PARTITION ASYMPTOTICS; ZEROS OF ZETA FUNCTIONS; AND APÉRY-LIKE NUMBERS

BY

AMITA MALIK

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Doctoral Committee:

Professor Emeritus, Harold G. Diamond, Chair
Professor Bruce C. Berndt, Director of Research
Professor Alexandru Zaharescu, Director of Research
Professor Kevin Ford

Abstract

PART I

G. H. Hardy and S. Ramanujan established an asymptotic formula for the number of unrestricted partitions of a positive integer, and claimed a similar asymptotic formula for the number of partitions into perfect kth powers, which was later proved by E. M. Wright. Recently, R. C. Vaughan provided a simpler asymptotic formula in the case $k = 2$. In the first part of the thesis, we study the number of partitions into parts from a specific set $A_k(a_0, b_0) := \{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\}$, for fixed positive integers k, a_0, and b_0. Using the Hardy-Littlewood circle method, we give an asymptotic formula for the number of such partitions, thus generalizing the aforementioned results of Wright and Vaughan. We also consider the parity problem for such partitions and prove that the number of such partitions is even (odd) infinitely often, which generalizes O. Kolberg's theorem for the ordinary partition function. This material builds on the joint work with B. C. Berndt and A. Zaharescu.

PART II

The Riemann Hypothesis implies that the zeros of all the derivatives of the Riemann-ξ function lie on the critical line. Results on the proportion of zeros on the critical line of derivatives of $\xi(s)$ have been investigated before by B. Conrey, and I. Rezvyakova. The percentage of zeros of $\xi^{(k)}(s)$ on the critical line approaches 100% percent as k increases. The second part of this thesis builds on the joint work with S. Chaubey, N. Robles, and A. Zaharescu. We study the zeros of combinations of derivatives of $\xi(s)$. Although such combinations do not always have all their zeros on the critical line, we show that the pro-

portion of zeros on the critical line still tends to 1.

PART III

The third part of this thesis focuses on the work on Apéry-like numbers joint with Armin Straub. In 1982, Gessel showed that the Apéry numbers associated to the irrationality of $\zeta(3)$ satisfy Lucas congruences. Our main result is to prove corresponding congruences for all known sporadic Apéry-like sequences. In several cases, we are able to employ approaches due to McIntosh, Samol–van Straten and Rowland–Yassawi to establish these congruences. However, for the sequences labeled s_{18} and (η) we require a finer analysis. As an application, we investigate modulo which numbers these sequences are periodic. In particular, we show that the Almkvist–Zudilin numbers are periodic modulo 8, a special property which they share with the Apéry numbers. We also investigate primes which do not divide any term of a given Apéry-like sequence.

To

my ma, papa, Prem and Mahinder;

&

loving sisters, Indu and Manju

Acknowledgments

I am grateful to my thesis advisors Professors Bruce C. Berndt and Alexandru Zaharescu for their tremendous support, inspiration, passion, and patience. I owe them a debt of gratitude for not only teaching me number theory but also making graduate school an enjoyable experience for me. I thank them for those numerous long and enriching mathematical discussions which have immensely improved my understanding of the subject.

I would like to express my deep gratitude to Professors Kevin Ford and Harold Diamond for agreeing to be on my thesis committee and for their invaluable suggestions at various stages during my graduate studies. I have greatly benefited from the breadth and depth of knowledge of the number theory group here at Illinois. Many thanks to Professors Jayadev Athreya and Elena Fuchs for broadening out my mathematical interests and their encouragement; to Professors George Andrews and Ae Ja Yee for their guidance and keeping me enthusiastic about combinatorics, and partitions, in particular; to Professors Scott Ahlgren, Hal Schenck, Alex Yong, and Rinat Kedem, and Rick Laugesen for their precious time and advice; to Armin Straub for enlightening me about Apéry numbers as well as for being a great friend. I am obliged to the department, and the number theory group for rewarding me with some precious off-teaching time through fellowships which allowed me to focus better on my research. My sincere thanks goes to all my co-authors for collaboration and making research more enjoyable than I had thought; and Alex Miller, Xin Zhang and Arindam (Biswas & Roy) for stimulating discussions. I would also like to thank the efficient and ever helpful department administrative staff: Marci, Paula, Ann, Tori, Tammy, and Broderick.

I convey my token of great respect and appreciation to my teachers in high school and in St. Stephen's who made me believe that math could be fun; and to Professors Sankaran Viswanath, Gautam Bharali, Ravindra Girivaru, Gadadhar Misra, and Sundaram Thangavelu, "Guru Droncharya" who continue to be a tremendous source of inspiration and guidance.

A special thanks to Sarah Brown for being my "local guardian" from the day I set my foot in the States: from teaching me Indian cooking to settling my graduation cap and hood. I thank Jackie Lang, Anush Tserunyan and Patrick Allen for their awesome company, advice, and the interesting discussions over the dinner table (ah, and my unsuccessful dosa attempts)! Thanks to Pritha, Neha, Juan, Sepideh, Mike, James, Gagan, Sarah, Albert, Eliana, Neriman, Iftikhar, Sneha, Ashish, Nicolas, Atul for making CU a fun place; the fruit lady at Farmers' market for the super delicious peaches; Purvi, Rishika, Laura for those "deep talks"; Ambika, Risha, Neha, and Sharmishthaa for their ever cheerful conversations; Sudha, Shweta, Debarati, Karm, Mayuresh, and Anshuman, Sayan, and Prahllad for their friendship; Hema anuty for her love; Jonathan and Abhinav for being my to-go friends who were always ready to give "gyan" and made every effort to make all my problems seem "trivial".

Many thanks to kaki Rajpati and Sangeeta aunty for being there for my parents at more than one occasions when I couldnt; to my awesome niece: Chhavi and nephews: Chiku and Daksh for being so adorable and making my summer breaks memorable at a different level; and to my brother-in-laws for their care. Lastly, I wish to express my indebtedness to my parents and sisters for their unconditional love and support. Thank you, mom and dad for believing in me and providing me more than I could imagine. A thank you bear hug to my absolutely fantastic sisters who have always thought of the best for me before I could.

Table of Contents

List of Symbols

σ	Real part of the complex number s.
t	Imaginary part of the complex number s.
$\lfloor X \rfloor$	The greatest integer less than or equal to X.
$[X]$	The greatest integer less than or equal to X.
$\mathbb{N}$	Set of all positive integers.
$\mathbb{Z}$	Set of all integers.
$\mathbb{Q}$	Set of all rational numbers.
$\mathbb{R}^+$	Set of all positive real numbers.
$\mathbb{R}$	Set of all real numbers.
$\mathbb{C}$	Set of all complex numbers.
$d(n)$	The number of divisor of n.
$\sigma_A(n)$	$\sum_{d\mid n, d\in A} d$.
$B_n(x)$	nth Bernoulli polynomial.
$\Gamma(s)$	Gamma function.

Chapter 1

Introduction

1.1 Partitions

The number of partitions of a natural number has been studied extensively for over a century but there remain some open problems.

For a positive integer n, partition $\lambda \vdash n$ is a non-increasing sequence of positive integers $\lambda_1 \geqslant \cdots \geqslant \lambda_m$ for some positive integer m such that

$$n = \lambda_1 + \cdots + \lambda_m.$$

The integers λ_js are referred as parts of the partition. The partition function $p(n)$ counts the number of ways n can be represented in this form with the convention that $p(0) = 1$. Thus, for example, for $n = 5$, we have 7 such representations given below

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1, \tag{1.1}$$

so $p(5) = 7$. In 1921, S. Ramanujan [48] studied the partition function modulo $5, 7$, and 11, and showed that

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

For certain generalizations, it is natural to restrict the set of parts for the partitions. For any non-negative integer n, and $A \subseteq \mathbb{N}$, let $p_A(n)$ denote the number of partitions of n with parts in the set A. Note that for $A = \mathbb{N}$, the quantity $p_A(n)$ counts the number of unrestricted partitions of n, denoted by $p(n)$. Some of the interesting special cases studied before include A as the set of primes or the set of kth powers.

In the first part of this thesis, we study such restricted partitions. For fixed positive integers k, a_0, and b_0, define the subset $A_k(a_0, b_0)$ of positive integers by $A_k(a_0, b_0) := \{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\}$. Denote by $p_{A_k(a_0,b_0)}(n)$ the number of partitions of n where the parts are taken from the subset $A_k(a_0, b_0)$. As an example, let us consider $k = 2, a_0 = 1, b_0 = 1$, and $n = 5$ i.e. we consider the partitions of 5 into squares. From all the 7 possible partitions in (1.1), the only 2 partitions which are now valid are

$$4+1+1, \qquad 1+1+1+1+1,$$

and with $k = 2, a_0 = 1, b_0 = 2$, we have only one partition given by $1+1+1+1+1$. However, there are no partitions of 5 into into even squares. So, $p_{A(2,2)}(5) = 0$ since in this case a_0 and b_0 are not coprime. We will discuss this in more details in the remarks stated after the main results.

Even though, the partition function $p(n)$ has been studied extensively, it was not known that $p(n)$ takes even (odd) values infinitely often until 1959, when Kolberg [31] established these facts. Other proofs of Kolberg's theorem were later found by M. Newman [38], and by J. Fabrykowski and M. V. Subbarao [25]. It is conjectured that $p(n)$ is even (odd) approximately half the time. Even though many results have been proved in this direction; for example, see K. Ono [43], J. L. Nicolas, I. Z. Ruzsa and A. Sárközy [40], and S. Ahlgren [1], the best known results are far from the estimates expected.

We focus our attention on the general function $p_{A_k(a_0,b_0)}(n)$. In Chapter 2, we work in the ring of formal power series in one variable over the field of two elements $\mathbb{Z}/2\mathbb{Z}$. Using elementary differential equations and algebraic tools such as Hensel's lemma, we develop a new method to prove that this partition function assumes even (odd) values for infinitely many positive integers. This generalizes the result of Kolberg [31] for the ordinary partition function $p(n)$. In fact, our method works in more generality and can be applied to certain other restricted partition functions, including plane partitions for which there are no congruence results known, to obtain corresponding parity results. A. J. Yee along with the first and third authors [9] obtained a similar result in the case of $k = 1$ using more advanced tools such as the Prime Number Theorem for arithmetic progressions and properties of Dirichlet L-functions.

Now we state our main result on these partitions. We reiterate definitions from the second paragraph above. For fixed positive integers k, a_0, and b_0, define $A_k(a_0, b_0) \subseteq \mathbb{N}$ by

$$A_k(a_0, b_0) := \left\{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\right\}. \tag{1.2}$$

Also, let

$$p_{A_k(a_0,b_0)}(n) := \#\{\text{partitions of } n \text{ into parts from } A_k(a_0, b_0)\}. \tag{1.3}$$

The first result is about the parity of $p_{A_k(a_0,b_0)}(n)$.

Theorem 1.1.1. *Let k, a_0, and b_0 be fixed positive integers satisfying $a_0 \leqslant b_0$, and $(a_0, b_0) = 1$. Let $A_k(a_0, b_0)$ and $p_{A_k(a_0,b_0)}(n)$ be defined as in (1.2) and (1.3), respectively. Then, there are infinitely many positive integers n such that $p_{A_k(a_0,b_0)}(n)$ is even, and there are infinitely many positive integers m for which $p_{A_k(a_0,b_0)}(m)$ is odd.*

Hardy and Ramanujan [29] initiated the study of $p(n)$ from an analytic point of view. They proved an asymptotic formula for $p(n)$, as n approaches infinity, and stated (without proof) a similar result for $p_{A_k(1,1)}(n)$, the number of partitions of n into perfect kth powers, for any $k \geqslant 2$. Later a proof was supplied for the case $k \geqslant 2$ by Wright [63] in 1934. His proof uses the ideas of Hardy and Ramanujan for the case $k = 1$, but relies heavily on a transformation for the generating function of $p_{A_k(1,1)}(n)$ involving generalized Bessel functions. In the case $k = 2$, a simpler aymptotic formula has recently been given by Vaughan [61], and has been generalized for any integer $k \geqslant 2$ by A. Gafni [27]. For asymptotics of some other restricted partitions, the reader is referred to [41] and [42]. In Chapter 3, we provide an asymptotic expansion for $p_{A_k(a_0,b_0)}(n)$, as n approaches infinity. This extends results of Hardy and Ramanujan [29], Vaughan [61], and Wright [63]. Our proof is based on the Hardy-Littlewood circle method. A fine analysis and modification of results pertaining to exponential sums help us overcome the complications posed by the general arithmetic progression $a_0 \pmod{b_0}$ when $b_0 > 1$. Moreover, following similar arguments as in the proof of Theorem 1.1.2, one can also obtain asymptotics for the difference of the number of such partitions of two consecutive positive integers as they grow large.

In the next result, we show that

$$p_{A_k(a_0,b_0)}(n) \sim \mathcal{L} \exp\left(\mathcal{M} n^{\frac{1}{k+1}}\right) n^{-\frac{b_0+b_0k+2a_0k}{2b_0(k+1)}},$$

where $\mathcal{L}$ and $\mathcal{M}$ are constants depending on the parameters a_0, b_0 and $k \geqslant 2$.

Theorem 1.1.2. *Fix positive integers k, a_0, and b_0 with $k \geqslant 2$, $a_0 \leqslant b_0$, and $(a_0, b_0) = 1$, let $A_k(a_0, b_0)$ and $p_{A_k(a_0,b_0)}(n)$ be defined as in (1.2) and (1.3), respectively. Set $\beta_0 = a_0/b_0$, and let $\zeta(s)$ and $\zeta(s, \beta_0)$ denote the Riemann zeta function and the Hurwitz zeta function, respectively. Let M be a fixed positive integer with*

$$M \leqslant \frac{1}{2016k^2} \left(\frac{1}{b_0k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)\right)^{-\frac{k}{k+1}} n^{\frac{1}{k+1}}.$$

Then, for any positive integer J, there exist constants $\mu_1, \ldots, \mu_{J-1}$ such that as $n \to \infty$,

$$p_{A_k(a_0,b_0)}(n) = \frac{\exp\left(\frac{k+1}{b_0 k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{\frac{1}{k}} + \zeta(0,\beta_0)(1-\log b_0^k) + k\zeta'(0,\beta_0)\right)}{2\sqrt{\pi}\sqrt{Y}X^{1-\zeta(0,\beta_0)}}$$
$$\times \exp\left(\sum_{m=1}^{M-1}\frac{b_0^{2mk}}{(2m)!}(1-2m)\zeta(1-2m)\zeta(-2mk,\beta_0)X^{-2m}\right)$$
$$\times\left(1+\sum_{j=1}^{J-1}\frac{\mu_j}{Y^j} + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right) + \mathrm{O}_{k,a_0,b_0}\left(X^{-2M+1}\right)\right),$$

where X and Y satisfy

$$\frac{n}{X} = \frac{1}{b_0k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} + \zeta(0,\beta_0) - \frac{b_0^k}{2}\zeta(-k,\beta_0)\frac{1}{X}$$
$$-\sum_{m=1}^{M}\frac{b_0^{2mk}}{(2m-1)!}\zeta(-2m+1)\zeta(-2km,\beta_0)\frac{1}{X^{2m}}, \tag{1.4}$$
$$Y = \frac{k+1}{2b_0k^3}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} + \frac{\zeta(0,\beta_0)}{2}$$
$$+\sum_{m=1}^{M}\frac{m(2m-1)b_0^{2mk}\zeta(-2m+1)\zeta(-2mk,\beta_0)}{(2m)!X^{2m}}, \tag{1.5}$$

and the terms (including the error term) involving M occur only when $\beta_0 \neq 1/2, 1$.

Remarks:

- For $(a_0, b_0) = d_0 > 1$, the number $p_{A_k(a_0,b_0)}(n)$ is zero unless n is a multiple of d_0^k. In fact, $p_{A_k(a_0,b_0)}(n) = p_{A_k(a_0/d_0,b_0/d_0)}(n/d_0^k)$. Also, note that a_0/d_0 and b_0/d_0 are relatively prime. Therefore, it is sufficient to consider only those integers a_0, b_0 which are coprime to each other and satisfy $1 \leqslant a_0 \leqslant b_0$.

- Note that in Theorem 1.1.2, $X \sim \mathcal{S}Y^k \sim \mathcal{T}n^{k/(k+1)}$, for some constants $\mathcal{S}$ and $\mathcal{T}$. In fact, these constants can be computed explicitly from (1.4) and (1.5). Moreover, one can show that $M \leqslant (2(4\pi/5)^{k+1}X)^{1/k}/(4k^2)$, which is used in Section 4.

- Following the arguments in the proof of Theorem 1.1.2, one can obtain an asymptotic result for the difference $p_{A_k(a_0,b_0)}(n+1) - p_{A_k(a_0,b_0)}(n)$ as n approaches infinity.

- In the case $\beta_0 = 1$, we recover Gafni's result [27, Theorem 1], and if we further set $k = 2$, we recover Vaughan's result [61, Theorem 1.5]. In these cases, $\beta_0 = 1$, and therefore as mentioned in Theorem 1.1.2, the expression for $p_{A_k(1,1)}(n)$ becomes much simpler since all the terms involving M disappear.

1.1.1 Future directions

In this thesis, we are concerned with partitions into parts of the form $(a_0 + mb_0)^k$, for some fixed positive integers k, a_0, and b_0 with $(a_0, b_0) = 1$. It would be interesting to know whether there are versions of Theorems 1.1.1 and 1.1.2 for a more general partition function, say, where parts are of the form of a general polynomial, $\sum_{j=1}^{k} a_j m^j$ for some fixed positive integers a_j. Other similar problems of interest would be to consider partitions into primes, or partitions into binary quadratic forms.

Y. Yang [64] considered the partition function $p_\Lambda(n)$ given by

$$\sum_{n=1}^{\infty} p_\Lambda(n) x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-\Lambda(m)},$$

where $\Lambda(m)$ denotes the von Mangoldt function. Improving an asymptotic formula of L. B. Richmond [51] for $p_\Lambda(n)$, Yang proved that the Riemann Hypothesis holds if and only if the error term in Richmond's theorem can be improved to a certain order. One may ask if Theorem 1.1.2 can be used to provide further insight into representations of integers as sums of kth powers, in analogy with Yang's theorem.

Several lower bounds have been obtained for the number of times the ordinary partition function $p(n)$ is even (odd) for $n \leqslant N$, as N approaches infinity (for example, see Ono [44], and Nicolas [39]). With regard to Theorem 1.1.1, it would be nice to obtain similar results for the function $p_{A_k(a,b_0)}(n)$ studied in this thesis. In fact, numerical experiments suggest that like $p(n)$, this function also assumes even values about half the time in almost all the cases, as explained below.

For positive integers n up to 100000, and for certain values of a_0, b_0, and k, we provide two tables with the number of times $p_{A_k(a,b_0)}(n)$ is even, and odd, respectively.

k	a_0	b_0	even	odd	k	a_0	b_0	even	odd	k	a_0	b_0	even	odd
1	1	1	49800	50200	1	5	6	49850	50150	1	1	9	50133	49867
1	1	2	99484	516	1	1	7	50103	49897	1	2	9	50040	49960
1	1	3	49991	50009	1	2	7	49845	50155	1	4	9	50356	49644
1	2	3	50082	49918	1	3	7	49861	50139	1	5	9	50306	49694
1	1	4	49815	50185	1	4	7	50048	49952	1	7	9	49899	50101
1	3	4	49945	50055	1	5	7	50050	49950	1	8	9	50129	49871
1	1	5	49715	50285	1	6	7	50009	49991	1	1	10	49801	50199
1	2	5	50044	49956	1	1	8	49867	50133	1	3	10	50231	49769
1	3	5	50066	49934	1	3	8	50007	49993	1	7	10	50246	49754
1	4	5	49668	50332	1	5	8	50130	49870	1	9	10	49852	50148
1	1	6	50021	49980	1	7	8	50104	49896	1	1	11	49929	50071

Table 1.1: Counting the number of even and odd values of selected partition functions

k	a_0	b_0	even	odd	k	a_0	b_0	even	odd	k	a_0	b_0	even	odd
2	1	1	50299	49701	2	1	7	50362	49638	3	1	5	49606	50394
2	1	2	49696	50304	2	2	7	49971	50029	3	2	5	50475	49525
2	1	3	49581	50419	2	3	7	50110	49890	3	3	5	51020	48980
2	2	3	50013	49987	2	4	7	50333	49667	3	4	5	54063	45937
2	1	4	50059	49941	2	5	7	50201	49879	4	1	1	50084	49916
2	3	4	50001	49999	2	6	7	50695	49305	4	1	2	50235	49765
2	1	5	50333	49667	3	1	1	50286	49714	4	1	3	49385	50614
2	2	5	49809	50191	3	1	2	50066	49934	4	2	3	54628	45372
2	3	5	50043	49957	3	1	3	49931	50069	5	1	1	50202	49798
2	4	5	50540	49460	3	2	3	50459	49541	5	1	2	48596	51404
2	1	6	50134	49866	3	1	4	50283	49717	6	1	1	49869	50131
2	5	6	50174	49826	3	3	4	52350	47650	7	1	1	50456	49544

Table 1.2: Counting the numbers of even and odd values of selected partition functions.

We pose the following two conjectures.

Conjecture 1.1.3. *For positive integers* $a_0 \leqslant b_0$ *with* $(a_0, b_0) = 1$*, let* $p_{A_1(a_0,b_0)}(n)$ *be as in* (1.3) *with* $A_1(a_0, b_0)$ *defined in* (1.2)*. Then, for* $b_0 \neq 2$, $p_{A_1(a_0,b_0)}$ *is even (odd) approximately half the time, i.e. for* $N \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leqslant n \leqslant N : p_{A_1(a_0,b_0)}(n) \text{ is even}\} = \frac{1}{2}. \tag{1.6}$$

It is clear from Table 1.1 that for $b_0 = 2$ (hence $a_0 = 1$) and $k = 1$, (1.6) is nowhere close to being true. In fact, in this case by first applying Euler's theorem (number of partitions into distinct parts equals number of partitions into odd parts), and then Euler's pentagonal number theorem (modulo 2), we obtain that for some positive constant ν,

$$\#\{1 \leqslant n \leqslant N : p_{A_1(1,2)} \text{ is odd}\} \sim \nu\sqrt{N},$$

as N tends to infinity.

Conjecture 1.1.4. *For positive integers* a_0, b_0 *and* k *with* $(a_0, b_0) = 1, a_0 \leqslant b_0$*, and* $k \geqslant 2$*, let* $p_{A_k(a_0,b_0)}(n)$ *be as in* (1.3) *with* $A_k(a_0, b_0)$ *defined in* (1.2)*. Then,* $p_{A_k(a_0,b_0)}$ *is even approximately half the time.*

Notice that $p_{A_k(a_0,b_0)}(n)$ equals zero for all n with $1 < n < a_0^k$. Thus for "large" a_0k, one needs to compute this function for n up to a "large" number N before one can start to witness this phenomenon, as is clear from the two tables above.

Note that after Conjecture 1.1.3 we discussed a case for which (1.6) is invalid. However, Theorem 1.1.1 has no such exceptions, and our proof is uniform for all k and for all arithmetic progressions.

1.2 Zeros

Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, for $s = \sigma + it$, $\sigma > 1$ and $t \in \mathbb{R}$, denote the Riemann zeta-function. The analytic continuation of $\zeta(s)$ to a meromorphic function is achieved by the functional equation

$$\xi(s) = \xi(1-s),$$

where, for any $s \in \mathbb{C}$, the Riemann ξ-function is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Indeed, this continuation shows that $\zeta(s)$ has only one simple pole at $s = 1$ with residue equal to 1. Moreover, it shows that $\zeta(s) = 0$ for $s = -2n$ for $n \in \mathbb{N}$. These are called the trivial zeros of $\zeta(s)$. For $\sigma > 1$, the Euler product is

$$\zeta(s) = \prod_{p}(1 - p^{-s})^{-1},$$

where the product is taken over all primes p. This links the Riemann zeta-function to multiplicative number theory [58, §1 and §2]. It is well understood from the work of Riemann and von Mangoldt that the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ are located inside the critical strip $0 < \beta < 1$; see [58, §3]. From the fact that Γ has no zeros, and has simple poles at the trivial zeros of $\zeta(s)$, it follows that the zeros of ξ are the same as the non-trivial zeros of ζ. The Riemann hypothesis states that all the non-trivial zeros of $\zeta(s)$ lie on the vertical line $\mathrm{Re}(s) = 1/2$.

Let $N(T)$ denote the number of zeros of $\xi(s)$ in the rectangle $0 < \sigma < 1$ and $0 < t \leq T$, each zero counted with multiplicity. It is well-known that

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \tag{1.7}$$

where

$$S(T) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right) \ll \log T,$$

as $T \to \infty$; see [58, §9]. Let us now define $N^{(0)}(T)$ to be the number of zeros of $\zeta(s)$ with $\beta = \frac{1}{2}$ on $0 < t \leq T$, where each zero is counted with multiplicity. Another reformulation of the Riemann hypothesis is that $N^{(0)}(T) = N(T)$ for all values of T. We further set

$$\kappa = \liminf_{T \to \infty} \frac{N^{(0)}(T)}{N(T)}.$$

In 1942, Selberg [54] showed that $\kappa > 0$, and later Levinson [33] showed that $\kappa > 0.34$. This was improved by Conrey [19] to $\kappa > 0.4088$. The history of these results and the current best bound can be found in [13, 26].

For a positive integer k, let $\xi^{(k)}(s)$ denote the kth derivative of the Riemann ξ-function. The Riemann hypothesis implies that for any positive integer k, all the zeros of $\xi^{(k)}(s)$ lie on the critical line. Suppose, in analogy to the above, that $N_k(T)$ denotes the number of zeros $\beta + i\gamma$ of $\xi^{(k)}(s)$ in the rectangle $0 < \beta < 1$ and $0 < \gamma \leq T$ and that $N_k^{(0)}(T)$ denotes the number of zeros of $\xi^{(k)}(s)$ with $\beta = \frac{1}{2}$ and $0 < \gamma \leq T$. A result of Conrey [17] states that if T is positive and sufficiently large, $L = \log \frac{T}{2\pi}$ and $U = TL^{-10}$, then

$$\liminf_{T \to \infty} \kappa_k(T, U) = 1 + O(k^{-2}) \tag{1.8}$$

as $k \to \infty$, where

$$\kappa_k(T, U) := \frac{N_k^{(0)}(T+U) - N_k^{(0)}(T)}{N_k(T+U) - N_k(T)}.$$

Moreover, in [18], following the observations from Anderson [6] and Heath-Brown [30], Conrey showed that these zeros are simple. The coefficient of k^{-2} was computed in [17] for zeros with multiplicity and in [18] for simple zeros. It was remarked that the proportion of simple zeros is always a bit smaller than that of zeros with potential multiplicity. This is due to the fact that a polynomial of degree one was used in the computation of κ, as it will be argued below. Nonetheless, from (1.8) as the order of the derivative of ξ increases, the proportion of zeros on the critical line increases to one.

Rezvyakova [49, 50] computed the coefficients of k^{-2} in 2005 and her result holds uniformly for the parameters T and k. In particular, she showed that the coefficient of k^{-2} could be taken to be $\frac{3}{5}$ for both simple as well as higher order zeros.

In the late 1990's, Selberg considered combinations of Dirichlet L-functions on the critical line. More specifically, let

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

be a Dirichlet L-function of modulus q and where χ denotes a primitive character. The functional equation of $L(s,\chi)$ is given by

$$\phi(s,\chi) = \varepsilon \pi^{-s/2} q^{s/2} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi) = \overline{\phi(1-\bar{s},\chi)},$$

where

$$\mathfrak{a} = \frac{1-\chi(-1)}{2} \quad \text{and} \quad |\varepsilon| = 1;$$

see e.g. [22]. If we have n distinct even characters (a similar result holds for odd characters) and form the function

$$F(s) = \sum_{j=1}^{n} c_j \varepsilon_j q_j^{s/2} L(s,\chi_j),$$

for real $c_j \neq 0$, then

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) F(s)$$

is real for $s = \frac{1}{2} + it$. In a series of his unpublished lectures, Selberg proved a beautiful result on the zeros of $F(s)$ in which he derived a formula analogous to (1.7) for $F(s)$, and also showed that $N^{(0)}(T,F) > c(n) N(T,F)$ for $T > T_0(F)$, where $c(n)$ is a positive constant that depends on n only, and $N(T,F), N^0(T,F)$ are defined for F in the same spirit as before.

Moreover, in those lectures, he mentions a conjecture that almost all the zeros have real part equal to $\frac{1}{2}$.

To state our results, we need to introduce some further notation. For a fixed positive integer M, let us fix a vector $\vec{c} = (c_0, c_1, \ldots, c_M)$ such that $c_j \in \mathbb{R}$ for all j and define

$$c^* := \sum_{j=0}^{M} \frac{(-1)^j c_j}{4^j}. \tag{1.9}$$

For all large numbers T, we set

$$L = \log \frac{T}{2\pi} \quad \text{and} \quad U = TL^{-10}.$$

For each positive integer a, consider the function

$$G_{\vec{c},a,T}(s) := \sum_{j=0}^{M} \frac{c_j(-1)^j}{L^{2j}} \xi^{(a+2j)}(s). \tag{1.10}$$

The presence of L^j has the effect of balancing the size of $\xi^{(j)}(s)$ in $G_{\vec{c},a,T}$(s), so that no one particular term dominates the entire combination.

Inspired by the earlier mentioned results of Selberg and the techniques of Levinson and Conrey, our object of study here is the number of zeros of $G_{\vec{c},a,T}(s)$ on the critical line $\sigma = \frac{1}{2}$ with imaginary part between T and $T + U$. With this in mind, we define the counting functions $N_{\vec{c},a}(T)$ and $N^{(0)}_{\vec{c},a}(T)$ by

$$N_{\vec{c},a}(T) = \sum_{\substack{G_{\vec{c},a,T}(\rho)=0 \\ 0<\operatorname{Im}\rho\leq T}} 1, \quad \text{and} \quad N^{(0)}_{\vec{c},a}(T) = \sum_{\substack{G_{\vec{c},a,T}(\rho)=0 \\ \operatorname{Re}\rho=1/2 \\ 0<\operatorname{Im}\rho\leq T}} 1.$$

Moreover, the proportion of zeros of $G_{\vec{c},a,T}(s)$ in the above rectangle on the critical line is given by the quotient

$$\kappa_{\vec{c},a,T} := \frac{N^{(0)}_{\vec{c},a}(T+U) - N^{(0)}_{\vec{c},a}(T)}{N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T)}. \tag{1.11}$$

Now we are ready to state our main result.

Theorem 1.2.1. *For any positive integer M, fix a vector $\vec{c} = (c_1, \cdots, c_M)$ with real components such that c^* as defined in* (1.9) *is nonzero. Also, for $G_{\vec{c},a,T}(s)$ defined in* (1.10)*, let $\kappa_{\vec{c},a,T}$ be as in* (1.11)*. Then*

$$\kappa_{\vec{c},a,T} \geq 1 - \frac{e^2+2}{16a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right), \tag{1.12}$$

as a and T tend to infinity such that

$$a \leq \frac{1}{2}\frac{\log\log T}{\log\log\log T}.$$

The above result maintains the uniformity achieved in [49, 50] and it shows that up to imposing that c^* defined in (1.9) is nonzero, the first two leading terms on the right-hand side of (1.12) are not affected by the specific coefficients c_j of the combination.

1.3 Apéry numbers

In his surprising proof [7], [59] of the irrationality of $\zeta(3)$, R. Apéry introduced the sequence

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{1.13}$$

which has since been referred to as the Apéry sequence. It was shown by I. Gessel [28, Theorem 1] that, for any prime p, these numbers satisfy the *Lucas congruences*

$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p}, \tag{1.14}$$

where $n = n_0 + n_1 p + \cdots + n_r p^r$ is the expansion of n in base p. Initial work of F. Beukers [12] and D. Zagier [65], which was extended by G. Almkvist, W. Zudilin [5] and S. Cooper [20], has complemented the Apéry numbers with a, conjecturally finite, set of sequences, known as Apéry-like, which share (or are believed to share) many of the remarkable properties of the Apéry numbers, such as connections to modular forms [56], [11], [3] or supercongruences [10], [21], [15], [45], [46], [47]. After briefly reviewing Apéry-like sequences in Section 5.1, we prove in Sections 5.2 and 5.3 our main result that all of these sequences also satisfy the Lucas congruences (1.14). For all but two of the sequences, we establish these congruences in Section 5.2 by extending a general approach provided by R. McIntosh [36]. The main difficulty, however, lies in establishing these congruences for the sequence (η). For this sequence, and to a lesser extent for the sequence s_{18}, we require a much finer analysis, which is given separately in Section 5.3. This is based on the joint work [35] with Armin Straub.

In the approaches of Gessel and McIntosh, binomial sums, like (1.13), are used to derive Lucas congruences. Other known approaches to proving Lucas congruences for a sequence $C(n)$ are based on expressing $C(n)$ as the constant terms of powers of a Laurent polynomial or as the diagonal coefficients of a multivariate algebraic function. However, neither of these approaches is known to apply, for instance, to the sequence (η). In the first approach, one seeks a Laurent polynomial $\Lambda(\boldsymbol{x}) = \Lambda(x_1, \ldots, x_d)$ such that $C(n)$ is the constant term of $\Lambda(\boldsymbol{x})$. In that case, we write $C(n) = \operatorname{ct} \Lambda(\boldsymbol{x})^n$ for brevity. If the Newton polyhedron of $\Lambda(\boldsymbol{x})$ has the origin as its only interior integral point, the results of K. Samol and D. van Straten

[53] (see also [37]) apply to show that $C(n)$ satisfies the *Dwork congruences*

$$C(p^r m+n)C(\lfloor n/p\rfloor) \equiv C(p^{r-1}m+\lfloor n/p\rfloor)C(n) \pmod{p^r} \tag{1.15}$$

for all primes p and all integers $m, n \geqslant 0$, $r \geqslant 1$. The case $r = 1$ of these congruences is equivalent to the Lucas congruences (1.14) for the sequence $C(n)$. For instance, in the case of the Apéry numbers (1.13), we have [57, Remark 1.4]

$$A(n) = \operatorname{ct}\left[\frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}\right]^n,$$

from which one may conclude that the Apéry numbers satisfy the congruences (1.15), generalizing (1.14). Similarly, for the sequence (η), one may derive from the binomial sum (5.15), using G. Egorychev's method of coefficients [24], that its nth term is given by $\operatorname{ct}\Lambda(x,y,z)^n$, where

$$\Lambda(x,y,z) = \left(1-\frac{1}{xy(1+z)^5}\right)\frac{(1+x)(1+y)(1+z)^4}{z^3}.$$

However, $\Lambda(x,y,z)$ is not a Laurent polynomial, and it is unclear if and how one could express the sequence (η) as constant terms of powers of an appropriate Laurent polynomial. As a second general approach, E. Rowland and R. Yassawi [52] show that Lucas congruences hold for a certain class of sequences that can be represented as the diagonal Taylor coefficients of $1/Q(\boldsymbol{x})^{1/s}$, where $s \geqslant 1$ is an integer and $Q(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ is a multivariate polynomial. Again, while such representations are known for some Apéry-like sequences, see, for instance, [57], no suitable representations are available for the sequences (η) or s_{18}.

