

© 2017 by Mi Young Jang. All rights reserved.

ON THE SUPER HILBERT SCHEME OF CONSTANT HILBERT POLYNOMIALS

BY

MI YOUNG JANG

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2017

Urbana, Illinois

Doctoral Committee:

Professor Thomas Nevins, Chair
Professor Sheldon Katz, Director of Research
Professor William Haboush
Professor Henry Schenck

Abstract

In this thesis we mainly consider supermanifolds and super Hilbert schemes.

In the first part of this dissertation, we construct the Hilbert scheme of 0-dimensional subspaces on dimension $1|1$ supermanifolds. By using a flattening stratification, we compute the local defining equation for the super Hilbert scheme. From local defining equations, we conclude that the Hilbert scheme of constant Hilbert polynomials on dimension $1|1$ supermanifolds is smooth.

The second part of this thesis concerns the smoothness and the non smoothness of 0-dimensional subspaces on some supermanifolds of higher dimensions, which is related with the future study chapter.

The last part is devoted to the splitness of the Hilbert scheme. The non-splitness of supermanifolds can be deduced from the non vanishing of some cohomology class, called the obstruction class. We find examples of both split and non-split super Hilbert schemes. For the split case, we find a split model which is isomorphic to $\text{Hilb}^{1|1}(\Pi\mathcal{O}_{\mathbb{P}^1}(k))$ for any k . For the non-split case, we compute the obstruction class of the super Hilbert scheme $\text{Hilb}^{2|1}(\Pi\mathcal{O}_{\mathbb{P}^1}(k))$ and show that this class is not vanishing for $k \neq 0$ and vanishing for $k = 0$. Moreover, since the odd dimension of this Hilbert scheme is 2, we can see that $\text{Hilb}^{2|1}(\Pi V)$ is projected for $k = 0$ and not projected for all $k \neq 0$.

To my family who supported me along the way.

Acknowledgments

I would like to express my deep gratitude to my advisor Sheldon Katz. I always honor him as a mathematician and as a mentor. I appreciate his constant guidance and encouragement through out my study. I would also like to thanks to my committee members, Thomas Nevins, William Haboush and Henry Schenck, for being my committee and helpful comments.

I acknowledge financial support from NSF award DMS-12-01089.

Thanks to my friends who spent quality time with me. Special thanks go to my parents for their faith in me and consistant support. Thanks to my husband Sangjun for his love and for being a great part of my life.

Table of Contents

Chapter 1	Introduction	1
Chapter 2	Supergeometry	4
2.1	Review of Superalgebra	4
2.1.1	Modules	4
2.1.2	Matrices	5
2.2	Superspaces	7
2.2.1	Supergrassmannians	8
2.2.2	Obstruction Class and Splitting	11
Chapter 3	Review of the Hilbert scheme	15
3.1	Hilbert scheme	15
3.2	Hilbert scheme of points on a plane	16
Chapter 4	The Super Hilbert Scheme	17
4.1	Flattening Stratifications	18
4.1.1	The super Hilbert scheme of $\mathbb{C}^{1 1}$	19
4.1.2	The super Hilbert scheme of a dimension $1 1$ supermanifold	24
Chapter 5	(Non)Smoothness of the Hilbert scheme $\text{Hilb}^{p q}(\mathbb{C}^{1 2})$	25
5.1	Smoothness of $\text{Hilb}^{p 0}(\mathbb{C}^{1 n})$	26
5.2	Smoothness of $\text{Hilb}^{p 1}(\mathbb{C}^{1 n})$	27
5.3	Smoothness of $\text{Hilb}^{1 2}(\mathbb{C}^{1 2})$	28
5.4	Nonsmoothness of $\text{Hilb}^{2 2}(\mathbb{C}^{1 2})$	29
Chapter 6	(Non)splitness of the super Hilbert scheme	32
6.1	A split super Hilbert scheme	32
6.2	The super Hilbert scheme $\text{Hilb}^{2 1}(\Pi\mathcal{O}_{\mathbb{P}^1}(k))$	33
6.2.1	(Non)splitness of the super Hilbert scheme $\text{Hilb}^{2 1}(\Pi\mathcal{O}(k))$	36
Chapter 7	Future works	47
7.1	(Non)smoothness of $\text{Hilb}^{p q}(\mathbb{C}^{1 2})$	47
References		48

Chapter 1

Introduction

Moduli spaces are geometric objects which parametrize geometric objects with certain properties. [Gr2, HM]. In other words, they can be viewed as solutions of classification problems. Moreover, the moduli space itself has lots of interesting properties and rich structure as a geometric object. The Hilbert scheme is one of the most important examples of moduli spaces. It classifies closed subschemes of a given space. The existence of the Hilbert scheme of projective spaces has been shown [Gr]. It also has been shown that the Hilbert scheme of projective spaces are also projective and closed subschemes of Grassmannians.

One of the main topics of this thesis is the Hilbert scheme $\text{Hilb}^n(X)$ of n points on X . It parametrizes 0-dimensional subschemes of X with length n . If we have distinct points in X , then the closed subspace of X corresponding to those points has dimension 0 and length n . All the information that we have to know to specify those n distinct points is encoded in the n th symmetric product $\text{Sym}^n(X)$ of X minus the diagonal. But when two (or more) points collide, we need more than that.

Suppose X is a smooth surface and we have two points that collide. Then we have the point where they meet and a tangent vector at that point.

According to [Fo], there is a birational map $\phi : \text{Hilb}^n(X) \rightarrow \text{Sym}^n(X)$ such that ϕ is a resolution of singularities along the diagonal of $\text{Sym}^n(X)$. Therefore, $\text{Hilb}^n(X)$ is smooth and has dimension $2n$.

Supergeometry is a \mathbb{Z}_2 -graded extension of ordinary geometry. The idea of this extension comes from supersymmetry in physics. Instead of a sheaf of commutative rings, we consider a sheaf of supercommutative rings. In this regard, an ordinary manifold has a \mathbb{Z}_2 -graded generalization called a supermanifold. As in the ordinary case, a supermanifold has a local coordinate system. That is, for any complex supermanifold of dimension $m | n$, there is a local coordinate ring $\mathbb{C}[x_1, \dots, x_m | \theta_1, \dots, \theta_n]$ where x_i 's are commuting variables

and θ_j 's are anti-commuting variables. Hence, the coordinate ring $\mathbb{C}[x_1, \dots, x_m | \theta_1, \dots, \theta_n]$ has relations

$$x_i x_j = x_j x_i$$

$$x_i \theta_j = \theta_j x_i$$

$$\theta_i \theta_j = -\theta_j \theta_i$$

for all i and j . Details about supergeometry can be found in [Ma, Be].

In Chapter 2, we review basic properties of superalgebras, superspaces and super Grassmannians. We also explain that the obstruction to splitting can be characterized by cohomology classes which we call obstruction classes.

In Chapter 4, we define a super version of the Hilbert scheme, which we call the super Hilbert scheme, and construct the super Hilbert scheme of constant Hilbert polynomials over dimension $1|1$ supermanifolds.

The main result in Chapter 4 is about the smoothness of the Hilbert scheme $\text{Hilb}^{p|q}(S)$.

Theorem 1.0.1. *Let S be a supermanifold of dimension $1|1$. Then the super Hilbert scheme $\text{Hilb}^{p|q}(S)$ is smooth and has dimension $p|p$.*

This theorem can be viewed as an analogue of the result for the ordinary Hilbert scheme.

Proposition 1.0.2. *(Fogarty [Fo]) Let X be a projective space. Then the Hilbert scheme $\text{Hilb}^n(X)$ of n points on X is connected. Moreover, if X is a smooth surface then $\text{Hilb}^n(X)$ is smooth and has dimension $2n$.*

In Chapter 5, conditions for $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ to be smooth are specified.

Theorem 1.0.3. *$\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is smooth only when $p + q \leq 3$ or $q \leq 1$.*

Superstring perturbative theory in physics can be described as an integration over the moduli space \mathfrak{M}_g of super Riemann surfaces. Consider a distribution $\mathcal{D} \subset TS$ generated by v . Then we say \mathcal{D} is *everywhere non-integrable* if v^2 is everywhere independent of v .

Definition 1.0.4. A *Super Riemann Surface* is a pair (S, \mathcal{D}) such that $S = (C, \mathcal{O}_S)$ is a complex analytic supermanifold of dimension $1|1$ and $\mathcal{D} \subset TS$ is everywhere non-integrable.

The moduli space of super Riemann surfaces of genus g is denoted by \mathfrak{M}_g . If \mathfrak{M}_g is projected, we have a projection to the moduli space \mathcal{M}_g of ordinary Riemann surfaces. Then we can integrate over \mathfrak{M}_g via the pushforward of the map $\mathfrak{M}_g \rightarrow \mathcal{M}_g$. However, \mathfrak{M}_g is not projected in general. It has been proved recently in [DW] that the supermoduli space \mathfrak{M}_g is not projected for $g \geq 5$.

Splitness is another important property of superspaces. Let (S, \mathcal{O}_S) be a superspace and let $\mathcal{E} = (\mathcal{J}/\mathcal{J}^2)^\vee$ be the locally free sheaf where $\mathcal{J} \subset \mathcal{O}_S$ is the ideal generated by nilpotents. We say S is split if it can be recovered from the bosonic part and a locally free sheaf \mathcal{E} in a precise way, which we describe in Section 2.2. Note that every split superspace has a natural projection map to its bosonic space, so is projected.

Chapter 6 is devoted to the splitness of some super Hilbert schemes.

Theorem 1.0.5. *Let $V = \mathcal{O}_{\mathbb{P}^1}(k)$ be a line bundle on \mathbb{P}^1 . Then the Hilbert scheme $\text{Hilb}^{1|1}(\text{IV})$ is split for all k and $\text{Hilb}^{2|1}(\text{IV})$ is split for $k = 0$ and not split for all $k \neq 0$.*

Since every superspace is locally isomorphic to the split model, each of them can be described by specifying transition maps. Those maps allow us to find an obstruction class. We show the non splitness of the Hilbert scheme $\text{Hilb}^{2|1}(\text{IV})$ by showing that the second obstruction class is nonvanishing for $k \neq 0$.

Chapter 2

Supergeometry

2.1 Review of Superalgebra

Algebraic geometry relies on commutative ring theory. Likewise, supergeometry is based on superalgebra. This section is devoted to reviewing some basic properties of superalgebras. ([Ma],[La])

Let $A = A_0 \oplus A_1$ be a \mathbb{Z}_2 -graded ring. Then it has the property $A_i A_j \subset A_{i+j}$ where all subscripts are in \mathbb{Z}_2 . Elements of A_i are called even if $i = 0$ and called odd if $i = 1$. An element $a \in A$ is called homogeneous if $a \in A_i$ for some i . Let's denote $|a| = i$.

Definition 2.1.1. A *supercommutative ring* $A = A_0 \oplus A_1$ is a \mathbb{Z}_2 -graded ring with the property that $ab = (-1)^{|a||b|}ba$ for all homogeneous elements a and b in A .

Example 2.1.1. Let R be a commutative ring and M be an R -module. The exterior algebra $A = \bigwedge^\bullet M = \bigoplus_{i \geq 0} \bigwedge^i M$ over R is a supercommutative ring with even part $A_0 = \bigoplus_{i \geq 0} \bigwedge^{2i} M$ and odd part $A_1 = \bigoplus_{i \geq 0} \bigwedge^{2i+1} M$.

Example 2.1.2. Consider the k -algebra $A = k[x_1, \dots, x_m | \theta_1, \dots, \theta_n]$ with relations $x_i x_j = x_j x_i$, $x_i \theta_j = \theta_j x_i$ and $\theta_i \theta_j = -\theta_j \theta_i$. Then A is a supercommutative ring.

2.1.1 Modules

Let $A = A_0 \oplus A_1$ be a supercommutative ring. A (left) A -module M is an abelian group with a decomposition $M = M_0 \oplus M_1$ such that the (left) module structure $A \times M \rightarrow M$ preserves the \mathbb{Z}_2 -grading $A_i M_j \subset M_{i+j}$ where all subscripts are in \mathbb{Z}_2 . Every left module has a natural bimodule structure defined by $m \cdot a = (-1)^{|m||a|} a \cdot m$ for homogeneous elements $a \in A$ and $m \in M$, i.e. for non-homogeneous $m = m_0 \oplus m_1$ and $a = a_0 \oplus a_1$,

$$\begin{aligned} a \cdot m &= (a_0 \oplus a_1) \cdot (m_0 \oplus m_1) \\ &= m_0 a_0 + m_0 a_1 + m_1 a_0 - m_1 a_1 \end{aligned}$$

An A -module homomorphism $f : M \rightarrow N$ is an additive A -linear map. The additive group $\text{Hom}_A(M, N)$ of A -module homomorphisms from M to N has a natural A -module structure with the decomposition $\text{Hom}_A(M, N) = \text{Hom}_A(M, N)_0 \oplus \text{Hom}_A(M, N)_1$ where $\text{Hom}_A(M, N)_0$ is the additive group of A -module homomorphisms which preserve the \mathbb{Z}_2 -grading and $\text{Hom}_A(M, N)_1$ is the additive group of A -module homomorphisms which reverse the \mathbb{Z}_2 -grading.

The *Jacobson radical* of a ring A is defined to be the intersection of all maximal ideals of A .

Proposition 2.1.2. (*Nakayama's Lemma [La]*) *Let M be a module over a supercommutative ring A and let $J \subset A$ be the Jacobson radical. For any finitely generated A -module M , $JM = M$ implies $M = 0$.*

Corollary 2.1.3. *Let M be a finitely generated A -module. Let x_1, \dots, x_n be elements in M such that the images $\bar{x}_1, \dots, \bar{x}_n$ in M/JM generate M/JM . Then x_1, \dots, x_n generate M .*

The parity change functor Π is the functor between modules such that ΠM (a) is the same as M as an additive group and (b) has the reverse parity $(\Pi M)_0 = M_1$ and $(\Pi M)_1 = M_0$ and (c) it has an A -module structure given by $a(\Pi m) = (-1)^{|a||m|} \Pi(am)$. Practically we can think of the parity of the parity change functor as odd.

A free A -module of rank $(p|q)$ is an A -module isomorphic to $A^p \oplus (\Pi A)^q$.

A derivation can be defined in an analogous way.

Definition 2.1.4. Let A be a supercommutative ring and let M be an A -module. A derivation of A into M is an A -module homomorphism $d : A \rightarrow M$ satisfying the Leibniz rule

$$d(ab) = (da)b + (-1)^{|a||d|} a(db)$$

where $|d|$ is the parity of d as a homomorphism of A -modules.

2.1.2 Matrices

Let R be a commutative ring. A m by p matrix with elements from R can be viewed as an R -module homomorphism from R^p to R^m . However, when we consider a morphism between free modules over a supercommutative ring, it is crucial to remember information about the parity.

Consider a supercommutative ring A . Consider two indexing sets with the decomposition $I = I_0 \cup I_1$ and

$J = J_0 \cup J_1$ where each of them are defined as

$$\begin{aligned} I_0 &= \{1, 2, \dots, m\} \\ I_1 &= \{m+1, m+2, \dots, m+n\} \\ J_0 &= \{1, 2, \dots, p\} \\ J_1 &= \{p+1, p+2, \dots, p+q\} \end{aligned}$$

A matrix in the given format $I \times J$ with values in A is defined as a matrix with four blocks

$$M = \begin{matrix} & \begin{matrix} p & q \end{matrix} \\ \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \end{matrix}$$

such that each M_i is a matrix with entries in A , M_1 is $m \times p$, M_2 is $m \times q$, M_3 is $n \times p$ and M_4 is $n \times q$. We say M is *even* if all the entries of M_1 and M_4 are even and entries of M_2 and M_3 are odd.

The set of all matrices with the format I and J over A is denoted by $M(m|n; p|q; A)$. Observe that $M(m|n; p|q; A)$ can be identified with the set of A -module homomorphisms $\text{Hom}_A(A^{p|q}, A^{m|n})$.

Lemma 2.1.5. *Let $X \in M(m|n; m|n; A)$ be an invertible matrix and let $\Gamma \in M(m|n; m|n; A)$ be a matrix such that all entries are odd. Then the matrix $X + \Gamma$ is (left) invertible.*

Proof. Claim that any odd matrix $\Gamma \in M(m|n; m|n; A)$ is nilpotent. Set $\Gamma = (\gamma_{ij})$. Then each entry of $\Gamma^{(m+n)^2+1} = 0$ has the form of $\sum \left(\prod_{k=1}^{(m+n)^2+1} \gamma_{i_k j_k} \right)$, and $\prod_{k=1}^{(m+n)^2+1} \gamma_{i_k j_k} = 0$.

