence relations necessarily preserves Becker embeddability on an invariant dense G_δ set. From this we extract in Theorem 3.3.4 a dynamical condition which answers Problem 1.2.1 for the class of CLI groups.

Theorem 3.3.4. *Suppose that X is a Polish G-space. If for any G-invariant dense G_δ subset C of X there exist $x, y \in C$ with different G-orbits such that x Becker-embeds in y, then for any G-invariant dense G_δ subset C of X the relation E_C^X is not CLI-classifiable.*

We then apply it to show that the Friedman-Stanley jump of equality $=^+$ is not Borel reducible to the orbit equivalence relation induced by a Borel action of a CLI group. The only proof of this fact that we are aware of relies on meta-mathematical reasoning and involves the theory of pinned equivalence relations; see [Kan08]. A natural reduction from this relation to the relations of unitary equivalence of bounded unitary or selfajdoint operators on an infinite-dimensional Hilbert space, shows that the latter relations are also not classifiable by the orbits of a CLI group actions. We note that it is still an open question whether the unitary group $\mathcal{U}(\mathcal{H})$ of the separable infinite dimensional Hilbert space $\mathcal{H}$ can produce under some action on a Polish space an orbit equivalence relation that is universal for orbit equivalence relations induced by continuous Polish group actions. Our results show that the complexity of possible orbit equivalence relations of $\mathcal{U}(\mathcal{H})$-

actions is not bounded from above by the complexity of orbit equivalence relations induced by continuous CLI group actions.

We conclude by discussing how all the results of the present paper admit natural generalizations from Polish group actions to Polish groupoids. Turbulence theory for Polish groupoids has been developed in [HL]. Applications of this more general framework to classification problems in operator algebras have also been presented in [HL].

In Section 1.2, Section 3.2 and Section 3.3 we present the results regarding Becker-embeddability and CLI groups. In Section 3.5 we present the proof of Hjorth's turbulence theorem within our context. Finally in Section 3.6 we recall the fundamental notions about Polish groupoids and explain how the main results of this paper can be adapted to this more general setting.

1.3 Reversing the arrows

Projective Fraïssé structures were introduced in [IS06] and since then they have been used by several authors in the study of the dynamics of various compact spaces such as the pseudo-arc, the Lelek fan and the Cantor space; see [BK13], [Kwi12], [Kwi14].

There is a standard process that is implemented in all these papers. One starts with a compact metrizable space Y where Y or $\mathrm{Homeo}(Y)$ is under investigation. Then one defines an appropriate class $\mathcal{K}$ of finite model theoretic $\mathcal{L}$-structures which "approximate" Y. If this class $\mathcal{K}$ satisfies the *projective Fraïssé axioms* then the projective Fraïssé limit $\boldsymbol{K}$ of $\mathcal{K}$ is uniquely defined and it is a compact, zero-dimensional, metrizable space K together with closed interpretations for the model-theoretic contained $\mathcal{L}$; see [IS06], or Section 4.2. In all cases above, whenever Y is not totally disconnected [IS06, BK13], the language $\mathcal{L}$ contains a special binary predicate $\mathfrak{r}$ whose interpretation in all finite structures of $\mathcal{K}$ is a reflexive, symmetric graph, and whose interpretation in $\boldsymbol{K}$ is a closed equivalence relation. Moreover, the space $K/\mathfrak{r}$ is homeomorphic to Y and the quotient projection map

$$K \mapsto Y$$

induces a continuous group embedding $\mathrm{Aut}(\boldsymbol{K}) \hookrightarrow \mathrm{Homeo}(Y)$ whose image is dense in $\mathrm{Homeo}(Y)$. This correspondence between $\boldsymbol{K}$ and Y allows one to study Y and $\mathrm{Homeo}(Y)$ through the combinatorial properties of the finite structures in $\mathcal{K}$.

The purpose of Chapter 4 is to turn these heuristics into a general theorem. The price we have to pay is that we have to introduce dual predicates and endow our structures further with dual semantics.

Dual predicates and structures were first introduced by S. Solecki in [Sol10, Sol12], where they were used to institute a uniform structural approach to various Ramsey theoretic results. Dual predicates in $\mathcal{L}$ quantify over dual tuples (finite clopen partitions). We will call a $\mathcal{L}$-structure purely dual if $\mathcal{L}$ contains only dual predicates. In Chapter 4.3 we use a standard orbit completion argument to get the following proposition.

Proposition 4.3.1. *Let G be a closed subgroup of* $\mathrm{Homeo}(K)$ *where K is zero-dimensional, compact, metrizable space. Then there is a purely dual projective Fraïssé structure $\boldsymbol{K}$ on domain K such that* $\mathrm{Aut}(\boldsymbol{K}) = G$.

Proposition 4.3.1 is essentially the dual of the statement that every closed subgroup of S_∞ is the automorphism group of a Fraïssé limit on domain $\mathbb{N}$.

In Section 4.4 we show how to turn any topological $\mathcal{L}$-structure into a purely dual one without losing any information. We also show that the above theorem is false if we do not allow our structures to support a dual structure. Therefore the context of dual structures is strictly more general than the context of direct structures.

In Section 4.5 we fix a special binary relation symbol $\mathfrak{r}$ whose interpretation will always be a reflexive and symmetric closed relation. This should be paralleled with the metric Fraïssé theory where the symbol d is reserved as a signifier for the metric. A formal relational language $\mathcal{L}$ will be decorated with the subscript $\mathfrak{r}$ whenever $\mathfrak{r}$ belongs to $\mathcal{L}$. Therefore, we always have $\mathfrak{r} \in \mathcal{L}_\mathfrak{r}$ and $\mathcal{L}_\mathfrak{r}$-structures are, in particular, reflexive $\mathfrak{r}$-graphs. We say that an $\mathcal{L}_\mathfrak{r}$-structure $\boldsymbol{K}$ is a pre-space if $\mathfrak{r}^{\boldsymbol{K}}$ is moreover transitive and therefore an equivalence relation. We apply Proposition 4.3.1 to get the following result.

Proposition 4.5.2. *Let G be a closed subgroup of* $\mathrm{Homeo}(Y)$*, for some compact metrizable space Y. Then there is a projective Fraïssé pre-space $\boldsymbol{K}$ such that $K/\mathfrak{r}^{\boldsymbol{K}}$ is homeomorphic to Y, and the quotient projection*

$$K \mapsto Y$$

induces a continuous group embedding $\mathrm{Aut}(\boldsymbol{K}) \hookrightarrow G$*, with dense image in G.*

We shall note here that R. Camerlo characterized all possible quotients $M/\mathfrak{r}^{\boldsymbol{M}}$ of Fraïssé structures in the language $\{\mathfrak{r}\}$ to be certain combinations of singletons, Cantor spaces and pseudo-arcs [Cam10].

Chapter 2

Extendability of automorphisms of generic substructures

2.1 The Banach Mazur game $G^{**}(E, X)$

Let X be a topological space and let A be a subset of X. The set A is *meager* if it is a countable union of nowhere dense in X sets. The collection of all meager sets of X forms a σ-ideal. Therefore, meager sets can be thought of as topologically small sets and their complements, the so called *comeager* sets, can be thought of as topologically large sets provided that the ambient space X is a Baire space. A topological space is a *Baire space* if no open subset U of X, with $U \neq \emptyset$, is meager. In what follows the ambient space X will always be a Polish space and Polish spaces are Baire. If X is a Baire space then we can equivalently define comeager sets as exactly those subsets of X which contain a dense in X, G_δ subset of X. A useful technique used to prove that a subset A of a topological space X is comeager involves an infinite game known as Banach Mazur game $G^{**}(A, X)$. Here, we are going to review in short the Banach Mazur game. For a more detailed exposure on the notion of meager and comeager sets, Baire spaces, Banach Mazur games as well as the proof of the main theorem of this chapter see [Kec95].

Let X be a topological space and let A be a subset of X. The *Banach Mazur game* $G^{**}(A, X)$ is a game played with 2 players, Player I and Player II. Player I starts by choosing an open subset U_0 of X and then Player II replies with an open subset V_0 of U_0. Then, Player I plays further a new open set U_1 with $U_1 \subset V_0$ and so on. The game continues this way with the two Players alternating turns and together defining a decreasing sequence of open sets. A run of the game looks as follows:

$$U_0 \supset V_0 \supset U_1 \supset V_1 \supset \ldots \supset U_m \supset V_m \supset \ldots,$$

and Player II wins this run of the game if and only if $\bigcap_n V_n (= \bigcap U_n) \subset A$. A winning strategy for Player II is roughly a pre-established rule that tells Player II which open set V_n to reply given an initial segment $(U_0, V_0, \ldots, U_n)$ of any possible run of the game and that, moreover, this rule leads always to victory for Player II. The following theorem is the main result that we are going to use regarding the $G^{**}(A, X)$ game.

Theorem 2.1.1 (Banach-Mazur, Oxtoby). *Let X be a nonempty topological space. Then A is comeager if and only if Player II has a winning strategy in $G^{**}(A, X)$.*

2.2 The countable discrete case

The main result of this section is Theorem 2.2.19. In what follows, $\mathcal{L}$ will always be a countable, relational language and $\boldsymbol{M}$ will always be a countable $\mathcal{L}$-structure. We write Age($\boldsymbol{M}$) for the class of all finite $\mathcal{L}$-structures that can be embedded in $\boldsymbol{M}$. We will use lightface letters for the subsets of the domain of $\boldsymbol{M}$ and boldface letters for the induced substructures. For example, M denotes the domain of $\boldsymbol{M}$ and for every $A \subset M$, we write $\boldsymbol{A} = \langle A \rangle_{\boldsymbol{M}}$ for the substructure of $\boldsymbol{M}$ generated by A. We will also use the notation $\boldsymbol{A}^c$ for the structure $\langle A^c \rangle_{\boldsymbol{M}}$. Notice that due to the fact that $\mathcal{L}$ is relational, we have a bijective correspondence between subsets of M and substructures of $\boldsymbol{M}$. Without any loss of generality we assume from now on that the domain of $\boldsymbol{M}$ is the set of natural numbers $\mathbb{N}$. We now have a natural bijective correspondence between substructures $\boldsymbol{A}$ of $\boldsymbol{M}$ and points in the Cantor space $\mathcal{C} = 2^{\mathbb{N}}$ given by the characteristic function χ_A of A.

The Polish group S_∞ is the group of all bijections of the domain of $\boldsymbol{M}$ endowed with the pointwise convergence topology. We denote with Aut($\boldsymbol{M}$), the group of all automorphisms of the structure $\boldsymbol{M}$. The group Aut($\boldsymbol{M}$) is a closed subgroup of S_∞ and therefore a Polish group inheriting the topology from S_∞ . If $\boldsymbol{A}, \boldsymbol{B}$ are substructures of $\boldsymbol{M}$, we write Iso($\boldsymbol{A}, \boldsymbol{B}$) to denote the space of all isomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. Again, endowed with the pointwise convergence topology, Iso($\boldsymbol{A}, \boldsymbol{B}$) is a Polish space.

An $\mathcal{L}$-structure $\boldsymbol{M}$ is called *ultrahomogeneous* if every isomorphism between finite substructures $\boldsymbol{A}, \boldsymbol{B}$ of $\boldsymbol{M}$ extends to a full automorphism of $\boldsymbol{M}$. Countable ultrahomogeneous structures are also known as *Fraïssé structures* or *Fraïssé limits*. We will further assume here that Fraïssé structures are always of non-finite cardinality. We will review some basic facts regarding Fraïssé structures. For a more detailed exposition someone may want to consult [Hod93]. If $\boldsymbol{M}$ is a Fraïssé structure and $\mathcal{K}$ = Age($\boldsymbol{M}$), then $\mathcal{K}$ has the following properties:

(i) Hereditary Property(HP): if $\boldsymbol{A} \in \mathcal{K}$ and $\boldsymbol{B}$ is a substructure of $\boldsymbol{A}$, then $\boldsymbol{B} \in \mathcal{K}$;

(ii) Joint Embedding Property(JEP): if $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, there is $\boldsymbol{C}$ in $\mathcal{K}$ such that both $\boldsymbol{A}$ and $\boldsymbol{B}$ embed in $\boldsymbol{C}$;

(iii) Amalgamation Property (AP): if $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ in $\mathcal{K}$ and $f : \boldsymbol{A} \to \boldsymbol{B}$, $g : \boldsymbol{A} \to \boldsymbol{C}$ embeddings, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $i : \boldsymbol{B} \to \boldsymbol{D}$, $j : \boldsymbol{C} \to \boldsymbol{D}$, such that $if = jg$;

(iv) every subclass of pairwise non-isomorphic structures of $\mathcal{K}$ is at most countable and

(v) $\mathcal{K}$ contains structures of arbitrary large, finite size.

If $\mathcal{K}$ is a class of finite $\mathcal{L}$-structures and has the properties (i)-(v), we say that $\mathcal{K}$ is a *Fraïssé class*. Fraïssé's theorem [Fra54] establishes the converse direction: if $\mathcal{K}$ is a Fraïssé class then there is an $\mathcal{L}$-structure $\boldsymbol{M} = \boldsymbol{M}(\mathcal{K})$, unique up to isomorphism such that $\boldsymbol{M}$ is countably infinite, ultrahomogeneous and $\mathcal{K} = \mathrm{Age}(\boldsymbol{M})$.

We say that an $\mathcal{L}$-structure $\boldsymbol{M}$ has no *algebraicity* if the pointwise stabilizer in $\mathrm{Aut}(\boldsymbol{M})$ of an arbitrary finite subset A of M has no finite orbits in its natural action on $M \setminus A$. Here we are going to work only with Fraïssé structures $\boldsymbol{M}$ that have no algebraicity. A Fraïssé structure $\boldsymbol{M}$ has no algebraicity if and only if the associated Fraïssé class $\mathcal{K}$ satisfies the strong amalgamation property defined as follows:

(SAP) we say that $\mathcal{K}$ has the *strong amalgamation property* if for every $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and $f : \boldsymbol{A} \to \boldsymbol{B}$, $g : \boldsymbol{A} \to \boldsymbol{C}$ embeddings there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $i : \boldsymbol{B} \to \boldsymbol{D}$, $j : \boldsymbol{C} \to \boldsymbol{D}$, such that $if = jg$ and

$$i(B) \cap j(C) = if(A) = jg(A).$$

For the interested reader a proof of this fact can be found in [Cam90].

Example 2.2.1. The list of countable ultrahomogeneous structures with no algebraicity includes the following examples.

- $\boldsymbol{M}_1 = (\mathbb{N})$, the empty-language, countable structure.
- $\boldsymbol{M}_2 = (\sqcup_{i \in \mathbb{N}} G_i, R)$, the disjoint union of countably many countable complete graphs ($\boldsymbol{M}_2 \models R(a, b)$ if and only if $a, b \in G_i$ for some i).
- $\boldsymbol{M}_3 = (\mathbb{G}, R)$, the random graph.
- $\boldsymbol{M}_4 = (\mathbb{Q}, \leq)$, the countable dense linear order without endpoints.
- $\boldsymbol{M}_5 = (Q\mathbb{U}, \{d_i\}_{i \in \mathbb{Q}^+})$, the rational Urysohn metric space.

Returning to the main question that concerns us here, notice that if we pick $\boldsymbol{M}$ to be any structure among $\boldsymbol{M}_2, \boldsymbol{M}_3, \boldsymbol{M}_4, \boldsymbol{M}_5$, and $\boldsymbol{N}$ some infinite substructure of $\boldsymbol{M}$, there are always ways of embedding $\boldsymbol{N}$ in $\boldsymbol{M}$ so that every automorphism of $\boldsymbol{N}$ extends to a global automorphism of $\boldsymbol{M}$ and ways of embedding $\boldsymbol{N}$ in $\boldsymbol{M}$ so that not every automorphism of $\boldsymbol{N}$ extends to a global automorphism of $\boldsymbol{M}$. Consider for example the random graph $\boldsymbol{M}_3 = (\mathbb{G}, R)$ and take $\boldsymbol{N}$ to be the structure that remains if we remove from $\mathbb{G}$ one point x. Using a back and forth system we can create an automorphism f of $\boldsymbol{N}$ that sends all points connected to x to the points not connected to x and vice versa. Then, f cannot be extended to an automorphism of $\boldsymbol{M}_3$. On the other hand, using the Katětov tower construction for graphs as done in [BM13], we can embed any countable graph $\boldsymbol{N}$ in $\boldsymbol{M}_3$ in such a way that every automorphism of $\boldsymbol{N}$ extends to an automorphism of $\boldsymbol{M}_3$.

Definition 2.2.2. *Let $\boldsymbol{A}, \boldsymbol{B}$ be two isomorphic substructures of $\boldsymbol{M}$. We write $\mathcal{E}(\boldsymbol{A})$ to denote the set of all self-isomorphisms of $\boldsymbol{A}$ that extend to a global isomorphism of $\boldsymbol{M}$ and by $\mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ we denote the set of all isomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ that extend to a global isomorphism of $\boldsymbol{M}$.*

We say that $\boldsymbol{A}$ is a global substructure *if $\mathcal{E}(\boldsymbol{A}) = \mathrm{Iso}(\boldsymbol{A})$, a* non-global substructure *if $\mathcal{E}(\boldsymbol{A}) \subsetneq \mathrm{Iso}(\boldsymbol{A})$, or a* strongly non-global substructure *if $\mathcal{E}(\boldsymbol{A})$ is a meager subset of* $\mathrm{Iso}(\boldsymbol{A})$. *Similarly, we say that the pair $\boldsymbol{A}, \boldsymbol{B}$ is a* global pair *if $\mathcal{E}(\boldsymbol{A}, \boldsymbol{B}) = \mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$, a* non-global pair *if $\mathcal{E}(\boldsymbol{A}, \boldsymbol{B}) \subsetneq \mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$, or a* strongly non-global pair *if $\mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ is a meager subset of* $\mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$.

Recall now that we have identified with $2^{\mathbb{N}}$ the space of all substructures of $\boldsymbol{M}$. We say that for a *generic substructure* $\boldsymbol{A}$ of $\boldsymbol{M}$ a certain property holds if and only if the set of all substructures $\boldsymbol{A}$ of $\boldsymbol{M}$ that have this property is a comeager subset of $2^{\mathbb{N}}$. Similarly, we say that for a *generic pair of substructures of* $\boldsymbol{M}$ a certain property holds if and only if the set of pairs $\boldsymbol{A}, \boldsymbol{B}$ of substructures of $\boldsymbol{M}$ that have this property form a comeager subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. In what follows, we are going to see that for a Fraïssé limit without algebraicity the generic substructure of $\boldsymbol{M}$ as well as the generic pair of substructures of $\boldsymbol{M}$ is either global or strongly non global. We are also going to reflect this dichotomy to the satisfiability or non-satisfiability of a certain property of the Fraïssé class $\mathcal{K}$ corresponding to $\boldsymbol{M}$. We begin by giving an example to offer some intuition regarding the forthcoming Definition 2.2.4.

Example 2.2.3. Let $\mathcal{K}_3$ be the Fraïssé class of all finite graphs. Let $\boldsymbol{D}$ be any finite graph and let $c \in D$. Let also $D_1 = D \cup \{w\}$, where $w \notin D$, and consider any graph $\boldsymbol{D}_1 \in \mathcal{K}_3$ with domain D_1 such that $\boldsymbol{D}_1 \restriction D = \boldsymbol{D}$. Notice that whatever $\boldsymbol{D}_1$ is chosen to be, we can find another graph $\boldsymbol{D}_2 \in \mathcal{K}_3$ on the same domain $D_2 = D_1$, such that:

- $\boldsymbol{D}_2 \restriction D = \boldsymbol{D}$;
- $\boldsymbol{D}_2 \restriction \big(D_2 \setminus \{c\}\big) = \boldsymbol{D}_1 \restriction \big(D_1 \setminus \{c\}\big)$, and
- $\boldsymbol{D}_2 \models R(c, w)$ if and only if $\boldsymbol{D}_1 \models \neg R(c, w)$.

This basically says that $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are not isomorphic, but the only way to witness this fact is by checking the relations between c and w. Notice, moreover, that the same is not true for the Fraïssé class $\mathcal{K}_2$ corresponding to $\boldsymbol{M}_2$ from Example 2.2.1 above: if $\boldsymbol{D}$ is any graph from $\mathcal{K}_2$ with $c, u \in D$ connected and $\boldsymbol{D}' \in \mathcal{K}_2$ is extending $\boldsymbol{D}$, then relationship between c and any new point w of $\boldsymbol{D}'$ is uniquely determined by the relation between u and w.

Definition 2.2.4. *Let $\mathcal{K}$ be a Fraïssé class, and let $\boldsymbol{C} \in \mathcal{K}$. We say that $\boldsymbol{C}$* splits *$\mathcal{K}$ if for every $\boldsymbol{D} \in \mathcal{K}$ and for every embedding $i : \boldsymbol{C} \to \boldsymbol{D}$ there are structures $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathcal{K}$, embeddings $j_1 : \boldsymbol{D} \to \boldsymbol{D}_1$ and $j_2 : \boldsymbol{D} \to \boldsymbol{D}_2$ and a bijection $f : D_1 \to D_2$, such that:*

- *$f \circ j_1 = j_2$;*
- *$f\restriction_{D_1 \setminus C}$ is an isomorphism between $\langle D_1 \setminus C\rangle_{\boldsymbol{D}_1}$ and $\langle D_2 \setminus C\rangle_{\boldsymbol{D}_2}$;*
- *f is not an isomorphism between $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$.*

Where, in order to keep the notation simple we write $D_1 \setminus C$ instead of $D_1 \setminus j_1(i(C))$, etc. We say that $\mathcal{K}$ splits *if there is a $\boldsymbol{C} \in \mathcal{K}$ that splits $\mathcal{K}$.*

Remark 2.2.5. If $\mathcal{K}_1, \ldots, \mathcal{K}_5$ are the Fraïssé classes that correspond to the structures $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_5$ of the Example 2.2.1 then $\mathcal{K}_3, \mathcal{K}_4$ and $\mathcal{K}_5$ split and $\mathcal{K}_1, \mathcal{K}_2$ do not split.