It was conjectured by S. Chowla, J. Cowles and M. Cowles [16] and subsequently proven by I. Gessel [28] that

$$A(n) \equiv \begin{cases} 1, & \text{if } n \text{ is even}, \\ 5, & \text{if } n \text{ is odd}, \end{cases} \pmod{8}. \tag{1.16}$$

The congruences (1.16) show that the Apéry numbers are periodic modulo 8, and it was recently demonstrated by E. Rowland and R. Yassawi [52] that they are not eventually periodic modulo 16, thus answering a question of Gessel. The Apéry numbers are also periodic modulo 3 (see (5.40)) and their values modulo 9 are characterized by an extension of the Lucas congruences [28]; see also the recent generalizations [32] of C. Krattenthaler and T. Müller, who characterize generalized Apéry numbers modulo 9. As an application of the Lucas congruences established in Sections 5.2 and 5.3, we address in Section 5.4 the natural question to which extent results like (1.16) are true for Apéry-like numbers in general. In particular, we show in Theorem 5.4.3 that the Almkvist–Zudilin numbers are periodic modulo 8 as well.

The primes $2, 3, 7, 13, 23, 29, 43, 47, \ldots$ do not divide any Apéry number $A(n)$, and E. Rowland and R. Yassawi [52] pose the question whether there are infinitely many such primes. While this question remains open, we offer numerical and heuristic evidence that a positive proportion of the primes, namely, about $e^{-1/2} \sim 0.6065$, do not divide any Apéry number. In Section 5.5, we investigate the analogous question for other Apéry-like numbers, and prove that Cooper's sporadic sequences [20] behave markedly differently. Indeed, for any given prime p, a fixed proportion of the last of the first p terms of these sequences is divisible by p. In the case of sums of powers of binomial coefficients, such a result has been proven by N. Calkin [14].

Chapter 2

Parity for partitions into powers of a fixed residue class

In this chapter, we give a proof of Theorem 1.1.1. The chapter consists of two sections, with the first one containing two results needed to prove the main theorem in the later section.

2.1 Auxiliary results

In the first section, we prove two propositions which are later used in the proof, but are also interesting in their own right. For brevity, we also set $A = A_k(a_0, b_0)$, defined in (1.2).

For any positive integer l, and any set $A \subseteq \mathbb{N}$, define

$$\sigma_A(l) := \sum_{\substack{d \mid l \\ d \in A}} d.$$

Proposition 2.1.1. *Let c be an odd positive integer such that $c \equiv a_0 \pmod{b_0}$. Suppose that for any positive integer B, there are distinct primes $q_1, \ldots, q_B$, and a positive integer l_j such that for each $j = 1, \ldots, B$,*

$$\begin{aligned} &q_j \geqslant B + 1, \quad q_j^{l_j} \equiv 1 \pmod{b_0}, \\ &c^{2k} + j \equiv 0 \pmod{q_j^{k(2l_j - 1)}}, \ \textit{and}\ c^{2k} + j \not\equiv 0 \pmod{q_j^{2k(2l_j-1)}}. \end{aligned} \tag{2.1}$$

Then, $\sigma_A(c^{2k})$ is odd, and $\sigma_A(c^{2k} + j)$ is even for all $j = 1, \ldots, B$.

Proof. Note that

$$\begin{aligned}\sigma_A(l) &= \sum_d d : d \in \mathbb{N}, d|l, d = m^k, m \equiv a_0 \pmod{b_0} \\ &\equiv \#\left\{d \in \mathbb{N} : d|l,\ d \text{ is odd and } d = m^k, m \equiv a_0 \pmod{b_0}\right\} \pmod{2} \\ &\equiv \#\left\{m \in \mathbb{N} : m^k|l, m \text{ is odd and } m \equiv a_0 \pmod{b_0}\right\} \pmod{2}. \end{aligned} \tag{2.2}$$

Also, let l have the prime factorization

$$l = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r},$$

where $p_1, \ldots, p_r$ are distinct odd primes, $\alpha_1, \ldots, \alpha_r$ are positive integers, and α_0 is a non-negative integer. We consider the function $f_k : \mathbb{N} \to \mathbb{N}$ given by

$$f_k(l) := p_1^{[\alpha_1/k]} \cdots p_r^{[\alpha_r/k]}.$$

Therefore, using (2.2), we can rewrite $\sigma_A(l)$ as

$$\sigma_A(l) \equiv \#\{m \in \mathbb{N} : m \equiv a_0 \pmod{b_0}, m|f_k(l)\} \pmod{2}. \tag{2.3}$$

First, we show that $\sigma_A(c^{2k} + j)$ is even for each $j = 1, \ldots, B$. Note that (2.1) implies the exponent of q_j in $c^{2k} + j$ is at least $k(2l_j - 1)$ but at most $2k(2l_j - 1) - 1$, i.e., the exponent of q_j in $f_k(c^{2k} + j)$ is exactly $2l_j - 1$. In other words, for fixed $j \in \{1, \ldots, B\}$, there exists a positive integer m_j, coprime to q_j, such that

$$f_k(c^{2k} + j) = m_j q_j^{2l_j - 1}. \tag{2.4}$$

Let d_j be any divisor of $f_k(c^{2k}+j)$ satisfying $d_j \equiv a_0 \pmod{b_0}$. Therefore,

$$d_j = \tilde{d}_j q_j^{\beta_j} \equiv a_0 \pmod{b_0},$$

for some $\tilde{d}_j$ coprime to q_j, and $0 \leqslant \beta_j < 2l_j$. If $\beta_j < l_j$, then from (2.4), we see that $\tilde{d}_j q_j^{\beta_j + l_j}$ also divides $f_k(c^{2k}+j)$, and

$$\tilde{d}_j q_j^{\beta_j + l_j} \neq \tilde{d}_j q_j^{\beta_j}, \text{ and } \tilde{d}_j q_j^{\beta_j + l_j} = \tilde{d}_j q_j^{\beta_j} q_j^{l_j} \equiv a_0 \pmod{b_0}.$$

Similarly, if $\beta_j > l_j$, then $d_j q_j^{\beta_j - l_j} \equiv a_0 \pmod{b_0}$, and is a factor of $f_k(c^{2k}+j)$. Thus the divisors congruent to $a_0 \pmod{b_0}$ of $f_k(c^{2k}+j)$ appear in pairs, and from (2.3) we conclude that $\sigma_A(c^{2k}+j)$ is even for all $j = 1, \ldots, B$.

Next, we show that $\sigma_A(c^{2k})$ is odd. Note that since c is odd, $f_k(c^{2k}) = c^2$. Let u be any divisor of $f_k(c^{2k})$ so that $u \equiv a_0 \pmod{b_0}$. Then, $uv = c^2$ for some $v \in \mathbb{N}$. Also,

$$a_0^2 \equiv c^2 = uv \equiv a_0 v \pmod{b_0}.$$

Since $(a_0, b_0) = 1$, we conclude that $v_0 = a_0 \pmod{b_0}$. Moreover, $u \neq v$ unless $u = c$. Therefore, once again by (2.3), we deduce that $\sigma_A(c^{2k})$ is odd. This completes the proof of the proposition. $\square$

Proposition 2.1.2. *For fixed positive integers k, a_0, and b_0 such that $(a_0, b_0) = 1$, let $A := \left\{m^k : m \in \mathbb{N}, m \equiv a_0 \pmod{b_0}\right\}$. For any positive integer l, let*

$$\sigma_A(l) := \sum_{\substack{d \mid l \\ d \in A}} d.$$

Then, for any fixed positive integer B, there exists an odd positive integer l_B such that $\sigma_A(l_B)$ is odd, and $\sigma_A(l_B + j)$ is even for $j = 1, \ldots, B$.

Proof. Notice that once the existence of c, q_j for $j = 1, \ldots, B$, as in Proposition 2.1.1, are established, we can simply let $l_B = c^{2k}$, and conclude the proof by invoking Proposition 2.1.1. Therefore, we only need to show that for each $j = 1, \ldots, B$, there exist distinct primes $q_1, \ldots, q_B$, and positive integers l_j satisfying

$$q_j \geqslant B + 1, \quad q_j^{l_j} \equiv 1 \pmod{b_0}, \quad c \equiv a_0 \pmod{b_0}, \tag{2.5}$$

$$c^{2k} + j \equiv 0 \pmod{q_j^{k(2l_j-1)}}, \text{ and } c^{2k} + j \not\equiv 0 \pmod{q_j^{2k(2l_j-1)}}. \tag{2.6}$$

We construct q_j's inductively. For a fixed $j \in \{1, \ldots, B\}$, assume $q_1, \ldots, q_{j-1}$ are already chosen, and set $q_0 := 1$. Define

$$K_j := \prod_{\substack{p_r \nmid j \\ p_r - \text{prime} \\ p_r | B! 2k b_0 q_1 \ldots q_{j-1}}} p_r.$$

Fix any prime factor q_j of $K_j^{2k} + j$. Then,

- $(q_j, j) = 1$, for if $q_j | j$, then $q_j | K_j$, which further implies $q_j = p_r \nmid j$,
- $q_j \geqslant B + 1$, as $q_j \leqslant B$ implies q_j divides K_j, and hence j,
- $q_j \notin \{q_1, \ldots, q_{j-1}\}$,
- $q_j \nmid k$,
- $(q_j, b_0) = 1$.

Thus, for each $j = 1, \ldots, B$, the congruence $x^{2k} + j \equiv 0 \pmod{q_j}$ has a solution; for example, one can take $x = K_j$. Also, let $l_j \in \mathbb{N}$ so that $q_j^{l_j} \equiv 1 \pmod{b_0}$.

Next, for a fixed $j \in \{1, \ldots, B\}$, we define a polynomial $g_j(x) \in \mathbb{Z}[x]$ by

$$g_j(x) := x^{2k} + j + q_j^{k(2l_j-1)}. \tag{2.7}$$

Then,

$$g_j(K_j) \equiv 0 \pmod{q_j}, \text{ and } g_j'(K_j) = 2kK_j^{2k-1} \not\equiv 0 \pmod{q_j}.$$

Therefore, by Hensel's Lemma, for $m_j \in \mathbb{N}$, there exists $\beta_{j,m_j} \in \mathbb{Z}$ such that

$$\beta_{j,m_j} \equiv K_j \pmod{q_j}, \text{ and } g_j(\beta_{j,m_j}) \equiv 0 \pmod{q_j^{m_j}}.$$

In particular, set $m_j = k(2l_j - 1) + 1$. Thus using (2.7) in the last congruence above, we deduce that $\beta_{j,k(2l_j-1)+1}$ satisfies

$$\beta_{j,k(2l_j-1)+1}^{2k} + j \equiv 0 \pmod{q_j^{k(2l_j-1)}}, \text{ and } \beta_{j,k(2l_j-1)+1}^{2k} + j \not\equiv 0 \pmod{q_j^{k(2l_j-1)+1}}.$$

Using the Chinese Remainder Theorem, choose a positive integer c such that for all $j = 1, \ldots, B$,

- $c \equiv 1 \pmod{2}$,
- $c \equiv a_0 \pmod{b_0}$,
- $c \equiv \beta_{j,k(2l_j-1)+1} \pmod{q_j^{k(2l_j-1)+1}}$.

Note that if b_0 is even, a_0 must be odd, and therefore $c \equiv a_0 \pmod{b_0}$ implies that $c \equiv 1 \pmod{2}$, and thus the Chinese Remainder Theorem does apply here. This implies that there exists an odd positive integer c such that $c \equiv a_0 \pmod{b_0}$, and for $j = 1, \ldots, B$,

$$c^{2k} + j \equiv \beta_{k(2l_j-1)+1}^{2k} + j \pmod{q_j^{k(2l_j-1)+1}}.$$

This shows the existence of c and q_j's as claimed in (2.5) and (2.6). From the discussion in the beginning of the proof, we are done. $\square$

2.2 Proof of Theorem 1.1.1

Now, we give a proof of Theorem 1.1.1.

Proof. Recall that the generating function for $p_A(n)$ is given by

$$F_A(q) := \sum_{n=0}^{\infty} p_A(n) q^n = \prod_{m \in A} \frac{1}{(1-q^m)}.$$

Consider the formal power series $F(X)$ in the variable X defined as

$$F(X) := \prod_{m \in A} \frac{1}{1-X^m}.$$

Taking the logarithmic derivative of $F(X)$ and then multiplying both sides by X, we obtain

$$\begin{aligned} X\frac{F'(X)}{F(X)} &= \sum_{m \in A} m \sum_{n=1}^{\infty} X^{mn} \\ &= \sum_{l=1}^{\infty} X^l \sum_{\substack{m|l \\ m \in A}} m \\ &= \sum_{l=1}^{\infty} \sigma_A(l) X^l \\ &=: H(X), \end{aligned} \tag{2.8}$$

where for any positive integer l, $\sigma_A(l) := \sum_{d|l, d \in A} d$. Therefore,

$$XF'(X) = F(X)H(X). \tag{2.9}$$

Claim 2.2.1. *$p_A(n)$ is odd for infinitely many $n \in \mathbb{N}$.*

Proof. Assume the contrary, and let, if possible, $p_A(n)$ be odd only for $n_i, i = 1, \ldots, r$, for some fixed positive integer r. Also, without loss of generality, we can assume that

$n_1 < \cdots < n_r$. Therefore,

$$F(X) \equiv \sum_{j=1}^{r} X^{n_j} \pmod 2.$$

Using this in (2.9), and the definition of $H(X)$ in (2.8), we see that

$$\sum_{j=1}^{r} n_j X^{n_j} \equiv \sum_{j=1}^{r} X^{n_j} \sum_{l=1}^{\infty} \sigma_A(l) X^l \pmod 2. \tag{2.10}$$

For $B = n_r$ in Proposition 2.1.2, we obtain a positive integer l_{n_r} such that $\sigma_A(l_{n_r})$ is odd, and $\sigma_A(l_{n_r} + j)$ is even for all $j = 1, \ldots, n_r$. Therefore, comparing the coefficients of $X^{l_{n_r}+n_r}$ on both sides of (2.10) yields

$$\begin{aligned} 0 &\equiv \sum_{\substack{j=1 \\ l+n_j = l_{n_r}+n_r}}^{r} \sigma_A(l) \pmod 2 \\ &\equiv \sum_{j=1}^{r} \sigma_A(l_{n_r} + n_r - n_j) \pmod 2 \\ &\equiv \sigma_A(l_{n_r}) \equiv 1 \pmod 2, \end{aligned}$$

which is a contradiction. This completes the proof of Claim 2.2.1. □

Claim 2.2.2. *$p_A(n)$ is even for infinitely many $n \in \mathbb{N}$.*

Proof. Assume that $p_A(n)$ is even only for $n = m_1 < \cdots < m_v$ for some fixed positive integer v. Therefore,

$$F(X) \equiv \sum_{\substack{j=1 \\ n \neq m_j}}^{v} X^n \pmod 2.$$

In other words,

$$F(X) \equiv \sum_{n=0}^{\infty} X^n + \sum_{j=1}^{v} X^{m_j} \pmod 2.$$

This implies

$$(1 - X)F(X) \equiv 1 - (1 - X)\sum_{j=1}^{v} X^{m_j} \pmod 2. \tag{2.11}$$

Differentiating, and then multiplying both sides by $(1 - X)$, we observe that

$$(1 - X)^2 F'(X) - (1 - X)F(X) \equiv (1 - X)\sum_{j=1}^{v} X^{m_j} - (1 - X)^2 \sum_{j=1}^{v} m_j X^{m_j - 1} \pmod 2.$$

Using (2.11), we find that the above congruence becomes

$$(1 - X)^2 F'(X) \equiv 1 - (1 - X)^2 \sum_{j=1}^{v} m_j X^{m_j - 1} \pmod 2. \tag{2.12}$$

Also, recall from (2.9),

$$X(1 - X)^2 F'(X) = (1 - X)^2 F(X)H(X).$$

Employing this along with (2.8), (2.11) and (2.12), we obtain

$$\begin{aligned} X - (1 - X)^2 \sum_{j=1}^{v} m_j X^{m_j} &\equiv (1 - X)\left\{1 - (1 - X)\sum_{j=1}^{v} X^{m_j}\right\}\sum_{l=1}^{\infty} \sigma_A(l)X^l \pmod 2 \\ &\equiv \left\{1 + X + \sum_{j=1}^{v} X^{m_j} + \sum_{j=1}^{v} X^{m_j+2}\right\}\sum_{l=1}^{\infty} \sigma_A(l)X^l \pmod 2. \end{aligned}$$

Let $B = m_v + 2$ in Proposition 2.1.2. So, we can find a positive integer l_{m_v+2} so that $\sigma_A(l_{m_v+2})$ is odd, while $\sigma_A(l_{m_v+2} + j)$ is even for $j = 1, \ldots, m_v + 2$. Hence, a comparison of coefficients of $X^{l_{m_v+2}+m_v+2}$ on both sides above yields

$$
\begin{aligned}
0 &\equiv \sigma_A(l_{m_v+2} + m_v + 2) + \sigma_A(l_{m_v+2} + m_v + 1) + \sum_{j=1}^{v} \sigma_A(l_{m_v+2} + m_v + 2 - m_j) \\
&\quad + \sum_{j=1}^{v} \sigma_A(l_{m_v+2} + m_v - m_j) \pmod 2 \\
&\equiv \sigma_A(l_{m_v+2}) \pmod 2,
\end{aligned}
$$

which is a contradiction. Thus, $p_A(n)$ is even for infinitely many positive integers n, which completes the proof of Claim 2.2.2. □

From Claim 2.2.1 and Claim 2.2.2, we obtain Theorem 1.1.1. □

Chapter 3

Asymptotics for partitions into powers of a fixed residue class

In this chapter, we prove two lemmas to be used in the following section in order to compute an asymptotic formula for $p_{A_k(a_0,b_0)}(n)$, as $n \to \infty$. Recall that for a fixed integer $k \geqslant 2$, $p_{A_k(a_0,b_0)}(n)$ denotes the number of partitions of n with parts in $A_k(a_0, b_0)$, where for integers k, a_0, and b_0 satisfying $0 < a_0 \leqslant b_0$, $(a_0, b_0) = 1$, and $k \geqslant 2$,

$$A := A_k(a_0, b_0) = \left\{m^k \in \mathbb{N} : m \equiv a_0 \pmod{b_0}\right\}. \tag{3.1}$$

Also, recall that the generating function $\Psi(z; A)$ is given by

$$\Psi(z; A) = \sum_{n=0}^{\infty} p_A(n) z^n = \prod_{m \in A} \frac{1}{1 - z^m}, \quad |z| < 1. \tag{3.2}$$

For $|z| < 1$, define the function $\Phi(z; A)$ by

$$\Psi(z; A) = \exp(\Phi(z; A)). \tag{3.3}$$

Therefore,

$$\Phi(z; A) = \sum_{j=1}^{\infty} \sum_{m \in A} \frac{z^{jm}}{j}, \quad |z| < 1. \tag{3.4}$$

Throughout the remainder of the chapter, we use the standard notation $e(x)$ for $\exp(2\pi i x)$ for any real number x.

3.1 Auxiliary results

In the current section, we state and prove the results used in the proof of Theorem 1.1.2 in the next section.

Lemma 3.1.1. *For each sufficiently large positive real number X, and $\Theta \in [-3/(8\pi X), 3/(8\pi X)]$, define*

$$\Delta = (1 + 4\pi^2\Theta^2X^2)^{-1/2}.$$

Let $R := R(k, X, \theta)$ be defined as

$$R = \frac{(2(\pi\Delta)^{k+1}X)^{1/k}}{2k^2}. \tag{3.5}$$

Let $\rho = e^{-1/X}$, and for $a_0, b_0 \in \mathbb{N}$ with $1 \leqslant a_0 \leqslant b_0$ and $(a_0, b_0) = 1$, let $\beta_0 = a_0/b_0$. For any complex number $s = \sigma + it$, let $\zeta(s)$ and $\zeta(s, \beta_0)$ denote the Riemann zeta function and the Hurwitz zeta function, respectively. Then, for A and $\Phi(z; A)$ defined in (3.1) and (3.4), respectively, as $X \to \infty$,

$$\begin{aligned}\Phi(\rho e^{2\pi i\Theta}; A) = {}& \frac{1}{b_0 k}\zeta(1 + 1/k)\Gamma(1/k)\left(\frac{X}{1 - 2\pi iX\Theta}\right)^{1/k} + \zeta(0, \beta_0)\log\left(\frac{b_0^{-k}X}{1 - 2\pi iX\Theta}\right)\\ & + k\zeta'(0, \beta_0) + \frac{b_0^k}{2}\zeta(-k, \beta_0)\left(\frac{1 - 2\pi iX\Theta}{X}\right) + \sum_{m=1}^{\lfloor R/2\rfloor}\frac{b_0^{2mk}}{(2m)!}\zeta(-2m + 1)\\ & \times \zeta(-2km, \beta_0)\left(\frac{1 - 2\pi iX\Theta}{X}\right)^{2m} + \mathrm{O}_{k,a_0,b_0}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\right),\end{aligned}$$

where the expression immediately before the error term occurs only when $\beta_0 \neq 1/2, 1$.

Proof. The series for the Riemann zeta function $\zeta(s + 1)$ and the Hurwitz zeta function $\zeta(ks, \beta_0)$ converge absolutely and uniformly for $\operatorname{Re} s > 1/k + \delta$ for any fixed positive number

δ. Therefore, using Mellin's transform, we have, for a real number $c > 1/k$,

$$\begin{aligned}
\Phi(\rho e(\Theta); A) &= \sum_{j=1}^{\infty} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \frac{\rho^{jm^k} e(jm^k\Theta)}{j} \\
&= \sum_{j=1}^{\infty} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \frac{1}{j} \exp\left(\frac{-jm^k}{X} + 2\pi i jm^k\Theta\right) \\
&= \sum_{j=1}^{\infty} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \frac{1}{j} \exp\left(-jm^k \, \frac{1-2\pi i X\Theta}{X}\right) \\
&= \frac{1}{2\pi i} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{\substack{m=1,\\ m\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} \int_{c-i\infty}^{c+i\infty} \left(jm^k \frac{1-2\pi i X\Theta}{X}\right)^{-s} \Gamma(s)\, ds \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} b_0^{-ks} \left(\frac{1-2\pi i X\Theta}{X}\right)^{-s} \Gamma(s) \sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \sum_{m=0}^{\infty} \frac{1}{(m+a_0/b_0)^{ks}}\, ds.
\end{aligned}$$

We notice that the series above can be written in terms of the Riemann and the Hurwitz zeta functions. Thus,

$$\begin{aligned}
\Phi(\rho e(\Theta); A) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} b_0^{-ks} \zeta(s+1)\zeta(ks, \beta_0)\Gamma(s) \left(\frac{X}{1-2\pi i X\Theta}\right)^{s} ds \\
&=: \frac{1}{2\pi i} \left(\int_{c-i\infty}^{c-iR} + \int_{c-iR}^{c+iR} + \int_{c+iR}^{c+i\infty}\right) \mathcal{J}_s \, ds, \qquad (3.6)
\end{aligned}$$

where R is defined in (3.5). We compute these integrals using the residue theorem. For the middle integral on the far right side of (3.6), consider the rectangle $\mathcal{R}_m$ with vertices $-R \pm iR$ and $c \pm iR$. Therefore, by the residue theorem,

$$\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \mathcal{J}_s \, ds = \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s - \left(\int_{c+iR}^{-R+iR} + \int_{-R+iR}^{-R-iR} + \int_{-R-iR}^{c-iR}\right) \mathcal{J}_s \, ds. \qquad (3.7)$$

In order to compute the first integral on the far right side of (3.6), for any real number $L > 0$, we define the rectangle $\mathcal{R}_L$ with vertices $-R - i(R+L), -R - iR, c - iR$, and $c - i(R+L)$.

Thus, by the residue theorem,

$$\frac{1}{2\pi i}\int_{c-i(R+L)}^{c-iR} \mathcal{J}_s\, ds = \sum_{\substack{\text{poles}\\ \text{in } \mathcal{R}_L}} \operatorname{Res} \mathcal{J}_s - \left(\int_{c-iR}^{-R-iR} + \int_{-R-iR}^{-R-i(R+L)} + \int_{-R-i(R+L)}^{c-i(R+L)}\right) \mathcal{J}_s\, ds. \quad (3.8)$$

Finally, for the last integral on the far right side of (3.6), for any real number $U > 0$, we define the rectangle $\mathcal{R}_U$ given by the vertices $-R+iR, -R+i(R+U), c+i(R+U), c+iR$. Once again, by the residue theorem,

$$\frac{1}{2\pi i}\int_{c+iR}^{c+i(R+U)} \mathcal{J}_s\, ds = \sum_{\substack{\text{poles}\\ \text{in } \mathcal{R}_U}} \operatorname{Res} \mathcal{J}_s - \left(\int_{c+i(R+U)}^{-R+i(R+U)} + \int_{-R+i(R+U)}^{-R+iR} + \int_{-R+iR}^{c+iR}\right) \mathcal{J}_s\, ds. \quad (3.9)$$

The rectangles R_m, R_L, and R_U are chosen so that the integrand has no zeros on their sides.

The only possible poles of the integrand $\mathcal{J}_s$ are at $s = 1/k, 0, -1$, and $-2j$, for each positive integer j. Thus, all the poles are real, which means $\mathcal{J}_s$ is holomorphic within the rectangles R_L and R_U. Therefore, the sum of the residues in (3.8) and (3.9) is zero. Thus, by letting L and U tend to infinity, we have

$$\int_{c-i\infty}^{c-iR} \mathcal{J}_s\, ds = -\int_{c-iR}^{-R-iR} \mathcal{J}_s\, ds - \lim_{L\to\infty}\left(\int_{-R-iR}^{-R-i(R+L)} + \int_{-R-i(R+L)}^{c-i(R+L)}\right) \mathcal{J}_s\, ds, \quad (3.10)$$

and

$$\int_{c+iR}^{c+i\infty} \mathcal{J}_s\, ds = -\int_{-R+iR}^{c+iR} \mathcal{J}_s\, ds - \lim_{U\to\infty}\left(\int_{c+i(R+U)}^{-R+i(R+U)} + \int_{-R+i(R+U)}^{-R+iR}\right) \mathcal{J}_s\, ds. \quad (3.11)$$

Using (3.7), (3.10), and (3.11) in (3.6), we deduce that

$$\begin{aligned}\Phi(\rho e(\Theta); A) = &-\lim_{L\to\infty}\left(\int_{-R-iR}^{-R-i(R+L)} + \int_{-R-i(R+L)}^{c-i(R+L)}\right) \mathcal{J}_s\, ds + \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s \\ &- \int_{-R+iR}^{-R-iR} \mathcal{J}_s\, ds - \lim_{U\to\infty}\left(\int_{c+i(R+U)}^{-R+i(R+U)} + \int_{-R+i(R+U)}^{-R+iR}\right) \mathcal{J}_s\, ds. \quad (3.12)\end{aligned}$$

Now, we show that the two integrals along the horizontal lines above approach zero as U and L approach infinity. Also,

$$\left|\left(\frac{X}{1-2\pi iX\Theta}\right)^s\right| = (X\Delta)^\sigma \exp(-t\phi),$$

where ϕ is the argument of $X^s/(1-2\pi iX\Theta)^s$. Therefore, $\tan(\pi/2-\phi) = 1/(2\pi X\Theta)$, and $\sin(\pi/2-\phi) = \Delta$. Using estimates for the sine function, we see that $\pi/2-\phi > \Delta$, and therefore,

$$\left|\left(\frac{X}{1-2\pi iX\Theta}\right)^s\right| \leqslant (X\Delta)^\sigma \exp(|t|(\pi/2-\Delta)). \tag{3.13}$$

Also, by Stirling's formula in a vertical strip, for $s = \sigma + it$ and $\alpha \leqslant \sigma \leqslant \beta$,

$$|\Gamma(s)| \ll |s|^{\sigma-1/2}\exp(-\pi|t|/2). \tag{3.14}$$

Combining this with (3.13) and standard bounds for $\zeta(s)$ and $\zeta(s,\beta_0)$ (for example, see [58, p. 81], [8, p. 270]), we deduce that there exist constants B and C such that

$$\int_{-R-i(R+L)}^{c-i(R+L)} \mathcal{J}_s\, ds \ll (R+L)^B e^{-(R+L)\Delta+R},$$

and

$$\int_{c+i(R+U)}^{-R+i(R+U)} \mathcal{J}_s\, ds \ll (R+U)^C e^{-(R+U)\Delta+R},$$

which both tend to zero as L and U approach infinity, since R and Δ are both fixed, positive real numbers. Therefore, from (3.12),

$$\begin{aligned}\Phi(\rho e(\Theta);A) &= -\int_{-R-iR}^{-R-i\infty} \mathcal{J}_s\,ds + \sum_{\text{poles in }\mathcal{R}_m} \operatorname{Res}\mathcal{J}_s - \int_{-R+iR}^{-R-iR} \mathcal{J}_s\,ds - \int_{-R+i\infty}^{-R+iR} \mathcal{J}_s\,ds \\ &= \sum_{\text{poles in }\mathcal{R}_m} \operatorname{Res}\mathcal{J}_s + \left(\int_{-R-i\infty}^{-R-iR} + \int_{-R-iR}^{-R+iR} + \int_{-R+iR}^{-R+i\infty}\right)\mathcal{J}_s\,ds. \qquad (3.15)\end{aligned}$$

Next, we find bounds for the integrand $\mathcal{J}_s$ in order to estimate the integrals in (3.15). Using the functional equation (in its asymmetric form) for the Riemann zeta function [22, p. 73], [58, p. 16] and the functional equation for the Hurwitz zeta function [22, p. 72], [58, p. 37], we have

$$\begin{aligned}\zeta(s+1)\zeta(ks,\beta_0) = 4\left\{\sin\left(\frac{\pi ks}{2}\right)\sum_{m=1}^{\infty}\frac{\cos(2m\pi\beta_0)}{m^{1-ks}} + \cos\left(\frac{\pi ks}{2}\right)\sum_{m=1}^{\infty}\frac{\sin(2m\pi\beta_0)}{m^{1-ks}}\right\} \\ \times (2\pi)^{(k+1)s-1}\cos(\pi s/2)\zeta(-s)\Gamma(-s)\Gamma(1-ks). \qquad (3.16)\end{aligned}$$

Using the functional equation and reflection formula for the gamma function,

$$\Gamma(1-ks) = -ks\ \Gamma(-ks), \quad -s\sin(\pi s)\Gamma(-s)\Gamma(s) = \pi,$$

we can write (3.16) in the form

$$\begin{aligned}\zeta(s+1)\zeta(ks,\beta_0)\Gamma(s) = 2k(2\pi)^{(k+1)s}\zeta(-s)\Gamma(-ks)\frac{\cos(\pi s/2)}{\sin(\pi s)} \\ \times \sum_{m=1}^{\infty}\left\{\sin\left(\frac{\pi ks}{2}\right)\frac{\cos(2m\pi\beta_0)}{m^{1-ks}} + \cos\left(\frac{\pi ks}{2}\right)\frac{\sin(2m\pi\beta_0)}{m^{1-ks}}\right\}.\end{aligned}$$

(3.17)

Also note that

$$\frac{\cos(\pi s/2)}{\sin(\pi s)}\sin(\pi ks/2) = \frac{\sin(\pi ks/2)}{2\sin(\pi s/2)} \ll e^{(k-1)|t|\pi/2},$$

and

$$\frac{\cos(\pi s/2)}{\sin(\pi s)}\cos(\pi k s/2)=\frac{\cos(\pi k s/2)}{2\sin(\pi s/2)}\ll e^{(k-1)|t|\pi/2}.$$

Using these bounds along with (3.13), (3.14), and (3.17), for the integrand $\mathcal{J}_s$ in (3.15), we find that

$$\begin{aligned}\mathcal{J}_s &\ll (2\pi)^{(k+1)\sigma}k^{-k\sigma}|s|^{-1/2-k\sigma}(X\Delta)^{\sigma}e^{-\Delta|t|}\\ &=\left(\frac{k^k}{(2\pi)^{k+1}X\Delta}\right)^{R}|R+it|^{-1/2+kR}e^{-\Delta|t|},\end{aligned}\tag{3.18}$$

since $\sigma=-R$ here. For the middle integral on the far right side of (3.15), we have $|t|\leqslant R$. Therefore, using the foregoing estimates for the integrand $\mathcal{J}_s$, we arrive at

$$\begin{aligned}\int_{-R-iR}^{-R+iR}\mathcal{J}_s\,ds &\ll \left(\frac{k^k}{2\pi^{k+1}X\Delta}\right)^{R}R^{-1/2+kR}\int_0^R e^{-\Delta t}\,dt\\ &\ll \left(\frac{(2k^2)^k}{2\pi^{k+1}X\Delta}\right)^{R}R^{-1/2+kR}\\ &\ll \exp\left(-\frac{(2(\pi\Delta)^{k+1}X)^{1/k}}{2k}\right),\end{aligned}\tag{3.19}$$

where in the penultimate step above, we have used the definition of R. For the first and the last integrals in (3.15), we have the inequality $|t|>R$. Therefore, invoking (3.18), and making a change of variable $y = \Delta t$, we deduce that

$$\begin{aligned}\left(\int_{-R-i\infty}^{-R-iR} + \int_{-R+iR}^{-R+i\infty}\right) \mathcal{J}_s \, ds &\ll \left(\frac{k^k}{2\pi^{k+1} X \Delta}\right)^R \int_R^\infty t^{-1/2+kR} e^{-\Delta t} \, dt \\ &\ll \left(\frac{k^k}{2(\pi\Delta)^{k+1} X}\right)^R \int_{\Delta R}^\infty y^{-1/2+kR} e^{-y} \, dy \\ &\ll \left(\frac{k^k}{2(\pi\Delta)^{k+1} X}\right)^R \Gamma(kR + 1/2) \\ &\ll \left(\frac{(2k^2)^k}{2(\pi\Delta)^{k+1} X}\right)^R e^{-kR} R^{kR} \\ &\ll \exp\left(-\frac{(2(\pi\Delta)^{k+1} X)^{1/k}}{2k}\right), \end{aligned} \tag{3.20}$$

where in the penultimate step above, Stirling's formula is invoked, and in the last step, we have used the definition of R. Using the fact that $\Delta \geqslant 4/5$ when Θ lies in the interval $[-3/(8\pi X), 3/(8\pi X)]$, and the estimates in (3.19) and (3.20) in (3.15), we obtain

$$\Phi(\rho e(\Theta); A) = \sum_{\text{poles in } \mathcal{R}_m} \operatorname{Res} \mathcal{J}_s + \mathrm{O}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1} X)^{1/k}}{2k}\right)\right). \tag{3.21}$$

Now, we compute the residues in the above sum. From [62, p. 267], we know that for any non-negative integer m,

$$\zeta(-m, \beta_0) = -\frac{B_{m+1}(\beta_0)}{m+1}, \tag{3.22}$$

where $B_m(x)$ denotes the Bernoulli polynomial of degree m, and in particular, $B_m(0)$ is the mth Bernoulli number. In particular,

$$\zeta(0, \beta_0) = \frac{1}{2} - \beta_0. \tag{3.23}$$

Nörlund showed that $B_{2m+1}(x)$ has only two real zeros, $1/2$ and 1, in the interval $(0, 1]$. Therefore, for any positive integer m, (3.22) implies that $\zeta(-2m, \beta_0)$ is zero if and only if β_0

equals 1/2 or 1.

Therefore, for $\beta_0 = 1/2, 1$, and any positive integer m, because $\Gamma(s)$ has a simple pole and $\zeta(s, \beta_0)$ has a simple zero at $s = -2m$, the product $\Gamma(s)\zeta(s, \beta_0)$ has a removable singularity at $s = -2m$. Moreover, from (3.23), we know that $\zeta(0, \beta_0) = 0$ if and only if $\beta_0 = 1/2$.

