RELATIVE WARING RANK OF BINARY FORMS

BY

NERIMAN TOKCAN

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Doctoral Committee:

Professor Maarten Bergvelt, Chair
Professor Bruce Reznick, Director of Research
Professor Sheldon Katz
Professor Thomas Nevins

Abstract

Suppose $f(x,y)$ is a binary form of degree d with coefficients in a field $K \subseteq \mathbb{C}$. The *K-rank of f* is the smallest number of d-th powers of linear forms over K of which f is a K-linear combination. We prove that for $d \geq 5$, there always exists a form of degree d with at least three different ranks over various fields. We also study the relation between the relative rank and the algebraic properties of the underlying field. In particular, we show that K-rank of a form f (such as x^3y^2) may depend on whether -1 is a sum of two squares in K. We provide lower bounds for the $\mathbb{C}$-rank (Waring rank) and for the $\mathbb{R}$-rank (real Waring rank) of binary forms depending on their factorization. We also give the rank of quartic and quintic binary forms based on their factorization over $\mathbb{C}$. We investigate the structure of binary forms with unique $\mathbb{C}$-minimal representation.

Acknowledgments

The chapters of mathematical results that follow represent the closing remarks of about five years as a Ph.D. student at the University of Illinois at Urbana-Champaign. These years have been ticked by discoveries, friendships, liters of coffee and at times by frustrations and uncertainty. The feeling in the end is that everything was complementary: I would have never lived my proudest moments without enduring downfalls. Because the mysterious path that made me a doctor was far from being straight, it would be unimaginable to list all the people that directly or indirectly routed me through that bumpy, twisted road. Nonetheless, I would like to express few special notes of gratitude for those people that were fundamental for the completion of this work.

First and foremost, I wish to thank my advisor, Professor Bruce Reznick. I have been told that being a good professor is hard, and that being a good advisor is even harder. Bruce was for me an excellent professor as his passion for mathematics was not only evident, but also contagious; he was an outstanding advisor as he trusted me at a time when a number of individuals decided to turn their backs on me; and an irreplaceable collaborator as he allowed me to progress toward my goals at an unprecedented pace.

Next, my gratitude goes to the members of my Ph.D. thesis committee, who contributed to drastically improve the quality of this work. They are: Professor Sheldon Katz, Professor Thomas Nevins, and Professor Maarten Bergvelt. Being Sheldon's student has been a great honor, and much that I learned from him has been crucial for this work. Similarly, learning from Thomas has been an experience of irreplaceable value. On top of their contribution to my preparation and my thesis, Thomas and Sheldon have also played a decisive role in determining the next step in my career. For that I am especially thankful. Last but not least, I wish to thank Maarten for his help and advice: substantial refinements to this work were possible thanks to his detailed feedback.

Clearly, a special place in this section belongs to my family. My father and my mother are the people that supported my studies, strongly backed my decision to live overseas, and relentlessly encouraged me through tough times. If this thesis had one word for each time

I dreamed about having a picnic on a grassy hill in Tarsus with my father, it would span through thousands of pages. If I were to count all the times that my mother suppressed her desire to tell me come back, I would use a logarithmic scale. I am also thankful to my brothers, Timur and Zulkif, for covering for my prolonged absence as much as they could. A thank also to my little nephew Alparslan: while it took five times less for him to come to this world, he was born around the same time as this thesis. His colorful smile has enlightened many mornings.

There have been a countless number of friends who became integral part of my life as a Ph.D. student in UIUC. Some of them will stick around for a little more, while others are spread around the continents. I am thankful to each and every of them for leaving invaluable memories. Among my friends, I consider Ebru Toprak as a Ph.D. sister. Ebru was the invariant of my UIUC experience: no matter what I went through, Ebru was there with a tea and a smile. Similarly, I met Fardin Abdi during the first months of my life in Urbana-Champaign. Since then, he has demonstrated endless generosity, patience, and kindness. Meeting Fardin made me remarkably more sociable. On one hand because he is the quintessence of a good friend, on the other side due to the sheer number of times he introduced me to someone. Finally, I want to deeply thank Huseyin Boler for being a one-of-a-kind friend. Way too many times Huseyin put aside his troubles to discuss mine; and way too many times he offered his help before I would even realize that I needed it. I would further like to thank my fellow graduate students Caglar Uyanik, Darlayne Addabbo, Sneha Chaubey, Amita Malik and Wafaa Albar for their support and friendship.

Of course no acknowledgments would be complete without giving thanks to my life partner, Renato Mancuso. He has been a constant source of happiness and motivation during the bumpy roads of grad school. I always admire his wit, intelligence and his never-ending energy. Thanks to him this period was not only a stressed student life; it was full of great memories, bundle of laughs, road trips across the states, and the joy of growing up and exploring together. He made me a happier person and through his eyes I have seen myself as a strong woman who pushes forward, even when all the odds are against her.

Table of Contents

Chapter 1

Introduction

The main results of this work concern the Waring problem for forms, sometimes called the symmetric tensor decomposition. Recently, the symmetric tensor decomposition has attracted attention with its applications in signal processing, statistics, neuroscience, chemometrics, data mining and machine learning [13, 20, 22, 35, 41].

The Waring problem for forms has several different versions. In this thesis, we consider the classical Waring decomposition of homogeneous polynomials into the powers of linear forms over intermediate fields of $\mathbb{C}/\mathbb{Q}$.

Let $H_d(K^2)$ denote the vector space of binary forms of degree d with coefficients in the field $K \subseteq \mathbb{C}$. Given a binary form $f \in H_d(K^2)$, the K-rank of f, $L_K(f)$, is the smallest r for which there exist $\lambda_j, \alpha_j, \beta_j \in K$ such that

$$f(x,y) = \sum_{j=1}^{r} \lambda_j \big(\alpha_j x + \beta_j y\big)^d.$$

Note that the K-rank of a binary form f is also called the *relative rank of f with respect to K*. The $\mathbb{C}$-rank is commonly called the *Waring rank* and the $\mathbb{R}$-rank is known as the *real Waring rank*. Sylvester [37, 38] presented an algorithm to compute the Waring rank of binary forms in 1851 and gave a lower bound for the real Waring rank in 1864. The Waring rank of binary forms has been studied extensively [1, 4, 9, 12, 21, 23, 32]. Recently the real Waring rank of binary forms has been investigated [5, 7, 10, 11, 14]. The relative ranks of binary forms over some intermediate fields of $\mathbb{C}/\mathbb{Q}$ were analyzed in [32, 34].

Let f be real binary form of degree d. If $L_K(f) = d$, we say that f has *full rank* over the field K. The case for $K = \mathbb{C}$ has been completely analyzed (Theorem 2.3.10). In the last years, the case $K = \mathbb{R}$ has been considered in different works [11, 14, 32], and a final result is given by Blekherman and Sinn; see Theorem 2.4.8.

We study binary forms with multiple ranks over different fields. We also study the properties of a binary form that appear to have a determining role in its rank. Some examples of such properties are the degree of the form; degree of the field extensions; algebraic properties of the underlying field; factorization of the form; number of real roots; and irreducibility. We explore the structure of the binary forms with unique $\mathbb{C}$-minimal representation. We also investigate whether there are fields besides $\mathbb{C}$ and $\mathbb{R}$ where forms with a given set of properties have full rank.

Traditionally, the study of the rank of binary forms has been restricted to complex and real numbers. The distinctive aspect and main contribution of this work consists in the extension of this study to intermediate fields of $\mathbb{C}/\mathbb{Q}$ as a continuation of [32].

We now outline the structure of the thesis.

In Chapter 2, after briefly discussing apolarity, we discuss the existence of the decomposition of forms into the sums of powers of linear forms and present its connection with apolarity. In Section 2.2, we recall Sylvester's 1851 theorem and collect some results on the K-rank of binary forms. Section 2.3 focuses on the set of possible different ranks of a form over different fields. Section 2.4 discusses real Waring rank of binary forms and presents recent improvements on the topic.

Chapter 3 is based on the paper [34] *Binary forms with three different relative ranks* by B. Reznick and N. Tokcan, which has been accepted for publication in *Proceedings of the American Mathematical Society*. In Section 3.1, we first give examples of binary forms with multiple ranks (Example 3.1.2). We then show that if $d \geq 5$, then there exist a binary form of degree d which takes at least three different ranks (Theorem 3.2.7). In particular, let ζ_m denote a primitive m-th root of unity. We prove that (Theorems 3.2.3 and 3.2.5) if $k \geq 3$ and $p_{2k-1}(x,y) = \binom{2k-1}{k} x^{k-1} y^{k-1}(x-y)$, then

$$L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k-1}) = k, \quad L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) = k+1, \quad L_{\mathbb{R}}(p_{2k-1}) = 2k-1 > k+1.$$

Similarly, if $k \geq 3$ and $p_{2k}(x,y) = \binom{2k}{k} x^k y^k$, then

$$L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k}) = k+1, \quad L_{\mathbb{Q}(\zeta_k)}(p_{2k}) = k+2, \quad L_{\mathbb{R}}(p_{2k}) = 2k > k+2.$$

Section 3.3 investigates the relation between the relative rank of a binary form and the algebraic properties of the underlying field. The *Stufe* of a non-real field F, $s(F)$, is the smallest integer n such that -1 can be written as a sum of n squares in F. It is already known that $L_{\mathbb{C}}(x^3y^2) = 4$ (from [9, Prop.3.1]) and $L_{\mathbb{R}}(x^3y^2) = 5$ (from [7, Prop.4.4]). We show in Theorem 3.3.3 that $L_K(x^3y^2) = 4$ if and only if $s(K) \leq 2$ and $L_K(x^3y^2) = 5$ otherwise. We show in Theorem 3.3.4 that if m is a square-free positive integer and $f(x,y) = 6x^5y - 20x^3y^3$, then $L_{\mathbb{Q}(\sqrt{-m})}(f) = 4$ if and only if $s(\mathbb{Q}(\sqrt{-m})) \leq 2$ if and only if $m \not\equiv 7 \mod 8$ (see [25, 39]), and $L_{\mathbb{Q}(\sqrt{-7})}(f) = 5$.

Chapter 4 is based on Section 2 of the paper [40] *On the Waring rank of binary forms* by N. Tokcan, which has been accepted for publication in *Linear Algebra and Its Applications*. In Section 4.1, we first show that if f is a binary form of degree d, not a d-th power, and $(x - \alpha_i y)$ is a factor of multiplicity m_i of f, then $L_{\mathbb{C}}(f) \geq m_i + 1$ (Theorem 4.1.1). It directly follows that $L_{\mathbb{C}}({\ell_0}^{d-2}\ell_1\ell_2) = L_{\mathbb{R}}({\ell_0}^{d-2}q) = d - 1$ where the ℓ_i's are distinct binary linear forms and q is an irreducible quadratic (Corollaries 4.1.2 and 4.1.3). Theorem 4.1.1 combines with [32, Theorem 3.2] into Corollary 4.1.5: if f is a real binary form of degree d, not a d-th power, with τ real linear factors (counting multiplicities), and $(x-\alpha_i y)$ is a factor of multiplicity m_i of f, then $L_{\mathbb{R}}(f) \geq \max(\tau, m_i + 1)$. In Section 4.2, we give the Waring rank of binary quartics and quintics based on their factorization. In Section 4.3, we show that if $f_\lambda(x, y) = x^{2k} + \binom{2k}{k}\lambda x^k y^k + y^{2k},\ \lambda \neq 0 \in \mathbb{R},\ k \geq 3$, then $L_{\mathbb{R}}(f_\lambda) \in \{2k - 2, 2k - 1\}$ (Theorem 4.3.2). The minimal representations of f_λ are parameterized in Theorem 4.3.3.

In Chapter 5, we study binary forms with unique $\mathbb{C}$-minimal representation. Sections 5.1 and 5.2 are adapted from Section 3 of the paper [40] *On the Waring rank of binary forms* by N. Tokcan. We show that if $f \in H_d(K^2)$ and $L_{\mathbb{C}}(f) = r < \frac{d+2}{2}$, then there exist a field extension S/K such that $L_S(f) = r$ and $[S : K]$ divides $r!$ (Theorem 5.1.4). We then look at the special case when the underlying field K is a real closed field and $r = 3$ (Corollary 5.1.10). In Section 5.2, we give examples of binary forms of Waring rank 3 by considering different cases corresponding to field extensions given in Theorem 5.1.4 and an additional example of a binary quartic form with infinitely many minimal representations of length 3 (Example 5.2.6). In Section 5.3, we introduce Sylvester fields: $K \subseteq \mathbb{C}$ is a *Sylvester field* if every binary form of degree $d \geq 2$ which splits over K and not a d-th power has full rank over K. We then present some preliminary results on the Sylvester fields: Any subfield of a given real closed field is a Sylvester field (Theorem 5.3.6 and Corollary 5.3.7); if K is an algebraically closed field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$, then K is not a Sylvester field (Theorem 5.3.8); for any $n \geq 3,\ \mathbb{Q}(\zeta_n)$ is not a Sylvester field (Theorem 5.3.9).

Chapter 2

Preliminaries

2.1 Apolarity and decomposition of forms

In this section we recall the main results in the theory of apolarity and show the connection between the decomposition of forms and apolarity. We refer the reader to [28, 30, 33] for additional background and detailed proofs of the results given in this section.

Let $H_d(K^n)$ denote the $N(n,d) = \binom{n+d-1}{d}$-dimensional vector space of forms of degree d in n variables with coefficients in a field K of characteristic zero. Let $I(n,d)$ denote the index set for monomials in $H_d(K^n)$:

$$I(n,d) = \{(i_1,\dots,i_n) : \ 0 \le i_k \in \mathbb{Z}, \ \sum_i i_k = d\}. \tag{2.1.1}$$

The multinomial abbreviation x^i means $x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}$ and $c(i) = \frac{d!}{\prod i_k!}$ is the associated multinomial coefficient for $i = (i_1,\dots,i_n) \in I(n,d)$. If $p \in H_d(K^n)$, then we write

$$p(x_1,\dots,x_n) = \sum_{i \in I(n,d)} c(i)a(p;i)x^i, \quad a(p;i) \in K. \tag{2.1.2}$$

The identification of p with $N(n,d)$-tuple $(a(p;i))$ shows that $H_d(K^n) \approx K^{N(n,d)}$ as a vector space. For $p,q \in H_d(K^n)$, we define the *symmetric bilinear form*:

$$[p,q] = \sum_{i \in I(n,d)} c(i)a(p;i)a(q;i) \in K. \tag{2.1.3}$$

For $\alpha \in K^n$, define $(\alpha.)^d \in H_d(K^n)$ by

$$(\alpha.)^d(x) = \left(\sum_{j=1}^n \alpha_j x_j\right)^d = \sum_{i \in I(n,d)} c(i)\alpha^i x^i. \tag{2.1.4}$$

It follows that for $\alpha \in K^n$,

$$[p, (\alpha.)^d] = \sum_{i \in I(n,d)} c(i)a(p;i)\alpha^i = p(\alpha). \tag{2.1.5}$$

Let U be a vector subspace of $H_d(K^n)$, and let

$$U^\perp = \{v \in H_d(K^n) \ : \ [u, v] = 0 \text{ for all } u \in U\}. \tag{2.1.6}$$

Clearly, $U^\perp$ is a subspace of $H_d(K^n)$.

Lemma 2.1.1. [30, Lemma 2.5]
If U is a vector subspace of $H_d(K^n)$, then $(U^\perp)^\perp = U$.

Definition 2.1.2. A field K is *formally real* if one of the following equivalent conditions is satisfied:

- -1 is not a sum of squares in K.
- If any sum of squares of elements of K equals zero, then each of those elements must be zero.

The theory of formally real closed fields is due to Artin and Schreier; see [17, Chapter VI] for more details.

Lemma 2.1.3.
If K is a formally real field, then the bilinear form gives an inner product on $H_d(K^n)$.

Proof. If K is formally real, then $char(K) = 0$. Assume that $[p, p] = 0$. Then $\sum_i c(i)a(p;i)^2 = 0$ and $a(p;i) = 0$ for all $i \in I(n, d)$ since K is formally real. Therefore, $p = 0$ and the bilinear form is an inner product. □

Lemma 2.1.3 does not hold if K is not formally real. However, if $K = \mathbb{C}$, then $(p, q) = [p, \bar{q}]$ turns $H_d(\mathbb{C}^n)$ into a inner product space over $\mathbb{C}$. In particular, $||p, p|| = [p, \bar{p}]$ is known as the Bombieri norm; see [2, 3, 29].

Suppose $p \in H_d(K^n)$. The differential operator associated to p is given by

$$p(D) = \sum_{i \in I(n,d)} c(i)a(p;i) \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}. \tag{2.1.7}$$

Let D^i denote the monomial differential operator

$$D^i = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}. \tag{2.1.8}$$

If $i \neq j \in I(n,d)$, then $D^i x^j = 0$ since $i_k > j_k$ for some k; otherwise, $D^i x^i = \prod_k i_k! = \frac{d!}{c(i)}$. Since $\frac{\partial}{\partial x_k}$ and $\frac{\partial}{\partial x_l}$ commute, $D^i D^j = D^{i+j} = D^j D^i$ for $i \in I(n,d)$ and $j \in I(n,e)$. Therefore, $(pq)(D) = p(D)q(D) = q(D)p(D)$ for any forms (possibly at different degree) by multilinearity.

The following theorem gives the connection between the differential operator and the bilinear product.

Theorem 2.1.4.
Let $p,\ q \in H_d(K^n)$, then $p(D)q = q(D)p = d![p,q]$.

Proof. It follows from bilinearity and (2.1.7) that

$$p(D)q = \sum_{i\in I(n,d)} c(i)a(p;i)D^i \left(\sum_{j\in I(n,d)} c(j)a(p;j)x^j\right) =$$

$$\sum_{i\in I(n,d)} \sum_{j\in I(n,d)} c(i)c(j)a(p;i)a(q;j)D^i x^j = \sum_{i\in I(n,d)} c(i)c(i)a(p;i)a(q;i)D^i x^i =$$

$$\sum_{i\in I(n,d)} c(i)^2 a(p;i)a(q;i)\frac{d!}{c(i)} = d![p,q] = d![q,p] = q(D)p.$$

□

Lemma 2.1.5.
If $d > e$ and $p \in H_d(K^n),\ q \in H_e(K^n)$, then $p(D)q = 0$.