Let $\boldsymbol{M} = \boldsymbol{M}(\mathcal{K})$ be the Fraïssé structure associated to $\mathcal{K}$ and let X be a finite subset of M. We denote by $\mathcal{L}_X$ the language obtained by adding to $\mathcal{L}$ a constant c_x for every $x \in X$.

Definition 2.2.6. *Let $\boldsymbol{M}$ be a Fraïssé structure and let $\boldsymbol{X}$ be a finite substructure of $\boldsymbol{M}$. By a* realized quantifier free type *(rqf-type) $p = p(y)$ over $\boldsymbol{X}$ we mean a set of quantifier free $\mathcal{L}_X$-formulas ϕ in one variable y for which there is a $z \in M$ such that*

$$\phi \in p \Leftrightarrow \boldsymbol{M} \models \phi(z).$$

As a slight abuse of notation, we will not exclude the possibility of X being the empty set in this definition. Finally, we say that the rqf-type p over $\boldsymbol{X}$ is non-trivial *if for all $x \in X$, the formula $\phi(y) \equiv (y = c_x)$ does not belong to p.*

Notice that if $\boldsymbol{M}$ is a Fraïssé structure then p is a rqf-type over a finite substructure $\boldsymbol{X}$ of $\boldsymbol{M}$ if and only if $\langle X, z\rangle_M \in \mathcal{K}$. So, it makes sense to talk about realized quantifier free types over a finite structure $\boldsymbol{X}$, whenever $\boldsymbol{X} \in \mathcal{K}$.

Definition 2.2.7. *Let $\boldsymbol{X}, \boldsymbol{X}' \in \mathcal{K}$ be $\mathcal{L}$-structures of the same size, and let $f : X \to X'$ be a bijection between their domains. Let also $p = p(y)$ be a realized quantifier free type over $\boldsymbol{X}$. We define $f[p] = f[p](y)$ to be following set of $\mathcal{L}_{X'}$-formulas:*

$$\varphi(y, c_{x_1}, \ldots, c_{x_n}) \in f[p] \;\Leftrightarrow\; \varphi(y, c_{f^{-1}(x_1)}, \ldots, c_{f^{-1}(x_n)}) \in p.$$

Notice that if p is a rqf-type and f is an isomorphism between $\boldsymbol{X}$ and $\boldsymbol{X}'$, then the set of quantifier free formulas $f[p]$ is a rqf-type over $\boldsymbol{X}'$.

Definition 2.2.8. *Let $\boldsymbol{A}$ be a substructure of a Fraïssé structure $\boldsymbol{M}$. We say that $\boldsymbol{A}$* absorbs points *if for every finite subset X of M and for every non-trivial rqf-type p over $\boldsymbol{X}$ there is an $a \in A$ such that $\boldsymbol{M} \models p(a)$.*

Notice that if $\boldsymbol{A}$ absorbs points in $\boldsymbol{M}$ then A is not empty. In particular, A is infinite and $\boldsymbol{A}$ is isomorphic to $\boldsymbol{M}$.

For the general Fraïssé structure we cannot hope that we can find even one subset A of M such that both $\boldsymbol{A}$ and $\boldsymbol{A}^c$ absorb points. For example, take any Fraïssé structure $\boldsymbol{M}$. Extend the language $\mathcal{L}$ to $\mathcal{L}' = \mathcal{L} \cup \{u\}$ so that it includes a new unary predicate u and turn $\boldsymbol{M}$ into a $\mathcal{L}'$ structure $\boldsymbol{M'}$ by letting for some $x_0 \in M$ the following:

$$\boldsymbol{M'} \models u(x_0) \quad \text{and} \quad \boldsymbol{M'} \models \forall x\big((x \neq x_0) \Rightarrow \neg u(x)\big).$$

This new structure is a Fraïssé limit of a new class $\mathcal{K}'$, but for no subset A of M' both $\boldsymbol{A}$ and $\boldsymbol{A}^c$ absorb points since then both A and A^c should contain a point that satisfies u. However, if we assume that our Fraïssé structure has no algebraicity or equivalently if the corresponding Fraïssé class has SAP, then we get the following result.

Lemma 2.2.9. *Let $\boldsymbol{M}$ be the Fraïssé limit of the Fraïssé class $\mathcal{K}$. Assume moreover that $\mathcal{K}$ has* SAP*. Then, for a generic substructure $\boldsymbol{A}$ we have that both $\boldsymbol{A}$ and $\boldsymbol{A}^c$ absorb points.*

Proof. Recall that we identify the domain M with the set of natural numbers. We will show that the set $\mathcal{A}$, of all the subsets A of M for which both $\boldsymbol{A}$ and $\boldsymbol{A}^c$ absorb points, is a dense G_δ subset of 2^M. Let

$$\mathcal{I} = \{\, (X, p) \,:\, X \subset M, \text{ finite}, \ p \text{ a non trivial rqf-type over } \boldsymbol{X}\},$$

and notice that $\mathcal{I}$ is countable. Let $\{i_m : m \in \mathbb{N}\}$ be an enumeration of $\mathcal{I}$. For fixed $i = (X, p)$, let N_i be the subset of M, of all elements n such that $\boldsymbol{M} \models p(n)$. We have that

$$\mathcal{A} = \bigcap_{i \in \mathcal{I}} \ \bigcup_{\substack{n \neq m \\ n, m \in N_i}} \big\{x \in 2^M : x(n) = 1, x(m) = 0\big\}.$$

Therefore $\mathcal{A}$ is a G_δ subset of 2^M. To see that $\mathcal{A}$ is also dense in 2^M, notice that since $\boldsymbol{M}$ has no algebraicity, for every finite substructure $\boldsymbol{X}$ of $\boldsymbol{M}$ and every rqf-type p over $\boldsymbol{X}$ there are infinitely many points $a \in M$ with $\boldsymbol{M} \models p(a)$. □

Corollary 2.2.10. *Let $\boldsymbol{M}$ be a Fraïssé structure. Then, $\boldsymbol{M}$ has no algebraicity if and only if for a generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$, the structure $\boldsymbol{A}$ is isomorphic to $\boldsymbol{M}$.*

Proof. First assume that $\boldsymbol{M}$ has no algebraicity. Then, using the fact that the generic $\boldsymbol{A}$ absorbs points (Lemma 2.2.9) we built a back and forth system between $\boldsymbol{A}$ and $\boldsymbol{M}$.

For the converse, notice that if $\boldsymbol{M}$ has algebraicity then there is a finite subset B of M and some $a \in M \setminus B$ so that the orbit of a under the pointwise stabilizer $\mathrm{Aut}(\boldsymbol{M})_B$ of B is finite. Let now $\mathcal{B} \subset 2^M$ be the set of all subsets of M that include B but exclude every point lying in the aforementioned orbit. Then $\mathcal{B}$ is an open subset of 2^M and every substructure $\boldsymbol{A}_x$ of $\boldsymbol{M}$ induced by the support of an $x \in \mathcal{B}$ cannot be isomorphic to $\boldsymbol{M}$. □

It is also immediate from Lemma 2.2.4 above and the fact that Cartesian product of comeager sets is comeager that under the assumptions of Lemma 2.2.4, for a generic pair $\boldsymbol{A}, \boldsymbol{B}$, of substructures of $\boldsymbol{M}$, all $\boldsymbol{A}, \boldsymbol{A}^c, \boldsymbol{B}, \boldsymbol{B}^c$ absorb points.

A *partial isomorphism* of $\boldsymbol{M}$ is a map $f : N \to M$ with $N \subset M$ which happens to be an isomorphism between $\boldsymbol{N}$ and $\langle f(N) \rangle_{\boldsymbol{M}}$. We will write $\mathrm{dom}(f)$ to denote the domain of f. We say that f is a *finite partial isomorphism* if f is a partial isomorphism with finite domain. If f_1, f_2 are two partial isomorphisms of $\boldsymbol{M}$, we say that f_1 and f_2 are *compatible* if there is a partial isomorphism f of $\boldsymbol{M}$ that extends both f_1 and f_2.

Lemma 2.2.11. *Let $\mathcal{K}$ be a Fraïssé class that splits and has the* SAP *and let $\boldsymbol{M} = \boldsymbol{M}(\mathcal{K})$ be the Fraïssé limit of $\mathcal{K}$. Let also $A, B \subset M$ such that all $\boldsymbol{A}, \boldsymbol{A}^c$, and $\boldsymbol{B}, \boldsymbol{B}^c$, absorb points. Then, $\mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ is a meager subset of* $\mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$.

Proof. Let A, A^c and B, B^c as above and let $\mathcal{N} = \mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B}) \setminus \mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$. We will show that there is a winning strategy for Player II in the Banach-Mazur game $G^{**}\big(\mathcal{N}, \mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})\big)$. Therefore, by Theorem 2.1.1, we will have that $\mathcal{N}$ is comeager subset of $\mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$.

Take $\boldsymbol{C} \in \mathcal{K}$ such that $\boldsymbol{C}$ splits $\mathcal{K}$. Since $\boldsymbol{A}^c$ absorbs points we can realize $\boldsymbol{C}$ inside $\boldsymbol{A}^c$. Let $\boldsymbol{C}_*$ be any such a realization. Let also $\{g_k : k \in \mathbb{N}\}$ be an enumeration of all finite partial isomorphisms of $\boldsymbol{M}$ which are compatible with some $h \in \mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ and whose domain includes C_*. Obviously, every $h \in \mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ is compatible with some g_k in the above list. Player II will pick his moves so that no matter what Player I does the resulting map h of the play will belong in $\mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$ and, moreover, h will not be compatible with any g_k. Therefore by the above observation, h will belong in $\mathcal{N}$.

For the first task, notice that by incorporating additionally in the moves of Player II a "back and forth" system between $\boldsymbol{A}$ and $\boldsymbol{B}$ we can assume without the loss of generality that the result of the play will indeed be an isomorphism h from $\boldsymbol{A}$ to $\boldsymbol{B}$. Assume now that the game is in its n-th step, with $n \geq 0$, and Player I has played an open set $U_n \subset \mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$ which is identified with an partial isomorphism h_n between finite substructures of $\boldsymbol{A}$ and $\boldsymbol{B}$. Player II will proceed as follows: let k_n be the minimum index so that g_{k_n} is compatible with h_n and let g be any finite partial isomorphism compatible with some $h \in \mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ so that g extends both g_{k_n} and h_n. Let D be the domain of g and notice that $C_* \subset D$. By Definition 2.2.4, and because $\boldsymbol{C}$ splits $\mathcal{K}$, there are structures $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathcal{K}$, embeddings $j_1 : \boldsymbol{D} \to \boldsymbol{D}_1$ and $j_2 : \boldsymbol{D} \to \boldsymbol{D}_2$ and a bijection $f : D_1 \to D_2$, such that:

- $f \circ j_1 = j_2$;
- $f\restriction_{D_1 \setminus C}$ is an isomorphism between $\langle D_1 \setminus C\rangle_{\boldsymbol{D}_1}$ and $\langle D_2 \setminus C\rangle_{\boldsymbol{D}_2}$;
- f is not an isomorphism between $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$.

Since A absorbs points we can extend D to $\tilde{D}_1 \subset M$ so that $\tilde{\boldsymbol{D}}_1 \simeq \boldsymbol{D}_1$ and all points of $\tilde{D}_1 \setminus D$ lie inside A. Similarly, since $\langle g(D)\rangle_{\boldsymbol{M}} \simeq \boldsymbol{D}$ and since B absorbs points we can extend $g(D)$ to $\tilde{D}_2 \subset M$ so that $\tilde{\boldsymbol{D}}_2 \simeq \boldsymbol{D}_2$ and all points of $\tilde{D}_2 \setminus g(D)$ lie inside B. The function f can be now realized as a bijection $\tilde{f} : \tilde{D}_1 \to \tilde{D}_2$ which extends g. The function $\tilde{f}$ is not a partial isomorphism of $\boldsymbol{M}$, however, if E is any subset of the domain of $\tilde{f}$ that excludes C_*, $\tilde{f}\restriction_E$ is a partial isomorphism of $\boldsymbol{M}$.

Player II will now reply in his n-th round with the open set V_n, given by the partial isomorphism $\tilde{h}_n = \tilde{f}\restriction_{\mathrm{dom}(\tilde{f}) \cap A}$. Notice that any extension of $\tilde{h}_n$ to an $h \in \mathcal{E}(\boldsymbol{A}, \boldsymbol{B})$ is not compatible with g_{k_n}. Hence, the game will end with an isomorpism $h = \cup h_n = \cup \tilde{h}_n$ between $\boldsymbol{A}$ and $\boldsymbol{B}$, which cannot be further extended to include C_* in its domain and therefore $h \in \mathcal{N}$. □

Together with the Lemma 2.2.9, Lemma 2.2.11 proves the one direction of Theorem 2.2.19. For the other direction we need first some lemmas.

Let $\boldsymbol{M}$ be the Fraïssé limit of a class $\mathcal{K}$ that does not split. Then, for every $c \in M$ there is a finite $D \subset M$ with $c \in D$ such that for every finite $D_1, D_2 \supset D$ and every bijection $f : D_1 \to D_2$ we have that: if $f\restriction_{D_1/\{c\}}$ is an isomorphism between $\langle D_1/\{c\}\rangle_{\boldsymbol{M}}$ and $\langle D_2/\{c\}\rangle_{\boldsymbol{M}}$, then f is an isomorphism between $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$.

In other words, for every $c \in M$ there is a finite $K \subset M$ ($K = D/\{c\}$ above) so that the rqf-type of c over $\boldsymbol{K}$ completely determines the rqf-type of c over any finite extension $\boldsymbol{F}$ of $\boldsymbol{K}$. It will be convenient to settle on the following definition.

Definition 2.2.12. *Let $\boldsymbol{M}$ be a Fraïssé structure, $c \in M$ and $K \subset M$ finite with $c \notin K$. We say that $\boldsymbol{K}$* controls *c if for every $F_1, F_2 \subset M$ finite with $K \subset F_1, F_2$ and every bijection $f : F_1 \cup \{c\} \to F_2 \cup \{c\}$ with $f\restriction_{K \cup \{c\}} = \mathrm{id}$ we have that: if $f\restriction_{F_1}$ is an isomorphism between $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ then f is an isomorphism between $\langle F_1 \cup \{c\}\rangle_{\boldsymbol{M}}$ and $\langle F_2 \cup \{c\}\rangle_{\boldsymbol{M}}$.*

In the following lemma we record some trivial facts regarding the above notion.

Lemma 2.2.13. *Let $\mathcal{K}$ be a Fraïssé class that does not split and let $\boldsymbol{M}$ be the Fraïssé limit of $\mathcal{K}$. Let also $c \in M$. Then:*

1. *there is a finite $K \subset M$ so that $\boldsymbol{K}$ controls c;*

2. *if $\boldsymbol{K}$ controls c, L is a finite subset of M with $K \subset L$ and $c \notin L$ then $\boldsymbol{L}$ also controls c;*

3. *if $\boldsymbol{K}$ controls c and $f \in \mathrm{Aut}(\boldsymbol{M})$, then $\langle f(K) \rangle_{\boldsymbol{M}}$ controls $f(c)$.*

Proof. All three statements follow directly from the definition 2.2.12. □

Lemma 2.2.14. *Let $\mathcal{K}$ be a Fraïssé class that does not split and let $\boldsymbol{M}$ be the Fraïssé limit of $\mathcal{K}$. Assume $F \subset M$ is finite with $K \subset F$ so that $\boldsymbol{K}$ controls the points $c_1, \ldots, c_n \in F^c$. Let p_i be the rqf-type of c_i over $\boldsymbol{K}$. If $f : F \cup \{c_1, \ldots, c_n\} \to M$ is an injective map with so that $f \restriction F$ in an embedding of $\boldsymbol{F}$ in $\boldsymbol{M}$ and the rqf-type of $f(c_i)$ over $f(K)$ is $f[p_i]$. Then f is also an embedding.*

Proof. We will prove this by induction. For $n = 1$, let $g \in \mathrm{Aut}(\boldsymbol{M})$ with $g\restriction_{K \cup \{c_1\}} = f\restriction_{K \cup \{c_1\}}$. The map $h : F \cup \{c_1\} \to g^{-1} f\big(F \cup \{c_1\}\big)$ is an injection fixing $K \cup \{c_1\}$ with $h\restriction F$ being an isomorphism. Since $\boldsymbol{K}$ controls c_1 the map h (and therefore the map f) is an isomorphism.

Assume now that the statement holds for every n with $n \leq k$ and let $f : F \cup \{c_1, \ldots, c_k, c_{k+1}\} \to M$. By the inductive hypothesis we can enlarge F to include $\{c_1, \ldots, c_k\}$, reducing the problem again to the $n = 1$ case. □

For $a, b \in M$ we say that a is *equivalent* to b and we write $a \sim_{\boldsymbol{M}} b$ if there is a finite $K \subset M$ such that $\boldsymbol{K}$ controls a and a, b have the same rqf-type over $\boldsymbol{K}$. Notice then that as a consequence of Lemma 2.2.13(3), $\boldsymbol{K}$ controls b as well.

Lemma 2.2.15. *Let $\mathcal{K}$ be a Fraïssé class that does not split and let $\boldsymbol{M}$ be the Fraïssé limit of $\mathcal{K}$. Let also $a, b \in M$ with $a \sim_{\boldsymbol{M}} b$. Then, for every finite subset F of M with $a, b \notin F$, the points a and b share the same rqf-type over $\boldsymbol{F}$.*

Proof. Let $f : F \cup \{a, b\} \to F \cup \{a, b\}$ be the function that fixes F and exchanges a with b and use Lemma 2.2.14. □

In the additional presence of SAP we now have the following results.

Lemma 2.2.16. *Let $\mathcal{K}$ be a Fraïssé class with* SAP *that does not split and let $\boldsymbol{M}$ be the Fraïssé limit of $\mathcal{K}$. Let also $c_1, \ldots c_n \in M$. Then there is $K \subset M$ so that $\boldsymbol{K}$ controls c_i for every $i \in \{1, \ldots, n\}$.*

Proof. From Lemma 2.2.13(1) let $K_i \subset M$ so that $\boldsymbol{K}_i$ that controls c_i. Using SAP and Lemma 2.2.13(3) we can arrange $K_1, \ldots, K_n$ in such a way that $K_1, \ldots, K_n, \{a, b, c\}$ are all pairwise disjoint. Let $K = \bigcup_{i=1}^n K_i$ $\square$

Corollary 2.2.17. *Let $\mathcal{K}$ be a Fraïssé class with* SAP *that does not split and let $\boldsymbol{M}$ be the Fraïssé limit of $\mathcal{K}$. Then, $\sim_{\boldsymbol{M}}$ is an equivalence relation on M.*

Proof. Reflexivity and symmetry follow directly from the definition. Transitivity, follows from Lemma 2.2.16 and Lemma 2.2.15. $\square$

Lemma 2.2.18. *Let $\mathcal{K}$ be a Fraïssé class with* SAP *that does not split and let $\boldsymbol{M}$ be the Fraïssé limit of $\mathcal{K}$. Let also $A, B \subset M$ such that all $\boldsymbol{A}, \boldsymbol{A}^c$ and $\boldsymbol{B}, \boldsymbol{B}^c$ absorb points. Then, every g in* $\mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$ *can be extended to an automorphism $\tilde{g} \in \mathrm{Aut}(\boldsymbol{M})$.*

Proof. Let $\{N_i : i \in I\}$ be an enumeration of all equivalence classes of $\sim_{\boldsymbol{M}}$ in M and notice that since all $\boldsymbol{A}, \boldsymbol{A}^c, \boldsymbol{B}, \boldsymbol{B}^c$ absorb points the sets $N_i \cap A, N_i \cap A^c, N_i \cap B, N_i \cap B^c$ are all infinite for every $i \in I$.

Let now $c \in A^c$ and pick $K \subset M$ so that $\boldsymbol{K}$ controls c. Using Lemma 2.2.13(3) and the fact that $\boldsymbol{A}$ absorbs points we can assume that $K \subset A$. Notice then that by Lemma 2.2.15 $\boldsymbol{K}$ controls every other point $c' \in M$ with $c \sim_{\boldsymbol{M}} c'$ and $c' \notin K$. In particular, if $c \in N_i$ for some $i \in I$, $\boldsymbol{K}$ controls every point $c' \in N_i \cap A^c$ as well as cofinitely many points $c' \in N_i \cap A$. So, for every $i \in I$ we can pick a finite subset K_i of A and some $a_i \in N_i \cap A$ so that $\boldsymbol{K}_i$ controls a_i as well as every $c \in N_i \cap A^c$.