Since $X + \Gamma = X(I + X^{-1}\Gamma)$, it is enough to show that $I + X^{-1}\Gamma$ is invertible. Observe that

$$(X^{-1}\Gamma)^{(m+n)^2+1} = 0$$

Therefore, the inverse is given as

$$(I + X^{-1}\Gamma)^{-1} = I - X^{-1}\Gamma + (X^{-1}\Gamma)^2 + \dots + (-1)^{(m+n)^2} (X^{-1}\Gamma)^{(m+n)^2}$$

□

2.2 Superspaces

Supergeometry is a \mathbb{Z}_2 -graded generalization of ordinary geometry. It is motivated from the theory of supersymmetry in theoretical physics. For simplicity, we will only consider analytic superspaces over the complex numbers \mathbb{C} hereafter. Details about superspaces and supermanifolds can be found in [Ma], [Be], [DW].

We review major definitions in this section.

Definition 2.2.1. A *superspace* is a pair (S, \mathcal{O}_S) consisting of a topological space S and a sheaf of supercommutative rings $\mathcal{O}_S = \mathcal{O}_S^0 \oplus \mathcal{O}_S^1$ such that for each $s \in S$ the stalk $\mathcal{O}_{S,s}$ is a local ring.

Each inclusion map $V \hookrightarrow U$ of open sets of S gives the restriction map $\mathcal{O}_S(U) \rightarrow \mathcal{O}_S(V)$ which is a supercommutative ring homomorphism.

Let $\mathcal{J} \subset \mathcal{O}_S$ be the ideal generated by the odd part \mathcal{O}_S^1 . Define the *bosonic space* S_b of S as the closed subspace $S_b = (S, \mathcal{O}_S/\mathcal{J})$ of S . Note that the bosonic space is an ordinary space (analytic, algebraic, etc).

As we defined a free module over a supercommutative ring, we can define a *free* sheaf of rank $(p|q)$ on a superspace S and denote it as $\mathcal{O}_S^p \oplus \Pi(\mathcal{O}_S^q)$. A *locally free* sheaf of rank $(p|q)$ on S is a sheaf on S which is locally isomorphic to $\mathcal{O}_S^p \oplus \Pi(\mathcal{O}_S^q)$.

Definition 2.2.2. A superspace (S, \mathcal{O}_S) is said to be *split* if there is a locally free sheaf \mathcal{E} on S_b such that (S, \mathcal{O}_S) is isomorphic to $S(S_b, \mathcal{E}) := (S_b, \wedge^\bullet \mathcal{E}^\vee)$. The dimension of a split superspace $S(S_b, \mathcal{E}) := (S_b, \wedge^\bullet \mathcal{E}^\vee)$ is defined as $(m|n)$ where m is the dimension of the ordinary scheme S_b and n is the rank of \mathcal{E} .

The simplest example of a split superspace is *analytic affine superspace* $\mathbb{C}^{m|n}$. Consider an ordinary (analytic) affine space $(\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m})$. We define $\mathbb{C}^{m|n}$ as a split superspace

$$\mathbb{C}^{m|n} = (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^{m|n}}) = S(\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m}^n)$$

Let $\theta_1, \dots, \theta_n$ be coordinates on fiber $\mathcal{O}_{\mathbb{C}^m}^n$. Then the structure sheaf is given by

$$\mathcal{O}_{\mathbb{C}^{m|n}}(U) = \mathcal{O}_{\mathbb{C}^m}(U)[\theta_1, \dots, \theta_n]$$

with relations $\theta_i \theta_j = -\theta_j \theta_i$ for all i and j .

If a superspace (S, \mathcal{O}_S) is split then there is a natural projection map $S \rightarrow S_b$. Let \mathcal{E} be a locally free sheaf on S_b such that $(S, \mathcal{O}_S) \simeq S(S_b, \mathcal{E})$, then S is just the total space of a parity reversed vector bundle on S_b and the projection map can be defined by forgetting all odd variables where odd variables are fiber coordinates on \mathcal{E} .

We say S is *projected* if it has a projection map from S to its bosonic part S_b so that \mathcal{O}_S endowed with a \mathcal{O}_{S_b} -module structure.

Definition 2.2.3. Consider an open subset $U \subset \mathbb{C}^m$. Let $I \subset \mathcal{O}_{\mathbb{C}^m|n}(U)$ be an ideal and let $Z(I) \subset \mathbb{C}^m$ be the closed subset defined by the zero set $Z(I \cap \mathcal{O}_{\mathbb{C}^m}(U))$. The *analytic subspace* defined by I on U is the superspace $(Z(I), \mathcal{O}_Z := \mathcal{O}_U/I)$. We say a superspace (S, \mathcal{O}_S) is an *analytic superspace* if it is locally isomorphic to some analytic subspace of some open subset $U \subset \mathbb{C}^n$.

An analytic superspace (S, \mathcal{O}_S) is called *smooth* if it is locally isomorphic to an open subspace $\mathbb{C}^{m|n}|_U$ of an affine space. We have a term to indicate a smooth superspace.

Definition 2.2.4. A *supermanifold* (S, \mathcal{O}_S) is a locally split analytic superspace such that the bosonic space S_b is isomorphic to some ordinary manifold.

2.2.1 Supergrassmannians

In this section we review the definition of the supergrassmannian [Ma].

The supergrassmannian functor $\mathcal{G}r(k|l; m|n)$ is the functor from the category of (analytic) superspaces to the category of sets defined by

$$\mathcal{G}r(k|l; m|n)(S) = \left\{ \begin{array}{l} \mathcal{O}_S^m \oplus \mathcal{O}_S^n \rightarrow Q \rightarrow 0 \\ \left| \begin{array}{l} Q \text{ is locally free} \\ \text{of rank } k|l \end{array} \right. \end{array} \right\}$$

Let e_1, \dots, e_m and $\epsilon_{m+1}, \dots, \epsilon_{m+n}$ be the (even and odd) standard generators of $\mathcal{O}_S^m \oplus \Pi\mathcal{O}_S^n$. Let

$$I = I_0 \times I_1 = \{i_1, i_2, \dots, i_k\} \times \{j_1, j_2, \dots, j_l\} \subset \{1, \dots, m\} \times \{m+1, \dots, m+n\}$$

be a subset of the indexing set. Define a natural inclusion map for each indexing set I as

$$\begin{aligned} i_I : (\oplus_{i \in I_0} e_i \cdot \mathcal{O}_S) \oplus (\oplus_{j \in I_1} \epsilon_j \cdot \mathcal{O}_S) &\hookrightarrow \mathcal{O}_S^m \oplus \Pi\mathcal{O}_S^n \\ e_i &\mapsto e_i \\ \epsilon_j &\mapsto \epsilon_j \end{aligned}$$

Definition 2.2.5. The subfunctor $\mathcal{G}r(k|l; m|n)_I$ of the supergrassmannian functor $\mathcal{G}r(k|l; m|n)$ is defined by the following property:

For any $[\mathcal{O}_S^m \oplus \Pi\mathcal{O}_S^n \xrightarrow{\mu} Q \rightarrow 0] \in \mathcal{G}r(k|l; m|n)_I(S) \subset \mathcal{G}r(k|l; m|n)(S)$, Q is canonically

isomorphic to $\mathcal{O}_S^k \oplus \Pi\mathcal{O}_S^l$ via the map

$$\mathcal{O}_S^k \oplus \Pi\mathcal{O}_S^l \xrightarrow{i_I} \mathcal{O}_S^m \oplus \Pi\mathcal{O}_S^n \xrightarrow{\mu} \mathcal{Q}$$

Let \mathcal{K} be the kernel of the map μ . Then \mathcal{K} is isomorphic to $\mathcal{O}_S^{m-k} \oplus \Pi\mathcal{O}_S^{n-l}$. Let k_i 's and κ_j 's be the canonical even and odd generators of $\mathcal{K} \simeq \mathcal{O}_S^{m-k} \oplus \Pi\mathcal{O}_S^{n-l}$. For each $i \notin I_0$, we can find unique even and odd sections a_{ij} and α_{ij} of \mathcal{O}_S such that

$$k_i := e_i + \sum_{j=1}^k a_{ij}e_j + \sum_{j=1}^l \alpha_{ij}\epsilon_j$$

Also, for each $i \notin I_1$, we can find unique even and odd sections b_{ij} and β_{ij} of \mathcal{O}_S such that

$$\kappa_i := \epsilon_i + \sum_{j=1}^k \beta_{ij}e_j + \sum_{j=1}^l b_{ij}\epsilon_j$$

Therefore, for all analytic superspaces S and for all $[\mathcal{O}_S^m \oplus \Pi\mathcal{O}_S^n \rightarrow \mathcal{Q} \rightarrow 0] \in \mathcal{G}r(k|l; m|n)_I(S)$, we get $a_{ij}, b_{ij}, \alpha_{ij}$'s and β_{ij} 's. Identifying $(a_{ij}, b_{ij} | \alpha_{ij}, \beta_{ij})$ with coordinates on $\mathbb{C}^{k(m-k)+l(n-l) | l(m-k)+k(n-l)}$, we get the map

$$\mathcal{G}r(k|l; m|n)_I(S) \rightarrow \mathbb{C}^{k(m-k)+l(n-l) | l(m-k)+k(n-l)}$$

In fact, $\mathcal{G}r(k|l; m|n)_I$ is representable by the affine space $\mathbb{C}^{k(m-k)+l(n-l) | l(m-k)+k(n-l)}$, and the universal quotient is defined as

$$\mathcal{O}^m \oplus \Pi\mathcal{O}^n \rightarrow \mathcal{Q} \rightarrow 0$$

where $\mathcal{Q} = \mathcal{O}^m \oplus \Pi\mathcal{O}^n / K$ is the quotient of $\mathcal{O}^m \oplus \Pi\mathcal{O}^n$ by K where K is the free module generated by

$$k_i := e_i + \sum_{j=1}^k a_{ij}e_j + \sum_{j=1}^l \alpha_{ij}\epsilon_j$$

and

$$\kappa_j := \epsilon_j + \sum_{l=1}^k \beta_{jl}e_l + \sum_{l=1}^l b_{jl}\epsilon_l$$

for $i \notin I_0$ and $J \notin I_1$.

For each indexing subset

$$I = I_0 \times I_1 = \{i_1, i_2, \dots, i_k\} \times \{j_1, j_2, \dots, j_l\}$$

of the indexing set let $U_I = \mathbb{C}^{k(m-k)+l(n-l) | l(m-k)+k(n-l)}$ denote the affine space defined as above. Then we can glue all of such U_I 's as follows.

Define a matrix $Z_I \in M(k|l; m|n)$ as follows:

- i) The submatrix obtained by taking columns of Z_I with indices in I is the identity matrix and
- ii) all other columns are filled with even variables x_I^{pq} and odd variables θ_I^{pq} as below

$$Z_I = \begin{array}{c} \begin{array}{cccc} & m-k & k & l & n-l \\ \begin{array}{c} k \\ \\ \\ l \end{array} & \left(\begin{array}{c|c|c|c} & & & \\ & 1 & & \\ & x_I^{pq} & \ddots & 0 \\ & & & 1 \\ \hline & \theta_I^{pq} & 0 & \ddots \\ & & & & 1 \\ & & & & x_I^{pq} \end{array} \right) \end{array}$$

Think of $(x_I^{pq} | \theta_I^{pq})$ as coordinates on $\mathbb{C}^{k(m-k)+l(n-l) | l(m-k)+k(n-l)}$. Label this affine (open) space $U_I = \mathbb{C}^{k(m-k)+l(n-l) | l(m-k)+k(n-l)}$.

Let B_{IJ} be the submatrix of Z_I formed by taking columns with indices in J . Let $U_{IJ} \subset U_I$ be the subspace where the matrix B_{IJ} is invertible, then since it is an open condition we know that U_{IJ} is open. Then we glue U_I and U_J along U_{IJ} and U_{JI} via $Z_J = B_{IJ}^{-1} Z_I$.

The supergrassmannian $\mathbb{G}r(k|l; m|n)$ is defined as $\bigcup_I U_I / \sim$ with the relations \sim induced by coordinate changes.

In conclusion, the supergrassmannian functor is representable by $\mathbb{G}r(k|l; m|n) := \bigcup_I U_I / \sim$.

Example 2.2.1. Projective superspace can be defined by using supergrassmannian

$$\mathbb{P}^{m|n} = \mathbb{G}r(1|0; m+1|n)$$

For each indexing set $I = i \times \emptyset \subset \{1, \dots, m+1\} \times \{m+1+1, \dots, m+1+n\}$, we have an open affine subspace $U_I = \mathbb{C}^{m|n}$ and a matrix $Z_I \in M(k|l; m|n)$.

$$Z_I = \left(\begin{array}{c|c} x_i^1 & \dots & x_i^{i-1} & 1 & x_i^{i+1} & \dots & x_i^{m+1} \\ \hline \theta_i^1 & \dots & \theta_i^n \end{array} \right)$$

Let $I = \{i\} \times \emptyset$ and $J = \{j\} \times \emptyset$. Then $B_{IJ} = \left(x_i^j \right)$ and U_{IJ} is defined by $x_j \neq 0$. The gluing map on each

U_{IJ} is given by

$$\begin{aligned} & \left(\begin{array}{cccccc|ccc} x_j^1 & \cdots & x_j^{j-1} & 1 & x_j^{j+1} & \cdots & x_j^{m+1} & \theta_j^1 & \cdots & \theta_j^n \end{array} \right) \\ & = (x_i^j)^{-1} \left(\begin{array}{cccccc|ccc} x_i^1 & \cdots & x_i^{i-1} & 1 & x_i^{i+1} & \cdots & x_i^{m+1} & \theta_i^1 & \cdots & \theta_i^n \end{array} \right) \end{aligned}$$

On the other hand, as in the ordinary case, we can use homogeneous coordinates $[x_0; \cdots; x_m | \theta_1; \cdots; \theta_n]$ on $\mathbb{P}^{m|n}$.

In fact, the projective space $\mathbb{P}^{m|n}$ is split.

Proposition 2.2.6. *The projective space $\mathbb{P}^{m|n}$ is canonically isomorphic to its split model*

$$S(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}^n \otimes \mathcal{O}_{\mathbb{P}^m}(1))$$

Proof. On each U_I , we identify θ_i^k with a generator of $(\mathcal{O}_{\mathbb{P}^m}^n)^\vee$ and identify x_i^l with a section in $\mathcal{O}_{\mathbb{P}^m}(1)$. Then the transition maps for $\mathbb{P}^{m|n}$ and $S(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}^n \otimes \mathcal{O}_{\mathbb{P}^m}(1))$ agree. \square

2.2.2 Obstruction Class and Splitting

Let (S, \mathcal{O}_S) be a supermanifold and let $J \subset \mathcal{O}_S$ be the ideal generated by \mathcal{O}_S^1 . We can recover the underlying ordinary manifold $\mathcal{O}_M := \mathcal{O}_S/J$ and define a locally free sheaf \mathcal{E} on M where \mathcal{E} is defined by $\mathcal{E}^\vee = J/J^2$. From these two ingredients, we can construct a split supermanifold $S(M, \mathcal{E})$. Then we say (S, \mathcal{O}_S) is modeled on M and \mathcal{E} . Let $\text{Gr}_J(\mathcal{O}_S)$ be the sheaf on S defined as $\bigoplus_{i=0}^{\infty} J^i/J^{i+1}$.

For the rest of this section, we will see the how to classify all supermanifolds modeled on M and \mathcal{E} and define an obstruction class such that non-vanishing of this class guarantees non-splitness, mostly following [DW].

Let $\text{Isom}(S(M, \mathcal{E}), S)$ be a sheaf of local isomorphisms on M defined by relating an open subset $U \subset M$ to the isomorphisms from $S(M, \mathcal{E})|_U$ to $S|_U$. Since (S, \mathcal{O}_S) has the reduced space M and has odd dimension n , S and $S(M, \mathcal{E})$ are locally isomorphic. Hence, $\text{Isom}(S(M, \mathcal{E}), S)$ is locally isomorphic to $\text{Aut}(S(M, \mathcal{E}))$ as a sheaf. Therefore, if we are given a supermanifold S which is modeled on M and \mathcal{E} , we get an element in $H^1(M, \text{Aut}(\wedge^\bullet \mathcal{E}))$ under the identification $\text{Aut}(S(M, \mathcal{E})) = \text{Aut}(\wedge^\bullet \mathcal{E}^\vee) \simeq \text{Aut}(\wedge^\bullet \mathcal{E})$.

Since S is modeled on M and \mathcal{E} , it induces an automorphism on $\wedge^\bullet \mathcal{E}$ which fixes \mathcal{E} and M . Let G be the

set of all automorphisms on $S(M, \mathcal{E})$ which preserve M and \mathcal{E} . Then G is the kernel of the map

$$\begin{aligned} \text{Aut}(\wedge^\bullet \mathcal{E}) &\rightarrow \text{Aut}(\mathcal{E}) \\ f &\mapsto f|_{\text{Aut}(\mathcal{E})} \end{aligned}$$

and we have the short exact sequence

$$0 \rightarrow G \rightarrow \text{Aut}(\wedge^\bullet \mathcal{E}) \rightarrow \text{Aut}(\mathcal{E}) \rightarrow 0$$

Define $S^{(i)}$ to be the superspace $(M, \mathcal{O}_S/J^{i+1})$. Then we have a filtration of S

$$S^{(0)} \subset S^{(1)} \subset \dots \subset S^{(n)} = S$$

where $n = \text{rank}(\mathcal{E})$.