Proof. Note that $D^i x^j = 0$ for every $i \in I(n,d)$ and $j \in I(n,e)$. As in the proof of Theorem 2.1.4,

$$p(D)q = \sum_{i\in I(n,d)} \sum_{j\in I(n,e)} c(i)c(j)a(p;i)a(q;j)\underbrace{D^i x^j}_{0}.$$

Thus, $p(D)q = 0$. □

Definition 2.1.6. Suppose $p \in H_d(K^n)$ and $q \in H_e(K^n)$. Then p and q are *apolar* if $p(D)q = q(D)p = 0$.

The following lemma is essential and trivial.

Lemma 2.1.7.
Assume that $X = Span\{q_j : q_j \in H_d(K^n)\}$. Then $X = H_d(K^n)$ if and only if there is no $0 \neq p \in H_d(K^n)$ which is apolar to each of the q_j's.

Proof. We first show that the bilinear form is non-degenerate. Assume that $p \in H_d(K^n)$ such that $[p, q] = 0$ for every $q \in H_d(K^n)$. Therefore $[p, x^i] = 0$ for every $i \in I(n, d)$. This implies that $a(p; i) = 0$ for every $i \in I(n, d)$, i.e., $p = 0$ and the bilinear form is non-degenerate. Since $H_d(K^n)$ is a finite dimensional space, $dim(X) + dim(X^\perp) = dim(H_d(K^n))$. Therefore, $X = H_d(K^n)$ if and only if $X^\perp = \{0\}$. □

The following corollary shows that every form in $H_d(K^n)$ can be written as a $\mathbb{C}$ -linear combination of d-th powers of linear forms over K.

Corollary 2.1.8.
The vector space $H_d(K^n)$ is spanned by $\{(\alpha.)^d : \alpha \in K^n\}$.

Proof. Assume that $q \in H_d(K^n)$ such that $0 = [q, (\alpha.)^d] = q(\alpha)$ for all $\alpha \in K^n$. Since K is a field of characteristic zero, this implies $q = 0$. The rest follows from Lemma 2.1.7. □

The following properties can be easily derived from the basic definitions of the differential operator and the bilinear form; see, [28, 30] for details and proofs.

Theorem 2.1.9.

(i) If $p \in H_{d-e}(K^n)$, $q \in H_e(K^n)$ and $h \in H_d(K^n)$, then

$$[pq, h] = \frac{(d-e)!}{d!}[p, q(D)h] \tag{2.1.9}$$

Thus, h is apolar to every polynomial multiple of q if and only if h and q are apolar.

(ii) If $p \in H_d(K^n)$, then $[p, x_j(\alpha.)^{d-1}] = \frac{1}{d}\frac{\partial p}{\partial x_j}(\alpha)$. Thus, p is apolar to $(\alpha.)^{d-1}$ if and only if p is singular at α. More generally, p is apolar to $(\alpha.)^{d-e}$ if and only if p vanishes to e-th order at α.

(iii) If $p \in H_d(K^n)$ and $q \in H_e(K^n)$ with $d \leq e$, then

$$p(D)q = \frac{e!}{(e-d)!}\sum_{\ell \in I(n,e-d)} c(\ell)\left(\sum_{i \in I(n,d)} c(i)a(p;i)a(q;i+l)\right)x^\ell \tag{2.1.10}$$

Let K be a field of characteristic zero and $H(K^n) = \bigcup_d H_d(K^n)$.

Definition 2.1.10. Let $p \in H_d(K^n)$. The *apolar ideal* of p, which is denoted by $p^\perp$, is the set of forms in $H(K^n)$ whose differential operator kills p, that is,

$$p^\perp = \{h \in H(K^n) : h(D)p = 0\}. \tag{2.1.11}$$

This is a homogeneous ideal with the decomposition

$$\begin{aligned} p^\perp &= \bigoplus_{e \geq 0} (p^\perp)_e, \\ (p^\perp)_e &= \{h \in H_e(K^n) : \ h(D)p = 0\}. \end{aligned}$$

Suppose $p \in H_d(K^n)$ and r is fixed. Define

$$\begin{aligned} V_{p,r} &= \{q \in H_r(K^n) : \ p(D)q = 0\}, \\ X_{p,r} &= \{f \in H_r(K^n) : p \mid f\}. \end{aligned}$$

Clearly, $V_{p,r}$ and $X_{p,r}$ are both subspaces of $H_r(K^n)$.

Theorem 2.1.11. [30, Theorem 2.18]
Assume that p is a form in $H_e(K^n)$. Then the orthogonal subspace $(V_{p,d})^\perp = X_{p,d}$.

Proof. If $e > d$, then $p(D)f = 0$ for every $f \in H_d(K^n)$, then $V_{p,d} = H_d(K^n)$ and $(V_{p,d})^\perp = \{0\}$, so the result is trivial. Assume that $e \leq d$. Then $f \in X_{p,d}$ if and only if $f = pg$ for $g \in H_{d-e}(K^n)$. On the other side, it follows from Theorem 2.1.9(i) that

$$\begin{aligned} q \in (X_{p,d})^\perp &\Leftrightarrow [pg, q] = 0 \text{ for all } g \in H_{d-e}(K^n), \\ &\Leftrightarrow [px^j, q] = 0 \text{ for all } j \in I(n, d-e), \\ &\Leftrightarrow p(D)q = 0. \end{aligned}$$

Therefore, $(X_{p,d})^\perp = V_{p,d}$ and $(V_{p,d})^\perp = X_{p,d}$ by Lemma 2.1.1. $\square$

Suppose K is an algebraically closed field. Let S be a set of polynomials in $K[x_1, \ldots, x_n]$. The *zero locus* of S is the set

$$Z(S) = \{S = 0\} = \{\alpha \in K^n : \ f(\alpha) = 0 \text{ for all } f \in S\}. \tag{2.1.12}$$

A subset V of K^n is called an *affine algebraic set* if $V = Z(S)$ for some S. For any subset $Y \subseteq K^n$, let us define the ideal of Y in $K[x_1, \dots, x_n]$ by

$$I(Y) = \{f \in K[x_1, \dots, x_n] : \; f(\alpha) = 0 \text{ for all } \alpha \in Y\}. \tag{2.1.13}$$

The radical of an ideal is defined by

$$\sqrt{I} = \{f \in K[x_1, \dots, x_n] : f^r \in I \text{ for some } r > 0\}. \tag{2.1.14}$$

We now recall Hilbert's Nullstellensatz.

Theorem 2.1.12 (Hilbert's Nullstellensatz)**.**
Suppose K is an algebraically closed field and J is an ideal in $A = K[x_1, \dots, x_n]$. Let $f \in A$ be a polynomial which vanishes at all points of $Z(J)$. Then $f^r \in J$ for some integer $r > 0$, i.e., $I(Z(J)) = \sqrt{J}$.

The following is a variation of the Nullstellensatz.

Theorem 2.1.13. [30, Proposition 3.1]
Suppose $p \in H_e(K^n)$, $e \geq 1$, is irreducible and $f \in H_d(K^n)$, $0 \leq k \leq d$. Then $D^i f(\alpha) = 0$ for all $i \in I(n,k)$ and all $\alpha \in Z(p)$ if and only if $p^{k+1} | f$.

The following theorem gives the essence of the connection between apolarity theory and decomposition of forms.

Theorem 2.1.14. [30, Theorem 4.1]
Let K be an algebraically closed field of characteristic 0. Suppose $p \in H_e(K^n)$ has the factorization $p = \prod_{i=1}^{m} p_i^{m_i}$ into distinct non-constant irreducible forms over K. Suppose $q \in H_d(K^n)$ and let $m_i' = \min(m_i, d+1)$. Then $p(D)q = 0$ if and only if there exist $h_{ik} \in H_{m_i'-1}(K^n)$ and $\alpha_{ik} \in Z(p_i)$ so that

$$q = \sum_{i=1}^{m} \sum_{k=1}^{N(n,d)} h_{ik}(\alpha_{ik}\cdot)^{d-m_i'+1}. \tag{2.1.15}$$

Proof. We define another subspace of $H_d(K^n)$ based on the representation given in (2.1.15):

$$W_{p,d} = \Big\{ \sum_{i=1}^{m} \sum_{k=1}^{N(n,d)} h_{ik}(\alpha_{ik}\cdot)^{d-m_i'+1} : \; h_{ik} \in H_{m_i'-1}(K^n), \; \alpha_{ik} \in Z(p_i) \Big\}. \tag{2.1.16}$$

We denote the set of permissible summands in (2.1.16) by $W_{p,d}(s)$, i.e.,

$$W_{p,d}(s) = \{h_i(\alpha_i.)^{d-m_i'+1} : h_i \in H_{m_i'-1}(K^n), \alpha_i \in Z(p_i), 1 \leq i \leq m\}. \tag{2.1.17}$$

We shall show that $W_{p,d} = V_{p,d}$. First suppose $m_i \geq d+1$ for some i, then $deg(p) > deg(q)$ and $m_i' = d+1$. Therefore, $V_{p,d} = H_d(K^n) = W_{p,d}$. We now assume that $m_i \leq d$ for all i and write m_i for m_i'.

$$\begin{aligned}
f \in (W_{p,d})^{\perp} \ &\Leftrightarrow\ [f,g] = 0 \text{ for all } g \in W_{p,d}(s),\\
&\Leftrightarrow\ [f, x^j(\alpha_i.)^{d-m_i+1}] = 0 \text{ for all } j \in I(n, m_i - 1) \text{ and } \alpha_i \in Z(p_i),\ 1 \leq i \leq m,\\
&\Leftrightarrow\ D^j f(\alpha_i) = 0 \text{ for all } j \in I(n, m_i-1) \text{ (Theorem 2.1.9(ii))},\\
&\Leftrightarrow\ p_i^{m_i} | f \text{ for } 1 \leq i \leq m \text{ (Theorem 2.1.13)},\\
&\Leftrightarrow\ p|f \text{ since } p_i\text{'s are distinct irreducibles},\\
&\Leftrightarrow\ f \in X_{p,d}.
\end{aligned}$$

Thus, $(W_{p,d})^{\perp} = X_{p,d}$ and it follows from Theorem 2.1.11 that $W_{p,d} = V_{p,d}$. □

Theorem 2.1.14 was first proved for $n = 2$ in 1851 by Sylvester in the case of no multiple factors, and in 1886 Gundelfinger proved it fully for $n = 2$. The history for $n \geq 3$ is very complicated; see e.g. [27].

We are mostly interested in the special case of Theorem 2.1.14 where $n = 2$ and q is a square-free binary form, i.e., $m_i = 1$ for all $1 \leq i \leq m$.

It is well known that any bivariate apolar ideal is a complete intersection.

Theorem 2.1.15. [16, Theorem 1.44(iv)]
Let $p(x,y) \in H_d(\mathbb{C}^n)$. Then $p^{\perp}$ is a complete intersection ideal over $\mathbb{C}$, i.e. $p^{\perp} = \langle f, g\rangle$ such that $deg(f) + deg(g) = d+2$ and $Z_{\mathbb{C}}(f,g) = \{(0,0)\}$. Also, any two such binary forms f, g generate an ideal $p^{\perp}$ for a binary form p of degree $deg(f) + deg(g) - 2$.

Example 2.1.16. Let n, m be positive integers. Then $(x^n y^m)^{\perp} = \langle x^{n+1},\ y^{m+1}\rangle$, $(x^n - y^n)^{\perp} = \langle xy,\ x^n + y^n\rangle$ and $((x+y)^n)^{\perp} = \langle x - y,\ y^{n+1}\rangle$.

The following corollary follows from Theorem 2.1.15. It shows that if p is a binary form of degree d, then any apolar form of degree $< \frac{d+2}{2}$ for p is unique (up to a scalar multiple).

Corollary 2.1.17.
Let $p(x,y)$ be a nonzero binary form in $H_d(K^2)$, not a d-th power, and suppose that $k < \frac{d+2}{2}$ is the smallest number such that $(p^\perp)_k \neq \{0\}$. Then there exists a projectively unique binary form $h(x,y) \in H_k(K^2)$ such that $(p^\perp)_k = \langle h \rangle$. Thus, $p(x,y)$ has at most one minimal representation of length k.

We give the proof of the corollary in Section 5.1.

2.2 The K-rank of binary forms

In Section 2.1 we discussed the existence of the decomposition of forms into the sums of powers of linear forms and presented its connection with apolarity; see Corollary 2.1.8, Theorem 2.1.14. In this section, we mostly focus on optimizing the decomposition of binary forms. A more detailed account and proofs of the results given in this section can be found in [32].

Let K be a field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. Given a binary form $f \in H_d(K^2)$, the K*-rank* of f, $L_K(f)$, is the smallest r for which there exist $\lambda_j, \alpha_j, \beta_j \in K$ such that

$$f(x,y) = \sum_{j=1}^{r} \lambda_j \left(\alpha_j x + \beta_j y\right)^d . \tag{2.2.1}$$

In case $K = \mathbb{C}$ or $\mathbb{R}$, the K-rank is commonly called the *Waring rank* or the *real Waring rank*. A representation such as (2.2.1) is called K*-minimal* if $r = L_K(f)$. Two linear forms are called *distinct* if they (or their d-th powers) are not proportional. A representation such as (2.2.1) is *honest* if the summands are pairwise distinct; that is, if $\lambda_i\lambda_j(\alpha_i\beta_j - \alpha_j\beta_i) \neq 0$ whenever $i \neq j$. Two honest representations are *different* if the ordered set of summands are not rearrangements of each other; we do not distinguish between ℓ^d and $(\zeta_d^k \ell)^d$ where $\zeta_d = e^{\frac{2\pi i}{d}}$.

Suppose f is a form in $H_d(\mathbb{C}^2)$. We let E_f denote the field generated by the coefficients of f over $\mathbb{Q}$; $L_K(f)$ is defined for fields K satisfying $E_f \subseteq K \subseteq \mathbb{C}$. Of course, if $K \subseteq F \subseteq \mathbb{C}$, then $f \in F[x,y]$ as well, and one may consider $L_F(f)$ to be the *relative rank* of f with respect to F. The following properties are immediate:

- Any minimal representation is necessarily honest.
- Rank is invariant under invertible linear change of variables, i.e., if g is obtained from f by an invertible linear change of variables over K, then $L_K(f) = L_K(g)$.

- If we order fields by inclusion, then the relative rank is order reversing, that is,

$$K_1 \subseteq K_2 \Rightarrow L_{K_1}(f) \geq L_{K_2}(f). \tag{2.2.2}$$

The Waring rank of binary forms has been studied extensively [1, 4, 9, 12, 21, 23, 32]. Sylvester [38, 37] presented an algorithm to compute $L_{\mathbb{C}}(f)$ in 1851.

Theorem 2.2.1 (Sylvester's 1851 Theorem)**.**
Suppose

$$p(x,y) = \sum_{i=1}^{d} \binom{d}{i} a_i x^{d-i} y^i \in H_d(\mathbb{C}^2) \tag{2.2.3}$$

and suppose $r \leq d, \alpha_j, \beta_j \in \mathbb{C}$ *and*

$$h(x,y) = \sum_{t=0}^{r} c_t x^{r-t} y^t = \prod_{j=1}^{r} (-\beta_j x + \alpha_j y) \tag{2.2.4}$$

is a product of pairwise distinct linear factors. Then there exist $\lambda_k \in \mathbb{C}$ *such that*

$$p(x,y) = \sum_{k=1}^{r} \lambda_k (\alpha_k x + \beta_k y)^d \tag{2.2.5}$$

if and only if

$$\begin{pmatrix} a_0 & a_1 & \dots & a_r \\ a_1 & a_2 & \dots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-r} & a_{d-r+1} & \dots & a_d \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \tag{2.2.6}$$

that is, if and only if

$$\sum_{i=0}^{r} a_{i+m} c_i = 0, \qquad m = 0, 1, \dots, d-r. \tag{2.2.7}$$

Proof. The proof of Sylvester's Theorem in [30] is based on apolarity. If h and p are given by (2.2.4) and (2.2.3), then $h(D) = \prod_{j=1}^{r}(-\beta_j \frac{\partial}{\partial x} + \alpha_j \frac{\partial}{\partial y})$, and

$$h(D)p = \sum_{m=0}^{d-r} \frac{d!}{(d-r-m)!m!} \left(\sum_{i=0}^{r} a_{i+m} c_i \right) x^{d-r-m} y^m. \tag{2.2.8}$$

Therefore, (2.2.7) is equivalent to $h(D)p = 0$. The rest follows from Theorem 2.1.14. ☐

The $(d-r+1)\times(r+1)$ Hankel matrix in (2.2.6) will be denoted by $H_r(p)$.

Definition 2.2.2. Suppose (p,h) satisfies the criterion of Theorem 2.2.1; that is, $h(D)p=0$ and h is square free. Then we shall say that h is a *Sylvester form* for p. If every Sylvester form of degree r is a constant multiple of h, then we say that h is the (projectively) *unique* Sylvester form of degree r for p.

In the proof of Theorem 2.2.1, we show that (2.2.7) is equivalent to $h(D)p=0$. Therefore, Sylvester's algorithm can be restated based on an apolarity argument.

Theorem 2.2.3 (Apolarity Lemma)**.**
Let $p\in H_d(\mathbb{C}^2)$. Then p can be written as $p(x,y)=\sum_{i=1}^{r}\lambda_i(\alpha_i x+\beta_i y)^d$ if and only if the vanishing ideal of the set $\{(\alpha_i,\beta_i),1\le i\le r\}$ is contained in the apolar ideal $p^\perp$.

It follows from Apolarity Lemma that if p has rank r, then the apolar ideal $p^\perp$ contains the vanishing ideal of r distinct points.

Corollary 2.2.4.
If h is a Sylvester form of degree r for p, then $L_{\mathbb{C}}(p)\le r$. Moreover, if p does not have a Sylvester form of degree $r-1$, then $L_{\mathbb{C}}(p)=r$.

The following properties are easily established.

Theorem 2.2.5.

(i) Any square free polynomial multiple of a Sylvester form is also a Sylvester form.

(ii) $L_{\mathbb{C}}(p)\le deg(p)+1$.

(iii) If p has a unique Sylvester form of degree r, then $L_{\mathbb{C}}(p)=r$. Moreover, if $K\subseteq\mathbb{C}$, then $L_K(p)\ge r$.

Proof. (i) If h is apolar to p, then any polynomial multiple of h is also apolar to p since $(hf)(D)=h(D)f(D)=f(D)h(D)$. We should only make sure that the multiple is square free.