Given now any $g \in \mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$ we have by Lemma 2.2.13(3) that $\langle g(K_i) \rangle_{\boldsymbol{M}}$ controls $g(a_i)$. For every $i \in I$, let gi be the unique $j \in I$ with $g(a_i) \in N_j$ and pick a bijection $h_i : N_i \cap A^c \to N_{gi} \cap B^c$. We extend g to an automorphism $\tilde{g} \in \mathrm{Aut}(\boldsymbol{M})$ setting $\tilde{g}(c) = h_i(c)$ whenever $c \in A^c$ with $c \in N_i$. To see that $\tilde{g}$ is indeed an automorphism, notice that by Lemma 2.2.14 the restriction of $\tilde{g}$ to any finite substructure is a partial isomorphism. $\square$

Theorem 2.2.19. *Let $\boldsymbol{M}$ be a Fraïssé structure that has no algebraicity and let $\mathcal{K}$ be the corresponding Fraïssé class.*

1. *If $\mathcal{K}$ splits, then the generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$ is a strongly non-global substructure and the generic pair $\boldsymbol{A}, \boldsymbol{B}$ in $\boldsymbol{M}$ is a strongly non-global pair.*

2. *If $\mathcal{K}$ does not split, then the generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$ is a global substructure and the generic pair $\boldsymbol{A}, \boldsymbol{B}$ in $\boldsymbol{M}$ is a global pair.*

Proof. We have from Lemma 2.2.9 that for a generic substructure $\boldsymbol{A}$ of $\boldsymbol{M}$, both $\boldsymbol{A}, \boldsymbol{A}^c$ absorb points and that for a generic pair $\boldsymbol{A}, \boldsymbol{B}$ in $\boldsymbol{M}$, all $\boldsymbol{A}, \boldsymbol{A}^c, \boldsymbol{B}, \boldsymbol{B}^c$ absorb points. The result follows from Lemma 2.2.11 in case that $\mathcal{K}$ splits and from Lemma 2.2.18 in case that $\mathcal{K}$ does not split. $\square$

2.3 Some structural consequences

Let $\mathcal{K}$ be a Fraïssé class with SAP, and let $\boldsymbol{M}$ be the corresponding Fraïssé limit. If A is a subset of M, we denote with $S_n^{\boldsymbol{M}}(A)$ the Stone space of all (complete) n-types over A. For every finite subset C of M we denote by $\mathrm{tp}(C|A)$ the type of C over A. Let now $C_0 \subset M$ so that $\boldsymbol{C}_0$ splits $\mathcal{K}$ and let n be the size of C_0. The fact that $\boldsymbol{C}_0 \in \mathcal{K}$ splits $\mathcal{K}$ can be rephrased as follows: for every finite $K \subset M$ with $K \cap C_0 = \emptyset$ there is a finite $F \supset K$ and a second copy $\boldsymbol{C}_1$ of $\boldsymbol{C}_0$ in $\boldsymbol{M}$ such that $F \cap C_0 = \emptyset$, $F \cap C_1 = \emptyset$ and $\mathrm{tp}(C_0|F) \neq \mathrm{tp}(C_1|F)$. Iterating this fact we produce a Cantor scheme in the compact metric space $S_n^{\boldsymbol{M}}(M)$ which results in an embedding of the Cantor set $2^{\mathbb{N}}$ into $S_n^{\boldsymbol{M}}(M)$. We have just proved the following proposition.

Proposition 2.3.1. *Let $\mathcal{K}$ be a Fraïssé class with SAP. If the corresponding Fraïssé limit $\boldsymbol{M}$ is ω-stable, then $\mathcal{K}$ does not split.*

Someone would hope that the above result could turn into a characterization of ω-stability for Fraïssé limits without algebraicity. This, however, is not the case as the following example exhibits.

Example 2.3.2. Let $\mathcal{L} = \{R, S\}$ where R, S are both binary relational symbols and let M be the disjoint union of a countable family of countable sets $\{N_i : i \in \mathbb{N}\}$. We define $\boldsymbol{M}$ to be an $\mathcal{L}$-structure with domain M, where the symbols of $\mathcal{L}$ are interpreted as follows. For S, let

$$\boldsymbol{M} \models S(a,b) \Leftrightarrow \exists i \in \mathbb{N} \;\; a, b \in N_i$$

To interpret R, equip first the set of indices $\{i : i \in \mathbb{N}\}$ with a structure $\mathbb{G}$ isomorphic to the random graph in the language $\mathcal{L} = \{R'\}$ of one binary symbol. Let

$$\boldsymbol{M} \models R(a,b) \quad \Leftrightarrow \quad a \in N_i, \;\; b \in N_j \;\text{ and }\; \mathbb{G} \models R'(i,j)$$

It is not difficult to see that $\boldsymbol{M}$ is a Fraïssé limit without algebraicity. For every $a \in M$, a is controlled by $\langle b \rangle_{\boldsymbol{M}}$ for any $b \neq a$ that lies in the same N_i with a. From that if follows that the corresponding Fraïssé class $\mathcal{K}$ does not split. Notice, however, that $\boldsymbol{M}$ is not ω-stable because the structure $\mathbb{G}$ of the random graph is not ω-stable.

We will now see that Example 2.3.2 is an archetype of how Fraïssé limits of classes $\mathcal{K}$ which have SAP and do not split look like. Recall that in Section 2.2 we defined a relation $\sim_{\boldsymbol{M}}$ between points a Fraïssé limit $\boldsymbol{M}$ whose Fraïssé class $\mathcal{K}$ does not split. If moreover $\mathcal{K}$ has SAP, we proved in Lemma 2.2.17 that $\sim_{\boldsymbol{M}}$ is an equivalence relation on M. Another useful observation is that $\sim_{\boldsymbol{M}}$ is $\mathrm{Aut}(\boldsymbol{M})$-invariant. This follows directly from Lemma 2.2.13(3).

Let $\{N_i : i \in I\}$ be the partition of M into the equivalence classes of $\sim_{\boldsymbol{M}}$, where I is a countable (possibly finite) set of indices. From the fact that $\mathcal{K}$ has SAP, it is straight forward that for each $i \in I$, N_i is infinite. Since $\sim_{\boldsymbol{M}}$ is $\mathrm{Aut}(\boldsymbol{M})$-invariant, we have a natural action of $\mathrm{Aut}(\boldsymbol{M})$ on the set of indices I: for every $i, j \in I$ let

$$g \cdot i = j \quad \Leftrightarrow \quad \exists a \in N_i \;\; g(a) \in N_j \quad \Leftrightarrow \quad \forall a \in N_i \;\; g(a) \in N_j.$$

Let G_0 be the kernel of the action $\mathrm{Aut}(\boldsymbol{M}) \curvearrowright I$ and let $H = \mathrm{Aut}(\boldsymbol{M})/G_0$. H is a subgroup of S_I, group of all permutations on the set I. From the analysis above it follows that the automorphism group $\mathrm{Aut}(\boldsymbol{M})$ is a subgroup of the unrestricted Wreath product $G = \mathrm{S}_\infty \operatorname{Wr}_I H = \big(\prod_{i \in I} \mathrm{S}_\infty\big) \rtimes H$ where the i-th copy of S_∞ is the permutation group of N_i; see for example [Cam90]. Moreover, by Lemma 2.2.14 it follows that $\mathrm{Aut}(\boldsymbol{M})$ lies densely in G. Therefore since $\mathrm{Aut}(\boldsymbol{M})$ is a closed subgroup of S_∞, the groups $\mathrm{Aut}(\boldsymbol{M})$ and G are actually equal.

So, we have shown that if $\boldsymbol{M}$ is the Fraïssé limit of a class $\mathcal{K}$ that has SAP and does not split then $\mathrm{Aut}(\boldsymbol{M}) = \big(\prod_{i \in I} \mathrm{S}_\infty\big) \rtimes H$ where H is a subgroup of S_I and I is finite or countably infinite. Notice also that H is a closed subgroup of S_I. Working now towards the opposite direction, let I be a finite or countably infinite set and let H be a closed subgroup of S_I. Let $\boldsymbol{I}_H$ be the canonical $\mathcal{L}_H$-structure with domain I where the language $\mathcal{L}_H = \{R^n_j\}$ has one distinct n-ary symbol R^n_j for each orbit $\mathcal{O}_j$ of the action of H on I^n; see for example [Cam90].

Let $\mathcal{L} = \mathcal{L}_H \cup \{S\}$, where S is a new binary symbol and let $M = I \times \mathbb{N}$. Consider the $\mathcal{L}$-structure $\boldsymbol{M}_H$ on M, where the interpretation is done as follows.

For every for every $R^n_j \in \mathcal{L}_H$ we have

$$\boldsymbol{M}_H \models R^n_j\big((i_1, m_1), \ldots, (i_n, m_n)\big) \quad \Leftrightarrow \quad \boldsymbol{I}_H \models R^n_j(i_1, \ldots, i_n)$$

and

$$\boldsymbol{M}_H \models S\big((i_1, m_1), (i_2, m_2)\big) \quad \Leftrightarrow \quad i_1 = i_2.$$

Let also $\mathcal{K}_H = \mathrm{Age}(\boldsymbol{M}_H)$. It is easy to check that $\boldsymbol{M}_H$ is a countable, ultrahomogeneous structure without algebraicity and that $\mathrm{Aut}(\boldsymbol{M}_H) = \big(\prod_{i \in I} \mathrm{S}_\infty\big) \rtimes H$. To see that $\mathcal{K}_H$ does not split let $C \subset M$ finite with $C = \{(i_1, m_1), \ldots, (i_k, m_k)\}$ where, some ordered couples might share the same index $i \in I$. Let $m = \max\{m_j + 1 : 1 \leq j \leq k\}$ and let $K = \{(i_1, m), \ldots, (i_k, m)\}$. Then, if $D = K \cup C$, the inclusion embedding $i : \boldsymbol{C} \to \boldsymbol{D}$ shows that $\boldsymbol{C}$ does not split $\mathcal{K}$.

We collect in the following theorem the above results.

Theorem 2.3.3. *Let $\boldsymbol{M}$ be a Fraïssé structure without algebraicity. If* $\mathrm{Age}(\boldsymbol{M})$ *does not split then* $\mathrm{Aut}(\boldsymbol{M}) = \left(\prod_{i\in I} \mathrm{S}_\infty\right) \rtimes H$*, where I is a countable possibly finite set of indices, H is a closed subgroup of* S_I*, and H acts on I in the natural way. Moreover, if G is a group isomorphic to* $\left(\prod_{i\in I} \mathrm{S}_\infty\right) \rtimes H$ *where I and H as above, then there is a countable ultrahomogeneous structure without algebraicity $\boldsymbol{M}_H$ such that* $\mathrm{Age}(\boldsymbol{M}_H)$ *does not split and such that G is isomorphic to* $\mathrm{Aut}(\boldsymbol{M}_H)$.

The question of whether the complete inverse of the first statement of Theorem 2.3.3 holds remains open.

Question 2.3.4. *Let $\boldsymbol{M}$ be a Fraïssé structure without algebraicity. Assume moreover that* $\mathrm{Aut}(\boldsymbol{M})$ *is isomorphic as a topological group to* $\left(\prod_{i\in I} \mathrm{S}_\infty\right) \rtimes H$*, where H and I as above. Is it the case that* $\mathrm{Age}(\boldsymbol{M})$ *does not split?*

2.4 The Wijsman hyperspace topology

In Section 2.2 and Section 2.3 we worked with countable relational structures. Given such a structure $\boldsymbol{M}$, we viewed the set of all substructures of $\boldsymbol{M}$ as a Polish space. Namely, the cantor space 2^M. In the next chapter we are going to work with the Urysohn space and the Urysohn sphere. Both are complete metric space of the size of the continuum. Like in Section 2.2, we will need a natural Polish space whose elements correspond to the substructures of the space under consideration.

Let (X, d) be some metric space and let $\mathcal{F}(X)$ be the set of all closed subsets of X. The Wijsman topology on $\mathcal{F}(X)$, introduced in [Wij64], is the weakest topology on $\mathcal{F}(X)$ that makes continuous the family of distance functionals $\{d_x\}_{x\in X}$, where $d_x : \mathcal{F}(X) \to \mathbb{R}$ with $d_x(F) = d(x, F)$. The proof of the following theorem can be found in [Bee91].

Theorem 2.4.1 (Beer)**.** *If the space (X, d) is complete and separable, then the Wijsman topology of $\mathcal{F}(X)$ is Polish.*

From now on $\mathcal{F}(X)$ will always be equipped with the Wijsman topology. We will say that for a *generic* subspace of X a certain property holds if the set $\mathcal{A} \subset \mathcal{F}(X)$ of all F which have this property is comeager in $\mathcal{F}(X)$. Similarly we say that for a *generic* pair of subspaces of X a certain property holds if the set $\mathcal{A} \subset \mathcal{F}(X) \times \mathcal{F}(X)$ of all (A, B) which have this property is comeager in $\mathcal{F}(X) \times \mathcal{F}(X)$.

2.5 The Urysohn space $\mathbb{U}$ and the Urysohn sphere $\mathbb{S}$

Our aim here is to prove Theorem 2.5.9. The (separable) *Urysohn metric space* $(\mathbb{U}, \rho)$ is the unique, up to isometry, Polish metric space that satisfies the following properties:

- (ultrahomogeneity) for every two finite isometric subspaces $A, B \subset \mathbb{U}$ and for every isometry $f : A \to B$, f extends to a full isometry $\tilde{f}$ of $\mathbb{U}$;
- (universality) every Polish metric space is isometric to a subspace of $\mathbb{U}$.

The Urysohn space was introduced by Urysohn in [Ury27] but the interest in this space was revived through the work of Katětov and Uspenskij [Kat86, Usp90]. Here, in the next couple of paragraphs, we record some definitions and facts commonly used in the study of Urysohn space. For a more detailed exposition the reader may want to consult Melleray ([Mel07] or [Mel08]).

Let $(A, d_A), (B, d_B)$ be isometric Polish metric spaces. We are going to denote with $\mathrm{Iso}(A, B)$ the space of all (bijective) isometries from A to B and with $\mathrm{Iso}(A)$ the space of all (bijective) isometries from A to A. A basic open set U in $\mathrm{Iso}(A, B)$ can be thought of as a couple $[f, \delta]$, where f is a partial isometry from A to B with $\mathrm{dom}(f)$ finite and $\delta > 0$. If $U \subset \mathrm{Iso}(A, B)$ is a basic open set corresponding to the couple $[f, \delta]$, then $g \in \mathrm{Iso}(A, B)$ belongs to U if for every $a \in \mathrm{dom}(f)$ we have $d_B(f(a), g(a)) < \delta$.

Definition 2.5.1. *Let (X, d) be a metric space. A map $g : X \to \mathbb{R}$ is a* Katětov map *on X if*

$$\forall x, y \in X \;\; g(x) - g(y) \leq d(x, y) \leq g(x) + g(y).$$

If moreover $\mathrm{range}(g) \subset \mathbb{Q}$, we say that g is a *rational Katětov map* on X. We denote by $E(X)$ the set of all Katětov maps on X, and by $E^{\mathbb{Q}}(X)$ the set of all rational Katětov maps on X. Following Katětov, we introduce a distance between any pair of maps g_1, g_2 that belong to $E(X)$ given by

$$d_X(g_1, g_2) = \sup\{|g_1(x) - g_2(x)| : x \in X\}.$$

This distance renders $(E(X), d_X)$ a complete metric space which extends (X, d) via the identification $x \to g_x(y) = d(x, y) \in E(X)$ for every $x \in X$. We are interested in a specific subset of $E(X)$ which plays important role in the study of Urysohn space. Let Y be a subset of X and g a Katětov map on Y. We can extend $g \in E(Y)$ to $\tilde{g} \in E(X)$ by letting

$$\tilde{g}(x) = \inf\{g(y) + d(x, y) : y \in Y\}.$$

We call $\tilde{g}$ the *Katětov extension* of g to X. If $f \in E(X)$ and $Y \subset X$ are such that f is the Katětov extension of $f\restriction_Y$ to X, we say that Y is a support of f. The set

$$E(X, \omega) = \{g \in E(X) : g \text{ has finite support}\}$$

should be thought of as the set of all the rqf-types over every finite substructure of a structure that we saw in Section 2.2. Katětov maps on X with finite support can be approximated by functions that belong to

$$E^{\mathbb{Q}}(X,\omega) = \{g \in E^{\mathbb{Q}}(X) : g \text{ has finite support}\}.$$

An easy observation that can be found also in [GK03] is that if X is separable and $D \subseteq X$ is countable and dense in X, then $E^{\mathbb{Q}}(D,\omega)$ is a countable dense subset of $E(X,\omega)$.

There is another useful characterization of Urysohn space. Let X be a metric space. We say that X has the *approximate extension property* if for every finite A subset of X every $g \in E(A)$ and every $\varepsilon > 0$, there is a $z \in X$ such that for every $a \in A$ we have $|d(z,a) - g(a)| \leq \varepsilon$. We say that X has the *extension property* if we can take $\varepsilon = 0$ in the above definition. For a complete separable metric space X the following are equivalent:

- X is isometric to the Urysohn space.
- X has the extension property.
- X has the approximate extension property.

Uspenskij in [Usp90] used Katětov's tower construction [Kat86] to prove that for every Polish metric space X we can find subspaces B of $\mathbb{U}$ isometric to X for which every isometry of B extends to a full isometry of $\mathbb{U}$. Therefore there are subspaces of $\mathbb{U}$, of infinite cardinality, which are global. The following example shows that $\mathbb{U}$ contains non-global subspaces too (see also [Mel07] for non-global embeddings of every non-compact space).

Example 2.5.2. Let x_0, x_1, x_2 be points in $\mathbb{U}$ such that $\rho(x_0,x_1) = 2, \rho(x_1,x_2) = 1, \rho(x_0,x_2) = 3$ let also ε with $0 < \varepsilon < 1$. Set $A = \mathbb{U} \setminus B(x_0,\varepsilon)$. Then it is easy to see that A has the extension property and therefore it is isomorphic to $\mathbb{U}$. So, there is an isometry $f : A \to A$ sending x_1 to x_2. This isometry cannot be extended to a full isometry of $\mathbb{U}$. Moreover, the choice of ε ensures that there is a uniform lower bound bigger than zero (say $1-\varepsilon$) between $g \restriction A$ and f for every isometry g of $\mathbb{U}$.

Definition 2.5.3. *Let $D \subseteq \mathbb{U}$. We say that D* absorbs points *if $D \neq \emptyset$, and for every finite $X = \{x_1, \ldots, x_k\} \subset \mathbb{U}$, for every $g \in E(X)$ with $\rho(x_i, D) < g(x_i)$ there is a $z \in D$ such that $\rho(z, x_i) = g(x_i)$.*

Remark 2.5.4. Notice that if F is a closed subset of $\mathbb{U}$ and F absorbs points, then F is complete, separable and has the extension property. Therefore it is isomorphic to the Urysohn space.

Lemma 2.5.5. *For a generic $F \in \mathcal{F}(\mathbb{U})$, F absorbs points.*

Proof. Let P be a countable dense subset of $\mathbb{U}$. Consider the set

$$\mathcal{A} = \big\{F \in \mathcal{F}(\mathbb{U}) : \forall \text{ finite } A = \{a_1, \dots, a_k\} \subset P \quad \forall g \in E^{\mathbb{Q}}(A) \quad \forall m \in \mathbb{N}$$
$$\rho(a_1, F) \geq g(a_1) \vee \dots \vee \rho(a_k, F) \geq g(a_k) \quad \text{or} \quad \exists z \in U$$
$$\text{such that } \rho(a_1, z) = g(a_1) \wedge \dots \wedge \rho(a_k, z) = g(a_k) \wedge \rho(z, F) < \frac{1}{m}\big\}.$$

Closed conditions in a Polish space are also G_δ. So $\mathcal{A}$ a G_δ subset of $\mathcal{F}(\mathbb{U})$. We now show that $\mathcal{A}$ is also dense in $\mathcal{F}(\mathbb{U})$. Let $F \in \mathcal{F}(\mathbb{U})$. We will find a sequence $\{F_n\}_{n\in\mathbb{N}}$ of sets from $\mathcal{A}$ that converges to F in the Wijsman topology. If $F = \mathbb{U}$, then the sequence $F_n = \mathbb{U}$ for every n lies in $\mathcal{A}$ and converges to $\mathbb{U}$. If $F \neq \mathbb{U}$, let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of F^c and $d_n = \rho(x_n, F) > 0$. Let

$$F_n = \big\{x \in \mathbb{U} : \rho(x, x_1) \geq d_1 \cdot (1 - \frac{1}{n}), \dots, \rho(x, x_n) \geq d_n \cdot (1 - \frac{1}{n})\big\}.$$

Clearly $\{F_n\}_{n\in\mathbb{N}}$ converges to F. Fix now a $n \in \mathbb{N}$. We will show that F_n belongs to $\mathcal{A}$. Let $A = \{a_1, \dots, a_k\} \subset P$, $g \in E^{\mathbb{Q}}(A)$ and $m \in \mathbb{N}$. Assume moreover that $\rho(a_i, F) < g(a_i)$ for every i. Let $\tilde{g}$ be the Katětov extension of g to $A \cup \{x_1, \dots x_n\}$. The extension property of the Urysohn space gives us a point $z \in \mathbb{U}$ that realizes the distances given by $\tilde{g}$. Moreover, for every $j \in \{1, \dots, n\}$ we have that

$$\tilde{g}(x_j) = \inf\{\rho(x_j, a_i) + g(a_i) : 1 \leq i \leq k\} >$$
$$\inf\{\rho(x_j, a_i) + \rho(a_i, F) : 1 \leq i \leq k\} \geq$$
$$\rho(x_j, F) > d_j \cdot (1 - \frac{1}{n}).$$

Therefore $0 = \rho(z, F_n) < \frac{1}{m}$, which proves that F_n belongs to $\mathcal{A}$ for every $n \in \mathbb{N}$.