Let $G^i \subset G$ be the set of automorphisms on $S(M, \mathcal{E})$ which preserve $S(M, \mathcal{E})^{(i-1)}$. Then we have a filtration of G

$$G^1 = G \supset G^2 \supset \dots \supset G^n$$

Definition 2.2.7. Let S be an analytic superspace. The tangent sheaf \mathcal{T}_S is defined as the sheaf of derivations of \mathcal{O}_S . Then the restriction of the tangent sheaf $\mathcal{T}_S|_M$ to the reduced space M of S is split and has even and odd subsheaf $\mathcal{T}_+S := \mathcal{T}_M$ and $\mathcal{T}_-S := V$. Here V is the locally free sheaf on M such that $S(M, V) = \text{Gr}(\mathcal{O}_S)$.

Remark 2.2.8. G^i/G^{i+1} can be identified with $T_{(-)^i}S \otimes \wedge^i \mathcal{E}^\vee$. One way to see this isomorphism is to identify G with its Lie algebra \mathfrak{g} . Since \mathfrak{g} is nilpotent, the exponential map from \mathfrak{g} to G is a bijection. Let $(x_1, \dots, x_m | \theta_1, \dots, \theta_n)$ be local coordinates. Set $\bar{x} = (x_1, \dots, x_m)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\theta^\alpha = \theta_1^{\alpha_1} \dots \theta_n^{\alpha_n}$ and $\alpha_i \in \{0, 1\}$. Note that every generator of \mathfrak{g} (locally) has the form

$$f_{a,\alpha}(\bar{x})\theta^\alpha \frac{\partial}{\partial x_a} \tag{2.1}$$

or

$$f_{b,\beta}(\bar{x})\theta^\beta \frac{\partial}{\partial \theta_b} \tag{2.2}$$

where $\sum_i \alpha_i$ is even and $\sum_j \beta_j$ is odd. With this identification, we can check that all the elements in G^i have the form of (2.1) or (2.2) where $\sum_j \alpha_j$ and $\sum_j \beta_j$ are greater than or equal to i .

Let S be a superspace which is modeled on M and \mathcal{E} . That is, S is locally isomorphic to $S(M, \mathcal{E})$. Then there is an open cover $\cup_{i \in I} U_i$ of M such that for each i ,

$$S|_{U_i} \simeq S(M, \mathcal{E})|_{U_i}.$$

Therefore, the transition maps on each $U_i \cap U_j$ define a class $\phi \in H^1(M, G)$ via the Čech cohomology construction.

We define $\phi^{(i)}$ in a similar manner. Suppose $S^{(i)}$ is isomorphic to $S(M, \mathcal{E})^{(i)}$. Then local isomorphisms between $S^{(i+1)}$ and $S(M, \mathcal{E})^{(i+1)}$ define the cohomology class $\phi^{(i)} \in H^1(M, G^{i+1})$.

From the identity $G^i/G^{i+1} \simeq T_{(-)^i} S \otimes \wedge^i \mathcal{E}^\vee$, we can think of G^i as even derivations valued in $\wedge^i \mathcal{E}^\vee$ and have the short exact sequence

$$0 \rightarrow G^{i+1} \hookrightarrow G^i \rightarrow T_{(-)^i} S \otimes \wedge^i \mathcal{E}^\vee \rightarrow 0$$

Consider the induced long exact sequence

$$\cdots \rightarrow H^1(M, G^{i+1}) \rightarrow H^1(M, G^i) \xrightarrow{\omega} H^1(M, T_{(-)^i} S \otimes \wedge^i \mathcal{E}^\vee) \rightarrow \cdots$$

We call $\omega_i := \omega(\phi^{(i-1)})$ the i th obstruction class. Note that if S is (globally) isomorphic to $S(M, \mathcal{E})$, then every $\phi^{(i-1)}$ is the image of $\phi^{(i)}$, and thus ω_i is vanishing. In conclusion, a non-vanishing obstruction class ω_2 guarantees the non-splitness of a supermanifold S . We will use this fact to show the non splitness of the Hilbert scheme in Chapter 6.

Corollary 2.2.9. *A supermanifold of odd dimension 1 is always split.*

Proof. Consider a supermanifold S modeled on M and \mathcal{E} . Then $J = \mathcal{O}_S^1$ and $J^i = 0$ for all $i \geq 2$. Therefore, $\text{Gr}_J(\mathcal{O}_S) = \mathcal{O}_M \oplus J$ and

$$S \simeq (M, \text{Gr}_J(\mathcal{O}_S)) = \mathcal{O}_M \oplus J \simeq S(M, \mathcal{E}).$$

□

Remark 2.2.10. If all obstruction classes ω_i for a supermanifold S are vanishing, then S is split.

Corollary 2.2.11. *Any differentiable supermanifold is split.*

This corollary can be shown by observing the property that for any locally free sheaf \mathcal{E} on a differentiable supermanifold S , $H^p(S, \mathcal{E}) = 0$ for all $p \geq 1$. Therefore, any supermanifold locally isomorphic to S has zero obstruction classes.

Corollary 2.2.12. *[DW] A supermanifold S with odd dimension 2 is split if and only if it is projected.*

If S is a supermanifold of dimension $(m|2)$, then the obstruction to splitting is the obstruction to projection. In fact, any supermanifold modeled on M and \mathcal{E} with odd dimension 2 is determined up to isomorphism by a cohomology class $\omega_2 \in H^1(M, T_M \otimes \wedge^2 \mathcal{E}^\vee)$. [Ma]

Chapter 3

Review of the Hilbert scheme

This chapter is devoted to reviewing the ordinary Hilbert scheme and some results about the smoothness of the Hilbert scheme of points in the plane.

3.1 Hilbert scheme

The Hilbert scheme is one of the most important examples of moduli spaces. The Hilbert scheme is the parameter space of closed subschemes of a given scheme.

Definition 3.1.1. Let \mathcal{C} be a category and let X be an object in \mathcal{C} . The *functor of points* h_X defined by X is the contravariant functor from the category \mathcal{C} to the category of sets defined as follows:

- i) For an object B in \mathcal{C} , $h_X(B) = \text{Hom}_{\mathcal{C}}(B, X)$
- ii) For a morphism $f : B \rightarrow C$ of \mathcal{C} , $h_X(f) : h_X(C) \rightarrow h_X(B)$ is defined by $\phi \mapsto \phi \circ f$

Given a contravariant functor \mathcal{F} from the category of schemes over k to the category of sets, a scheme X over k is said to *represent* the functor \mathcal{F} if there is a natural isomorphism between two functors \mathcal{F} and the functor of points h_X .

Remark 3.1.2. If a functor \mathcal{F} is representable by X , then there is a universal family $\mathcal{U} \rightarrow X$ corresponding to the identity element of $\text{Hom}(X, X)$. The universal family has the property that for any scheme S and for any family $\mathcal{Y} \rightarrow S$ in $\mathcal{F}(S)$, there is a unique map $f : S \rightarrow X$ such that \mathcal{Y} is the pullback of \mathcal{U} via f .

Definition 3.1.3. Let X be a projective scheme over a field k and let $p(n)$ be a polynomial. The Hilbert functor $\mathcal{H}_{X,p(n)}$ is the contravariant functor from the category of schemes over k to the category of sets defined by:

$$i) \mathcal{H}_{X,p(n)}(B) = \left\{ \begin{array}{c} \begin{array}{ccc} \mathcal{Z} & \longrightarrow & X \times B \\ \downarrow \pi & \swarrow & \\ B & & \end{array} \\ \left| \begin{array}{l} \mathcal{Z} \text{ is a closed subscheme of } X \times B, \\ \text{for all } b \in B \text{ the fiber } \mathcal{Z}_b \text{ has the Hilbert} \\ \text{polynomial } p(n) \text{ and } \pi \text{ is flat} \end{array} \right. \end{array} \right\}$$

ii) morphisms are defined by the pullback

Proposition 3.1.4. (*[Gr]*) *The Hilbert functor $\mathcal{H}_{X,p(n)}$ is representable by a projective scheme $\text{Hilb}^{p(n)}(X)$.*

We call $\text{Hilb}^{p(n)}(X)$ the Hilbert scheme.

3.2 Hilbert scheme of points on a plane

There is a well-known theorem about the smoothness of the Hilbert scheme of points in the plane.

Proposition 3.2.1. (*Fogarty [Fo]*) *The Hilbert scheme of n points on a smooth surface is connected and smooth of dimension $2n$.*

Let X be a smooth surface. Then, there is a natural map $\phi : \text{Hilb}^n(X) \rightarrow \text{Sym}^n(X)$. This map ϕ is an isomorphism away from the diagonal. In fact, the Hilbert scheme $\text{Hilb}^n(X)$ is a resolution of the singularity of $\text{Sym}^n(X)$ along the diagonal.

However, we can find that Hilbert schemes are singular even for naive examples.

Example 3.2.1. The Hilbert scheme $\text{Hilb}^4(\mathbb{A}^3)$ of four points on \mathbb{A}_k^3 is not smooth. Let (x, y, z) be coordinates on \mathbb{A} and let $m = (x, y, z)$ be the maximal ideal at the origin. Identify points on the Hilbert scheme with ideals in $k[x, y, z]$ with $\text{length}(I) = 4$. Then the Hilbert scheme is not smooth at $I := m^2 \in \text{Hilb}^4(\mathbb{A}^3)$. It can be checked by observing that the dimension of the tangent space $\mathcal{T}_I(\text{Hilb}^4\mathbb{A}^3)$ at I is $\dim \text{Hom}(I, k[x, y, z]/I) = 18 \neq \dim \text{Hilb}^4(\mathbb{A}^3) = 12$. The dimension of the Hilbert scheme $\text{Hilb}^4(\mathbb{A}^3)$ can be computed from the fact that the dimension of $\text{Hilb}^4(\mathbb{A}^3) - \Delta \simeq \text{Sym}^4(\mathbb{A}^3) - \Delta$ is 3×4 and the Hilbert scheme $\text{Hilb}^4(\mathbb{A}^3)$ is irreducible.

We might expect that Hilbert schemes can have bad singularities. The following is a law that supports this expectation.

Murphy's law for Hilbert schemes [Va] *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

Chapter 4

The Super Hilbert Scheme

The definition of the super Hilbert scheme is analogous to the definition of the ordinary Hilbert scheme.

Definition 4.0.1.

- I. Let S be an analytic superspace. The *super Hilbert functor* $\mathcal{H}_S^{p|q}$ is the contravariant functor from the category \mathfrak{S} of analytic superspaces to the category of sets defined as below:

For $X, Y \in \mathfrak{S}$ and a morphism $f : Y \rightarrow X$

$$\mathcal{H}_S^{p|q}(X) = \left\{ \begin{array}{c} \mathcal{Z} \hookrightarrow S \times X \\ \downarrow \pi \quad \swarrow \\ X \end{array} \left| \begin{array}{l} \mathcal{Z} \text{ is a closed subspace of } S \times X \text{ such that} \\ \pi_* \mathcal{O}_{\mathcal{Z}} \text{ is a locally free } \mathcal{O}_X\text{-module of rank } (p|q) \\ \text{and } \mathcal{Z} \text{ is finite over } X. \end{array} \right. \right\}$$

The morphism is defined by the pullback

$$\mathcal{H}_S^{p|q}(f) = f^* : \mathcal{H}_S^{p|q}(X) \rightarrow \mathcal{H}_S^{p|q}(Y)$$

Note that the condition that $\pi_* \mathcal{O}_{\mathcal{Z}}$ is locally free guarantees the flatness of the map π .

- II. If the super Hilbert functor $\mathcal{H}_S^{p|q}$ is representable by the analytic superspace $\text{Hilb}^{p|q}(S)$, We call this the *super Hilbert scheme*.

Let's look at the one of the simplest examples of super Hilbert schemes.

Example 4.0.1. Let $\mathcal{U} \subset \mathbb{C}_{x|\theta}^{1|1} \times \mathbb{C}_{b|\beta}^{1|1}$ be the closed subscheme defined by the ideal $(x + b + \beta\theta)$ where $(x|\theta)$ are coordinates of the first component $\mathbb{C}^{1|1}$ and $(b|\beta)$ are coordinates of the second component.

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \mathbb{C}_{x|\theta}^{1|1} \times \mathbb{C}_{b|\beta}^{1|1} \\ & \searrow \pi & \downarrow \\ & & \mathbb{C}_{b|\beta}^{1|1} \end{array}$$

Then we claim that $\mathbb{C}^{1|1}_{b|\beta}$ represents the functor $\mathcal{H}_{\mathbb{C}^{1|1}}^{1|1}$ and \mathcal{U} is the universal family.

Consider any analytic superspace S and a family in $\mathcal{H}_{\mathbb{C}^{1|1}}^{1|1}(S)$.

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathbb{C}^{1|1} \times S \\ & \searrow p & \downarrow \\ & & S \end{array}$$

The pushforward $p_*\mathcal{O}_{\mathcal{Y}}$ is a free \mathcal{O}_S -module of dimension $(1|1)$. Since 1 and x are linearly dependent, there are $a, b \in \Gamma(S, \mathcal{O}_S)$ such that $ax + b = 0$. If $a = 0$ at any $s \in S$, then $p_*\mathcal{O}_{\mathcal{Y}}$ has even dimension 0 at s . Hence, a is invertible and we have $x + a^{-1}b \cdot 1 = 0$ and $(p_*\mathcal{O}_{\mathcal{Y}})^0$ is generated by 1. Similarly, we can see that $(p_*\mathcal{O}_{\mathcal{Y}})^1$ is generated by θ . Then we can find a relation $x + a + \alpha\theta = 0$ for $a \in \Gamma(S, \mathcal{O}_S)^0$ and $\alpha \in \Gamma(S, \mathcal{O}_S)^1$. Define a map from S to $\mathbb{C}^{1|1}_{b|\beta}$ via $b \mapsto a$ and $\beta \mapsto \alpha$. Then p is the pullback of the universal family.

4.1 Flattening Stratifications

Flattening stratifications provide a key step for proving the existence of the ordinary Hilbert scheme. We demonstrate a super-version of the flattening stratification that is needed in our situation.

Proposition 4.1.1. (*Flattening Stratification*) *Let X and Y be analytic superspaces. Let \mathcal{F} be a coherent sheaf of modules on $Y \times X$ such that the restriction of the support of \mathcal{F} to each fiber of the projection $Y \times X \rightarrow X$ is zero dimensional. Then for each $(p, q) \in \mathbb{N} \times \mathbb{N}$ we have a locally closed subspace $X_{(p,q)} \subset X$ with the following properties:*

i) $X = \dot{\cup}_{(p,q)} X_{(p,q)}$

ii) $\pi_*\mathcal{F}|_{(X_{(p,q)})}$ is locally free of rank $(p|q)$

iii) for any $f : C \rightarrow X$, $f^*\mathcal{F}$ is flat if and only if f factors through $C \rightarrow X_{(p,q)} \hookrightarrow X$ for some $(p, q) \in \mathbb{N} \times \mathbb{N}$

Proof. Pick $x \in X_b$. Then there are $p, q \in \mathbb{N}$ such that $\dim_{k(x)} \mathcal{F}_x \times_{\mathcal{O}_{X,x}} k(x) = (p|q)$. Using Corollary 2.1.3, find a neighborhood U of x such that generators of \mathcal{F}_x also generate \mathcal{F} on U . Then $\mathcal{F}|_U$ has p even and q odd generators as an $\mathcal{O}_X|_U$ -module and we have the surjection

$$\mathcal{O}_U^p \oplus \Pi\mathcal{O}_U^q \xrightarrow{\zeta} \mathcal{F}|_U \rightarrow 0$$

Since \mathcal{F} is coherent, $\ker \zeta$ is also coherent and hence it is finitely generated. By shrinking U , if necessary,

we have an exact sequence

$$\mathcal{O}_U^s \oplus \Pi \mathcal{O}_U^t \xrightarrow{\sigma} \mathcal{O}_U^p \oplus \Pi \mathcal{O}_U^q \xrightarrow{\zeta} \mathcal{F}|_U \rightarrow 0$$

where the image of σ is the kernel of ζ .

Consider a map $f : C \rightarrow X|_U$ and the induced exact sequence

$$\mathcal{O}_C^s \oplus \Pi \mathcal{O}_C^t \xrightarrow{f^* \sigma} \mathcal{O}_C^p \oplus \Pi \mathcal{O}_C^q \xrightarrow{f^* \zeta} f^*(\mathcal{F}|_U) \rightarrow 0$$

Observe that

$$f^* \mathcal{F}|_U \text{ is free of rank } (p|q) \Leftrightarrow f^* \sigma = 0 \Leftrightarrow f^* \sigma_{ij} = 0 \text{ for all } i, j \Leftrightarrow f \text{ factors through } X_\sigma$$

with the matrix representation $\sigma = (\sigma_{ij})_{i,j} \in M(s|t; p|q; \mathcal{O}_C)$ and $X_\sigma \subset X$ closed subspace defined by the ideal $I = (\sigma_{ij})_{i,j}$.