(ii) If $deg(h)>deg(p)$, then $h(D)p=0$ automatically. That is if p is a binary form of degree d, then any square free binary form of degree $d+1$ is a Sylvester form for p.

(iii) We shall show that p does not have a Sylvester form of degree $r-1$. Assume the contrary. Let h be a Sylvester form of degree $r-1$ for p. Let $\gamma_1\ne\gamma_2\in\mathbb{C}-Z(h)$, then $h(x,y)(x-\gamma_1 y)$ and $h(x,y)(x-\gamma_2 y)$ are two non-proportional Sylvester forms of degree r for p by (i). This contradiction implies that p does not have a Sylvester form of degree $r-1$. Hence, $L_{\mathbb{C}}(p)=r$ by Corollary 2.2.4. If $K\subseteq\mathbb{C}$, then $L_K(f)\ge r$ by (2.2.2). □

Remark 2.2.6. The upper bound given in Theorem 2.2.5(ii) can be generalized to n-ary forms, $n \geq 1$. Suppose $f(x, y) \in H_d(\mathbb{C}^n)$. It follows from Corollary 2.1.8 that $L_{\mathbb{C}}(f) \leq \binom{n+d-1}{n-1}$. Landsberg and Teitler sharpened this upper bound to $L_{\mathbb{C}}(f) \leq \binom{n+d-1}{n-1} - (n-1)$; see [23, Corollary 5.2].

Lemma 2.2.7.
Let g and h be binary forms such that $Z_{\mathbb{C}}(g, h) = \{(0, 0)\}$ and $deg(g) = deg(h)$. Then there exist $\lambda \in \mathbb{Q}$ such that $h + \lambda g$ is square free.

Proof. The discriminant $\Delta(h + \lambda g)$ is a non-zero polynomial of λ. Therefore, it is zero for finitely many λ and there exists $\lambda \in \mathbb{Q}$ such that discriminant is non-zero and $h + \lambda g$ has distinct roots. $\square$

Theorem 2.2.8.
Let $p(x, y) \in H_d(\mathbb{C}^2)$ with $p^{\perp} = \langle g, h \rangle$ such that $deg(g) \leq deg(h)$. Then

$$L_{\mathbb{C}}(p) = \begin{cases} deg(g) & \text{if } g \text{ is square free} \\ deg(h) & \text{otherwise} \end{cases}$$

Proof. If g is square free, then g is the minimal degree Sylvester form and $L_{\mathbb{C}}(f) = deg(g)$. Assume that g is not square free. Every apolar form of degree $< deg(h)$ is a multiple of g, so $L_{\mathbb{C}}(p) \geq deg(h)$. Let ℓ_i's be distinct linear forms so that

$$\tilde{g}(x, y) = g\ell_0 \dots \ell_i, \quad deg(\tilde{g}) = deg(h) \text{ and } Z_{\mathbb{C}}(\tilde{g}, h) = \{(0, 0)\}. \tag{2.2.9}$$

By Lemma 2.2.7, there exist $\lambda \in \mathbb{Q}$ such that $h + \lambda\tilde{g}$ is square free, i.e., $h + \lambda\tilde{g}$ is a Sylvester form of degree $deg(h)$. Thus, $L_{\mathbb{C}}(p) = deg(h)$. $\square$

Example 2.2.9. Let $p(x, y) = 5x(x^2 + y^2)^2$. Then in (2.2.3), $a_0 = 5,\ a_1 = a_3 = a_5 = 0$ and $a_2 = a_4 = 1$. First with $r = 2$, we see that the linear system (2.2.6) has only trivial solution:

$$\begin{pmatrix} 5 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $L_{\mathbb{C}}(p) \geq 3$. On taking $r = 3$, (2.2.6) becomes:

$$\begin{pmatrix} 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.2.10}$$

The system in (2.2.10) implies $c_0 = c_2 = 0$ and $c_1 = -c_3$. Thus, $h(x, y) = x^2y - y^3$ is the projectively unique Sylvester form of degree 3 for p. This directly implies that $L_{\mathbb{C}}(p) = 3$.

Therefore, $p(x, y)$ can be written as a $\mathbb{C}$-linear combination of 3 distinct 5-th powers of linear forms:

$$\begin{array}{rccccc} h(x,y) = & y & * & (x-y) & * & (x+y) \\ & \downarrow & & \downarrow & & \downarrow \\ p(x,y) = & \lambda_1 x^5 & + & \lambda_2(x+y)^5 & + & \lambda_3(x-y)^5. \end{array}$$

Indeed, $\lambda_2 = \lambda_3 = \frac{1}{2}$ and $\lambda_1 = 4$ and

$$p(x, y) = 4x^5 + \frac{1}{2}(x + y)^5 + \frac{1}{2}(x - y)^5. \tag{2.2.11}$$

If $K \subseteq \mathbb{C}$, then $L_K(p) \geq 3$ by (2.2.2). Notice that $L_{\mathbb{Q}}(p) = 3$ since (2.2.11) is also a representation over $\mathbb{Q}$. Then $L_K(p) = 3$ for any $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. It can be easily checked that $x^4 - 5y^4 \in (p^{\perp})_4$ and $p^{\perp} = \langle x^2y - y^3, x^4 - 5y^4 \rangle$.

The following theorem is a generalization of Sylvester's 1851 Theorem to the subfields of $\mathbb{C}$.

Theorem 2.2.10. [32, Corollary 2.2]
Assume that $p \in H_d(K^2)$. Then $L_K(p)$ is the minimal degree of a Sylvester form for p which completely splits over K.

Proof. Assume that the representation in (2.2.5) is a minimal representation over K, i.e., $\lambda_k, \alpha_k, \beta_k \in K$. Then the corresponding Sylvester form $h(x, y)$ completely splits over K by (2.2.4). Conversely, assume that h is a Sylvester form for p of minimal degree r over K, i.e., satisfying (2.2.4) with $\alpha_k, \beta_k \in K$. Then the representation in (2.2.5) holds for some $\lambda_k \in \mathbb{C}$. Thus, for every $1 \leq i \leq d$, the linear system

$$a_i = \sum_{k=1}^{r} \alpha_k^{d-i} \beta_k^i X_k \tag{2.2.12}$$

has a solution $\{X_k = \lambda_k\}$ over $\mathbb{C}$. Then it also has a solution over K and p has a K-representation of length r; thus, $L_K(p) \le r$. It follows from minimality that $L_K(p) = r$. $\square$

2.3 The cabinet of a binary form

In this section we leverage the arguments given in Section 2 and we study the possible different ranks of a binary form. Suppose p is a binary form in $H_d(K^2)$. Let $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. Then we have

$$L_{\mathbb{Q}}(p) \ge L_K(p) \ge L_{\mathbb{C}}(p). \tag{2.3.1}$$

Definition 2.3.1. The *cabinet* of p, $\mathcal{C}(p)$, is the set of all possible ranks of p over different fields, that is,

$$\mathcal{C}(p) = \{L_K(p) : \ E_p \subseteq K \subseteq \mathbb{C}\}.$$

If $p = 5x(x^2 + y^2)^2$, then (2.3.1) is an equality, i.e., $L_{\mathbb{Q}}(p) = L_{\mathbb{C}}(p)$, so $|\mathcal{C}(p)| = 1$; see Example 2.2.9. However, the following example shows that if $p(x, y) = 10x^3y^2 - 5xy^4$, then $|\mathcal{C}(p)| = 3$. Therefore, inequality and equality in (2.3.1) are both possible.

Example 2.3.2. Let $p(x, y) = 10x^3y^2 - 5xy^4$, so that in (2.2.3), $a_2 = 1$, $a_4 = -1$ and $a_i = 0$ otherwise. The vector subspace $(p^{\perp})_2$ is trivial, and therefore, $L_{\mathbb{C}}(p) \ge 3$. If we take $r = 3$, then the linear system in (2.2.6) is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_0 = c_2 = 0, \ c_1 = c_3.$$

Therefore, up to a multiple, $h(x, y) = x^2y + y^3 = y(x + iy)(x - iy)$. It follows from Theorem 2.2.10 that $L_K(p) = 3$ if and only if $\mathbb{Q}(i) \subseteq K$. In particular, $L_{\mathbb{Q}(i)}(p) = 3$ and

$$p(x, y) = x^5 - \frac{1}{2}(x + iy)^5 - \frac{1}{2}(x - iy)^5. \tag{2.3.2}$$

Now set $r = 4$, then (2.2.6) becomes:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.3.3}$$

The linear system in (2.3.3) implies that $c_1 = c_3$ and $c_2 = c_4$, but it places no condition on c_0. In particular, we may choose $c_0 = c_1 = c_2 = c_3 = c_4 = 1$ to get a Sylvester form over $\mathbb{Q}(\zeta_5)$:

$$h(x,y) = x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \prod_{j=1}^{4}(x - \zeta_5^j y).$$

Since $\zeta_5 \notin \mathbb{Q}(i)$, it follows that $L_{\mathbb{Q}(\zeta_5)}(p) = 4$ by Theorem 2.2.10. We shall see in Example 2.4.6 that $L_{\mathbb{R}}(p) = 5$.

Suppose $p(x,y)$ is a binary form of degree d with the coefficient field E_p. Let $E_p \subseteq K \subseteq \mathbb{C}$. It follows from Corollary 2.1.8 that $L_K(p) \leq d+1$. In this section we present an improvement of this upper bound from $d+1$ to d, which was originally given in [32]. We also address the following questions to understand the cabinet of binary forms:

1. Assume that $E_p \subseteq K_1, K_2$ and $L_{K_1}(p) \neq L_{K_2}(p)$. What can be said about $L_{K_1}(p) + L_{K_2}(p)$?
2. Is there any relation between $|\mathcal{C}(p)|$ and the degree of p?

The following theorem shows that if $p \in H_d(\mathbb{C}^2)$ is a d-th power, then the cabinet $\mathcal{C}(p) = \{1\}$.

Theorem 2.3.3. [32, Theorem 4.1]
If $p \in H_d(K^2)$, then $L_K(p) = 1$ if and only if $L_{\mathbb{C}}(p) = 1$.

Proof. If $L_K(p) = 1$, then $L_{\mathbb{C}}(p) = 1$ since rank is order reversing. For the other direction, assume that $L_{\mathbb{C}}(p) = 1$. Then $p(x,y) = (\alpha x + \beta y)^d$ with $\alpha, \beta \in \mathbb{C}$. We should show that p is a power over K. If $\alpha = 0$, then $p(x,y) = \beta^d y^d$ with $\beta^d \in K$. If $\alpha \neq 0$, then $p(x,y) = \alpha^d(x + (\beta/\alpha)y)^d$. The coefficients of x^d and $x^{d-1}y$ are α^d and $d\alpha^{d-1}\beta$ respectively. Therefore, α^d and $\alpha^{d-1}\beta$ are in K. Then α^d and $\beta/\alpha = (\alpha^{d-1}\beta)/\alpha^d$ are both in K and p is a d-th power over K with $L_K(p) = 1$. $\square$

The vector space of complex forms in $H_d(\mathbb{C}^n)$ in n variables of degree d is spanned by the set of linear forms taken to the d-th power; see Corollary 2.1.8. The following theorem provides a basis for $H_d(\mathbb{C}^2)$.

Theorem 2.3.4. [32, Theorem 4.2]
Any set $\{(\alpha_j x + \beta_j y)^d : 0 \leq j \leq d\}$ of pairwise distinct d-th powers is linearly independent and spans the vector space of binary forms of degree d.

Proof. Let A be the representation matrix of this set with respect to the basis $\binom{d}{i}x^{d-i}y^i$. Then the entry $A_{i,j} = \alpha_j^{d-i}\beta_j^i,\ 0 \leq i, j \leq d$. The determinant of the matrix is Vandermonde:

$$\prod_{0\leq j<k\leq d} (\alpha_j\beta_k - \alpha_k\beta_j). \tag{2.3.4}$$

The determinant is a product of non-zero terms since the $(\alpha_j x + \beta_j y)$'s are distinct forms by the hypothesis.

□

In order to understand the structure of the cabinet of a binary form, we shall first explore the relation between two different honest representations of p.

Theorem 2.3.5. [32, Corollary 4.3]
Assume that $p \in H_d(\mathbb{C}^2)$ has two different honest representations:

$$p(x, y) = \sum_{i=1}^{s} \lambda_i \left(\alpha_i x + \beta_i y\right)^d = \sum_{j=1}^{t} \mu_j \left(\gamma_j x + \sigma_j y\right)^d. \tag{2.3.5}$$

Then $s+t \geq d+2$. If $s+t = d+2$, then the combined set of linear forms, $\{\alpha_i x+\beta_i y, \gamma_j x+\sigma_j y\}$, is pairwise distinct.

Proof. We may assume without loss of generality that the summands in (2.3.5) are distinct. If (2.3.5) holds, then the combined set of d-th powers of linear forms, $\{(\alpha_i x + \beta i y)^d, (\gamma_j x + \sigma_j y)^d,\ 1 \leq i \leq s,\ 1 \leq j \leq t\}$ is linearly dependent. Therefore, $s + t \geq d + 2$ by Theorem 2.3.4.

Let $s + t = d + 2$ and assume that there exist i, j so that the binary forms $(\alpha_i x + \beta_i y)^d$ and $(\gamma_j x + \sigma_j y)^d$ are proportional; that is, $\alpha_i\sigma_j = \beta_i\gamma_j$. Without loss of generality, we can assume that $i = j = 1$. Then the combined set of linear forms $\{\alpha_1 x+\beta_1 y, \alpha_i x+\beta_i y, \gamma_j x+\sigma_j y,\ 2 \leq i \leq s,\ 2 \leq j \leq t\}$ is a linearly dependent set of size $d+1$. This is impossible, unless the dependence is trivial. □

Corollary 2.3.6.
Let $p(x, y)$ be a binary form of degree d with coefficient field $E_p = E$. Suppose $L_E(p) = r$.

(i) If $r \leq \frac{d}{2} + 1$, then $L_{\mathbb{C}}(p) = r$, so $\mathcal{C}(p) = \{r\}$.

(ii) If $r < \frac{d}{2} + 1$, then p has a unique $\mathbb{C}$-minimal representation.

Proof. (i) First notice that $L_{\mathbb{C}}(p) \leq r$ since rank is order reversing. It follows from Theorem 2.3.5 that $L_E(p) + L_{\mathbb{C}}(p) \geq d + 2$, and so $L_{\mathbb{C}}(p) \geq r$. Therefore, $L_{\mathbb{C}}(p) = r$. If $E \subseteq K \subseteq \mathbb{C}$, then $r = L_E(p) \geq L_K(p) \geq L_{\mathbb{C}}(p) = r$. This implies that $\mathcal{C}(p) = \{r\}$.

(ii) If $r < \frac{d}{2} + 1$, then the length of any other honest representation is greater then r by Theorem 2.3.5. Thus, p has a unique $\mathbb{C}$-minimal representation. □

As we saw in Remark 2.2.6, Landsberg and Teitler [23, Corollary 5.2] prove that if $p \in H_d(\mathbb{C}^n)$, then $L_{\mathbb{C}}(p) \leq \binom{n+d-1}{n-1} - (n-1)$. It reduces to the following theorem for the binary case.

Theorem 2.3.7.
If $p \in H_d(\mathbb{C}^2)$, then $L_{\mathbb{C}}(p) \leq d$.

Proof. We shall show that p has a Sylvester form of degree d. We write p as in (2.2.3). By a change of variables, we may assume that neither x nor y divide p. Hence, $a_0 a_d \neq 0$ and $h(x, y) = a_d x^d - a_0 y^d$ is a Sylvester form. □

Notice that $h(x, y) = a_d x^d - a_0 y^d$ usually does not split over K for $K \subsetneq \mathbb{C}$, and so, the proof of Theorem 2.3.7 does not apply to $L_K(f)$ for $K \neq \mathbb{C}$. We now give the generalization of the theorem.

Theorem 2.3.8. [32, Theorem 4.10]
If $p \in H_d(K^2)$, then $L_K(p) \leq d$.

We refer the reader to [32] for a detailed proof.

Theorem 2.3.9. [32, Corollary 5.1]
Suppose $p \in H_d(\mathbb{C}^2)$.

(i) If $L_{\mathbb{C}}(p) = r$, then $\mathcal{C}(p) \subseteq \{r\} \cup \{d + 2 - i : \; 2 \leq i \leq r\}$.

(ii) If $L_{\mathbb{C}}(p) = 2$, then $\mathcal{C}(p)$ is either $\{2\}$ or $\{2, d\}$.

(iii) If $|\mathcal{C}(p)| \geq k$, then $d \geq 2k - 1$.

(iv) If $d = 3$, then $\mathcal{C}(p) = \{1\}, \{2\}, \{3\}$ or $\{2, 3\}$.

(v) If $d = 4$, then $\mathcal{C}(p) = \{1\}, \{2\}, \{3\}, \{4\}, \{2, 4\}$ or $\{3, 4\}$.

Proof. The proof immediately follows from Theorem 2.3.5 and Theorem 2.3.8. □

The following theorem gives all the binary forms of degree $d \geq 3$ with Waring rank d. It is well known that $L_{\mathbb{C}}(x^{d-1}y) = d$; however, to the best of our knowledge, the converse has been proven only later in [4, Corollary 3] and [15, Ex.11.35].

Theorem 2.3.10.
If $d \geq 3$, then $L_{\mathbb{C}}(p) = d$ if and only if there are two distinct linear forms ℓ_0 and ℓ_1 so that $p = {\ell_0}^{d-1}\ell_1$.

The following theorem gives the complete classification of K-rank of binary cubics.

Theorem 2.3.11. [32, Theorem 5.2]
Suppose $p(x, y)$ in $E_p[x, y]$, is a cubic form with discriminant Δ and suppose $E_p \subseteq K \subseteq \mathbb{C}$.

(i) If p is a cube, then $L_K(p) = 1$ and $\mathcal{C}(p) = \{1\}$.

(ii) If p has a repeated linear factor, but is not a cube, then $L_K(p) = 3$ and $\mathcal{C}(p) = \{3\}$.

(iii) If p does not have a repeated factor, then $L_K(p) = 2$ if $\sqrt{-3\Delta} \in K$ and $L_K(p) = 3$ otherwise, so either $\mathcal{C}(p) = \{2\}$ or $\mathcal{C}(p) = \{2, 3\}$.

Proof. (i) If p is a cube, then $\mathcal{C}(p) = \{1\}$ by Theorem 2.3.3.