We showed so far that $\mathcal{A}$ is a comeager subset of $\mathcal{F}(\mathbb{U})$. We now proceed to prove that for every F that belongs to $\mathcal{A}$, F absorbs points. Let $F \in \mathcal{A}$ and $X = \{x_1, \dots, x_k\} \subset \mathbb{U}$. Let also $g \in E(X)$, with $\rho(x_i, F) < g(x_i)$ for every $i \in \{1, \dots, k\}$. Urysohn space is complete, so it suffice to find a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{U}$ such that for every $n \in \mathbb{N}$ we have:

- $|\rho(z_n, x_i) - g(x_i)| < 2^{-n}$ for every $i \in \{1 \dots, k\}$;
- $\rho(z_n, F) < 2^{-n}$, and
- $\rho(z_{n+1}, z_n) < 2^{1-n}$.

Let $d = \min\{g(x_i) - \rho(x_i, F) : 1 \leq i \leq k\} > 0$ and let $\{\delta_n\}, \{\delta'_n\}$ be two sequences of positive real numbers with $\delta_n, \delta'_n < \min\{d, 1\} \cdot 2^{-(n+1)}$. For $n = 1$ let $P_1 = \{p^1_1, p^1_2, \dots, p^1_k\} \subset P$ with $\rho(p^1_i, x_i) < \delta_1$ for all

$i \in \{1, \dots k\}$ and let $g_1 \in E^{\mathbb{Q}}(P_1)$ with $|g_1(p_i^1) - g(x_i)| < \delta_1'$ for all $i \in \{1, \dots, k\}$. These conditions imply that $g_1(p_i^1) > \rho(p_i^1, F)$. Therefore, by definition of $\mathcal{A}$, we can find a z_1 such that

- $|\rho(z_1, x_i) - g(x_i)| \leq \delta_1 + \delta_1' < 2^{-1}$ for every $i \in \{1 \dots, k\}$, and
- $\rho(z_1, F) < 2^{-1}$.

Suppose now that we have defined $z_1, \dots, z_n$ fulfilling the above properties. Let $f_n \in E(\{x_1, \dots, x_k\})$ with $f_n(x_i) = d(x_i, z_n)$. Then $d_X(f_n, g) = \sup\{|f_n(x_i) - g(x_i)| : 1 \leq i \leq k\} < 2^{-n}$.

We define now $P_{n+1} = \{p_1^{n+1}, p_2^{n+1}, \dots, p_k^{n+1}\} \cup \{p_*^{n+1}\} \subset P$ with $\rho(p_*^{n+1}, z_n) < \delta_{n+1}$, $\rho(p_i^{n+1}, x_i) < \delta_{n+1}$ for all $i \in \{1, \dots k\}$ and $g_{n+1} \in E^{\mathbb{Q}}(P_{n+1})$ with $|g_{n+1}(p_*^{n+1}) - d_X(f_n, g)| < \delta_{n+1}'$ and $|g_{n+1}(p_i^{n+1}) - g(x_i)| < \delta_{n+1}'$ for all $i \in \{1, \dots, k\}$. Again these conditions imply that $g_{n+1}(p_i^{n+1}) > \rho(p_i^{n+1}, F)$ so we get a z_{n+1} such that

- $|\rho(z_{n+1}, x_i) - g(x_i)| \leq \delta_{n+1} + \delta_{n+1}' < 2^{-(n+1)}$ for every $i \in \{1 \dots, k\}$;
- $\rho(z_{n+1}, F) < 2^{-(n+1)}$, and
- $\rho(z_{n+1}, z_n) \leq d_X(f_n, g) + \delta_{n+1} + \delta_{n+1}' < 2^{-n} + 2^{-(n+1)} < 2^{1-n}$.

This proves that every $F \in \mathcal{A}$ absorbs points and therefore the set of all closed subsets F of $\mathbb{U}$ that absorb points is a comeager subset of $\mathcal{F}(\mathbb{U})$. □

By the Lemma 2.5.5 and Remark 2.5.4 we have the following corollary. For a result similar in spirit but on different context; see [Ver04].

Corollary 2.5.6. *For a generic $F \in \mathcal{F}(\mathbb{U})$, F is isometric to $\mathbb{U}$.*

With the following lemma we establish in relation to the Definition 2.2.4 that the class of all finite metric spaces "splits" in a uniform way.

Lemma 2.5.7. *Let d_* be a positive real number. Then for every $\varepsilon < d_*$ and every finite metric space (X, d) with $X = \{a_1, \dots, a_n, c\}$ such that $d(c, a_i) \geq d_*$, for all $i \in \{1, \dots, n\}$, there are $g_1, g_2 \in E(X)$ such that*

- *$g_1(a_i) = g_2(a_i)$ for all $i \in \{1, \dots, n\}$;*
- *$g_1(c), g_2(c) \geq d_*$, and*
- *$|g_1(c) - g_2(c)| > \varepsilon$.*

Proof. Let $D = \operatorname{diam}(X)$ and pick any $\delta \in (\varepsilon, d_*)$. Define $g_1, g_2 : X \to \mathbb{R}$ with $g_1(a_i) = g_2(a_i) = 2D$ for every $i \in \{0, \dots, n\}$, $g_1(c) = 2D$ and $g_2(c) = 2D - \delta$. □

Lemma 2.5.8. *Let $A, B \in \mathcal{F}(\mathbb{U})$ such that A, B absorb points and $A, B \neq \mathbb{U}$. Then, $\mathcal{E}(A, B)$ is a meager subset of* $\mathrm{Iso}(A, B)$.

Proof. The proof is essentially the same as in Lemma 2.2.11. Therefore we will skip the details. Players I and II take turns playing open sets in the Banach Mazur game $G^{**}(\mathcal{N}, \mathrm{Iso}(A, B))$ where $\mathcal{N} = \mathrm{Iso}(A, B) \setminus \mathcal{E}(A, B)$. Let $c_* \in A^c$, let $d_* \in \mathbb{R}$ with $0 < d_* < \rho(c_*, A)$ and let ε with $0 < \varepsilon < d_*$. Consider the set

$$C = \{y \in B^c : \exists h \in \mathcal{E}(A, B) \text{ with } h(c_*) = y\},$$

and let $\{W_i\}_{i \in \mathbb{N}}$ be an open covering of C with $\mathrm{diam}(W_i) < \frac{\varepsilon}{2}$ for every $i \in \mathbb{N}$.

Assume that in the n-th step Player I has played the open set $U_n = [f_n, \delta_n] \subset \mathrm{Iso}(\mathbb{U})$, where f_n is an isometry between the finite subspaces $A_n \subset A$ and $B_n \subset B$. Assume also that i_n is the smallest index for which there is an $h \in \mathcal{E}(A, B)$ with $h(c_*) \in W_{i_n}$ and let $y_n = h(c_*)$ for any h as above. Let also $h_n : A_n \cup \{c_*\} \to B_n \cup \{y_n\}$ be the unique partial isometry that extends f_n to $A_n \cup \{c_*\}$.

Player II will make use of Lemma 2.5.7 to get $g_1, g_2 \in E(A_n \cup \{c_*\})$ with $g_1 \restriction_{A_n} = g_2 \restriction_{A_n}$ and $|g_1(c_*) - g_2(c_*)| > \varepsilon$. Moreover, by adding if necessary the same constant function to g_1 and g_2, he can arrange so that $g_i(c_*) > \rho(c_*, A)$ for $i \in \{1, 2\}$. Due to the fact that both A and B absorb points, he will find points $z_n \in A$ and $z'_n \in B$ such that $\rho(x, z_n) = g_1(x)$ for every $x \in A_n \cup \{c_*\}$ and $\rho(x, z'_n) = g_2 \circ h_n^{-1}(x)$ for every $x \in B_n \cup \{y_n\}$. Player II will play his n-th move $V_n = [f'_n, \delta'_n]$ where

$$f'_n : A_n \cup \{z_n\} \to B_n \cup \{z'_n\} \quad \text{with} \quad f'_n \restriction_{A_n} = f_n,$$

$f'_n(z_n) = z'_n$ and $\delta'_n = \min\{\delta_n, \frac{\varepsilon}{2}, 2^{-n}\}$. As in Lemma 2.2.11 this leads to a winning strategy for Player II and by Theorem 2.1.1 we have that $\mathcal{E}(A, B)$ is a meager subset of $\mathrm{Iso}(A, B)$. ☐

Summarizing the above results we have the following theorem:

Theorem 2.5.9. *Let $\mathbb{U}$ be the Urysohn space. Then for a generic $F \in \mathcal{F}(\mathbb{U})$, the generic isometry $f \in \mathrm{Iso}(F)$ cannot be extended to an isometry $\tilde{f} \in \mathrm{Iso}(\mathbb{U})$. Moreover, for a generic pair $(A, B) \in \mathcal{F}(\mathbb{U})^2$ the generic isometry $f \in \mathrm{Iso}(A, B)$ cannot be extended to an isometry $\tilde{f} \in \mathrm{Iso}(\mathbb{U})$.*

Proof. By Theorem 2.5.5, we have that for a generic subspace $F \in \mathcal{F}(\mathbb{U})$ and for a generic pair $A, B \in \mathcal{F}(\mathbb{U})$ all F, A, B absorb points. Moreover, for generic F, A, B, $F, A, B \neq \mathbb{U}$. Lemma 2.5.8 proves the rest. ☐

We want to point out here that the situation with the Urysohn sphere $\mathbb{S}$ is not much different. Analogous statements to Lemmas 2.5.5, 2.5.7, 2.5.8 and Theorem 2.5.9 follow easily for $\mathbb{S}$ if we make the obvious changes. There is also a final remark that should be made. The theory of Fraïssé limits grafted with ideas from continuous logic can be naturally generalized to the context of complete separable metric structures; see for example [Yaa15]. In this context, the Urysohn space and the Urysohn sphere are just examples of the general theory. A natural question arises, namely whether a dichotomy similar to the one in Section 2.2 could possibly hold for metric Fraïssé structures. The following example suggests which types of metric structures would belong to the “global” side of the dichotomy. However, the methods developed here face some obstacles when we try to apply them into this context. For example, the general theory of metric Fraïssé structures is developed for *approximately ultrahomogeneous* structures rather than ultrahomogeneous structures and moreover, to my current understanding, a natural and convenient notion of SAP does not seem to exist.

Example 2.5.10. Let $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ be the Baire space endowed with the ultrametric d with

$$d(\alpha, \beta) = \frac{1}{m} \quad \text{where} \quad m = \min\{n : \alpha(n) \neq \beta(n)\},$$

if $\alpha \neq \beta$ and $d(\alpha, \beta) = 0$ otherwise. The metric structure $(\mathcal{N}, d)$ is a metric Fraïssé structure that happens to be ultrahomogeneous. For a generic subspace $F \in \mathcal{F}(\mathcal{N})$, F is the body of a pruned tree T on $\mathbb{N}$ such that for every $n \in \mathbb{N}$ there are infinitely many $s \in T$ and infinitely many $s \notin T$ of length n. It is easy now to see that for a generic $F \in \mathcal{F}(\mathcal{N})$, F is a global substructure and that the generic pair $A, B \in \mathcal{F}(\mathcal{N})$ is also global.

Chapter 3

Games and obstructions to Borel reducibility

3.1 The Becker-embedding game

Recall that a CLI group is a Polish group that admits a compatible *complete left-invariant metric*. It is easy to see that a Polish group is CLI if and only if it admits a compatible right-invariant metric; see [Bec98, 3.A.2. Proposition]. Suppose that G is a Polish group, and X is a Polish G-space. The main goal of this section is to provide a dynamical criterion of a Polish G-space X which implies that the corresponding orbit equivalence relation is not Borel reducible to the orbit equivalence relation induced by a Borel action of a CLI group.

We recall the notion of ι-embeddability for points of X from [Bec98, Definition 3.D.1].

Definition 3.1.1. *Let d be a left-invariant metric on G. If $x, y \in X$, then x is ι-embeddable into y if there exists a sequence $(h_n)_{n\in\omega}$ in G such that $h_n x \to y$ and $(h_n)_{n\in\omega}$ is Cauchy with respect to the metric d.*

By [Bec98, Proposition 3.B.1] the definition does not depend on the choice of the left-invariant metric d on G. Similarly one can consider the following similar notion of "embedding" that is also defined, but not extensively studied in [Bec98].

Definition 3.1.2. *Let d be a right-invariant metric on G. If $x, y \in X$, then x is Becker-embeddable into y if there exists a sequence $(h_n)_{n\in\omega}$ in G such that $h_n y \to x$ and $(h_n)_{n\in\omega}$ is Cauchy with respect to the metric d.*

We now consider a game between two players, which captures the notion of Becker-embeddability from Definition 3.1.2.

Definition 3.1.3. *Suppose that X is a Polish G-space, and $x, y \in X$. We consider the* Becker-embedding game $\mathrm{Emb}(x, y)$ *played between two players as follows. Set $U_0 = X$ and $V_0 = G$.*

1. *In the first turn, Player I plays an open neighborhood U_1 of x, and an open neighborhood V_1 of the identity of G. Player II replies with an element g_0 in V_0.*

2. *In the second turn, Player I then plays an open neighborhood U_2 of x, and an open neighborhood V_2 of the identity of G, and Player II replies with an element g_1 in V_1.*

(n) *At the n-th turn, Player I plays an open neighborhood U_n of x, and an open neighborhood V_n of the identity of G, and Player II responds with an element g_{n-1} in V_{n-1}.*

The game proceed in this way, producing a sequence (g_n) of elements of G, a sequence (U_n) of open neighborhoods of x in X, and a sequence (V_n) of open neighborhoods of the identity in G. Player II wins the game if for every $n > 0$, $g_{n-1} \cdots g_0 y \in U_n$. We say that x is Becker embeddable *into y—and write $x \preccurlyeq_{\mathcal{B}} y$—if Player II has a winning strategy for the game* $\mathrm{Emb}(x, y)$.

Remark 3.1.4. It is not difficult to see that, if Player II has a winning strategy for the Becker-embedding game as described in Definition 3.1.3, then it also has a winning strategy for the same game with the additional winning conditions that g_n belongs to some given comeager subset of V_n, and $g_{n-1} \cdots g_0 y$ belongs to some given comeager subset X_0 of X, provided that the set of $g \in G$ such that $gy \in X_0$ is comeager. This is a consequence of the following version of the Kuratowski-Ulam theorem: suppose that X, Y are Polish spaces and $f : X \to Y$ is a continuous open map. Then a Baire-measurable subset A of X is comeager if and only if the set $\left\{y \in Y : A \cap f^{-1}\{y\} \text{ is comeager in } f^{-1}\{y\}\right\}$ is comeager; see [MT13, Theorem A.1]. One can then apply this fact to the continuous and open map $G \times X \to X$, $(g, x) \mapsto gx$. This observation can be equivalently phrased in terms of properties of the Vaught transform for Polish group actions; see [Gao09, Section 3.2].

The following Lemma is an immediate consequence of the above definitions3.1.2[1].

Lemma 3.1.5. *Let X be a Polish G-space. If x, y are points of X, then the following statements are equivalent:*

1. *$x \preccurlyeq_{\mathcal{B}} y$;*

2. *x is Becker-embeddable in y.*

Suppose that X is a Polish G-space. It follows from Lemma 3.1.5 and [Bec98, Proposition 3.D.4] that the relation of Becker-embeddability is a preorder. Furthermore, if $x \preccurlyeq_{\mathcal{B}} y$, x' belongs to the G-orbit of x, and y' belongs to the G-orbit of y, then $x' \preccurlyeq_{\mathcal{B}} y'$.

We let X/G be the space of G-orbits of points of X. The Becker-embeddability preorder defines a directed graph structure on X/G obtained by declaring that there is an arrow from the orbit $[x]$ of x to the orbit $[y]$

[1]We would like to thank here Alex Kruckman for pointing out a mistake in an early version of this draft, where we were using ι-embeddings instead of r-embeddings.

of y if and only if $x \preccurlyeq_{\mathcal{B}} y$. We will call this the *Becker digraph* $\mathcal{B}(X/G)$ of the Polish G-space X. Similarly, for a G-invariant subset X_0 of X we let $\mathcal{B}(X_0/G)$ the induced subgraph of $\mathcal{B}(X/G)$ only containing vertices corresponding to orbits from X_0. Suppose that G, H are Polish groups, X is a Polish G-space, and Y is a Polish H-space. Any $\left(E_G^X, E_H^Y\right)$-homomorphism $f : X \to Y$ induces a function $[f] : X/G \to Y/H$, $[x] \mapsto [f(x)]$. We will show below that, when f is Baire-measurable, such a function is *generically* a digraph homomorphism with respect to the Becker digraph structures on X/G and Y/H.

We now recall an example from [Bec98], describing the notion of Becker-embedding in case of Polish G-spaces arising from classes of countable models. Suppose that $\mathcal{L} = (R_i)_{i \in I}$ is a countable first order relational language, where R_i is a relation symbol with arity n_i. Let $\mathrm{Mod}(\mathcal{L})$ be the space of countable $\mathcal{L}$-structures having $\mathbb{N}$ as support, F be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and S_∞ be the group of permutations of $\mathbb{N}$. As usual, one can regard $\mathrm{Mod}(\mathcal{L})$ as the product $\prod_{i \in I} 2^{(\mathbb{N}^{n_i})}$. Any $\mathcal{L}_{\omega_1,\omega}$ formula φ with parameters from $\mathbb{N}$ defines a subset $[\varphi]$ of $\mathrm{Mod}(\mathcal{L})$ of the structures that satisfy φ. The fragment F defines a topology t_F on $\mathrm{Mod}(\mathcal{L})$ having the collections of sets of the form $[\varphi]$, where φ ranges among the formulas in F with parameters, as a clopen basis. The canonical action $S_\infty \curvearrowright \mathrm{Mod}(\mathcal{L})$ turns $(\mathrm{Mod}(\mathcal{L}), t_F)$ into a Polish G-space [Bec98, Proposition 2.D.2]. In the case where F is the fragment consisting of all first-order formulas and $x, y \in \mathrm{Mod}(\mathcal{L})$, then x ι-embeds in y, if and only if there is an injective function $f : \mathbb{N} \to \mathbb{N}$ that witnesses an elementary embedding of x in y [Bec98, Proposition 3.D.2]. It is easy to see that in the case where when F is the fragment consisting of atomic first-order formulas and $x, y \in \mathrm{Mod}(\mathcal{L})$, then x Becker-embeds in y, if and only if there is an injective function $f : \mathbb{N} \to \mathbb{N}$ that witnesses a simple embedding of x in y as an $\mathcal{L}$-structure.

3.2 The orbit continuity lemma

Recall that if E, F are equivalence relations on Polish spaces X, Y respectively, then a (E, F)-homomorphism is a function $f : X \to Y$ mapping E-classes to F-classes. In this subsection we isolate a lemma to be used in the rest of the paper. It states that a Baire-measurable homomorphism between orbit equivalence relations admits a restriction to a dense G_δ set which is continuous at the level of orbits, in a suitable sense. Variations of such a lemma are well known. The starting point is essentially [Hjo00, Lemma 3.17] modified as in the beginning of the proof of [Hjo00, Theorem 3.18]; see also [Gao09, Lemma 10.1.4 and Theorem 10.4.2].

Lemma 3.2.1. *Suppose that G, H are Polish groups, X is a Polish G-space, and Y is a Polish H-space. Let $f : X \to Y$ be a Baire-measurable $\left(E_G^X, E_H^Y\right)$-homomorphism. Then there exists a dense G_δ subset C of X such that*

- *the restriction of f to C is continuous;*
- *for any $x \in C$, $\{g \in G : gx \in C\}$ is a comeager subset of G;*
- *for any $x_0 \in C$ and for any open neighborhood W of the identity in H there exists an open neighborhood U of x_0 and an open neighborhood V of the identity of G such that for any $x \in U \cap C$ and for a comeager set of $g \in V$, one has that $f(gx) \in Wf(x)$ and $gx \in C$.*

Proof. In the course of the proof, we will use the category quantifier $\forall^* x \in U$ for the statement "for a comeager set of $x \in U$"; see [Gao09, Section 3.2]. Fix a neighborhood W_0 of the identity in H. We first prove the following claim: $\forall x_0 \in X\ \forall^* g_0 \in G$, there is an open neighborhood V of the identity in G such that $\forall^* g_1 \in V$, $f(g_1 g_0 x_0) \in W_0 f(g_0 x_0)$.

Fix a neighborhood W of the identity of H such that $WW^{-1} \subset W_0$. Let (h_n) be a sequence in H such that $\{Wh_n : n \in \mathbb{N}\}$ is a cover of H. Since $Wh_n f(x_0)$ is analytic, the set of elements x of the orbit of x_0 such that $f(x) \in Wh_n f(x_0)$ has the Baire property. Therefore we can find a sequence (O_n) of open subsets of G with dense union O and a comeager subset D of O such that $\forall g \in D \cap O_n$, $f(gx_0) \in Wh_n f(x_0)$. Suppose now that $g_0 \in D$. Let $n \in \mathbb{N}$ be such that $g_0 \in O_n$. Then there exists a neighborhood V of the identity of G such that $Vg_0 \subset O_n$. Observe that $(D \cap O_n)\, g_0^{-1} \cap V$ is a comeager subset of V. If $g_1 \in (D \cap O_n)\, g_0^{-1} \cap V$, then we have

$$f(g_1 g_0 x_0) \in Wh_n f(x_0) \quad \text{and} \quad f(g_0 x_0) \in Wh_n f(x_0).$$

Therefore

$$f(g_1 g_0 x_0) \in WW^{-1} f(g_0 x_0) \subset W_0 f(g_0 x_0).$$

This concludes the proof of the claim.