Therefore, X_σ represents the functor \mathcal{G}_U defined by

$$\mathcal{G}_U(f : C \rightarrow X|_U) = \{f^* \mathcal{F} \rightarrow C \text{ is flat of rank } (p|q)\}$$

We can glue all X_σ 's with fixed $(p|q)$ by the universality and $X_{(p,q)} := \cup_\sigma X_\sigma$ satisfies the required properties above. □

A flattening stratification plays a pivotal role in constructing the super Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^{1|1})$ in the next section.

The structure of Hilbert schemes get complicated rapidly when the base space or the Hilbert polynomial becomes complicated. In this section, we stick with the super Hilbert scheme of dimension $1|1$ supermanifolds with a constant Hilbert polynomial.

4.1.1 The super Hilbert scheme of $\mathbb{C}^{1|1}$

Let's fix coordinates $x|\theta$ on $\mathbb{C}^{1|1}$.

To study the Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^{1|1})$, we have to look at families in $\mathcal{H}_{\mathbb{C}^{1|1}}^{p|q}(X)$.

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & \mathbb{C}^{1|1} \times X \\ & \searrow \pi & \downarrow \\ & & X \end{array}$$

The definition of the Hilbert functor gives us a condition that the pushforward $\pi_*\mathcal{O}_{\mathcal{Z}}$ is a locally free sheaf of \mathcal{O}_X -modules. It turns out that this locally free sheaf $\pi_*\mathcal{O}_{\mathcal{Z}}$ is actually free.

Lemma 4.1.2. *Let $\mathcal{Y} \subset \mathbb{C}^{1|1}$ be a closed subspace with $\dim_{\mathbb{C}} \Gamma(\mathbb{C}^{1|1}, \mathcal{O}_{\mathcal{Y}}) = p|q$. Then $1, x, \dots, x^{p-1}$ and $\theta, x\theta, \dots, x^{q-1}\theta$ form a basis of the vector space $\Gamma(\mathbb{C}^{1|1}, \mathcal{O}_{\mathcal{Y}})$.*

Proof. Let $I \subset \mathbb{C}[x|\theta]$ be the ideal defined by \mathcal{Y} . Then $\mathbb{C}[x|\theta]/I \simeq \Gamma(\mathbb{C}^{1|1}, \mathcal{O}_{\mathcal{Y}})$. By the given condition, we have p even generators and q odd generators for $\mathbb{C}[x|\theta]/I$ as a \mathbb{C} -vector space.

Observe that every element in the even part of $\mathbb{C}[x|\theta]/I$ has the form $\sum_{i \geq 0} a_i x^i$ and an element of the odd part of $\mathbb{C}[x|\theta]/I$ has the form $\sum_{j \geq 0} b_j x^j \theta$ for $a_i, b_j \in \mathbb{C}$.

Since $\dim_{\mathbb{C}} (\mathbb{C}[x|\theta]/I)^0 = p$, $1, x, \dots, x^p \in (\mathbb{C}[x|\theta]/I)^0$ are linearly dependent. Therefore, we can find c_i 's in \mathbb{C} not all zero such that $\sum_{i=0}^p c_i x^i = 0$. Let k be the largest number such that $c_k \neq 0$, $k \leq p$. For any $f \in (\mathbb{C}[x|\theta]/I)^0$, by applying long division by $\sum_{i=0}^p c_i x^i = 0$ if needed, we can assume that degree of f is less than k . Then f is a linear combination of $1, x, \dots, x^{k-1}$. Since the dimension of $(\mathbb{C}[x|\theta]/I)^0$ must be less than k , we have $p = k$. Therefore, $1, x, \dots, x^{p-1}$ generate $(\mathbb{C}[x|\theta]/I)^0$.

Similarly, we can show that $\theta, x\theta, \dots, x^{q-1}\theta$ form an odd basis and hence $1, x, \dots, x^{p-1}, \theta, x\theta, \dots, x^{q-1}\theta$ is a basis. □

Proposition 4.1.3. *Let $\mathcal{Z} \xrightarrow{\pi} Y$ be an element in $\mathcal{H}_{\mathbb{C}^{1|1}}^{p|q}(Y)$. Then the sheaf $\pi_*\mathcal{O}_{\mathcal{Z}}$ is free.*

Proof. Let $R = \mathbb{C}[x|\theta]$ and let $I \subset R$ be an ideal such that $\dim_{\mathbb{C}} R/I = p|q$. Lemma 4.1.2 says that $1, x, \dots, x^{p-1}, \theta, x\theta, \dots, x^{q-1}\theta$ generate R/I as a \mathbb{C} -vector space.

Pick $y \in Y$ and let \mathcal{I} be the ideal sheaf of \mathcal{Z} . Then $I = \mathcal{I}_y/m_y \mathcal{O}_{Y,y}$ can be viewed as an ideal in R where m_y is the maximal ideal of the local ring $\mathcal{O}_{Y,y}$. Then $\frac{(\pi_*\mathcal{O}_{\mathcal{Z}})_y}{m_y(\pi_*\mathcal{O}_{\mathcal{Z}})_y}$ is isomorphic to R/I and has rank $p|q$ as

a \mathbb{C} -vector space. Therefore, $\frac{(\pi_*\mathcal{O}_{\mathcal{Z}})_y}{m_y(\pi_*\mathcal{O}_{\mathcal{Z}})_y}$ is generated by $1, x, \dots, x^{p-1}, \theta, x\theta, \dots, x^{q-1}\theta$. By corollary 2.1.3, there is an open neighborhood U of y such that $\pi_*\mathcal{O}_{\mathcal{Z}}|_U$ is generated by $1, x, \dots, x^{p-1}, \theta, x\theta, \dots, x^{q-1}\theta$ as an $\mathcal{O}_Y|_U$ -module.

Hence, $\pi_*\mathcal{O}_{\mathcal{Z}}$ is a \mathcal{O}_Y -module with free generators $1, x, \dots, x^{p-1}, \theta, x\theta, \dots, x^{q-1}\theta$. □

Let $\mathcal{Y} \subset \mathbb{C}^{1|1}_{x|\theta} \times \mathbb{C}^{p+q|p+q}_{a,b|\alpha,\beta}$ be the closed subspace defined by the ideal

$$\tilde{I} = \left(x^p + \sum_{i=0}^{p-1} a_i x^i + \sum_{i=0}^{q-1} \alpha_i x^i \theta, x^q \theta + \sum_{i=0}^{q-1} b_i x^i \theta + \sum_{i=0}^{p-1} \beta_i x^i \right)$$

Consider the diagram with the projection π

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathbb{C}^{1|1} \times \mathbb{C}^{p+q|p+q} \\ \pi \downarrow & & \swarrow \\ \mathbb{C}^{p+q|p+q} & & \end{array}$$

Lemma 4.1.4. $\mathbb{C}_{(p,q)}^{(p+q|p+q)}$ is isomorphic to $\mathbb{C}^{p|p}$.

Proof. Set

$$f := x^p + \sum_{i=0}^{p-1} a_i x^i + \sum_{i=0}^{q-1} \alpha_i x^i \theta, \quad g := x^q \theta + \sum_{i=0}^{q-1} b_i x^i \theta + \sum_{i=0}^{p-1} \beta_i x^i$$

Apply long division by $x^q + \sum_{i=0}^{q-1} b_i x^i$ to f and g

$$\begin{aligned} f &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} c'_i x^i) + \sum_{i=0}^{q-1} d'_i x^i + \sum_{i=0}^{q-1} \gamma_i x^i \theta \\ g &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(\theta + \sum_{i=0}^{p-q-1} \delta_i x^i) + \sum_{i=0}^{q-1} \epsilon_i x^i \end{aligned}$$

Change coordinates on $\mathbb{C}_{a,b,c,|\alpha,\beta,\gamma}^{p+q|p+q}$ so that we have the following form

$$\begin{aligned} f &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) + \sum_{i=0}^{q-1} c_i x^i + \sum_{i=0}^{q-1} \beta_i x^i (\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) \\ g &= (x^q + \sum_{i=0}^{q-1} b_i x^i)(\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) + \sum_{i=0}^{q-1} \gamma_i x^i \end{aligned}$$

Denote $\sum_{i=0}^{p-q-1} a_i x^i, \sum_{i=0}^{q-1} b_i x^i, \dots$ by a, b, \dots for simplicity.

According to the proof in Lemma 4.1.1, there is an open subset $U \subset \mathbb{C}^{p+q|p+q}$ and an exact sequence

$$\mathcal{O}_U^s \oplus \Pi \mathcal{O}_U^t \xrightarrow{\sigma} \mathcal{O}_U^p \oplus \Pi \mathcal{O}_U^q \xrightarrow{\phi} \pi_* \mathcal{O}_{\mathcal{Y}}|_U \rightarrow 0 \quad (4.1)$$

such that $\mathbb{C}_{(p,q)}^{p+q|p+q}$ is generated by $I = (\sigma_{ij})$. Note that the map is defined as $(A_i | \mathcal{A}_j)_{i,j} \mapsto \sum_{i=0}^p A_i x^i + \sum_{j=0}^q \mathcal{A}_j x^j \theta$.

Let's compute some elements in the kernel of ϕ .

$$f(\theta + \alpha) - g(x^{p-q} + a) \tag{4.2}$$

$$= c(\theta + \alpha) - \gamma(x^{p-q} + a) \tag{4.3}$$

$$= \left(\sum_{i=0}^{q-1} c_i x^i \right) \theta + \left(\sum_{i=0}^{q-1} c_i x^i \right) \left(\sum_{i=0}^{p-q-1} \alpha_i x^i \right) - \sum_{i=0}^{q-1} \gamma_i x^i (x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) \tag{4.4}$$

$$g(\theta + \alpha) \tag{4.5}$$

$$= \gamma(\theta + \alpha) \tag{4.6}$$

$$= \left(\sum_{i=0}^{q-1} \gamma_i x^i \right) \theta + \left(\sum_{i=0}^{q-1} \gamma_i x^i \right) \left(\sum_{i=0}^{p-q-1} \alpha_i x^i \right) \tag{4.7}$$

Since \mathcal{Y} is defined by the ideal generated by f and g , (4.4) and (4.7) give two elements in the kernel of ϕ .

$$h := ((c_0 \alpha_0 - a_0 \gamma_0, \dots, \gamma_{q-1}, \overbrace{0, \dots, 0}^{p-q}), (c_0, \dots, c_{q-1}))$$

$$k := ((\gamma_0 \alpha_0, \dots, \gamma_{q-1} \alpha_{p-q-1}, \overbrace{0, \dots, 0}^q), (\gamma_0, \dots, \gamma_{q-1}))$$

Therefore, $\mathbb{C}_{(p,q)}^{p+q|p+q}$ is contained in the closed subspace $\mathcal{H} := Z\left(\{c_i, \gamma_i\}_{i=0}^{q-1}\right) \cap U \subset U$. By restricting (4.1) to \mathcal{H} we get

$$\mathcal{O}_{\mathcal{H}}^{s'} \oplus \Pi \mathcal{O}_{\mathcal{H}}^{t'} \xrightarrow{\sigma_{\mathcal{H}}} \mathcal{O}_{\mathcal{H}}^p \oplus \Pi \mathcal{O}_{\mathcal{H}}^q \xrightarrow{\phi_{\mathcal{H}}} \pi_* \mathcal{O}_{\mathcal{Y}}|_{\mathcal{H}} \rightarrow 0$$

Claim: $\phi_{\mathcal{H}}$ is an isomorphism.

Let $(A_1, \dots, A_p | \mathcal{A}_1, \dots, \mathcal{A}_q)$ be an element in a section of $\ker \phi_{\mathcal{H}}$. We can find C_i 's and \mathcal{D}_j 's in $\Gamma(\mathbb{C}^{1|1} \times U, \mathcal{O}_{\mathbb{C}^{1|1} \times \mathbb{C}^{p+q|p+q}})$ such that

$$\begin{aligned} & \sum_{i=0}^{p-1} A_i x^i + \theta \sum_{i=0}^{q-1} \mathcal{A}_i x^i \\ &= C f + \mathcal{D} g \\ &= C(x^q + b)(x^{p-q} + a) + C\beta\alpha + C\beta\theta + \mathcal{D}\theta(x^q + b) + \mathcal{D}\alpha(x^q + b) \end{aligned}$$

I.e.,

$$\begin{aligned} \sum_{i=0}^{p-1} A_i x^i = C(x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) + C(\sum_{i=0}^{q-1} \beta_i x^i) & \left(\sum_{i=0}^{p-q-1} \alpha_i x^i \right) \\ & + \mathcal{D} \left(\sum_{i=0}^{p-q-1} \alpha_i x^i \right) \left(x^q + \sum_{i=0}^{q-1} b_i x^i \right) \end{aligned} \quad (4.8)$$

and

$$\sum_{i=0}^{q-1} \mathcal{A}_i x^i = C \left(\sum_{i=0}^{q-1} \beta_i x^i \right) + \mathcal{D} \left(x^q + \sum_{i=0}^{q-1} b_i x^i \right) \quad (4.9)$$

Comparing the highest degree terms in (4.8), we see that $C = 0$. Similarly, from (4.9) we can figure out that $\mathcal{D} = 0$. Hence, A_i and \mathcal{A}_j vanish for all i and j .

Therefore, $\phi_{\mathcal{H}}$ is an isomorphism and thus $\mathbb{C}_{(p,q)}^{p+q|p+q} = \mathcal{H}$ where \mathcal{H} is defined by the ideal

$$(c_0, \dots, c_{q-1}, \gamma_0, \dots, \gamma_{q-1})$$

Furthermore, $\mathbb{C}_{(p,q)}^{p+q|p+q}$ is isomorphic to $\mathbb{C}^{p|p}$. □

Let $\tilde{\mathcal{Y}}$ be the pullback of \mathcal{Y} to $\mathbb{C}_{(p,q)}^{p+q|p+q}$. With Lemma 4.1.4 above, we can prove the following theorem.

Theorem 4.1.5. *The super Hilbert scheme functor $\mathcal{H}_{\mathbb{C}^1|1}^{p|q}$ is representable by $\mathbb{C}^{p|p}$.*

Proof. Consider a flat family

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathbb{C}^1|1 \times X \\ & \searrow p & \downarrow \\ & & X \end{array}$$

in $\mathcal{H}_{\mathbb{C}^1|1}^{p|q}(X)$.

By Proposition 4.1.3 there are $c_i, d_i \in (H^0(X, \mathcal{O}_X))^0$, $\gamma_i, \delta_i \in (H^0(X, \mathcal{O}_X))^1$ such that \mathcal{Z} is defined by the ideal

$$\left(x^p + \sum_{i=0}^{p-1} c_i x^i + \sum_{i=0}^{q-1} \gamma_i x^i \theta, x^q \theta + \sum_{i=0}^{q-1} d_i x^i \theta + \sum_{i=0}^{p-1} \delta_i x^i \right)$$

Then there is a unique map $X \xrightarrow{\varphi} \mathbb{C}^{p+q|p+q}$ such that the pullback gives a flat family in $\mathcal{H}_{\mathbb{C}^1|1}^{p|q}(\mathbb{C}^{p+q|p+q})$. Since p is flat, φ factors through $\mathbb{C}_{(p,q)}^{p+q|p+q}$, i.e, there is a unique map from X to $\mathbb{C}_{(p,q)}^{p+q|p+q}$ such that $\mathcal{Z} \xrightarrow{p} X$ is the pullback of $\tilde{\mathcal{Y}} \xrightarrow{\pi} \mathbb{C}_{(p,q)}^{p+q|p+q}$. □

For the rest section of the thesis, we fix coordinates

$$(a_0, \dots, a_{p-q-1}, b_0, \dots, b_{q-1} \mid \alpha_0, \dots, \alpha_{p-q-1}, \beta_0, \dots, \beta_{q-1}) \quad (4.10)$$

on the super Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^{1|1}) \simeq \mathbb{C}^{p|p}$, so that the ideal of the universal family is

$$\left((x^q + \sum_{i=0}^{q-1} b_i x^i)(x^{p-q} + \sum_{i=0}^{p-q-1} a_i x^i) + \sum_{i=0}^{q-1} \beta_i x^i (\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i), \right. \\ \left. (x^q + \sum_{i=0}^{q-1} b_i x^i)(\theta + \sum_{i=0}^{p-q-1} \alpha_i x^i) \right)$$

4.1.2 The super Hilbert scheme of a dimension $1|1$ supermanifold

By applying Theorem 4.1.5 to appropriate open subfunctors of $\mathcal{H}_S^{p|q}$ which are representable, we conclude that Hilbert functor of dimension $1|1$ supermanifold is representable.