(ii) If $p = {\ell_0}^2\ell_1$ and not a cube, then $L_{\mathbb{C}}(p) = 3$ by Theorem 2.3.10. If $K \subseteq \mathbb{C}$, then $L_K(p) \geq 3$ since rank is order reversing and $L_K(p) \leq 3$ by Theorem 2.3.8. Therefore, $\mathcal{C}(p) = \{3\}$.

(iii) Suppose $p(x, y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$ is square free. Then the discriminant of p, $\Delta(p) \neq 0$. We consider the linear system:

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.3.6}$$

This system has rank 2 and the unique Sylvester form of degree 3 is

$$h(x, y) = (a_1a_3 - a_2^2)x^2 + (a_1a_2 - a_0a_3)xy + (a_0a_2 - a_1^2)y^2, \tag{2.3.7}$$

which is the Hessian of p. Now we compare the discriminant of h and p.

$$\Delta(h) = (a_1a_2 - a_0a_3)^2 - 4(a_1a_3 - a_2^2)(a_0a_2 - a_1^2) = -\frac{\Delta(p)}{27} = -\frac{3\Delta(p)}{9^2}. \tag{2.3.8}$$

Thus, h splits over K if and only if $\sqrt{-3\Delta(p)} \in K$. This implies that $L_K(p) = 2$ if $\mathbb{Q}\big(\sqrt{-3\Delta(p)}\,\big) \subseteq K$, and 3 otherwise. □

Example 2.3.12. The binary forms x^3, x^3+y^3, x^2y and $(x+iy)^3+(x-iy)^3$ have the cabinets enumerated in Theorem 2.3.9(iv). It follows from Theorem 2.3.11 that if $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$, then $L_K(x^3) = 1,\ L_K(x^3+y^3) = 2,\ L_K(x^2y) = 3$ and $L_K((x+iy)^3+(x-iy)^3) = 2$ if $i \in K$, and 3 otherwise.

Definition 2.3.13. We call a binary form $p \in H_d(\mathbb{R}^2)$ *hyperbolic* if all its roots are real, i.e., it splits into linear factors over $\mathbb{R}$.

We can determine the real Waring rank of binary cubic forms based on their factorization over $\mathbb{R}$. Assume that ℓ_i's are distinct real binary linear forms and $q(x,y)$ is an irreducible quadratic. The following table follows from Theorem 2.3.11.

$p(x,y)$	$L_{\mathbb{R}}(p(x,y))$
$\ell_0(x,y)^3$	1
$\ell_0(x,y)^2\ell_1(x,y)$	3
$\ell_0(x,y)q(x,y)$	2
$\ell_0(x,y)\ell_1(x,y)\ell_2(x,y)$	3

We can conclude that if p is a real cubic, and not a cube, then $L_{\mathbb{R}}(p) = 3$ if and only if p is hyperbolic.

Reznick showed that $L_{\mathbb{C}}(x^ky^k) = k+1$ [32, Theorem 5.5]. The following example shows that $\mathcal{C}(x^2y^2) = \{3,4\}$.

Example 2.3.14. Let $p(x,y) = x^2y^2$. It follows from Theorem 2.2.1 that $(p^\perp)_2 = \{0\}$ and $(p^\perp)_3 = \langle x^3,\ y^3\rangle$, so the possible Sylvester forms of degree 3 have the shape $h(x,y) = \alpha x^3 + \beta y^3,\ \alpha\beta \neq 0$. There is no Sylvester form of degree 3 which completely splits over $\mathbb{R}$. Therefore, $L_{\mathbb{C}}(p) = 3$ and $L_{\mathbb{R}}(p) = 4$ by Theorem 2.2.10.

The binary forms $x^4, x^4+y^4, x^4+y^4+(x+y)^4, x^3y, (x+iy)^4+(x-iy)^4$ and x^2y^2 have the cabinets enumerated in Theorem 2.3.9 (v).

2.4 The real Waring rank of binary forms

Recently the real Waring rank of binary forms has been investigated [5, 7, 10, 11, 14]. The relation between the number of real roots and the real Waring rank of binary forms has also received substantial attention. Extending the work of Sylvester, Reznick showed that if $p(x, y)$ is a binary form of degree d, not a d-th power, with τ real roots (counting multiplicities), then $L_{\mathbb{R}}(f) \geq \tau$ [32, Theorem 3.2].

We begin with Theorem 2.4.1, which was discovered by Sylvester [36] in 1864 as a part of proving Isaac Newton's conjectural variation on Descartes' Rule of Signs. This theorem also appears in [19, 26, 27].

Theorem 2.4.1. [Sylvester's 1864 Theorem]
Suppose $\lambda_k \neq 0$ for all k and $\gamma_1 < \cdots < \gamma_r,\ r \geq 2$ are real numbers such that

$$Q(t) = \sum_{k=1}^{r} \lambda_k (t - \gamma_k)^d \tag{2.4.1}$$

does not vanish identically. Suppose Q is not a d-th power and suppose that the sequence $(\lambda_1, \cdots, \lambda_r, (-1)^d \lambda_1)$ has C changes of sign and Q has Z zeros, counting multiplicity. Then $Z \leq C$.

The following theorem is an equivalent homogenized version of Theorem 2.4.1. The equivalence is discussed in [30, 31].

Theorem 2.4.2. [32, Theorem 3.2]
Suppose $p(x, y)$ is a non-zero real form of degree d, not a d-th power, with τ linear factors (counting multiplicity) and

$$p(x, y) = \sum_{j=1}^{r} \lambda_j (\cos\theta_j x + \sin\theta_j y)^d \tag{2.4.2}$$

where $-\frac{\pi}{2} < \theta_1 < \cdots < \theta_r \leq \frac{\pi}{2}, r \geq 2$ and $\lambda_j \neq 0$. If there are σ sign changes in the tuple $(\lambda_1, \lambda_2, \cdots, \lambda_r, (-1)^d \lambda_1)$, then $\tau \leq \sigma$. In particular, $\tau \leq r$.

The following is just a restatement of Theorem 2.4.2 and gives a lower bound for the real Waring rank.

Theorem 2.4.3.
Suppose that p is a real binary form of degree d, and not a d-th power. If p has τ real linear factors, counting multiplicity, then $L_{\mathbb{R}}(p) \geq \tau$.

An improvement of Theorem 2.4.3 is given in Chapter 4; see Corollary 4.1.5.

Corollary 2.4.4.
If $p \in H_d(\mathbb{R}^2)$ is hyperbolic, and not a d-th power, then $L_{\mathbb{R}}(p) = d$.

Proof. It is immediate from Theorem 2.4.3 that $L_{\mathbb{R}}(p) \geq d$. On the other hand, $L_{\mathbb{R}}(p) \leq d$ by Theorem 2.3.8.

$\square$

The following result on the real Waring rank of monomials is immediate from Corollary 2.4.4. A different proof can be found in [7].

Corollary 2.4.5.
Suppose $a, b \geq 1$ and $p(x, y) = x^a y^b$. Then $L_{\mathbb{R}}(p) = a + b$.

Example 2.4.6. Let $p(x, y) = 10x^3y^2 - 5xy^4$. It was shown in Example 2.3.2 that $L_{\mathbb{Q}(i)}(p) = 3$ and $L_{\mathbb{Q}(\zeta_5)}(p) = 4$. Since p is hyperbolic, then $L_{\mathbb{R}}(p) = 5$ by Corollary 2.4.4. Therefore, $\mathcal{C}(p) = \{3, 4, 5\}$.

Example 2.4.7. Let $p(x, y) = 15x^2y^2(x^2+y^2)$. It follows from Theorem 2.4.3 that $L_{\mathbb{R}}(p) \geq 4$. On taking $r = 4$, (2.2.6) becomes:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.4.3}$$

Clearly, $(c_0, c_1, c_2, c_3, c_4) = (0, 1, 0, -1, 0)$ is a solution for (2.4.3), so that $h(x, y) = xy(x^2-y^2)$ is a Sylvester form for p. It follows from Theorem 2.2.10 that $\text{Ł}_{\mathbb{R}}(p) = L_{\mathbb{Q}}(p) = 4$. Taking $r = 3$, we note that the Hankel matrix

$$H_3(p) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.4.4}$$

is non-singular, hence there are no representations of rank 3 and $L_{\mathbb{C}}(p) = 4$. If $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$, then $L_K(p) = 4$ by (2.3.1), and so $|\mathcal{C}(p)| = \{4\}$.

The converse of Corollary 2.4.4 was conjectured and proved for $d \leq 4$ in [32]. Causa and Re [11] and Comon and Ottaviani [14] showed that the conjecture holds for any square-free binary form, and recently Blekherman and Sinn [5] proved that the conjecture is true for any binary form.

Theorem 2.4.8. [5, Theorem 2.2]
Let $p(x, y)$ be a binary form of degree $d \geq 3$ and suppose that p is not a d-th power. The real Waring rank of p is d if and only if p is hyperbolic.

Example 2.4.9. Suppose $d \geq 3$ and $r \neq 0 \in \mathbb{R}$ such that

$$p(x, y) = (x + \sqrt{r}y)^d + (x - \sqrt{r}y)^d \in H_d(\mathbb{R}^2). \tag{2.4.5}$$

Then $L_{\mathbb{R}}(p) = 2$ if $\sqrt{r} \in \mathbb{R}$ and d otherwise by Theorem 2.3.9. Therefore, it follows from Theorem 2.4.8 that p is hyperbolic if and only if r is negative.

Corollary 2.4.10.
Suppose $p \in H_d(\mathbb{R}^2)$ is a hyperbolic form and not a d-th power. If h is a Sylvester form for p, with deg$(h) < d$, then h is non-hyperbolic.

Chapter 3

Binary forms with 3 different relative ranks

In this chapter we study binary forms with multiple ranks. We show that for every $d \geq 5$, there always exists a binary form of degree d with at least three different ranks over various fields. We also study the relation between the relative rank and the algebraic properties of the underlying field. In particular, we show that the K-rank of a binary form may depend on whether -1 is a sum of two squares in K. This chapter is adapted from the paper [34] *Binary forms with three different relative ranks* by B. Reznick and N. Tokcan, which has been accepted for publication by *Proceedings of the American Mathematical Society*.

3.1 Binary forms with multiple ranks

Let $p(x, y) \in H_d(\mathbb{C}^2)$ be a nonzero binary form. One relation between the number of different ranks and the degree of a form is given by Theorem 2.3.9 (iii): If p has k different ranks, then $d \geq 2k - 1$.

Forms with cabinet size one are abundant in $H(\mathbb{C}^2)$. For every $d \geq 1$, there exist a binary form $p \in H_d(\mathbb{C}^2)$ so that $|\mathcal{C}(p)| = 1$. We now give some examples:

- Suppose p is a d-th power, i.e., $p(x, y) = (\alpha x + \beta y)^d$, $\alpha\beta \neq 0 \in \mathbb{C}$. Then $\mathcal{C}(p) = \{1\}$ by Theorem 2.3.3.
- If $p(x, y) = x^{d-1}y$, then $\mathcal{C}(p) = \{d\}$ by Theorem 2.3.10.
- If $L_{\mathbb{Q}}(p) = k \leq \frac{d+2}{2}$, then $\mathcal{C}(p) = \{k\}$ by Theorem 2.3.5 and (2.3.1). For example, if $p(x, y) = 5x(x^2 + y^2)^2$, then $\mathcal{C}(p) = \{3\}$; see Example 2.2.9.

If $d \geq 3$, then it is possible to have a binary form with three different ranks. The following is a central example which illustrates the phenomenon of multiple ranks over different fields.

Example 3.1.1. Suppose $p(x, y) = (x + \sqrt{2}\ y)^d + (x - \sqrt{2}\ y)^d \in \mathbb{Q}[x, y]$, $d \geq 3$. Then it follows from Theorem 2.3.5 that $L_K(f) = 2$ if $\mathbb{Q}(\sqrt{2}) \subseteq K$ and d otherwise. Therefore, $\mathcal{C}(p) = \{2, d\}$.

The following is a generalized version of Example 3.1.1.

Example 3.1.2. Suppose there exists $\gamma \in \mathbb{Q}$ with $\sqrt{\gamma} \notin \mathbb{Q}$ so that

$$p_d(x,y) = \sum_{0 \le 2i \le d} \binom{d}{2i} \gamma^i x^{d-2i} y^{2i}, \ d \ge 3. \tag{3.1.1}$$

Then $p_d(x,y)$ is a binary form in $\mathbb{Q}[x,y]$ of Waring rank 2 with the following projectively unique representation:

$$p_d(x,y) = \frac{1}{2}(x + \sqrt{\gamma}y)^d + \frac{1}{2}(x - \sqrt{\gamma}y)^d. \tag{3.1.2}$$

Notice that the summands in (3.1.2) are conjugates of each other in $\mathbb{Q}(\sqrt{\gamma})$. It follows from Corollary 2.3.6(ii) that p has a unique minimal representation of length 2. Then $L_K(p_d) = 2$ if $\mathbb{Q}(\sqrt{\gamma}) \subseteq K$ and d otherwise. Hence, $\mathcal{C}(p) = \{2, d\}$.

By Theorem 2.3.9(iii), it is possible to have a binary form p of degree $d \ge 5$ with 3 different ranks over different fields; see Example 2.4.6.

The first explicit example of a binary form with three different ranks was given by Reznick [32].

Example 3.1.3. Suppose $p(x,y) = 3x^5 - 20x^3y^2 + 10xy^4$. We can write p as

$$\begin{aligned} p(x,y) \ &= \ \binom{5}{0} \cdot 3x^5 + \binom{5}{1} \cdot 0x^4y + \binom{5}{2} \cdot (-2)x^3y^2 \\ &+ \ \binom{5}{3} \cdot 0x^2y^3 + \binom{5}{4} \cdot 2xy^4 + \binom{5}{5} \cdot 0y^5. \end{aligned}$$

The vector subspace $(p^\perp)_2$ is trivial, so $L_\mathbb{C}(p) \ge 3$. By taking $d = 5, \ r = 3$, the linear system in (2.2.6) becomes

$$\begin{pmatrix} 3 & 0 & -2 & 0 \\ 0 & -2 & 0 & 2 \\ -2 & 0 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.1.3}$$

The only solution to (3.1.3) is $c_1 = c_3$ and $c_0 = c_2 = 0$, so that p has a unique Sylvester form of degree 3:

$$h(x,y) = y(x^2 + y^2) = y(x - iy)(x + iy).$$

It can be easily checked that the representation of p corresponding to h is

$$p(x,y) = x^5 + (x+iy)^5 + (x-iy)^5. \tag{3.1.4}$$

It follows that $L_K(p) = 3$ if and only if $\mathbb{Q}(i) \subseteq K$.

Let $d = 5,\ r = 4$. If $(c_0, c_1, c_2, c_3, c_4)$ is a solution of the 2×5 system in (2.2.6), then $(c_0, c_1, c_2, c_3, c_4) = r_1(2,0,3,0,0) + r_2(0,1,0,1,0) + r_3(0,0,1,0,1)$. Then a Sylvester form of degree 4 is of the following form:

$$h_r(x,y) = r_1 x^2(2x^2 + 3y^2) + y(x^2 + y^2)(r_2 x + r_3 y). \tag{3.1.5}$$

We need to consider different choices of $r = (r_1, r_2, r_3)$ and the splitting field of the corresponding h_r; see [32, Example 2.1] for details. It follows that $L_{\mathbb{Q}(\sqrt{-d})}(p) = 4$ for several values of $d \geq 2$. Since p is hyperbolic, $L_{\mathbb{R}}(p) = 5$ by Corollary 2.4.4.

3.2 Binary forms with three different ranks

In this section we show that the three-rank phenomenon occurs in all degrees $d \geq 5$ by Theorem 3.2.7. We prove that if $k \geq 3$ and $p_{2k-1}(x,y) = x^{k-1}y^{k-1}(x-y)$, then

$$L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k-1}) = k, \quad L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) = k+1, \quad L_{\mathbb{R}}(p_{2k-1}) = 2k-1 > k+1.$$

Similarly, if $k \geq 3$ and $p_{2k}(x,y) = x^k y^k$, then

$$L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k}) = k+1, \quad L_{\mathbb{Q}(\zeta_k)}(p_{2k}) = k+2, \quad L_{\mathbb{R}}(p_{2k}) = 2k > k+2.$$

We need to consider the rank of a hyperbolic binary form over cyclotomic fields. The following theorem is an elementary result in algebraic number theory. We include the proof for completeness (See [8, p.158(Lemma 3)] for a different proof). Let $\zeta_d = e^{\frac{2\pi i}{d}}$.

Theorem 3.2.1.
Suppose m, n are integers. Then $\zeta_m \in \mathbb{Q}(\zeta_n)$ if and only if $m \mid n$ or n is odd and $m \mid 2n$.

Proof. Note that $\zeta_m = \zeta_{mt}^t$. If n is odd and m divides $2n$ but not n, then $m = 2u$ and $n = tu$ with odd t, u, so $\zeta_m = \zeta_{2n}^t = -\zeta_{2n}^{t+tu} = -\zeta_n^{t(u+1)/2} \in \mathbb{Q}(\zeta_n)$. Conversely, let $g = gcd(m,n)$ so that $m = gr, n = gs$, where $gcd(r,s) = 1$, and let $q = grs = lcm(m,n)$. Then $\zeta_m = \zeta_q^s$ and $\zeta_n = \zeta_q^r$. Now choose integers e, f so that $es + fr = 1$. We have $\zeta_m^e \zeta_n^f = \zeta_q^{es+fr} = \zeta_q$.

Since $\zeta_m \in \mathbb{Q}(\zeta_n)$, it follows that $\zeta_q \in \mathbb{Q}(\zeta_n)$, so $\mathbb{Q}(\zeta_q) \subseteq \mathbb{Q}(\zeta_n)$, but since $n \mid q$, the converse inclusion holds as well, and so $\mathbb{Q}(\zeta_q) = \mathbb{Q}(\zeta_n)$. This in turn implies that $\Phi(n) = \Phi(q)$. Since $n \mid q$, this implies that $n = q$ (and $gs = grs$, so $r = 1$ and $m \mid n$) or n is odd and $q = 2n$ (and $grs = 2gs$, so $r = 2$ and $m \mid 2n$). □

Corollary 3.2.2.
If $m \geq 3$, then $\zeta_m \notin \mathbb{Q}(\zeta_{m\pm 1})$.