From the claim and the Kuratowski-Ulam theorem, one deduces that there exists a dense G_δ subset C_0 of X such that for every $x \in C_0$ there exists an open neighborhood V of the identity of G such that $\forall^* g \in V$, $f(gx) \in Wf(x)$. Since f is Baire-measurable, we can furthermore assume that the restriction of f to C_0 is continuous.

Fix now a countable basis (W_k) of open neighborhoods of the identity of H and a countable basis (V_n) of open neighborhoods of the identity in G. Let $N : X \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ be the function that assigns to (x, k) the least $n \in \mathbb{N}$ such that $\forall^* g \in V_n$, $f(gx) \in W_k f(x)$ if such an n exists and $x \in C_0$, and ∞ otherwise. Then N is an analytic function, and hence one can find a dense G_δ subset C_1 of X contained in C_0 such that $N|_{C_1 \times \mathbb{N}}$ is continuous. By [Gao09, Proposition 3.2.5 and Theorem 3.2.7] the set $C := \{x \in C_1 : \forall^* g \in G, gx \in C_1\}$ is a dense G_δ subset of X such that $\forall x \in C$, $\forall^* g \in G$, $gx \in C$. Therefore C satisfies the desired conclusions. $\square$

3.3 Generic homomorphisms between Becker graphs

In this section we use the Becker-embedding game and the orbit continuity lemma to address Problem 1.2.1 for the class of CLI groups.

Definition 3.3.1. *An equivalence relation E on a Polish space X is CLI-classifiable if it is Borel reducible to E_H^Y for some CLI group H and Polish H-space Y.*

We will obtain below an obstruction to CLI-classiability in terms of the Becker digraph. This will be based upon the following properties of the Becker digraph:

1. the Becker digraph contains only loops in the case of CLI group actions (Lemma 3.3.2), and

2. a Baire-measurable homomorphism between orbit equivalence relations induces, after restricting to an invariant dense G_δ set, a homomorphism at the level of Becker digraphs (Proposition 3.3.3).

Lemma 3.3.2. *If Y is a Polish H-space and H is a CLI group, then the Becker digraph $\mathcal{B}(Y/H)$ contains only loops.*

Proof. Fix a compatible complete right-invariant metric d on H. For a subset A of H we let $\mathrm{diam}\,(A)$ be the diameter of A with respect to d. Let x, y be elements of Y with different H-orbits. We show that Player I has a winning strategy in $\mathrm{Emb}(x, y)$. In the n-th round Player I plays some symmetric open neighborhood V_{n+1} of the identity of H with $\mathrm{diam}\,(V_{n+1}) < 2^{-n}$ and an open neighborhood U_n of x such that the sequence (U_n) forms a decreasing basis of neighborhoods of x. Let (g_n) be the sequence of group elements chosen by Player II, and set $h_n := g_n \cdots g_0$. We claim that such a sequence does not satisfy the winning condition for Player II in the Becker-embedding game. Suppose by contradiction that this is the case, and hence $\lim_n h_n y = x$. For every $n > m$ we have by right invariance of d that

$$d(h_n, h_m) = d(g_n \cdots g_{m+1}, 1) \leq d(g_n, 1) + d(g_{n-1}, 1) + \cdots + d(g_{m+1}, 1) < 2^{-m}.$$

Therefore h_n is a d-Cauchy sequence with respect to d. Since by assumption d is complete, h_n converges to some $h \in H$. From $\lim_n h_n y = x$ and continuity of the action, we deduce that $hy = x$. This contradicts the assumption that the H-orbits of x and y are different. □

Using the orbit continuity lemma (Lemma 3.2.1) one can then show that a Baire-measurable homomorphism preserves Becker embeddability on a comeager set. This is the content of the following proposition.

Proposition 3.3.3. *Suppose that G, H are Polish groups, X is a Polish G-space, and Y is a Polish H-space. Let $f : X \to Y$ be a Baire-measurable $\left(E_G^X, E_H^Y\right)$-homomorphism. Then there exists a G-invariant dense G_δ subset X_0 of X such that the function $[f] : X_0/G \to Y/H$, $[x] \mapsto [f(x)]$ is a digraph homomorphism from the Becker digraph $\mathcal{B}(X_0/G)$ to the Becker digraph $\mathcal{B}(Y/G)$.*

Proof. As in the proof of Lemma 3.2.1, we will use the category quantifier $\forall^* x \in U$ for the statement "for a comeager set of $x \in U$". Let C be dense G_δ subsets of X obtained from f as in Lemma 3.2.1. Set $X_0 := \{x \in X : \forall^* g \in G, gx \in C\}$, which is a G-invariant dense G_δ set by [Gao09, Proposition 3.2.5 and Theorem 3.2.7]. We claim that $[f] : X_0/G \to Y/H$, $[x] \mapsto [f(x)]$ is a digraph homomorphism from the Becker digraph $\mathcal{B}(X_0/G)$ to the Becker digraph $\mathcal{B}(Y/G)$.

Fix $x_0, y_0 \in X_0$ such that $x_0 \preccurlyeq_{\mathcal{B}} y_0$. We want to prove that $f(x_0) \preccurlyeq_{\mathcal{B}} f(y_0)$. Observe that $\forall^* g \in G$, $gx_0 \in C \cap X_0$. Therefore after replacing x_0 with gx_0 for a suitable $g \in G$ we can assume that $x_0 \in C \cap X_0$. Let us consider thus the Becker-embedding game $\mathrm{Emb}(f(x_0), f(y_0))$. At the same time we consider the Becker-embedding game $\mathrm{Emb}(x_0, y_0)$ and use the fact that Player II has a winning strategy for such a game.

In the first turn of $\mathrm{Emb}(f(x_0), f(y_0))$, Player I plays an open neighborhood $\widehat{U}_1$ of $f(x_0)$ and an open neighborhood $\widehat{V}_1$ of the identity of H. Consider an open neighborhood U_1 of x_0 and an open neighborhood V_1 of the identity of G such that for any $x \in U_1 \cap C \cap X_0$ and a comeager set of $g \in V_1$ one has that $f(gx) \in \widehat{V}_1 f(x)$. Consider now the round of the game $\mathrm{Emb}(x_0, y_0)$ where, in the first turn, Player I plays the neighborhood U_1 of x_0 and the neighborhood V_1 of the identity of G. Since by assumption Player II has a winning strategy for $\mathrm{Emb}(x_0, y_0)$, we can consider an element g_0 of V_1 which is obtained from such a winning strategy. By Remark 3.1.4, we can also insist that g_0 belongs to the comeager set of $g \in V_1$ such that $gy_0 \in U_1 \cap C \cap X_0$ and $f(gy_0) \in \widehat{V}_1 f(x)$. We can then let Player II play, in the first turn of the game $\mathrm{Emb}(f(x_0), f(y_0))$, an element h_0 of $\widehat{V}_1$ such that $f(g_0 y_0) = h_0 f(y_0)$.

At the n-th turn of $\mathrm{Emb}(f(x_0), f(y_0))$, Player I plays an open neighborhood $\widehat{U}_n$ of $f(x_0)$ and an open neighborhood $\widehat{V}_n$ of the identity of H. Consider now an open neighborhood U_n of x_0 and an open neighborhood V_n of the identity of G such that for any $x \in U_n \cap C \cap X_0$ and a comeager set of $g \in V_n$ one has that $f(gx) \in \widehat{V}_n f(x)$. Let Player I play, in the n-turn of $\mathrm{Emb}(x_0, y_0)$, the open neighborhoods U_n of x_0 and V_n of the identity of G. Let $g_{n-1} \in V_n$ be obtained from a winning strategy for Player II. By Remark 3.1.4 we can insist that g_{n-1} belongs to the comeager set of $g \in V_n$ such that $gg_{n-2} \cdots g_1 g_0 y \in U_n \cap C \cap X_0$ and $f(gx) \in \widehat{V}_1 f(x)$. Therefore we can let Player II play, in the n-th turn of the game $\mathrm{Emb}(f(x_0), f(y_0))$, an element $h_{n-1} \in \widehat{V}_{n-1}$ such that $f(g_{n-1} \cdots g_0 y) = h_{n-1} f(g_{n-1} \cdots g_0 y) = h_{n-1} \cdots h_0 y \in \widehat{U}_n$. Such a construction witness that Player II has a winning strategy for the game $\mathrm{Emb}(f(x_0), f(y_0))$. □

From Lemma 3.3.2 and Proposition 3.3.3 one can immediately deduce the following criterion for showing that the orbit equivalence relation of a Polish group action is not Borel reducible to the orbit equivalence relation of CLI group action.

Theorem 3.3.4. *Suppose that X is a Polish G-space. If for any G-invariant dense G_δ subset C of X there exist $x, y \in C$ with different G-orbits such that $x \preccurlyeq_{\mathcal{B}} y$, then for any G-invariant dense G_δ subset C of X the relation E_C^X is not CLI-classifiable.*

Proof. Suppose that H is a CLI group, and Y is a Polish H-space. Suppose that D is a G-invariant dense G_δ subset of X, and $f : D \to Y$ is a Borel $\left(E_G^D, E_H^Y\right)$-homomorphism. Then by Proposition 3.3.3 there exists a G-invariant dense G_δ subset C of D such that $[f] : C/G \to Y/H$ is a digraph homomorphisms for the Becker digraphs $\mathcal{B}(C/G)$ and $\mathcal{B}(Y/H)$. By assumption there exist elements x, y of C with different G-orbits such that $x \preccurlyeq_{\mathcal{B}} y$. Therefore $f(x) \preccurlyeq_{\mathcal{B}} f(y)$. Since H is CLI we have by Lemma 3.3.2 that $f(x)$ and $f(y)$ belong to the same H-orbit. Therefore f is not a reduction from E_G^D to E_H^Y. □

3.4 Applications

Suppose that E is an equivalence relation on a Polish space X. Recall that the Friedman–Stanley jump E^+ of E [Gao09, Definition 8.3.1]—see also [FS89]—is the equivalence relation on the standard Borel space $X^{\mathbb{N}}$ of sequences of elements of X defined by $(x_n)\, E^+ \,(y_n)$ if and only if $\{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\}$.

In particular one can start with the relation $=$ of equality on a perfect Polish space X. The corresponding Friedman–Stanley jump is the relation $=^+$ on X^ω defined by $(x_n) =^+ (y_n)$ if and only if the sequences (x_n) and (y_n) have the same range. With respect to Borel reducibility, $=^+$ is the most complicated (essentially) $\mathbf{\Pi}_3^0$ equivalence relation [Gao09, Theorem 12.5.5]; see also [HKL98].

Hjorth has proven in [Hjo98, Theorem 5.19] that $=^+$ is not Borel reducible to the orbit equivalence relation of a continuous action of an *abelian* Polish group. As remarked in [Hjo98, page 663], Hjorth's proof uses a metamathematical argument involving forcing and Stern's absoluteness principle . Similar methods are used in [Kan08, Theorem 17.1.3] to prove that $=^+$ is not Borel reducible to the orbit equivalence relation of a Borel action of a CLI group. This is obtained as a consequence of a general result concerning pinned equivalence relations; see [Kan08, Definition 17.1.2]. To our knowledge, the argument below provides the first entirely classical proof of this result.

Let $\sigma : X^{\mathbb{N}} \to X^{\mathbb{N}}$ be the *unilateral shift* $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. We consider the restriction of $=^+$ to the dense G_δ subset Y of $X^{\mathbb{N}}$ that consists of injective sequences. Observe that this is the orbit equivalence relation of the canonical action of S_∞ on $X^{\mathbb{N}}$ obtained by permuting the indices.

Theorem 3.4.1. *Let $Z \subset Y$ be a nonempty S_∞-invariant G_δ set such that $\sigma[Z] = Z$. The restriction of $=^+$ to any S_∞-invariant dense G_δ subset of Z is not Borel reducible to a Borel action of a CLI group on a standard Borel space.*

Proof. Let E be the restriction of $=^+$ to Z. As observed before, E is the orbit equivalence relation of the canonical action $S_\infty \curvearrowright Z \subset Y \subset X^{\mathbb{N}}$ given by permuting the coordinates. We apply Proposition 3.3.4. Let C be an S_∞-invariant dense G_δ subset of Z. We need to prove that there exist $x, y \in C$ with different orbits such that $x \preccurlyeq_{\mathcal{B}} y$. For $x = (x_n) \in Y$ we let $\mathrm{Ran}(x)$ be the set $\{x_n : n \in \mathbb{N}\}$. It is not difficult to see directly that, for $x, y \in Y$, $x \preccurlyeq_{\mathcal{B}} y$ if and only if $\mathrm{Ran}(x) \subset \mathrm{Ran}(y)$. When X is the Cantor space $2^{\mathbb{N}}$, this assertion is a particular instance of the discussion at the end of Section 3.1. Indeed, in this case $X^{\mathbb{N}}$ can be seen as the space $\mathrm{Mod}(\mathcal{L})$ of $\mathcal{L}$-structures endowed with the topology t_F, where $\mathcal{L}$ is the language containing a countably infinite collection of unitary relations and F is the fragment consisting of atomic formulas. Observe that $\sigma : Z \to Z$ is continuous, open, and surjective. Therefore, since C is a dense G_δ subset of Z, we have that there exists a comeager subset C_0 of C such that, for every $x \in C_0$, $\sigma^{-1}(x) \cap C$ is a comeager subset of $\sigma^{-1}(x)$; see [MT13, Theorem A.1]. Pick now $x \in C_0$ and $y \in \sigma^{-1}(x) \cap C$. It is clear that $x \preccurlyeq_{\mathcal{B}} y$ and x, y lie in different S_∞-orbits. This concludes the proof. □

We now apply Theorem 3.4.1 to obtain information about the orbit equivalence relation of some canonical actions of the uniteray group $\mathcal{U}(\mathcal{H})$. Let $\mathcal{H}$ be the separable infinite-dimensional Hilbert space, and let $\mathcal{U}(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$. This is a Polish group when endowed with the weak operator topology; see [Bla06, Proposition I.3.2.9]. The group $\mathcal{U}(\mathcal{H})$ admits a canonical action by conjugation on itself and on the space $\mathcal{B}(\mathcal{H})_{sa}$ of selfadjoint operators.

Theorem 3.4.2. *The following relations are not Borel reducible to a Borel action of a CLI group on a standard Borel space:*

1. *unitary equivalence of unitary operators;*

2. *unitary equivalence of selfadjoint operators.*

Proof. As in Theorem 3.4.1 we consider the equivalence relation $=^+$ on the set $X^{\mathbb{N}}$ of sequences of elements of a perfect Polish space X. Fix an orthonormal basis (e_n) of $\mathcal{H}$. Let X be the circle group $\mathbb{T}$, and $Y \subset \mathbb{T}^{\mathbb{N}}$ be the set of injective sequences. The map $f : Y \to \mathcal{U}(\mathcal{H})$ which sends an element $(\lambda_n) \in Y$ to the unitary operator

$$(e_n) \mapsto (\lambda_n e_n)$$

is a Borel reduction from $=^+ |_Y$ to unitary equivalence of unitary operators. The proof of selfadjoint operators is the same, where one replaces $\mathbb{T}$ with $[0,1]$. □

3.5 A game-theoretic approach to turbulence

Suppose that $\mathcal{L} = (R_i)_{i\in I}$ is a countable first order relational language, where R_i is a relation symbol with arity n_i. We denote as above by $\mathrm{Mod}(\mathcal{L})$ the Polish S_∞-space of $\mathcal{L}$-structures with support $\mathbb{N}$. Recall that a Polish group G is called *non-Archimedean* if it admits a neighborhood basis of the identity of open subgroups or, equivalently, it is isomorphic to a closed subgroup of S_∞; see [BK96, Theorem 1.5.1]. A relation E is *classifiable by countable structures* if it is Borel reducible to the isomorphism relation in $\mathrm{Mod}(\mathcal{L})$ for some countable first order relational language $\mathcal{L}$. This is equivalent to the assertion that E is Borel reducible to the orbit equivalence relation of a Borel action of a non-Archimedean Polish group G on a standard Borel space by [BK96, Theorem 5.1.11] and [Gao09, Theorem 3.5.2, Theorem 11.3.8].

Turbulence is a dynamical condition on a Polish G-space X which is an obstruction of classifiability of E_G^X by countable structures. We now recall here the fundamental notions of the theory of turbulence, developed by Hjorth in [Hjo00]. Suppose that X is a Polish G-space, $x \in X$, U is a neighborhood of x, and V is a neighborhood of the identity in G. The local orbit $\mathcal{O}(x, U, V)$ is the smallest subset of U with the property that $x \in \mathcal{O}(x, U, V)$, and if $g \in V$, $x \in \mathcal{O}(x, U, V)$, and $gx \in U$, then $gx \in \mathcal{O}(x, U, V)$. A point $x \in X$ is called *turbulent* if it has dense orbit and, for any neighborhood U of x and neighborhood V of the identity in G, the closure of $\mathcal{O}(x, U, V)$ is a neighborhood of x. A Polish G-space X is *preturbulent* if every point $x \in X$ is turbulent, and *turbulent* if every point $x \in X$ is turbulent and has meager orbit.

An equivalence relation E on a Polish space X is generically S_∞-ergodic if, for any Polish S_∞-space Y and Baire-measurable $\left(E, E_{S_\infty}^Y\right)$-homomorphism, there exists a comeager subset of X that is mapped by f to a single S_∞-orbit. By [Gao09, Theorem 3.5.2, Theorem 11.3.8], this is equivalent to the assertion that, for any non-Archimedean Polish group H, Polish H-space Y, and Baire measurable $\left(E, E_H^Y\right)$-homomorphisms, there exists a comeager subset of X that is mapped by f to a single H-orbit. The following is the main result in Hjorth's turbulence theory, providing a dichotomy for preturbulent Polish G-spaces.

Theorem 3.5.1 (Hjorth)**.** *Suppose that X is a preturbulent Polish G-space. Then the associated orbit equivalence relation E_G^X is generically S_∞-ergodic. In particular, either X has a dense G_δ orbit, or the restriction of E_G^X to any comeager subset of X is not classifiable by countable structure.*

In this section, for each Polish G-space X, we define a graph structure $\mathcal{H}(X/G)$ with domain the quotient $X/G = \{[x] \colon x \in X\}$ of X via the action of G. We call this the Hjorth graph associated with the G-space

X. An (induced) subgraph of $\mathcal{H}(X/G)$ is of the form $\mathcal{H}(C/G)$, where C is an invariant subset of X. We view Hjorth's turbulence theorem as a corollary of the following facts:

1. $\mathcal{H}(X/G)$ contains only loops if G is non-Archimedean;
2. $\mathcal{H}(X/G)$ is a clique if the action of G on X is preturbulent;
3. given a Polish G-space X and a Polish H-space Y, a Baire measurable (E_G^X, E_H^Y)-homomorphism f induces, after restricting to an invariant dense G_δ set, a graph homomorphism between the corresponding Hjorth graphs.

We start by defining a game associated with points of a given Polish G-space, which captures isomorphism in the case of Polish S_∞-spaces.

Definition 3.5.2. *Suppose that X is a Polish G-space, and $x, y \in X$. We consider the* Hjorth-isomorphism game $\mathrm{Iso}(x, y)$ *played between two players as follows. Set $x_0 := x$, $y_0 := y$, $U_0^y := X$, and $V_0^y = G$.*

1. In the first turn, Player I plays an open neighborhood U_0^x of x_0 and an open neighborhood V_0^x of the identity in G. Player II replies with an element g_0^y in G.

2. In the second turn, Player I then plays an open neighborhood U_1^y of $y_1 := g_0^y y_0$ and an open neighborhood V_1^y of the identity of G, and Player II replies with an element g_0^x in G.

(2n+1) At the $(2n+1)$-st turn, Player I plays an open neighborhood U_n^x of $x_n := g_{n-1}^x x_{n-1}$ and an open neighborhood V_n^x of the identity of G, and Player II responds with an element g_n^y of G.

(2n+2) At the $(2n+2)$-nd turn, Player I plays an open neighborhood U_{n+1}^y of $y_{n+1} := g_n^y y_n$ and an open neighborhood V_{n+1}^y of the identity of G, and Player II responds with an element g_n^x of G.

The game proceed in this way, producing sequences (x_n) and (y_n) of elements of X, sequences (g_n^x) and (g_n^y) of elements of G, sequences (U_n^x) and (U_n^y) of open subsets of X, and sequences (V_n^x) and (V_n^y) of open neighborhoods of the identity in G. Player II wins the game if, for every $n \geq 0$,

- *$y_{n+1} \in U_n^x$ and $x_n \in U_n^y$,*
- *$g_n^y = h_k \cdots h_0$ for some $k \geq 0$ and $h_0, \ldots, h_k \in V_n^y$ such that $h_i \cdots h_0 y_n \in U_n^y$ for $i \leq k$,*
- *$g_n^x = h_k \cdots h_0$ for some $k \geq 0$ and $h_0, \ldots, h_k \in V_n^x$ such that $h_i \cdots h_0 x_n \in U_n^x$ for $i \leq k$.*

We write $x \sim_{\mathcal{H}} y$ and we say that x, y are Hjorth-isomorphic if Player II has a winning strategy for the Hjorth game $\mathcal{H}(x, y)$.