Theorem 4.1.6. *Let S be a supermanifold of dimension $1|1$. The super Hilbert functor $\mathcal{H}_S^{p|q}$ is representable by a smooth superspace $\text{Hilb}^{p|q}(S)$ which has dimension $p|p$.*

Proof. Let $U = \dot{\bigcup}_i U_i \subset S$ be a finite disjoint union of open subspaces of S such that U_i is isomorphic to some nonempty open subspace of $\mathbb{C}^{1|1}$. Let $\mathcal{H}_{S,U}^{p|q}$ be the open subfunctor of $\mathcal{H}_S^{p|q}$ defined as $\prod_{\sum_{i=1}^p p_i=p} \prod_{\sum_{i=1}^q q_i=q} \mathcal{H}_{U_i}^{p_i|q_i}$

Then the Hilbert functor $\mathcal{H}_S^{p|q}$ is the union of open subfunctors $\bigcup_U \mathcal{H}_{S,U}^{p|q}$, and each $\mathcal{H}_{S,U}^{p|q}$ is representable by a smooth superspace of dimension $(p|p)$ as an application of theorem 4.1.5.

To be specific, let's consider any family $\mathcal{Z} \subset S \times X$ in $\mathcal{H}_S^{p|q}(X)$. For each $x \in X$, we can find a neighborhood V of x such that the support of $\mathcal{Z}|_{\pi^{-1}(V)}$ is contained in $U \times X$ for some $U = \dot{\bigcup}_i U_i \subset S$. By the argument above, there is a map from V to $\text{Hilb}^{p|q}(U)$ such that $\mathcal{Z}|_{\pi^{-1}(V)}$ is the pullback of the universal family. Then the universality of the Hilbert scheme guarantees that we can glue them for all V .

Let $X = \cup_\alpha V_\alpha$ be an open covering of X constructed as above. To check $\text{Hilb}^{p|q}(X)$ is Hausdorff, pick $Z_1, Z_2 \in \text{Hilb}^{p|q}(X)$ such that $Z_1 \neq Z_2$. Then there exists V_1 and V_2 such that $Z_i \in \text{Hilb}^{p|q}(V_i)$. If $\text{Supp}(Z_1) = \text{Supp}(Z_2)$, then $Z_2 \in \text{Hilb}^{p|q}(V_1)$ which is Hausdorff by the construction. If $\text{Supp}(Z_1) \neq \text{Supp}(Z_2)$, then shrink V_i 's enough so that $\text{Hilb}^{p|q}(V_1)$ and $\text{Hilb}^{p|q}(V_2)$ are disjoint.

Therefore, the Hilbert functor $\mathcal{H}_S^{p|q}$ is representable by a dimension $(p|p)$ smooth superspace. □

Chapter 5

(Non)Smoothness of the Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^1|2)$

Lemma 5.0.1. *Let $R = \mathbb{C}[x_1, \dots, x_m | \theta_1, \dots, \theta_n]$ be a polynomial ring. Let $I \subset R$ be an ideal such that $\dim_{\mathbb{C}} R/I = (p|q)$. Then R/I is generated by monomials as a \mathbb{Z}_2 -graded \mathbb{C} -vector space.*

Proof. If $p = 0$, then $R/I = \mathbb{C}$ and nothing to prove.

Consider $p \geq 1$. First observe that $1 \neq 0$ in R/I .

Let $N = N_0 \cup N_1$ be a set of even and odd monomials of R such that 1) elements in N are linearly independent in R/I and 2) N is maximal.

Pick any $f = \sum_{ij} a_{ij} x^{\alpha_i} \theta^{\beta_j} \in R$. Then by the maximality of N , for each term $f_{ij} := a_{ij} x^{\alpha_i} \theta^{\beta_j}$, the image $\overline{f_{ij}}$ of f_{ij} in R/I must be a linear combination of elements in N . Hence, N generates R/I and $(|N_0| + |N_1|) = (p|q)$.

□

Then now we have a generalized version of Proposition 4.1.3.

Lemma 5.0.2. *Let X be an analytic supermanifold. Consider a closed subspace $\mathcal{Z} \subset \mathbb{C}^{m|n}_{x_1, \dots, x_m | \theta_1, \dots, \theta_n} \times X$ and a flat family*

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & \mathbb{C}^{m|n}_{x_1, \dots, x_m | \theta_1, \dots, \theta_n} \times X \\ & \searrow \pi & \downarrow \\ & & X \end{array}$$

such that the pushforward $\pi_ \mathcal{O}_{\mathcal{Z}}$ is a locally free \mathcal{O}_B -module of rank $(p|q)$. Then $\pi_* \mathcal{O}_{\mathcal{Z}}$ is locally freely generated by some monomials in $R = \mathbb{C}[x_1, \dots, x_m | \theta_1, \dots, \theta_n]$*

Proof. Pick a point x from the bosonic part of X . Then we can identify $\mathcal{I}_x/m_x \mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}/m_x \mathcal{O}_{X,x}$ with some ideal $I \subset R$ where $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^{m|n} \times X}$ is the sheaf of ideals defining \mathcal{Z} and $m_x \subset \mathcal{O}_{X,x}$ is the maximal ideal

of the local ring. Then $\frac{(\pi_* \mathcal{O}_{\mathcal{Z}})_x}{m_x (\pi_* \mathcal{O}_{\mathcal{Z}})_x}$ is isomorphic to R/I . By Lemma 5.0.1, we can find monomials f_i 's and η_j 's from R^0 and R^1 , respectively, so that R/I is generated by $\{f_1, \dots, f_p | \eta_1, \dots, \eta_q\}$ as a \mathbb{C} -vector space.

By Lemma 2.1.5, $(\pi_* \mathcal{O}_{\mathcal{Z}})_x$ is generated by f_i 's and η_j 's as a $\mathcal{O}_{X,x}$ -module. Then by Nakayama's lemma

(Lemma 2.1.2), we can find some neighborhood $U \subset X$ of x such that $\pi_* \mathcal{O}_{\mathcal{Z}}|_U$ is generated by f_i 's and η_j 's as an $\mathcal{O}_X|_U$ -module. Therefore, it is locally free. \square

5.1 Smoothness of $\text{Hilb}^{p|0}(\mathbb{C}^{1|n})$

Example 5.1.1. Let $(a, b, c | \alpha, \beta, \gamma, \delta, \epsilon, \eta)$ be coordinates of $\mathbb{C}^{3|6}$ and let $(x | \theta_1, \theta_2)$ be coordinates on $\mathbb{C}^{1|2}$. Define a closed subspace $\mathcal{Z} \subset \mathbb{C}^{3|6} \times \mathbb{C}^{1|2}$ corresponding to the ideal

$$I = (x^3 + ax^2 + bx + c, \theta_1 + \alpha x^2 + \beta x + \gamma, \theta_2 + \delta x^2 + \epsilon x + \eta)$$

Consider the surjection

$$\begin{aligned} 1 \cdot \mathcal{O}_{\mathbb{C}^{3|6}} \oplus x \cdot \mathcal{O}_{\mathbb{C}^{3|6}} \oplus x^2 \cdot \mathcal{O}_{\mathbb{C}^{3|6}} \oplus \xrightarrow{\phi} \pi_* \mathcal{O}_{\mathcal{Z}} \rightarrow 0 \\ 1 \mapsto 1 \\ x \mapsto x \\ x^2 \mapsto x^2 \end{aligned}$$

For any local sections A, B, C of $\mathcal{O}_{\mathbb{C}^{3|6}}$, $Ax^2 + Bx + C = 0$ in $\pi_* \mathcal{O}_{\mathcal{Z}}$ implies $A = B = C = 0$. According to the flattening stratification, $\mathbb{C}_{(3,0)}^{3|6}$ is defined by the zeros of the kernel and hence we get $\mathbb{C}_{(3,0)}^{3|6} = \mathbb{C}^{3|6}$. By checking the universality, we conclude that $\text{Hilb}^{3|0}(\mathbb{C}^{1|2}) \simeq \mathbb{C}^{3|6}$.

We can apply the same technique as above to find the Hilbert scheme $\text{Hilb}^{p|0}(\mathbb{C}^{1|n})$

Lemma 5.1.1. $\text{Hilb}^{p|0}(\mathbb{C}^{1|n}) \simeq \mathbb{C}^{p|np}$

Proof. Fix coordinates $(a_0, \dots, a_{p-1} | \alpha_{10}, \alpha_{11}, \dots, \alpha_{1(p-1)}, \alpha_{20}, \dots, \alpha_{n(p-1)})$ on $\mathbb{C}^{p|np}$ and $(x | \theta_1, \theta_2)$ on $\mathbb{C}^{1|2}$. Let $\mathcal{Z} \subset \mathbb{C}^{p|np} \times \mathbb{C}^{1|2}$ be the closed subspace corresponding to the ideal

$$\left(x^p + \sum_{i=0}^{p-1} a_i x^i, \theta_1 + \sum_{i=0}^{p-1} \alpha_{1i} x^i, \dots, \theta_n + \sum_{i=0}^{p-1} \alpha_{ni} x^i \right)$$

Then we have an isomorphism

$$\bigoplus_{i=0}^{p-1} x^i \cdot \mathcal{O}_{\mathbb{C}^{p|np}} \rightarrow \pi_* \mathcal{O}_{\mathcal{Z}}.$$

After checking the universality, we have

$$\text{Hilb}^{p|0}(\mathbb{C}^{1|n}) = \mathbb{C}_{(p,0)}^{p|np} = \mathbb{C}^{p|np}$$

Therefore, the result follows. □

As a corollary we show the smoothness of the Hilbert scheme $\text{Hilb}^{p|0}(\mathbb{C}^{1|n})$.

5.2 Smoothness of $\text{Hilb}^{p|1}(\mathbb{C}^{1|n})$

Proposition 5.2.1. *The Hilbert scheme $\text{Hilb}^{p|1}(\mathbb{C}^{1|n})$ is smooth for all p and n .*

Proof. Let $(x|\theta_1, \dots, \theta_n)$ be coordinates on $\mathbb{C}^{1|n}$. Consider a flat family

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathbb{C}^{1|n} \times X \\ & \searrow \pi & \downarrow \\ & & X \end{array}$$

such that $\pi_*\mathcal{O}_{\mathcal{Z}}$ is a locally free \mathcal{O}_X -module of rank $(p|1)$.

Let's observe the following:

- i) According to the proof of Lemma 5.0.2, $\pi_*\mathcal{O}_{\mathcal{Z}}$ is locally freely generated by some monomials.
- ii) For any ideal $I \subset \mathbb{C}[x|\theta_1, \dots, \theta_n]$, if the \mathbb{Z}_2 -graded dimension of $\mathbb{C}[x|\theta_1, \dots, \theta_n]/I$ as a \mathbb{C} -vector space is $(p|1)$ then $\mathbb{C}[x|\theta_1, \dots, \theta_n]/I$ is generated by $1, x, \dots, x^{p-1}$ and θ_i for some i (i does not need to be determined uniquely).

By combining these facts, we can find an open cover $\cup_i U_i$ of X such that $\pi_*\mathcal{O}_{\mathcal{Z}}|_{U_i}$ is freely generated by $1, x, \dots, x^{p-1}$ and θ_i (U_i possibly be empty).

Consider the natural surjection

$$1 \cdot \mathcal{O}_X \oplus x \cdot \mathcal{O}_X \oplus \dots \oplus x^{p-1} \cdot \mathcal{O}_X \oplus \theta_1 \cdot \mathcal{O}_X \oplus \dots \oplus \theta_n \cdot \mathcal{O}_X \xrightarrow{\phi} \pi_*\mathcal{O}_{\mathcal{Z}} \rightarrow 0 \quad (5.1)$$

defined by $x^i \mapsto x^i$ and $\theta_j \mapsto \theta_j$.

Then the quotient (5.1) can be viewed as an element in $\mathcal{G}r(p|1; p|n)(X)$. By the universality of the Grassmannian functor, we get the maps

$$\tau : \mathcal{H}_{\mathbb{C}^{1|n}}^{p|1}(X) \rightarrow \mathcal{G}r(p|1; p|n)(X) \quad (5.2)$$

which determine a map $\mathcal{H}_{\mathbb{C}^{1|n}}^{p|q} \rightarrow \mathcal{G}r(p|1; p|n)$.

For each i , the image of the open subset U_i is $\tau(U_i) = U_I \subset \mathbb{G}r(p | 1; p | n)$, where

$$U_I = \mathbb{C}_X^{(n-1)|p(n-1)} \subset \mathbb{G}r(p | 1; p | n)(X)$$

is the open subset of the Grassmannian corresponding to the indexing set

$$I = I_0 \times I_1 = \{1, \dots, p\} \times \{p+i\} \subset \{1, \dots, p\} \times \{p+1, \dots, p+n\}.$$

Since $1, x, \dots, x^{p-1}$ and θ_i are generators, we have $a_0, \dots, a_p \in \Gamma(X, \mathcal{O}_X)^0$ and $\alpha_0, \dots, \alpha_p \in \Gamma(X, \mathcal{O}_X)^1$ such that

$$\begin{aligned} x^p + \sum_{i=0}^{p-1} a_i x^i + \alpha_p \theta_i &= 0 \\ x\theta_i + a_p \theta_i + \sum_{i=0}^{p-1} \alpha_i x^i &= 0 \end{aligned}$$

Let $(b_0, \dots, b_p | \beta_0, \dots, \beta_p)$ be coordinates on $\mathbb{C}^{p+1|p+1}$. Define the map $X \xrightarrow{r} \mathbb{C}^{p+1|p+1}$ via $b_j \mapsto a_j$ and $\beta_j \mapsto \alpha_j$. Since π is flat, the map r factors through $\mathbb{C}_{(p|1)}^{p+1|p+1} = \mathbb{C}^{p|p} = \text{Hilb}^{p|1}(\mathbb{C}^{1|1})$.

By combining two maps τ and r , we get the map $U_i \xrightarrow{\tau \times r} U_I \times \text{Hilb}^{p|1}(\mathbb{C}^{1|1})$ and \mathcal{Z} is obtained by the pullback of the universal family of the Grassmannian and the Hilbert scheme. Therefore, $\text{Hilb}^{p|1}(\mathbb{C}^{1|n})|_{U_i}$ is isomorphic to $U_I \times \text{Hilb}^{p|1}(\mathbb{C}^{1|1})$ and $\text{Hilb}^{p|1}(\mathbb{C}^{1|m})$ is smooth. □

Note that the dimension of $\text{Hilb}^{p|1}(\mathbb{C}^{1|m})$ is $(p|p) + (n-1|p(n-1)) = (p+n-1|pn)$.

5.3 Smoothness of $\text{Hilb}^{1|2}(\mathbb{C}^{1|2})$

Let $(x | \theta_1, \theta_2)$ be coordinates on $\mathbb{C}^{1|2}$ and let $R := \mathbb{C}[x | \theta_1, \theta_2]$. Consider a family

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & \mathbb{C}^{1|2} \times X \\ & \searrow \pi & \downarrow \\ & & X \end{array}$$

such that $\pi_* \mathcal{O}_{\mathcal{Z}}$ is locally free of rank $1|2$.

Observe that, for any ideal $I \subset R = \mathbb{C}[x | \theta_1, \theta_2]$ such that $\dim_{\mathbb{C}} R/I = (1|2)$, R/I is generated by $1, \theta_1$ and θ_2 . Therefore, $\pi_* \mathcal{O}_{\mathcal{Z}}$ is free \mathcal{O}_X -module generated by $1, \theta_1, \theta_2$.

We can embed $\text{Hilb}^{1|2}(\mathbb{C}^{1|2})$ in $\mathbb{C}^{2|4}$ as

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad} & \mathbb{C}^{1|2} \times \mathbb{C}^{2|4} \\ & \searrow \pi & \downarrow \\ & & \mathbb{C}^{2|4} \end{array}$$

where \mathcal{Y} is defined by the ideal $(x + a + \alpha\theta_1 + \beta\theta_2, \theta_1\theta_2 + \gamma\theta_1 + \delta\theta_2 + b)$.

Let's denote $f = x + a + \alpha\theta_1 + \beta\theta_2$ and $g = \theta_1\theta_2 + \gamma\theta_1 + \delta\theta_2 + b$

$$g\theta_1 = \delta\theta_2\theta_1 + b\theta_1 \tag{5.3}$$

$$\begin{aligned} (5.3) + \delta g &= (-\delta\theta_1\theta_2 + b\theta_1) + (\delta\theta_1\theta_2 + \delta\gamma\theta_1 + \delta b) \\ &= (b + \delta\gamma)\theta_1 + \delta b \end{aligned} \tag{5.4}$$

Therefore, we have $b = -\delta\gamma$. Applying the same technique that we used in Lemma 4.1.4 and the flattening stratification, we can check that $\text{Hilb}^{1|2}(\mathbb{C}^{1|2}) \simeq \mathbb{C}^{1|4}$.

5.4 Nonsmoothness of $\text{Hilb}^{2|2}(\mathbb{C}^{1|2})$

For the ordinary Hilbert scheme, we saw that $\text{Hilb}^4(\mathbb{C}^3)$ is not smooth at $I = m^2$ where m is the maximal ideal at the origin.