Let h be a binary form of degree $r \geq 3$. Corollary 3.2.2 guarantees that if the smallest splitting field of h is $\mathbb{Q}(\zeta_{m\pm 1})$, then h does not split over $\mathbb{Q}(\zeta_m)$. Therefore, it becomes possible to find a binary form $p \in H_d(\mathbb{C}^2)$ such that $L_{\mathbb{Q}(\zeta_{m\pm 1})}(p) \neq L_{\mathbb{Q}(\zeta_m)}(p)$.

Our approach to find a binary form with three different ranks is simple and straightforward. Suppose $d = 2k - 1$ is odd. Choosing $r = k$, we see that (2.2.6) is a $k \times (k+1)$ linear system, which in general has a unique solution. We consider a form p of degree $2k-1$ which is hyperbolic so that $L_{\mathbb{R}}(p) = 2k - 1$. We also choose K to be the field generated by the coefficients of this unique representation of p over $\mathbb{C}$, so $L_K(p) = k$; necessarily, $p \in K[x, y]$. Finally, we somehow find a representation of rank between k and $2k-1$ over a non-real field which does not contain the rank k representation. If $d = 2k$, the same approach applies, but there will be, in general, infinitely many representations of rank $k + 1$. In certain cases though, each of these representations must contain a specific non-real root of unity ζ.

Theorem 3.2.3.
Suppose $k \geq 3$ and $p_{2k-1}(x, y) = \binom{2k-1}{k} x^{k-1} y^{k-1}(x - y)$. Then the cabinet $\mathcal{C}(p_{2k-1}) \supseteq \{k, k+1, 2k-1\}$.

Proof. Let $p_{2k-1}(x, y) = \binom{2k-1}{k} x^{k-1} y^{k-1}(x - y)$, so that in (2.2.3), $a_{k-1} = 1$, $a_k = -1$ and $a_i = 0$ otherwise. First, with $r = k - 1$, we see that (2.2.6) has a trivial null space:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It follows that $L_{\mathbb{C}}(p_{2k-1}) > k-1$. On taking $r = k$, (2.2.6) becomes:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & -1 & \cdots & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.2.1}$$

Clearly, the only solution to (3.2.1) has $c_i = c$ for all i, so that up to multiple,

$$h(x,y) = \sum_{t=0}^{k} x^{k-t} y^t = \frac{x^{k+1} - y^{k+1}}{x-y} = \prod_{j=1}^{k} (x - \zeta_{k+1}^j y),$$

and so $L_K(p_{2k-1}) = k$ if and only if $\zeta_{k+1} \in K$; in particular, $L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k-1}) = k$. Since p_{2k-1} is hyperbolic, it follows from Corollary 2.4.4 that $L_{\mathbb{R}}(p_{2k-1}) = 2k-1$.

Now set $r = k+1$, so that (2.2.6) becomes:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.2.2}$$

The system (3.2.2) implies $c_1 = \cdots = c_k$, but places no conditions on c_0 and c_{k+1} . In particular, we may choose $c_0 = c_{k+1} = 0$ and $c_1 = \cdots = c_k = 1$, to get a Sylvester polynomial over $\mathbb{Q}(\zeta_k)$:

$$h(x,y) = \sum_{t=1}^{k} x^{k+1-t} y^t = xy \left(\frac{x^k - y^k}{x-y} \right) = xy \prod_{j=1}^{k-1} (x - \zeta_k^j y).$$

It follows that $L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) \leq k+1$. Since $\zeta_{k+1} \notin \mathbb{Q}(\zeta_k)$ by Corollary 3.2.2, it follows that $L_{\mathbb{Q}(\zeta_k)}(p_{2k-1}) = k+1$.

Since

$$c_0 x^{k+1} + c_1(x^k y + \cdots + xy^k) + c_{k+1} y^{k+1} =$$
$$(c_0 x + (c_1 - c_0)y)(x^k + \cdots + y^k) + (c_{k+1} - c_1 + c_0) y^{k+1},$$

it is not hard to show that the apolar ideal of p_{2k-1} is generated by $\frac{x^{k+1}-y^{k+1}}{x-y}$ and y^{k+1}; note that $k + (k+1) = (2k-1) + 2$. It seems to be a quite difficult question to determine which fields K have the property that, for a suitable choice of c_i's, a form such as this is square-free and splits over K. □

The following example gives the explicit representations of $p_5,\ p_7$ as sums of powers of linear forms.

Example 3.2.4. For $k = 3$, the following two formulas may be directly verified (as usual, $\omega = \zeta_3$ and $i = \zeta_4$):

$$\begin{aligned} p_5(x,y) &= 10x^2y^2(x-y) \\ &= \tfrac{1}{4}\cdot\left((-1-i)(x+iy)^5 + 2(x-y)^5 + (-1+i)(x-iy)^5\right) \in \mathbb{Q}(\zeta_4)[x,y] \\ &= x^5 - y^5 + \tfrac{1}{\omega-\omega^2}\cdot\left(\omega^2(x+\omega y)^5 - \omega(x+\omega^2 y)^5\right) \in \mathbb{Q}(\zeta_3)[x,y]. \end{aligned}$$

The expressions seem to get more complicated for larger values of k. For example,

$$(1+2\zeta_5+3\zeta_5^2-\zeta_5^3)p_7(x,y) =$$
$$\zeta_5^4(x+\zeta_5 y)^7 - \zeta_5^2(1+\zeta_5+\zeta_5^2)(x+\zeta_5^2 y)^7 + \zeta_5(1+\zeta_5+\zeta_5^2)(x+\zeta_5^3 y)^7 - \zeta_5(x+\zeta_5^4 y)^7.$$

Here, $1+2\zeta_5+3\zeta_5^2-\zeta_5^3 = i\sqrt{\frac{5}{2}(5+\sqrt{5})} \approx 4.25i$.

Theorem 3.2.5.
Suppose $k \geq 3$ and $p_{2k}(x,y) = \binom{2k}{k}x^k y^k$. Then $\mathcal{C}(p_{2k}) \supseteq \{k+1, k+2, 2k\}$.

Proof. Suppose that $p_{2k}(x,y) = \binom{2k}{k}x^k y^k$, so $a_k = 1$ and $a_i = 0$ otherwise. (This example is also discussed in [32, Theorem 5.5].) Taking $r = k$, we note that the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \tag{3.2.3}$$

is non-singular, hence there are no representations of rank k. For $r = k+1$,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Longrightarrow c_1 = \cdots = c_k = 0. \tag{3.2.4}$$

Thus every Sylvester form of degree $k+1$ has the shape $h(x,y) = \alpha x^{k+1} - \beta y^{k+1}$ and the apolar ideal of p_{2k} is generated by x^{k+1} and y^{k+1}. If h has distinct factors, then $\alpha\beta \neq 0$ and

$$h(x,y) = \alpha \prod_{j=0}^{k} (x - \zeta_{k+1}^{j} u y),$$

where $\alpha u^{k+1} = \beta$. If h splits over K, then $u, \zeta_{k+1} u \in K$, hence $\zeta_{k+1} \in K$ and $\mathbb{Q}(\zeta_{k+1}) \subseteq K$. In particular, by taking $\alpha = \beta = 1$, we see that $x^{k+1} - y^{k+1}$ is a Sylvester form for p_{2k} over $\mathbb{Q}(\zeta_{k+1})$, and so $L_{\mathbb{Q}(\zeta_{k+1})}(p_{2k}) = k+1$. Since $x^k y^k$ is hyperbolic, $L_{\mathbb{R}}(p_{2k}) = 2k$.

Any expression of rank $k+2$ over K would have a Sylvester form of shape

$$(\alpha x + \beta y) x^{k+1} + (\gamma x + \delta y) y^{k+1}.$$

In particular, $xy(x^k - y^k) = x^{k+1} y - x y^{k+1}$ splits over $\mathbb{Q}(\zeta_k)$, which does not contain ζ_{k+1} and so we have $L_{\mathbb{Q}(\zeta_k)}(p_{2k}) = k+2$.

□

Example 3.2.6 gives the representation of p_6 as sums of powers of linear forms.

Example 3.2.6. The representations of p_{2k} of rank $k+1$ are given in [32, Theorem 5.5]. For $k = 3$, taking $w = 1$ in [32, (5.6)], we obtain after some simplification,

$$\begin{aligned} p_6(x,y) &= 20x^3y^3 \\ &= \tfrac{1}{4} \cdot \left((x+y)^6 + i(x+iy)^6 - (x-y)^6 - i(x-iy)^6 \right) \in \mathbb{Q}(\zeta_4)[x,y] \\ &= \tfrac{1}{3} \cdot \left((x+y)^6 + (x+\omega y)^6 + (x+\omega^2 y)^6 - 3x^6 - 3y^6 \right) \in \mathbb{Q}(\zeta_3)[x,y]. \end{aligned}$$

The evident patterns shown above are easily proved, using the methods of [32]; see also Thoeorem 4.2.3.

Theorem 3.2.7.
If $d \geq 5$, then there exists a binary form p_d of degree d which takes at least three different ranks.

Proof. The proof follows from Theorem 3.2.3 and 3.2.5. □

3.3 Relative rank and the Stufe

In this section we show that the relative rank can depend on the algebraic properties of the underlying field.

Definition 3.3.1. The *Stufe* of a non-real field F, $s(F)$, is the smallest integer n such that -1 can be written as a sum of n squares in F.

We show in Theorem 3.3.3 that $L_K(x^3y^2) = 4$ if and only if $s(K) \leq 2$ and $L_K(x^3y^2) = 5$ otherwise. We show in Theorem 3.3.4 that if m is a square-free positive integer and $f(x, y) = 6x^5y - 20x^3y^3$, then $L_{\mathbb{Q}(\sqrt{-m})}(f) = 4$ if and only if $s(\mathbb{Q}(\sqrt{-m})) \leq 2$ if and only if $m \not\equiv 7 \mod 8$ and $L_{\mathbb{Q}(\sqrt{-7})}(f) = 5$.

The following theorem is based on the works of Nagell and Szymiczek; see [25, 39].

Theorem 3.3.2.
Suppose $F = \mathbb{Q}(\sqrt{-m})$, where m is a square-free positive integer. Then there exist solutions to either of the equations

$$r^2 + s^2 = -1, \qquad rs(r^2 - s^2) \neq 0, \quad r, s \in F \tag{3.3.1}$$

$$t^2 + u^2 = -2, \qquad tu(t^2 - u^2) \neq 0, \quad t, u \in F \tag{3.3.2}$$

if and only if $m \not\equiv 7 \mod 8$.

Proof. First note that if (3.3.1) holds and $(t, u) = (r+s, r-s)$, then $t^2+u^2 = 2(r^2+s^2) = -2$ and $tu(t^2 - u^2) = 4rs(r^2 - s^2)$, so (3.3.2) holds. This argument goes the other way with $(r, s) = (\frac{t+u}{2}, \frac{t-u}{2})$, and so it suffices to prove the theorem for (3.3.1).

Nagell [25] proves that $s(\mathbb{Q}(\sqrt{-m})) \leq 2$ (that is, there is a solution to $r^2 + s^2 = -1$ in $\mathbb{Q}(\sqrt{-m})$) if and only if $m \not\equiv 7 \mod 8$, so all we need to do is consider the additional condition $rs(r^2 - s^2) \neq 0$. If $r^2 + s^2 = -1$ and $rs(r^2 - s^2) = 0$, then up to permutation, $(r, s) = (\pm i, 0)$ or $(\pm\frac{\sqrt{-2}}{2}, \pm\frac{\sqrt{-2}}{2})$. These solutions are relevant to $\mathbb{Q}(\sqrt{-m})$ only when $m = 1, 2$, in which case the following alternatives suffice:

$$\mathbb{Q}(\sqrt{-1}): \quad \left(\tfrac{3}{4}\right)^2 + \left(\tfrac{5i}{4}\right)^2 = -1, \qquad \mathbb{Q}(\sqrt{-2}): \quad 7^2 + (5\sqrt{-2})^2 = -1.$$

□

Theorem 3.3.3.
Suppose that $f(x, y) = \binom{5}{2}x^3y^2$. Then $L_K(f) = 4$ iff $s(K) \leq 2$; otherwise, $L_K(f) = 5$.

Proof. We already know from [9] that $L_{\mathbb{C}}(f) = 4$, hence $L_K(f) \geq 4$. This can also be shown directly via Theorem 2.2.1. We omit the details. Suppose now that $L_K(f) = 4$. Then,

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow c_1 = c_2 = 0. \tag{3.3.3}$$

and $h(x, y) = c_0x^4 + c_3xy^3 + c_4y^4$ is a Sylvester form for f over K. Thus, we are led to the question: for which choices of c_i and which fields K can such a square-free form split into distinct factors over K?

If $c_0 = 0$ then h is not square-free, so we scale to take $c_0 = 1$. Then $L_K(f) = 4$ if and only if there exist distinct $r_i \in K$ so that

$$x^4 + c_3xy^3 + c_4y^4 = (x - r_1y)(x - r_2y)(x - r_3y)(x - r_4y);$$

that is, if and only if the Diophantine system

$$r_1 + r_2 + r_3 + r_4 = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = 0 \tag{3.3.4}$$

has a solution in K with distinct r_i's.

We solve (3.3.4), first ignoring the restriction to distinct elements. Putting $r_4 = -(r_1 + r_2 + r_3)$ into the second equation yields

$$\begin{gathered} r_1^2 + r_2^2 + r_3^2 + r_1r_2 + r_1r_3 + r_2r_3 = 0 \Longrightarrow \\ r_3 = -\frac{r_1 + r_2}{2} \pm \frac{\sqrt{-3r_1^2 - 2r_1r_2 - 3r_2^2}}{2}. \end{gathered}$$

Choose $r_1, r_2 \in K$. We see that $r_3 \in K$ (and so $r_4 \in K$) if and only if

$$-3r_1^2 - 2r_1r_2 - 3r_2^2 = -2(r_1 + r_2)^2 - (r_1 - r_2)^2 = w^2$$

is a non-zero square in K. Let $(X, Y, Z) = (w, r_1 - r_2, r_1 + r_2) \in K^3$. We have (as in the proof of Theorem 3.3.2)

$$-2Z^2 - Y^2 = X^2 \Longrightarrow \left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right)^2 = -2 \Longrightarrow \left(\frac{X+Y}{2Z}\right)^2 + \left(\frac{X-Y}{2Z}\right)^2 = -1.$$

Thus, if $L_K(f) = 4$, then $s(K) \leq 2$. The converse is almost immediate.

If (3.3.4) has repeated r_i's, we may assume without loss of generality that $r_1 = r_2$, hence $r_3, r_4 = r_1(-1 \pm \sqrt{-2})$. The only fields in which this solution might occur contain $\sqrt{-2}$, so if we can find an alternate solution to (3.3.4) in $\mathbb{Q}(\sqrt{-2})$, we will be done. It may be checked that

$$\{r_1, r_2\} = \{5\sqrt{-2} \pm 6\}, \qquad \{r_3, r_4\} = \{-5\sqrt{-2} \pm 8\}$$

is such an alternate solution to (3.3.4) with distinct r_i. $\square$

The following result presents another sextic with three different ranks.

Theorem 3.3.4.
Suppose $f(x, y) = \binom{6}{1}x^5y - \binom{6}{3}x^3y^3 = 2x^3y(3x^2 - 10y^2)$. *Then* $L_K(f) = 4$ *if and only if* $s(K) \leq 2$. *In particular, if* m *is a positive square-free integer, and* $m \not\equiv 7 \mod 8$, *then* $L_{\mathbb{Q}(\sqrt{-m})}(f) = 4$. *Further,* $L_{\mathbb{Q}(\sqrt{-7})}(f) = 5$.

Proof. Again, taking (2.2.6) for $r = 3$ gives a non-singular matrix

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.3.5}$$

so $L_{\mathbb{C}}(f) > 3$. Moving up one,

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow c_0 = c_2, \quad c_1 = c_3 = 0, \tag{3.3.6}$$

so the possible Sylvester polynomials over K have the shape $h(x, y) = c_0x^4 + c_0x^2y^2 + c_4y^4$. If $c_0c_4 = 0$, then h is not square-free, so we may scale to $c_0 = 1$. Since h is an even polynomial, if $x - ry$ is a factor with $r \neq 0$ (since $c_4 \neq 0$), then so is $x + ry$, hence if h splits over K, then there exist $r, s \in K$ ($r^2 \neq s^2 \neq 0$) so that

$$x^4 + x^2y^2 + c_4y^4 = (x^2 - r^2y^2)(x^2 - s^2y^2).$$

Thus, $L_K(f) = 4$ if and only if K is a field in which the equation

$$r^2 + s^2 = -1 \tag{3.3.7}$$

has a solution, $r^2 \neq -\frac{1}{2}, 0, -1$. As we have seen in the proof of Theorem 3.3.2 , this is true precisely when $s(K) \leq 2$, so if $K = \mathbb{Q}(\sqrt{-m})$, precisely when $m \not\equiv 7 \mod 8$.

Since f is hyperbolic, $L_{\mathbb{R}}(f) = 6$. The previous paragraph shows that the apolar ideal for f is generated by $x^4 + x^2y^2$ and y^4. We now wish to find at least one field K for which $L_K(f) = 5$. Since K must be non-real with $s(K) > 2$, we take $K = \mathbb{Q}(\sqrt{-7})$ and look for a representation with relative rank 5. To this end, observe that $y(x^4 + x^2y^2) - 2xy^4 = x^4y + x^2y^3 - 2xy^4 = xy(x-y)\left(x + \frac{1+\sqrt{-7}}{2}y\right)\left(x + \frac{1-\sqrt{-7}}{2}y\right)$ splits over $\mathbb{Q}(\sqrt{-7})$. $\square$

Chapter 4

Rank with respect to factorization of forms

In this chapter we give a lower bound for the Waring rank of binary forms based on their factorization over $\mathbb{C}$ and improve the lower bound for the real Waring rank of binary forms which is given by Theorem 2.4.3. We also give the rank of quartics and quintics based on their factorization over $\mathbb{C}$. This chapter is adapted from Section 2 of the paper [40] *On the Waring rank of binary forms* by N. Tokcan, which has been accepted for publication by *Linear Algebra and Its Applications*.

4.1 A lower bound for the rank

The following theorem shows the relation between the $\mathbb{C}$-rank of a binary form and the factorization of the form over $\mathbb{C}$.