Also, for any positive integer m, the product $\Gamma(s)\zeta(s+1)$ has a removable singularity at $s = -2m + 1$ because of the trivial zeros of the Riemann zeta function at the negative even integers.

Thus for $\beta_0 = 1/2$, and the integrand $\mathcal{J}_s$, defined in (3.6), the only poles are at $s = 1/k, 0, -1$, and all are simple. For $\beta_0 = 1$, there is a double pole at $s = 0$, and there are simple poles at $s = 1/k$ and -1. And for $\beta_0 \neq 1/2, 1$, there is a double pole at $s = 0$, and there are simple poles at $s = 1/k, -1$, and $-2l \leqslant R$, where l is any positive integer.

The residue of $\mathcal{J}_s$ for the pole at $s = 1/k$ is given by

$$\operatorname{Res} \mathcal{J}_s|_{s=1/k} = \frac{\zeta(1 + 1/k)}{b_0 k}\Gamma(1/k)\left(\frac{X}{1 - 2\pi i X\Theta}\right)^{1/k}. \tag{3.24}$$

The function $\zeta(s+1)\Gamma(s)$ has a Laurent expansion of the form, (see [58, p. 16])

$$\left(\frac{1}{s} - \gamma + \sum_{j=1}^{\infty} a_j s^j\right)\left(\frac{1}{s} + \gamma + \sum_{j=1}^{\infty} b_j s^j\right) = \frac{1}{s^2} + \sum_{j=0}^{\infty} c_j s^j,$$

where a_j, b_j and c_j are constants, and γ is Euler's constant. Thus, the residue of $\mathcal{J}_s$ for the pole at $s = 0$ can be written as

$$\operatorname{Res} \mathcal{J}_s|_{s=0} = \zeta(0, \beta_0) \log\left(\frac{X}{1 - 2\pi i X\Theta}\right) - k\zeta(0, \beta_0) \log b_0 + k\zeta'(0, \beta_0).$$

Also, the residue at $s = -1$ is given by

$$\operatorname{Res} \mathcal{J}_s|_{s=-1} = -b_0^k \zeta(0)\zeta(-k, \beta_0)\left(\frac{1 - 2\pi i X\Theta}{X}\right).$$

Lastly, for each positive integer $l \leqslant M$, where M is defined in the statement of this lemma, the residue at the pole at $s = -2l$ lying inside the rectangle R_m is given by

$$\operatorname{Res} \mathcal{J}_s|_{s=-2l} = \frac{b_0^{2lk}}{(2l)!}\zeta(-2l+1)\zeta(-2kl, \beta_0)\left(\frac{1-2\pi i X\Theta}{X}\right)^{2l}. \tag{3.25}$$

Recall from the discussion above that $\zeta(-2kl, \beta_0)$ equals zero for $\beta_0 = 1/2, 1$. Therefore, for these values of β_0, the expression on the right side of (3.25) has the value zero. With this in mind, and using (3.24)–(3.25) in (3.21), we obtain the desired result. □

Lemma 3.1.2. *For any two natural numbers q and l with $(q, l) = 1$, define*

$$S(k; q, l) := \sum_{m=1}^{q} e(l^k m/q).$$

Suppose that $X, \Theta \in \mathbb{R}, X > 1, u \in \mathbb{Z}, q \in \mathbb{N}, (u, q) = 1$ and $\theta = \Theta - u/q$. Then, for any $\epsilon > 0$, and $\Phi(z; A)$ defined in (3.4), as $X \to \infty$,

$$\Phi(\rho e(\Theta); A) = \frac{1}{b_0}\,\Gamma(1+1/k)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k}\sum_{j=1}^{\infty}\frac{S(k; q_j, u_j)}{j^{1+1/k}q_j}$$
$$+ O_\epsilon\left(q^{1/2+\epsilon}(1+|\theta|^{1/2}X^{1/2})\log X\right),$$

where $q_j = q/(q, j)$, $u_j = uj/(q, j)$.

Proof. Recall the definition of $\Phi(z; A)$ given in (3.4),

$$\Phi(\rho e(\Theta); A) = \sum_{j=1}^{\infty}\frac{1}{j}\sum_{\substack{n=1\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}}^{\infty} e^{-jn^k/X}e(jn^k\Theta).$$

Employing

$$e^{-jn^k/X} = \int_n^{\infty} kx^{k-1}jX^{-1}e^{-jx^k/X}\,dx \tag{3.26}$$

in the above sum, we obtain

$$\Phi(\rho e(\Theta); A) = \sum_{j=1}^{\infty} \frac{1}{j} \int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X} \sum_{\substack{n\leqslant x \\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\ dx. \tag{3.27}$$

Using trivial bounds, integrating by parts, and lastly, making the substitution $y = jx^k/X$, we obtain

$$\begin{aligned} &\int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X} \sum_{\substack{n\leqslant x \\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\ dx \\ &\ll \int_0^{\infty} x(kx^{k-1} jX^{-1} e^{-jx^k/X})\ dx \\ &= \int_0^{\infty} e^{-jx^k/X}\ dx = \left(\frac{X}{j}\right)^{1/k} \int_0^{\infty} e^{-y^k}\ dx \ll \left(\frac{X}{j}\right)^{1/k}. \end{aligned}$$

Therefore, invoking the estimates above in (3.27), for a fixed positive integer N, we find that

$$\begin{aligned} \sum_{j=N+1}^{\infty} \frac{1}{j} \int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X} &\sum_{\substack{n\leqslant x \\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\ dx \\ &\ll \sum_{j=N+1}^{\infty} \frac{1}{j} \left(\frac{X}{j}\right)^{1/k} \ll \left(\frac{X}{N}\right)^{1/k}. \end{aligned}$$

Using this in (3.27), we have

$$\Phi(\rho e(\Theta); A) = \Sigma_N + \mathrm{O}\left(\left(\frac{X}{N}\right)^{1/k}\right), \tag{3.28}$$

where

$$\Sigma_N := \sum_{j=1}^{N} \frac{1}{j} \int_0^{\infty} kx^{k-1} jX^{-1} e^{-jx^k/X} \sum_{\substack{n\leqslant x \\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\ dx. \tag{3.29}$$

By a variation of Theorem 4.1 [60, p. 43], (which can be justified using the Euler-Maclaurin summation formula and standard techniques), we can write, for any real number $\epsilon > 0$,

$$\sum_{\substack{n \leqslant x \\ n \equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k \Theta) = \frac{S(k; q_j, u_j)}{b_0 q_j} \int_0^x e(j\gamma^k \theta)\ d\gamma + \mathrm{O}_\epsilon \left(q_j^{1/2+\epsilon} (1 + x^k j |\theta|)^{1/2} \right).$$

Employing the above estimate in (3.29), and applying (3.26) after interchanging the order of integration below, we obtain

$$\begin{aligned} \Sigma_N &= \frac{1}{b_0} \sum_{j=1}^{N} \frac{S(k; q_j, u_j)}{j q_j} \int_0^\infty k x^{k-1} j X^{-1} e^{-jx^k/X} \int_0^x e(j\gamma^k \theta)\ d\gamma\ dx + \mathrm{O}_\epsilon \left(E_N(X) \right) \\ &= \frac{1}{b_0} \sum_{j=1}^{N} \frac{S(k; q_j, u_j)}{j q_j} \int_0^\infty e^{-j\gamma^k/X} e(j\gamma^k \theta)\ d\gamma + \mathrm{O}_\epsilon \left(E_N(X) \right), \end{aligned} \tag{3.30}$$

with

$$\begin{aligned} E_N(X) &\coloneqq \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} \int_0^\infty k x^{k-1} j X^{-1} e^{-jx^k/X} (1 + x^k j |\theta|)^{1/2}\ dx \\ &\ll \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} \left(1 + \int_0^\infty \frac{j|\theta| k x^{k-1}}{2\sqrt{x^k j |\theta|}} \frac{e^{-x^k j/X}}{\sqrt{1 + 1/(x^k j |\theta|)}}\ dx \right) \\ &\ll \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j} \left(1 + \frac{k}{2} \sqrt{j|\theta|} \int_0^\infty x^{k/2-1} e^{-x^k j/X}\ dx \right), \end{aligned} \tag{3.31}$$

where in the second step, we have integrated by parts. Using the substitution $y = jx^k/X$ in the integral above, we deduce that

$$\begin{aligned} \int_0^\infty x^{k/2-1} e^{-x^k j/X}\ dx &= (X/j)^{1/k} k^{-1} \int_0^\infty (yX/j)^{1/2-1/k} e^{-y} y^{1/k-1}\ dy \\ &= \frac{1}{k} \left(\frac{X}{j} \right)^{1/2} \int_0^\infty y^{-1/2} e^{-y}\ dy = \frac{1}{k} \sqrt{\frac{\pi X}{j}}. \end{aligned}$$

Using this in (3.31), we see that

$$E_N(X) \ll \sum_{j=1}^{N} \frac{q_j^{1/2+\epsilon}}{j}(1+\sqrt{\pi X|\theta|/4})$$
$$\ll q^{1/2+\epsilon}(1+\sqrt{|\theta|X})\log N. \tag{3.32}$$

We now turn our attention to the main term of the expression on the far right side of (3.30). First, we rewrite the integrand there as

$$e^{-j\gamma^k/X}e(j\gamma^k\theta) = \exp(-j\gamma^k X^{-1}(1-2\pi iX\theta)),$$

and set

$$z = (j\gamma^k X^{-1}|1-2\pi iX\theta|e^{i\phi})^{1/k},$$

where ϕ is the argument of $1-2\pi iX\theta$, and $|\phi| \leqslant \pi/2$. This gives

$$\int_0^\infty e^{-j\gamma^k/X}e(j\gamma^k\theta))\;d\gamma = \int_0^\infty \exp(-j\gamma^k X^{-1}e(j\gamma^k X\theta))\;d\gamma$$
$$= \left(\frac{X}{j(1-2\pi iX\theta)}\right)^{1/k}\int_{\mathcal{L}} e^{-z^k}\;dz, \tag{3.33}$$

where $\mathcal{L}$ is the ray $\{z = ue^{i\phi/k} : 0 \leqslant u < \infty\}$. By Cauchy's theorem, the integral along $\mathcal{L}$ is given by

$$\int_{\mathcal{L}} e^{-z^k}\;dz = \int_0^\infty e^{-u^k}\;du$$
$$= \frac{1}{k}\int_0^\infty t^{\frac{1}{k}-1}e^{-t}\;dt = \frac{1}{k}\Gamma\left(\frac{1}{k}\right).$$

Combining this evaluation along with (3.33), (3.30), and (3.32), we obtain

$$\Sigma_N = \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k}\sum_{j=1}^{N}\frac{S(k;q_j,u_j)}{j^{1+1/k}q_j} + \mathrm{O}_\epsilon\left(q^{1/2+\epsilon}(1+\sqrt{|\theta|X})\log N + \left(\frac{X}{N}\right)^{1/k}\right). \tag{3.34}$$

Since $|S(k;q_j,u_j)| \leqslant q_j$, for each j, we have

$$\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k}\sum_{j=N}^{\infty}\frac{S(k;q_j,u_j)}{j^{1+1/k}q_j} \ll \left(\frac{X}{N}\right)^{1/k}.$$

Using this in (3.34), we conclude that

$$\Sigma_N = \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\theta}\right)^{1/k}\sum_{j=1}^{\infty}\frac{S(k;q_j,u_j)}{j^{1+1/k}q_j} + \mathrm{O}_\epsilon\left(q^{1/2+\epsilon}(1+\sqrt{|\theta|X})\log N + \left(\frac{X}{N}\right)^{1/k}\right).$$

Setting $N = \lfloor X \rfloor$ in the above expression and invoking (3.28), we obtain the desired result.

□

To obtain an upper bound for the contribution from the minor arcs, we first prove the following lemma.

Lemma 3.1.3. *For X defined in* (1.4)*, and $\mathfrak{m}$ in* (3.38)*, let $\rho = e^{-1/X}$, $\Theta \in \mathfrak{m}$. Then for $\Phi(z;A)$, defined in* (3.4)*,*

$$\Phi(\rho e(\Theta);A) \ll_\epsilon X^{1/k-2^{1-k}/k+\epsilon}.$$

Proof. Let K be a positive integer. As in the proof of Lemma 3.1.2, we have

$$\begin{aligned}\Phi(\rho e(\Theta)) &= \sum_{j=1}^{K} \frac{1}{j} \int_0^\infty kx^{k-1} jX^{-1} e^{-x^k j/X} \sum_{\substack{n\leqslant x\\ n\equiv a_0 \,(\mathrm{mod}\, b_0)}} e(jn^k\Theta)\, dx + \mathrm{O}\left((X/K)^{1/k}\right)\\ &= \sum_{j=1}^{K} \frac{1}{j} \int_0^\infty kx^{k-1} jX^{-1} e^{-x^k j/X} \sum_{m=0}^{m=\lfloor (x-a_0)/b_0\rfloor} e(j\Theta(a_0+b_0m)^k)\, dx\\ &\quad + \mathrm{O}\left((X/K)^{1/k}\right). \end{aligned} \tag{3.35}$$

For each j, we use Dirichlet's approximation theorem to choose $u_j \in \mathbb{Z}_{\geqslant 0}, q_j \in \mathbb{N}$, so that

$$\left| jb_0^k m^k \Theta - \frac{u_j}{q_j} \right| \leqslant q_j^{-1} X^{1/k-1}, \text{ and } q_j \leqslant X^{1-1/k}.$$

By Weyl's inequality [60, Lemma 2.5],

$$\sum_{m=0}^{m=\lfloor (x-a_0)/b_0\rfloor} e(j\Theta(a_0+b_0m)^k) \ll_\epsilon x^{1+\epsilon-2^{-(k+1)}} + x^{1+\epsilon} q_j^{-2^{-(k-1)}} + x^{1+\epsilon}(q_j/x^k)^{2^{-(k-1)}}.$$

Note that for any $\lambda > 0$, an integration by parts gives

$$\int_0^\infty x^\lambda (jkx^{k-1}X^{-1}e^{-x^k j/X})\, dx \ll \left(\frac{X}{j}\right)^{\lambda/k}. \tag{3.36}$$

Also, since $\Theta \notin \mathfrak{M}$, we have $jb_0^k m^k q_j > X^{1/k}$. Furthermore, recall that $q_j \leqslant X^{1-1/k}$. Invoking

(3.36), and using these bounds for q_j in (3.35), we conclude that

$$\begin{aligned}
&\sum_{j=1}^{K} \frac{1}{j} \int_1^\infty k x^{k-1} j X^{-1} e^{-x^k j/X} \sum_{m=0}^{m=\lfloor (x-a_0)/b_0 \rfloor} e(j\Theta(a_0+b_0 m)^k)\ dx \\
&\quad \ll_e \sum_{j=1}^{K} \frac{1}{j} \left(\left(\frac{X}{j}\right)^{\frac{1+\epsilon}{k}-\frac{1}{k2^{k-1}}} + \left(\frac{X}{j}\right)^{\frac{1+\epsilon}{k}} q_j^{-2^{-(k-1)}} + \left(\frac{X}{j}\right)^{\frac{1+\epsilon}{k}-\frac{1}{2^{k-1}}} q_j^{2^{-(k-1)}} \right) \\
&\quad \ll_e X^{\frac{1+\epsilon}{k}-\frac{1}{k2^{k-1}}} \sum_{j=1}^{K} \left(j^{-1-\frac{1+\epsilon}{k}+\frac{1}{k2^{k-1}}} + j^{-1-\frac{1+\epsilon}{k}+\frac{1}{k2^{k-1}}} \right) + \left(\frac{X}{K}\right)^{1/k} \\
&\quad \ll_e X^{\frac{1}{k}+\epsilon-\frac{1}{k2^{k-1}}} + \left(\frac{X}{K}\right)^{1/k}.
\end{aligned}$$

Letting K approach infinity, we obtain the desired bounds. □

3.2 Proof of Theorem 1.1.2

In this section, we give a proof of Theorem 1.1.2. The proof relies on the Hardy-Littlewood circle method. First, we write the function $p_A(n)$ as an integral, i.e., by (3.2), (3.3), and Cauchy's theorem,

$$\begin{aligned}
p_A(n) &= \int_0^1 \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\ d\Theta \\
&= \int_{\mathcal{U}} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\ d\Theta, \qquad (3.37)
\end{aligned}$$

where in the last step, using the periodicity of the integrand, we have replaced the unit interval $(0,1]$ by the unit interval $\mathcal{U} = (-X^{-1+1/k}, 1 - X^{-1+1/k}]$, with X as in (1.4). Now, we define the major and the minor arcs. For $u, q \in \mathbb{N}$ with $(u,q)=1$, define the major arcs by

$$\mathfrak{M}(q,u) = \{\Theta \in \mathcal{U} : |\Theta - u/q| \leqslant q^{-1} X^{1/k-1}\},$$

and let

$$\mathfrak{M} = \cup_{1 \leqslant u \leqslant q \leqslant X^{1/k}} \mathfrak{M}(q, u).$$

The minor arcs $\mathfrak{m}$ are defined to be the complement of the major arcs in the interval $\mathcal{U}$, i.e.,

$$\mathfrak{m} = \mathcal{U} \setminus \mathfrak{M}. \tag{3.38}$$

3.2.1 Main contribution from part of a major arc

First, we compute the integral in (3.37) over the sub-interval $[-3/(8\pi X), 3/(8\pi X)]$, a portion of the major arc $\mathfrak{M}(1, 0)$, i.e., we consider

$$\int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n \Theta) \, d\Theta. \tag{3.39}$$

By Lemma 3.1.1,

$$\rho^{-n} \exp(\Phi(\rho e(\Theta); A)) = \rho^{-n} \exp(\widetilde{\Xi}(\rho e(\Theta); A)) \left(1 + \mathrm{O}_{k,a_0,b_0}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1} X)^{1/k}}{2k}\right)\right)\right), \tag{3.40}$$

where

$$\begin{aligned}\widetilde{\Xi}(\rho e(\Theta); A) &= \frac{1}{b_0 k}\zeta(1+1/k)\Gamma(1/k)\left(\frac{X}{1-2\pi i X\Theta}\right)^{1/k} + \zeta(0,\beta_0)\log\left(\frac{b_0^{-k}X}{1-2\pi i X\Theta}\right)\\ &\quad + k\zeta'(0,\beta_0) + \frac{b_0^k}{2}\zeta(-k,\beta_0)\left(\frac{1-2\pi i X\Theta}{X}\right) + \sum_{m=1}^{\lfloor R/2\rfloor}\frac{b_0^{2mk}}{(2m)!}\zeta(-2m+1)\\ &\quad \times \zeta(-2km,\beta_0)\left(\frac{1-2\pi i X\Theta}{X}\right)^{2m}\\ &= \frac{1}{b_0 k}\zeta(1+1/k)\Gamma(1/k)\left(\frac{X}{1-2\pi i X\Theta}\right)^{1/k} + \zeta(0,\beta_0)\log\left(\frac{b_0^{-k}X}{1-2\pi i X\Theta}\right)\\ &\quad + k\zeta'(0,\beta_0) + \frac{b_0^k}{2}\zeta(-k,\beta_0)\left(\frac{1-2\pi i X\Theta}{X}\right) + \sum_{m=1}^{M-1}\frac{b_0^{2mk}}{(2m)!}\zeta(-2m+1)\\ &\quad \times \zeta(-2km,\beta_0)\left(\frac{1-2\pi i X\Theta}{X}\right)^{2m} + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X^{2M-1}}\right)\\ &=: \Xi(\rho e(\Theta); A) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X^{2M-1}}\right), \end{aligned} \tag{3.41}$$

with

$$R = \frac{(2(\pi\Delta)^{k+1}X)^{1/k}}{2k^2}, \quad \Delta = (1+4\pi^3\Theta^2X^2)^{-1/2},$$

and a fixed positive integer M satisfying $M \leqslant R/2$. This can be seen by combining the fact that $\Delta \geqslant 4/5$ for $\Theta \in [-3/(8\pi X), 3/(8\pi X)]$, and that $M \leqslant (2(4\pi/5)^{k+1}X)^{1/k}/(4k^2)$ as per the remark following the statement of Theorem 1.1.2. Also, from Lemma 3.1.1, note that the terms (including the error term) in (3.41) involving M disappear when β_0 equals 1/2 or 1.

Thus, using (3.41), we can rewrite $\exp(\Phi(\rho e(\Theta); A))$ in (3.40) as

$$\begin{aligned}\rho^{-n}\exp(\Phi(\rho e(\Theta); A)) = \rho^{-n}\exp(\Xi(\rho e(\Theta); A))\Bigg(1 + \mathrm{O}\left(\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\right)\\ + \mathrm{O}_{k,a_0,b_0}\left(1/X^{2M-1}\right)\Bigg),\end{aligned} \tag{3.42}$$

Also,

$$\frac{X}{1-2\pi iX\Theta}=X\Delta e^{i\phi},$$

where $\phi=\arg(1+2\pi iX\Theta)$. Note that $0<|\phi|\leqslant\pi/2$, so $0<\cos(\phi/k)<1$. Hence,

$$\left|\left(\frac{X}{1-2\pi iX\Theta}\right)^{1/k}\right|=(X\Delta)^{1/k}. \tag{3.43}$$

Therefore, for the first error term in (3.42), we note that, by (3.41) and (3.43),

$$\begin{aligned}
&\rho^{-n}\exp(\Xi(\rho e(\Theta);A))\exp\left(-\frac{(2(4\pi/5)^{k+1}X)^{1/k}}{2k}\right)\\
&=X^{\zeta(0,\beta_0)}\exp\left(\frac{n}{X}+\frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)(X\Delta)^{1/k}+\frac{b_0^k}{2}\zeta(-k,\beta_0)(X\Delta)^{-1}\right.\\
&\quad\left.+\sum_{m=1}^{M}\frac{b_0^{2mk}}{(2m)!}\zeta(-2m+1)\zeta(-2km,\beta_0)(X\Delta)^{-2m}-\frac{1}{2k}(2(4\pi/5)^{k+1}X)^{1/k}\right)\\
&\ll X^{\zeta(0,\beta_0)}\exp\left(\frac{n}{X}+\frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k}-\delta X^{1/k}\right),
\end{aligned} \tag{3.44}$$

where $\delta:=\frac{1}{2k}(2(4\pi/5)^{k+1})^{1/k}>0$. Similarly, for the second error term in (3.42), we have

$$\rho^{-n}\exp(\Xi(\rho e(\Theta);A))X^{-2M+1}\ll_{k,a_0,b_0}X^{-2M+1}\exp\left(\frac{n}{X}+\frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k}\right). \tag{3.45}$$

Therefore, by (3.44), (3.45), (3.39), and (3.42), we deduce that

$$\begin{aligned}
&\int_{-3/(8\pi X)}^{3/(8\pi X)}\rho^{-n}\exp(\Phi(\rho e(\Theta);A))-2\pi in\Theta)\,d\Theta\\
&\qquad=\rho^{-n}\int_{-3/(8\pi X)}^{3/(8\pi X)}\exp(\Xi(\rho e(\Theta)))-2\pi in\Theta)\,d\Theta\\
&\qquad\quad+\mathrm{O}_{k,a_0,b_0}\left(X^{\zeta(0,\beta_0)}\exp\left(\frac{n}{X}+\frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k}-\delta X^{1/k}\right)\right)\\
&\qquad\quad+\mathrm{O}_{k,a_0,b_0}\left(X^{-2M+1}\exp\left(\frac{n}{X}+\frac{1}{b_0k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k}\right)\right).
\end{aligned} \tag{3.46}$$

We now turn our attention to the main term in (3.46). Since $|\Theta| < 1/(2\pi X)$, we can rewrite the expression

$$\Xi(\rho e(\Theta); A) - 2\pi i n\Theta$$

in the integrand as a power series in Θ by expanding the terms in (3.41) using the binomial formula and the Taylor series expansion for the logarithm. Using the definition of X in (1.4), we note that the coefficient of Θ in this power series is equal to zero. Hence, with Y defined in (1.5), the main term in (3.46) is given by

$$\begin{aligned}&\rho^{-n}\int_{-3/(8\pi X)}^{3/(8\pi X)} \exp(\Xi(\rho e(\Theta); A)) - 2\pi i n\Theta)\; d\Theta\\&= \rho^{-n}e^{\mathcal{C}}\int_{-3/(8\pi X)}^{3/(8\pi X)} \exp(-Y(2\pi X\Theta)^2 + G(\Theta))\; d\Theta\\&=: \mathcal{I},\end{aligned} \tag{3.47}$$

where

$$\begin{aligned}\mathcal{C} &:= \frac{1}{b_0 k}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)X^{1/k} + \zeta(0,\beta_0)\log(b_0^{-k}X) + k\zeta'(0,\beta_0) + \frac{b_0^k}{2}\zeta(-k,\beta_0)X^{-1}\\&\quad+\sum_{m=1}^{M}\frac{b_0^{2km}}{(2m)!}\zeta(-2m+1)\zeta(-2km,\beta_0)X^{-2m},\end{aligned} \tag{3.48}$$

and

$$G(\Theta) := \sum_{j=3}^{\infty}(a_jY + b_j)(2\pi i X\Theta)^j,$$

with

$$\begin{aligned}a_j &:= \binom{j-1+1/k}{j}\binom{1+1/k}{2}^{-1},\\b_j &:= \zeta(0,\beta_0)\left(\frac{1}{j} - \frac{a_j}{2}\right) + \sum_{m=1}^{M}\frac{b_0^{2mk}\zeta(-2m+1)\zeta(-2mk,\beta_0)}{(2m)!X^{2m}}\left(\binom{2m}{j} - a_j\binom{2m}{2}\right).\end{aligned}$$

Note that since X is large, and M is a fixed positive integer,

$$b_j = \zeta(0,\beta_0)\left(\frac{1}{j} - \frac{a_j}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right).$$

Thus,

$$\frac{b_j}{a_j} = \zeta(0,\beta_0)\left(\frac{1}{ja_j} - \frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right).$$

Also, $ja_j \geqslant 1$ for any $j \geqslant 3$, and since $\zeta(0,\beta_0) = 1/2 - \beta_0$ [8, p. 264], we deduce that $|\zeta(0,\beta_0)| \leqslant 1/2$, since $0 < \beta_0 \leqslant 1$. Hence,

$$\left|\zeta(0,\beta_0)\left(\frac{1}{ja_j} - \frac{1}{2}\right)\right| \leqslant \frac{3}{4}.$$

Therefore, for X large, we conclude that $|b_j/a_j| \leqslant 1$, i.e., $|b_j| \leqslant |a_j|$ for all $j \geqslant 3$.

We rewrite the integral on the right side in (3.47) as

$$\begin{aligned}
\rho^n e^{-\mathcal{C}}\mathcal{I} &= \int_{-3/(8\pi X)}^{3/(8\pi X)} \exp(-Y(2\pi X\Theta)^2 + G(\Theta))\, d\Theta \\
&= \int_0^{3/(8\pi X)} (\exp(G(\Theta)) + \exp(G(-\Theta)))\exp(-Y(2\pi X\Theta)^2)\, d\Theta \\
&= 2\int_0^{3/(8\pi X)} \Re\exp(G(\Theta) - Y(2\pi X\Theta)^2)\, d\Theta \\
&= \frac{1}{2\pi X\sqrt{Y}} \int_0^{9Y/16} t^{-1/2}e^{-t}\, \Re\exp(H(t))\, dt, \qquad (3.49)
\end{aligned}$$

where in the last step, we made the substitution $t = Y(2\pi X\Theta)^2$, and where

$$\begin{aligned}
H(t) &:= \sum_{j=3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \\
&= \sum_{j=3}^{2J+2} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} + \sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \\
&=: H_J(t) + \sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2}, \qquad (3.50)
\end{aligned}$$

for any fixed positive integer J. Note that for $j \geqslant 2$,

$$|b_{2j}| \leqslant a_{2j} \leqslant a_4 = \frac{6k^2+5k+1}{12k^2}.$$

Therefore, for $0 \leqslant t \leqslant 9Y/16$, where Y is sufficiently large, and $k \geqslant 2$,

$$\begin{aligned}
\Re H_J(t) &= \Re \sum_{j=3}^{2J+2} i^j (a_j + b_j Y^{-1}) t^{j/2} Y^{1-j/2} = \sum_{j=2}^{J+1} (-1)^j (a_{2j} Y + b_{2j}) t^j Y^{-j} \\
&\leqslant a_4 (Y+1) \sum_{j=2}^{2J+2} \left(\frac{t}{Y}\right)^j \leqslant a_4 (Y+1) \sum_{j=2}^{\infty} \left(\frac{t}{Y}\right)^j \\
&= \frac{6k^2+5k+1}{12k^2} (Y+1) \frac{(t/Y)^2}{1-t/Y} \\
&\leqslant \frac{9}{7} \times \frac{6k^2+5k+1}{12k^2} (1+Y^{-1}) t = \frac{18k^2+15k+3}{28k^2} (1+Y^{-1}) t \\
&= \left(\frac{18}{28} + \frac{15}{28k} + \frac{3}{28k^2}\right) (1+Y^{-1}) t < \left(\frac{18}{28} + \frac{15}{56} + \frac{3}{112}\right) (1+Y^{-1}) t \\
&= \frac{105}{112} (1+Y^{-1}) t < \frac{105}{112} \left(1 + \frac{1}{105}\right) t < \left(1 - \frac{1}{2016}\right) t,
\end{aligned} \tag{3.51}$$

where now we assume, at least, that $Y > 105$. Note that by the definition of $H_J(t)$ in (3.50), as we let $J \to \infty$, $H_J(t)$ approaches $H(t)$. Thus, for a fixed positive real number $Z < 9Y/16$, by (3.51),

$$\begin{aligned}
\int_Z^{9Y/16} t^{-1/2} e^{-t} \Re \exp(H(t))\, dt &\leqslant \int_Z^{9Y/16} t^{-1/2} e^{-t} e^{\Re H(t)}\, dt \\
&\leqslant \int_Z^{9Y/16} t^{-1/2} e^{-t} \exp\left(t - \frac{t}{2016}\right)\, dt \\
&\ll Z^{-1/2} \int_Z^{9Y/16} e^{-t/(2016)}\, dt \ll Z^{-1/2} e^{-Z/(2016)}.
\end{aligned}$$

We let $Z = 2016 J \log Y$ in the above estimates to obtain

$$\int_Z^{9Y/16} t^{-1/2} e^{-t}\, \Re \exp(H(t))\, dt \ll Y^{-J}.$$

This, combined with (3.49) and (3.50), gives

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\ \Re\exp(H(t))\ dt + \mathrm{O}\left(Y^{-J}\right)$$
$$= \int_0^Z t^{-1/2}e^{-t}\ \Re\exp\left(H_J(t) + \sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2}\right)\ dt$$
$$+ \mathrm{O}\left(Y^{-J}\right). \tag{3.52}$$

For $0 \leqslant t \leqslant Z$,

$$\sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \ll \frac{t^{J+3/2}Y^{-1/2-J}}{1-(t/Y)^{1/2}} \ll t^{J+3/2}Y^{-1/2-J}.$$

Therefore,

$$\exp\left(\sum_{j=2J+3}^{\infty} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2}\right) = 1 + \mathrm{O}\left(t^{J+3/2}Y^{-1/2-J}\right).$$

Employing this in (3.52), we deduce that

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\Re\exp(H_J(t))\big(1 + \mathrm{O}\left(t^{J+3/2}Y^{-1/2-J}\right)\big)\ dt. \tag{3.53}$$

From (3.51), we see that

$$\Re H_J(t) < \left(1 - \frac{1}{2016k^2}\right) t.$$

So, for the error term in (3.53), we find that

$$\int_0^Z t^{J+1}e^{-t}\ \Re\exp(H_J(t))Y^{-1/2-J}\ dt \ll Y^{-1/2-J}\int_0^\infty e^{-t/(2016k^2)}t^{J+1}\ dt \ll Y^{-J}.$$

Using this in (3.53), we find that

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\ \Re\exp(H_J(t))\ dt + \mathrm{O}\left(Y^{-J}\right)$$
$$= \int_0^Z t^{-1/2}e^{-t}\ \Re\sum_{j=0}^{\infty}\frac{H_J(t)^j}{j!}\ dt + \mathrm{O}\left(Y^{-J}\right). \tag{3.54}$$

Next, for $0 \leqslant t \leqslant Z = 2016J\log Y$,

$$H_J(t) = \sum_{j=3}^{2J+2} i^j(a_j + b_jY^{-1})t^{j/2}Y^{1-j/2} \ll \sum_{j=3}^{\infty} Y(t/Y)^{j/2} \ll Y^{-1/2}t^{3/2} \leqslant Y^{-1/4}.$$

Therefore,

$$\left|\int_0^Z t^{-1/2}e^{-t}\ \Re(H_J(t)^j)\ dt\right| \leqslant Y^{-j/4}.$$

This yields

$$\sum_{j=4J+4}^{\infty}\frac{1}{j!}\int_0^Z t^{-1/2}e^{-t}\Re(H_J(t)^j)\ dt \ll Y^{-J}.$$

Using this in (3.54), we obtain

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \int_0^Z t^{-1/2}e^{-t}\ \Re\sum_{j=0}^{4J+3}\frac{1}{j!}H_J(t)^j\ dt + \mathrm{O}\left(Y^{-J}\right). \tag{3.55}$$

Recall from (3.50) that

$$H_J(t) = \sum_{l=3}^{2J+2}(a_l + b_lY^{-1})Y^{1-l/2}(it^{1/2})^l = \sum_{l=3}^{2J+2}\left(a_l(Y^{-1/2})^{l-2} + b_l(Y^{-1/2})^l\right)(it^{1/2})^l. \tag{3.56}$$

So, for a fixed Y, $H_J(t)$ can be viewed as a polynomial in $it^{1/2}$ of degree $2J+2$ with real coefficients. Therefore,

$$\sum_{j=0}^{4J+3}\frac{1}{j!}(H_J(t))^j = \sum_{r=0}^{(2J+2)(4J+3)} f_r(Y^{-1/2})(it^{1/2})^r, \tag{3.57}$$

where $f_r(x)$ is a real polynomial in x of degree not larger than r. Also, from (3.56) it is not hard to verify that

$$f_0(x) = 1,\ f_1(x) = f_2(x) = 0, \quad \text{and} \quad f_r(0) = 0,$$

for $r \geqslant 3$, and the polynomial $f_r(x)$ is even (odd) when r is even (odd). So, $f_r(x)$ is indeed a polynomial in Y^{-1} when r is even. Using these facts and (3.57) in (3.55), and replacing r by $2r$ below, we conclude that

$$\begin{aligned} 2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} &= \int_0^Z t^{-1/2}e^{-t}\ \mathfrak{R}\sum_{r=0}^{(2J+2)(4J+3)} f_r(Y^{-1/2})(it^{1/2})^r\ dt + \mathrm{O}\left(Y^{-J}\right) \\ &= \int_0^Z t^{-1/2}e^{-t}\sum_{r=0}^{(J+1)(4J+3)}(-1)^r f_{2r}(Y^{-1/2})t^r\ dt + \mathrm{O}\left(Y^{-J}\right) \\ &= \sum_{r=0}^{(J+1)(4J+3)}(-1)^r f_{2r}(Y^{-1/2})\int_0^Z t^{r-1/2}e^{-t}\ dt + \mathrm{O}\left(Y^{-J}\right) \\ &= \sum_{r=0}^{(J+1)(4J+3)}\alpha_r Y^{-r}\int_0^Z t^{r-1/2}e^{-t}\ dt + \mathrm{O}\left(Y^{-J}\right), \end{aligned} \tag{3.58}$$

for certain real numbers α_r, with $\alpha_0 = 1$. Note that, since $Z = 2016k^2 J\log Y$,

$$\begin{aligned} \int_0^Z t^{r-1/2}e^{-t}\ dt &= \int_0^\infty t^{r-1/2}e^{-t}\ dt - \int_Z^\infty t^{r-1/2}e^{-t}\ dt \\ &= \Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}\left(e^{-Z/2}\int_Z^\infty t^{r-1/2}e^{-t/2}\ dt\right) \\ &= \Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}\left(e^{-Z/2}\Gamma\left(r+\frac{1}{2}\right)\right) \\ &= \Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}\left(Y^{-J}\right), \end{aligned}$$

since J is fixed. Using this in (3.58), we conclude that

$$2\pi X\sqrt{Y}\rho^n e^{-\mathcal{C}}\mathcal{I} = \sum_{r=0}^{(J+1)(4J+3)} \left(\alpha_r Y^{-r}\left(\Gamma(r+1/2) + \mathrm{O}\left(Y^{-J}\right)\right) + \mathrm{O}\left(Y^{-J}\right)\right)$$
$$= \sum_{r=0}^{J-1} \alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right)$$
$$= \sqrt{\pi} + \sum_{r=1}^{J-1} \alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right), \tag{3.59}$$

as $\alpha_0 = 1$. Since $\rho = e^{-1/X}$, we deduce from (3.59) that

$$\mathcal{I} = \frac{1}{2\pi X\sqrt{Y}} \exp\left(\frac{n}{X} + \mathcal{C}\right)\left(\sqrt{\pi} + \sum_{r=1}^{J-1} \alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(Y^{-J}\right)\right).$$