We have the analogous result for the super Hilbert scheme that the Hilbert scheme $\text{Hilb}^{2|2}(\mathbb{C}^{1|2})$ is not smooth.

Proposition 5.4.1. *The Hilbert scheme $\text{Hilb}^{2|2}(\mathbb{C}^{1|2})$ is not smooth.*

Proof. Let $(x | \theta_1, \theta_2)$ be coordinates on $\mathbb{C}^{1|2}$ and let $R := \mathbb{C}[x | \theta_1, \theta_2]$.

Let $\Delta \subset \text{Hilb}^{2|2}(\mathbb{C}^{1|2})$ be the diagonal. Consider $(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) - \Delta)$. Observe that

- 1) $\text{Hilb}^{0|2}(\mathbb{C}^{1|2})$ is empty.
- 2) $\text{Hilb}^{1|1}(\mathbb{C}^{1|2}) \times \text{Hilb}^{1|1}(\mathbb{C}^{1|2}) - \Delta$ has dimension $2 \times (2|2) = (4|4)$
- 3) $\text{Hilb}^{1|0}(\mathbb{C}^{1|2}) \times \text{Hilb}^{1|2}(\mathbb{C}^{1|2}) - \Delta$ has dimension $(1|2) + (1|2)$

Since $(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) - \Delta)$ is the union of 2) and 3), the bosonic part of $(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) - \Delta)$ has dimension ≤ 4 .

Observe that Δ is contained in the open subset $U \subset \text{Hilb}^{2|2}(\mathbb{C}^{1|2})$ such that the pushforward of the universal family is generated by $1, x, \theta_1$ and θ_2 . Then U corresponds to the ideal generated by

$$\begin{aligned} x^2 + ax + b \\ x\theta_1 + \alpha x + \beta \\ x\theta_2 + \gamma x + \delta \\ \theta_1\theta_2 + cx + d \end{aligned}$$

Therefore, the dimension of Δ_b is less than or equal to 4.

In conclusion, the dimension of reduced part of the Hilbert scheme is less than or equal to 4.

$$\dim (\text{Hilb}^{2|2}\mathbb{C}^{1|2})_{red} \leq 4$$

Let's compute the dimension of the tangent space $\mathcal{T}_I \left(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) \right)$ at I .

$$\begin{aligned} \dim \mathcal{T}_I \left(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) \right) &= \dim \text{Hom}_R(I, R/I) \\ &= \dim \text{Hom}_R(m^2, R/m^2) \end{aligned}$$

Observe that $\text{Hom}_R(m^2, R/m^2) = \text{Hom}_R(m^2, m/m^2)$.

Moreover, for any $\varphi \in \text{Hom}_R(m^2, m/m^2)$, the image of m^3 must vanish.

$$\varphi(m^3) \subset m \cdot \varphi(m^2) \subset m^2/m^2 = 0$$

Therefore, we have

$$\begin{aligned} \dim \mathcal{T}_I \left(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) \right) &= \dim \text{Hom}_R(m^2, R/m^2) \\ &= \dim \text{Hom}_{\mathbb{C}}(m^2/m^3, R/m^2) \end{aligned}$$

As a \mathbb{C} -vector space, m^2/m^3 is generated by $x^2, x\theta_1, x\theta_2, \theta_1\theta_2$ and has dimension $2|2$. Also, m/m^2 has generators x, θ_1, θ_2 and has dimension $1|2$.

Then the dimension of the tangent space $\mathcal{T}_I \left(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) \right)$ is given by

$$\begin{aligned} \dim \text{Hom}_{\mathbb{C}}(m^2/m^3, m/m^2) &= \dim (m^2/m^3)^\vee \otimes m/m^2 \\ &= (2 \cdot 1 + 2 \cdot 2 \mid 2 \cdot 2 + 2 \cdot 1) = (6 \mid 6) \end{aligned}$$

Therefore,

$$\dim (\text{Hilb}^{2|2}(\mathbb{C}^{1|2})|_U)_{red} < \dim \left(\mathcal{T}_I \left(\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) \right) \right)_{red}$$

and the Hilbert scheme $\text{Hilb}^{2|2}(\mathbb{C}^{1|2})$ is not smooth at I .

□

Chapter 6

(Non)splitness of the super Hilbert scheme

In Chapter 4, we found local defining equations and gluing maps of the Hilbert scheme $\text{Hilb}^{p|q}C$ where C is a supermanifold of dimension $1|1$. We use these to construct a nonzero obstruction class and show (non) splitness of the Hilbert scheme.

6.1 A split super Hilbert scheme

Let $V = \mathcal{O}_{\mathbb{P}^1}(k)$ be a line bundle on \mathbb{P}^1 . Then the parity reversed bundle ΠV is a supermanifold of dimension $1|1$ and the Hilbert scheme $\text{Hilb}^{p|q}(\Pi V)$ is smooth and has dimension $p|p$.

Consider the case $p = q = 1$. Note that the bosonic part of the Hilbert scheme $\text{Hilb}^{1|1}(\Pi V)$ is \mathbb{P}^1 . This can be checked from the fact that two open subfunctors $\mathcal{H}_{\Pi V|U_0}^{1|1}$ and $\mathcal{H}_{\Pi V|U_1}^{1|1}$ cover $\text{Hilb}^{1|1}(\Pi V)$. Since $\Pi V|_{U_i} \simeq \mathbb{C}^{1|1}$ for each i , we can use (4.10) to find local coordinates and glue. The bosonic space \mathbb{P}^1 is just obtained from modding out the super Hilbert scheme by the odd parts. Let $[z_0, z_1]$ be coordinates of \mathbb{P}^1 and let $U_i \subset \mathbb{P}^1$ be the standard open set defined by $z_i \neq 0$. Assign affine coordinates on ΠV as

$$\Pi V|_{U_0} \simeq \mathbb{C}_{x,\theta}^{1|1}$$

$$\Pi V|_{U_1} \simeq \mathbb{C}_{y,\psi}^{1|1}$$

Due to Theorem 4.1.5, we can assign coordinates on each Hilbert scheme restricted to an open set U_i

$$\left(\text{Hilb}^{1|1}(\Pi V) \right) \Big|_{U_0} = \text{Hilb}^{1|1}(\Pi V|_{U_0}) \simeq \mathbb{C}_{a,\alpha}^{1|1}$$

$$\left(\text{Hilb}^{1|1}(\Pi V) \right) \Big|_{U_1} = \text{Hilb}^{1|1}(\Pi V|_{U_1}) \simeq \mathbb{C}_{b,\beta}^{1|1}$$

As we can see in Example 4.0.1, we have relations $x = a + \alpha\theta$, $y = b + \beta\psi$ and $y = 1/x$, $\psi = \theta/x^k$ on the intersection $U_0 \cap U_1$. Combining all of these relations on $U_0 \cap U_1$, we get $b = \frac{1}{a}$ and $\beta = -a^{k-2}\alpha$. Note that the odd dimension of the Hilbert scheme $\text{Hilb}^{1|1}(\Pi V)$ is 1. Therefore, $\text{Hilb}^{1|1}(\Pi V)$ is split and there is a line

bundle W on \mathbb{P}^1 such that $\Pi W = \text{Hilb}^{1|1}(\Pi V)$. Then there is an integer d such that $W \simeq \mathcal{O}_{\mathbb{P}^1}(d)$ and this integer can be determined by the gluing map. Therefore, the Hilbert scheme $\text{Hilb}^{1|1}(\Pi V)$ is isomorphic to the parity reversed bundle ΠW where $W = \mathcal{O}(-k+2) = \mathcal{O}(2) \otimes V^\vee$.

6.2 The super Hilbert scheme $\text{Hilb}^{2|1}(\Pi \mathcal{O}_{\mathbb{P}^1}(k))$

Let $V = \mathcal{O}_{\mathbb{P}^1}(k)$ be a line bundle on \mathbb{P}^1 .

Consider $\text{Hilb}^{2|1}(\Pi V)$. Observe that the bosonic part of the Hilbert scheme $(\text{Hilb}^{2|1}(\Pi V))_b$ is $\mathbb{P}^1 \times \mathbb{P}^1$. Note that the Hilbert scheme of two points on \mathbb{P}^1 is $\text{Hilb}^2(\mathbb{P}^1) = \text{Sym}^2(\mathbb{P}^1)$ and there is no distinction between two points. However, for the super Hilbert scheme $\text{Hilb}^{2|1}(\Pi V)$, two points (corresponding to supports) are distinguished by the odd part.

Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal. Let $U_{ij} = U_i \times U_j \subset \mathbb{P}^1_{[z_0; z_1]} \times \mathbb{P}^1_{[w_0; w_1]}$ be the open subset where U_i is defined by $z_i \neq 0$ and U_j is defined by $w_j \neq 0$.

Consider the open cover $\mathbb{P}^1 \times \mathbb{P}^1 = \bigcup_{k=1}^4 V_k$ where $V_1 := U_{00}$, $V_2 := U_{10} - \Delta$, $V_3 := U_{01} - \Delta$ and $V_4 := U_{11}$.

Since $\text{Hilb}^{1|1}(\Pi V|_{U_i})$ is affine, there is a projection map to its reduced part. Let p_{10} and p_{01} be the projections from $\text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0})$ and $\text{Hilb}^{1|1}(\Pi V|_{U_0}) \times \text{Hilb}^{1|0}(\Pi V|_{U_1})$ to the reduced parts

$$p_{10} : \text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0}) \rightarrow U_1 \times U_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$$

$$p_{01} : \text{Hilb}^{1|1}(\Pi V|_{U_0}) \times \text{Hilb}^{1|0}(\Pi V|_{U_1}) \rightarrow U_0 \times U_1 \subset \mathbb{P}^1 \times \mathbb{P}^1$$

Let $\Delta^* := p^* \Delta$ be the pullback of the diagonal for each $p = p_{10}, p_{01}$.

Note that we can naturally identify

$$\text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0}) - \Delta^* \xrightarrow{\sim} \text{Hilb}^{2|1}(\Pi V)|_{V_2}$$

and

$$\text{Hilb}^{1|1}(\Pi V|_{U_0}) \times \text{Hilb}^{1|0}(\Pi V|_{U_1}) - \Delta^* \xrightarrow{\sim} \text{Hilb}^{2|1}(\Pi V)|_{V_3}$$

Assign coordinates as in (4.10)

$$\mathrm{Hilb}^{2|1}(\Pi V)|_{V_1} \simeq \mathbb{C}_{a_1, a_2 | \alpha_1, \alpha_2}^{2|2} \quad (6.1)$$

$$\mathrm{Hilb}^{2|1}(\Pi V)|_{V_2} \simeq \mathrm{Hilb}^{1|1}(\Pi V|_{U_1}) \times \mathrm{Hilb}^{1|0}(\Pi V|_{U_0}) - \Delta^* \quad (6.2)$$

$$\simeq \mathbb{C}_{b_1 | \beta_1}^{1|1} \times \mathbb{C}_{b_2 | \beta_2}^{1|1} - \tilde{\Delta} \quad (6.3)$$

$$\mathrm{Hilb}^{2|1}(\Pi V)|_{V_3} \simeq \mathrm{Hilb}^{1|1}(\Pi V|_{U_0}) \times \mathrm{Hilb}^{1|0}(\Pi V|_{U_1}) - \Delta^* \quad (6.4)$$

$$\simeq \mathbb{C}_{c_1 | \gamma_1}^{1|1} \times \mathbb{C}_{c_2 | \gamma_2}^{1|1} - \tilde{\Delta} \quad (6.5)$$

$$\mathrm{Hilb}^{2|1}(\Pi V)|_{V_4} \simeq \mathbb{C}_{d_1, d_2 | \delta_1, \delta_2}^{2|2} \quad (6.6)$$

where $\tilde{\Delta}$ is defined by $c_1 c_2 = 1$ in (6.5) and $b_1 b_2 = 1$ in (6.3).

Remark 6.2.1. The Hilbert scheme $\mathrm{Hilb}^{2|1}(\Pi V)$ can be covered by four open subsets

$$\mathrm{Hilb}^{2|1}(\Pi V)|_{V_1} \cup \mathrm{Hilb}^{2|1}(\Pi V)|_{V_2} \cup \mathrm{Hilb}^{2|1}(\Pi V)|_{V_3} \cup \mathrm{Hilb}^{2|1}(\Pi V)|_{V_4}$$

To check how to glue them, first consider V_1 and V_3 and let us compute the gluing map on the intersection.

Assign local coordinates on ΠV .

$$\Pi V|_{U_0} \simeq \mathbb{C}_{x, \theta}^{1|1}$$

$$\Pi V|_{U_1} \simeq \mathbb{C}_{y, \psi}^{1|1}$$

On $V_1 \cap V_3$, we have $c_2 \neq 0$ and identities $y = \frac{1}{x}$ and $\psi = \frac{\theta}{x^k}$. Observe that the isomorphism (6.5) is given by

$$\begin{aligned} \mathbb{C}_{c_1 | \gamma_1}^{1|1} \times \mathbb{C}_{c_2 | \gamma_2}^{1|1} - \tilde{\Delta} &\rightarrow \mathrm{Hilb}^{1|1}(\Pi V|_{U_0}) \times \mathrm{Hilb}^{1|0}(\Pi V|_{U_1}) - \Delta^* \\ ((c_1 | \gamma_1), (c_2 | \gamma_2)) &\mapsto \langle x + c_1 + \gamma_1 \theta \rangle \times \langle y + c_2, \psi + \gamma_2 \rangle \\ &\rightarrow \mathrm{Hilb}^{2|1}(\Pi V)|_{V_3} \\ &\mapsto \langle (x + c_1 + \gamma_1 \theta)(y + c_2), (x + c_1 + \gamma_1 \theta)(\psi + \gamma_2) \rangle \end{aligned}$$

On the intersection, we have identities

$$\begin{aligned}
& \langle (x + c_1 + \gamma_1 \theta)(y + c_2), (x + c_1 + \gamma_1 \theta)(\psi + \gamma_2) \rangle \\
&= \left\langle (x + c_1 + \gamma_1 \theta) \left(x + \frac{1}{c_2}\right), (x + c_1 + \gamma_1 \theta) \left(\theta + \frac{\gamma_2}{(-c_2)^k}\right) \right\rangle \\
&= \langle (x + c_1 - \gamma_1 \gamma_2 (-c_2)^{-k}) (x + c_2^{-1}) + \gamma_1 (c_2^{-1} - c_1) (\theta + \gamma_2 (-c_2)^{-k}), \\
&\quad (x + c_1 - \gamma_1 \gamma_2 (-c_2)^{-k}) (\theta + \gamma_2 (-c_2)^{-k}) \rangle
\end{aligned} \tag{6.7}$$

Therefore, the gluing map on $V_1 \cap V_3$ is given by

$$(a_1, a_2 \mid \alpha_1, \alpha_2) = \left(c_1 - \gamma_1 \gamma_2 (-c_2)^{-k}, \frac{1}{c_2} \left| \gamma_1 \left(\frac{1}{c_2} - c_1 \right), \gamma_2 (-c_2)^{-k} \right. \right) \tag{6.8}$$

To find the gluing map on $V_1 \cap V_2$, consider the isomorphism (6.3)

$$\begin{aligned}
\mathbb{C}_{b_1|\beta_1}^{1|1} \times \mathbb{C}_{b_2|\beta_2}^{1|1} - \tilde{\Delta} &\rightarrow \text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0}) - \Delta^* \\
((b_1|\beta_1), (b_2|\beta_2)) &\mapsto \langle y + b_1 + \beta_1 \psi \rangle \times \langle x + b_2, \theta + \beta_2 \rangle \\
&\rightarrow \text{Hilb}^{2|1}(\Pi V) \Big|_{V_2} \\
&\mapsto \langle (y + b_1 + \beta_1 \psi) (x + b_2), (y + b_1 + \beta_1 \psi) (\theta + \beta_2) \rangle
\end{aligned}$$

By using identification $\langle y + b_1 + \beta_1 \psi \rangle = \langle x + b_1^{-1} - \beta_1 (-b_1)^{k-2} \theta \rangle$, we get

$$\begin{aligned}
& \langle (y + b_1 + \beta_1 \psi) (x + b_2), (y + b_1 + \beta_1 \psi) (\theta + \beta_2) \rangle \\
&= \langle (x + b_1^{-1} - \beta_1 (-b_1)^{k-2} \theta) (x + b_2), (x + b_1^{-1} - \beta_1 (-b_1)^{k-2} \theta) (\theta + \beta_2) \rangle \\
&= \left\langle \left(x + \frac{1}{b_1} + \beta_1 (-b_1)^{k-2} \beta_2 \right) (x + b_2) - \beta_1 (-b_1)^{k-2} \left(b_2 - \frac{1}{b_1} \right) (\theta + \beta_2), \right. \\
&\quad \left. \left(x + \frac{1}{b_1} + \beta_1 (-b_1)^{k-2} \beta_2 \right) (\theta + \beta_2) \right\rangle
\end{aligned}$$

Therefore, we get the gluing map given by

$$(a_1, a_2 \mid \alpha_1, \alpha_2) = \left(\frac{1}{b_1} + \beta_1 \beta_2 (-b_1)^{k-2}, b_2 \left| -\beta_1 (-b_1)^{k-2} \left(b_2 - \frac{1}{b_1} \right), \beta_2 \right. \right) \tag{6.9}$$

By using symmetry, we can compute the transition map on $V_2 \cap V_4$ by using the transition map on $V_1 \cap V_3$.