Theorem 4.1.1.
Let $f(x,y)$ be a nonzero binary form of degree d with the factorization

$$f(x,y) = \prod_{i=0}^{r} \ell_i(x,y)^{m_i} \tag{4.1.1}$$

where $r \geq 1$ and the ℓ_i's are distinct linear forms and $m_0 \geq m_1 \geq \ldots \geq m_r$. Then $L_{\mathbb{C}}(f) \geq m_0 + 1$.

Proof. We use the fact that rank is invariant under invertible linear change of variables. After a linear change of variables we may assume $\ell_0 = y$, then we have

$$\tilde{f}(x,y) = y^{m_0} g(x,y) \text{ such that } y \nmid g(x,y). \tag{4.1.2}$$

The first m_0 coefficients of $\tilde{f}$ are zero, i.e., $a_0 = \ldots = a_{m_0-1} = 0$ and $a_{m_0} \neq 0$. Note that

$deg(\tilde{f}) \geq m_0 + 1$, so by setting $r = m_0$, (2.2.6) becomes:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_{m_0} \\ 0 & 0 & \dots & a_{m_0} & a_{m_0+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ \vdots \\ c_{m_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence, $a_{m_0}c_{m_0} = a_{m_0}c_{m_0-1} + a_{m_0+1}c_{m_0} = 0$. It follows that $c_{m_0-1} = c_{m_0} = 0$ and every apolar form of degree m_0 is divisible by x^2 and $L_{\mathbb{C}}(f) \geq m_0 + 1$ by Theorem 2.2.1. □

Landsberg and Teitler [23, Corollary 4.5] and Boij, Carlini and Geramita [7] have separately computed that $L_{\mathbb{C}}(x^a y^b) = \max(a+1, b+1)$ if $a, b \geq 1$.

Corollary 4.1.2.
Let $f(x,y) = \ell_0(x,y)^{d-2}\ell_1(x,y)\ell_2(x,y)$ such that $d \geq 3$ and the ℓ_i's are distinct binary linear forms. Then $L_{\mathbb{C}}(f) = d-1$.

Proof. It follows from Theorem 4.1.1 that $d-1 \leq L_{\mathbb{C}}(f)$ and $L_{\mathbb{C}}(f) \leq d-1$ by Theorem 2.3.10. Thus, $L_{\mathbb{C}}(f) = d-1$. □

By using Corollary 4.1.2 and Corollary 2.4.4, we can generate forms with two different relative ranks. For example, let $r \neq 0 \in \mathbb{R}$ and $d \geq 3$, then $L_{\mathbb{C}}(x^{d-2}y(x+ry)) = d-1$ and $L_{\mathbb{R}}(x^{d-2}y(x+ry)) = d$.

Corollary 4.1.3.
Suppose $f(x,y) = \ell(x,y)^{d-2}q(x,y)$ is a real binary form of degree $d \geq 3$ where $\ell(x,y)$ is a real linear form and $q(x,y)$ is an irreducible real quadratic form. Then $L_{\mathbb{R}}(f) = d-1$.

Proof. The Waring rank of f is $d-1$ by Corollary 4.1.2; therefore, $d-1 \leq L_{\mathbb{R}}(f)$. On the other hand, it follows from Theorem 2.4.8 that $L_{\mathbb{R}}(f) \leq d-1$. □

Remark 4.1.4. Notice that if $f(x,y) = \ell(x,y)^{d-2}q(x,y)$ as in Corollary 4.1.3, then the real rank and complex rank of f coincide, i.e., $L_{\mathbb{R}}(f) = L_{\mathbb{C}}(f) = d-1$.

Theorem 4.1.1 combines with Theorem 2.4.3 into Corollary 4.1.5.

Corollary 4.1.5.
Let $f(x,y)$ be a nonzero real binary form of degree d and not a d-th power, with the factorization

$$f(x,y) = \prod_{i=0}^{r} \ell_i(x,y)^{m_i} \prod_{k=0}^{s} p_k(x,y)^{n_k}, \tag{4.1.3}$$

where the ℓ_i's are distinct real binary linear forms and p_k's are distinct irreducible real quadratics. Then $L_{\mathbb{R}}(f) \geq \max\big(\sum_{i=0}^{r} m_i, max(m_0, \ldots, m_r, n_0, \ldots, n_s) + 1\big)$.

Example 4.1.6. Let $p(x,y) = x^6 + 15x^2y^4 + 3y^6$, then p is square-free and irreducible over $\mathbb{R}$. It can be checked that $(p^\perp)_3 = \{0\}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff (c_0, c_1, c_2, c_3, c_4) = (c_0, c_1, 3c_0, 0, -c_0)$$

and $(p^\perp)_4$ consists of all the scalar multiples of $h(x,y) = x^4 + rx^3y + 3x^2y^2 - y^4$. Note that a square-free hyperbolic form must have a positive discriminant. The discriminant of $h(x,y)$ is $-27r^4 + 540r^2 - 2704 \ = -27(r^2 - 10)^2 - 4$ and it is negative for all $r \in \mathbb{R}$. Then h does not split over $\mathbb{R}$ and $L_{\mathbb{R}}(p) \geq 5$. Since p is non-hyperbolic, $L_{\mathbb{R}}(p) = 5$ by Theorem 2.4.8.

4.2 Rank of quartic and quintic binary forms

Theorem 2.3.11 gives the rank of binary cubics based on their factorizations:

$p(x,y)$	$L_{\mathbb{C}}(p(x,y))$
$\ell_0(x,y)^3$	1
$\ell_0(x,y)^2\ell_1(x,y)$	3
$\ell_0(x,y)\ell_1(x,y)\ell_2(x,y)$	2

Theorem 2.3.10 and Theorem 4.1.1 suggest a unique Waring rank for binary quartics with repeating roots based on their factorization. However, these theorems do not suggest a unique Waring rank for square-free binary quartics.

Theorem 4.2.1.
Suppose $p_\lambda(x,y) = x^4 + 6\lambda x^2y^2 + y^4$. *Then* $L_{\mathbb{C}}(p_\lambda) = 2$ *if* $\lambda = 0, \pm 1$ *and* 3 *otherwise.*

Proof. Notice that $p_0(x,y) = x^4 + y^4$, $p_1(x,y) = \frac{1}{2}(x+y)^4 + \frac{1}{2}(x-y)^4$ and $p_{-1}(x,y) = \frac{1}{2}(x+iy)^4 + \frac{1}{2}(x-iy)^4$; therefore, $L_{\mathbb{C}}(p_\lambda) = 2$ for $\lambda = 0, \pm 1$. If $\lambda \neq 0, \pm 1$, then the Hankel matrix

$$\begin{pmatrix} 1 & 0 & \lambda \\ 0 & \lambda & 0 \\ \lambda & 0 & 1 \end{pmatrix} \tag{4.2.1}$$

is non-singular and $L_{\mathbb{C}}(p_\lambda) \geq 3$. It follows from Theorem 2.3.10 that $L_{\mathbb{C}}(p_\lambda) = 3$. $\square$

Theorem 4.2.2.
Let $p(x,y) = 4ax^3y + 6bx^2y^2 + 4cxy^3 = xy(4ax^2 + 6bxy + 4cy^2)$, $ac \neq 0 \in \mathbb{C}$. *Then* $L_{\mathbb{C}}(p) = 2$ *if* $b^2 = 2ac$ *and 3 otherwise.*

Proof. We already know that $L_{\mathbb{C}}(p) \geq 2$ since p is not a fourth power. Then the Hankel matrix $H_3(p)$

$$\begin{pmatrix} 0 & a & b \\ a & b & c \\ b & c & 0 \end{pmatrix} \tag{4.2.2}$$

is non-singular if $b^2 \neq 2ac$. If $b^2 = 2ac$, then the Sylvester form of degree 2 is $h(x,y) = cx^2 - \sqrt{2ac}xy + ay^2$ and $L_{\mathbb{C}}(p) = 2$.

Note that p does not have a root of multiplicity 3, therefore $L_{\mathbb{C}}(p) \leq 3$ by Theorem 2.3.10. Hence, if $b^2 \neq 2ac$, then $L_{\mathbb{C}}(p) = 3$. $\square$

$\mathbb{C}$-rank of binary quartics: Assume that ℓ_i's are distinct binary linear forms. The first three case of the following table directly follows from Theorem 2.3.10 and Theorem 4.1.1. We provide supporting examples for the last case.

$p(x,y)$	$L_{\mathbb{C}}(p(x,y))$
$\ell_0(x,y)^4$	1
$\ell_0(x,y)^3\ell_1(x,y)$	4
$\ell_0(x,y)^2\ell_1(x,y)^2$	3
$\ell_0(x,y)^2\ell_1(x,y)\ell_2(x,y)$	3
$\ell_0(x,y)\ell_1(x,y)\ell_2(x,y)\ell_3(x,y)$	2,3

Theorem 4.2.1 and Theorem 4.2.2 show that a square-free binary quartic form can have Waring rank 2 or 3. In particular, $L_{\mathbb{C}}(4x^3y + 6x^2y^2 + 4xy^3) = L_{\mathbb{C}}(x^4 + 4x^2y^2 + y^4) = 3$ and $L_{\mathbb{C}}(8x^3y + 36x^2y^2 + 36xy^3) = L_{\mathbb{C}}(x^4 + y^4) = 2$.

It follows from the above table and Theorem 2.4.8 that $L_{\mathbb{C}}(p(x,y)^2) = L_{\mathbb{R}}(p(x,y)^2) = 3$ where $p(x,y)$ is an irreducible real quadratic. This result is also a consequence of the following theorem by Reznick.

Theorem 4.2.3. [32, Corollary 5.6]
For $k \geq 2$, $L_{\mathbb{C}}(x^2+y^2)^k = k+1$, *and* $L_K(x^2+y^2)^k = k+1$ *if and only if* $\tan\frac{\pi}{k+1} \in K$. *The* $\mathbb{C}$*-minimal representations of* $(x^2+y^2)^k$ *are given by*

$$\binom{2k}{k}(x^2+y^2)^k = \frac{1}{k+1}\sum_{j=0}^{k}\left(\cos\left(\frac{j\pi}{k+1}+\theta\right)x + \sin\left(\frac{j\pi}{k+1}+\theta\right)y\right)^{2k}, \ \theta \in \mathbb{C}. \quad (4.2.3)$$

First note that after an invertible linear change of variables, we may assume that if p is an irreducible quadratic, then $p(x,y) = (x^2+y^2)$. Then, it is immediate from Theorem 4.2.3 that if $k \geq 2$, then $L_{\mathbb{C}}(p(x,y)^k) = L_{\mathbb{R}}(p(x,y)^k) = k+1$.

We now have enough tools at our disposal to determine the Waring rank of a binary quintic based on its factorization over $\mathbb{C}$.

Example 4.2.4. Suppose $p(x,y) = \ell_0(x,y)^2\ell_1(x,y)^2\ell_2(x,y)$ such that the ℓ_i's are distinct binary linear forms. Then $3 \leq L_{\mathbb{C}}(p) \leq 4$ by Theorem 2.3.10 and Theorem 4.1.1. After an invertible linear change of variables, we may assume that $p(x,y) = 10x^2y^2(x+\alpha y),\ \alpha \neq 0 \in \mathbb{C}$. If we let $d = 5$ and $r = 3$, then the linear system (2.2.6) becomes:

$$\begin{pmatrix} 0 & 0 & 1 & \alpha \\ 0 & 1 & \alpha & 0 \\ 1 & \alpha & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.2.4)$$

Therefore, $(c_0, c_1, c_2, c_3) = s(\alpha^3, -\alpha^2, \alpha, -1),\ s \in \mathbb{C}$, so that up to a multiple,

$$\begin{aligned} h(x,y) &= \alpha^3x^3 - \alpha^2x^2y + \alpha xy^2 - y^3 \\ &= (\alpha x - y)(\alpha x + iy)(\alpha x - iy). \end{aligned} \quad (4.2.5)$$

The corresponding unique $\mathbb{C}$-minimal representation is

$$10x^2y^2(x+\alpha y) = \frac{-1}{4\alpha^2}\left(-2(x+\alpha y)^5 + (1-i)(x+\alpha iy)^5 + (1+i)(x-\alpha iy)^5\right).$$

Therefore, $L_K(p) = 3$ if only if $\mathbb{Q}(\alpha, i) \subseteq K$; in particular, $L_{\mathbb{C}}(p) = 3$.

The following is a generalization of Example 4.2.4.

Theorem 4.2.5.
Suppose $p(x,y) = \ell_0(x,y)^k\ell_1(x,y)^k\ell_2(x,y)$ such that the ℓ_i's are distinct linear forms. Then $L_{\mathbb{C}}(p) = k+1$.

Proof. As in the example above, after invertible linear change of variables, we may assume that $p(x,y) = \binom{2k+1}{k} x^k y^k (x + \alpha y)$. Theorem 4.1.1 suggests that $L_{\mathbb{C}}(p) \geq k+1$. It follows from Theorem 2.2.1 that $h(x,y) = \sum_{i=0}^{k+1} (-1)^i (\alpha x)^{k+1-i} y^i$ is a Sylvester form of degree $k+1$ for p. Therefore, $L_{\mathbb{C}}(p) = k+1$. $\square$

$\mathbb{C}$-rank of binary quintics: Assume that the ℓ_i's are distinct binary linear forms. The following table gives the rank of binary quintics based on their factorization over $\mathbb{C}$.

$p(x,y)$	$L_{\mathbb{C}}(p(x,y))$
$\ell_0(x,y)^5$	1
$\ell_0(x,y)^4 \ell_1(x,y)$	5
$\ell_0(x,y)^3 \ell_1(x,y)^2$	4
$\ell_0(x,y)^3 \ell_1(x,y) \ell_2(x,y)$	4
$\ell_0(x,y)^2 \ell_1(x,y)^2 \ell_2(x,y)$	3
$\ell_0(x,y)^2 \ell_1(x,y) \ell_2(x,y) \ell_3(x,y)$	3,4
$\ell_0(x,y) \ell_1(x,y) \ell_2(x,y) \ell_3(x,y) \ell_4(x,y)$	2,3,4

For the first four case, we get a unique rank directly from Theorem 2.3.10 and Theorem 4.1.1. However, these theorems do not suggest a unique rank for the last three factorizations. Example 4.2.4 shows that $L_{\mathbb{C}}(\ell_0(x,y)^2 \ell_1(x,y)^2 \ell_2(x,y)) = 3$. We leverage Sylvester's algorithm and provide examples for the last two cases:

- $L_{\mathbb{C}}(x^5 + 10x^2y^3) = 3$ and $L_{\mathbb{C}}(x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3) = 4$.
- $L_{\mathbb{C}}(x^5 + y^5) = 2$, $L_{\mathbb{C}}(3x^5 + 20x^3y^2 + 10xy^4) = 3$ and $L_{\mathbb{C}}(x^5 - 5xy^4) = 4$.

The following theorem shows that the real Waring rank is not invariant under small perturbation of the coefficients of a binary form.

Theorem 4.2.6.
Suppose $r \neq 0 \in \mathbb{Q}$ and $p(x,y) = 10x^2(x+y)(rx^2+y^2)$. Let r_0 be the real root of $2r^3 - 77r^2 - 16r - 1 = 0$, $r_0 \approx 38.707$. Then $L_{\mathbb{C}}(p) = 3$ and the real Waring rank depends on r:

(i) If $r > r_0$, then $L_{\mathbb{R}}(p) = 3$.

(ii) If $r \in (0, r_0)$, then $L_{\mathbb{R}}(p) = 4$.

(iii) If $r < 0$, then $L_{\mathbb{R}}(p) = 5$.

Proof. It is immediate from Theorem 4.1.1 that $L_{\mathbb{C}}(p) \geq 3$. We can show by Theorem 2.2.1 that p has a unique Sylvester form of degree 3:

$$h_r(x, y) = x^3 - x^2y + (1 - 2r)xy^2 - (1 + 6r)y^3. \tag{4.2.6}$$

We check the discriminant of h to understand the behavior of the roots. The discriminant $\Delta(h_r) = 16(2r^3 - 77r^2 - 16r - 1)$. First notice that $\Delta(h)$ does not have any rational root, and so $L_{\mathbb{C}}(p) = 3$. If r_0 is the unique real root of the discriminant, then $\Delta(h_r)$ is positive when $r > r_0$. Therefore, h splits over $\mathbb{R}$ if $r > r_0$ and $L_{\mathbb{R}}(p) = 3$. Note that p is hyperbolic if and only if $r < 0$, then (ii) and (iii) are immediate from Theorem 2.4.8. □

Lemma 4.2.7. *Suppose $p(x, y) = \ell_0(x, y)q_1(x, y)^2$ such that ℓ_0 is a real linear form and $q(x, y)$ is a real irreducible quadratic. Then $L_{\mathbb{R}}(p) = 3$.*

Proof. The proof follows from Corollary 4.1.5 and Theorem 2.4.8. □

Comon and Ottaviani studied the real rank of square-free binary quintics; see [14]. A hyperbolic binary quintic, which is not a 5-th power, has real rank 5 by Corollary 2.4.4. We now give the real rank of non-hyperbolic binary quintic forms with repeating roots depending on their factorizations.

Assume that the ℓ_i's are distinct real linear forms and $q(x, y)$ is an irreducible quadratic. The following table follows from Lemma 4.2.7, Corollary 4.1.3 and Theorem 4.2.6.

$p(x, y)$	$L_{\mathbb{R}}(p(x, y))$
$\ell_0(x, y)q(x, y)^2$	3
$\ell_0(x, y)^3q(x, y)$	4
$\ell_0(x, y)^2\ell_1(x, y)q(x, y)$	3,4

Notice that if $p(x, y) = \ell_0(x, y)^2\ell_1(x, y)q(x, y)$, then after an invertible linear change of variables, we may assume that $p(x, y) = 10x^2(x + y)(rx^2 + y^2),\ r > 0$. Thus, $L_{\mathbb{R}}(p) \in \{3, 4\}$ by Theorem 4.2.6.

4.3 The real rank of positive definite forms

Let $p(x, y)$ be a square-free positive definite binary form. Theorem 4.1.1 and Corollary 4.1.5 do not suggest a lower bound for the complex and real rank of p.

Let $f_\lambda(x,y) = x^{2k} + \binom{2k}{k}\lambda x^k y^k + y^{2k},\ \lambda \neq 0 \in \mathbb{R}$. If $|\binom{2k}{k}\lambda| < 2$, then f_λ is a square-free definite form. In the following theorem, arguments employing Descartes' Rule of Signs provide a lower bound for $L_{\mathbb{R}}(f_\lambda)$.