Therefore, by (3.46) and (3.47),

$$\int_{-3/(8\pi X)}^{3/(8\pi X)} \rho^{-n}\exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta$$
$$= \frac{1}{2\pi X\sqrt{Y}} \exp\left(\frac{n}{X} + \mathcal{C}\right)\left(\sqrt{\pi} + \sum_{r=1}^{J-1} \alpha_r Y^{-r}\Gamma\left(r+\frac{1}{2}\right) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{Y^J}\right)\right.$$
$$\left. + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X^{2M-1}}\right)\right). \tag{3.60}$$

3.2.2 Estimates over remaining major arcs and minor arcs

The remainder of the proof consists of showing that the contributions from the remaining major arcs and the minor arcs are negligible. In other words, it suffices to show that

$$\int_{\mathcal{U}\setminus[-3/(8\pi X), 3/(8\pi X)]} \rho^{-n}\exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta \ll_{k,a_0,b_0} \exp\left(\frac{n}{X} + \mathcal{C}\right)\frac{1}{XY^{J+1/2}}.$$

Suppose that $\Theta \in \mathfrak{M}(1,0) \setminus [-3/(8\pi X), 3/(8\pi X)]$, so that $\Theta > 3/(8\pi X)$. Thus,

$$\Delta = (1 + 4\pi^2\Theta^2 X^2)^{-1/2} \leqslant 4/5.$$

Invoking Lemma 3.1.2 with $q = 1$ and $u = 0$, we see that, for $\Theta \in \mathfrak{M}(1,0)$,

$$\theta = \Theta,\ |\Theta| \leqslant X^{1/k-1},\ q_j = 1, u_j = 0,\ S(k; q_j, u_j) = 1,$$

and

$$\begin{aligned}\Phi(\rho e(\Theta); A) &= \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\Theta}\right)^{1/k}\zeta\left(1+\frac{1}{k}\right) + \mathrm{O}_\epsilon\left((1+X^{1/2}|\theta|^{1/2})\log X)\right) \\ &= \frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)(X\Delta)^{1/k} + \mathrm{O}_\epsilon\left(X^{1/(2k)+\epsilon}\right),\end{aligned}$$

where $\epsilon > 0$. Therefore, for $\rho = e^{-1/X}$,

$$\begin{aligned}\exp(\Phi(\rho e(\Theta); A)) &= \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)(X\Delta)^{1/k}\right)\left(1+\mathrm{O}_\epsilon\left(X^{\frac{1}{2k}+\epsilon}\right)\right) \\ &\ll \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)(X\Delta)^{1/k}\right) \\ &\ll \exp\left(\left(\frac{4}{5}\right)^{1/k}\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)X^{1/k}\right). \end{aligned} \tag{3.61}$$

In other words,

$$\exp(\Phi(\rho e(\Theta); A)) \ll \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right)X^{1/k} - \gamma_k X^{1/k}\right) =: \exp(\mathcal{D}),$$

where

$$\gamma_k := \left(1-(4/5)^{1/k}\right)\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\zeta\left(\frac{k+1}{k}\right) > 0.$$

Now we rewrite $\mathcal{D}$ in terms of $\mathcal{C}$ defined in (3.48) as follows:

$$\begin{aligned}\mathcal{D} &= \mathcal{C} - \gamma_k X^{1/k} - \zeta(0,\beta_0)\log(b_0^{-k}X) - k\zeta'(0,\beta_0) - \frac{b_0^k}{2}\zeta(-k,\beta_0)X^{-1} \\ &\quad - \sum_{m=1}^{M} \frac{b_0^{2km}}{(2m)!}\zeta(-2m+1)\zeta(-2km,\beta_0)X^{-2m} \\ &= \mathcal{C} - \gamma_k X^{1/k} - \zeta(0,\beta_0)\log(b_0^{-k}X) - k\zeta'(0,\beta_0) + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\exp(\mathcal{D}) &= \exp\left(\mathcal{C} - \gamma_k X^{1/k} - \zeta(0,\beta_0)\log(b_0^{-k}X) - k\zeta'(0,\beta_0)\right)\left(1 + \mathrm{O}_{k,a_0,b_0}\left(\frac{1}{X}\right)\right) \\ &\ll X^{\zeta(0,\beta_0)}\exp\left(\mathcal{C} - \gamma_k X^{1/k}\right) \ll X^{-\mathcal{L}}\exp(\mathcal{C}),\end{aligned}$$

for any $\mathcal{L} > 0$, since X is large. Using this in (3.61), we have

$$\exp(\Phi(\rho e(\Theta);A)) \ll X^{-\mathcal{L}}\exp(\mathcal{C}). \tag{3.62}$$

Choose $\mathcal{L}$ large enough so that $X^{-\mathcal{L}} \ll X^{-1}Y^{-J-1/2}$, which is possible since X can be written as a monomial in Y. Therefore, for $\rho = e^{-1/X}$, we can conclude that

$$\int_{\mathfrak{M}(1,0)\setminus[-3/(8\pi X),3/(8\pi X)]} \rho^{-n}\exp(\Phi(\rho e(\Theta);A) - 2\pi i n\Theta)\, d\Theta \ll \frac{\exp\left(\frac{n}{X} + \mathcal{C}\right)}{XY^{J+1/2}}. \tag{3.63}$$

We now investigate the integral on the remaining major arcs. Let $\Theta \in \mathfrak{M}(q,u)$ with $q > 1$. So, $q \leqslant X^{1/k}$, and $\theta := \Theta - u/q$ satisfies $|\theta| \leqslant q^{-1}X^{1/k-1}$. This gives

$$\begin{aligned}q^{1/2+\epsilon}(1 + X^{1/2}|\theta|^{1/2})\log X &\ll q^{1/2+\epsilon}X^{1/2+\epsilon}|\theta|^{1/2} \\ &\ll q^{1/2+\epsilon}X^{1/2+\epsilon}q^{-1/2}X^{1/(2k)-1/2} \\ &\ll q^{\epsilon}X^{1/(2k)+3\epsilon} \ll X^{1/(2k)+3\epsilon}.\end{aligned}$$

Once again, by an application of Lemma 3.1.2, we have

$$\exp\left(\Phi(\rho e(\Theta);A)\right) = \exp\left(\frac{1}{b_0 k}\Gamma\left(\frac{1}{k}\right)\left(\frac{X}{1-2\pi i X\Theta}\right)^{1/k}\sum_{j=1}^{\infty}\frac{S(k;q_j,u_j)}{j^{1+1/k}q_j}\right)$$
$$\times\left(1+\mathrm{O}_\epsilon\left(X^{1/(2k)+\epsilon}\right)\right). \tag{3.64}$$

Recall, from Lemma 3.1.2 the notation

$$S(k;q_j,u_j) = \sum_{l=1}^{q_j} e(u_j^k l/q_j), \quad q_j = q/(q,j), \quad u_j = uj/(q,j).$$

If $q|j$, then we have $q=(q,j)$, i.e., $q_j = 1$ and $S(k;q_j,u_j)=1$. On the other hand, if $q\nmid j$, then $q_j > 1$ and it is not difficult to see ([27, Lemma 1]) that there is a constant $\delta_k > 0$ such that $|S(k;q_j,u_j)| \leqslant (1-\delta_k)q_j$. Thus,

$$\begin{aligned}
\sum_{j=1}^{\infty}\frac{|S(k;q_j,u_j)|}{j^{1+1/k}q_j} &= \sum_{\substack{j=1\\ q|j}}^{\infty}\frac{|S(k;q_j,u_j)|}{j^{1+1/k}q_j} + \sum_{\substack{j=1\\ q\nmid j}}^{\infty}\frac{|S(k;q_j,u_j)|}{j^{1+1/k}q_j}\\
&\leqslant \sum_{\substack{j=1\\ q|j}}^{\infty}\frac{1-\delta_k}{j^{1+1/k}} + \sum_{\substack{j=1\\ q\nmid j}}^{\infty}\frac{1}{j^{1+1/k}}\\
&= (1-\delta_k)(1-q^{-(k+1)/k})\zeta(1+1/k) + q^{-(k+1)/k}\zeta(1+1/k)\\
&= (1-\delta_k+\delta_k q^{-(k+1)/k})\zeta(1+1/k)\\
&< (1-\delta_k/2)\zeta(1+1/k),
\end{aligned}$$

where in the third step, we have used the fact that $\sum_{q|j} j^{-\alpha} = q^{-\alpha}\zeta(\alpha)$, $\alpha > 1$. Employing this in (3.64), we have

$$\begin{aligned}
\exp\left(\Phi(\rho e(\Theta);A)\right) &\ll \exp\left(\frac{1}{b_0 k}(1-\delta_k/2)\zeta(1+1/k)\Gamma(1/k)X^{1/k}\right)\\
&\ll \exp\left(\mathcal{C}\right) X^{-1}Y^{-J-1/2},
\end{aligned}$$

which can be justified using the arguments as in (3.61) leading up to (3.62). Let $\tilde{\mathfrak{M}} = \mathfrak{M} \setminus \mathfrak{M}(1,0)$. Then, the bounds above imply that for $\rho = e^{-1/X}$,

$$\int_{\tilde{\mathfrak{M}}} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta) d\Theta \ll \frac{\exp\left(\frac{n}{X} + \mathcal{C}\right)}{XY^{J+1/2}}. \tag{3.65}$$

For $\Theta \in \mathfrak{m}$, by Lemma 3.1.3,

$$\Phi(\rho e(\Theta); A) \ll X^{\frac{1}{k} - \frac{1}{2^{2017k}}}.$$

Therefore, for some positive constant $\nu < 1$,

$$\Phi(\rho e(\Theta); A) \leqslant \nu \frac{1}{b_0 k} \Gamma\left(\frac{1}{k}\right) \zeta\left(\frac{k+1}{k}\right) X^{1/k}.$$

Using the argument in (3.61) leading to (3.62), we conclude that

$$\int_{\mathfrak{m}} \rho^{-n} \exp(\Phi(\rho e(\Theta); A) - 2\pi i n\Theta)\, d\Theta \ll \exp\left(\frac{n}{X} + \mathcal{C}\right) X^{-1} Y^{-J-1/2}. \tag{3.66}$$

Combining (3.37), (3.60), (3.63), (3.65), and (3.66), we deduce that

$$p_A(n) = \frac{1}{2\pi X\sqrt{Y}} \exp\left(\frac{n}{X} + \mathcal{C}\right) \left(\sqrt{\pi} + \sum_{r=1}^{J-1} \Gamma(r+1/2) \frac{\alpha_r}{Y^r} + \mathrm{O}\left(\frac{1}{Y^J}\right) + \mathrm{O}\left(\frac{1}{X^{2M-1}}\right)\right).$$

Substituting the values of $\mathcal{C}$ and n/X from (3.48) and (1.4), respectively, in the foregoing expression, we obtain the desired bounds for $p_A(n)$. The remark about the disappearance of the terms involving M follows from our earlier discussion after (3.41). This completes the proof of Theorem 1.1.2.

Chapter 4

Zeros of normalized combinations of the Riemann ξ-function on the critical line

In this chapter, we prove Theorem 1.2.1 about the proportion of the zeros on the critical line for the function $G_{\vec{c},a,T}(s)$. Recall the definition

$$G_{\vec{c},a,T}(s) := \sum_{j=0}^{M} \frac{c_j(-1)^j}{L^{2j}} \xi^{(a+2j)}(s).$$

Observe that the zeros of $G_{\vec{c},a,T}(s)$ are the same as the zeros of the function

$$F_{\vec{c},a,T}(s) := (i/L)^a G_{\vec{c},a,T}(s),$$

where $L := \log(T/2\pi)$. We use the above function $F_{\vec{c},a,T}(s)$ to perform all our computations. Let $N_{\vec{c},a}(T)$ and $N^{(0)}_{\vec{c},a}(T)$ be defined by

$$N_{\vec{c},a}(T) := \{\rho = \beta + i\gamma : F_{\vec{c},a,T}(\rho) = 0 \text{ and } 0 < \gamma \leq T\} \tag{4.1}$$

and

$$N^{(0)}_{\vec{c},a}(T) := \{\rho = 1/2 + i\gamma : F_{\vec{c},a,T}(\rho) = 0 \text{ and } 0 < \gamma \leq T\}. \tag{4.2}$$

4.1 Zero free region and $N_{\vec{c},a}(T)$

Lemma 4.1.1. *The function $F_{\vec{c},a,T}(s)$ satisfies the functional equation given below*

$$F_{\vec{c},a,T}(s) = (-1)^a F_{\vec{c},a,T}(1-s). \tag{4.3}$$

Moreover, if all the zeros of $F_{\vec{c},a,T}(s)$ satisfy $\sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2$ for some $\sigma_1, \sigma_2 \in \mathbb{R}$, then so do the zeros of all its higher order derivatives. In particular, if $F_{\vec{c},a,T}(s)$ satisfies the Riemann Hypothesis, then the same is true for all its higher order derivatives.

Proof. For any non-negative integer k, using the functional equation for $\xi(s) = \xi(1-s)$ for $\xi(s)$, and differentiating it k times, we obtain

$$\xi^{(k)}(s) = (-1)^k \xi^{(k)}(1-s).$$

Using this in the definition of $F_{\vec{c},a,T}(s)$, we have

$$\begin{aligned} F_{\vec{c},a,T}(s) &= \left(\frac{i}{L}\right)^a \sum_{j=0}^{M} \frac{c_j(-1)^j}{L^{2j}} \xi^{(a+2j)}(s) \\ &= \left(\frac{-i}{L}\right)^a \sum_{j=0}^{M} \frac{c_j(-1)^j}{L^{2j}} \xi^{(a+2j)}(1-s) \\ &= (-1)^a F_{\vec{c},a,T}(1-s). \end{aligned}$$

This yields the functional equation (4.3) for $F_{\vec{c},a,T}(s)$.

We use induction on the order of the derivative of $F_{\vec{c},a,T}(s)$ to prove the statement on zeros. By our assumption, the statement is true for the zeroth derivative. Let us assume it is true for a derivative $k = m$ for some nonnegative integer m. We now show that it also holds for $m+1$. Using the Weierstrass product formula [22] for $F^{(m)}_{\vec{c},a,T}(s)$, which is an entire function of order one, we can write its logarithmic derivative in terms of its zeros $\rho = \beta + i\gamma$

as

$$\frac{F_{\vec{c},a,T}^{(m+1)}(s)}{F_{\vec{c},a,T}^{(m)}(s)} = \sum_{\rho} \frac{1}{s-\rho}.$$

Let $\beta_1 + i\gamma_1$ be a zero of the quotient on the left side of the above equation. For any zero $\rho_1 = \beta_1 + i\gamma_1$ of $F_{\vec{c},a,T}^{(m)}(s)$ with real part less than σ_1, comparing the real parts on both sides of the above equality, we see that

$$0 = \sum_{\rho=\beta+i\gamma} \frac{\beta_1-\beta}{(\beta_1-\beta)^2+(\gamma_1-\gamma)^2} \leqslant (\beta_1-\sigma_1) \sum_{\rho=\beta+i\gamma} \frac{1}{(\beta_1-\beta)^2+(\gamma_1-\gamma)^2} < 0,$$

which is absurd. This implies that $\beta_1 \geqslant \sigma_1$. In a similar way, we can prove that $\beta_1 \leqslant \sigma_2$. Hence, all zeros of $F_{\vec{c},a,T}^{(m+1)}(s)$ have real part between σ_1 and σ_2. This completes the induction argument.

For $\sigma_1 = \sigma_2 = 1/2$, we obtain the particular case of the Riemann Hypothesis. □

Next, we obtain a zero free region for the function $F_{\vec{c},a,T}(s)$ before we find an asymptotic formula for the number of zeros $N_{\vec{c},a}(T)$.

Lemma 4.1.2. *Let $s = \sigma + it$ with $\sigma > 1$ and $T \leqslant t \leqslant T+U$. For some $\beta_{\vec{c}} \gg a$,*

$$F_{\vec{c},a,T}(s) \neq 0 \quad \text{whenever} \quad \sigma_{\vec{c}} > \beta_{\vec{c}} \quad \textit{or} \quad \sigma_{\vec{c}} < 1-\beta_{\vec{c}}.$$

Proof. Writing $\xi(s) = \zeta(s)H(s)$ with

$$H(s) := \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right),$$

we can rewrite $F_{\vec{c},a,T}(s)$ as

$$F_{\vec{c},a,T}(s) = \sum_{j=0}^{M} \frac{c_j i^{a+2j}}{L^{a+2j}} \sum_{l=0}^{a+2j} \binom{a+2j}{l} \zeta^{(a+2j-l)}(s) H^{(l)}(s),$$

Using the relation [49, Lemma 2]

$$\frac{H^{(k)}(s)}{H(s)} = \left(\frac{1}{2}\log\frac{s}{2\pi} + \mathrm{O}\left(\frac{1}{|t|}\right)\right)^k + \mathrm{O}\left(\left|\frac{1}{2}\log\frac{s}{2\pi}\right|^{k-2}|t|^{-1}\right), \tag{4.4}$$

gives

$$\begin{aligned}
F_{\vec{c},a,T}(s) &= H(s)\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{L^{a+2j}}\sum_{l=0}^{a+2j}\binom{a+2j}{l}\zeta^{(a+2j-l)}(s) \\
&\quad\times\left(\left(\frac{1}{2}\log\frac{s}{2\pi}+O\Big(\frac{1}{t}\Big)\right)^l + O\Big(\left|\frac{1}{2}\log\frac{s}{2\pi}\right|^{l-2}\Big)\right) \\
&= H(s)\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{L^{a+2j}}\sum_{l=0}^{a+2j}\binom{a+2j}{l}\zeta^{(a+2j-l)}(s)\left(\frac{1}{2}\log\frac{s}{2\pi}\right)^l\left(1+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right) \\
&= H(s)\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{L^{a+2j}}\left(\frac{1}{2}\log\frac{s}{2\pi}\right)^{a+2j}\sum_{l=0}^{a+2j}\binom{a+2j}{l}\zeta^{(l)}(s)\Big(\frac{1}{2}\log\frac{s}{2\pi}\Big)^{-l} \\
&\quad\times\left(1+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right) \\
&= H(s)\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{L^{a+2j}}\left(\frac{1}{2}\log\frac{s}{2\pi}\right)^{a+2j}\big(1+R_{a+2j,\vec{c}}(s)\big) \\
&= H(s)\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\left(1+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right)(1+R_{a+2j,\vec{c}}(s)) \\
&= H(s)\frac{i^a}{2^a}\sum_{j=0}^{M}\frac{c_j(-1)^j}{4^j}\left(1+R_{a+2j,\vec{c}}(s)+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right) \\
&= H(s)\frac{i^a}{2^a}c^*\left(1+\frac{1}{c^*}\sum_{j=0}^{M}\frac{c_j(-1)^j}{4^j}R_{a+2j,\vec{c}}(s)+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right),
\end{aligned} \tag{4.5}$$

where the remainder term $R_{a+2j,\vec{c}}$ is given by

$$R_{a+2j,\vec{c}}(s) = \zeta(s) - 1 + \sum_{l=1}^{a+2j}\binom{a+2j}{l}\left(\frac{1}{2}\log\frac{s}{2\pi}\right)^{-l}\zeta^{(l)}(s)\Big(1+O\Big(\frac{1}{\log^2 t}\Big)\Big)+O\Big(\frac{1}{\log^2 t}\Big).$$

Also note that

$$
\begin{aligned}
|R_{a+2j,\vec{c}}(s)| &\leqslant \sum_{l=0}^{a+2j} \binom{a+2j}{l} \left|\frac{1}{2}\log\frac{s}{2\pi}\right|^{-l} \frac{(\log 2)^l}{2^{\beta_{\vec{c}}-2}} \left(1+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right) + O\left(\frac{1}{\log^2 t}\right) \\
&\leqslant \frac{1}{2^{\beta_{\vec{c}}-2}} \left(1+\frac{2\log 2}{\log(t/2\pi)}\right)^{a+2j} \left(1+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right) + O\left(\frac{1}{\log^2 t}\right) \\
&\leqslant 2^{a+2j-\beta_{\vec{c}}+2} \left(1+\mathrm{O}\left(\frac{1}{\log^2 t}\right)\right) + O\left(\frac{1}{\log^2 t}\right) < \frac{1}{2},
\end{aligned}
$$

for $\beta_{\vec{c}} \gg a$. Using this in (4.5), we conclude that

$$
F_{\vec{c},a,T}(s) \neq 0 \quad \text{for} \quad \mathrm{Re}(s) > \beta_{\vec{c}},
$$

The functional equation (4.3) yields,

$$
F_{\vec{c},a,T}(s) = (-1)^a F_{\vec{c},a,T}(1-s) \neq 0 \quad \text{for} \quad \mathrm{Re}(s) < 1-\beta_{\vec{c}}.
$$

This completes the proof of the lemma. □

In the following lemma, we compute the number of zeros of $F_{\vec{c},a,T}(s)$ with imaginary part between 0 and T for some fixed large real number T. More precisely, we have the following lemma.

Lemma 4.1.3. *For any positive real number T, let $N_{\vec{c},a}(T)$ be defined by* (4.1)*. For large T,*

$$
N_{\vec{c},a}(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O_{\vec{c}}\left(a\log T\right).
$$

Proof. The proof follows from the Hardy-Littlewood lemma using the standard arguments as for the zeros of the Riemann zeta function $\zeta(s)$, and we omit the details. □

We now prove an inequality involving zeros of a certain arithmetic function $V(s)$ used in the proof of Theorem 1.2.1 later.

Lemma 4.1.4. *Let T be large and $L = \log(T/2\pi)$, $U = TL^{-10}$, and $N_{\vec{c},a}(T), N^{(0)}_{\vec{c},a}(T)$ be as defined in* (4.1) *and* (4.2)*, respectively. Then,*

$$N^{(0)}_{\vec{c},a}(T+U) - N^{(0)}_{\vec{c},a}(T) \geq N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T) - 2N + O_{\vec{c}}\left(a \log T\right),$$

where N denotes the number of zeros of $V(s)$ inside the rectangle $\mathcal{R}$ with vertices $1/2 + iT, \mathcal{B} + iT, \mathcal{B} + i(T+U)$ and $1/2 + i(T+U)$, for some $\mathcal{B} > \beta_{\vec{c}}$, of the function $V(s)$ defined by

$$V(s) := \sum_{j=0}^{M} \frac{c_j i^{a+2j}}{L^{a+2j}} V_{a+2j}(s) \quad \textit{and} \quad V_{a+2j}(s) := Q_{a+2j}(s)/H(s), \tag{4.6}$$

where

$$\begin{aligned} Q_k(s) := &\sum_{m=0}^{k} \binom{k}{m} H^{(m)}(s) \int_{0 \swarrow 1} \frac{z^{-s} e^{\pi i z^2}}{2i \sin(\pi z)} (-\log z)^{k-m} \left(1 - \frac{\log z}{L}\right) \, dz \\ &+ \sum_{m=0}^{k} (-1)^m \binom{k}{m} H^{(m)}(1-s) \int_{0 \searrow 1} \frac{z^{s-1} e^{-\pi i z^2}}{2i \sin(\pi z)} (\log z)^{k-m} \frac{\log z}{L} \, dz, \end{aligned}$$

and, as before, $H(s)$ is given by

$$H(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right).$$

The notation $\int_{0 \searrow 1}$ denotes an integral along a line directed from the upper right to lower left which is inclined at an angle of $\pi/4$ to the real axis and intersects it between 0 *and* 1*; see [55] and [58, §2.10].*

Proof. For $T \leq t \leq T+U$ and $0 \leq \sigma \leq A \log\log T$ for some constant A, define

$$P(s) := H(s)V(s)$$

with $V(s)$ and $H(s)$ as defined in the statement of the lemma. Recall that

$$F_{\vec{c},a,T}(s) = \sum_{j=0}^{M} \frac{c_j i^{a+2j}}{L^{a+2j}} \xi^{(a+2j)}(s).$$

From [49, Lemma 8] and differentiating, we can rewrite $F_{\vec{c},a,T}(s)$ as

$$F_{\vec{c},a,T}(s) = P(s) + \overline{P(1-\bar{s})}, \tag{4.7}$$

We now use (4.4) and apply Lemma 9 from [49] to estimate the integrals appearing in the sums in $Q_{a+2j}(s)$. Using (4.7), we find

$$N_{\vec{c},a}^{(0)}(T+U) - N_{\vec{c},a}^{(0)}(T) \geq \frac{1}{\pi}\left(\arg P\left(\frac{1}{2} + i(T+U)\right) - \arg P\left(\frac{1}{2} + iT\right)\right). \tag{4.8}$$

Note that from (4.6), $\arg P(s) = \arg H(s) + \arg V(s)$. Lemmas 2 and 7 from [49] yield

$$\frac{1}{\pi}\left(\arg H\left(\frac{1}{2} + i(T+U)\right) - \arg H\left(\frac{1}{2} + iT\right)\right) = N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T) + O_{\vec{c}}\left(a \log T\right).$$

Combining this along with (4.8), we have

$$\begin{aligned} N_{\vec{c},a}^{(0)}(T+U) - N_{\vec{c},a}^{(0)}(T) \geq N_{\vec{c},a}(T+U) - N_{\vec{c},a}(T) + \frac{1}{\pi} \arg V\left(\frac{1}{2} + i(T+U)\right) \\ - \frac{1}{\pi} \arg V\left(\frac{1}{2} + iT\right) + O_{\vec{c}}\left(a \log T\right). \end{aligned} \tag{4.9}$$

From the argument principle applied to $V(s)$ in the rectangular contour defined by $\mathcal{R}$, we obtain

$$\frac{1}{\pi} \operatorname{var} \arg V\left(\frac{1}{2} + it\right)\Big|_{t=T}^{T+U} = -2N + O_{\vec{c}}\left(a \log T\right), \tag{4.10}$$

where N is the number of zeros of $V(s)$ inside the contour $\mathcal{R}$ and on its upper side, excluding the point $1/2+i(T+U)$. Combining (4.9) and (4.10), we complete the proof of the lemma. □

4.2 An upper bound for N

In this section, we give an upper bound on N, the number of zeros of $V(s)$ inside the contour $\mathcal{R}$ as in Lemma 4.1.4.

Lemma 4.2.1. *Let N be as in the previous lemma, Lemma 4.1.4. Then the following inequality holds*

$$N \leq \frac{UL}{4\pi} \log\left(\frac{J}{U}\right) + O_{\vec{c}}(aU),$$

where $d^* := (i/2)^a c^*$, and

$$J = \frac{1}{|d^*|^2} \sum_{j,l=0}^{M} \sum_{r,m=\tilde{a}}^{\tilde{b}} \frac{i^{a+2j}(-i)^{a+2l} c_j c_l \tilde{c}_r \tilde{c_m}}{4^{a+j+l}}$$
$$\times \int_T^{T+U} \left(B_j\psi_r\overline{B_l\psi_m} + |\chi^*|^2 D_j\psi_r\overline{D_l\psi_m} + \chi^* D_j\psi_r\overline{B_l\psi_m} + B_j\psi_r\overline{\chi^* D_l\psi_m}\right)(\sigma_a + it)\, dt,$$

and for all $a \leq k \leq b$,

$$B_k(s) := \sum_{n \leq \sqrt{T/(2\pi)}} \frac{b_k(n)}{n^s} := \sum_{n \leq \sqrt{\frac{T}{2\pi}}} \left(1 - \frac{\log n}{L}\right)\left(1 + \frac{\pi i}{2L} - \frac{2\log n}{L}\right)^k n^{-s},$$

$$D_k(s) := \sum_{n \leq \sqrt{T/(2\pi)}} \frac{d_k(n)}{n^{1-s}} := \sum_{n \leq \sqrt{\frac{T}{2\pi}}} \frac{\log n}{L}\left(\frac{2\log n}{L} + \frac{\pi i}{2L} - 1\right)^k n^{s-1},$$
$$\chi^*(t) = e e^{i\left(\frac{\pi}{4} - t\log\left(\frac{t}{2\pi e}\right)\right)},$$

with $y = T^{1/2}L^{-20}$ and constants $\tilde{c}_r$ so that $\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c}_r = 1$, and

$$\psi_u(s) := \sum_{n \leq y} \frac{a_u(n)}{n^s} := \sum_{n \leq y} \frac{\mu(n)}{n^{1/L}}\left(\frac{\log y/n}{\log y}\right)^u n^{-s} \text{ for all } \tilde{a} \leq u \leq \tilde{b}. \tag{4.11}$$

Proof. We define the mollifier $\psi(s)$ by

$$\psi(s) := \sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c_r}\psi_r(s),$$

with $\psi_r(s)$ defined as in (4.11). Then, N, the number of zeros of $V(s)$ inside the contour $\mathcal{R}$, is bounded by the number of zeros of $\frac{1}{d*}V(s)\psi(s)$ therein where $\psi(s)$ is a mollifying function which on average approximates the behavior of the inverse of the function $F_{\vec{c},a,T}(s)$. Therefore, in order to bound N, we bound the number of zeros of $\frac{1}{d*}V(s)\psi(s)$. We apply the Hardy-Littlewood lemma to $\frac{1}{d*}V(s)\psi(s)$ on the rectangle Ω with vertices $\sigma_a + iT, \sigma_1 + iT, \sigma_1 + i(T+U)$ and, $\sigma_a + i(T+U)$, where $\sigma_1 = \log L/\log 2$ and $\sigma_a = 1/2 - 1/L$. This gives

$$2\pi i \sum_{\rho=\beta+i\gamma} (\beta - \sigma_a) = -\oint_\Omega \log\left(\frac{1}{d^*}\psi(s)V(s)\right) ds, \tag{4.12}$$

where the summation is performed over all the zeros ρ of $V(s)\psi(s)$ inside Ω and on its upper side. Using the definitions of $V(s)$ from (4.6), d^*, and c^*, the fact that $\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c_r} = 1$, and estimates for the integral

$$\oint_\Omega \log\left(\frac{2}{L}\right)^k \psi_k(s)V_k(s)\ ds$$

from [49, §3], we get approximations for the integrals in (4.12) along the right and horizontal sides of the contour Ω. In fact,

$$\int \log\left(\frac{1}{d^*}\psi(s)V(s)\right) ds = O_{\vec{c}}\left(aUL^{-1}\right)$$

along the right vertical line, and satisfies $O_{\vec{c}}(\sigma_1^2 L)$ along the horizontal lines of Ω. Now the number of zeros of the product $V(s)\psi(s)$ in a larger domain Ω is greater than or equal to the number of zeros of $V(s)$ in a smaller domain $\mathcal{R}$. Thus the imaginary part of the left hand side of (4.12) is at least N/L. Putting all these facts together, we conclude

$$N \leq \frac{L}{2\pi} \int_T^{T+U} \log \left| \frac{1}{d^*} \psi(\sigma_a + it) V(\sigma_a + it) \right| \, dt + O_{\vec{c}}(aU)$$

Using log-concavity in the above integral, we get

$$N \leq \frac{UL}{2\pi} \log \left(\frac{I}{U} \right) + O_{\vec{c}}(aU), \tag{4.13}$$

where

$$I := \frac{1}{|d^*|} \int_T^{T+U} \left| \psi(\sigma_a + it) V(\sigma_a + it) \right| \, dt.$$

Now, we concentrate on the integral I. As in [49], with

$$\chi(s) := \frac{H(1-s)}{H(s)},$$

we first write $V(\sigma_a + it)$ as

$$V(\sigma_a + it) = B(\sigma_a + it) + \chi(\sigma_a + it) D(\sigma_a + it),$$

where

$$B(\sigma_a + it) := \sum_{j=0}^{M} \frac{c_j i^{a+2j}}{L^{a+2j}} \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \left(1 - \frac{\log n}{L}\right) n^{-\sigma_a - it}$$
$$\times \left(\frac{1}{2} \log \left(\frac{\sigma_a + it}{2\pi} \right) - \log n + \mathrm{O}\left(\frac{1}{|t|} \right) \right)^{a+2j},$$

and

$$D(\sigma_a+it):=\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{L^{a+2j}}\sum_{n\le\sqrt{\frac{t}{2\pi}}}\frac{\log n}{L}n^{\sigma_a+it-1}$$
$$\times\left(\log n-\frac{1}{2}\log\left(\frac{1-\sigma_a-it}{2\pi}\right)+\mathrm{O}\left(\frac{1}{|1-t|}\right)\right)^{a+2j}.$$

We rewrite $B(\sigma_a+it)$ as

$$B(\sigma_a+it)=\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{L^{a+2j}}\sum_{n\le\sqrt{\frac{t}{2\pi}}}\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}$$
$$\times\left(\frac{1}{2}\log\left(\frac{t}{2\pi}\right)+\frac{i\pi}{4}-\log n+\mathrm{O}\left(\frac{1}{t}\right)\right)^{a+2j}+O_{\vec{c}}\left((a+2j)T^{-3/4}\left(\frac{L}{2}\right)^{a+2j}\right).$$

Next we break the sum over n in the expressions for $B(\sigma_a+it)$ above into three parts and use approximations from [49, §3] to estimate each of them. Extracting $\frac{1}{2}\log\left(\frac{t}{2\pi}\right)$ out of the sum over n and using the fact that $L=\log(T/2\pi)$, we get

$$\begin{aligned}B(\sigma_a+it)=&\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\sum_{n\le\sqrt{\frac{T}{2\pi}}}\left(1+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\\&+\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\sum_{n\le\sqrt{\frac{T}{2\pi}}}\left(\left(\frac{\log(t/2\pi)}{L}+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\right.\\&\left.-\left(1+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\times\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\right)\\&+\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\sum_{\sqrt{\frac{T}{2\pi}}\le n\le\sqrt{\frac{t}{2\pi}}}\left(\frac{\log(t/2\pi)}{L}+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\\&\times\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}+O_{\vec{c}}\left(aT^{-3/4}\right).\end{aligned}\tag{4.14}$$