Remark 3.5.3. As in the case of the Becker-embedding game—see Remark 3.1.4—it is not difficult to see that, if Player II has a winning strategy for the Hjorth game as described above, then it also has a winning strategy for the same game with the additional winning conditions that $g_n^x = h_k \cdots h_0$ for some $h_0, \ldots, h_k$ from a given comeager subset of V_n^x such that $h_i \cdots h_0 x_n$ belongs to a given comeager subset X_0 of X for $i = 0, \ldots, k$, provided that the set of $h \in G$ such that $hx \in X_0$ is comeager. Similarly one can add the winning conditions that $g_n^y = h_k \cdots h_0$ for some $h_0, \ldots, h_k$ from a given comeager subset of V_n^y such that $h_i \cdots h_0 y_n$ belongs to a given comeager subset X_0 of X, provided that the set of $h \in G$ such that $hy \in X_0$ is comeager.

The relation $\sim_{\mathcal{H}}$ is an equivalence relation on X which we call *Hjorth isomorphism.* It is clear that Hjorth isomorphism is a coarsening of the orbit equivalence relation E_G on G. Furthermore if $x \sim_{\mathcal{H}} y$, x' belongs to the G-orbit of x, and y' belongs to the G-orbit of y, then $x' \sim_{\mathcal{H}} y'$. Let as before X/G be the space of G-orbits of elements of X. The Hjorth-graph $\mathcal{H}(X/G)$ associated with the Polish G-space X is symmetric, reflexive graph on X/G given by declaring that there exists an edge between the orbit $[x]$ of x and the orbit $[y]$ of y if and only if $x \sim_{\mathcal{H}} y$. We call $\mathcal{H}(X/G)$ the Hjorth graph associated with the Polish G-space X. One can similarly define the Hjorth graph $\mathcal{H}(C/G)$ for any invariant subset C of X. A comeager subgraph $\mathcal{G}$ of $\mathcal{H}(X/G)$ is a graph of the form $\mathcal{H}(C/G)$, for some invariant comeager subset C of X.

We now proceed to the proof of the properties of Hjorth graphs stated at the end of Section 3.5. In the following, for a subset V of G and $k \in \mathbb{N}$ let V^k be the set of elements of G that can be written as the product of k elements from V.

Lemma 3.5.4. *Suppose that H is a non-Archimedean Polish group, and Y is a Polish H-space. Then the Hjorth graph $\mathcal{H}(Y/H)$ contains only loops.*

Proof. Suppose that G is a non-Archimedean Polish group. Fix a compatible complete metric d on X, and a compatible complete metric d_G on G. We denote by $\mathrm{diam}(A)$ the diameter of a subset A of X with respect to the metric d, and by $\mathrm{cl}(A)$ the closure of A. Suppose that Player II has a winning strategy for the Hjorth-isomorphism game $\mathrm{Iso}(x, y)$. We want to show that x and y belong to the same orbit. This can be seen by letting Player I play open subsets U_n^x and U_n^y of X such that $\mathrm{cl}(U_{n+1}^y) \subset U_n^x$, $\mathrm{cl}(U_n^x) \subset U_n^y$, $\mathrm{diam}(U_n^x) \leq 2^{-n}$, $\mathrm{diam}\left(U_{n+1}^y\right) \leq 2^{-n}$, and open subgroups V_n^x and V_n^y of G such that

$$\begin{aligned} V_n^x &\subset \left\{g \in G : d_G\left(g g_{n-1}^x \cdots g_0^x, g_{n-1}^x \cdots g_0^x\right) < 2^{-n}\right\} \\ V_n^y &\subset \left\{g \in G : d_G\left(g g_{n-1}^y \cdots g_0^y, g_{n-1}^y \cdots g_0^y\right) < 2^{-n}\right\}. \end{aligned}$$

Let then (x_n) and (y_n) be the sequences of elements of X and (g_n^x) and (g_n^y) be the sequences of elements of G obtained from the corresponding round of the Hjorth game. Then the assumptions on U_n^x and U_n^y guarantee that the sequences (x_n) and (y_n) converge to the same point z of X. The assumptions on V_n^x and V_n^y guarantee that the sequences $\left(g_n^x g_{n-1}^x \cdots g_0^x\right)_{n\in\omega}$ and $\left(g_n^y g_{n-1}^y \cdots g_0^y\right)_{n\in\omega}$ converge in H to elements g_∞^x and g_∞^y such that $g_\infty^x x = z$ and $g_\infty^y y = z$. This shows that x and y belong to the same orbit. □

Lemma 3.5.5. *Suppose that X is a preturbulent Polish G-space. Then the Hjorth graph $\mathcal{H}(X/G)$ is a clique.*

Proof. Suppose that X is a preturbulent Polish G-space. Fix $x, y \in X$. We want to prove that Player II has a winning strategy for the Hjorth game $\mathcal{H}(x, y)$. We begin with a preliminary observation. Suppose that $z \in X$, U is an open neighborhood of z, and V is an open neighborhood of the identity in G. Let $\mathcal{I}(z, U, V)$ be the interior of the closure of the local orbit $\mathcal{O}(z, U, V)$. Since z is turbulent, $\mathcal{I}(z, U, V)$ contains z. It is not difficult to see that, for any $w \in \mathcal{I}(z, U, V)$, the local orbit $\mathcal{O}(w, \mathcal{I}(z, U, V), V)$ is dense in $\mathcal{I}(z, U, V)$. We use this observation to conclude that Player II has a winning strategy, which we proceed to define. As in the definition of the Hjorth game, we let $x_0 = x$, $y_0 = y$, $U_0^y = X$, and $V_0^y = G$. At the $(2n+1)$-st turn Player II plays an element $g_n^y = h_k \cdots h_0 \in (V_n^y)^k$ for some $k \geq 1$ such that $y_{n+1} = g_n^y y_n \in \mathcal{I}(x_n, U_n^x, V_n^x)$ and $h_i \cdots h_0 y_n \in U_n^y$ for $i \leq k$, while at the $(2n+2)$-nd turn Player II plays an element $g_n^x = h_k \cdots h_0 \in (V_n^x)^k$ for some $k \geq 1$ such that $x_{n+1} = g_n^x x_n \in \mathcal{I}\left(y_{n+1}, U_{n+1}^y, V_{n+1}^y\right)$ and $h_i \cdots h_0 x_n \in U_n^x$ for $i \leq k$. Such a choice is possible at the 1-st turn since y has dense orbit. It is possible at the $(2n+2)$-nd turn $(n \geq 0)$ since $y_{n+1} \in \mathcal{I}(x_n, U_n^x, V_n^x)$ and for every $w \in \mathcal{I}(x_n, U_n^x, V_n^x)$ the local orbit $\mathcal{O}(w, \mathcal{I}(x_n, U_n^x, V_n^x), V_n^x)$ is dense in $\mathcal{I}(x_n, U_n^x, V_n^x)$. It is possible at the $(2n+1)$-st turn $(n \geq 1)$ since $x_n \in \mathcal{I}(y_n, U_n^y, V_n^y)$ and for any $w \in \mathcal{I}(y_n, U_n^y, V_n^y)$ the local orbit $\mathcal{O}(w, \mathcal{I}(y_n, U_n^y, V_n^y), V_n^y)$ is dense in $\mathcal{I}(y_n, U_n^y, V_n^y)$. This concludes the proof that Player II has a winning strategy for the Hjorth game $\mathcal{H}(x, y)$. □

Proposition 3.5.6. *Suppose that G, H are Polish groups, X is a Polish G-space, and Y is a Polish H-space. If f is a Baire-measurable $\left(E_G^X, E_H^Y\right)$-homomorphism, then there exists a G-invariant dense G_δ subset X_0 of X such that the function $X_0/G \to Y/H$, $[x] \mapsto [f(x)]$ is a homomorphism from the Hjorth graph $\mathcal{H}(X_0/G)$ to the Hjorth graph $\mathcal{H}(Y/H)$.*

Proof. We proceed as in the proof of Proposition 3.3.3. Let C be dense G_δ subsets of X obtained from f as in Lemma 3.2.1. Set $X_0 := \{x \in X : \forall^* g \in G, gx \in C\}$, which is a G-invariant dense G_δ set by [Gao09, Proposition 3.2.5 and Theorem 3.2.7]. We claim that $X_0/G \to Y/H$, $[x] \mapsto [f(x)]$ is a graph homomorphism from the Hjorth graph $\mathcal{H}(X_0/G)$ to the Hjorth graph $\mathcal{H}(Y/H)$.

Fix $x_0, y_0 \in X_0$ such that $x_0 \sim_{\mathcal{H}} y_0$. We want to prove that $f(x_0) \sim_{\mathcal{H}} f(y_0)$. Observe that $\forall^* g \in G$, $gx_0 \in C \cap X_0$. Therefore after replacing x_0 with gx_0 for a suitable $g \in G$ we can assume that $x_0 \in C \cap X_0$.

In this case one can define, similarly as in the proof of Proposition 3.3.3, a winning strategy for Player II for $\mathrm{Iso}(f(x_0), f(y_0))$ from a winning strategy for Player II for $\mathrm{Iso}(x_0, y_0)$ using Remark 3.5.3 and the choice of C. □

It is now easy to see that Theorem 3.5.1 is an immediate consequence of Lemma 3.5.4 and Lemma 3.5.5 together with Proposition 3.5.6.

3.6 Groupoids

The goal of this section is to observe that the proofs above apply equally well in the setting of Polish groupoids as introduced in [Ram00, Ram90, Lup]. A *groupoid* G is a small category where every morphism (also called arrow) is invertible. By identifying any object with the corresponding identity arrow, one can regard the set G^0 of objects of G as a subset of G. The source and range maps $s, r : G \to G^0$ assign to every arrow in G its domain (or source) and codomain (or range). The set G^2 of composable arrows is the set of pairs (γ, ρ) of arrows from G such that $s(\gamma) = r(\rho)$. Composition of arrows is a function $G^2 \to G$, $(\gamma, \rho) \to \gamma\rho$. If $A, B \subset G$, then we denote by AB the set $\{\gamma\rho : (\gamma, \rho) \in G^2 \cap (A \times B)\}$. If $x \in G^0$ and $A \subset G$, then we let $Ax := A\{x\} = \{\gamma \in G : s(\gamma) = x\}$ and $xA := \{x\} A = \{\gamma \in G : r(\gamma) = x\}$.

A *Polish groupoid* is a groupoid G endowed with a topology such that

(1) there exists a countable basis $\mathcal{B}$ of Polish open sets,

(2) composition and inversion of arrows are continuous and open,

(3) the sets Gx and xG are Polish subspaces for every $x \in G^0$, and

(4) the set of objects G^0 is a Polish subspace.

A Polish groupoid is not required to be globally Hausdorff. Many Polish groupoids arising in the applications, such as the locally compact groupoids associated with foliations of manifolds, are not Hausdorff; see [Pat99, Chapter 2].

Suppose that H is a Polish group. One can associate with any Polish H-space X a Polish groupoid $H \ltimes X$—the *action groupoid*—that completely encodes the action. Such a groupoid has the Cartesian product $H \times X$ as set of arrows (endowed with the product topology), and $\{(1_H, x) : x \in X\}$ as set of objects. Source and range maps are defined by $s(h, x) = (1_H, x)$ and $r(h, x) = (1_H, hx)$. Composition is given by $(h, x)(h', y) = (hh', y)$ whenever $x = h'y$. In this way one can regard continuous actions of Polish groups on Polish spaces as a particular instance of Polish groupoids. One can also consider continuous actions of Polish groupoids on Polish spaces, but these can be in turn regarded as Polish groupoids via a similar construction as the one described above. The class of Polish groupoids is also closed under taking restrictions. If X is a G_δ subset of the set of objects of a Polish groupoid G, then the *restriction* $G|_X$ is the collection of arrows of G with source and range in X, endowed with the induced Polish groupoid structure. More information about Polish groupoids can be found in [Lup].

Given a Polish groupoid G, the orbit equivalence relation E_G is the equivalence relation on G^0 defined by setting xE_Gy if and only if x, y are source and range of an arrow from G. The orbit of an object in G is the E_G-class of x.

The notion of (pre)turbulence for Polish groupoid has been considered in [HL, Section 4]. Suppose that G is a Polish groupoid, x is an object of G, and U is a neighborhood of x in G. The local orbit $\mathcal{O}(x, U)$ is the smallest subset of $U \cap G^0$ with the property that $x \in \mathcal{O}(x, U)$, and if $\gamma \in U$ is such that $s(\gamma) \in \mathcal{O}(x, U)$, then $r(\gamma) \in \mathcal{O}(x, U)$. An object x is called turbulent if it has orbit dense in G^0 and, for any neighborhood U of x, the closure of $\mathcal{O}(x, U)$ is a neighborhood of x in G^0. A Polish groupoid is preturbulent if every object is turbulent, and turbulent if every object is turbulent and has orbit meager in G^0. It is not difficult to see that these definitions are consistent with the ones for Polish group actions, when a Polish group action is identified with its associated action groupoid.

Suppose that G is a Polish groupoid, and $x, y \in G^0$ are two objects of G. The Hjorth-isomorphism game $\mathrm{Iso}(x, y)$ can be defined similarly as in Definition 3.5.2. Set $x_0 := x$, $y_0 := y$, $U_0^y = G$, and $V_0^y = G$. In this case, in the first turn Player I plays an open neighborhood U_0^x of x_0 in G and Player II replies with an element γ_0^y of G with $s\left(\gamma_0^y\right) = y_0$. In the second turn, Player I plays an open neighborhood U_1^y of $y_1 := r\left(\gamma_0^y\right)$ in G and an element γ_0^x of G with $s\left(\gamma_0^x\right) = x_0$. At the $(2n+1)$-st turn, Player I plays an open neighborhood U_n^x of $x_n := r\left(\gamma_{n-1}^x\right)$ in G, and Player II responds with an element γ_n^y of G with $s\left(\gamma_n^y\right) = y_n$. At the $(2n+2)$-nd turn, Player I plays an open neighborhood U_{n+1}^y of $y_{n+1} := r\left(\gamma_n^y\right)$ in G, and Player II responds with an element γ_n^x of G.

The game then produces sequences $(x_n), (y_n)$ of objects of G, sequences $(\gamma_n^x), (\gamma_n^y)$ of arrows in G, and sequences $(U_n^x), (U_n^y)$ of open subsets of G. Player II wins the game if, for every $n \geq 0$,

- $y_{n+1} \in U_n^x$ and $x_n \in U_n^y$,
- $\gamma_n^y = \rho_1^y \rho_2^y \cdots \rho_k^y$ for some $k \geq 1$ and $\rho_i^y \in V_n^x$ for $i = 1, 2, \ldots, k$, and $\gamma_n^x = \rho_1^x \cdots \rho_k^x$ for some $k \geq 1$ and $\rho_i^x \in V_n^y$ for $i = 1, 2, \ldots, k$.

As in the case of Polish group actions, this defines an equivalence relation $\sim_{\mathcal{H}}$ (Hjorth-isomorphism) on the set of objects of G, by letting $x \sim_{\mathcal{H}} y$ whenever Player II has a winning strategy for the Hjorth-isomorphism game $\mathrm{Iso}(x,y)$. Adding to the winning conditions in the Hjorth-isomorphism game the requirement that $r(\gamma_n^x)$ belongs to a given comeager subset X of G^0 and that γ_n^x belongs to a given comeager subset of Gx_n yields an equivalent game, provided that the set of $\gamma \in Gx$ such that $r(\gamma) \in X$ is comeager. The same applies to y. The Hjorth-isomorphism relation on G^0 defines a graph structure $\mathcal{H}(G)$ on the space of G-orbits, which we call the Hjorth graph of G. The same proof as Lemma 3.5.5 shows that if G is a preturbulent Polish groupoid, then the Hjorth graph $\mathcal{H}(G)$ is a clique. The analogue of Lemma 3.2.1 for Polish groupoids has been proved in [HL, Lemma 4.5]. Using this one can then prove the analog of Proposition 3.5.6 and deduce the following result.

Theorem 3.6.1. *Suppose that G is a preturbulent Polish groupoid. Then the associated orbit equivalence relation E_G is generically S_∞-ergodic.*

Theorem 3.6.1 recovers [HL, Theorem 4.3], and can be seen as the groupoid version of Theorem 3.5.1 for Polish groupoids.

Since the operations in the groupoid G are continuous and open, one can reformulate the Hjorth-isomorphism game $\mathrm{Iso}(x,y)$ as presented above by letting Player II play open sets rather than groupoid elements. Fix a countable basis $\mathcal{B}$ of Polish open subsets of G. In this formulation of the game, Player I plays elements U_n^x, U_{n+1}^y of $\mathcal{B}$ for $n \geq 0$ and player II plays elements W_n^x, W_n^y of $\mathcal{B}$ for $n \geq 0$. The winning conditions are then, setting $U_0^y = G$,

- $r\left[W_{n+1}^y\right] \subset U_n^x$ and $r\left[W_n^x\right] \subset U_n^y$,
- $W_n^y \subset (U_n^y)^k$ for some $k \geq 1$ and $W_n^x \subset (U_n^y)^k$ for some $k \geq 1$,
- $y \in s\left[W_n^y \cdots W_0^y\right]$ and $x \in s\left[W_n^x \cdots W_0^x\right]$.

Such a version of the Hjorth-isomorphism game fits in the framework of Borel games as described in [Kec95, Section 2.A]. In fact, this is an *open* game for Player I and closed for Player II, which allows one to define an ω_1-valued *rank* for strategies of Player I [Kec95, Exercise 20.2]. Insisting that Player I only has winning strategies of rank at least $\alpha \in \omega_1$ (or no winning strategy at all) gives a hierarchy of equivalence relations $\sim_\alpha$ indexed by countable ordinals, whose intersection is the Hjorth isomorphism relation.

Similarly as for the Hjorth-isomorphism game, the Becker-embedding game $\mathrm{Emb}(x,y)$ can be defined whenever x, y are objects in a Polish groupoid G. This gives a notion of Becker embedding for objects G, by letting $x \preccurlyeq_{\mathcal{B}} y$ if and only if Player II has a winning strategy for $\mathrm{Emb}(x,y)$. In turn this induces a digraph structure $\mathcal{B}(G)$ on the space of G-orbits.

One can prove the groupoid analog of Proposition 3.3.3 in a similar fashion, by replacing Lemma 3.2.1 with [HL, Lemma 4.5]. One can then deduce the following generalization of Theorem 3.3.4 to Polish groupoids.

Theorem 3.6.2. *Suppose that G is a Polish groupoid. If for any invariant dense G_δ subset C of G^0 there exist $x, y \in C$ with different orbits such that $x \preccurlyeq_{\mathcal{B}} y$, then the orbit equivalence relation E_G is not CLI-classifiable.*

As for the case of the Hjorth-isomorphism game, one can also describe the Becker-embedding game $\mathrm{Emb}(x, y)$ for objects x, y in a Polish groupoid G as an open game for Player I and closed for Player II. This allows one to define an ω_1-valued rank for strategies for Player I. Again, insisting that Player I only has winning stategies of rank at least $\alpha \in \omega_1$ gives a hierarchy or preorder relations $\preccurlyeq_\alpha$ indexed by countable ordinals, whose intersection is the Becker-embeddability preorder.

Chapter 4

Compact spaces as projective Fraïssé limits

4.1 Preliminaries

In what follows, K will always denote a zero-dimensional, compact, metrizable space. Our main objects of study will be spaces K as above, which support both the usual model-theoretic structure as well as dual structure. To make this precise, following S. Solecki [Sol10, Sol12], we consider the following two types of tuples.

- In the classical model-theoretic context, a tuple of size $n > 0$ in K corresponds to an injection

  $$i : \{0, \dots, n-1\} \to K$$

  We will call this kind of tuple a *direct tuple*. We denote the set of all direct tuples in K of size $n > 0$ by $K^{[n]}$.

- In the dual context, a tuple of size n in K corresponds to a surjection

  $$e : K \to \{0, \dots, n-1\}$$

  Since our intention is to work with "topological" structures, we endow $\{0, \dots, n-1\}$ with the discrete topology and we impose a further regularity condition, that e is also continuous. We will call this kind of tuple a *dual tuple*. We denote the set of all dual tuples in K of size n by $[n]^K$.

Notice that for K zero-dimensional, compact, metrizable space, the set $[n]^K$ is at most countably infinite and moreover these functions suffice to separate points of K, i.e., for every $x_1, x_2 \in K$ there is an $e \in [n]^K$ such that $e(x_1) \neq e(x_2)$. Notice also that the set $[n]^K$ of all dual tuples naturally corresponds to the set $\mathrm{CP}_n(K)$ of all clopen, ordered, n-partitions of K, i.e.,

$$\mathrm{CP}_n(K) = \big\{(\Delta_0, \dots, \Delta_{n-1}) : \Delta_i \subset K \ \text{clopen}, \ \ \Delta_i \cap \Delta_j = \emptyset \ \ \cup_i \Delta_i = K\big\}.$$

Whenever it is convenient for notational purposes we will not distinguish between the set $\{\Delta_0, \dots, \Delta_{n-1}\}$ and the tuple $(\Delta_0, \dots, \Delta_{n-1})$. If for example $P \in \mathrm{CP}_n(K)$ and Δ is a clopen set appearing some of the n entries of P then we will write $\Delta \in P$.