The next tool is an application of Descartes' Rule of Signs; see [27, Problem 49, p.43].

Theorem 4.3.1. *Let $a_0 \neq 0, a_n \neq 0$, and assume that $2m$ consecutive coefficients of the polynomial $a_0 + a_1 t + \ldots + a_n t^n$ vanish, where m is an integer, $m \geq 1$. Then the polynomial has at least $2m$ non-real zeros.*

Theorem 4.3.2.
Let $f_\lambda(x,y) = x^{2k} + \binom{2k}{k}\lambda x^k y^k + y^{2k}$, where $\lambda \neq 0 \in \mathbb{R}$ and $k \geq 3$. Then $L_{\mathbb{R}}(f_\lambda) \in \{2k-2, 2k-1\}$.

Proof. First notice that if $k \geq 3$, then $f_\lambda(x,y)$ is not hyperbolic by Theorem 4.3.1. Thus, it follows from Theorem 2.4.8 that $L_{\mathbb{R}}(f_\lambda) \leq 2k-1$. We then let $r = k+j$, $0 \leq j \leq k-1$ and look for a Sylvester form of degree r. If $k = 4,\ j = 1$, then (2.2.6) becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, $(c_0, c_1, c_2, c_3, c_4, c_5) = (-\lambda c_4, c_1, 0, 0, c_4, -\lambda c_1)$. Instead if $k = 5,\ j = 2$, then

$$(c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (-\lambda c_5, c_1, c_2, 0, 0, c_5, c_6, -\lambda c_2)$$

is the solution of the linear system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In general for $r = k+j$, we can see that if $(c_0, c_1, \ldots, c_{k+j})$ is a solution for (2.2.6), then

$$c_i = 0, \quad j+1 \leq i \leq k-1,$$
$$c_0 = -\lambda c_k,\ c_{k+j} = -\lambda c_j.$$

Therefore, $h_{k+j}(x,y)$, the corresponding Sylvester form of degree $k+j$, has at least $k-j-1$ consecutive missing coefficients. If h_{k+j} splits over $\mathbb{R}$, then $k-j \le 2$ by Theorem 4.3.1; thus, $2k-2 \le L_{\mathbb{R}}(f_\lambda)$. $\square$

The following theorem gives a parametrization for a $\mathbb{C}$-minimal representation of $f_\lambda(x,y)$ as λ varies over all non-zero complex numbers.

Theorem 4.3.3.
Suppose $f_\lambda(x,y) = x^{2k} + \binom{2k}{k}\lambda x^k y^k + y^{2k},\ \lambda \neq 0$. *Then* $L_{\mathbb{C}}(f_\lambda) = k$ *if* $\lambda = \pm 1$ *and* $k+1$ *otherwise. The following is a minimal representation of* f_λ,

$$x^{2k} + \binom{2k}{k}\lambda x^k y^k + y^{2k} = (1-\lambda^2)y^{2k} + \frac{1}{k}\sum_{i=0}^{k-1}(x + \lambda^{\frac{1}{k}}\zeta_k^i y)^{2k}. \tag{4.3.1}$$

Proof. We first evaluate the right-hand side of (4.3.1):

$$(1-\lambda^2)y^{2k} + \frac{1}{k}\sum_{i=0}^{k-1}(x + \lambda^{\frac{1}{k}}\zeta_k^i y)^{2k} = (1-\lambda^2)y^{2k} + \frac{1}{k}\sum_{j=0}^{2k}\binom{2k}{j}x^{2k-j}y^j\lambda^{\frac{j}{k}}\Big(\sum_{i=0}^{k-1}\zeta_k^{ij}\Big). \tag{4.3.2}$$

The sum $\sum_{i=0}^{k-1}\zeta_k^{ij} = 0$ unless $k \mid j$, in which case it equals to k. The only multiples of k in the set $\{j : 0 \le j \le 2k\}$ are $0, k, 2k$. The right-hand side of (4.3.2) reduces to the left-hand side of (4.3.1). If we let $r = k-1$, then the linear system in (2.2.6) has only the trivial solution, so $k \le L_{\mathbb{C}}(f_\lambda) \le k+1$.

If $\lambda \in \{1,-1\}$, then the first summand in (4.3.2) is zero and f_λ has a unique minimal representation which is given by (4.3.1) and $L_{\mathbb{C}}(f_\lambda) = k$.

Let $\lambda \neq \pm 1$ and $r = k$, then the matrix in (2.2.6) is non-singular, so $L_{\mathbb{C}}(f_\lambda) = k+1$. (We search that the minimal representation given by (4.3.1) is not necessarily unique.) $\square$

Chapter 5

$\mathbb{C}$-minimal representations and Sylvester fields

Suppose $\mathbb{Q} \subseteq E_p \subseteq \mathbb{C}$. Let p be a binary form of degree d with the coefficient field E_p. If $L_{\mathbb{C}}(p) < \frac{d+2}{2}$, then p has a unique $\mathbb{C}$-minimal representation by Corollary 2.3.6 (ii). Assume that F/E_p is a finite degree field extension such that $L_F(p) = L_{\mathbb{C}}(p)$. In this chapter we study the relation between $[F : E_p]$ and $L_{\mathbb{C}}(p)$ and investigate the structure of unique $\mathbb{C}$-minimal honest representations of p. Sections 5.1 and 5.2 are adapted from Section 3 of the paper [40] *On the Waring rank of binary forms* by N. Tokcan, which has been accepted for publication by *Linear Algebra and Its Applications*.

5.1 Structure of $\mathbb{C}$-minimal representations

If $p \in H_d(K^2)$, then any apolar form of minimal degree $k < \frac{d+2}{2}$ is unique (up to a scalar multiple) by Corollary 2.1.17. We now provide a proof of the corollary.

Corollary 5.1.1. (Corollary 2.1.17)
Let $p(x, y)$ be a nonzero binary form in $H_d(K^2)$, not a d-th power, and suppose that $k < \frac{d+2}{2}$ is the smallest number such that $(p^{\perp})_k \neq \{0\}$. Then there exists a projectively unique binary form $h(x, y) \in H_k(K^2)$ such that $(p^{\perp})_k = \langle h \rangle$. Thus, $p(x, y)$ has at most one minimal representation of length k.

Proof. We first prove uniqueness: If $g(x, y)$ is a binary form which is apolar to p and non-proportional to h, then $deg(g) > k$ by Theorem 2.1.15. It follows that $(p^{\perp})_k$ has a unique element (up to a scalar multiple).

We now prove that $h \in K[x, y]$: If we take $r = k$, then the linear system in (2.2.6) has at least one nonzero solution over $\mathbb{C}$, since $h(x, y)$ corresponds to a solution. Thus, it must have a solution over K as well and by uniqueness $h(x, y) \in K[x, y]$. □

The following is a restatement of Theorem 2.2.10.

Theorem 5.1.2.
Suppose $h(x,y)$ is a Sylvester form of degree r for $p(x,y)$. If S is a splitting field of h, then $L_S(p) \le r$. If furthermore there is no Sylvester form of degree $r-1$, then $L_S(p) = r$.

Proof. The length of the shortest representation of p over S is $L_S(p)$. If h splits over S, then it follows from Theorem 2.2.1 that there exist $\lambda_k, \alpha_k, \beta_k \in S$ such that

$$p(x,y) = \sum_{k=1}^{r} \lambda_k (\alpha_k x + \beta_k y)^d. \tag{5.1.1}$$

Therefore, $L_S(p) \le r$. If there is no Sylvester form of degree $r-1$, then (5.1.1) is a $\mathbb{C}$-minimal representation and $L_S(p) = L_{\mathbb{C}}(p) = r$. □

We shall need the following result on the splitting fields [18, Ex.3, p.30]. Let $[L:K]$ denote the degree of the field extension L/K.

Theorem 5.1.3.
Let p be a polynomial of degree d with coefficients in K. Let L be the splitting field of p over K. Then $[L:K]$ is a divisor of $d!$.

Theorem 5.1.4.
Suppose $r < \frac{d+2}{2}$ and $p(x,y) \in H_d(K^2)$ with $L_{\mathbb{C}}(p) = r$. Then there exists a field extension S/K such that $L_S(p) = r$ and $[S:K]$ divides $r!$.

Proof. Let $h(x,y)$ be a Sylvester form of degree r for p. Then $h(x,y) \in K[x,y]$ by Corollary 2.1.17. There is no Sylvester form of degree $r-1$ since $L_{\mathbb{C}}(p) = r$. If S is a splitting field of h, then $L_S(p) = r$ by Theorem 5.1.2. Moreover, it follows from Theorem 5.1.3 that $[S:K] \mid r!$. □

Suppose $p \in H_d(K^2)$, $d \ge 3$ with $L_{\mathbb{C}}(p) = 2$ and $L_K(p) \ge 3$. Then there exist a field extension S/K such that $[S:K] = 2$ with $L_S(p) = 2$ by Theorem 5.1.4. The classification of binary forms with Waring rank 2 is presented in [32].

Theorem 5.1.5. [32, Theorem 4.6]
Let $p(x,y)$ be a nonzero binary form of degree $d \ge 3$, and not a d-th power, with $\lambda_i, \alpha_i, \beta_i \in \mathbb{C}$ so that

$$p(x,y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2(\alpha_2 x + \beta_2 y)^d \in K[x,y]. \tag{5.1.2}$$

If (5.1.2) is honest and $L_K(p) > 2$, then there exists $u \in K$ with $\sqrt{u} \notin K$ so that $L_{K(\sqrt{u})}(p) = 2$. The summands in (5.1.2) are conjugates of each other in $K(\sqrt{u})$.

It was shown in Example 3.1.2 that if $d \geq 3$ and γ is a square-free rational, then $p_d(x, y) = (x + \sqrt{\gamma}y)^d + (x - \sqrt{\gamma}y)^d$ has a unique $\mathbb{C}$-minimal representation with $L_{\mathbb{Q}(\sqrt{\gamma})}(p_d) = 2$. Notice that in this example $E_p = \mathbb{Q}$ and $[\mathbb{Q}(\sqrt{\gamma}) : \mathbb{Q}] = 2$.

In the following example, the coefficient field E_p is a quadratic field.

Example 5.1.6. Let $p(x, y) = 2(x^3 - 3\sqrt{5}x^2y + 36xy^2 - 26\sqrt{5}y^3) \in \mathbb{Q}(\sqrt{5})[x, y]$. Then it follows from Theorem 2.2.1 that

$$p(x, y) = (x - (\sqrt{5} - \sqrt{7})y)^3 + (x - (\sqrt{5} + \sqrt{7})y)^3. \tag{5.1.3}$$

By Corollary 2.3.6(ii), p has a unique representation of rank 2. Thus, $L_K(p) = 2$ if and only if $\mathbb{Q}(\sqrt{5}, \sqrt{7}) \subseteq K$. In particular, $L_{\mathbb{Q}(\sqrt{5},\sqrt{7})}(p) = 2$ and $[\mathbb{Q}(\sqrt{5}, \sqrt{7}) : \mathbb{Q}(\sqrt{5})] = 2$. Moreover, $L_{\mathbb{Q}(\sqrt{5})}(p) = 3$ by Theorem 2.3.8.

Lemma 5.1.7.
Suppose $d \geq 5$ and there exist nonzero $\lambda_i, \alpha_1, \beta_1 \in \mathbb{C}$ so that

$$p(x, y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2 x^d + \lambda_3 y^d \in K[x, y]. \tag{5.1.4}$$

Then $L_K(p) = 3$, and (5.1.4) is the projectively unique representation of p of length 3.

Proof. The Sylvester form corresponding to (5.1.4) is $h(x, y) = (\beta_1 x - \alpha_1 y)yx$ by Theorem 2.2.1. It follows from Corollary 2.1.17 that $h \in K[x, y]$; thus, h splits over K and $L_K(p) = 3$. $\square$

Corollary 5.1.10 concerns a special case of Theorem 5.1.4 where $r = 3$ and K is a real closed field. We first give a simple property of real closed fields.

Definition 5.1.8. A *real closed field* F is a real field that has no non-trivial real algebraic extension $F_1 \supset F$.

Theorem 5.1.9. [6, Theorem 1.2.2]
Let F be a real closed field. Then every odd-degree polynomial of $F[x]$ has a root in F.

Corollary 5.1.10.
Suppose $d \geq 5$, $K \subseteq \mathbb{C}$ is real closed field and there exist $\lambda_i, \alpha_i, \beta_i \in \mathbb{C}$ such that

$$p(x,y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2(\alpha_2 x + \beta_2 x)^d + \lambda_3(\alpha_3 x + \beta_3 y)^d \in K[x,y] \tag{5.1.5}$$

is a honest representation and $L_K(p) > 3$. Suppose

$$u = (\beta_1\alpha_2 - \beta_2\alpha_1)^2(\beta_1\alpha_3 - \beta_3\alpha_1)^2(\beta_2\alpha_3 - \ \beta_3\alpha_2)^2.$$

Then $L_{K(\sqrt{u})}(p) = 3$. One of the summands in (5.1.5) is in $K[x,y]$ whereas the other two summands are conjugates of each other in $K(\sqrt{u})[x,y]$.

Proof. It follows from Theorem 2.2.1 and Corollary 2.1.17 that the projectively unique Sylvester form of degree 3 for p is given by

$$h(x,y) = (\beta_1 x - \alpha_1 y)(\beta_2 x - \alpha_2 y)(\beta_3 x - \alpha_3 y) \in K[x,y].$$

Notice that u equals the discriminant of $h(x,1)$. By the hypothesis h does not split over K; therefore, $\sqrt{u} \notin K$. Since h is an odd degree form over a real closed field, it must have a factor in $K[x,y]$ by Theorem 5.1.9. Thus, only one factor of $h(x,y)$ is in $K[x,y]$ and the other two are conjugates of each other in $K(\sqrt{u})$. Note that every field automorphism which fixes K permutes the summands in (5.1.5). If we consider the conjugation with respect to $\sqrt{u}$, then (5.1.5) has two summands which are conjugates of each other in $K(\sqrt{u})[x,y]$ and a summand in $K[x,y]$. □

The following theorem in [32] gives the structure of the summands of a unique $\mathbb{C}$-minimal representation.

Theorem 5.1.11. [32, Corollary 4.8]
Suppose K is an extension field of E_p, $r < \frac{d+2}{2}$, and

$$p(x,y) = \sum_{i=1}^{r} \lambda_i(\alpha_i x + \beta_i y)^d \tag{5.1.6}$$

with $\lambda_i, \alpha_i, \beta_i \in K$. Then every automorphism of K which fixes E_p permutes the summands in (5.1.6).

Proof. If σ is an automorphism of K which fixes E_p, then $\sigma(\lambda(\alpha x + \beta y)^d) = \sigma(\lambda)(\sigma(\alpha)x + \sigma(\beta)y)^d$. Since $\sigma(p) = p$, the action of σ gives a representation of p. It follows from Corollary

2.3.6(ii) that p has a unique minimal representation of length r, and so σ permutes the summands of p. $\square$

In the above results, we have $p \in H_d(K^2)$ with $L_{\mathbb{C}}(p) = r < \frac{d+2}{2}$, i.e., with a unique $\mathbb{C}$-minimal representation. In Theorem 5.1.13 we remove the condition $r < \frac{d+2}{2}$ and set $\lambda_i = \alpha_i = 1$. We first need to give simple definitions and properties concerning symmetric functions.

Definition 5.1.12. The *k-th elementary symmetric polynomial* in n variables $x_1, \ldots, x_n$ is defined by

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} \ldots x_{i_k}, \quad 0 \leq k \leq n.$$

The *k-th power sum symmetric polynomial* in n variables is defined by

$$p_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} x_i^k.$$

The following property is well known:

$$\mathbb{Q}[e_1, \ldots, e_n] = \mathbb{Q}[p_1, \ldots, p_n]. \tag{5.1.7}$$

The coefficients of a polynomial can be given in terms of elementary symmetric polynomials:

$$\prod_{i=1}^{d} (x - \alpha_i) = \sum_{i=0}^{d} (-1)^i e_i(\alpha_1, \ldots, \alpha_d) x^{d-i}. \tag{5.1.8}$$

More details about symmetric functions and the proofs of above mentioned properties can be found in [24, Chapter 1].

Theorem 5.1.13.
If the polynomial $p(x, y) = \sum_{j=1}^{k} (x + \gamma_j y)^d \in K[x, y]$, then $\gamma_1, \ldots \gamma_k$ are roots of a polynomial in $K[x]$.

Proof. Let

$$p(x, y) = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i, \quad a_i \in K.$$

Then the coefficients of $p(x, y)$ can be written as, $0 \leq i \leq d$,

$$a_i = \sum_{j=1}^{k} \gamma_j^i := p_i(\gamma_1, \ldots, \gamma_k) \in K. \tag{5.1.9}$$

It follows from (5.1.7) that elementary symmetric polynomials can be written as $\mathbb{Q}$-linear combinations of power sum symmetric polynomials. Hence, $e_i(\gamma_1, \ldots, \gamma_k) \in K$ for $0 \leq i \leq d$. Then by (5.1.8),

$$\prod_{i=1}^{k}(x - \gamma_i) = \sum_{i=0}^{k}(-1)^i e_i(\gamma_1, \ldots, \gamma_k)x^{k-i} \in K[x]. \tag{5.1.10}$$

$\square$

5.2 Binary Forms of Waring rank 3

Let f be a binary form of degree $d \geq 5$ in $K[x, y]$ and $L_{\mathbb{C}}(f) = 3$. Then it follows from Theorem 5.1.4 that there exist a field extension S/K with $L_S(f) = 3$. Either $S = K$ or else S/K has degree 2, 3, or 6. In this section, we give examples for each case of S/K and additional one showing that if $d \leq 4$, there can be infinitely many representations of length 3.

Example 5.2.1. Let $d \geq 5, \lambda \neq 0, \gamma \in \mathbb{Q}$ with $\sqrt{\gamma} \notin \mathbb{Q}$ such that

$$p(x, y) = \sum_{0 \leq 2i \leq d} \binom{d}{2i} \gamma^i x^{d-2i} y^{2i} + \lambda x^d \in \mathbb{Q}[x, y]. \tag{5.2.1}$$

Then p has a representation of length 3 by Example 3.1.2:

$$p(x, y) = \frac{1}{2}(x + \sqrt{\gamma}y)^d + \frac{1}{2}(x - \sqrt{\gamma}y)^d + \lambda x^d. \tag{5.2.2}$$

Any other honest representation of p has length at least $d - 1 \geq 4$ by Theorem 2.3.5; thus, $L_{\mathbb{C}}(p) = L_{\mathbb{Q}(\sqrt{\gamma})}(p) = 3$. Notice that $[\mathbb{Q}(\sqrt{\gamma}) : \mathbb{Q}] = 2$.