Note that the first sum over n in the above expression is indeed $B_j(\sigma_a+it)$ as defined in the statement of the lemma. From [49, §3], the squares of the integrals of the latter two inner sums over n inside $B(\sigma_a+it)$ are estimated as

$$\int_T^{T+U}\Bigg|\sum_{n\le\sqrt{\frac{T}{2\pi}}}\Bigg(\left(\frac{\log(t/2\pi)}{L}+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}-\left(1+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\Bigg)$$
$$\times\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\Bigg|^2\,dt\ll(a+2j)^2\frac{U}{L^{19}},$$

and

$$\int_T^{T+U}\Bigg|\sum_{\sqrt{\frac{T}{2\pi}}\le n\le\sqrt{\frac{t}{2\pi}}}\left(\frac{\log(t/2\pi)}{L}+\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{a+2j}\left(1-\frac{\log n}{L}\right)n^{-\sigma_a-it}\Bigg|^2\,dt\ll\frac{U}{L^{10}},$$

as well as

$$\int_T^{T+U}|\psi_j(\sigma_a+it)|^2\,dt\ll UL.$$

Hence, on applying the Cauchy-Schwarz inequality for the latter two sums in (4.14), we obtain

$$\int_T^{T+U}\psi(\sigma_a+it)B(\sigma_a+it)\,dt \tag{4.15}$$
$$=\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\int_T^{T+U}\psi_r(\sigma_a+it)B_j(\sigma_a+it)\,dt+O_{\vec{c}}\left(\frac{U}{L^{9/2}}\right).$$

We perform an argument similar to $B(\sigma_a+it)$ for $D(\sigma_a+it)$ and consequently obtain,

$$\int_T^{T+U}\chi^*(t)\psi(\sigma_a+it)D(\sigma_a+it)\,dt$$
$$=\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}\sum_{j=0}^{M}\frac{c_j i^{a+2j}}{2^{a+2j}}\int_T^{T+U}\chi^*(t)\psi_r(\sigma_a+it)D_j(\sigma_a+it)\,dt+O_{\vec{c}}\left(\frac{U}{L^{9/2}}\right). \tag{4.16}$$

Now for the other term in $V(\sigma_a+it)$, we first estimate the function $\chi(\sigma_a+it)$ by adding and subtracting to it

$$\chi^*(t) = e e^{1+i\left(\frac{\pi}{4} - t\log\left(\frac{t}{2\pi e}\right)\right)}.$$

This gives,

$$|\chi(\sigma_a + it)| \leq |\chi^*(t)| + |\chi(\sigma_a + it) - \chi^*(t)| \leq |\chi^*(t)| + \mathrm{O}\left(L^{-11}\right).$$

the Cauchy-Schwarz inequality along with (4.15) and (4.16), yields

$$\begin{aligned} I &= \frac{1}{|d^*|}\int_T^{T+U} \Big|\psi(\sigma_a + it)V(\sigma_a + it)\Big|\, dt \\ &= \frac{1}{|d^*|}\int_T^{T+U} \Bigg|\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c_r} \sum_{j=0}^{M} \frac{c_j i^{a+2j}}{2^{a+2j}} \psi_r(\sigma_a + it)\left(B_j(\sigma_a + it) + \chi^*(t) D_j(\sigma_a + it)\right)\Bigg|\, dt \\ &\quad + O_{\vec{c}}\left(\frac{U}{L^{9/2}}\right). \end{aligned}$$

Denote by J the double sum given by

$$J := \frac{U}{|d^*|^2} \sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l}}{4^{a+j+l}} A_{j,l}, \tag{4.17}$$

where

$$\begin{aligned} A_{j,l} &:= \frac{1}{U}\int_T^{T+U} \Bigg((B_j + \chi^* D_j) \sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c_r}\psi_r(\sigma_a + it)\Bigg)\Bigg(\left(\overline{B_l} + \overline{\chi^* D_l}\right) \sum_{m=\tilde{a}}^{\tilde{b}} \tilde{c_m}\overline{\psi_m}(\sigma_a + it)\Bigg)\, dt \\ &=: \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c_r}\tilde{c_m} E_{j,l,r,m}, \end{aligned} \tag{4.18}$$

with

$$E_{j,l,r,m} := \frac{1}{U}\int_T^{T+U} \left(B_j\psi_r\overline{B_l\psi_m} + \overline{\chi^*}\chi^* D_j\psi_r\overline{D_l\psi_m} + B_j\psi_r\overline{\chi^* D_l\psi_m} + \chi^* D_j\psi_r\overline{B_l\psi_m}\right)\, dt. \tag{4.19}$$

Employing the Cauchy-Schwarz inequality and using the above notations, we get

$$I \leq \sqrt{JU} + O_{\vec{c}}\left(\frac{U}{L^{9/2}}\right).$$

Thus, using the above inequality and (4.13), we conclude that the number of zeros is bounded by

$$N \leq \frac{UL}{4\pi}\log\left(\frac{J}{U}\right) + O_{\vec{c}}\left(aU\right),$$

which completes the proof of the lemma. □

4.3 Asymptotics for J/U

Lemma 4.3.1. *With the notations from Lemma 4.2.1,*

$$\frac{J}{U} = 1 + \frac{e^2+2}{16a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

Proof. First, we consider the following integral appearing in J in Lemma 4.2.1, i.e.

$$\begin{aligned}\int_T^{T+U} & B_j\psi_r\overline{B_l\psi_m}(\sigma_a+it)\,dt \\ &= U\sum_{\substack{n_1,n_2,n_3,n_4\\ n_1,n_2\le\sqrt{T/2\pi}\\ n_3,n_4\le y}}\frac{b_j(n_1)\overline{b_l}(n_2)a_r(n_3)\overline{a_m}(n_4)}{(n_1n_2n_3n_4)^{\sigma_a}}\int_T^{T+U}\left(\frac{n_2n_4}{n_1n_3}\right)^{it}\,dt \\ &= U\sum_{\substack{n_1,n_2,n_3,n_4\\ n_2n_4=n_1n_3\\ n_1,n_2\le\sqrt{T/2\pi}\\ n_3,n_4\le y}}\frac{b_j(n_1)\overline{b_l}(n_2)a_r(n_3)\overline{a_m}(n_4)}{(n_2n_4)^{2\sigma_a}} \\ &\quad+\sum_{\substack{n_1,n_2,n_3,n_4\\ n_2n_4\neq n_1n_3\\ n_1,n_2\le\sqrt{T/2\pi}\\ n_3,n_4\le y}}\frac{b_j(n_1)\overline{b_l}(n_2)a_r(n_3)\overline{a_m}(n_4)}{(n_1n_2n_3n_4)^{\sigma_a}}\left(\frac{(\frac{n_2n_4}{n_1n_3})^{i(T+U)}-(\frac{n_2n_4}{n_1n_3})^{iT}}{\log(\frac{n_2n_4}{n_1n_3})}\right).\end{aligned} \tag{4.20}$$

Note that, $b_j(n_1), b_l(n_2), a_r(n_3), a_m(n_4) \ll 1$, and $\sigma_a = 1/2 - 1/L$. This implies that the second sum above in the last equality can be bounded by a constant multiple of

$$\sum_{\substack{n_1,n_2,n_3,n_4\\ n_2n_4\neq n_1n_3\\ n_1,n_2\le\sqrt{T/2\pi}\\ n_3,n_4\le y}}\frac{1}{(n_1n_2n_3n_4)^{1/2}}\frac{1}{|\log\left(\frac{n_2n_4}{n_1n_3}\right)|}.$$

Rewriting $n_1n_3 = m$ and $n_2n_4 = n$, we obtain that this sum can further be bounded as

$$\sum_{\substack{n_1,n_2,n_3,n_4\\ n_2n_4\neq n_1n_3\\ n_1,n_2\le\sqrt{T/2\pi}\\ n_3,n_4\le y}}\frac{1}{(n_1n_2n_3n_4)^{1/2}}\frac{1}{|\log\left(\frac{n_2n_4}{n_1n_3}\right)|}\ll\sum_{\substack{m,n\le y\sqrt{\frac{T}{2\pi}}\\ m\neq n}}\frac{d(m)d(n)}{(mn)^{1/2}}\frac{1}{|\log(m/n)|},$$

where $d(n)$ denotes the number of positive divisors of n. Also, note that for any positive $\epsilon > 0$, we have $d(m) = \mathrm{O}(m^\epsilon) = \mathrm{O}(T^\epsilon)$. Let $h = n - m$. One may assume without loss of generality that $m < n$. This gives,

$$\sum_{\substack{m,n\le y\sqrt{\frac{T}{2\pi}}\\ m\ne n}} \frac{d(m)d(n)}{(mn)^{1/2}|\log(m/n)|} \ll T^{2\epsilon} \sum_{1\le m<n\le T^{1/2+\theta}} \frac{1}{(mn)^{1/2}|\log(m/n)|}$$

$$\ll T^{2\epsilon} \sum_{\substack{2\le n\le T^{1/2+\theta}\\ 1\le h\le n-1}} \frac{1}{(n(n-h))^{1/2}|\log(1-\frac{h}{n})|}$$

$$\ll T^{2\epsilon} \sum_{\substack{2\le n\le T^{1/2+\theta}\\ 1\le h\le n-1}} \frac{n^{1/2}}{h(n-h)^{1/2}}$$

$$\ll T^{2\epsilon} \sum_{1\le h\le T^{1/2+\theta}} \frac{1}{h} \sum_{1+h\le n\le T^{1/2+\theta}} \frac{1}{\left(1-\frac{h}{n}\right)^{1/2}}$$

$$\ll T^{2\epsilon+1/2+\theta}\log^2 T.$$

The first sum in the second equality in (4.20) can be written as

$$\sum_{\substack{n_1,n_2,n_3,n_4\\ n_2n_4=n_1n_3=n\\ n_1,n_2\le\sqrt{T/2\pi}\\ n_3,n_4\le y}} \frac{b_j(n_1)\overline{b_l}(n_2)a_r(n_3)\overline{a_m}(n_4)}{(n_2n_4)^{2\sigma_a}} \tag{4.21}$$

$$= \sum_{n\le y\sqrt{\frac{T}{2\pi}}} \frac{1}{n^{2\sigma_a}}\Bigg(\sum_{\substack{k_1|n\\ n\sqrt{\frac{2\pi}{T}}\le k_1\le y}} a_r(k_1)b_j\left(\frac{n}{k_1}\right)\Bigg)\Bigg(\overline{\sum_{\substack{k_2|n\\ n\sqrt{\frac{2\pi}{T}}\le k_2\le y}} a_m(k_2)b_l\left(\frac{n}{k_2}\right)}\Bigg).$$

We now reverse the order of summation so that the partial sums of $b_j(x)b_l(x)$ on the inside are fairly easier to handle. In order to do so, we let $d=\gcd(k_1,k_2)$ and $n=\frac{k_1k_2}{d}w$, where

$$w \le \frac{\sqrt{\frac{T}{2\pi}}d}{\max(k_1,k_2)}.$$

Using the above relations and interchanging the sums over k_1, k_2 and n, we can rewrite the right-hand side of equation (4.21) as

$$\sum_{k_1,k_2\le y}\frac{a_r(k_1)\overline{a_m(k_2)}}{(k_1k_2)^{2\sigma_a}}d^{2\sigma_a}\sum_{w\le\sqrt{\frac{T}{2\pi}}\frac{d}{\max(k_1,k_2)}}b_j\left(\frac{k_2}{d}w\right)\overline{b_l\left(\frac{k_1}{d}w\right)}w^{-2\sigma_a}.$$

Therefore, employing this in (4.20), we obtain

$$\begin{aligned}&\int_T^{T+U}B_j\psi_r\overline{B_l\psi_m}(\sigma_a+it)\ dt \\ &=U\sum_{k_1,k_2\le y}\frac{a_r(k_1)\overline{a_m(k_2)}}{(k_1k_2)^{2\sigma_a}}d^{2\sigma_a}\sum_{w\le\sqrt{\frac{T}{2\pi}}\frac{d}{\max(k_1,k_2)}}b_j\left(\frac{k_2}{d}w\right)\overline{b_l\left(\frac{k_1}{d}w\right)}w^{-2\sigma_a} \\ &+\mathrm{O}\left(T^{\epsilon+1/2+\theta}\log^2 T\right).\end{aligned}\tag{4.22}$$

In a similar fashion, we can estimate the second term of the integral in J in Lemma 4.2.1 to obtain

$$\begin{aligned}&\int_T^{T+U}\overline{\chi^*}\chi^*D_j\psi_r\overline{D_l\psi_m}(\sigma_a+it)\ dt \\ &=e^2U\sum_{r_1,r_2\le y}\frac{a_r(r_1)\overline{a_m(r_2)}}{r_1r_2}D^{2-2\sigma_a} \\ &\times\sum_{w\le\frac{\sqrt{\frac{T}{2\pi}}D}{\max(r_1,r_2)}}d_j\left(\frac{r_2}{D}w\right)\overline{r_l\left(\frac{r_1}{D}w\right)}w^{2\sigma_a-2}+\mathrm{O}\left(T^{\epsilon+1/2+\theta}\log^2 T\right),\end{aligned}\tag{4.23}$$

where $D=\gcd(r_1,r_2)$. Next, one can see that the following integral from J in Lemma 4.2.1

becomes

$$\begin{aligned}
&\int_T^{T+U} \chi^* D_j \psi_r \overline{B_l \psi_m}(\sigma_a + it)\ dt \\
&\quad = e^{1+i\pi/4} \sum_{\substack{n_1,n_2,n_3,n_4 \\ n_1,n_2 \le \sqrt{T/2\pi} \\ n_3,n_4 \le y}} \frac{d_j(n_2)\overline{b_l(n_1)} a_r(n_3) \overline{a_m(n_4)}}{(n_1 n_3 n_4)^{\sigma_a} (n_2)^{1-\sigma_a}} \int_T^{T+U} e^{-it\log\left(\frac{t}{2\pi e}\right)} \left(\frac{n_1 n_2 n_4}{n_3}\right)^{it}\ dt \\
&\quad = e^{1+i\pi/4} \sum_{\substack{n_1,n_2,n_3,n_4 \\ n_1,n_2 \le \sqrt{T/2\pi} \\ n_3,n_4 \le y}} \frac{d_j(n_2)\overline{b_l(n_1)} a_r(n_3) \overline{a_m(n_4)}}{(n_1 n_3 n_4)^{\sigma_a} (n_2)^{1-\sigma_a}} \int_T^{T+U} e^{-it\log\left(\frac{t}{2\pi e}\left(\frac{n_3}{n_1 n_2 n_4}\right)\right)}\ dt \\
&\quad = e^{1+i\pi/4} \sum_{\substack{n_3,n_4 \\ n_3,n_4 \le y}} \frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3 n_4)^{\sigma_a}} \sum_{\substack{n_1,n_2 \\ n_1,n_2 \le \sqrt{\frac{T}{2\pi}}}} \frac{d_j(n_2)\overline{b_l(n_1)}}{n_1^{\sigma_a} n_2^{1-\sigma_a}} \int_T^{T+U} e^{-it\log\left(\frac{t}{2\pi e}\left(\frac{n_3}{n_1 n_2 n_4}\right)\right)}\ dt.
\end{aligned} \tag{4.24}$$

Using Lemma 3.4 from [33] for

$$\frac{T}{2\pi} \leqslant \frac{n_1 n_2 n_4}{n_3} \le \frac{T+U}{2\pi},$$

we have

$$\int_T^{T+U} e^{-it\log\left(\frac{t}{2\pi e}\left(\frac{n_3}{n_1 n_2 n_4}\right)\right)}\ dt = 2\pi \left(\frac{n_1 n_2 n_4}{n_3}\right)^{1/2} e^{2\pi i\left(\frac{n_1 n_2 n_4}{n_3}\right)} e^{-i\pi/4} + E, \tag{4.25}$$

where for $T' = T$ or $T' = T+U$

$$E = \begin{cases} \mathrm{O}\left(1\right), & \text{if } \frac{2\pi n_1 n_2 n_4}{n_3} \le \frac{3T}{4} \text{ or } \frac{2\pi n_1 n_2 n_4}{n_3} \ge \frac{5T}{4}. \\ \mathrm{O}\left(\frac{T'}{T' - 2\pi r + T^{1/2}}\right), & \text{otherwise.} \end{cases}$$

As in [33] and [49], when $E = \mathrm{O}(1)$, the four sums in (4.24) over n_1, n_2, n_3, n_4 times E gives a contribution $\ll L^{-10}$ and for the other case, it is $\ll L^{-7}$. Substituting these error estimates in (4.24) and using (4.25), we have

$$\int_T^{T+U} \chi^* D_j \psi_r \overline{B_l \psi_m}(\sigma_a + it)\, dt$$
$$= 2\pi e \sum_{\substack{n_3,n_4 \\ n_3,n_4 \le y}} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3^{\sigma_a+1/2} n_4^{\sigma_a-1/2}} \sum_{\substack{n_1,n_2 \\ n_1,n_2 \le \sqrt{\frac{T}{2\pi}} \\ \frac{T}{2\pi} \leqslant \frac{n_1 n_2 n_4}{n_3} \le \frac{T+U}{2\pi}}} \frac{d_j(n_2)\overline{b_l(n_1)}}{n_1^{\sigma_a-1/2} n_2^{1/2-\sigma_a}} e^{2\pi i\left(\frac{n_1 n_2 n_4}{n_3}\right)} + O_{\vec{c}}\left(\frac{aU}{L^7}\right). \tag{4.26}$$

For the next step we split the above main term into two sums depending on whether $\dfrac{n_2 n_4}{n_3} \in \mathbb{N}$ or $\dfrac{n_2 n_4}{n_3} \notin \mathbb{N}$. The latter case contributes an error of $\mathrm{O}\left(L^{-8}\right)$ to the sums over n_1, n_2, n_3 and n_4. The former case can be examined in the following way. Let m^* denote the gcd of n_3 and n_4. When $\dfrac{n_2 n_4}{n_3} \in \mathbb{N}$, one has

$$n_2 \equiv 0 \mod \frac{n_3}{m^*}.$$

Let $n_2 = n^* \frac{n_3}{m^*}$ for some $n^* \in \mathbb{N}$. We first find the range of n^* and n_1. Since $\frac{n_1 n_2 n_4}{n_3} \le \frac{T+U}{2\pi}$ and $n_1 \le \sqrt{\frac{T}{2\pi}}$, we have

$$n_1 \le \frac{T+U}{2\pi} \frac{n_3}{n_2 n_4} = \frac{(T+U)}{2\pi} \frac{m^*}{n^* n_4}.$$

Thus,

$$n_1 \le \min\left\{\sqrt{\frac{T}{2\pi}}, \frac{(T+U)}{2\pi} \frac{m^*}{n^* n_4}\right\},$$

and

$$n_1 \ge \frac{T}{2\pi} \frac{n_3}{n_2 n_4} = \frac{T}{2\pi} \frac{m^*}{n^* n_4}.$$

Using the above inequalities and the bounds $n_1, n_2 \le \sqrt{\frac{T}{2\pi}}$, we obtain the following inequalities

$$\sqrt{\frac{T}{2\pi}} \frac{m^*}{n_4} \le \frac{T}{2\pi} \frac{m^*}{n_1 n_4} \le \frac{n_2}{n_3} m^* = n^* = \frac{n_2}{n_3} m^* \le \sqrt{\frac{T}{2\pi}} \frac{m^*}{n_3}.$$

The sum over n_2 for which $\frac{T+U}{2\pi} \frac{m^*}{n^* n_4} > \sqrt{\frac{T}{2\pi}}$ contributes an error term bounded by $\mathrm{O}\left(L^{-7}\right)$

and so we consider only the terms for which

$$n_1 \le \min\left(\sqrt{\frac{T}{2\pi}}, \frac{(T+U)}{2\pi}\frac{m^*}{n^* n_4}\right) = \frac{(T+U)}{2\pi}\frac{m^*}{n^* n_4}.$$

We rewrite the sum in equation (4.26) as a sum over n^* and n_1 with the above bounds and substitute $n_2 = n^* \frac{n_3}{m^*}$. This yields

$$\begin{aligned}
&\int_T^{T+U} \chi^* D_j \psi_r \overline{B_l \psi_m}(\sigma_a + it)\, dt \\
&\quad = 2\pi e \sum_{\substack{n_3, n_4 \\ n_3, n_4 \le y}} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3^{\sigma_a+1/2} n_4^{\sigma_a-1/2}} \sum_{\substack{n_1, n_2 \\ n_1, n_2 \le \sqrt{\frac{T}{2\pi}} \\ \frac{T}{2\pi} \leqslant \frac{n_1 n_2 n_4}{n_3} \le \frac{T+U}{2\pi}}} \frac{d_j(n_2)\overline{b_l(n_1)}}{n_1^{\sigma_a-1/2} n_2^{1/2-\sigma_a}} e^{2\pi i\left(\frac{n_1 n_2 n_4}{n_3}\right)} \\
&\quad = 2\pi e \sum_{\substack{n_3, n_4 \le y \\ n_3 \ge n_4}} \frac{a_r(n_3)\overline{a_m(n_4)} {m^*}^{1/2-\sigma_a}}{n_3^{\sigma_a+1/2} n_4^{\sigma_a-1/2}} \sum_{\sqrt{\frac{T}{2\pi}}\frac{m^*}{n_4} \le n^* \le \sqrt{\frac{T}{2\pi}}\frac{m^*}{n_3}} \frac{d_j\left(\frac{n^* n_3}{m^*}\right)}{{n^*}^{1/2-\sigma_a} {n_3}^{1/2-\sigma_a}} \\
&\qquad \times \sum_{\frac{T}{2\pi}\frac{m^*}{n^* n_4} \le n_1 \le \frac{(T+U)}{2\pi}\frac{m^*}{n^* n_4}} \frac{\overline{b_l(n_1)}}{{n_1}^{\sigma_a-1/2}} + \mathrm{O}\left(\frac{aU}{L^7}\right).
\end{aligned} \tag{4.27}$$

Applying the mean value theorem for the function $\frac{b_l(x)}{x^{\sigma_a-1/2}}$ on the interval $[n_1, \frac{T}{2\pi}\frac{m^*}{n^* n_4}]$, we get

$$\frac{b_l(n_1)}{{n_1}^{\sigma_a-1/2}} = b_l\left(\frac{T}{2\pi}\frac{m^*}{n^* n_4}\right)\left(\frac{T}{2\pi}\frac{m^*}{n^* n_4}\right)^{1/2-\sigma_a} + \mathrm{O}\left(\frac{n_1 n^* n_4}{TLm^*}\right).$$

Now,

$$\left(\frac{T}{2\pi}\right)^{1/2-\sigma_a} = e \quad \text{and} \quad b_l\left(\frac{T}{2\pi}\frac{m^*}{n^* n_4}\right) = d_l\left(\frac{n^* n_4}{m^*}\right).$$

Therefore,

$$\sum_{\frac{T}{2\pi}\frac{m^*}{n^* n_4} \le n_1 \le \frac{(T+U)}{2\pi}\frac{m^*}{n^* n_4}} \frac{b_l(n_1)}{{n_1}^{\sigma_a-1/2}} = e\frac{U}{2\pi}\left(\frac{m^*}{n^* n_4}\right)^{3/2-\sigma_a} d_l\left(\frac{n^* n_4}{m^*}\right) + \mathrm{O}\left(\frac{n^* n_4}{m^* L^{11}}\right).$$

Substituting these in equation (4.27), we obtain

$$\int_T^{T+U} \chi^* D_j\psi_r\overline{B_l\psi_m}(\sigma_a+it)\ dt \tag{4.28}$$
$$= e^2U \sum_{\substack{n_3,n_4\le y\\ n_3\ge n_4}} \frac{a_r(n_3)\overline{a_m(n_4)}m^{*2-2\sigma_a}}{n_3n_4} \sum_{\sqrt{\frac{T}{2\pi}}\frac{m^*}{n_4}\le n^*\le\sqrt{\frac{T}{2\pi}}\frac{m^*}{n_3}} \frac{d_j\left(\frac{n^*n_3}{m^*}\right)\overline{d_l\left(\frac{n^*n_4}{m^*}\right)}}{n^{*2-2\sigma_a}} + \mathrm{O}\left(\frac{aU}{L^7}\right).$$

A similar computation enables us to write

$$\int_T^{T+U} \overline{\chi^*} B_j\psi_r\overline{D_l\psi_m}(\sigma_a+it)\ dt \tag{4.29}$$
$$= e^2U \sum_{\substack{n_3,n_4\le y\\ n_3\le n_4}} \frac{a_r(n_3)\overline{a_m(n_4)}m^{*2-2\sigma_a}}{n_3n_4} \sum_{\sqrt{\frac{T}{2\pi}}\frac{m^*}{n_3}\le n^*\le\sqrt{\frac{T}{2\pi}}\frac{m^*}{n_4}} \frac{d_j\left(\frac{n^*n_3}{m^*}\right)\overline{d_l\left(\frac{n^*n_4}{m^*}\right)}}{n^{*2-2\sigma_a}} + \mathrm{O}\left(\frac{aU}{L^7}\right).$$

Combining (4.28) and (4.29), we obtain

$$\int_T^{T+U} \left(\chi^* D_j\psi_r\overline{B_l\psi_m} + \overline{\chi^*} B_j\psi_r\overline{D_l\psi_m}\right)(\sigma_a+it)\ dt \tag{4.30}$$
$$= e^2U \sum_{n_3,n_4\le y} \frac{a_r(n_3)\overline{a_m(n_4)}m^{*2-2\sigma_a}}{n_3n_4} \sum_{\sqrt{\frac{T}{2\pi}}\frac{m^*}{\max(n_3,n_4)}\le n^*\le\sqrt{\frac{T}{2\pi}}\frac{m^*}{\min(n_3,n_4)}} \frac{d_j\left(\frac{n^*n_4}{m^*}\right)\overline{d_l\left(\frac{n^*n_3}{m^*}\right)}}{n^{*2-2\sigma_a}}$$
$$+ \mathrm{O}\left(\frac{aU}{L^7}\right).$$

The three integrals in (4.22), (4.23) and (4.30) upon substitution in (4.19) give the following equality for $E_{j,l,r,m}$

$$E_{j,l,r,m} = \sum_{k_1,k_2 \le y} \frac{a_r(k_1)\overline{a_m(k_2)}}{(k_1k_2)^{2\sigma_a}} d^{2\sigma_a} \sum_{w \le \frac{\sqrt{\frac{T}{2\pi}}d}{\max(k_1,k_2)}} b_j\left(\frac{k_2}{d}w\right)\overline{b_l\left(\frac{k_1}{d}w\right)} w^{-2\sigma_a}$$
$$+ e^2 \sum_{r_1,r_2 \le y} \frac{a_r(r_1)\overline{a_m(r_2)}}{r_1r_2} D^{2-2\sigma_a} \sum_{w \le \frac{\sqrt{\frac{T}{2\pi}}D}{\max(r_1,r_2)}} d_j\left(\frac{r_2}{D}w\right)\overline{d_l\left(\frac{r_1}{D}w\right)} w^{2\sigma_a-2}$$
$$+ e^2 \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)} m^{*2-2\sigma_a}}{n_3n_4} \sum_{\sqrt{\frac{T}{2\pi}}\frac{m^*}{\max(n_3,n_4)} \le n^* \le \sqrt{\frac{T}{2\pi}}\frac{m^*}{\min(n_3,n_4)}} \frac{d_j\left(\frac{n^*n_4}{m^*}\right)\overline{d_l\left(\frac{n^*n_3}{m^*}\right)}}{n^{*2-2\sigma_a}}$$
$$+ \mathrm{O}\left(\frac{a}{L^7}\right),$$

where $d = \gcd(k_1,k_2), D = \gcd(r_1,r_2)$ and $m^* = \gcd(n_3,n_4)$. After combining the second and third sums in the above estimate, we obtain

$$E_{j,l,r,m} = \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}} m^{*2\sigma_a} \sum_{n^* \le \frac{\sqrt{\frac{T}{2\pi}}m^*}{\max(n_3,n_4)}} b_j\left(\frac{n^*n_4}{m^*}\right)\overline{b_l\left(\frac{n^*n_3}{m^*}\right)} n^{*-2\sigma_a}$$
$$+ e^2 \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3n_4} m^{*2-2\sigma_a} \sum_{n^* \le \frac{\sqrt{\frac{T}{2\pi}}m^*}{\min(n_3,n_4)}} d_j\left(\frac{n^*n_4}{m^*}\right)\overline{d_l\left(\frac{n^*n_3}{m^*}\right)} n^{*2\sigma_a-2}$$
$$+ O_{\vec{c}}\left(\frac{a}{L^7}\right).$$

Using Euler's summation formula for the sums over n^* in the above expression, and after the change of variable

$$\Theta_1 = \frac{1}{\Theta}\left(\frac{\frac{T}{2\pi}(m^*)^2}{n_3n_4}\right)$$

and $d_k\left(\frac{T}{2\pi\Theta}\right) = b_k(\Theta)$, the integral containing d_j's can be written in terms of b_j's. All these simplifications amount to

$$
\begin{aligned}
E_{j,l,r,m} &= \sum_{n_3,n_4\le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}} m^{*2\sigma_a} \Bigg(\int_1^{\frac{\sqrt{\frac{T}{2\pi}}m^*}{\max(n_3,n_4)}} b_j\left(\frac{xn_4}{m^*}\right) \overline{b_l\left(\frac{xn_3}{m^*}\right)} x^{-2\sigma_a}\ dx \\
&\quad + \int_{\frac{\sqrt{\frac{T}{2\pi}}m^*}{\max(n_3,n_4)}}^{\frac{T}{2\pi}\frac{m^{*2}}{n_3n_4}} b_j\left(\frac{xn_4}{m^*}\right) \overline{b_l\left(\frac{xn_3}{m^*}\right)} x^{-2\sigma_a}\ dx \Bigg) + O_{\vec{c}}\left(\frac{a}{L^7}\right) \\
&= \sum_{n_3,n_4\le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}} m^{*2\sigma_a} \int_1^{\frac{T}{2\pi}\frac{m^{*2}}{n_3n_4}} b_j\left(\frac{xn_4}{m^*}\right) \overline{b_l\left(\frac{xn_3}{m^*}\right)} x^{-2\sigma_a}\ dx + O_{\vec{c}}\left(\frac{a}{L}\right). \quad (4.31)
\end{aligned}
$$

Note that for a polynomial $P(u)$ of degree k in the variable u, the following (which can be obtained using integration by parts) holds

$$
\int_A^B P(\log u)\ du = \sum_{n=1}^{k} (-1)^n \frac{P^{(n)}(\log B)}{(1+\alpha)^{n+1}} B^{1+\alpha} - \sum_{n=1}^{k} (-1)^n \frac{P^{(n)}(\log A)}{(1+\alpha)^{n+1}} A^{1+\alpha}.
$$

For $0 \le k \le M$, define

$$
\phi_k(x) := (1-x)\left(1 - 2x + \frac{\pi i}{2L}\right)^{a+2k}, \quad (4.32)
$$

which implies that

$$
b_k(x) = \phi_k\left(\frac{\log x}{L}\right)
$$

is a polynomial in $\log x/L$. On applying the above identity to $\phi_j(\log x/L)$ and $\phi_l(\log x/L)$, we express the integral in (4.31) as the sum

$$
E_{j,l,r,m} = S_1 - S_2 + O_{\vec{c}}\left(a/L\right),
$$

where

$$S_1 := \frac{Le^2}{2} \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3 n_4} m^{*2-2\sigma_a}$$
$$\times \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \sum_{v=0}^{n} \binom{n}{v} {\phi_j}^{(v)}\left(1 - \frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\right) \overline{{\phi_l}^{(n-v)}\left(1 - \frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\right)},$$

and

$$S_2 := \frac{L}{2} \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3 n_4)^{2\sigma_a}} m^{*2\sigma_a}$$
$$\times \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \sum_{v=0}^{n} \binom{n}{v} {\phi_j}^{(v)}\left(\frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\right) \overline{{\phi_l}^{(n-v)}\left(\frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\right)}.$$

The Taylor series expansions of ${\phi_j}^{(v)}(x)$ and ${\phi_l}^{(n-v)}(x)$ at $x = 1$ in S_1 and at $x = 0$ in S_2 give

$$S_1 = \frac{Le^2}{2} \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \sum_{v=0}^{n} \binom{n}{v} \Bigg\{ \phi_j^{(v)}(1)\overline{{\phi_l}^{(n-v)}(1)} \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3 n_4} m^{*2-2\sigma_a}$$
$$- \left({\phi_j}^{(v)}(1)\overline{{\phi_l}^{(n-v+1)}(1)} + {\phi_j}^{(v+1)}(1)\overline{{\phi_l}^{(n-v)}(1)}\right) \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3 n_4} \frac{m^{*2-2\sigma_a}}{L} \log\left(\frac{n_4}{m^*}\right)$$
$$+ {\phi_j}^{(v+1)}(1)\overline{{\phi_l}^{(n-v+1)}(1)} \sum_{n_3,n_4 \le y} \frac{a_r(n_3)\overline{a_m(n_4)}}{n_3 n_4} m^{*2-2\sigma_a} \frac{1}{L}\log\left(\frac{n_4}{m^*}\right) \frac{1}{L}\log\left(\frac{n_3}{m^*}\right) \Bigg\}$$
$$+ O_{\vec{c}}\left((2a)^{2a+4} \frac{\log^5 L}{L}\right),$$

and

$$S_2 = \frac{L}{2}\sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n}\sum_{v=0}^{n}\binom{n}{v}\Bigg\{\phi_j{}^{(v)}(0)\overline{\phi_l{}^{(n-v)}(0)}\sum_{n_3,n_4\le y}\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}m^{*2\sigma_a}$$

$$+\left(\phi_j{}^{(v)}(0)\overline{\phi_l{}^{(n-v+1)}(0)}+\phi_j{}^{(v+1)}(0)\right)\overline{\phi_l{}^{(n-v)}(0)}\Big)\sum_{n_3,n_4\le y}\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}\frac{m^{*2\sigma_a}}{L}\log\left(\frac{n_4}{m^*}\right)$$

$$+\phi_j{}^{(v+1)}(0)\overline{\phi_l{}^{(n-v+1)}(0)}\sum_{n_3,n_4\le y}\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}m^{*2\sigma_a}\frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\Bigg\}$$

$$+O_{\vec{c}}\left((2a)^{2a+4}\frac{\log^5 L}{L}\right).$$

In order to estimate the inner sums we use extensions of Lemmas 18 and 20 from [49].