Similarly, on $V_2 \cap V_4$, with the identification $\langle x + b_2 \rangle = \langle y + b_2^{-1} \rangle$ and $\langle \theta + \beta_2 \rangle = \langle \psi + \frac{\beta_2}{(-b_2)^k} \rangle$, we get

$$\begin{aligned}
\mathbb{C}_{b_1|\beta_1}^{1|1} \times \mathbb{C}_{b_2|\beta_2}^{1|1} - \tilde{\Delta} &\rightarrow \text{Hilb}^{1|1}(\Pi V|_{U_1}) \times \text{Hilb}^{1|0}(\Pi V|_{U_0}) - \Delta^* \\
((b_1|\beta_1), (b_2|\beta_2)) &\mapsto \langle y + b_1 + \beta_1\psi \rangle \times \langle x + b_2, \theta + \beta_2 \rangle \\
&\rightarrow \text{Hilb}^{2|1}(\Pi V)|_{V_2} \\
&\mapsto \langle (y + b_1 + \beta_1\psi)(x + b_2), (y + b_1 + \beta_1\psi)(\theta + \beta_2) \rangle \\
&= \left\langle (y + b_1 + \beta_1\psi)(y + b_2^{-1}), (y + b_1 + \beta_1\psi)\left(\psi + \frac{\beta_2}{(-b_2)^k}\right) \right\rangle \\
&= \left\langle \left(y + b_1 - \frac{\beta_1\beta_2}{(-b_2)^k}\right)(y + b_2^{-1}) + \beta_1(b_2^{-1} - b_1)\left(\psi + \frac{\beta_2}{(-b_2)^k}\right), \right. \\
&\quad \left. \left(y + b_1 - \frac{\beta_1\beta_2}{(-b_2)^k}\right)\left(\psi + \frac{\beta_2}{(-b_2)^k}\right) \right\rangle
\end{aligned}$$

Therefore, the transition map is given by

$$(d_1, d_2|\delta_1, \delta_2) = \left(b_1 - \frac{\beta_1\beta_2}{(-b_2)^k}, b_2^{-1} \mid \beta_1(b_2^{-1} - b_1), \frac{\beta_2}{(-b_2)^k} \right) \quad (6.10)$$

One can similarly compute gluing maps on each intersection $V_i \cap V_j$ for all i and j , and the transitivity can be checked.

6.2.1 (Non)splitness of the super Hilbert scheme $\text{Hilb}^{2|1}(\Pi\mathcal{O}(k))$

Let V be the line bundle $\mathcal{O}(k)$ on \mathbb{P}^1 and let W be the vector bundle defined by $W^\vee = \mathcal{J}/\mathcal{J}^2$ where $\mathcal{J} \subset \mathcal{O}_{\text{Hilb}^{2|1}(\Pi V)}$ is the ideal sheaf generated by all nilpotents. Then we have the following theorem.

Theorem 6.2.2. *Let V be the line bundle $\mathcal{O}(k)$ on \mathbb{P}^1 . The Hilbert scheme $\text{Hilb}^{2|1}(\Pi V)$ is not split for all $k \neq 0$ and it is split for $k = 0$.*

Observe that $\wedge^2 W^\vee$ is a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, and hence there are a and b such that

$$\wedge^2 W^\vee \simeq \mathcal{O}(a, b)$$

Lemma 6.2.3. *$a = k - 3$ and $b = -k - 1$*

Proof. First of all, to compute a , restrict $\wedge^2 W^\vee$ to $\mathbb{P}^1 \times \{0\}$.

$$\wedge^2 W^\vee|_{\mathbb{P}^1 \times \{0\}} \simeq \mathcal{O}_{\mathbb{P}^1}(a)$$

Then the transition map between V_1 and V_2 gives the transition map between U_0 and U_1 , where U_0 and U_1 are standard open sets on $\mathbb{P}^1 \simeq \mathbb{P}^1 \times \{0\}$. Changing coordinates on V_2 by $\beta_1(b_1b_2 - 1) \mapsto \beta_1$, then the gluing map (6.9) gives us

$$\alpha_1\alpha_2 = \beta_1\beta_2(-b_1)^{k-3} \quad (6.11)$$

Note that the section $\alpha_1\alpha_2$ generates the line bundle $\wedge^2 W^\vee$ on V_1 and $\beta_1\beta_2$ generates the line bundle $\wedge^2 W^\vee$ on V_2 . Therefore, (6.11) gives us $a = k - 3$.

To compute b , restrict the line bundle $\wedge^2 W^\vee$ to $\{0\} \times \mathbb{P}^1$. Then by plugging in $b_1 = 0$ to the transition map (6.10) on $V_2 \cap V_4$, we get

$$\begin{aligned} \delta_1 &\mapsto \frac{\beta_1}{b_2} \\ \delta_2 &\mapsto (-b_2)^{-k} \beta_2 \end{aligned}$$

Therefore, $\delta_1\delta_2 = -\beta_1\beta_2(-b_2)^{-k-1}$ and $b = -k - 1$.

□

Remark 6.2.4. Note that the obstruction class for $\text{Hilb}^{21}(\text{IV})$ lives in $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$. From Lemma 6.2.3, we have

$$\begin{aligned} &H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee) \\ &= H^1(\mathbb{P}^1 \times \mathbb{P}^1, (\mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2)) \otimes \mathcal{O}(k - 3, -k - 1)) \\ &= H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k - 1, -k - 1) \oplus \mathcal{O}(k - 3, -k + 1)) \\ &= H^1(\mathbb{P}^1 \otimes \mathbb{P}^1, \mathcal{O}(k - 1, -k - 1)) \oplus H^1(\mathbb{P}^1 \otimes \mathbb{P}^1, \mathcal{O}(k - 3, -k + 1)) \end{aligned}$$

Therefore, $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$ is nonzero if and only if $H^1(\mathbb{P}^1 \otimes \mathbb{P}^1, \mathcal{O}(k - 1, -k - 1)) \neq 0$ or $H^1(\mathbb{P}^1 \otimes \mathbb{P}^1, \mathcal{O}(k - 3, -k + 1)) \neq 0$.

Observe the fact that

$$\begin{aligned} &H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b)) \\ &= (H^0(\mathbb{P}^1, \mathcal{O}(a)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(b))) \oplus (H^1(\mathbb{P}^1, \mathcal{O}(a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(b))) \end{aligned}$$

and hence if $a \leq -2$ and $b \geq 0$ (or $a \geq 0$ and $b \leq -2$ by the symmetry) then $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b))$ is not equal to zero.

i) If $k \geq 1$ then $k - 1 \geq 0$ and $-k - 1 \leq -2$.

$$\begin{aligned}
& H^0(\mathbb{P}^1, \mathcal{O}(k-1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-k-1)) \neq 0 \\
& \Rightarrow (H^0(\mathbb{P}^1, \mathcal{O}(k-1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-k-1))) \oplus (H^1(\mathbb{P}^1, \mathcal{O}(k-1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(-k-1))) \neq 0 \\
& \Rightarrow H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k-1, -k-1)) \neq 0
\end{aligned}$$

ii) If $k \leq 1$ then $k - 3 \leq -2$ and $-k + 1 \geq 0$.

$$\begin{aligned}
& H^1(\mathbb{P}^1, \mathcal{O}(k-3)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(-k+1)) \neq 0 \\
& \Rightarrow (H^0(\mathbb{P}^1, \mathcal{O}(k-3)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-k+1))) \oplus (H^1(\mathbb{P}^1, \mathcal{O}(k-3)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(-k+1))) \neq 0 \\
& \Rightarrow H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(k-3, -k+1)) \neq 0
\end{aligned}$$

In conclusion, $H^1(\mathbb{P}^1 \otimes \mathbb{P}^1, \mathcal{O}(k-1, -k-1)) \oplus H^1(\mathbb{P}^1 \otimes \mathbb{P}^1, \mathcal{O}(k-3, -k+1))$ is non zero for all k .

We are now ready to prove the main theorem.

Proof. of Theorem 6.2.2.

To show the (non) splitness of the Hilbert scheme, it is enough to check whether the obstruction class $\omega_2 \in H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$ defined by Hilb^{2|1}(IIV) is vanishing or not. Note that $\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee$ is the sheaf of $\wedge^2 W^\vee$ -valued even derivations on $\mathbb{P}^1 \times \mathbb{P}^1$.

i) The transition map (6.9) on $V_{12} := V_1 \cap V_2$

$$\begin{aligned}
a_1 & \mapsto \frac{1}{b_1} + \beta_1 \beta_2 (-b_1)^{k-2} \\
a_2 & \mapsto b_2 \\
\alpha_1 & \mapsto -\beta_1 (-b_1)^{k-2} (b_2 - \frac{1}{b_1}) \\
\alpha_2 & \mapsto \beta_2
\end{aligned}$$

defines a section $\omega_2^{12} \in \Gamma(V_1 \cap V_2, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$ as

$$\omega_2^{12} = \beta_1 \beta_2 (-b_1)^{k-2} \frac{\partial}{\partial a_1} = -\frac{\alpha_1 \alpha_2}{a_2 - a_1} \frac{\partial}{\partial a_1}$$

Here the identification $\alpha_1\alpha_2 = -(-b_1)^{k-2}(b_2 - \frac{1}{b_1})\beta_1\beta_2$ is used.

ii) On $V_{13} := V_1 \cap V_3$

The transition map (6.8)

$$\begin{aligned} a_1 &\mapsto c_1 - \gamma_1\gamma_2(-c_2)^{-k} \\ a_2 &\mapsto \frac{1}{c_2} \\ \alpha_1 &\mapsto \gamma_1 \left(\frac{1}{c_2} - c_1 \right) \\ \alpha_2 &\mapsto \gamma_2(-c_2)^{-k} \end{aligned}$$

defines $\omega_2^{13} \in \Gamma(V_1 \cap V_3, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$

$$\omega_2^{13} = -(-c_2)^{-k}\gamma_1\gamma_2 \frac{\partial}{\partial a_1} = -\frac{\alpha_1\alpha_2}{a_2 - a_1} \frac{\partial}{\partial a_1}$$

iii) On $V_{23} := V_2 \cap V_3$.

We have $\omega_2^{23} = 0$ because $V_{23} \subset V_{12} \cap V_{13}$.

iv) The transition map (6.10) on $V_{24} := V_2 \cap V_4$ gives

$$\begin{aligned} \beta_1 &= \delta_1(b_2^{-1} - b_1)^{-1} = \delta_1(d_2 - d_1)^{-1} \\ \beta_2 &= \delta_2(-b_2)^k = \delta_2(-d_2)^{-k} \end{aligned}$$

and $\beta_1\beta_2 = \frac{\delta_1\delta_2}{(d_2-d_1)(-d_2)}$.

Hence, we have the transition map

$$\begin{aligned} b_1 &\mapsto d_1 + \frac{\beta_1\beta_2}{(-b_2)^k} = d_1 + \frac{\delta_1\delta_2}{(d_2 - d_1)} \\ b_2 &\mapsto d_2^{-1} \\ \beta_1 &\mapsto \delta(d_2 - d_1)^{-1} \\ \beta_2 &\mapsto \delta_2(-d_2)^{-k} \end{aligned}$$

which defines a section $\omega_2^{24} \in \Gamma(V_2 \cap V_4, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee)$ as

$$\omega_2^{24} = \frac{\beta_1 \beta_2}{(-b_2)^k} \frac{\partial}{\partial b_1} = \frac{\delta_1 \delta_2}{d_2 - d_1} \frac{\partial}{\partial d_1}$$

Then non-vanishing of the obstruction class ω_2 can be proven by showing that there is no element $\sigma = (\sigma_i)_i \in \prod_i \Gamma(V_i, \mathcal{T} \otimes \wedge^2 W^\vee)$ such that the boundary map sends $(\sigma_i)_i$ to $(\omega_2^{ij})_{ij}$.

Suppose not. Then there are $\sigma_i \in \Gamma(V_i, \mathcal{T} \otimes \wedge^2 W^\vee)$ such that $\omega_2^{ij} = \sigma_j - \sigma_i$ on each V_{ij} . More specifically, fix coordinates $([z_0; z_1], [w_0; w_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ and let $f(\frac{z_1}{z_0}, \frac{w_1}{w_0}), \bar{f}(\frac{z_1}{z_0}, \frac{w_1}{w_0}) \in \mathbb{C} \left[\frac{z_1}{z_0}, \frac{w_1}{w_0} \right]$, $g(\frac{z_0}{z_1}, \frac{w_0}{w_1}), \bar{g}(\frac{z_0}{z_1}, \frac{w_0}{w_1}) \in \mathbb{C} \left[\frac{z_0}{z_1}, \frac{w_0}{w_1} \right]$, $h(\frac{z_1}{z_0}, \frac{w_0}{w_1}), \bar{h}(\frac{z_1}{z_0}, \frac{w_0}{w_1}) \in \mathbb{C} \left[\frac{z_1}{z_0}, \frac{w_0}{w_1} \right]$ and $l(\frac{z_0}{z_1}, \frac{w_0}{w_1}), \bar{l}(\frac{z_0}{z_1}, \frac{w_0}{w_1}) \in \mathbb{C} \left[\frac{z_0}{z_1}, \frac{w_0}{w_1} \right]$ be polynomials such that

$$\begin{aligned} \sigma_1 &= f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \alpha_1 \alpha_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + \bar{f}\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \alpha_1 \alpha_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} \\ \sigma_2 &= g\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_0}{z_1})} + \bar{g}\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) \beta_1 \beta_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} \\ \sigma_3 &= h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) \gamma_1 \gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + \bar{h}\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) \gamma_1 \gamma_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} \\ \sigma_4 &= l\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) \delta_1 \delta_2 \frac{\partial}{\partial(\frac{z_0}{z_1})} + \bar{l}\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) \delta_1 \delta_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} \end{aligned} \tag{6.12}$$

Observe that

$$\begin{aligned} \omega_2^{12} &= \left(-\frac{z_0}{z_1}\right)^{k-2} \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} \\ &= \sigma_2 - \sigma_1 \\ &= -f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \alpha_1 \alpha_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} - \bar{f}\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \alpha_1 \alpha_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} \\ &\quad + g\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_0}{z_1})} + \bar{g}\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) \beta_1 \beta_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} \\ &= f \cdot \left(b_2 - \frac{1}{b_1}\right) (-b_1)^{k-2} \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} - g \cdot \left(\frac{z_1}{z_0}\right)^2 \beta_1 \beta_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + (-\bar{f} \alpha_1 \alpha_2 + \bar{g} \beta_1 \beta_2) \frac{\partial}{\partial(\frac{w_1}{w_0})} \end{aligned} \tag{6.13}$$

By comparing coefficients, we get

$$\left(-\frac{z_0}{z_1}\right)^k = -g\left(\frac{z_0}{z_1}, \frac{w_0}{w_1}\right) + f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) \left(-\frac{z_0}{z_1}\right)^k \tag{6.14}$$

Similarly, on V_{13}

$$\begin{aligned}
\omega_2^{13} &= -\frac{\alpha_1\alpha_2}{a_2 - a_1} \frac{\partial}{\partial(\frac{z_1}{z_0})} = -(-c_2)^{-k}\gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} \\
&= \sigma_3 - \sigma_1 \\
&= -f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)\alpha_1\alpha_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right)\gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} - \bar{f}\alpha_1\alpha_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} + \bar{h}\gamma_1\gamma_2 \frac{\partial}{\partial(\frac{w_0}{w_1})}
\end{aligned}$$

Hence, we have

$$-\left(-\frac{w_1}{w_0}\right)^k = h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) - f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)\left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right)\left(-\frac{w_1}{w_0}\right)^k \quad (6.15)$$

Finally, on V_{23} , we have

$$\begin{aligned}
\omega_2^{23} &= 0 = \sigma_3 - \sigma_2 \\
&= h \cdot \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + \bar{h} \cdot \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} - g \cdot \beta_1\beta_2 \frac{\partial}{\partial(\frac{z_0}{z_1})} - \bar{g} \cdot \beta_1\beta_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} \\
&= h \cdot \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + g \cdot (-c_2)^{-k}(-b_1)^{-k+2}\gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_0}{z_1})} + \bar{h} \cdot \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} - \bar{g} \cdot \beta_1\beta_2 \frac{\partial}{\partial(\frac{w_1}{w_0})} \\
&= h \cdot \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} - g \cdot \left(-\frac{w_0}{w_1}\right)^{-k} \left(-\frac{z_0}{z_1}\right)^{-k+2} \left(\frac{z_1}{z_0}\right)^2 \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{z_1}{z_0})} + \bar{h} \cdot \gamma_1\gamma_2 \frac{\partial}{\partial(\frac{w_0}{w_1})} - \bar{g} \cdot \beta_1\beta_2 \frac{\partial}{\partial(\frac{w_1}{w_0})}
\end{aligned}$$

and thus

$$h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right) - g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right)\left(-\frac{w_1}{w_0}\right)^k \left(-\frac{z_1}{z_0}\right)^k = 0 \quad (6.16)$$

Now we derive a contradiction for any k .