Example 5.2.2. Let $d \geq 5$ and

$$p(x, y) = 3 \sum_{0 \leq 3k \leq d} \binom{d}{3k} 2^k x^{d-3k} y^{3k} \in \mathbb{Q}[x, y]. \tag{5.2.3}$$

It follows from Theorem 2.2.1 that $L_{\mathbb{C}}(p) = 3$ and the unique $\mathbb{C}$-minimal representation is

$$p(x,y) = (x + \sqrt[3]{2}y)^d + (x + \sqrt[3]{2}\omega y)^d + (x + \sqrt[3]{2}\omega^2 y)^d, \quad \omega = e^{\frac{2\pi i}{3}}. \tag{5.2.4}$$

Thus, $L_K(p) = 3$ if and only if $\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3}) \subseteq K$. In particular, $L_{\mathbb{Q}(\sqrt[3]{2},\sqrt{-3})}(p) = 3$ and $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3}) : \mathbb{Q}] = 6$.

Example 5.2.3. This example displays a form falling into the case where S/K has degree 2.

Let $f(x,y) = (1+2\sqrt{2})x^5 - 25x^4y + (60\sqrt{2}+10)x^3y^2 - 170x^2y^3 + (90\sqrt{2}+5)xy^4 - 53y^5$.

First, with $r = 2$, the (2.2.6) becomes

$$\begin{pmatrix} 1+2\sqrt{2} & -5 & 6\sqrt{2}+1 \\ -5 & 6\sqrt{2}+1 & -17 \\ 6\sqrt{2}+1 & -17 & 18\sqrt{2}+1 \\ -17 & 18\sqrt{2}+1 & -53 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow (c_0, c_1, c_2) = (0,0,0).$$

Therefore, $L_{\mathbb{C}}(f) \geq 3$. On taking $r = 3$, the linear system in (2.2.6) becomes:

$$\begin{pmatrix} 1+2\sqrt{2} & -5 & 6\sqrt{2}+1 & -17 \\ -5 & 6\sqrt{2}+1 & -17 & 18\sqrt{2}+1 \\ 6\sqrt{2}+1 & -17 & 18\sqrt{2}+1 & -53 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.2.5}$$

The only solution to (5.2.5) is, up to a multiple, $(c_0, c_1, c_2, c_3) = (3, -3, -1, 1)$, so that

$$h(x,y) = 3x^3 - 3x^2y - xy^2 + y^3 = (y - \sqrt{3}x)(y + \sqrt{3}x)(y - x).$$

We arrive the following conclusion: $L_S(f) = 3$ if and only if $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq S$ with the corresponding representation:

$$f(x,y) = (\sqrt{2}+\sqrt{3})(x - \sqrt{3}y)^5 + (\sqrt{2}-\sqrt{3})(x + \sqrt{3}y)^5 + (x+y)^5.$$

Notice that the above representation has two summands which are conjugates of each other under the conjugation with respect to $\sqrt{3}$ in $\mathbb{Q}(\sqrt{2})$ and a summand in $\mathbb{Q}(\sqrt{2})[x,y]$. In particular, if $S = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, then $[S : \mathbb{Q}(\sqrt{2})] = 2$ and $L_S(f) = 3$.

Example 5.2.4. Let $f(x,y) = -15x^5 + 90x^4y - 30x^3y^2 + 60x^2y^3 + 3y^5$. If we set $r = 2$, then the solution to the linear system in (2.2.6) is trivial, so $L_{\mathbb{C}}(f) \geq 3$. If we set $r = 3$ in (2.2.6), then up to a scalar multiple $h(x, y) = x^3 - 3xy^2 + y^3$. We factor $h(x, y)$ by using the trigonometric identity $4\cos^3\theta - 3\cos\theta = \cos 3\theta$:

$$h(x, y) = (x - 2\cos\tfrac{2\pi}{9}y)(x - 2\cos\tfrac{4\pi}{9}y)(x - 2\cos\tfrac{8\pi}{9}y).$$

Therefore, $L_S(f) = 3$ if and only if $\mathbb{Q}(\cos\frac{2\pi}{9}) \subseteq S$ with the $\mathbb{C}$-minimal representation:

$$f(x, y) = (y + 2x\cos\tfrac{2\pi}{9})^5 + (y + 2x\cos\tfrac{4\pi}{9})^5 + (y + 2x\cos\tfrac{8\pi}{9})^5.$$

If we let $S = \mathbb{Q}(\cos\frac{2\pi}{9})$, then we have $[S : \mathbb{Q}] = 3$ and $L_S(f) = 3$.

Example 5.2.5. Let $f(x, y) = 3x^7 + 735x^4y^3 + 1029xy^6$. Then, $(f^\perp)_2$ is empty by Theorem 2.2.1; thus, the Waring rank of f is at least 3. If $r = 3$, then the linear system in (2.2.6) becomes

$$\begin{pmatrix} 3 & 0 & 0 & 21 \\ 0 & 0 & 21 & 0 \\ 0 & 21 & 0 & 0 \\ 21 & 0 & 0 & 147 \\ 0 & 0 & 147 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the projectively unique Sylvester form of degree 3 is

$$h(x, y) = y^3 - 7x^3 = (y - \sqrt[3]{7}x)(y - \sqrt[3]{7}\omega x)(y - \sqrt[3]{7}\omega^2 x),\ \omega = e^{\frac{2\pi i}{3}}.$$

Note that h splits over $\mathbb{Q}(\sqrt[3]{7}, \sqrt{-3})$. The $\mathbb{C}$-minimal representation of f is given by

$$f(x, y) = (x + \sqrt[3]{7}y)^7 + (x + \sqrt[3]{7}\omega y)^7 + (x + \sqrt[3]{7}\omega^2 y)^7.$$

Let $S = \mathbb{Q}(\sqrt[3]{7}, \sqrt{-3})$, then $[S : \mathbb{Q}] = 6$ and $L_S(f) = 3$.

If the degree of a binary form is less than $2k - 1$, then the Sylvester form of degree k does not need to be unique. A binary quartic can have infinitely many representations of length 3.

Example 5.2.6. Let $f(x, y) = (x^2 + y^2)^2$, then by Theorem 4.2.3, $L_S(f) = 3$ if and only if

$\mathbb{Q}(\sqrt{3}) \subseteq S$ with the infinitely many minimal representations

$$(x^2+y^2)^2 = \frac{1}{18}\sum_{j=0}^{2}(\cos(\tfrac{j\pi}{3}+\theta)x+\sin(\tfrac{j\pi}{3}+\theta)y)^4, \quad \theta \in \mathbb{C}.$$

5.3 Sylvester fields

Let f be real binary form of degree d. If $L_K(f) = d$, we say that f has *full rank* over the field K. The case for $K = \mathbb{C}$ has been fully studied; see Theorem 2.3.10. In the last years, the case $K = \mathbb{R}$ has been considered in different works [11, 14, 32]. A final result has been recently achieved by Blekherman and Sinn: $L_{\mathbb{R}}(f) = deg(f)$ if and only if f is hyperbolic and not a d-th power (Theorem 2.4.8).

It is natural to wonder whether there are fields besides $\mathbb{C}$ and $\mathbb{R}$ where forms with a given set of properties have full rank.

Definition 5.3.1. Let $K \subseteq \mathbb{C}$. If every binary form of degree $d \geq 2$ which splits over K and not a d-th power has K-rank d, then we say that K is a *Sylvester field.*

Proposition 5.3.2. [5, Proposition 2.6]
A binary form $f \in H_d(\mathbb{R}^2)$ of degree $d \geq 3$ is hyperbolic if and only if all its directional derivatives are hyperbolic.

Remark 5.3.3. The above proposition does not generalize to other fields, but this part does: if a real polynomial splits over $\mathbb{R}$, then its derivative does as well. We call this field property *Rolle's property.* Not all the fields have a notion of differentiable function, but the elements in the polynomial ring $K[x_1, \dots, x_n]$ can be symbolically differentiated.

The question of which fields satisfy Rolle's property was raised in [18]. It holds for any algebraically closed field and any real closed field (we are only interested in infinite fields). It does not hold for rational numbers. For example,

$$\begin{aligned} f(x,y) &= x^3 - 4xy^2 = x(x-2y)(x+2y) \text{ splits over } \mathbb{Q}, \text{ but} \\ \partial_x f &= 3x^2 - 4y^2 = (\sqrt{3}x-2y)(\sqrt{3}x+2y) \text{ does not.} \end{aligned}$$

We need the following extension of the first side of the Proposition 5.3.2.

Proposition 5.3.4.
Let K be a field with Rolle's property. If a binary form $f \in H_d(K^2)$ of degree $d \geq 3$ splits over K, then all its directional derivatives also split over K.

Proof. Let f be a binary form in $H_d(K^2)$ which splits over K. Let $D_v(f) = v_1\partial_x f + v_2\partial_y f$ and $v = (v_1, v_2) \in K^2$. After a change of coordinates and dehomogenizing, we can assume that f is a univariate polynomial of degree d and it splits over K. Thus, we can let D_v be the usual derivative f'. Then the derivative splits over K by Rolle's property. $\square$

Lemma 5.3.5.
Let K be a real closed field and f be a binary cubic form, not a cube, in $H_3(K^2)$. If f splits over K, then $L_K(f) = 3$.

Proof. If f has a repeated factor, then it follows from Theorem 2.3.11(ii) that $L_K(f) = 3$. Assume that f does not have a repeated factor. If f splits over K, then $\sqrt{\Delta} \in K$. Since K is a real closed field, $\sqrt{-3\Delta} \notin K$. Therefore, $L_K(f) = 3$ by Theorem 2.3.11(iii). $\square$

Theorem 5.3.6.
If K is a real closed field, then K is a Sylvester field.

Proof. Let $f \in H_d(K^2)$ be a product of d linear forms in K, and not a d-th power. We need to show that $L_K(f) = d$. We prove the theorem by induction on the degree of f. Notice that Lemma 5.3.5 proves the base case $d = 3$. Let f be a binary form of $d \geq 4$ and suppose that f splits over K. Then every directional derivative of f splits over K by Proposition 5.3.4. By induction, K-rank of directional derivatives is $d-1$. However, if the K-rank of f was less than d, then at least one of the directional derivatives would have K-rank less than $d-1$. Therefore, $L_K(f) = d$. $\square$

Corollary 5.3.7.
Let $F \subseteq K$ where K is a Sylvester field. Then F is also a Sylvester field.

Proof. Let f be a binary form in $H_d(K^2)$ and not a d-th power. It follows from (2.2.2) and Theorem 2.3.8 that $d \geq L_F(f) \geq L_K(f)$. If f splits over F, then it must also split over K. Then $L_K(f) = d$ since K is a Sylvester field. Hence, $L_F(f) = d$ and F is a Sylvester field. $\square$

Theorem 5.3.8.
Let K be an algebraically closed field such that $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. Then K is not a Sylvester field.

Proof. We prove the theorem with an example. Let $f(x, y) = x^3 + y^3$, then f splits over K. However, $L_K(f) = L_{\mathbb{Q}}(f) = 2$ by Theorem 2.2.10. $\square$

The following theorem shows that a cyclotomic field can not be Sylvester.

Theorem 5.3.9.
Let $n \geq 3$, then $\mathbb{Q}(\zeta_n)$ is not a Sylvester field.

Proof. Suppose $f(x, y) = x^n - y^n$ and $n \geq 3$. Then f splits over $\mathbb{Q}(\zeta_n)$. However, $L_{\mathbb{Q}(\zeta_n)}(f) = L_{\mathbb{Q}}(f) = 2$. $\square$

References

[1] J. Alexander and A. Hirschowitz. Polynomial interpolation in several variables. *Journal of Algebraic Geometry*, 4(2):201–222, 1995.

[2] B. Beauzamy, E. Bombieri, P. Enflo, and H.L. Montgomery. Products of polynomials in many variables. *Journal of Number Theory*, 36(2):219–245, 1990.

[3] B. Beauzamy and J. Dégot. Differential identities. *Transactions of the American Mathematical Society*, 347(7):2607–2619, 1995.

[4] A. Białynicki-Birula and A. Schinzel. Extreme binary forms. *Acta Arith*, 142:219–249, 2010.

[5] G. Blekherman and R. Sinn. Real rank with respect to varieties. *Linear Algebra and Its Applications*, 505:344–360, 2016.

[6] J. Bochnak, M. Coste, and M.F. Roy. *Real algebraic geometry.* Springer–Verlag, 1998.

[7] M. Boij, E. Carlini, and A. Geramita. Monomials as sums of powers: the real binary case. *Proceedings of the American Mathematical Society*, 139(9):3039–3043, 2011.

[8] Z.I. Borevich and I.R. Shafarevich. *Number theory.* Academic Press, New York, 1966.

[9] E. Carlini, M.V. Catalisano, and A.V. Geramita. The solution to the Waring problem for monomials and the sum of coprime monomials. *Journal of Algebra*, 370:5–14, 2012.

[10] E. Carlini, M. Kummer, A. Oneto, and E. Ventura. On the real rank of monomials. *Mathematische Zeitschrift*, 286(1-2):571–577, 2017.

[11] A. Causa and R. Re. On the maximum rank of a real binary form. *Annali di Matematica Pura ed Applicata*, 190(1):55–59, 2011.

[12] G. Comas and M. Seiguer. On the rank of a binary form. *Foundations of Computational Mathematics*, 11(1):65–78, 2011.

[13] P. Comon, G. Golub, L.H. Lim, and B. Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM Journal on Matrix Analysis and Applications*, 30(3):1254–1279, 2008.

[14] P. Comon and G. Ottaviani. On the typical rank of real binary forms. *Linear and Multilinear Algebra*, 60(6):657–667, 2012.

[15] J. Harris. *Algebraic geometry: A first course*, volume 133. Springer–Verlag, New York, 1992.

[16] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci.* Springer–Verlag, 1999.

[17] N. Jacobson. *Lectures in Abstract Algebra: III. Theory of Fields and Galois Theory*, volume 32. Springer–Verlag, 2012.

[18] I. Kaplansky. *Fields and rings.* University of Chicago Press, 1972.

[19] S. Karlin. *Total positivity*, volume 1. Stanford University Press, 1968.

[20] T.G. Kolda and B.W. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, 2009.

[21] J.P. Kung. Gundelfinger's theorem on binary forms. *Studies in Applied Mathematics*, 75(2):163–169, 1986.

[22] J.M. Landsberg. *Tensors: geometry and applications*, volume 128. American Mathematical Society, Providence, RI, 2012.

[23] J.M. Landsberg and Z. Teitler. On the ranks and border ranks of symmetric tensors. *Foundations of Computational Mathematics*, 10(3):339–366, 2010.

[24] I.G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 1998.

[25] T. Nagell. Sur la résolubilité de l'équation $x^2 + y^2 + z^2 = 0$ dans un corps quadratique. *Acta Arithmetica*, 21(1):35–43, 1972.

[26] G. Pólya and I.J. Schoenberg. Remarks on de la Vallée Poussin means and convex conformal maps of the circle. *Pacific Journal of Mathematics*, 8(2):295–334, 1958.

[27] G. Pólya and G. Szegö. *Problems and Theorems in Analysis II: Theory of Functions. Zeros. Polynomials. Determinants. Number Theory. Geometry.* Springer–Verlag, 1976.

[28] B. Reznick. Sum of even powers of real linear forms. *Memoirs of the American Mathematical Society*, 96(463), 1992.

[29] B. Reznick. An inequality for products of polynomials. *Proceedings of the American Mathematical Society*, 117(4):1063–1073, 1993.

[30] B. Reznick. Homogeneous polynomial solutions to constant coefficient PDE's. *Advances in Mathematics*, 117(2):179–192, 1996.

[31] B. Reznick. Laws of inertia in higher degree binary forms. *Proceedings of the American Mathematical Society*, 138(3):815–826, 2010.

[32] B. Reznick. On the length of binary forms. In *Quadratic and Higher Degree Forms*, volume 31, pages 207–232. Springer–Verlag, 2013.

[33] B. Reznick. Some new canonical forms for polynomials. *Pacific Journal of Mathematics*, 266(1):185–220, 2013.

[34] B. Reznick and N. Tokcan. Binary forms with three different relative ranks. *Proceedings of the American Mathematical Society*, 2017.

[35] N.D. Sidiropoulos, L. De Lathauwer, X. Fu, K. Huang, E.E. Papalexakis, and Christos Faloutsos. Tensor decomposition for signal processing and machine learning. *IEEE Transactions on Signal Processing*, 65(13):3551–3582, 2017.

[36] J. J. Sylvester. On an elementary proof and generalization of Sir Isaac Newton's hitherto undemonstrated rule for the discovery of imaginary roots. Proc. Lond. Math. Soc. **1** (1865/1866), 1–16; Paper 84 in *Mathematical Papers*, Vol.2, Chelsea, New York, 1973. Originally published by Cambridge University Press in 1908.

[37] J.J. Sylvester. An essay on canonical forms, supplement to a sketch of a memoir on elimination, transformation and canonical forms. Originally published by George Bell, Fleet Street, London, 1851; Paper 34 in *Mathematical Papers*, Vol. 1, Chelsea, New York, 1973. Originally published by Cambridge University Press in 1904.

[38] J.J. Sylvester. On a remarkable discovery in the theory of canonical forms and of hyperdeterminants. Originally in Phiosophical Magazine, vol. 2, 1851; Paper 42 in *Mathematical Papers*, Vol. 1, Chelsea, New York, 1973. Originally published by Cambridge University Press in 1904.

[39] K. Szymiczek. Note on a paper by T. Nagell. *Acta Arithmetica*, 25(3):313–314, 1974.

[40] N. Tokcan. On the waring rank of binary forms. *Linear Algebra and Its Applications*, 524:250–262, 2017.

[41] M.A. Vasilescu and D. Terzopoulos. Multilinear analysis of image ensembles: Tensorfaces. In *European Conference on Computer Vision*, pages 447–460. Springer–Verlag, 2002.