Lemma 4.3.2. *With the notations used earlier, the following bounds hold:*

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{n_3n_4}m^{*2-2\sigma_a}\right)=\frac{1}{L}\left(\left(\frac{1}{2(r+m+1)}\right)+1+\left(\frac{2rm}{r+m-1}\right)\right)$$
$$+O(L^{-2}\log^5 L).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{n_3n_4}m^{*2-2\sigma_a}\frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\right)=-\frac{1}{L}\left(\left(\frac{1}{2(r+m+1)}\right)+\left(\frac{r}{r+m}\right)\right)$$
$$+O(L^{-2}\log^5 L).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{n_3n_4}m^{*2-2\sigma_a}\frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\right)=\left(\frac{1}{2(r+m+1)}\right)\frac{1}{L}$$
$$+\mathrm{O}\left(L^{-2}\log^5 L\right).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{n_3n_4}m^{*2-2\sigma_a}\frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\right)=-\frac{1}{L}\left(\left(\frac{1}{2(r+m+1)}\right)+\left(\frac{m}{r+m}\right)\right)$$
$$+O(L^{-2}\log^5 L).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}m^{*2a}\right)=\frac{1}{L}\left(\left(\frac{1}{2(r+m+1)}\right)-1+\left(\frac{2rm}{r+m-1}\right)\right)$$
$$+\mathrm{O}\left(L^{-2}\log^5 L\right).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}m^{*2a}\frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\right)=-\frac{1}{L}\left(\left(\frac{-1}{2(r+m+1)}\right)+\left(\frac{r}{r+m}\right)\right)$$
$$+O(L^{-2}\log^5 L).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}m^{*2\sigma_a}\frac{1}{L}\log\left(\frac{n_4}{m^*}\right)\frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\right)=\left(\frac{1}{2(r+m+1)}\right)\frac{1}{L}$$
$$+\mathrm{O}\left(L^{-2}\log^5 L\right).$$

$$\sum_{n_3,n_4\le y}\left(\frac{a_r(n_3)\overline{a_m(n_4)}}{(n_3n_4)^{2\sigma_a}}m^{*2\sigma_a}\frac{1}{L}\log\left(\frac{n_3}{m^*}\right)\right)=-\frac{1}{L}\left(\left(\frac{-1}{2(r+m+1)}\right)+\left(\frac{m}{r+m}\right)\right)$$
$$+O(L^{-2}\log^5 L).$$

Proof. Following Lemmas 18 and 20 from [49], and employing the fact that $\log y = L/2 +$

$\mathrm{O}(\log L)$, it is not difficult to prove these estimates, and we omit the details here. $\square$

Using the above lemma, we have

$$
\begin{aligned}
&E_{j,l,r,m} = S_1 - S_2 + O_{\vec{c}}\left(\frac{a}{L}\right) \\
&= \frac{e^2}{2}\sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n}\left((\phi_j(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}\left(\frac{1}{2(r+m+1)}+1+\frac{2rm}{r+m-1}\right)\right. \\
&\quad +(\phi_j(x))\overline{\phi_l{}'(x)}+\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}\frac{1}{2(r+m+1)}+(\phi_j{}'(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=1}\frac{1}{2(r+m+1)} \\
&\quad \left.+(\phi_j(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=1}\frac{r}{r+m}+(\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}\frac{m}{r+m}\right) \\
&\quad -\frac{1}{2}\sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n}\left((\phi_j(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}\left(\frac{1}{2(r+m+1)}-1+\frac{2rm}{r+m-1}\right)\right. \\
&\quad +(\phi_j(x))\overline{\phi_l{}'(x)}+\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\frac{1}{2(r+m+1)}+(\phi_j{}'(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=0}\frac{1}{2(r+m+1)} \\
&\quad \left.-(\phi_j(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=0}\frac{r}{r+m}-(\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\frac{m}{r+m}\right) \\
&\quad + O_{\vec{c}}\left((2a)^{2a+4}\frac{\log^5 L}{L}\right).
\end{aligned}
$$

Denote by

$$
\begin{aligned}
\mathcal{U} &:= \frac{e^2}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\times \\
&\quad \times\left((\phi_j(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}+(\phi_j(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=1}\frac{r}{r+m}+(\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=1}\frac{m}{r+m}\right) \\
&\quad +\frac{1}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n} \\
&\quad \times\left((\phi_j(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=0}+(\phi_j(x))\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=0}\frac{r}{r+m}+(\phi_j{}'(x))\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\frac{m}{r+m}\right).
\end{aligned}
\tag{4.33}
$$

On rearranging and using the definition of $\mathcal{U}$ given above, we get

$$E_{j,l,r,m} = \frac{1}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\left(e^2(\phi_j(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=1} - (\phi_j(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\right)$$
$$\times\left(\frac{1}{2(r+m+1)} + \frac{2rm}{r+m-1}\right)$$
$$+\frac{1}{2}\sum_{n=0}^{j+l+2}\left(e^2(\phi_j(x)\overline{\phi_l{}'(x)} + \phi_j{}'(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=1} - (\phi_j(x)\overline{\phi_l{}'(x)} + \phi_j{}'(x)\overline{\phi_l(x)})^{(n)}\Big|_{x=0}\right)$$
$$\times\frac{1}{2(r+m+1)}$$
$$+\frac{1}{2}\sum_{n=0}^{j+l+2}\frac{(-1)^n}{2^n}\left(e^2(\phi_j{}'(x)\overline{\phi_l'(x)})^{(n)}\Big|_{x=1} - (\phi_j{}'(x)\overline{\phi_l{}'(x)})^{(n)}\Big|_{x=0}\right)\frac{1}{2(r+m+1)}$$
$$+\mathcal{U} + O_{\vec{c}}\left((2a)^{2a+4}\frac{\log^5 L}{L}\right).$$

Using Lemma 28 from [49] and the earlier estimates for $\mathcal{U}$, we have

$$\begin{aligned}E_{j,l,r,m} = &\left(\frac{1}{2(r+m+1)} + \frac{2rm}{r+m-1}\right)\int_0^1 e^{2x}\phi_j(x)\overline{\phi_l(x)}\ dx\\ &+\frac{1}{2(r+m+1)}\int_0^1 e^{2x}\phi_j{}'(x)\overline{\phi_l{}'(x)}\ dx + \frac{1}{2(r+m+1)}\int_0^1 e^{2x}(\phi_j(x)\overline{\phi_l'(x)}\\ &+\phi_j{}'(x)\overline{\phi_l(x)})\ dx + \mathcal{U} + O_{\vec{c}}\left((2a)^{2a+4}\frac{\log^5 L}{L}\right).\end{aligned} \tag{4.34}$$

Note that the last integrals can be written in terms of the first one as follows

$$\int_0^1 e^{2x}(\phi_j(x)\overline{\phi_l'(x)} + \phi_j{}'(x)\overline{\phi_l(x)})\ dx = -1 - 2\int_0^1 e^{2x}\phi_j(x)\overline{\phi_l(x)}\ dx + \mathrm{O}\left(aL^{-2}\right).$$

Therefore (4.34) becomes

$$\begin{aligned}E_{j,l,r,m} = &\left(\frac{2rm}{r+m-1} - \frac{1}{2(r+m+1)}\right)\int_0^1 e^{2x}\phi_j(x)\overline{\phi_l(x)}\ dx\\ &+\frac{1}{2(r+m+1)}\int_0^1 e^{2x}\phi_j{}'(x)\overline{\phi_l{}'(x)}\ dx - \frac{1}{2(r+m+1)} + \mathcal{U} + O_{\vec{c}}\left((2a)^{2a+4}\frac{\log^5 L}{L}\right).\end{aligned} \tag{4.35}$$

Using the definitions of the function ϕ_j and ϕ_l from (4.32) in the integral and using binomial expansions, we estimate the first two integrals in the above expression. For the first integral, we have

$$\int_0^1 e^{2x}\phi_j(x)\overline{\phi_l(x)}\ dx = \int_0^1 e^{2x}(1-x)^2(1-2x)^{2a+2j+2l}\ dx + \mathrm{O}\left(aL^{-2}\right).$$

Using integration by parts repeatedly, we rewrite the integral on the right-hand side as

$$\begin{aligned}
&\int_0^1 e^{2x}(1-x)^2(1-2x)^{2a+2j+2l}\ dx \\
&= \frac{1}{2(2a+2j+2l+1)} + \frac{e^2-1}{4(2a+2j+2l+1)(2a+2j+2l+2)(2a+2j+2l+3)} \\
&\quad - \frac{e^2+1}{4(2a+2j+2l+1)(2a+2j+2l+2)(2a+2j+2l+3)(2a+2j+2l+4)} \\
&\quad + \mathrm{O}\left(1/(a+j+l)^5\right) \\
&= \frac{1}{2(2a+2j+2l)} - \frac{1}{2(2a+2j+2l)^2} + \frac{(e^2+1)}{4(2a+2j+2l)^3} - \frac{9e^2-3}{4(2a+2j+2l)^4} + \mathrm{O}\left(1/a^5\right).
\end{aligned} \tag{4.36}$$

Similarly, for the second integral in (4.35), we have

$$
\begin{aligned}
&\int_0^1 e^{2x}\phi_j{}'(x)\overline{\phi_l{}'(x)}\ dx \\
&= \int_0^1 e^{2x}(1-2x)^{2a+2j+2l-2} \\
&\quad \times ((1+2a+4j)-(2a+4j+2)x)((1+2a+4l)-(2a+4l+2)x)\ dx \\
&= \frac{(-1)^{a+2j+l}e^2+(2a+4l+1)(2a+4l+1)}{2(2a+2j+2l-1)} \\
&\quad + \frac{-e^2(2a+2j+2l+3)-(2a+2j+2l+1)}{2(2a+2j+2l-1)(j+l)} \\
&\quad - \frac{3e^2+1}{4(2a+2j+2l+1)(2a+2j+2l+2)(2a+2j+2l+3)(2a+2j+2l+4)} \\
&\quad + \mathrm{O}\left(1/(a+j+l)^4\right) \\
&= \frac{2(a+2j)(a+2l)}{2a+2j+2l} + \frac{2(a+2j)(a+2l)}{(2a+2j+2l)^2} + 1 \\
&\quad + \left(\frac{(a+2j)(a+2l)}{(2a+2j+2l)^2} + 1 + \frac{(a+2j)(a+2l)((-1)^{2a+2j+2l}e^2-1)}{(2a+2j+2l)^2}\right)\frac{1}{2a+2j+2l} \\
&\quad + \left(\frac{-2((-1)^{2a+2j+2l}3e^2+1)(a+2j)(a+2l)}{(2a+2j+2l)^2} + (-1)^{2a+2j+2l}e^2 - 1 + \frac{4(a+2j)(a+2l)}{(2a+2j+2l)^2}\right) \\
&\quad \times \frac{1}{2(2a+2j+2l)^2} + O_{\vec{c}}\left(1/a^3\right). \qquad (4.37)
\end{aligned}
$$

We also expand the terms in r and m in (4.35) as follows

$$\frac{2rm}{r+m-1} = \frac{2rm}{r+m} + \frac{2rm}{(r+m)^2} + \frac{2rm}{(r+m)^3} + \frac{2rm}{(r+m)^4} + \frac{2rm}{(r+m)^4(m+r-1)}, \quad (4.38)$$

and

$$\frac{1}{r+m+1} = \frac{1}{r+m} - \frac{1}{(r+m)^2} + \frac{1}{(r+m)^3} - \frac{1}{(r+m)^4} + \frac{1}{(r+m)^4(r+m+1)}. \quad (4.39)$$

Substituting (4.36), (4.37), (4.38), and (4.39) in (4.35), we obtain

$$
\begin{aligned}
E_{j,l,r,m} &= \frac{(rm+(a+2j)(a+2l))}{(2a+2j+2l)(r+m)} + \left\{ \frac{(rm-(a+2j)(a+2l))}{(2a+2j+2l)(r+m)} \left(\frac{1}{r+m} - \frac{1}{2a+2j+2l} \right) \right\} \\
&+ \left\{ \frac{rm}{(2a+2j+2l)(r+m)^3} - \frac{rm}{(2a+2j+2l)^2(r+m)^2} + \frac{(e^2+1)rm}{2(2a+2j+2l)^3(r+m)} \right. \\
&- \frac{1}{4(2a+2j+2l)(r+m)} + \frac{(a+2j)(a+2l)}{(2a+2j+2l)(r+m)^3} - \left(\frac{2jl}{(2a+2j+2l)^2} + 1 \right) \frac{1}{2(r+m)^2} \\
&+ \left(\frac{2(a+2j)(a+2l)}{(2a+2j+2l)^2} + 1 + \frac{(a+2j)(a+2l)(e^2-1)}{(2a+2j+2l)^2} \right) \frac{1}{2(2a+2j+2l)(r+m)} \\
&\left. + \frac{1}{2(r+m)^2} \right\} + \mathcal{U} + O_{\vec{c}}\left(\frac{1}{a^3}\right) \\
&= \mathcal{U} + \frac{(rm+(a+2j)(a+2l))}{(2a+2j+2l)(r+m)} + \frac{e^2+2}{16}\frac{1}{a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right),
\end{aligned}
\tag{4.40}
$$

where in the last equality, we have used the facts

$$
\frac{rm-(a+2j)(a+2l)}{(2j+2l+2a)(r+m)} \left(\frac{1}{r+m} - \frac{1}{2j+2l} \right) = O_{\vec{c}}\left(\frac{1}{a^3}\right),
$$

and

$$
\begin{aligned}
&\frac{rm}{(2j+2l+2a)(r+m)^3} - \frac{rm}{(2j+2l+2a)^2(r+m)^2} + \frac{(e^2+1)rm}{(2j+2l+2a)^3 2(r+m)} \\
&\quad - \frac{1}{4(2j+2l+2a)(r+m)} + \frac{(a+2j)(a+2l)}{(2j+2l+2a)(r+m)^3} - \left(\frac{2(a+2j)(a+2l)}{(2j+2l+2a)^2} + 1 \right) \\
&\quad \times \frac{1}{2(r+m)^2} + \left(\frac{2(a+2j)(a+2l)}{(2j+2l+2a)^2} + 1 + \frac{(a+2j)(a+2l)(e^2-1)}{(2j+2l+2a)^2} \right) \\
&\quad \frac{1}{2(2j+2l+2a)(r+m)} + \frac{1}{2(r+m)^2} \\
&= \frac{e^2+2}{16}\frac{1}{a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).
\end{aligned}
$$

Recall from (4.18) that $A_{j,l} = \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c_r}\tilde{c_m} E_{j,l,r,m}$. First, we compute the term corresponding to $\mathcal{U}$ in $A_{j,l}$ separately below. Recall from the definition of $\mathcal{U}$ from (4.33),

$$\begin{aligned}
&\sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c}_r \tilde{c}_m \mathcal{U} \\
&= \frac{e^2}{2} \sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c}_r^2 \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{1}{2} \left((\phi_j(x)\overline{\phi_l}'(x))^{(n)} + (\phi_j{}'(x)\overline{\phi_l}(x))^{(n)} \right) \right) \Bigg|_{x=1} \\
&\quad + \frac{e^2}{2} \sum_{r=\tilde{a}}^{\tilde{b}} \sum_{\substack{m=\tilde{a} \\ m\neq r}}^{\tilde{b}} \tilde{c}_r \tilde{c}_m \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \\
&\quad \times \left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{r}{r+m} (\phi_j(x)\overline{\phi_l}'(x))^{(n)} + \frac{m}{r+m} (\phi_j{}'(x)\overline{\phi_l}(x))^{(n)} \right) \Bigg|_{x=1} \\
&\quad + \frac{1}{2} \sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c}_r^2 \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \\
&\quad \times \left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{1}{2} \left((\phi_j(x)\overline{\phi_l}'(x))^{(n)} + (\phi_j{}'(x)\overline{\phi_l}(x))^{(n)} \right) \right) \Bigg|_{x=0} \\
&\quad + \frac{1}{2} \sum_{r=\tilde{a}}^{\tilde{b}} \sum_{\substack{m=\tilde{a} \\ m\neq r}}^{\tilde{b}} \tilde{c}_r \tilde{c}_m \sum_{n=0}^{2a+2j+2l+2} \frac{(-1)^n}{2^n} \\
&\quad \times \left((\phi_j(x)\overline{\phi_l}(x))^{(n)} + \frac{r}{r+m} (\phi_j(x)\overline{\phi_l}'(x))^{(n)} + \frac{m}{r+m} (\phi_j{}'(x)\overline{\phi_l}(x))^{(n)} \right) \Bigg|_{x=0}. \qquad (4.41)
\end{aligned}$$

With the use Lemma 29 from [49], first and third term in the above expression combine as follows.

$$\frac{e^2}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}^2\sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n}\left((\phi_j(x)\overline{\phi_l}(x))^{(n)}+\frac{1}{2}\left((\phi_j(x)\overline{\phi_l}'(x))^{(n)}+(\phi_j'(x)\overline{\phi_l}(x))^{(n)}\right)\right)\Big|_{x=1}$$
$$+\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}^2\sum_{n=0}^{2a+2j+2l+2}\frac{(-1)^n}{2^n}\left((\phi_j(x)\overline{\phi_l}(x))^{(n)}+\frac{1}{2}\left((\phi_j(x)\overline{\phi_l}'(x))^{(n)}+(\phi_j'(x)\overline{\phi_l}(x))^{(n)}\right)\right)\Big|_{x=0}$$
$$=\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}^2\left(\frac{e^2}{2}\phi_j(1)\overline{\phi_l}(1)+\frac{1}{2}\phi_j(0)\overline{\phi_l}(0)\right)$$
$$=\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}^2\left(1+\frac{i\pi}{2L}\right)^{a+2j}\left(1-\frac{i\pi}{2L}\right)^{a+2l}$$
$$=\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}^2+O_{\vec{c}}\left(\frac{a}{L}\right),$$

where we obtain the last equality using Lemma 1 from [49]. Similarly, the second and fourth sum in (4.41) combine as

$$\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\sum_{\substack{m=\tilde{a}\\m\neq r}}^{\tilde{b}}\tilde{c_r}\tilde{c_m}\left(1+\frac{i\pi}{2L}\right)^{a+2j}\left(1-\frac{i\pi}{2L}\right)^{a+2l}$$
$$=\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\sum_{\substack{m=\tilde{a}\\m\neq r}}^{\tilde{b}}\tilde{c_r}\tilde{c_m}+O_{\vec{c}}\left(\frac{a}{L}\right).$$

Thus, employing the fact that $\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}=1$, we conclude

$$\sum_{r,m=\tilde{a}}^{\tilde{b}}\tilde{c_r}\tilde{c_m}\mathcal{U}=\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}^2+\frac{1}{2}\sum_{r=\tilde{a}}^{\tilde{b}}\sum_{\substack{m=\tilde{a}\\m\neq r}}^{\tilde{b}}\tilde{c_r}\tilde{c_m}+O_{\vec{c}}\left(\frac{a}{L}\right)=\frac{1}{2}\sum_{r,m=\tilde{a}}^{\tilde{b}}\tilde{c_r}\tilde{c_m}+O_{\vec{c}}\left(\frac{a}{L}\right)$$
$$=\frac{1}{2}\left(\sum_{r=\tilde{a}}^{\tilde{b}}\tilde{c_r}\right)^2+O_{\vec{c}}\left(\frac{a}{L}\right)=\frac{1}{2}+O_{\vec{c}}\left(\frac{a}{L}\right).$$

Employing this estimate in (4.40), we have

$$A_{j,l} = \sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c_r}\tilde{c_m} E_{j,l,r,m}$$
$$= \frac{1}{2} + \frac{e^2+2}{16}\frac{1}{a^2} + \sum_{r,m=\tilde{a}}^{\tilde{b}} \frac{\tilde{c_r}\tilde{c_m}(rm+(2j+a)(2l+a))}{(2j+2l+2a)(r+m)} + O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.42}$$

Let $r = a+u, m = a+v$ for some $-X \le u, v \le X$. Then,

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c_r}\tilde{c_m} \frac{rm+(2j+a)(2l+a)}{(2j+2l+2a)(r+m)}$$
$$= \sum_{-X\le u,v\le X} \tilde{c_{a+u}}\tilde{c_{a+v}} \frac{2a^2 + a(u+v+2j+2l)+(uv+4jl)}{4a^2(1+\frac{2j+2l}{2a})(1+\frac{u+v}{2a})}$$
$$= \sum_{-X\le u,v\le X} \tilde{c_{a+u}}\tilde{c_{a+v}} \left(\frac{1}{2} + \left(\frac{-(u+v)(2j+2l)}{4} + \frac{uv+4jl}{4}\right)\frac{1}{a^2}\right) + O_{\vec{c}}\left(\frac{1}{a^3}\right)$$
$$= \frac{1}{2}\sum_{-X\le u,v\le X} \tilde{c_{a+u}}\tilde{c_{a+v}} + \frac{jl}{a^2}\sum_{-X\le u,v\le X} \tilde{c_{a+u}}\tilde{c_{a+v}} - \frac{(j+l)}{2a^2}\sum_{-X\le u,v\le X} \tilde{c_{a+u}}\tilde{c_{a+v}}(u+v)$$
$$+ \frac{1}{4a^2}\sum_{-X\le u,v\le X} \tilde{c_{a+u}}\tilde{c_{a+v}}uv + O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.43}$$

Now let $\mathcal{S}$ denote the sum

$$\mathcal{S} := \sum_{-X\le t\le X} t\tilde{c_{a+t}}.$$

Using this notation and the fact $\sum_{r=\tilde{a}}^{\tilde{b}} \tilde{c_r} = 1$ in (4.43), we see that

$$\sum_{r,m=\tilde{a}}^{\tilde{b}} \tilde{c_r}\tilde{c_m} \frac{rm+(2j+a)(2l+a)}{(2j+2l+2a)(r+m)} = \frac{1}{2} + \frac{4jl - 2(j+l)\mathcal{S} + \mathcal{S}^2}{4a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

Using the above estimates in (4.42), we arrive at

$$A_{j,l} = 1 + \frac{e^2+2}{16}\frac{1}{a^2} + \frac{4jl - 2(j+l)\mathcal{S} + \mathcal{S}^2}{4a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

Also recall from (4.17) that

$$\frac{J}{U} = \frac{1}{|d^*|^2} \sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l}}{2^{2a+2j+2l}} A_{j,l}.$$

On substituting $A_{j,l}$ here and using the definition of d^*, we obtain

$$\frac{J}{U} = 1 + \frac{e^2+2}{16a^2} + \left(\sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l} 4jl}{2^{2a+2j+2l}} - \sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l} 2(j+l)}{2^{2a+2j+2l}} \mathcal{S} + |d^*|^2 \mathcal{S}^2 \right)$$
$$\times \frac{1}{|d^*|^2 4a^2} + O_{\vec{c}}\left(\frac{1}{a^3}\right). \tag{4.44}$$

Although one has flexibility in choosing $\mathcal{S}$, the expression inside the parenthesis on the right side of (4.44) cannot be decreased below zero. Its minimum is actually zero, and it is attained at

$$\mathcal{S} = \frac{1}{2|d^*|^2} \sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l}(2j+2l)}{2^{2a+2j+2l}}.$$

Also, the mollifier in this case does allow one to arrange for such a condition to hold: there exist coefficients $\tilde{c}_{a+t}$ such that the minimum is attained and such that $\tilde{c}_{a+t}$ also satisfies

$$\sum_{-X \le t \le X} \tilde{c_{a+t}} = 1.$$

Therefore after substituting the minimum value of $\mathcal{S}$ in (4.44) we arrive at

$$\frac{J}{U} = 1 + \frac{e^2+2}{16a^2} + \frac{1}{4a^2|d^*|^2} \sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l} 4jl}{2^{2a+2j+2l}}$$
$$- \frac{1}{4a^2} \left(\frac{1}{2|d^*|^2} \sum_{j,l=0}^{M} \frac{c_j c_l i^{a+2j}(-i)^{a+2l}(2j+2l)}{2^{2a+2j+2l}} \right)^2 + O_{\vec{c}}\left(\frac{1}{a^3}\right).$$

The two sums involving c_j, c_l cancel each other since

$$\begin{aligned}
&\left(\frac{1}{2|d^*|^2}\sum_{j,l=0}^{M}\frac{c_jc_li^{a+2j}(-i)^{a+2l}(2j+2l)}{2^{2a+2j+2l}}\right)^2\\
&\quad=\left(\frac{1}{|d^*|^2}\sum_{j,l=0}^{M}\frac{jc_jc_li^{2a+2j+2l}}{2^{2a+2j+2l}}+\frac{1}{|d^*|^2}\sum_{j,l=0}^{M}\frac{lc_jc_li^{2a+2j+2l}}{2^{2a+2j+2l}}\right)^2\\
&\quad=\frac{4}{|d^*|^4}\left(\sum_{j,l=0}^{M}\frac{jc_jc_li^{2a+2j+2l}}{2^{2a+2j+2l}}\right)^2=\frac{4}{|d^*|^4}\left(\sum_{j=0}^{M}\frac{jc_ji^{a+2j}}{2^{a+2j}}\right)^2\left(\sum_{l=0}^{M}\frac{c_li^{a+2l}}{2^{a+2l}}\right)^2\\
&\quad=\frac{4(-1)^a}{|d^*|^4}\left(\sum_{j=0}^{M}\frac{jc_ji^{a+2j}}{2^{a+2j}}\right)^2\left(\sum_{j=0}^{M}\frac{c_ji^{a+2j}}{2^{a+2j}}\sum_{l=0}^{M}\frac{c_l(-i)^{a+2l}}{2^{a+2l}}\right)\\
&\quad=\frac{4(-1)^a}{|d^*|^4}\left(\sum_{j=0}^{M}\frac{jc_ji^{a+2j}}{2^{a+2j}}\right)^2|d^*|^2=\frac{4(-1)^a}{|d^*|^2}\left(\sum_{j=0}^{M}\frac{jc_ji^{a+2j}}{2^{a+2j}}\right)^2\\
&\quad=\frac{1}{|d^*|^2}\sum_{j,l=0}^{M}\frac{c_jc_li^{a+2j}(-i)^{a+2l}4jl}{2^{2a+2j+2l}}.
\end{aligned}$$

Consequently, this yields

$$\frac{J}{U}=1+\frac{e^2+2}{16a^2}+O_{\vec{c}}\left(\frac{1}{a^3}\right),$$

which completes the proof of the lemma. □

4.4 Proof of Theorem 1.2.1

Proof of Theorem 1.2.1. Finally, putting Lemmas 4.1.4, 4.2.1, and Lemma 4.3.1 together, we obtain

$$\begin{aligned}
N^{(0)}_{\vec{c},a}(T+U)-&N^{(0)}_{\vec{c},a}(T)\\
&\geq N_{\vec{c},a}(T+U)-N_{\vec{c},a}(T)-2N+O_{\vec{c}}\left(a\log T\right)\\
&\geqslant N_{\vec{c},a}(T+U)-N_{\vec{c},a}(T)-\frac{UL}{2\pi}\log\left(\frac{J}{U}\right)+O_{\vec{c}}\left(aU\right)\\
&\geqslant N_{\vec{c},a}(T+U)-N_{\vec{c},a}(T)-\frac{UL}{2\pi}\log\left(1+\frac{e^2+2}{16a^2}+O_{\vec{c}}\left(\frac{1}{a^3}\right)\right)+O_{\vec{c}}\left(aU\right)\\
&\geqslant N_{\vec{c},a}(T+U)-N_{\vec{c},a}(T)-\frac{UL}{2\pi}\left(\frac{e^2+2}{16a^2}+O_{\vec{c}}\left(\frac{1}{a^3}\right)\right)+O_{\vec{c}}\left(aU\right).
\end{aligned}$$

From Lemma 4.1.3, one notes that

$$N_{\vec{c},a}(T+U)-N_{\vec{c},a}(T)=\frac{UL}{2\pi}+O_{\vec{c}}\left(U\right)=\frac{UL}{2\pi}\left(1+O_{\vec{c}}\left(\frac{a}{L}\right)\right).$$

Using this in the above inequality, we obtain

$$\begin{aligned}
\kappa_{\vec{c},a,T}&=\frac{N^{(0)}_{\vec{c},a}(T+U)-N^{(0)}_{\vec{c},a}(T)}{N_{\vec{c},a}(T+U)-N_{\vec{c},a}(T)}\\
&\geq 1-\frac{1}{1+O_{\vec{c}}\left(L^{-1}\right)}\left(\frac{e^2+2}{16a^2}+O_{\vec{c}}\left(a^{-3}\right)\right)+O_{\vec{c}}\left(\frac{a}{L}\right)\\
&\geq 1-\frac{e^2+2}{16a^2}+O_{\vec{c}}\left(\frac{1}{a^3}\right).
\end{aligned}$$

This completes the proof of Theorem 1.2.1.

□

Remark 4.4.1. *Using the same techniques, one can get an identical result on the proportion of simple zeros of $F_{\vec{c},a,T}(s)$ on the critical line.*

Chapter 5

Divisibility properties of sporadic Apéry-like numbers

In this chapter, we state and prove our results on sporadic Apéry-like numbers. We start with a review of the Apéry-like numbers.

5.1 Review of Apéry-like numbers

Recall form (1.13) that the Apéry numbers are defined as

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Along with the Apéry numbers $A(n)$, defined in (1.13), R. Apéry also introduced the sequence

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

which allowed him to (re)prove the irrationality of $\zeta(2)$. This sequence is the solution of the three-term recursion

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \tag{5.1}$$

with the choice of parameters $(a, b, c) = (11, 3, -1)$ and initial conditions $u_{-1} = 0$, $u_0 = 1$. Because we divide by $(n+1)^2$ at each step, it is not to be expected that the recursion (5.1) should have an integer solution. Inspired by F. Beukers [12], D. Zagier [65] conducted a systematic search for other choices of the parameters (a, b, c) for which the solution to (5.1), with initial conditions $u_{-1} = 0$, $u_0 = 1$, is integral. After normalizing, and apart from degenerate cases, he discovered four hypergeometric, four Legendrian as well as six sporadic solutions. It is still open whether further solutions exist or even that there should be only finitely many solutions. The six sporadic solutions are reproduced in Table 5.1. Note that each binomial sum included in this table certifies that the corresponding sequence indeed consists of integers.

Table 5.1: The six sporadic solutions of (5.1)

(a,b,c)	[65]	[4]	$A(n)$
$(7,2,-8)$	**A**	(a)	$\sum_k \binom{n}{k}^3$
$(11,3,-1)$	**D**	(b)	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}$
$(10,3,9)$	**C**	(c)	$\sum_k \binom{n}{k}^2 \binom{2k}{k}$
$(12,4,32)$	**E**	(d)	$\sum_k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(9,3,27)$	**B**	(f)	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$
$(17,6,72)$	**F**	(g)	$\sum_{k,l} (-1)^k 8^{n-k} \binom{n}{k}\binom{k}{l}^3$

Similarly, the Apéry numbers $A(n)$, defined in (1.13), are the solution of the three-term recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}, \tag{5.2}$$

with the choice of parameters $(a,b,c,d) = (17,5,1,0)$ and initial conditions $u_{-1} = 0$, $u_0 = 1$. Systematic computer searches for further integer solutions have been performed by G. Almkvist and W. Zudilin [5] in the case $d = 0$ and, more recently, by S. Cooper [20], who introduced the additional parameter d. As in the case of (5.1), apart from degenerate cases, only finitely many sequences have been discovered. In the case $d = 0$, there are again six sporadic sequences, which are recorded in Table 5.2. Moreover, by general principles (see [20, Eq. (17)]), each of the sequences in Table 5.1 times $\binom{2n}{n}$ is an integer solution of (5.2) with $d \neq 0$. Apart from such expected solutions, Cooper also found three additional sporadic solutions, including

$$s_{18}(n) = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right], \qquad (5.3)$$

for $n \geqslant 1$, with $s_{18}(0) = 1$, as well as s_7 and s_{10}, which are included in Table 5.2. Remarkably, these sequences are again connected to modular forms [20] (the subscript refers to the level) and satisfy supercongruences, which are proved in [47]. Indeed, it was the corresponding modular forms and Ramanujan-type series for $1/\pi$ that led Cooper to study these sequences, and the binomial expressions for s_7 and s_{18} were found subsequently by Zudilin (sequence s_{10} was well-known before).

5.2 Lucas congruences

It is a well-known and beautiful classical result of Lucas [34] that the binomial coefficients satisfy the congruences

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r} \pmod{p}, \qquad (5.4)$$

where p is a prime and n_i, respectively k_i, are the p-adic digits of n and k. That is, $n = n_0 + n_1 p + \cdots + n_r p^r$ and $k = k_0 + k_1 p + \cdots + k_r p^r$ are the expansions of n and k in base p. Correspondingly, a sequence $a(n)$ is said to satisfy *Lucas congruences*, if the congruences

$$a(n) \equiv a(n_0)a(n_1)\cdots a(n_r) \pmod{p} \qquad (5.5)$$

hold for all primes p. It was shown by I. Gessel [28, Theorem 1] that the Apéry numbers $A(n)$, defined in (1.13), satisfy Lucas congruences. E. Deutsch and B. Sagan [23, Theorem 5.9] show

Table 5.2: The sporadic solutions of (5.2)

(a,b,c,d)	[4]	[20]	$A(n)$
$(7,3,81,0)$	(δ)		$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125,0)$	(η)		$\sum_{k=0}^{\lfloor n/5 \rfloor} (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64,0)$	(α)		$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16,0)$	(ϵ)		$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27,0)$	(ζ)		$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1,0)$	(γ)		$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$
$(13,4,-27,3)$		s_7	$\sum_k \binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$
$(6,2,-64,4)$		s_{10}	$\sum_k \binom{n}{k}^4$
$(14,6,192,-12)$		s_{18}	defined in (5.3)

that the Lucas congruences (5.5) in fact hold for the family of generalized Apéry sequences

$$A_{r,s}(n) = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^s, \tag{5.6}$$

with r and s positive integers. This family includes the sequences (a), (b) from Table 5.1, and the sequences (γ), s_{10} from Table 5.2. The purpose of this section and Section 5.3 is to show that, in fact, all the Apéry-like sequences in Tables 5.1 and 5.2 satisfy the Lucas congruences (5.5). Using and extending the general framework provided by R. McIntosh [36, Theorem 6], which we review below, we are able to prove this claim for all of the sequences in the two tables, with the exception of the two sequences (η) and s_{18}, for which we require a much finer analysis, which is given in Section 5.3.

Theorem 5.2.1. *Each of the sequences from Tables 5.1 and 5.2 satisfies the Lucas congruences* (5.5)*.*

Remark 5.2.1. *The Lucas congruences* (5.5)*, in general, do not extend to prime powers. However, it is shown in [28], and generalized in [32], that the Lucas congruences modulo* 3 *for the Apéry numbers extend to hold modulo* 9*.*

On the other hand, numerical evidence suggests that all the Apéry-like sequences from Tables 5.1 and 5.2 in fact satisfy the Dwork congruences (1.15)*. While Theorem 5.2.1 proves the case* $r = 1$ *of these congruences, it would be desirable to establish the corresponding congruences modulo higher powers of primes.*

Following [36], we say that a function $L : \mathbb{Z}^2_{\geqslant 0} \to \mathbb{Z}$ has the *double Lucas property* (**DLP**) if $L(n, k) = 0$, for $k > n$, and if

$$L(n,k) \equiv L(n_0,k_0)L(n_1,k_1)\cdots L(n_r,k_r) \pmod{p}, \tag{5.7}$$

for every prime p. Here, as in (5.4), n_i and k_i are the p-adic digits of n and k, respectively. Equation (5.4) shows that the binomial coefficients $\binom{n}{k}$ are a **DLP** function. More generally, it is shown in [36, Theorem 6] that, for positive integers $r_0, r_1, \ldots, r_m$,

$$L(n,k) = \binom{n}{k}^{r_0}\binom{n+k}{k}^{r_1}\binom{n+2k}{k}^{r_2}\cdots\binom{n+mk}{k}^{r_m} \tag{5.8}$$

is a **DLP** function. For instance, choosing the exponents as $r_i = 1$, we find that the multinomial coefficient

$$\binom{n+mk}{k,k,\ldots,k,n-k} = \frac{(n+mk)!}{k!^{m+1}(n-k)!}$$

is a **DLP** function for any integer $m \geqslant 0$.