We will work with relational only languages $\mathcal{L}$. The structures we describe here, with additional dual function symbols, were introduced in [Sol10, Sol12]. To each relational symbol R in $\mathcal{L}$ corresponds some natural number $\mathrm{arity}(R) > 0$ which is the arity of the symbol R. Moreover, for every symbol in $\mathcal{L}$ we have predetermined our intention to use it in the direct or in the dual context. We will make the convention here of using lower case letters $r, p, q, \dots$ for direct relational symbols, and capital letters $R, P, Q, \dots$ for dual relational symbols. We will call the language $\mathcal{L}$ *purely direct* if it contains only direct symbols and *purely dual* if it contains only dual symbols

By a *topological $\mathcal{L}$-structure $\boldsymbol{K}$* we mean a zero-dimensional, metrizable, compact space K together with appropriate interpretations for every symbol in $\mathcal{L}$.

- If $r \in \mathcal{L}$ is a direct relation symbol of arity n, an appropriate interpretation for r is any closed subset $r^{\boldsymbol{K}}$ of $K^{[n]}$.

- If $R \in \mathcal{L}$ is a dual relation symbol of arity n, an appropriate interpretation for R is any subset $R^{\boldsymbol{K}}$ of $[n]^K$, or equivalently, any subset of $\mathrm{CP}_n(K)$.

We call a topological $\mathcal{L}$-structure *purely direct $\mathcal{L}$-structure* whenever $\mathcal{L}$ is purely direct and *purely dual $\mathcal{L}$-structure* whenever $\mathcal{L}$ is purely dual.

Following [IS06], we will be working with epimorphisms. The epimorphisms here will additionally preserve the dual structure. Such epimorphisms were introduced in [Sol10, Sol12]. Let $\boldsymbol{A}, \boldsymbol{B}$ be two dual topological $\mathcal{L}$-structure. By an *epimorphism* f from $\boldsymbol{A}$ to $\boldsymbol{B}$ we mean a continuous surjection $f : A \to B$ such that:

- for every $r \in \mathcal{L}$ of arity say m and every $\beta \in B^{[m]}$ we have

$$\beta \in r^{\boldsymbol{B}} \iff \exists \alpha \in r^{\boldsymbol{A}} \quad \beta = f \circ \alpha$$

- for every $R \in \mathcal{L}$ of arity say m and for every $\beta \in [m]^B$ we have

$$\beta \in R^{\boldsymbol{B}} \iff \beta \circ f \in R^{\boldsymbol{A}}$$

An isomorphism between $\boldsymbol{A}$ and $\boldsymbol{B}$ is a bijective epimorphism and an automorphism of $\boldsymbol{A}$ is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{A}$.

Let $\boldsymbol{K}$ be a topological $\mathcal{L}$-structure, let A be a zero-dimensional, metrizable, compact space and let $f : K \to A$ be a continuous surjection. Notice that there is a unique topological $\mathcal{L}$-structure $\boldsymbol{A}$ on domain A that renders f an epimorphism. We call this structure $\boldsymbol{A}$, the structure *induced by the map* f.

Let $f : \boldsymbol{K} \to \boldsymbol{A}, h : \boldsymbol{K} \to \boldsymbol{B}$ be epimorphisms. We say that f *factors through* h if there is an epimorphism $f_h : \boldsymbol{B} \to \boldsymbol{A}$ such that $f_h \circ h = f$.

Lemma 4.1.1. *Let $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{K}$ be topological $\mathcal{L}$-structures with $\boldsymbol{A}, \boldsymbol{B}$ finite. Let also $f : \boldsymbol{K} \to \boldsymbol{A}$ and $g : \boldsymbol{K} \to \boldsymbol{B}$ be epimorphisms. Then there is a finite topological $\mathcal{L}$-structure $\boldsymbol{C}$ and an epimorphism $h : \boldsymbol{K} \to \boldsymbol{C}$ such that both f and g factor through h.*

Proof. Let $C = (\Delta_0, \ldots, \Delta_{n-1})$ a clopen partition of K whose every entry Δ_i is a (nonempty) set of the form $f^{-1}(a) \cap g^{-1}(b)$, where $a \in A$ and $b \in B$. Let $h : K \to C$ be the inclusion map, i.e, $h(x) = \Delta_i$ if and only if $x \in \Delta_i$. This map is a continuous surjection, so, it induces a structure $\boldsymbol{C}$ on domain C. It is immediate now that both f and g factor through h. □

Given a sequence $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_i, \ldots$ of finite topological $\mathcal{L}$-structures together with epimorhisms $\pi_i : \boldsymbol{A}_{i+1} \to \boldsymbol{A}_i$, we can define a new structure $\boldsymbol{M}$ and epimorphisms $\pi_i^\infty : \boldsymbol{M} \to \boldsymbol{A}_i$ through an inverse limit construction. Let

$$M = \big\{(a_1, a_2, \ldots) \in \prod_{i \in \mathbb{N}} A_i : \ \forall\, i \geq 1 \ \ \pi_i(a_{i+1}) = a_i\big\}.$$

M is a closed subset of the compact space $\prod_{i \in \mathbb{N}} A_i$ and it will serve as the domain of $\boldsymbol{M}$. We define π_i^∞ to be the projection map from M to A_i.

For $r \in \mathcal{L}$ of arity say m, and $\beta \in M^{[m]}$ we let $\beta \in r^{\boldsymbol{M}}$ if and only if $\pi_i^\infty \circ \beta \in r^{\boldsymbol{A}_i}$ for all $i \in \mathbb{N}$. For $R \in \mathcal{L}$ of arity say m, and $\gamma \in [m]^M$, notice that there is an $i_0 \in \mathbb{N}$ such that γ factors through $\pi_{i_0}^\infty$. Let $\alpha \in [m]^A$ be such that $\gamma = \alpha \circ \pi_{i_0}^\infty$. We let $\gamma \in R^{\boldsymbol{M}}$ if and only if $\alpha \in R^{\boldsymbol{A}_{i_0}}$, which happens if and only if for every $i > i_0$ we have $(\alpha \circ \pi_{i_0} \circ \ldots \circ \pi_{i-1}) \in R^{\boldsymbol{A}_i}$.

This turns $\boldsymbol{M}$ into a topological $\mathcal{L}$-structure and every π_i^∞ to an epimorphism. We call $\boldsymbol{M}$ the *inverse limit* of the *inverse system* $\{(\boldsymbol{A}_i, \pi_i) : i \in \mathbb{N}\}$ and we write $\boldsymbol{M} = \varprojlim(\boldsymbol{A}_i, \pi_i)$.

4.2 Projective Fraïssé structures

In Chapter 7 of [Hod93], Hodges reviews the theory of Fraïssé limits of direct structures via direct morphisms (embeddings). Following Hodges and [IS06] we present here the theory of Fraïssé limits of topological $\mathcal{L}$-structures via dual morphisms. To avoid confusion, we should emphasize two things. What in [IS06] is called topological $\mathcal{L}$-structure, here it falls under the name purely direct topological $\mathcal{L}$-structure. Moreover, in contrast with the definition that we will be using here, a projective Fraïssé class in [IS06] is not bound to satisfy the hereditary property (HP).

We say that a topological $\mathcal{L}$-structure $\boldsymbol{M}$ is *projectively Fraïssé* or *projectively ultra-homogeneous* if for every two epimorphisms f_1, f_2 of $\boldsymbol{M}$ on some finite topological $\mathcal{L}$-structure $\boldsymbol{A}$ there is an automorphism g of $\boldsymbol{M}$ such that $f_1 \circ g = f_2$. For every topological $\mathcal{L}$-structure $\boldsymbol{M}$ we denote by $\mathrm{Age}(\boldsymbol{M})$ the class of all the finite topological $\mathcal{L}$-structures $\boldsymbol{A}$ such that $\boldsymbol{M}$ epimorphs on $\boldsymbol{A}$. We call a class $\mathcal{K}$ of topological $\mathcal{L}$-structures an *age* if $\mathcal{K} = \mathrm{Age}(\boldsymbol{M})$ for some topological $\mathcal{L}$-structure $\boldsymbol{M}$. It is immediate that if $\mathcal{K}$ is an age, then $\mathcal{K}$ is not empty, any subclass of $\mathcal{K}$ of pairwise non-isomorphic structures is at most countable, and the following two properties hold for $\mathcal{K}$.

- *Hereditary Property* (HP): if $\boldsymbol{A} \in \mathcal{K}$ and $\boldsymbol{A}$ epimorphs onto a structure $\boldsymbol{B}$, then $\boldsymbol{B} \in \mathcal{K}$.
- *Joint Surjecting Property* (JSP): if $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ then there is $\boldsymbol{C} \in \mathcal{K}$ that epimorphs onto both $\boldsymbol{A}$ and $\boldsymbol{B}$.

The converse is also true i.e. if $\mathcal{K}$ is a non empty class of finite topological $\mathcal{L}$-structures such that any subclass of $\mathcal{K}$ of pairwise non-isomorphic structures is at most countable and the above two properties hold for $\mathcal{K}$ then $\mathcal{K}$ is an age.

To see this, let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots$ be a list of structures in $\mathcal{K}$ that up to isomorphism exhaust $\mathcal{K}$. Using the JSP we can find a new list $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots$ of structures in $\mathcal{K}$ such that $\boldsymbol{B}_1 = \boldsymbol{A}_1$ and for $i > 1$, $\boldsymbol{B}_{i+1}$ epimorphs on both $\boldsymbol{A}_{i+1}$ and $\boldsymbol{B}_i$. Let $\pi_i : \boldsymbol{B}_{i+1} \to \boldsymbol{B}_i$ be such epimorphisms and let $\boldsymbol{M} = \varprojlim(\boldsymbol{B}_i, \pi_i)$. Then $\mathcal{K} = \mathrm{Age}(\boldsymbol{M})$ because by construction $\boldsymbol{M}$ epimorphs to every $\boldsymbol{A}_i$ and moreover, every epimorphism of $\boldsymbol{M}$ to some finite dual topological $\mathcal{L}$-structure $\boldsymbol{A}$ factors through an epimorphism from some $\boldsymbol{B}_i \in \mathcal{K}$ that was used in the inverse system so, by HP we have that $\boldsymbol{A} \in \mathcal{K}$.

Given now that the structure $\boldsymbol{M}$ is projectively ultrahomogeneous, it is easy to see that its age $\mathcal{K}$ satisfies moreover the following property.

- *Projective Amalgamation Property* (PAP): if $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and $f_A : \boldsymbol{A} \to \boldsymbol{C}$, $f_B : \boldsymbol{B} \to \boldsymbol{C}$ are epimorphisms, then there is $\boldsymbol{D} \in \mathcal{K}$ and epimorphisms $g_A : \boldsymbol{D} \to \boldsymbol{A}$, $g_B : \boldsymbol{D} \to \boldsymbol{B}$ such that $f_A \circ g_A = f_B \circ g_B$.

To check that this is true, notice first that since $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, there are epimorphisms $h_A : \boldsymbol{M} \to \boldsymbol{A}$ and $h_B : \boldsymbol{M} \to \boldsymbol{B}$. But then $f_A \circ h_A$ and $f_B \circ h_B$ are both epimorphisms from $\boldsymbol{M}$ to $\boldsymbol{C}$. So, by projective ultra-homogeneity of $\boldsymbol{M}$ there is $\phi \in \mathrm{Aut}(\boldsymbol{M})$ such that $f_A \circ h_A \circ \phi = f_B \circ h_B$. Using Lemma 4.1.1, we can find $\boldsymbol{D} \in \mathcal{K}$ and an epimorphism $h_D : \boldsymbol{M} \to \boldsymbol{D}$ such that h_A and $h_B \circ \phi^{-1}$ factor through h_D. Let $g_A : \boldsymbol{D} \to \boldsymbol{A}$

and $g_B : \boldsymbol{D} \to \boldsymbol{B}$ be the maps that close these diagrams, i.e., $g_A \circ h_D = h_A$ and $g_B \circ h_D = h_B \circ \phi^{-1}$. The functions g_A and g_B are the required epimorphisms from $\boldsymbol{D}$ to $\boldsymbol{A}$ and $\boldsymbol{B}$ in respect.

In Theorem 4.2.3 we will see that the converse is also true, i.e., if an age $\mathcal{K}$ has PAP then we can built from it a projective Fraïssé structure $\boldsymbol{M}$ with $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$. An age $\mathcal{K}$ that satisfies PAP is called *projective Fraïssé class.*

Let $\boldsymbol{M}$ be a topological $\mathcal{L}$-structure with $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$. We say that $\boldsymbol{M}$ has the *finite extension property* if for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ and $f : \boldsymbol{B} \to \boldsymbol{A}$, $g : \boldsymbol{M} \to \boldsymbol{A}$ epimorphisms, there is an epimorphism $h : \boldsymbol{M} \to \boldsymbol{B}$ such that $f \circ h = g$. We say that $\boldsymbol{M}$ has the *one point extension property* if the above holds when the size of B is one more than the size of A. Notice that for any topological $\mathcal{L}$-structure $\boldsymbol{M}$, $\boldsymbol{M}$ has the one point extension property if and only if $\boldsymbol{M}$ has the finite extension property.

Lemma 4.2.1. *Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be two topological $\mathcal{L}$-structure of the same age $\mathcal{K}$. Let $\boldsymbol{A} \in \mathcal{K}$ and let $f : \boldsymbol{M} \to \boldsymbol{A}$ and $g : \boldsymbol{N} \to \boldsymbol{A}$ be two epimorphisms. If $\boldsymbol{M}$ and $\boldsymbol{N}$ have the finite extension property then there is an isomorphism $h : \boldsymbol{M} \to \boldsymbol{N}$ such that $g \circ h = f$.*

Proof. We will use a back and forth type of argument. For every $n \in \mathbb{N}$ we will construct $\boldsymbol{A}_n \in \mathcal{K}$ and epimorphisms $f_n : \boldsymbol{M} \to \boldsymbol{A}_n$, $g_n : \boldsymbol{N} \to \boldsymbol{A}_n$, and for every $n > 0$ we will also construct an epimorphism $\pi_{n-1} : \boldsymbol{A}_n \to \boldsymbol{A}_{n-1}$. At the end of the construction $\boldsymbol{M}$ and $\boldsymbol{N}$ will be proven to be isomorphic to $\varprojlim(\boldsymbol{A}_n, \pi_n)$. By using these indirect isomorphisms we will get the desired isomorphism h. Let $\{e_n : n \in \mathbb{N}\}$ be an enumeration of dual tuples $[m]^M$ of M for every $m > 0$ and let $\{e'_n : n \in \mathbb{N}\}$ be an enumeration of dual tuples $[m]^N$ of N for every $m > 0$.

$n = 0$. Let $\boldsymbol{A}_0 = \boldsymbol{A}$, $f_0 = f$ and $g_0 = g$.

odd $n > 0$. Using Lemma 4.1.1 we can find a structure $\boldsymbol{A}_n$ and an epimorphism $f_n : \boldsymbol{M} \to \boldsymbol{A}_n$ such that both f_{n-1} and e_{n-1} factor through f_n. Let $\pi_{n-1} : \boldsymbol{A}_n \to \boldsymbol{A}_{n-1}$ be the epimorphism that closes the one diagram, i.e., $\pi_{n-1} \circ f_n = f_{n-1}$. Finally define $g_n : \boldsymbol{N} \to \boldsymbol{A}_n$ to be any map such that $\pi_{n-1} \circ g_n = g_{n-1}$. A map like this exists, since $\boldsymbol{N}$ satisfies the finite extension property.

even $n > 0$. Again, using Lemma 4.1.1 we can find a structure $\boldsymbol{A}_n$ and an epimorphism $g_n : \boldsymbol{N} \to \boldsymbol{A}_n$ such that both g_{n-1} and e'_{n-1} factor through g_n. Let $\pi_{n-1} : \boldsymbol{A}_n \to \boldsymbol{A}_{n-1}$ be the epimorphism that closes the one diagram, i.e., $\pi_{n-1} \circ g_n = g_{n-1}$. Finally define $f_n : \boldsymbol{M} \to \boldsymbol{A}_n$ to be any map such that $\pi_{n-1} \circ f_n = f_{n-1}$. A map like this exists, since $\boldsymbol{M}$ satisfies the finite extension property.

Let now $\boldsymbol{B} = \varprojlim(\boldsymbol{A}_n, \pi_n)$. The maps $\mu : \boldsymbol{M} \to \boldsymbol{B}$ with $\mu(x) = (f_0(x), f_1(x), \ldots)$ and $\nu : \boldsymbol{N} \to \boldsymbol{B}$ with $\nu(x) = (g_0(x), g_1(x), \ldots)$ are bijections since the families $\{e_n\}$ and $\{e'_n\}$ separate points of M and N in respect. It is moreover easy to see that μ and ν are actually isomorphisms. So, the map $h : \boldsymbol{M} \to \boldsymbol{N}$ with $h = \nu^{-1} \circ \mu$ is also an isomorphism which by construction satisfies the desired property $g \circ h = f$. □

Lemma 4.2.2. *Let $\boldsymbol{M}$ be a topological $\mathcal{L}$-structure with* $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$ *then the following are equivalent:*

(1) $\boldsymbol{M}$ is projectively ultrahomogeneous;

(2) $\boldsymbol{M}$ has the finite extension property;

(3) $\boldsymbol{M}$ has the one point extension property.

Proof. It is immediate that (2) and (3) are equivalent. We prove that (1) is also equivalent to (2).

(1) $\rightarrow$ (2) Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ and $f : \boldsymbol{B} \rightarrow \boldsymbol{A}$, $g : \boldsymbol{M} \rightarrow \boldsymbol{A}$ epimorphisms. Since $\boldsymbol{B} \in \mathcal{K} = \mathrm{Age}(\boldsymbol{M})$, there is an epimorphism $j : \boldsymbol{M} \rightarrow \boldsymbol{B}$. So, $f \circ j : \boldsymbol{M} \rightarrow \boldsymbol{A}$ is an epimorphism, and by the projective ultra-homogeneity of $\boldsymbol{M}$ there is $\phi \in \mathrm{Aut}(\boldsymbol{M})$ with $g \circ \phi = f \circ j$. Let $h = j \circ \phi^{-1}$. Then $h : \boldsymbol{M} \rightarrow \boldsymbol{B}$ is an epimorphism such that $f \circ h = g$.

(2) $\rightarrow$ (1) Let $f_1, f_2 : \boldsymbol{M} \rightarrow \boldsymbol{A}$ be epimorphisms for some $\boldsymbol{A} \in \mathcal{K}$. Then by Lemma 4.2.1, there is $g \in \mathrm{Aut}(\boldsymbol{M})$ such that $f_1 \circ g = f_2$. □

Theorem 4.2.3. *For every projective Fraïssé class $\mathcal{K}$ there is a unique, up to isomorphism, projectively ultra-homogeneous topological $\mathcal{L}$-structure $\boldsymbol{M}$ such that* $\mathrm{Age}(\boldsymbol{M}) = \mathcal{K}$.

Proof. First notice that is $\boldsymbol{M}_1, \boldsymbol{M}_2$ share the same age and are both projectively ultra-homogeneous, by Lemma 4.2.2 they have finite extension property. Let $\boldsymbol{A}$ any structure in $\mathcal{K}$. Since $\mathcal{K}$ is the age of both $\boldsymbol{M}_1, \boldsymbol{M}_2$, there are epimorphisms $f_1 : \boldsymbol{M}_1 \rightarrow \boldsymbol{A}$ and $f_2 : \boldsymbol{M}_2 \rightarrow \boldsymbol{A}$. Lemma 4.2.1 gives as then an isomorphism h between $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. □

4.3 Closed subgroups of $\mathrm{Homeo}(K)$

By definition and since K is compact, every automorphism of a topological $\mathcal{L}$-structure $\boldsymbol{K}$ is also a homeomorphism, therefore, $\mathrm{Aut}(\boldsymbol{K})$ can be seen as a subgroup of $\mathrm{Homeo}(K)$. We will view $\mathrm{Homeo}(K)$ as a topological group equipped with the compact-open topology τ_{co}. The collection of the sets

$$V(F, U) = \{g \in \mathrm{Homeo}(K) : g(F) \subset U\},$$

where F is a compact subset of K and U is an open subset of K, provide a subbase for τ_{co}. In this topology the group $\mathrm{Aut}(\boldsymbol{K})$ of automorphisms of a dual topological $\mathcal{L}$-structure $\boldsymbol{K}$ is a closed subgroup of $\mathrm{Homeo}(K)$. To check this, let $g \notin \mathrm{Aut}(\boldsymbol{K})$. We will find an open neighborhood V_g of g in $\mathrm{Homeo}(K)$ which does not intersect $\mathrm{Aut}(\boldsymbol{K})$. Since $g \notin \mathrm{Aut}(\boldsymbol{K})$, one of the following holds:

(1) there is $R \in \mathcal{L}$ of arity say m and a dual tuple $e \in [m]^K$ such that $\boldsymbol{K} \models R(e)$ if and only if $\boldsymbol{K} \not\models R(e \circ g^{-1})$, or

(2) there is $r \in \mathcal{L}$ of arity say m and a tuple $i \in K^{[m]}$ such that $\boldsymbol{K} \models r(i)$ but $\boldsymbol{M} \not\models r(g \circ i)$, or

(3) there is $r \in \mathcal{L}$ of arity say m and a tuple $i \in K^{[m]}$ such that $\boldsymbol{K} \not\models r(i)$ but $\boldsymbol{M} \models r(g \circ i)$.