Case I $k > 0$

If k is positive, $g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right)\left(-\frac{w_1}{w_0}\right)^k \left(-\frac{z_1}{z_0}\right)^k$ has a term with w_0 at the denominator for all nonzero g . Since $h\left(\frac{z_1}{z_0}, \frac{w_0}{w_1}\right)$ can not have w_0 at the denominator, to make the equality (6.16) true, g and h must vanish. Then the equation (6.15) implies

$$\begin{aligned}
\left(-\frac{w_1}{w_0}\right)^k &= f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)\left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right)\left(-\frac{w_1}{w_0}\right)^k \\
\Rightarrow 1 &= f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)\left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right)
\end{aligned}$$

which is a contradiction.

Case II . $k < 0$

Observe that $g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \cdot \left(-\frac{w_1}{w_0}\right)^k \left(-\frac{z_1}{z_0}\right)^k$ has z_1 at the denominator for any nonzero $g \neq 0$.

Hence, the equation (6.16) means that $h = g = 0$. Then, as in the case $k > 0$, the equation (6.15) implies $1 = f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right)$ which is a contradiction.

Case III . $k = 0$

If $k = 0$, the equation (6.16) becomes

$$h\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) = g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right).$$

By comparing variables, we can conclude that

$$h\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) = g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) = c \quad (6.17)$$

for some constant c . Then the equation (6.14) implies that

$$\begin{aligned} 1 &= f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) - g\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) \\ \Rightarrow \quad 1 + c &= f\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right) \cdot \left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) \end{aligned}$$

To make the right hand side a constant, f must vanish and thus c equals -1 .

Observe (6.13)

$$\begin{aligned} \omega_2^{12} &= \left(-\frac{z_0}{z_1}\right)^{-2} \beta_1 \beta_2 \frac{\partial}{\partial\left(\frac{z_1}{z_0}\right)} \\ &= f \cdot \left(b_2 - \frac{1}{b_1}\right) b_1^{-2} \beta_1 \beta_2 \frac{\partial}{\partial\left(\frac{z_1}{z_0}\right)} - g \cdot \left(\frac{z_1}{z_0}\right)^2 \beta_1 \beta_2 \frac{\partial}{\partial\left(\frac{z_1}{z_0}\right)} + (-\bar{f}\alpha_1\alpha_2 + \bar{g}\beta_1\beta_2) \frac{\partial}{\partial\left(\frac{w_1}{w_0}\right)} \\ &= \left(\frac{z_1}{z_0}\right)^2 \beta_1 \beta_2 \frac{\partial}{\partial\left(\frac{z_1}{z_0}\right)} + (-\bar{f}\alpha_1\alpha_2 + \bar{g}\beta_1\beta_2) \frac{\partial}{\partial\left(\frac{w_1}{w_0}\right)} \end{aligned}$$

Taking coefficients of $\frac{\partial}{\partial(\frac{w_1}{w_0})}$, we have

$$\begin{aligned} 0 &= -\bar{f}\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)\alpha_1\alpha_2 + \bar{g}\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right)\beta_1\beta_2 \\ &= -\bar{f}\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)b_1^{-2}\left(b_2 - \frac{1}{b_1}\right)\beta_1\beta_2 + \bar{g}\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right)\beta_1\beta_2 \end{aligned}$$

However, there is no nonzero \bar{f} which satisfies

$$-\bar{f}\left(\frac{z_1}{z_0}, \frac{w_1}{w_0}\right)\left(-\frac{z_0}{z_1}\right)^{-2}\left(\frac{w_1}{w_0} - \frac{z_1}{z_0}\right) + \bar{g}\left(\frac{z_0}{z_1}, \frac{w_1}{w_0}\right) = 0$$

Therefore, we get $\bar{f} = 0$ and $\bar{g} = 0$.

By using symmetry, we can also show that $\bar{h} = 0$ and $\bar{l} = 0$.

On $V_1 \cap V_4$, we have relations $x = \frac{1}{y}$ and $\theta = \frac{\phi}{y^0} = \phi$. Then

$$\begin{aligned} &= \langle (y + d_1)(y + d_2) + \delta_1(\phi + \delta_2), (y + d_1)(\phi + \delta_2) \rangle \\ &= \left\langle \left(\frac{1}{x} + d_1\right)\left(\frac{1}{x} + d_2\right) + \delta_1(\theta + \delta_2), \left(\frac{1}{x} + d_1\right)(\theta + \delta_2) \right\rangle \\ &= \langle (d_1x + 1)(d_2x + 1) + \delta_1(\theta + \delta_2)x^2, (d_1x + 1)(\theta + \delta_2) \rangle \\ &= \langle (x + d_1^{-1})(x + d_2^{-1}) + \delta_1d_1^{-1}d_2^{-1}(\theta + \delta_2)x^2, (x + d_1^{-1})(\theta + \delta_2) \rangle \end{aligned} \tag{6.18}$$

Note that $x(\theta + \delta_2) = -d_1^{-1}(\theta + \delta_2)$ modulo the above ideal. Therefore, we have

$$\begin{aligned} \delta_1d_1^{-1}d_2^{-1}x^2(\theta + \delta_2) &= -\delta_1d_1^{-2}d_2^{-1}x(\theta + \delta_2) \\ &= \delta_1d_1^{-3}d_2^{-1}(\theta + \delta_2) \end{aligned}$$

Then the ideal in (6.18) equals to

$$\langle (x + d_1^{-1})(x + d_2^{-1}) + \delta_1d_1^{-3}d_2^{-1}(\theta + \delta_2), (x + d_1^{-1})(\theta + \delta_2) \rangle$$

Hence, the transition map on $V_1 \cap V_4$ is

$$(a_1, a_2 \mid \alpha_1, \alpha_2) \mapsto \left(\frac{1}{d_1}, \frac{1}{d_2} \mid \delta_1d_1^{-3}d_2^{-1}, \delta_2 \right)$$

By using symmetry, we can easily find the transition map on V_{34} from the transition map on V_{12} .

$$(d_1, d_2 | \delta_1, \delta_2) \mapsto \left(\frac{1}{c_1} + \gamma_1 \gamma_2 (-c_1)^{-2}, c_2 \mid -\gamma_1 (-c_1)^{-2} \left(c_2 - \frac{1}{c_1} \right), \gamma_2 \right)$$

On $V_2 \cap V_3$, consider the following identities

$$\begin{aligned} (c_1, c_2 | \gamma_1, \gamma_2) &\mapsto \langle (x + c_1 + \gamma_1 \theta)(y + c_2), (x + c_1 + \gamma_1 \theta)(\phi + \gamma_2) \rangle \\ &= \langle (x + c_1 + \gamma_1 \theta)(x + c_2^{-1}), (x + c_1 + \gamma_1 \theta)(\theta + \gamma_2) \rangle \\ &= \langle (x + c_1 - \gamma_1 \gamma_2)(x + c_2^{-1}) + \gamma_1 (c_2^{-1} - c_1)(\theta + \gamma_2), \\ &\quad (x + c_1 - \gamma_1 \gamma_2)(\theta + \gamma_2) \rangle \end{aligned}$$

on V_3 and

$$\begin{aligned} (b_1, b_2 | \beta_1, \beta_2) &\mapsto \langle (y + b_1 + \beta_1 \phi)(x + b_2), (y + b_1 + \beta_1 \phi)(\theta + \beta_2) \rangle \\ &= \left\langle \left(\frac{1}{x} + b_1 + \beta_1 \theta \right) (x + b_2), \left(\frac{1}{x} + b_1 + \beta_1 \theta \right) (\theta + \beta_2) \right\rangle \\ &= \langle (x + b_1^{-1} - b_1^{-2} \beta_1 \theta)(x + b_2), (x + b_1^{-1} - b_1^{-2} \beta_1 \theta)(\theta + \beta_2) \rangle \\ &= \langle (x + b_1^{-1} + b_1^{-2} \beta_1 \beta_2 \theta)(x + b_2) - b_1^{-2} \beta_1 (b_2 - b_1^{-1})(\theta + \beta_2), \\ &\quad (x + b_1^{-1} + b_1^{-2} \beta_1 \beta_2 \theta)(\theta + \beta_2) \rangle \end{aligned}$$

on V_2 .

Therefore, we have

$$\begin{aligned} c_1 - \gamma_1 \gamma_2 &= b_1^{-1} + b_1^{-2} \beta_1 \beta_2 \\ c_2^{-1} &= b_2 \\ \gamma_1 (c_2^{-1} - c_1) &= -\beta_1 b_1^{-2} (b_2 - b_1^{-1}) \\ \gamma_2 &= \beta_2 \end{aligned}$$

Using identities $\gamma_1 \gamma_2 = -\beta_1 \beta_2 b_1^{-2}$ and $c_1 = b_1^{-1}$, the transition map on $V_2 \cap V_3$ is given as

$$(c_1, c_2 | \gamma_1, \gamma_2) = (b_1^{-1}, b_2^{-1} \mid -\beta_1 b_1^{-2}, \beta_2)$$

Then we can see that the Hilbert scheme gives the sections

$$\begin{aligned}
\omega_2^{12} &= \beta_1\beta_2(-b_1)^{-2} \frac{\partial}{\partial a_1} \\
&= -\beta_1\beta_2 \frac{\partial}{\partial b_1} \quad \in \Gamma(V_1 \cap V_2, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee) \\
\omega_2^{13} &= -\gamma_1\gamma_2 \frac{\partial}{\partial c_1} \quad \in \Gamma(V_1 \cap V_3, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee) \\
\omega_2^{23} &= 0 \\
\omega_2^{34} &= -\gamma_1\gamma_2(-c_1)^{-2} \frac{\partial}{\partial d_1} \\
&= \gamma_1\gamma_2 \frac{\partial}{\partial c_1} \quad \in \Gamma(V_3 \cap V_4, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee) \\
\omega_2^{24} &= \beta_1\beta_2 \frac{\partial}{\partial d_1} \quad \in \Gamma(V_2 \cap V_4, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \wedge^2 W^\vee) \\
\omega_2^{14} &= 0
\end{aligned} \tag{6.19}$$

From the assumption (6.12), we have the identity

$$\begin{aligned}
0 &= \omega_2^{14} \\
&= \sigma_4 - \sigma_1 \\
&= l \left(\frac{z_0}{z_1}, \frac{w_0}{w_1} \right) \delta_1 \delta_2 \frac{\partial}{\partial \left(\frac{z_0}{z_1} \right)} - f \left(\frac{z_1}{z_0}, \frac{w_1}{w_0} \right) \alpha_1 \alpha_2 \frac{\partial}{\partial \left(\frac{z_1}{z_0} \right)}
\end{aligned}$$

Since $f = 0$, we have $l = 0$ and $\sigma_4 = 0$.

To sum up, we have

$$\begin{aligned}
\sigma_1 &= 0 \\
\sigma_2 &= -\beta_1\beta_2 \frac{\partial}{\partial \left(\frac{z_0}{z_1} \right)} \\
\sigma_3 &= -\gamma_1\gamma_2 \frac{\partial}{\partial \left(\frac{z_1}{z_0} \right)} \\
\sigma_4 &= 0
\end{aligned}$$

and $\omega_2^{ij} = \sigma_j - \sigma_i$ for all i and j .

Therefore, the obstruction class is vanishing.

According to [Ma], every supermanifold (S, \mathcal{O}_S) of odd dimension 2 is defined up to isomorphism

by the pair (S_{red}, W) and a cohomology class $\omega \in H^1(S_{red}, \mathcal{T}_{S_{red}} \otimes \wedge^2 W^\vee)$. Therefore, the Hilbert scheme $\text{Hilb}^{2|1}(\Pi\mathcal{O}_{\mathbb{P}^1})$ is isomorphic to its split model $\wedge^\bullet W$ where $W = (\mathcal{J}/\mathcal{J}^2)^\vee$ and $\mathcal{J} \subset \mathcal{O}_{\text{Hilb}^{2|1}(\Pi\mathcal{O}_{\mathbb{P}^1})}$ is the ideal generated by nilpotents.

□

Chapter 7

Future works

7.1 (Non)smoothness of $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$

It is natural to ask that when $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is smooth. In Chapter 5, we already checked that the Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is smooth if $p + q \leq 3$ or $q \leq 1$. It remains to be checked whether $\text{Hilb}^{p|q}$ is smooth or not, for $p + q > 3$ and $q > 1$ (i.e., $p \geq 2$ and $q \geq 2$).

Remark 7.1.1.

- i) $\text{Hilb}^{1|q}(\mathbb{C}^{1|2})$ is empty for $q > 2$ and $\text{Hilb}^{0|q}(\mathbb{C}^{1|2})$ is empty for all $q > 0$.
- ii) $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is non empty if and only if $p \neq 0$ and $q \leq 2p$.

From already proved cases, we conjecture the following.

Conjecture 7.1.1. The Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is smooth if i) $p + q \leq 3$ or ii) $q \leq 1$, and singular otherwise.

For $p = 2$, $(p|q) = (2|3)$ and $(p|q) = (2|4)$ are the only nontrivial cases.

If $q \leq 2p - 2$ and $p > 2$, then the Hilbert scheme $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ contains an open subset which is isomorphic to $\text{Hilb}^{2|2}(\mathbb{C}^{1|2}) \times \text{Hilb}^{p-2|q-2}(\mathbb{C}^{1|2}) - \Delta$. Since $\text{Hilb}^{2|2}(\mathbb{C}^{1|2})$ is not smooth, the result follows.

If $q > 2p$ and $p > 2$, $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ is empty.

Therefore, Conjecture 7.1.1 can be proved by showing the non-smoothness of $\text{Hilb}^{p|q}(\mathbb{C}^{1|2})$ when $(p|q) = (2|3)$, $(p|q) = (2|4)$, $q = 2p - 1$ and $q = 2p$. More details will appear in another paper.

References

- [Ma] Y. Manin, *Gauge Field Theory and Complex Geometry*, Grundlehren 289, Berlin : Springer-Verlag, 1988.
- [Be] F. Berezin, *Introduction to Superanalysis*, Dordrecht : D. Reidel Pub. Co. ; 1987.
- [DW] R. Donagi, and E. Witten, “Supermoduli space is not projected”. String-math 2012. Vol. 90. American Mathematical Society, 2015.(R. Donagi, S. Katz, A. Klemm, and D. Morrison. eds.)
- [Gr] A. Grothendieck, “Techniques de construction et théorèmes d’existence en géométrie algébrique IV: Les schémas de Hilbert.” Séminaire Bourbaki 6 (1960): 249-276.
- [Fo] J. Fogarty, “Algebraic families on an algebraic surface.” American Journal of Mathematics 90, no. 2 (1968): 511-521.
- [La] T. Y. Lam, *A First Course In Noncommutative Rings*. New York : Springer-Verlag, 1991.
- [TR] P. Topiwala, and J. M. Rabin. “The super GAGA principle and families of super Riemann surfaces.” Proceedings of the American Mathematical Society (1991): 11-20.
- [Ha] R. Hartshorne, *Algebraic Geometry*. New York : Springer-Verlag, 1977.
- [Ni] N. Nitsure, “Construction of Hilbert and Quot schemes.” Fundamental Algebraic Geometry, Math.Surveys Monogr, vol. 123, 2005, pp. 105-137.
- [Ja] M. Jang, “Families of 0-dimensional submanifolds on supercurves”, arXiv:1610.04948 (2016).
- [HM] J. Harris, I. Morrison, *Moduli Of Curves*. New York : Springer, 1998.
- [Gr2] A. Grothendieck, “Techniques de construction en géométrie analytique. I. Description axiomatique de l’espace de Teichmüller et de ses variantes.” Séminaire Henri Cartan 13, no. 1 (1960): 1-33.
- [Na] H. Nakajima, *Lectures On Hilbert Schemes Of Points On Surfaces*. Providence, R.I. : American Mathematical Society, 1999.
- [Go] L. Göttsche, “The Betti numbers of the Hilbert scheme of points on a smooth projective surface.” Mathematische Annalen 286.1-3 (1990): 193-208.
- [Va] R. Vakil, “Murphys Law in Algebraic Geometry: Badly-Behaved Deformation Spaces.” *Inventiones Mathematicae*, vol. 164, no. 3, June 2006, pp. 569-590.
- [Wi] E. Witten, “Notes On Super Riemann Surfaces And Their Moduli”, arXiv:1209.2459 (2013)