Suppose that $L(n, k)$ is a **DLP** function and that $G(n)$ and $H(n)$ are **LP** functions, that is, the sequences $G(n)$ and $H(n)$ satisfy the Lucas congruences (5.5). Then, as shown in [36, Theorem 5],

$$F(n) = \sum_{k=0}^{n} L(n,k)G(k)H(n-k) \tag{5.9}$$

is an **LP** function. Note that (5.8) and (5.9) combined are already sufficient to prove that the generalized Apéry sequences, defined in (5.6), satisfy Lucas congruences. In order to apply this machinery more generally, and prove Theorem 5.2.1, our next results extend the repertoire of **DLP** functions. In fact, it turns out that we need a natural extension of the Lucas property to the case of three variables. We say that a function $M : \mathbb{Z}_{\geqslant 0}^3 \to \mathbb{Z}$ has the *triple Lucas property* (**TLP**) if $M(n,k,j) = 0$, for $j > n$, and if

$$M(n,k,j) \equiv M(n_0,k_0,j_0) \cdots M(n_r,k_r,j_r) \pmod{p},$$

for every prime p, where n_i, k_i and j_i are the p-adic digits of n, k and j, respectively. It is straightforward to prove the following analog of (5.9) for **TLP** functions.

Lemma 5.2.2. *If $M(n,k,j)$ is a **TLP** function, then*

$$L(n,k) = \sum_{j=0}^{n} M(n,k,j)$$

satisfies the double Lucas congruences (5.7)*. In particular, if $L(n,k) = 0$, for $k > n$, then $L(n,k)$ is a **DLP** function.*

Proof. Let p be a prime. It is enough to show that, for any nonnegative integers n_0, n', k_0, k' such that $n_0 < p$ and $k_0 < p$,

$$L(n_0 + n'p, k_0 + k'p) \equiv L(n_0,k_0)L(n',k') \pmod{p}.$$

Since the sum defining $L(n,k)$ is naturally supported on $j \in \{0, 1, \ldots, n\}$, we may extend it over all $j \in \mathbb{Z}$. Modulo p, we have

$$\begin{aligned} L(n,k) &= \sum_{j \in \mathbb{Z}} M(n,k,j) \\ &= \sum_{j_0=0}^{p-1} \sum_{j' \in \mathbb{Z}} M(n,k,j_0+j'p) \\ &\equiv \sum_{j_0 \in \mathbb{Z}} \sum_{j' \in \mathbb{Z}} M(n_0,k_0,j_0) M(n',k',j') \\ &= L(n_0,k_0) L(n',k'), \end{aligned}$$

which is what we had to prove. □

Lemma 5.2.3. *The function*

$$M(n,k,j) = \binom{n}{j}\binom{k+j}{n}$$

is a ***TLP*** *function.*

Proof. Clearly, $M(n,k,j) = 0$, for $j > n$. In order to show that $M(n,k,j)$ is a **TLP** function, we therefore need to show that, for any prime p,

$$M(n_0+n'p, k_0+k'p, j_0+j'p) \equiv M(n_0,k_0,j_0) M(n',k',j') \pmod{p}, \tag{5.10}$$

provided that $0 \leqslant n_0, k_0, j_0 < p$ and $n', k', j' \geqslant 0$. Observe that in the case $j_0 > n_0$ both sides of the congruence (5.10) vanish because of the Lucas congruences (5.4) for the binomial coefficients. We may therefore proceed under the assumption that $j_0 \leqslant n_0$.

Writing $[x^n]f(x)$ for the coefficient of x^n in the polynomial $f(x)$, we begin with the simple observation that

$$\binom{k+j}{n} = [x^n](1+x)^{k+j}.$$

Modulo p, we have

$$(1+x)^{k+j} = (1+x)^{k_0+j_0}(1+x)^{(k'+j')p} \equiv (1+x)^{k_0+j_0}(1+x^p)^{k'+j'} \pmod{p}.$$

Since $0 \leqslant k_0 + j_0 < 2p$, extracting the coefficient of $x^n = x^{n_0}(x^p)^{n'}$ from this product results in the congruence

$$\binom{k+j}{n} \equiv \binom{k_0+j_0}{n_0}\binom{k'+j'}{n'} + \binom{k_0+j_0}{n_0+p}\binom{k'+j'}{n'-1} \pmod{p}.$$

Note that, under our assumption that $j_0 \leqslant n_0$, the second term on the right-hand side of this congruence vanishes (since $n_0 + p \geqslant j_0 + p > j_0 + k_0$). This, along with (5.4), proves (5.10). □

Corollary 5.2.4. *The function*

$$L(n,k) = \binom{n}{k}\binom{2k}{n}$$

is a ***DLP*** *function.*

Proof. Set $j = k$ in Lemma 5.2.3. □

Lemma 5.2.5. *The function*

$$L(n,k) = 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}$$

is a ***DLP*** *function.*

Proof. Let p be a prime. As usual, we write $n = n_0 + n'p$ and $k = k_0 + k'p$ where $0 \leqslant n_0 < p$ and $0 \leqslant k_0 < p$. In light of (5.4) and (5.9), the simple observation

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2, \tag{5.11}$$

demonstrates that the sequence of central binomial coefficients is an **LP** function. We claim that

$$\frac{(3k)!}{k!^3} = \binom{3k}{k}\binom{2k}{k}$$

is an **LP** function as well. From the Lucas congruences for the central binomials, that is

$$\binom{2k}{k} \equiv \binom{2k_0}{k_0}\binom{2k'}{k'} \pmod{p},$$

we observe that $\binom{2k}{k}$ is divisible by p if $2k_0 \geqslant p$. Hence, we only need to show the congruences

$$\frac{(3k)!}{k!^3} \equiv \frac{(3k_0)!}{k_0!^3}\frac{(3k')!}{k'!^3} \pmod{p} \tag{5.12}$$

under the assumption that $k_0 < p/2$. Note that

$$\begin{aligned}\binom{3k}{k} &= [x^k](1+x)^{3k} \\ &\equiv [x^{k_0}(x^p)^{k'}](1+x)^{3k_0}(1+x^p)^{3k'} \pmod{p} \\ &= \binom{3k_0}{k_0}\binom{3k'}{k'} + \binom{3k_0}{k_0+p}\binom{3k'}{k'-1} + \binom{3k_0}{k_0+2p}\binom{3k'}{k'-2}.\end{aligned}$$

In the case $k_0 < p/2$, we have $k_0 + p > 3k_0$, so that the last two terms on the right-hand side vanish. This proves (5.12).

Next, we claim that

$$\binom{n}{3k}\frac{(3k)!}{k!^3} \equiv \binom{n_0}{3k_0}\frac{(3k_0)!}{k_0!^3}\binom{n'}{3k'}\frac{(3k')!}{k'!^3} \pmod{p}. \tag{5.13}$$

By congruence (5.12), both sides vanish modulo p if $3k_0 \geqslant p$. On the other hand, if $3k_0 < p$, then the usual argument shows that

$$\binom{n}{3k} \equiv [x^{3k_0}(x^p)^{3k'}](1+x)^{n_0}(1+x^p)^{n'} = \binom{n_0}{3k_0}\binom{n'}{3k'} \pmod{p}.$$

In combination with (5.12), this proves (5.13).

Finally, the congruences $L(n,k) \equiv L(n_0,k_0)L(n',k')$, that is

$$3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3} \equiv 3^{n_0-3k_0}\binom{n_0}{3k_0}\frac{(3k_0)!}{k_0!^3}3^{n'-3k'}\binom{n'}{3k'}\frac{(3k')!}{k'!^3} \pmod{p},$$

follow from Fermat's little theorem and the fact that both sides vanish if $3k_0 > n_0$ or $3k' > n'$. □

We are now in a comfortable position to prove Theorem 5.2.1 for all but two of the sporadic Apéry-like sequences. To show that sequences (η) and s_{18} satisfy Lucas congruences as well requires considerable additional effort, and the corresponding proofs are given in Section 5.3.

Proof of Theorem 5.2.1. Recall from (5.11) that the sequence of central binomial coefficients is an **LP** function. Further armed with (5.8) as well as Corollary 5.2.4 and Lemma 5.2.5, the claimed Lucas congruences for the sequences (a), (b), (c), (d), (f), (α), (ϵ), (γ), s_{10}, s_7 follow from (5.9). It remains to consider the sequences (g), (δ), (ζ) as well as (η) and s_{18}.

Sequence (g) can be written as

$$A_g(n) = \sum_{k=0}^{n}(-1)^k 8^{n-k}\binom{n}{k}F(k),$$

where $F(k) = \sum_{l=0}^{k}\binom{k}{l}^3$ are the Franel numbers (sequence (a)), which we already know to be an **LP** function. As a consequence of Fermat's little theorem, the sequence a^n is an **LP** function for any integer a. Hence, equation (5.9) applies to show that $A_g(n)$ is an **LP** function.

In order to see that sequence (δ) satisfies the Lucas congruences as well, it suffices to observe that $L(n,k) = \binom{n+k}{k}$ is almost a **DLP** function, that is, it satisfies the congruences

(5.7) but does not vanish for $k > n$. This is enough to conclude from Lemma 5.2.5 that

$$L(n,k) = 3^{n-3k}\binom{n}{3k}\binom{n+k}{k}\frac{(3k)!}{k!^3}$$

is a **DLP** function. Since this is the summand of sequence (δ), the desired Lucas congruences again follow from (5.9).

On the other hand, for sequence (ζ), we observe that

$$L(n,k) = \sum_{j=0}^{n}\binom{n}{j}\binom{k}{j}\binom{k+j}{n}$$

satisfies the congruences (5.7) by Lemma 5.2.2 because the summand is a **TLP** function by Lemma 5.2.3. Hence, $\binom{n}{k}^2 L(n,k)$ is a **DLP** function. Writing sequence (ζ) as

$$A_\zeta(n) = \sum_{k=0}^{n}\binom{n}{k}^2 L(n,k),$$

the claimed congruences once more follow from (5.9). □

5.3 Proofs for the two remaining sequences

The proof of the Lucas congruences in the previous section does not readily extend to the sequences (η) and s_{18} from Table 5.2, because, in contrast to the other cases, the known binomial sums for these sequences do not have the property that their summands satisfy the double Lucas property. Let us first note that the binomial sums for s_{18} and sequence (η), given in (5.3) and Table 5.2, can be simplified at the expense of working with binomial coefficients with negative entries. Namely, we have

$$s_{18}(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\binom{2n-3k}{n} \tag{5.14}$$

and

$$A_\eta(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}, \tag{5.15}$$

where, as usual, for any integer $m \geqslant 0$ and any number x, we define

$$\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}.$$

For instance, the equivalence between (5.3) and (5.14) is a simple consequence of the fact that, for integers $n \geqslant 0$ and $l = n-k$,

$$(-1)^k \binom{2n-3k}{n} = (-1)^{k+n} \binom{-n+3k-1}{n} = (-1)^l \binom{2n-3l-1}{n}. \tag{5.16}$$

For the first equality, we used that, for integers $b \geqslant 0$,

$$\begin{aligned} \binom{a}{b} &= \frac{a(a-1)\cdots(a-b+1)}{b!} \\ &= (-1)^b \frac{(-a)(-a+1)\cdots(-a+b-1)}{b!} = (-1)^b \binom{-a+b-1}{b}. \end{aligned} \tag{5.17}$$

The following result generalizes the Lucas congruences for the sequence $s_{18}(n)$.

Theorem 5.3.1. *Suppose that $B(n,k)$ is a **DLP** function with the property that $B(n,k) = B(n, n-k)$. Then, the sequence*

$$A(n) = \sum_{k=0}^{n} (-1)^k B(n,k) \binom{2n-3k}{n}$$

*is an **LP** function, that is, $A(n)$ satisfy the Lucas congruences* (5.5).

Proof. Let p be a prime and let $n \geqslant 0$ be an integer. Write $n = n_0 + n'p$ and $k = k_0 + k'p$, where $0 \leqslant n_0 < p$ and $0 \leqslant k_0 < p$ and n', k' are nonnegative integers. We have to show that

$$A(n) \equiv A(n_0) A(n') \pmod{p}. \tag{5.18}$$

In the sequel, we denote

$$C(n,k) = (-1)^k B(n,k)\binom{2n-3k}{n}.$$

For $k_0 \leqslant n_0/3$, we have $2n_0 - 3k_0 \geqslant n_0 \geqslant 0$ and $2n_0 - 3k_0 \leqslant 2n_0 < n_0 + p$. Hence, by the usual argument, we have

$$\begin{aligned}\binom{2n-3k}{n} &\equiv [x^{n_0}(x^p)^{n'}](1+x)^{2n_0-3k_0}(1+x^p)^{2n'-3k'} \pmod p\\ &\equiv \binom{2n_0-3k_0}{n_0}\binom{2n'-3k'}{n'} \pmod p.\end{aligned}$$

Hence, we find that, when $k_0 \leqslant n_0/3$,

$$C(n,k) \equiv C(n_0,k_0)C(n',k') \pmod p. \tag{5.19}$$

For $n_0/3 < k_0 < 2n_0/3$, we have $n_0 > 2n_0 - 3k_0 > 0$. By the same argument as above, we find that

$$\binom{2n-3k}{n} \equiv 0 \pmod p, \tag{5.20}$$

and hence $C(n,k) \equiv C(n_0,k_0) \equiv 0$ modulo p.

Finally, consider the case $n_0 \geqslant 1$ and $2n_0/3 \leqslant k_0 \leqslant n_0$. In that case, $-p < -n_0 \leqslant 2n_0 - 3k_0 \leqslant 0$ or, equivalently, $0 < 2n_0 - 3k_0 + p \leqslant p$. Hence, we have, modulo p,

$$\begin{aligned}\binom{2n-3k}{n} &\equiv [x^{n_0}(x^p)^{n'}](1+x)^{2n_0-3k_0+p}(1+x^p)^{2n'-3k'-1}\\ &\equiv \binom{2n_0-3k_0+p}{n_0}\binom{2n'-3k'-1}{n'}\\ &\equiv \binom{2n_0-3k_0}{n_0}\binom{2n'-3k'-1}{n'},\end{aligned} \tag{5.21}$$

because, for any integers A, B and m such that $0 \leqslant m < p$,

$$\begin{aligned}\binom{A+Bp}{m} &= \frac{1}{m!}(A+Bp)(A+Bp-1)\cdots(A+Bp-m+1)\\ &\equiv \frac{1}{m!}A(A-1)\cdots(A-m+1) = \binom{A}{m} \pmod p. \qquad (5.22)\end{aligned}$$

Set $l' = n' - k'$. Applying (5.16) to the second binomial factor in (5.21), we find that

$$\binom{2n-3k}{n} \equiv (-1)^{n'}\binom{2n_0-3k_0}{n_0}\binom{2n'-3l'}{n'} \pmod p.$$

In combination with the assumed symmetry of $B(n,k)$, we therefore have that, when $n_0 \geqslant 1$ and $2n_0/3 \leqslant k_0 \leqslant n_0$,

$$C(n,k) \equiv C(n_0,k_0)C(n',n'-k') \pmod p. \qquad (5.23)$$

We are now ready to combine all cases. First, suppose that $n_0 \geqslant 1$. Noting that $k \leqslant n/3$ implies $k' \leqslant n'/3$, and using (5.19), (5.20) and (5.23), we conclude that, modulo p,

$$\begin{aligned}A(n) &= \sum_{k_0=0}^{p-1}\sum_{k'=0}^{n'} C(n,k) \equiv \sum_{k_0=0}^{n_0}\sum_{k'=0}^{n'} C(n,k)\\ &\equiv \sum_{k_0=0}^{\lfloor n_0/3\rfloor}\sum_{k'=0}^{n'} C(n,k) + \sum_{k_0=\lceil 2n_0/3\rceil}^{n_0}\sum_{k'=0}^{n'} C(n,k)\\ &\equiv \sum_{k_0=0}^{\lfloor n_0/3\rfloor} C(n_0,k_0)\sum_{k'=0}^{n'} C(n',k') + \sum_{k_0=\lceil 2n_0/3\rceil}^{n_0} C(n_0,k_0)\sum_{k'=0}^{n'} C(n',n'-k')\\ &= \left[\sum_{k_0=0}^{\lfloor n_0/3\rfloor} C(n_0,k_0) + \sum_{k_0=\lceil 2n_0/3\rceil}^{n_0} C(n_0,k_0)\right]\sum_{k'=0}^{n'} C(n',k')\\ &= A(n_0)A(n'),\end{aligned}$$

which is what we wanted to prove. The case $n_0 = 0$ is simpler, and we only have to use (5.19) to again conclude that (5.18) holds. □

Corollary 5.3.2. *The sequence* $s_{18}(n)$ *satisfies the Lucas congruences* (5.5).

Proof. Recall from the discussion in Section 5.2 that

$$B(n,k) = \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$$

is a **DLP** function. Obviously, $B(n,k) = B(n, n-k)$. Hence, Theorem 5.3.1 applies to show that $s_{18}(n)$, in the form (5.14) satisfies the Lucas congruences (5.5). □

Next, we prove that the sequence (η), which corresponds to the choice $a = 3$ in Theorem 5.3.3, satisfies Lucas congruences as well.

Theorem 5.3.3. *Let* $a \in \{1, 3\}$. *Then, the sequence*

$$A(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \tag{5.24}$$

is an ***LP*** *function, that is,* $A(n)$ *satisfy the Lucas congruences* (5.5).

Proof. Let p be a prime and let $n \geqslant 0$ be an integer. As in the proof of Theorem 5.3.1, we write $n = n_0 + n'p$ and $k = k_0 + k'p$, where $0 \leqslant n_0 < p$ and $0 \leqslant k_0 < p$ and n', k' are nonnegative integers. Again, we have to show that

$$A(n) \equiv A(n_0)A(n') \pmod{p}. \tag{5.25}$$

Throughout the proof, let $d = \lfloor 3n_0/p \rfloor$.

If $k_0 \leqslant n_0/5$, then $4n_0 - 5k_0 \geqslant 3n_0 \geqslant 0$ and $4n_0 - 5k_0 \leqslant 4n_0 < 3n_0 + p$. Since $d = \lfloor 3n_0/p \rfloor$, we thus have $0 \leqslant 3n_0 - dp < p$ and $0 \leqslant 4n_0 - 5k_0 - dp < (3n_0 - dp) + p$. Therefore, modulo

p,

$$\begin{aligned}\binom{4n-5k}{3n} &\equiv [x^{3n_0-dp}(x^p)^{3n'+d}](1+x)^{4n_0-5k_0-dp}(1+x^p)^{4n'-5k'+d}\\ &\equiv \binom{4n_0-5k_0-dp}{3n_0-dp}\binom{4n'-5k'+d}{3n'+d}\\ &\equiv \binom{4n_0-5k_0}{3n_0}\binom{4n'-5k'+d}{3n'+d},\end{aligned}$$

where in the last step we used that, modulo p,

$$\binom{4n_0-5k_0-dp}{3n_0-dp} = \binom{4n_0-5k_0-dp}{n_0-5k_0} \equiv \binom{4n_0-5k_0}{n_0-5k_0} = \binom{4n_0-5k_0}{3n_0}, \tag{5.26}$$

which follows from (5.22) because $0 \leqslant n_0 - 5k_0 < p$. In particular, we have

$$\begin{aligned}&\sum_{k_0=0}^{\lfloor n_0/5\rfloor}\sum_{k'=0}^{n'}(-1)^k\binom{n}{k}^a\binom{4n-5k}{3n}\\ &\equiv \sum_{k_0=0}^{\lfloor n_0/5\rfloor}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0}\sum_{k'=0}^{n'}(-1)^{k'}\binom{n'}{k'}^a\binom{4n'-5k'+d}{3n'+d},\end{aligned} \tag{5.27}$$

and we observe that, for $d \in \{0,1\}$,

$$A(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k+d}{3n+d}. \tag{5.28}$$

To see this, note that the the sum of the k-th and $(n-k)$-th term does not depend on the value of $d \in \{0,1\}$. Indeed, using (5.17), Pascal's relation and (5.17) again, we deduce that

$$\begin{aligned}
& \binom{4n-5k+1}{3n+1} + (-1)^n \binom{4n-5(n-k)+1}{3n+1} \\
= \; & \binom{4n-5k+1}{3n+1} - \binom{4n-5k-1}{3n+1} \\
= \; & \left[\binom{4n-5k+1}{3n+1} - \binom{4n-5k}{3n+1}\right] + \left[\binom{4n-5k}{3n+1} - \binom{4n-5k-1}{3n+1}\right] \\
= \; & \binom{4n-5k}{3n} + \binom{4n-5k-1}{3n} \\
= \; & \binom{4n-5k}{3n} + (-1)^n \binom{4n-5(n-k)}{3n}.
\end{aligned}$$

Next, suppose that $n_0 \geqslant 1$ and $4n_0/5 \leqslant k_0 \leqslant n_0$. In that case, $-p < -n_0 \leqslant 4n_0 - 5k_0 \leqslant 0$ or, equivalently, $0 < 4n_0 - 5k_0 + p \leqslant p$. Hence, we have, modulo p,

$$\begin{aligned}
\binom{4n-5k}{3n} & \equiv [x^{3n_0-dp}(x^p)^{3n'+d}](1+x)^{4n_0-5k_0+p}(1+x^p)^{4n'-5k'-1} \\
& \equiv \binom{4n_0-5k_0+p}{3n_0-dp}\binom{4n'-5k'-1}{3n'+d}.
\end{aligned}$$

We rewrite the first binomial factor as follows, applying first (5.17) and then (5.22) twice, to find that, with $l_0 = n_0 - k_0$, modulo p,

$$\begin{aligned}
\binom{4n_0-5k_0+p}{3n_0-dp} & = (-1)^{n_0+d}\binom{4n_0-5l_0-(d+1)p-1}{3n_0-dp} \\
& \equiv (-1)^{n_0+d}\binom{4n_0-5l_0-dp-1}{3n_0-dp} \\
& = (-1)^{n_0+d}\binom{4n_0-5l_0-dp-1}{n_0-5l_0-1} \\
& \equiv (-1)^{n_0+d}\binom{4n_0-5l_0-1}{n_0-5l_0-1} \\
& = (-1)^{n_0+d}\binom{4n_0-5l_0-1}{3n_0}.
\end{aligned}$$

Here, we proceeded under the assumption that $n_0 - 5l_0 > 0$. It is straightforward to check that the final congruence also holds when $n_0 = 5l_0$, because then the binomial coefficients vanish modulo p. We conclude that, when $n_0 \geqslant 1$ and $4n_0/5 \leqslant k_0 \leqslant n_0$,

$$(-1)^k \binom{4n-5k}{3n} \equiv (-1)^{l_0} \binom{4n_0-5l_0-1}{3n_0} (-1)^{k'+d} \binom{4n'-5k'-1}{3n'+d} \pmod p.$$

In particular, we have

$$\begin{aligned}
&\sum_{k_0=\lceil 4n_0/5 \rceil}^{n_0} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \\
\equiv\ & \sum_{k_0=\lceil 4n_0/5 \rceil}^{n_0} (-1)^{l_0} \binom{n_0}{l_0}^a \binom{4n_0-5l_0-1}{3n_0} \sum_{k'=0}^{n'} (-1)^{k'+d} \binom{n'}{k'}^a \binom{4n'-5k'-1}{3n'+d} \\
=\ & \sum_{k_0=0}^{\lfloor n_0/5 \rfloor} (-1)^{k_0} \binom{n_0}{k_0}^a \binom{4n_0-5k_0-1}{3n_0} \sum_{k'=0}^{n'} (-1)^{k'+d} \binom{n'}{k'}^a \binom{4n'-5k'-1}{3n'+d},
\end{aligned} \tag{5.29}$$

and we observe that, for integers $d \geqslant 0$,

$$\sum_{k=0}^{n} (-1)^{k+d} \binom{n}{k}^a \binom{4n-5k-1}{3n+d} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a \binom{4n-5k+d}{3n+d}$$

because, by (5.17),

$$(-1)^k \binom{4n-5k+d}{3n+d} = (-1)^{(n-k)+d} \binom{4n-5(n-k)-1}{3n+d}.$$

Therefore, we can combine (5.27) and (5.29) into

$$\begin{aligned}
&\sum_{\substack{k_0=0 \\ k_0 \leqslant n_0/5 \text{ or } k_0 \geqslant 4n_0/5}}^{n_0} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \\
\equiv\ & A(n_0) \sum_{k'=0}^{n'} (-1)^{k'} \binom{n'}{k'}^a \binom{4n'-5k'+d}{3n'+d} \pmod p,
\end{aligned} \tag{5.30}$$

which holds for all $0 \leqslant n_0 < p$ (recall from the discussion at the beginning of this section that $A(n_0)$, like sequence (η), can be represented as in Table 5.2).

On the other hand, suppose that $n_0/5 < k_0 < 4n_0/5$. Set $f = \lfloor (4n_0 - 5k_0)/p \rfloor$. Since $0 < 4n_0 - 5k_0 < 3n_0 < 3p$, we have $f \in \{0, 1, 2\}$. The usual arguments show that, modulo p,

$$\begin{aligned} \binom{4n-5k}{3n} &\equiv [x^{3n_0-dp}(x^p)^{3n'+d}](1+x)^{4n_0-5k_0-fp}(1+x^p)^{4n'-5k'+f} \\ &\equiv \binom{4n_0-5k_0-fp}{3n_0-dp}\binom{4n'-5k'+f}{3n'+d} \\ &\equiv \binom{4n_0-5k_0}{3n_0-dp}\binom{4n'-5k'+f}{3n'+d}. \end{aligned} \tag{5.31}$$

We are now in a position to begin piecing everything together. To do so, we consider individually the cases corresponding to the value of $d \in \{0, 1, 2\}$.

First, suppose $d = 0$ or $d = 1$. Congruence (5.30) coupled with (5.28) implies that

$$\sum_{\substack{k_0=0 \\ k_0 \leqslant n_0/5 \text{ or } k_0 \geqslant 4n_0/5}}^{n_0} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \equiv A(n_0)A(n') \pmod{p}.$$

To conclude the desired congruence (5.25), it therefore only remains to show that

$$\sum_{k_0=\lfloor n_0/5 \rfloor+1}^{\lceil 4n_0/5 \rceil -1} \sum_{k'=0}^{n'} (-1)^k \binom{n}{k}^a \binom{4n-5k}{3n} \equiv 0 \pmod{p}. \tag{5.32}$$

This is easily seen in the case $d = 0$, because then each term of this sum vanishes modulo p. Equivalently, for $d = 0$, (5.31) vanishes whenever $n_0/5 < k_0 < 4n_0/5$ (because $0 \leqslant 4n_0 - 5k_0 - fp \leqslant 4n_0 - 5k_0 < 3n_0$). On the other hand, if $d = 1$, we claim that the sum (5.32) vanishes modulo p because the terms corresponding to (k_0, k') and $(k_0, n' - k')$ cancel each other. To see that, observe first that, for $d = 1$, (5.31) vanishes whenever $n_0/5 < k_0 < 4n_0/5$ and $f = \lfloor (4n_0 - 5k_0)/p \rfloor \neq 0$ (because $0 \leqslant 4n_0 - 5k_0 - fp \leqslant 4n_0 - 5k_0 - p < 3n_0 - p$ if $f \in \{1, 2\}$). Therefore, for the term corresponding to (k_0, k'),

$$(-1)^k \binom{4n-5k}{3n} \equiv (-1)^{k_0} \binom{4n_0-5k_0}{3n_0-p} (-1)^{k'} \binom{4n'-5k'}{3n'+1} \pmod{p},$$

while, for the term corresponding to $(k_0, n'-k')$ with $j = k_0 + (n'-k')p$,

$$\begin{aligned}(-1)^j \binom{4n-5j}{3n} &\equiv (-1)^{k_0}\binom{4n_0-5k_0}{3n_0-p}(-1)^{n'-k'}\binom{4n'-5(n'-k')}{3n'+1} \\ &\equiv (-1)^{k_0}\binom{4n_0-5k_0}{3n_0-p}(-1)^{k'+1}\binom{4n'-5k'}{3n'+1} \\ &\equiv -(-1)^k\binom{4n-5k}{3n} \pmod p,\end{aligned}$$

where we applied (5.17) for the second congruence. It is now immediate to see that the sum (5.32) indeed vanishes modulo p for $d = 1$.

It remains to prove the Lucas congruences (5.25) in the case $d = 2$. Using (5.30), we have

$$A(n) \equiv A(n_0)\sum_{k'=0}^{n'}(-1)^{k'}\binom{n'}{k'}^a\binom{4n'-5k'+2}{3n'+2} + M \pmod p,$$

where

$$M := \sum_{k_0=\lfloor n_0/5\rfloor+1}^{\lceil 4n_0/5\rceil-1}\sum_{k'=0}^{n'}(-1)^k\binom{n}{k}^a\binom{4n-5k}{3n}.$$

Combining this congruence with the identity

$$A(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\left[\binom{4n-5k+2}{3n+2} - \binom{4n-5k}{3n+2}\right],$$

which can be deduced along the same lines as (5.28), we find that

$$A(n) \equiv A(n_0)A(n') + A(n_0)\sum_{k'=0}^{n'}(-1)^{k'}\binom{n'}{k'}^a\binom{4n'-5k'}{3n'+2} + M \pmod p. \tag{5.33}$$

We have, by (5.31), modulo p,

$$\begin{aligned} M &\equiv \sum_{k_0=\lfloor n_0/5\rfloor+1}^{\lceil 4n_0/5\rceil-1}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0-2p}\sum_{k'=0}^{n'}(-1)^{k'}\binom{n'}{k'}^a\binom{4n'-5k'+f}{3n'+2} \\ &\equiv \sum_{k_0=\lfloor n_0/5\rfloor+1}^{\lceil 4n_0/5\rceil-1}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0-2p}\sum_{k'=0}^{n'}(-1)^{k'}\binom{n'}{k'}^a\binom{4n'-5k'}{3n'+2}, \end{aligned}$$

where the last congruence is a consequence of the identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k+1}{3n+2}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a\binom{4n-5k}{3n+2}$$

(which follows from (5.17) and replacing k with $n-k$) and the fact that (5.31) vanishes for $n_0/5<k_0<4n_0/5$ if $f=2$. Using this value of M in (5.33), we find that the desired Lucas congruence (5.25) follows, if we can show that

$$A(n_0)+\sum_{k_0=\lfloor n_0/5\rfloor+1}^{\lceil 4n_0/5\rceil-1}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0-2p}\equiv 0\ (\mathrm{mod}\ p). \tag{5.34}$$

Note that, if $k_0\leqslant n_0/5$, then, by (5.22) and (5.26),

$$\binom{4n_0-5k_0}{3n_0-2p}\equiv\binom{4n_0-5k_0-2p}{3n_0-2p}\equiv\binom{4n_0-5k_0}{3n_0}\ (\mathrm{mod}\ p). \tag{5.35}$$

A similar argument, combined with (5.17), shows that the congruence (5.35) also holds if $k_0\geqslant 4n_0/5$. We therefore find that (5.34) is equivalent to

$$\sum_{k_0=0}^{n_0}(-1)^{k_0}\binom{n_0}{k_0}^a\binom{4n_0-5k_0}{3n_0-2p}\equiv 0\ (\mathrm{mod}\ p).$$

The next lemma proves that this congruence indeed holds provided that $a\in\{1,3\}$. □

Lemma 5.3.4. *Let p be a prime, and $a\in\{1,2,3\}$. Then we have, for all n such that*

$2p/3 \leqslant n < p$,

$$\sum_{k=0}^{n} (-1)^{ak} \binom{n}{k}^{a} \binom{4n-5k}{3n-2p} \equiv 0 \pmod{p}.$$

Proof. To prove these congruences we employ N. Calkin's technique [14] for proving similar divisibility results for sums of powers of binomials (5.50). Denoting $r = p - n$, we have, by (5.17) and (5.22),

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^{ak} \binom{n}{k}^{a} \binom{4n-5k}{3n-2p} &= \sum_{k=0}^{p-r} (-1)^{ak} \binom{p-r}{k}^{a} \binom{4p-4r-5k}{p-3r} \\
&= \sum_{k=0}^{p-r} \binom{k-p+r-1}{k}^{a} \binom{4p-4r-5k}{p-3r} \\
&\equiv \sum_{k=0}^{p-r} \binom{k+r-1}{k}^{a} \binom{4p-4r-5k}{p-3r} \pmod{p}.
\end{aligned}$$

Clearly,

$$\binom{k+r-1}{k} = \frac{(k+1)(k+2)\cdots(k+r-1)}{(r-1)!} = \frac{(k+1)_{r-1}}{(r-1)!}, \tag{5.36}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$ denotes the Pochhammer symbol (in particular, $(x)_0 = 1$). Likewise,

$$\binom{4p-4r-5k}{p-3r} = \frac{(3p-r-5k+1)_{p-3r}}{(p-3r)!}$$

Since $(r-1)!$ and $(p-3r)!$ are not divisible by p, we have to show that

$$\sum_{k=0}^{p-r} (k+1)_{r-1}^{a} (3p-r-5k+1)_{p-3r} \equiv 0 \pmod{p}. \tag{5.37}$$

Since the polynomials $(x)_k, (x)_{k-1}, \ldots, (x)_0$ form an integer basis for the space of all polynomials with integer coefficients and degree at most k, there exist integers $c_0, c_1, \ldots, c_N$ with $N = (a-1)(r-1) + p - 3r$ so that

$$(k+1)_{r-1}^{a-1} (3p-r-5k+1)_{p-3r} = \sum_{j=0}^{N} c_j (k+r)_j.$$

Then the left-hand side of (5.37) becomes

$$\begin{aligned}\sum_{k=0}^{p-r}(k+1)_{r-1}\sum_{j=0}^{N}c_j(k+r)_j &= \sum_{j=0}^{N}c_j\sum_{k=0}^{p-r}(k+1)_{r-1}(k+r)_j \\ &= \sum_{j=0}^{N}c_j\sum_{k=0}^{p-r}(k+1)_{r+j-1} \\ &= \sum_{j=0}^{N}c_j\frac{(p-r+1)_{r+j}}{r+j}, \qquad (5.38)\end{aligned}$$

where we used

$$(x)_k-(x-1)_k=k(x)_{k-1}$$

to evaluate

$$\sum_{k=0}^{p-r}(k+1)_{r+j-1}=\sum_{k=0}^{p-r}\frac{(k+1)_{r+j}-(k)_{r+j}}{r+j}=\frac{(p-r+1)_{r+j}}{r+j}.$$

The desired congruence therefore follows if we can show that

$$\frac{(p-r+1)_{r+j}}{r+j}\equiv 0 \,(\mathrm{mod}\, p) \qquad (5.39)$$

for all $j=0,1,\ldots,N$. Since $r>0$ and $j\geqslant 0$, the numerator $(p-r+1)_{r+j}$ is always divisible by p. The congruences (5.39) thus follow if $r+j<p$ for all j, or, equivalently, $r+N<p$. Since

$$r+N=(a-1)(r-1)+p-2r,$$

we have $r+N<p$ if and only if

$$(a-1)(r-1)<2r.$$

Clearly, this inequality holds for all $r\geqslant 1$ if and only if $a\leqslant 3$. □

Remark 5.3.1. *Numerical evidence suggests that the values $a\in\{1,3\}$ in Theorem 5.3.3 are the only choices for which the sequence* (5.24) *satisfies Lucas congruences. In light of*

Lemma 5.3.4, it is natural to ask if there are additional values of a and ε, for which the sequence

$$\sum_{k=0}^{n}(-1)^{\varepsilon k}\binom{n}{k}^{a}\binom{4n-5k}{3n}$$

satisfies Lucas congruences. Empirically, this does not appear to be the case. In particular, for $a=2$ this sequence does not satisfy Lucas congruences for either $\varepsilon=0$ or $\varepsilon=1$.