In the first case notice that if $g \notin \mathrm{Aut}(\boldsymbol{K})$ then there is $R \in \mathcal{L}$ of arity say m and But then $V\big(e^{-1}(0), g \circ e^{-1}(0)\big) \cap \ldots \cap V\big(e^{-1}(m-1), g \circ e^{-1}(m-1)\big)$ is an open subset of $\mathrm{Homeo}(K)$ containing g and lying entirely out of $\mathrm{Aut}(\boldsymbol{K})$.

In the second case, because $r^{\boldsymbol{M}}$ is closed, we can find an open rectangle $U_0 \times \ldots \times U_{m-1}$ around $\big((g \circ i)(0), \ldots, (g \circ i)(m-1)\big)$ which does not intersect $r^{\boldsymbol{M}}$. Therefore, let $V_g = V\big(\{i(0)\}, U_0\big) \cap \ldots \cap V\big(\{i(m-1)\}, U_{m-1}\big)$.

For the last case, notice that if we let $(b_0, \ldots, b_{m-1}) = \big((g \circ i)(0), \ldots, (g \circ i)(m-1)\big)$, then, as in the previous case we can find open neighborhood $V_{g^{-1}}$ of g^{-1} such that for every $f \in V_{g^{-1}}$, $f(b_0, \ldots, b_{m-1}) \notin r^{\boldsymbol{M}}$. Let then $V_g = V_{g^{-1}}^{-1} = \{f^{-1} : f \in V_{g^{-1}}\}$. Using the continuity of the inversion operator $f \to f^{-1}$ in τ_{co} we have that V_g is open and moreover $g \in V_g \subset \mathrm{Aut}(\boldsymbol{K})^c$.

The following proposition says that the inverse of the above observation is true, i.e., for every closed subgroup G of $\mathrm{Homeo}(K)$ there is topological $\mathcal{L}$-structure $\boldsymbol{K}$ on K such that $G = \mathrm{Aut}(\boldsymbol{K})$. Moreover, $\boldsymbol{K}$ can be taken to be purely dual and projectively ultra-homogeneous.

Proposition 4.3.1. *Let G be a closed subgroup of* $\mathrm{Homeo}(K)$. *Then there is a purely dual projective Fraïssé structure $\boldsymbol{K}$ on domain K such that* $\mathrm{Aut}(\boldsymbol{K}) = G$.

Proof. For every $n > 0$, the group G acts on $[n]^K$ in a natural way: for $g \in G$ and $e \in [n]^K$ let

$$g \cdot e := e \circ g^{-1}.$$

We denote this action by $G \curvearrowright [n]^K$. Notice that this action corresponds to the following action $G \curvearrowright \mathrm{CP}_n$ of G on CP_n: for $g \in G$ and $P = (\Delta_0, \ldots, \Delta_{n-1}) \in \mathrm{CP}_n(K)$ let

$$g \cdot P := \big(g(\Delta_0), \ldots, g(\Delta_{n-1})\big).$$

For each $n > 0$ let $(\mathcal{O}_i^n : i \in I_n)$ be the collection of all orbits of $G \curvearrowright [n]^K$.

Consider now the language $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}^n$, where $\mathcal{L}^n$ in the language that consists of n-ary relational symbols $\{O_i^n : i \in I_n\}$, one for every orbit $\mathcal{O}_i^n$. We turn K into a topological $\mathcal{L}$-structure $\boldsymbol{K}$. For $e \in [m]^K$

we let

$$\boldsymbol{K} \models O_i^m(e) \quad \text{if and only if} \quad e \in \mathcal{O}_i^m.$$

It is immediate that $G \subseteq \mathrm{Aut}(\boldsymbol{K})$. We work now towards the converse inclusion.

Let $g \in \mathrm{Aut}(\boldsymbol{K})$ and let $V(F,U)$ be an open neighborhood of g in $\mathrm{Homeo}(K)$. We can assume that $U \neq K$. We will find $h \in G \cap V(F,U)$ which will prove that $G \supseteq \mathrm{Aut}(\boldsymbol{K})$. Because $g(F)$ is compact and U is a union of clopen sets, $g(F)$ can be covered with finitely many of them, so we can assume without loss of generality that U is clopen and $U \neq K$. Notice that $g \in V\big(g^{-1}(U), U\big) \subset V(F,U)$. Consider the following two dual tuples $e_1, e_2 \in 2^K$, with $e_1^{-1}(\{0\}) = g^{-1}(U)$, $e_1^{-1}(\{1\}) = K \setminus g^{-1}(U)$ and $e_2^{-1}(\{0\}) = U$, $e_2^{-1}(\{1\}) = K \setminus U$. Since g is an automorphism of $\boldsymbol{K}$ and since $e_1 = g \cdot e_2$, we have that e_1 and e_2 lie in the same orbit $\mathcal{O}_i^2$ for some $i \in I_2$. Therefore, there is an $h \in G$ that sends $g^{-1}(U)$ into U and therefore $h \in G \cap V(F,U)$, which proves that $G = \mathrm{Aut}(\boldsymbol{K})$.

We prove now that $\boldsymbol{K}$ is projectively Fraïssé. First notice that for every dual tuple $e \in [m]^K$, there is a unique $i \in I_m$ such that $\boldsymbol{K} \models O_i^m(e)$. Let $\boldsymbol{C} \in \boldsymbol{\mathcal{K}}$ and let f_1, f_2 be two epimorphisms of $\boldsymbol{K}$ onto $\boldsymbol{C}$. We can assume without the loss of generality that $C = \{0, \ldots, m-1\}$ for some $m > 0$ and therefore $f_1, f_2 \in [m]^K$. Because f_1 and f_2 induce the same structure $\boldsymbol{C}$, there is a unique $i \in I_m$ such that $\boldsymbol{K} \models O_i^m(f_1)$ and $\boldsymbol{K} \models O_i^m(f_2)$. Therefore, f_1 and f_2 lie in the same orbit $G \curvearrowright [m]^K$, so there is $g \in \mathrm{Aut}(\boldsymbol{K})$ such that $f_1 \circ g = f_2$, showing that $\boldsymbol{K}$ is projectively ultra-homogeneous. □

4.4 Turning a structure to a purely dual one

Here we show that it is always possible to translate the direct structure into a dual one without losing any information. We provide a counterexample to show that the converse is not always possible. Although purely dual structures are sufficient for the development of the general theory, in Section 4.5 it will be convenient to make use of direct relations. Moreover, there are many examples of structures whose most natural presentation would involve both direct and dual structure.

Let $\mathcal{L}$ be a language and $\boldsymbol{M}$ a topological $\mathcal{L}$-structure. Let also $s \in \mathcal{L}$ be a direct relation of arity n. For every k with $0 < k \leq n$ and for every $f \in [k]^n$ (f is therefore a surjection), we introduce a dual relational symbol R_s^f of arity $k+1$. Let

$$\mathcal{L}_{\not s} = \mathcal{L} \cup \{R_s^f : f \in [k]^n \text{ for some } 0 < k \leq n\} \setminus \{s\}.$$

We turn now $\boldsymbol{M}$ into an $\mathcal{L}_{\not s}$-structure $\boldsymbol{M}_{\not s}$ on the same domain M. We encode $s^{\boldsymbol{M}}$ using the new dual symbols as follows: for $f \in [k]^n$ we let $\boldsymbol{M}_{\not s} \models R^f_s(\Delta_0, \ldots, \Delta_k)$, if and only if there are $a_0, \ldots, a_{n-1} \in M$ such that

$$\boldsymbol{M} \models s(a_0, \ldots, a_{n-1}) \text{ and } a_i \in \Delta_{f(i)} \text{ for every } i \in n.$$

It can easily be checked that $\mathrm{Aut}(\boldsymbol{M})$ can be fully recovered from $\mathrm{Aut}(\boldsymbol{M}_{\not s})$, that $\boldsymbol{M}$ is projectively Fraïssé if and only if $\boldsymbol{M}_{\not s}$ is, and that $\mathrm{Aut}(\boldsymbol{M})$ and $\mathrm{Aut}(\boldsymbol{M}_{\not s})$ are equal as permutation groups on M.

There are cases of topological $\mathcal{L}$-structures which can be turned into purely direct structures. However, this is not the case always. The main observation is that if r is direct relation of arity k which belongs to $\mathcal{L}$ and $\boldsymbol{M}$ is a topological $\mathcal{L}$-structure then $r^{\boldsymbol{M}}$ is a set-wise invariant closed subset of M^k. Let now $K = 2^{\mathbb{N}}$ and let μ be the uniform probability measure on $2^{\mathbb{N}}$. The group $\mathrm{Aut}(K, \mu)$ of all continuous measure preserving bijections can be easily seen to be a closed proper subgroup of $\mathrm{Homeo}(K)$ which for every $n > 0$ leaves no proper subset of $K^{[n]}$ invariant. Therefore, the canonical Fraïssé structure given by an application of Proposition 4.3.1 on $\mathrm{Aut}(K, \mu)$ cannot be turned into a purely direct one.

4.5 Compact Polish spaces as quotients of dual Fraïssé structures

We fix a special binary relation symbol $\mathfrak{r}$ whose interpretation will always be a reflexive and symmetric closed relation. A formal relational language $\mathcal{L}$ will be decorated with the subscript $\mathfrak{r}$ whenever $\mathfrak{r} \in \mathcal{L}_{\mathfrak{r}}$. Therefore, an $\mathcal{L}_{\mathfrak{r}}$-structure is always going to be a reflexive $\mathfrak{r}$-graph perhaps with some extra structure. We say that an $\mathcal{L}_{\mathfrak{r}}$-structure $\boldsymbol{K}$ is a pre-space if $\mathfrak{r}^{\boldsymbol{K}}$ is moreover transitive and therefore an equivalence relation.

As we noted in the introduction T. Irwin and S. Solecki used purely direct Fraïssé theory to express the pseudo-arc P as a quotient of a projective Fraïssé $\{\mathfrak{r}\}$-structure $\mathbb{P}$ via $\mathfrak{r}^{\mathbb{P}}$. Moreover, through their construction, the group $\mathrm{Aut}(\mathbb{P})$ naturally embedded in $\mathrm{Homeo}(P)$ as a dense subgroup. In [Cam10], R. Camerlo characterized all different projective Fraïssé classes[1] of $\{\mathfrak{r}\}$-structures. Their limits are pre-spaces with quotients $M/\mathfrak{r}^{\boldsymbol{M}}$ which vary between certain combinations of singletons, Cantor spaces and pseudo-arcs [Cam10]. In [BK13], D. Bartošová and A. Kwiatkowska express the Lelek fan L as the quotient of a the projective Fraïssé limit $\mathbb{L}$ of a certain class of directed graphs. Their limit $\mathbb{L}$ can be seen again as pre-space in some $\mathcal{L}_{\mathfrak{r}}$. Here again the group $\mathrm{Aut}(\mathbb{L})$ naturally embedded in $\mathrm{Homeo}(L)$ as a dense subgroup.

In this section we show that under the notion of projective Fraïssé limit we developed here the same representation applies to every second-countable compact space Y. Since this is trivial for finite spaces, we will restrict ourselves to the case where Y is infinite.

The following proposition will be use in the proof of Theorem 4.5.2.

[1]He allows Fraïssé classes to lack hereditary property.

Proposition 4.5.1. *Let G, H be Polish groups and let S be a dense subgroup of G. Then any continuous homomorphism $f : S \to H$ extends to a continuous homomorphism $\tilde{f} : G \to H$.*

The proof of Proposition 4.5.1 is an easy exercise given that every Polish admits a compatible left-invariant metric d and given this metric we can define a new compatible complete metric D by $D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$. For more details, see page 6 of [BK96].

Theorem 4.5.2. *Let G be a closed subgroup of* $\mathrm{Homeo}(Y)$*, for some compact metrizable space Y. Then there is a projective Fraïssé pre-space $\boldsymbol{K}$ such that $K/\mathfrak{r}^{\boldsymbol{K}}$ is homeomorphic to Y, and the quotient projection*

$$K \mapsto Y$$

induces a continuous group embedding $\mathrm{Aut}(\boldsymbol{K}) \hookrightarrow G$*, with dense image in G.*

Proof. Let Y be an infinite compact Polish space and let H be a countable, dense subgroup of G. In what follows, we define a countable Boolean algebra $(\mathcal{F}, 0_{\mathcal{F}}, 1_{\mathcal{F}}, \wedge, \vee, ')$ of closed subsets of Y as well as an action of H on $\mathcal{F}$ via Boolean algebra automorphisms. Every set $F \in \mathcal{F}$ will be regular closed. Recall that an open set U is called regular open if $\mathrm{int}(\overline{U}) = U$ and a closed set F is called regular closed if $\overline{\mathrm{int}(F)} = F$. We define $0_{\mathcal{F}}, 1_{\mathcal{F}}$ and the operations $\wedge, \vee, '$ as follows:

- $0_{\mathcal{F}} = \emptyset$;
- $1_{\mathcal{F}} = Y$;
- $F_1 \wedge F_2 = \overline{\mathrm{int}(F_1 \cap F_2)}$;
- $F_1 \vee F_2 = F_1 \cup F_2$;
- $F' = \overline{F^c}$.

The boolean algebra axioms are satisfied by the above configuration since $\mathcal{F}$ consists of regular closed sets (see also [GH12] for the boolean algebra of regular open sets).

To construct the boolean algebra fix first a compatible complete metric d on Y. For every n chose a finite open cover $\{V_0^n, \ldots, V_{k_n}^n\}$ of Y such $\mathrm{diam}(V_i^n) < 1/n$ for every $i \in \{0, \ldots, k_n\}$. Since $\overline{V}$ is regular closed for every open V we have that the collection $\mathcal{J} = \{F_i^n : F_i^n = \overline{V_i^n},\ n \in \mathbb{N},\ 0 \leq i \leq k_n\}$ consists of regular closed sets. We define $\mathcal{F}$ to be the least family of closed subsets of Y such that:

(1) $\mathcal{J} \subset \mathcal{F}$;

(2) $\mathcal{F}$ is closed under the boolean operators $\wedge, \vee, '$ and

(3) $\mathcal{F}$ is closed under translation by elements of H, i.e., if $h \in H$ and $F \in \mathcal{F}$ then $h(F) \in \mathcal{F}$.

Notice that all these operations preserve regularity and since $\mathcal{J}$ and H are countable $\mathcal{F}$ is a countable family of regular closed sets. Notice that this implies that the only $F \in \mathcal{F}$ that has empty interior is the empty set. The group H is acting on $\mathcal{F}$ with Boolean algebra automorphisms: for every $h \in H$ and $F \in \mathcal{F}$ let

$$h \cdot F = h(F).$$

Let $K = \mathrm{S}(\mathcal{F})$ be the Stone space of all ultrafilters x on $\mathcal{F}$. This space comes with a topology whose basic clopen sets can be taken to be the sets of the form $\tilde{F} = \{x : F \in x\}$ for $F \in \mathcal{F}$. The space K is a compact, second-countable, and zero-dimensional. Let $p : K \to Y$ be the natural projection defined by:

$$\{p(x)\} = \bigcap_{F \in x} F.$$

The map p is continuous surjection with $p(\tilde{F}) = F$ for every $F \in \mathcal{F}$. We can turn now K to a $\{\mathfrak{r}\}$-structure $K_{\mathfrak{r}}$ by setting $K_{\mathfrak{r}} \models \mathfrak{r}(x_0, x_1)$ if and only if $p(x_0) = p(x_1)$. It is immediate that $K_{\mathfrak{r}}$ is a pre-space and that $K/\mathfrak{r}^{K_{\mathfrak{r}}} = Y$.

Notice now that H is acting on K with homeomorphisms: for every $h \in H$ and $x \in K$

$$h \cdot x = \big\{h(F) : F \in x\big\} \in K.$$

This action is faithful since for every pair $y_0, y_1 \in Y$ there are $F_0, F_1 \in \mathcal{F}$ such that $y_0 \in \mathrm{int}(F_0), y_1 \in \mathrm{int}(F_1)$ and $F_0 \cap F_1 = \emptyset$. Therefore, H embeds into $\mathrm{Homeo}(K)$. We will denote this copy of H inside $\mathrm{Homeo}(K)$ by H_K to distinguish it from H which is a subgroup of $\mathrm{Homeo}(Y)$ and we will denote by T_0 the inverse of this embedding, i.e.,

$$T_0 : H_K \to H \quad \text{with} \quad \widetilde{T_0(h)(F)} = h(\tilde{F}), \ \text{ for every } \ F \in \mathcal{F}.$$

The map T_0 is also continuous. To see that, let $h \in H_K$ and let $V(L, U)$ be an open neighborhood of $T_0(h)$ in $\mathrm{Homeo}(Y)$, i.e., $T_0(h)(L) \subset U$. Since the family $\{\mathrm{int}(F) : F \in \mathcal{J}\}$ constitutes a basis of Y and since $T_0(h)(L)$ is compact, we can find $F_1, \ldots, F_k \in \mathcal{J}$ such that $T_0(h)(L) \subseteq F_1 \cup \ldots \cup F_k \subseteq U$. Let $F_0 = F_1 \vee \ldots \vee F_k$, then both F_0 and $h^{-1}(F_0)$ belong to $\mathcal{F}$. Moreover, $V(\widetilde{h^{-1}(F_0)}, \tilde{F_0})$ is an open neighborhood of h in H_K that is mapped via T_0 completely inside $V(L, U)$, proving that T_0 is continuous at h.

By applying the Proposition 4.3.1, we can endow K with a topological Fraïssé structure $\boldsymbol{K}_0$ in a purely dual language $\mathcal{L}$, such that $\overline{H_K} = \mathrm{Aut}(\boldsymbol{K}_0)$ (the closure here is taken in $\mathrm{Homeo}(K)$). By Proposition 4.5.1

the map T_0 extends to a continuous homomorphism $T : \mathrm{Aut}(\boldsymbol{K}_0) \to G$. We denote the image of $\mathrm{Aut}(\boldsymbol{K}_0)$ under T by $\hat{H}$. Notice that $\hat{H}$ lies densely in $\mathrm{Homeo}(K)$ since $H < \hat{H} \leq \mathrm{Homeo}(K)$, and since the same is true for H. Moreover, by the density of H_K in $\overline{H_K}$ the continuity of T and the fact that every $F \in \mathcal{F}$ has non-empty interior we get that for every $h \in \overline{H_K}$ and for every $F \in \mathcal{F}$ the following equality holds

$$\widetilde{T(h)\big(F\big)} = h(\tilde{F}). \tag{4.5.1}$$

We combine now the structures $\boldsymbol{K}_0$ and $K_{\mathfrak{r}}$ into one $\mathcal{L}_{\mathfrak{r}}$-structure $\boldsymbol{K}$ on domain K, where $\mathcal{L}_{\mathfrak{r}} = \mathcal{L} \cup \{\mathfrak{r}\}$. Notice that $\mathfrak{r}^{\boldsymbol{K}}$ is invariant under $\mathrm{Aut}(\boldsymbol{K}_0)$ since $(x_0, x_1) \in \mathfrak{r}^{\boldsymbol{K}}$ if and only if for all $F_0, F_1 \in \mathcal{F}$ with $x_0 \in \tilde{F}_0$ and $x_1 \in \tilde{F}_1$ we have that $F_0 \cap F_1 \neq \emptyset$. Thus $\mathrm{Aut}(\boldsymbol{K}) = \mathrm{Aut}(\boldsymbol{K}_0) = \overline{H_K}$, every $\boldsymbol{A}_0 \in \mathrm{Age}(\boldsymbol{K}_0)$ uniquely extends to an $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{K})$ and $\boldsymbol{K}$ is a also a projective Fraïssé structure. The fact that $p(\tilde{F}) = F$ for every $F \in \mathcal{F}$ and the relation (4.5.1) above let us view $T : \mathrm{Aut}(\boldsymbol{K}) \to G$ as the homomorphism induced by the quotient $p : K \to Y$.

We are left to show that T is injective. Let $h \in \mathrm{Aut}(\boldsymbol{M})$ so that $h \neq \mathrm{id}_{\mathrm{Aut}(\boldsymbol{M})}$. By the continuity of h we can find a non-empty F in $\mathcal{F}$ so that $F \wedge h(F) = \emptyset$. Therefore, the interiors in Y of $p(F)$ and $p\big(h(F)\big)$ do not intersect and because the interior in Y of every non-empty F in $\mathcal{F}$ is non-empty we have that $T(h) \neq \mathrm{id}_{\mathrm{Homeo}(Y)}$. □

We should remark here that the image $\hat{H}$ of $\mathrm{Aut}(\boldsymbol{K})$ under T is in general a meager subset of G. This can be seen as follows: first notice that as a corollary of Pettis theorem we have that if $f : B \to D$ is a Baire-measurable homomorphism between Polish groups and $f(B)$ is not meager, then f is open (see for example Theorem 1.2.6 [BK96]). Now notice that for $F \in \mathcal{F}$ the set $V(\tilde{F}, \tilde{F})$ is open in $\mathrm{Homeo}(K)$ but the set $V(F, F)$ is rarely open in G (except if Y is zero-dimensional or if G contains very few homeomorphisms). Therefore T will fail in general to be an open map.

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