5.4 Periodicity of residues

The Apéry numbers satisfy

$$A(n) \equiv (-1)^n \pmod 3, \tag{5.40}$$

and so are periodic modulo 3. As in the case of the congruences (1.16), which show that the Apéry numbers are also periodic modulo 8, the congruences (5.40) were first conjectured in [16] and then proven in [28]. We say that a sequence $C(n)$ is *eventually periodic* if there exists an integer $M>0$ such that $C(n+M)=C(n)$ for all sufficiently large n. An initial numerical search suggests that each sporadic Apéry-like sequence listed in Tables 5.1 and 5.2 can only be eventually periodic modulo a prime p if $p \leqslant 5$. As an application of Theorem 5.2.1, we prove this claim next.

Corollary 5.4.1. *None of the sequences from Tables 5.1 and 5.2 is eventually periodic modulo p for any prime $p>5$.*

Proof. Gessel [28] shows that, if a sequence $C(n)$ satisfies the Lucas congruences (5.5) modulo p and is eventually periodic modulo p, then $C(n) \equiv C(1)^n$ modulo p for all $n=0,1,\ldots,p-1$.

For instance, let $C(n)$ be the Almkvist–Zudilin sequence (δ). Then, $C(1)=3$, $C(2)=9$ and $C(3)=3$. Suppose $C(n)$ was eventually periodic modulo p. Then p has to divide $C(3)-C(1)^3=-24$, which implies that $p\in\{2,3\}$.

In Table 5.3 we list, for each sequence, the primes dividing both $C(2)-C(1)^2$ and $C(3)-C(1)^3$. The fact, that all these primes are at most 5, proves our claim. □

Table 5.3: The primes dividing both $C(2)-C(1)^2$ and $C(3)-C(1)^3$, for each sequence $C(n)$ from Tables 5.1 and 5.2.

(a)	(b)	(c)	(d)	(f)	(g)	(δ)	(η)	(α)	(ϵ)	(ζ)	(γ)	(s_7)	(s_{10})	(s_{18})
$2,3$	$2,5$	$2,3$	2	$2,3$	$2,3$	$2,3$	$2,5$	$2,3$	$2,3$	$2,3$	$2,3$	2	2	$2,3$

As another simple consequence of Theorem 5.2.1, we observe that the Apéry-like sequences are in fact eventually periodic modulo each of the primes listed in Table 5.3.

Corollary 5.4.2. *Let $C(n)$ be any sequence from Tables 5.1 and 5.2.*

- $C(n) \equiv C(1) \pmod 2$ *for all* $n \geqslant 1$.
- $C(n) \equiv C(1) \pmod 3$ *for all* $n \geqslant 1$ *if* $C(n)$ *is one of* (c), (f), (g), (δ), (α), (ϵ), (ζ), s_{18}, *and* $C(n) \equiv (-1)^n \pmod 3$ *for all* $n \geqslant 0$ *if* $C(n)$ *is* (a) *or* (γ).
- $C(n) \equiv 3^n \pmod 5$ *for all* $n \geqslant 0$ *if* $C(n)$ *is* (b), *and* $C(n) \equiv 0 \pmod 5$ *for all* $n \geqslant 1$ *if* $C(n)$ *is* (η).

Proof. One can check that Table 5.3 does not change if we include only those primes p such that $C(n) - C(1)^n$ is divisible by p for all $n \in \{0, 1, 2, 3, 4\}$. For $n = 0$, this is trivial since $C(0) = 1$. Therefore, in each of the cases considered here, we have

$$C(n) \equiv C(1)^n \pmod p$$

for all $n \in \{0, 1, \ldots, p-1\}$. For any $n \geqslant 0$, let $n = n_0 + n_1 p + \cdots + n_r p^r$ be the p-adic expansion of n. Then, by Theorem 5.2.1, we have

$$\begin{aligned} C(n) &\equiv C(n_0)C(n_1)\cdots C(n_r) \pmod p \\ &\equiv C(1)^{n_0+n_1+\cdots+n_r} \pmod p \\ &\equiv C(1)^n \pmod p. \end{aligned}$$

For the final congruence we used Fermat's little theorem. All claimed congruences then follow from the specific initial values of $C(n)$ modulo p. $\square$

More interestingly, the congruences (1.16) show that the Apéry numbers (sequence (γ)) are periodic modulo 8. We offer the following corresponding result for the Almkvist–Zudilin sequence (δ).

Theorem 5.4.3. *The Almkvist–Zudilin numbers*

$$Z(n) = \sum_{k=0}^{n} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

satisfy the congruences

$$Z(n) \equiv \begin{cases} 1, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd,} \end{cases} \pmod{8}. \tag{5.41}$$

Proof. It is shown in [57] that the numbers $(-1)^n Z(n)$ are the diagonal Taylor coefficients of the multivariate rational function

$$F(x_1, x_2, x_3, x_4) = \frac{1}{1 - (x_1 + x_2 + x_3 + x_4) + 27 x_1 x_2 x_3 x_4}. \tag{5.42}$$

That is, if

$$F(x_1, x_2, x_3, x_4) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} C(n_1, n_2, n_3, n_4) x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$$

is the Taylor expansion of the rational function F, then $Z(n) = (-1)^n C(n, n, n, n)$.

Given such a rational function as well as a prime power p^r, Rowland and Yassawi [52] give an explicit algorithm for computing a finite state automaton, which produces the values of the diagonal coefficients modulo p^r. In the present case, this finite state automaton for the values $(-1)^n Z(n)$ modulo 8 turns out to be the same automaton as the one for the Apéry numbers modulo 8. Hence, the congruences (5.41) follow from the congruences (1.16). We refer to [52] for details on finite state automata and the algorithm to construct them from a multivariate rational generating function. We also remark that, due to the complexity of the algorithm, p^r should be reasonably small in practice (for instance, the implementation accompanying [52] takes several minutes to compute the finite state automaton for the coefficients of (5.42) modulo 2^5, and did not finish in reasonable time modulo 2^6). □

Empirically, Theorem 5.4.3 is the only other interesting set of congruences, apart from the congruences (1.16), which demonstrates that an Apéry-like sequence is periodic modulo a prime power. More precisely, numerical evidence suggests that none of the sequences in Tables 5.1 and 5.2 is eventually periodic modulo p^r, for some $r > 1$, unless $p = 2$. Moreover, the only other instances modulo a power of 2 appear to be the following, less interesting, ones: sequences (d) and (α) are eventually periodic modulo 4 because all their terms, except the first, are divisible by 4; likewise, sequences (ε) and s_7 are eventually periodic modulo 8 because all their terms, except the first, are divisible by 8. We do not attempt to prove these claims here. We remark, however, that these claims can be established by the approach used in the proof of Theorem 5.4.3, provided that one is able to determine a computationally accessible analog of (5.42) for the sequence at hand.

5.5 Primes not dividing Apéry-like numbers

Using the Lucas congruences proved in Theorem 5.2.1, it is straightforward to verify whether or not a given prime divides some Apéry-like number.

Example 5.5.1. *The values of Apéry numbers* $A(0), A(1), \ldots, A(6)$ *modulo* 7 *are* $1, 5, 3, 3, 3, 5, 1$. *Since* 7 *does not divide* $A(0), A(1), \ldots, A(6)$, *it follows from the Lucas congruences* (5.5) *that* 7 *does not divide any Apéry number.*

Arguing as in Example 5.5.1, one finds that the primes $2, 3, 7, 13, 23, 29, 43, 47, \ldots$ do not divide any Apéry number $A(n)$. E. Rowland and R. Yassawi [52] pose the question whether there are infinitely many such primes. Table 5.4 records, for each sporadic Apéry-like sequence, the primes below 100 which do not divide any of its terms, and the last column gives the proportion of primes below 10^4 with this property. Each Apéry-like sequence is specified by its label from [4], which is also used in Tables 5.1 and 5.2. The alert reader will notice that Cooper's sporadic sequences (the ones with $d \neq 0$ in Table 5.2) are missing from Table 5.4. That is because these sequences turn out to be divisible by all primes. A more precise result for these sequences is proved at the end of this section.

Table 5.4: Primes not dividing Apéry-like numbers

(a)	3, 11, 17, 19, 43, 83, 89, 97	0.2994
(b)	2, 5, 13, 17, 29, 37, 41, 61, 73, 89	0.2897
(c)	2, 7, 13, 37, 61, 73	0.2962
(d)	3, 11, 17, 19, 43, 59, 73, 83, 89	0.2815
(f)	2, 5, 13, 17, 29, 37, 41, 61, 73, 97	0.2994
(g)	5, 11, 29, 31, 59, 79	0.2929
(δ)	2, 5, 7, 11, 13, 19, 29, 41, 47, 61, 67, 71, 73, 89, 97	0.6192
(η)	2, 3, 17, 19, 23, 31, 47, 53, 61	0.2897
(α)	3, 5, 13, 17, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 83, 89	0.5989
(ϵ)	3, 7, 13, 19, 23, 29, 31, 37, 43, 47, 61, 67, 73, 83, 89	0.6037
(ζ)	2, 5, 7, 13, 17, 19, 29, 37, 43, 47, 59, 61, 67, 71, 83, 89	0.6046
(γ)	2, 3, 7, 13, 23, 29, 43, 47, 53, 67, 71, 79, 83, 89	0.6168

The primes below 100 not dividing Apéry-like numbers (sequence indicated in first column using the labels from [4]) as well as the proportion of primes (in the last column) below $10,000$ not dividing any term

Example 5.5.1 shows that the first 7 values of the Apéry numbers modulo 7 are palindromic. Our next result, which was noticed by E. Rowland, shows that this is true for all primes.

Lemma 5.5.2. *For any prime p, and integers n such that $0 \leqslant n < p$, the Apéry numbers $A(n)$ satisfy the congruence*

$$A(n) \equiv A(p-1-n) \pmod{p}.$$

Proof. For n such that $0 \leqslant n < p$, we employ (5.17) and (5.22) to arrive at

$$\begin{aligned} A(p-1-n) &= \sum_{k=0}^{p-1} \binom{p-1-n}{k}^2 \binom{p-1-n+k}{k}^2 \\ &\equiv \sum_{k=0}^{p-1} \binom{n+k}{k}^2 \binom{n}{k}^2 = A(n) \pmod{p}, \end{aligned}$$

as claimed. □

Theorem 5.2.1 and Lemma 5.5.2, considered together, suggest that $e^{-1/2} \approx 60.65\%$ of the primes do not divide any Apéry number. Indeed, let us make the empirical assumption that the values $A(n)$ modulo p, for $n = 0, 1, \ldots, (p-1)/2$, are independent and uniformly random. Since one of the values $A(n)$ is congruent to 0 modulo p with probability $1/p$, it follows that the probability that p does not divide any of the $(p+1)/2$ first values is

$$\left(1 - \frac{1}{p}\right)^{(p+1)/2}. \tag{5.43}$$

By the Lucas congruences, shown in Theorem 5.2.1, and Lemma 5.5.2, p does not divide any of the $(p+1)/2$ first values if and only if p does not divide any Apéry number. In the limit $p \to \infty$, the proportion (5.43) becomes $e^{-1/2}$. Observe that this empirical prediction matches the numerical data in Table 5.4 rather well. We therefore arrive at the following conjecture.

Conjecture 5.5.3. *The proportion of primes not dividing any Apéry number $A(n)$ is $e^{-1/2}$.*

While Lemma 5.5.2 does not hold for the other Apéry-like numbers $C(n)$ from Tables 5.1 and 5.2, we make the weaker observation that if a prime $p > 5$ divides $C(n)$, where $0 \leqslant n < p$, then p also divides $C(p-1-n)$. We expect that this empirical observation can be proven in the spirit of the proof of Lemma 5.5.2, but do not pursue this theme further. We only note that it allows us to extend the heuristic leading to Conjecture 5.5.3 to the Apéry-like sequences (δ), (α), (ϵ), (ζ) from Table 5.2. In other words, we conjecture that, for each

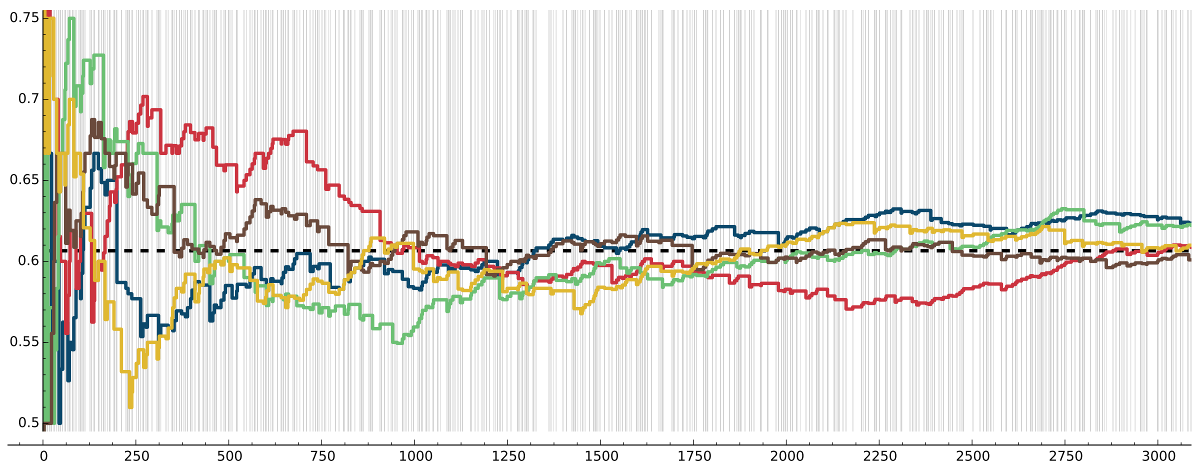

Figure 5.1: Proportion of primes (up to 3000) not dividing the sequences (δ), (α), (ϵ), (ζ), (γ), with the dotted line indicating $e^{-1/2}$. The Apéry sequence is plotted in blue. (We thank Arian Daneshvar for producing this plot.)

of these sequences, the proportion of primes not dividing any of the terms is again $e^{-1/2}$. Figure 5.1 visualizes some numerical evidence for this conjecture. On the other hand, for sequence (η) as well as the sequences from Table 5.1, the proportion of primes not dividing any of their terms appears to be about half of that, that is $e^{-1/2}/2 \approx 30.33\%$.

To explain this extra factor of 1/2, we note that, for the Apéry-like numbers

$$A_b(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{n}, \tag{5.44}$$

Stienstra and Beukers [56] proved that, modulo p,

$$A_b\left(\frac{p-1}{2}\right) \equiv \begin{cases} 4a^2 - 2p, & \text{if } p = a^2 + b^2,\ a \text{ odd}, \\ 0, & \text{if } p \equiv 3 \pmod 4) \end{cases} \tag{5.45}$$

(and conjectured that the congruence should hold modulo p^2, which was later proved by Ahlgren [2]; see also [3]). In particular, congruence (5.45) makes it explicit that every prime $p \equiv 3 \pmod 4$ divides a term of this Apéry-like sequence. Note that, by a classical congruence of Gauss, the congruences (5.45) are equivalent, modulo p, to the congruences

$$A_b\left(\left\lfloor\frac{p}{2}\right\rfloor\right) \equiv \begin{cases} \binom{\lfloor p/2\rfloor}{\lfloor p/4\rfloor}^2, & \text{if } p \equiv 1 \pmod 4, \\ 0, & \text{otherwise,} \end{cases} \tag{5.46}$$

which are valid for any prime $p \neq 2$. The more general result in [56] also includes the cases A_a and A_c. Similar divisibility results appear to hold for the other Apéry-like numbers from Table 5.1, and it would be interesting to make these explicit.

On the other hand, the extra factor of 1/2 in case of sequence (η) is explained by the following congruences, which resemble (5.46) remarkably well.

Theorem 5.5.4. *For any prime $p \neq 3$, we have that, modulo p,*

$$A_\eta\left(\left\lfloor\frac{p}{3}\right\rfloor\right) \equiv \begin{cases} (-1)^{\lfloor p/5\rfloor}\binom{\lfloor p/3\rfloor}{\lfloor p/15\rfloor}^3, & \text{if } p \equiv 1,2,4,8 \pmod{15}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.47}$$

Proof. Suppose that $p \equiv 2 \pmod 3$, and write $p = 3n+2$. The congruence (5.47) can be checked directly for $p = 2$ and $p = 5$, and so we may assume $p > 5$ in the sequel. Applying (5.36) to the definition of sequence (η) in Table 5.2, we have

$$\begin{aligned} A_\eta(n) &= \sum_{k=0}^{\lfloor n/5\rfloor} (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right) \\ &= \sum_{k=0}^{\lfloor n/5\rfloor} (-1)^k \binom{n}{k}^3 \left(\frac{(n-5k)_{3n}}{(3n)!} + \frac{(n-5k+1)_{3n}}{(3n)!}\right). \end{aligned} \tag{5.48}$$

Since $3n = p-2$ and $0 \leqslant k \leqslant n/5$, the term

$$\frac{(n-5k)_{3n}}{(3n)!} \tag{5.49}$$

is always divisible by p, unless $n - 5k \in \{1, 2\}$ (because, otherwise, one of the $p-2$ factors of $(n-5k)_{3n}$ is divisible by p, while $(3n)!$ is not). Note that $n-5k = 1$ and $n-5k = 2$ are equivalent to $k = (p-5)/15$ and $k = (p-8)/15$, respectively. However, $(p-5)/15$ cannot be an integer (since $p \neq 5$). We thus find that (5.49) vanishes modulo p unless $p \equiv 8 \pmod{15}$ and $k = \lfloor p/15 \rfloor$, in which case (5.49) is congruent to -1 modulo p. Combined with the analogous discussion for the corresponding second term in (5.48), we conclude that

$$\frac{(n-5k)_{3n}}{(3n)!} + \frac{(n-5k+1)_{3n}}{(3n)!} \equiv \begin{cases} 1, & \text{if } k = \lfloor p/15 \rfloor \text{ and } p \equiv 2 \pmod{15}, \\ -1, & \text{if } k = \lfloor p/15 \rfloor \text{ and } p \equiv 8 \pmod{15}, \\ 0, & \text{otherwise.} \end{cases}$$

Applying this to the sum (5.48) and combining the signs properly, we arrive at the congruences (5.47) when $p \equiv 2 \pmod 3$.

The case $p \equiv 1 \pmod 3$ is similar and a little bit simpler. □

In summary, we conjecture that the proportion of primes not dividing any term of the Apéry-like sequences in Tables 5.1 and 5.2 is as follows.

Conjecture 5.5.5.

- *Let $C(n)$ be one of the sequences of Table 5.1 or sequence (η). Then the proportion of primes not dividing any $C(n)$ is $\frac{1}{2}e^{-1/2}$.*
- *Let $C(n)$ be one of the sequences (δ), (α), (ϵ), (ζ), (γ) from Table 5.2. Then the proportion of primes not dividing any $C(n)$ is $e^{-1/2}$.*

In stark contrast, Cooper's sporadic sequences s_7, s_{10}, s_{18} from Table 5.2 are divisible by all primes. Indeed, let $C(n)$ denote any of these three sequences. Then,

$$C(p-1) \equiv 0 \pmod p$$

for all primes p. In fact, we can prove much more. For any given prime p, the last quarter (or third) of the first p terms of these sequences are divisible by p. In the case of sequence s_{10}, the sum of fourth powers of binomial coefficients, this is proved by N. Calkin [14]. Indeed, among other divisibility results on sums of powers of binomials, Calkin shows that, for all integers $a \geqslant 0$, the sums

$$\sum_{k=0}^{n} \binom{n}{k}^{2a} \tag{5.50}$$

are divisible by all primes p in the range

$$n < p < n + 1 + \frac{n}{2a - 1}.$$

In particular, in the case $a = 2$, we conclude that $s_{10}(n)$ is divisible by all primes p that satisfy $n < p < \frac{4n}{3} + 1$. Equivalently, we have

$$s_{10}(p - j) \equiv 0 \pmod{p}$$

whenever $1 \leqslant j \leqslant (p + 2)/4$. Our final result proves the same phenomenon for Cooper's sporadic sequences s_7, s_{18}. We note that in each case, empirically, the bounds on j cannot be improved (with the exception of the case $p = 3$ for s_{18}; see Remark 5.5.1).

Theorem 5.5.6. *For any prime p, we have*

$$s_7(p - j) \equiv 0 \pmod{p}$$

whenever $1 \leqslant j \leqslant (p + 1)/3$, *and*

$$s_{18}(p - j) \equiv 0 \pmod{p}$$

whenever $1 \leqslant j \leqslant (p + 2)/4$.

Proof. For the sequence s_7, we want to show

$$\sum_{k=0}^{p-j} \binom{p-j}{k}^2 \binom{p-j+k}{k} \binom{2k}{p-j} \equiv 0 \,(\operatorname{mod} p),$$

for $1 \leqslant j \leqslant (p+1)/3$. Note that for $2k < p-j$ or $k > p-j$ the summand is already zero. Therefore, we assume that $p-j \geqslant k \geqslant (p-j)/2$. We write the summand as

$$\binom{p-j}{k}^2 \binom{p-j+k}{k} \binom{2k}{p-j} = \frac{(p-j+k)!(2k)!}{k!^3(p-j-k)!^2(2k-p+j)!},$$

and observe that the denominator is not divisible by p if $j \geqslant 1$. On the other hand, the factorial $(p-j+k)!$ in the numerator is divisible by p since

$$p-j+k \geqslant p-j+\left\lceil \frac{p-j}{2} \right\rceil \geqslant p,$$

where we used $j \leqslant (p+1)/3$ to verify the final inequality. Thus, for $1 \leqslant j \leqslant (p+1)/3$, the congruences $s_7(p-j) \equiv 0$ hold modulo p, as claimed.

We proceed similarly for $s_{18}(p-j)$, which is given by

$$\sum_{k=0}^{\lfloor (p-j)/3 \rfloor} (-1)^k \binom{p-j}{k} \binom{2k}{k} \binom{2(p-j-k)}{p-j-k} \left\{ \binom{2(p-j)-3k-1}{p-j} + \binom{2(p-j)-3k}{p-j} \right\},$$

and, using (5.36), write the summand as

$$\frac{(-1)^k(2k)!(2(p-j-k))!}{k!^3(p-j-k)!^3}(p-j-3k+1)_{p-j-1}(3p-3j-6k). \tag{5.51}$$

None of the terms in the denominator is divisible by p since $j \geqslant 1$. On the other hand, $(2(p-j-k))!$ in the numerator is divisible by p since

$$2(p-j-k) \geqslant 2\left(p-j-\left\lfloor \frac{p-j}{3} \right\rfloor\right) \geqslant p,$$

where we used $j \leqslant (p+2)/4$ for the final inequality. Therefore, for $1 \leqslant j \leqslant (p+2)/4$, each of the terms in the sum for $s_{18}(p-j)$ is a multiple of p, and we obtain the desired congruences. □

Remark 5.5.1. *Employing* (5.51)*, we observe that* $s_{18}(n) \equiv 0 \pmod 3$ *for* $n \geqslant 1$*, which reaffirms Corollary 5.4.2 for this sequence.*

Finally, as noted in [20], each of the sequences in Table 5.1 times $\binom{2n}{n}$ is an integer solution of (5.2) with $d \neq 0$. Observe that $\binom{2n}{n}$ is divisible by a prime p for all n such that $n < p \leqslant 2n$. This results in a (weaker) analog of Theorem 5.5.6 for these Apéry-like sequences, and implies, in particular, that these sequences are again divisible by all prime numbers.

5.6 Conclusion and open questions

In Sections 5.2 and 5.3, we showed that all sporadic solutions of (5.1) and (5.2), given in Tables 5.1 and 5.2, uniformly satisfy Lucas congruences. However, for two of these sequences, especially sequence (η), we had to resort to a rather technical analysis. We therefore wonder if there is a representation of these sequences from which the Lucas congruences can be deduced more naturally, based on, for instance the approaches of [53] and [37], or [52]. More generally, it would be desirable to have a uniform approach to these congruences, possibly directly from the shape of the defining recurrences and associated differential equations. In another direction, it would be interesting to show that, as numerical evidence suggests, *all* of the Apéry-like sequences in fact satisfy the Dwork congruences (1.15).

The congruences (1.16) show that the Apéry numbers are periodic modulo 8, alternating between the values 1 and 5. As a consequence, the other residue classes $0, 2, 3, 4, 6, 7$ modulo 8 are never attained. On the other hand, the observations in Section 5.5 show that certain primes do not divide any Apéry number. This can be rephrased as saying that the residue class 0 is not attained by the Apéry numbers modulo these primes. This leads us to the question of which residue classes are not attained by Apéry-like numbers modulo prime powers p^α. In particular, are there interesting cases which are not explained by Sections 5.4 and 5.5?

The second part of congruence (5.45) makes it explicit that every prime $p \equiv 3 \pmod 4$ divides a term of the Apéry-like sequence (5.44). Is there a similarly explicit result which demonstrates that the Apéry numbers are divisible by infinitely many distinct primes?

Recall that Conjecture 5.5.3 predicts that the proportion of primes not dividing any Apéry number is $e^{-1/2}$. One of the referees raised the question whether there might be a connection between this conjectured proportion and classical divisbility questions of Bernoulli numbers and the notion of regular primes (for instance, C. L. Siegel conjectured that $e^{-1/2}$ of all prime numbers are regular).

Another interesting question was suggested by the second referee, who noted that the right-hand side of (5.45) is the p-th Fourier coefficient c_p of the modular form

$$\eta(4z)^6 := q \prod_{n=1}^{\infty} (1 - q^{4n})^6 = \sum_{n=1}^{\infty} c_n q^n, \quad q = e^{2\pi i z}.$$

With this observation, a natural question concerning Theorem 5.5.4 is whether there exists a modular form (with CM?) $f(z)$ whose p-th Fourier coefficient is related modulo p^2 to A_η.

References

[1] S. Ahlgren. Distribution of parity of the partition function in arithmetic progressions. *Indag. Math. (N.S.)*, 10(2):173–181, 1999.

[2] S. Ahlgren. Gaussian hypergeometric series and combinatorial congruences. In *Symbolic computation, number theory, special functions, physics and combinatorics (Gainesville, FL, 1999)*, volume 4 of *Dev. Math.*, pages 1–12. Kluwer Acad. Publ., Dordrecht, 2001.

[3] S. Ahlgren and K. Ono. A Gaussian hypergeometric series evaluation and Apéry number congruences. *J. Reine Angew. Math.*, 518:187–212, 2000.

[4] G. Almkvist, D. van Straten, and W. Zudilin. Generalizations of Clausen's formula and algebraic transformations of Calabi-Yau differential equations. *Proc. Edinb. Math. Soc. (2)*, 54(2):273–295, 2011.

[5] G. Almkvist and W. Zudilin. Differential equations, mirror maps and zeta values. In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 481–515. Amer. Math. Soc., Providence, RI, 2006.

[6] R. J. Anderson. Simple zeros of the Riemann zeta-function. *Journal of Number Theory*, (357):176–182, 1983.

[7] R. Apéry. Irrationalité de $\zeta(3)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

[8] T. M. Apostol. *Introduction to analytic number theory.* Springer-Verlag, New York-Heidelberg, 1976. Undergraduate Texts in Mathematics.

[9] B. C. Berndt, A. J. Yee, and A. Zaharescu. New theorems on the parity of partition functions. *J. Reine Angew. Math.*, 566:91–109, 2004.

[10] F. Beukers. Some congruences for the Apéry numbers. *J. Number Theory*, 21(2):141–155, 1985.

[11] F. Beukers. Another congruence for the Apéry numbers. *J. Number Theory*, 25(2):201–210, 1987.

[12] F. Beukers. On Dwork's accessory parameter problem. *Math. Z.*, 241(2):425–444, 2002.

[13] H. M. Bui, B. Conrey, and M. P. Young. More than 41% of the zeros of the zeta function are on the critical line. *Acta Arithmetica*, (150.1):35–64, 2011.

[14] N. J. Calkin. Factors of sums of powers of binomial coefficients. *Acta Arith.*, 86(1):17–26, 1998.

[15] H. H. Chan, S. Cooper, and F. Sica. Congruences satisfied by Apéry-like numbers. *Int. J. Number Theory*, 6(1):89–97, 2010.

[16] S. Chowla, J. Cowles, and M. Cowles. Congruence properties of Apéry numbers. *J. Number Theory*, 12(2):188–190, 1980.

[17] J. B. Conrey. Zeros of derivatives of the Riemann's ξ-function on the critical line. *Journal of Number Theory*, (16):49–74, 1983.

[18] J. B. Conrey. Zeros of derivatives of the Riemann's ξ-function on the critical line, II. *Journal of Number Theory*, (17):71–75, 1983.

[19] J. B. Conrey. More than two fifths of the zeros of the Riemann zeta function are on the critical line. *Journal für die reiene und angewandte Mathematik*, (399):1–26, 1989.

[20] S. Cooper. Sporadic sequences, modular forms and new series for $1/\pi$. *Ramanujan J.*, 29(1-3):163–183, 2012.

[21] M. J. Coster. Supercongruences. In *p-adic analysis (Trento, 1989)*, volume 1454 of *Lecture Notes in Math.*, pages 194–204. Springer, Berlin, 1990.

[22] H. Davenport. *Multiplicative Number Theory.* Number 72. Markham Publishing Company, 1967.

[23] E. Deutsch and B. E. Sagan. Congruences for Catalan and Motzkin numbers and related sequences. *J. Number Theory*, 117(1):191–215, 2006.

[24] G. P. Egorychev. *Integral representation and the computation of combinatorial sums*, volume 59 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1984. Translated from the Russian by H. H. McFadden, Translation edited by Lev J. Leifman.

[25] J. Fabrykowski and M. V. Subbarao. Some new identities involving the partition function $p(n)$. In *Number theory (Banff, AB, 1988)*, pages 139–146. de Gruyter, Berlin, 1990.

[26] S. Feng. Zeros of the Riemann zeta function on the critical line. *Journal of Number Theory*, (132):511–542, 2012.

[27] A. Gafni. Power partitions. *J. Number Theory*, 163:19–42, 2016.

[28] I. Gessel. Some congruences for Apéry numbers. *J. Number Theory*, 14(3):362–368, 1982.

[29] G. H. Hardy and S. Ramanujan. Asymptotic formulæ in combinatory analysis [Proc. London Math. Soc. (2) **17** (1918), 75–115]. In *Collected papers of Srinivasa Ramanujan*, pages 276–309. AMS Chelsea Publ., Providence, RI, 2000.

[30] D. R. Heath-Brown. Simple zeros of the Riemann zeta function on the critical line. *Bull. London Math. Soc.*, 11(1):17–18, 1979.

[31] O. Kolberg. Note on the parity of the partition function. *Math. Scand.*, 7:377–378, 1959.

[32] C. Krattenthaler and T. W. Müller. Generalised Apéry numbers modulo 9. *J. Number Theory*, 147:708–720, 2015.

[33] N. Levinson. More than One Third of Zeros of Riemann's Zeta-Function are on $\sigma = \frac{1}{2}$. *Advances in Mathematics*, (13):383–436, 1974.

[34] E. Lucas. Sur les congruences des nombres eulériens et les coefficients différentiels des functions trigonométriques suivant un module premier. *Bull. Soc. Math. France*, 6:49–54, 1878.

[35] A. Malik and A. Straub. Divisibility properties of sporadic Apéry-like numbers. *Res. Number Theory*, 2:Art. 5, 26, 2016.

[36] R. J. McIntosh. A generalization of a congruential property of Lucas. *Amer. Math. Monthly*, 99(3):231–238, 1992.

[37] A. Mellit and M. Vlasenko. Dwork's congruences for the constant terms of powers of a Laurent polynomial. *Int. J. Number Theory*, 12(2):313–321, 2016.

[38] M. Newman. Periodicity modulo m and divisibility properties of the partition function. *Trans. Amer. Math. Soc.*, 97:225–236, 1960.

[39] J.-L. Nicolas. Valeurs impaires de la fonction de partition $p(n)$. *Int. J. Number Theory*, 2(4):469–487, 2006.

[40] J.-L. Nicolas, I. Z. Ruzsa, and A. Sárközy. On the parity of additive representation functions. *J. Number Theory*, 73(2):292–317, 1998. With an appendix in French by J.-P. Serre.

[41] J.-L. Nicolas and A. Sárközy. On the asymptotic behaviour of general partition functions. *Ramanujan J.*, 4(1):29–39, 2000.

[42] J.-L. Nicolas and A. Sárközy. On the asymptotic behaviour of general partition functions. II. *Ramanujan J.*, 7(1-3):279–298, 2003. Rankin memorial issues.

[43] K. Ono. Parity of the partition function in arithmetic progressions. *J. Reine Angew. Math.*, 472:1–15, 1996.

[44] K. Ono. The partition function in arithmetic progressions. *Math. Ann.*, 312(2):251–260, 1998.

[45] R. Osburn and B. Sahu. Supercongruences for Apéry-like numbers. *Adv. in Appl. Math.*, 47(3):631–638, 2011.

[46] R. Osburn and B. Sahu. A supercongruence for generalized Domb numbers. *Funct. Approx. Comment. Math.*, 48(part 1):29–36, 2013.

[47] R. Osburn, B. Sahu, and A. Straub. Supercongruences for sporadic sequences. *Proc. Edinb. Math. Soc. (2)*, 59(2):503–518, 2016.

[48] S. Ramanujan. Congruence properties of partitions. *Math. Z.*, 9(1-2):147–153, 1921.

[49] I. S. Rezvyakova. On the zeros of derivatives of the Riemann ξ-function. *Izv. Ross. Akad. Nauk Ser. Mat.*, 69(3):109–178, 2005.

[50] I. S. Rezvyakova. On simple zeros of the derivatives of the Riemann ξ-function. *Izv. Ross. Akad. Nauk Ser. Mat.*, 70(2):57–68, 2006.

[51] B. Richmond. A general asymptotic result for partitions. *Canad. J. Math.*, 27(5):1083–1091 (1976), 1975.

[52] E. Rowland and R. Yassawi. Automatic congruences for diagonals of rational functions. *J. Théor. Nombres Bordeaux*, 27(1):245–288, 2015.

[53] K. Samol and D. van Straten. Dwork congruences and reflexive polytopes. *Ann. Math. Qué.*, 39(2):185–203, 2015.

[54] A. Selberg. On the zeros of Riemann's zeta-function. *Skr. Norske Vid. Akad. Oslo*, I(10):1–59, 1942.

[55] C. L. Siegel. *On Advanced Analytic Number Theory.* Tata Institute of Fundamental Research, 1959.

[56] J. Stienstra and F. Beukers. On the Picard-Fuchs equation and the formal Brauer group of certain elliptic $K3$-surfaces. *Math. Ann.*, 271(2):269–304, 1985.

[57] A. Straub. Multivariate Apéry numbers and supercongruences of rational functions. *Algebra Number Theory*, 8(8):1985–2007, 2014.

[58] E. C. Titchmarsh. *The theory of the Riemann zeta-function.* The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[59] A. van der Poorten. A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer*, 1(4):195–203, 1978/79. An informal report.

[60] R. C. Vaughan. *The Hardy-Littlewood method*, volume 125 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, second edition, 1997.

[61] R. C. Vaughan. Squares: additive questions and partitions. *Int. J. Number Theory*, 11(5):1367–1409, 2015.

[62] E. T. Whittaker and G. N. Watson. *A course of modern analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions, Reprint of the fourth (1927) edition.

[63] E. M. Wright. Asymptotic partition formulae. III. Partitions into k-th powers. *Acta Math.*, 63(1):143–191, 1934.

[64] Y. Yang. Partitions into primes. *Trans. Amer. Math. Soc.*, 352(6):2581–2600, 2000.

[65] D. Zagier. Integral solutions of Apéry-like recurrence equations. In *Groups and symmetries*, volume 47 of *CRM Proc. Lecture Notes*, pages 349–366. Amer. Math. Soc., Providence, RI, 2009.