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# MULTI-SOLVER SCHEMES FOR ELECTROMAGNETIC MODELING OF LARGE AND COMPLEX OBJECTS 

## BY

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## DISSERTATION

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## ABSTRACT

The work in this dissertation primarily focuses on the development of numerical algorithms for electromagnetic modeling of large and complex objects.

First, a GPU-accelerated multilevel fast multipole algorithm (MLFMA) is presented to improve the efficiency of the traditional MLFMA by taking advantage of GPU hardware advancement. The proposed hierarchical parallelization strategy ensures a high computational throughput for the GPU calculation. The resulting OpenMP-based multi-GPU implementation is capable of solving real-life problems with over one million unknowns with a remarkable speedup. The radar cross sections (RCS) of a few benchmark objects are calculated to demonstrate the accuracy of the solution. The results are compared with those from the CPU-based MLFMA and measurements. The capability and efficiency of the presented method are analyzed through the examples of a sphere, an aircraft, and a missile-like object. Compared with the 8-threaded CPU-based MLFMA, the OpenMP-CUDAMLFMA method can achieve from 5 to 20 times total speedup.

Second, an efficient and accurate finite element-boundary integral (FEBI ) method is proposed for solving electromagnetic scattering and radiation problems. A mixed testing scheme, in which the Rao-Wilton-Glisson and the Buffa-Christiansen functions are both employed as the testing functions, is first presented to improve the accuracy of the FE-BI method. An efficient absorbing boundary condition (ABC)-based preconditioner is then proposed to accelerate the convergence of the iterative solution. To further improve the efficiency of the total computation, a GPU-accelerated MLFMA is applied to the iterative solution. The RCSs of several benchmark objects are calculated to demonstrate the numerical accuracy of the solution and also to show that the proposed method not only is free of interior resonance corruption, but also has a better convergence than the conventional FE-BI methods. The capability and efficiency of the proposed method are analyzed through several
numerical examples, including a large dielectric coated sphere, a partial human body, and a coated missile-like object. Compared with the 8-threaded CPU-based algorithm, the GPU-accelerated FE-BI-MLFMA algorithm can achieve a total speedup up to 25.5 times.

Third, a multi-solver (MS) scheme based on combined field integral equation (CFIE) is proposed. In this scheme, an object is decomposed into multiple bodies based on its material property and geometry. To model bodies with complicated materials, the FE-BI method is applied. To model bodies with homogeneous or conducting materials, the method of moments is employed. Specifically, three solvers are integrated in this multi-solver scheme: the FE-BI(CFIE) for inhomogeneous objects, the CFIE for dielectric objects, and the CFIE for conducting objects. A mixed testing scheme that utilizes both the Rao-Wilton-Glisson and the Buffa-Christiansen functions is adopted to obtain a good accuracy of the proposed multi-solver algorithm. In the iterative solution of the combined system, the MLFMA is applied to accelerate computation and reduce memory costs, and an ABC-based preconditioner is employed to speed up the convergence. In the numerical examples, the individual solvers are first demonstrated to be well conditioned and highly accurate. Then the validity of the proposed multi-solver scheme is demonstrated and its capability is shown by solving scattering problems of electrically large missile-like objects.

Fourth, a MS scheme based on Robin transmission condition (RTC) is proposed. Different from the FE-BI method that applies BI equations to truncate the FE domain, this proposed multi-solver scheme employs both FE and BI equations to model an object along with its background. To be specific, the entire computational domain consisting of the object and its background is first decomposed into multiple non-overlapping subdomains with each modeled by either an FE or BI equation. The equations in the subdomains are then coupled into a multi-solver system by enforcing the RTC at the subdomain interfaces. Finally, the combined system is solved iteratively with the application of an extended ABC-based preconditioner and the MLFMA. To obtain an accurate solution, both the Rao-WiltonGlisson and the Buffa-Christiansen functions are employed as the testing functions to discretize the BI equations. This scheme is applied to a variety of benchmark problems and the scattering from an aircraft with a launched missile to demonstrate its accuracy, versatility, and capability. The proposed
scheme is compared with the MS-CFIE to illustrate the differences between the two schemes.

Fifth, to further improve the modeling capability, an accelerated MS method is developed on distributed computing systems to simulate the scattering from very large and complex objects. The parallelization strategy is to parallelize different subdomains individually, which is different from the parallelized domain decomposition methods, where the subdomains are handled in parallel. The multilevel fast multipole algorithm is parallelized to enable computation on many processors. The modeling strategy using the MS-RTC method is also discussed so that one can easily follow the guideline to model large and complex objects. Numerical examples are given to show the parallel efficiency of the proposed strategy and the modeling capability of the proposed method.

Finally, the specific absorption rate (SAR) in a human head at 5G frequencies is simulated by taking advantage of the MS-RTC method. Based on the strong skin effect, the human head model is first simplified to reduce the computation cost. Then the MS-RTC method is applied to model the human head. Numerical examples show that the MS method is very efficient in solving electromagnetic fields in the human head and the simplified human head model can be used in the SAR simulation with an acceptable accuracy.

To my parents, for their love and support.

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## LIST OF ABBREVIATIONS

| ABC | Absorbing Boundary Condition |
| :--- | :--- |
| AI | Approximate Inverse |
| BI | Buffa-Christiansen |
| BiCGSTAB | Biconjugate Gradient Stabilized Method |
| BIE | Boundary Integral Equation |
| CFIE | Combined Field Integral Equation |
| COO | Coordinates List |
| CPU | Central Processing Unit |
| CRWG | Curvilinear Rao-Wilton-Glisson |
| CUDA | Compute Unified Device Architecture |
| DDM | Domain Decomposition Method |
| DoF | Degree of Freedom |
| EFIE | Electric Filed Integral Equation |
| EM | Electromagnetic |
| FE | Finite Element |
| FEM | Finite Element Method |
| FFP | Far-Field Patterns |
| FETI-DP | Dual-Primal Finite Element Tearing and Interconnecting |
| FE-BI | Finite Element-Boundary Integral |


| FE-BI(EJ) | Finite Element-Boundary Integral with Electric Field and Current as System Unknowns |
| :---: | :---: |
| GMRES | Generalized Minimal Residual Method |
| GPU | Graphics Processing Unit |
| IBC | Impedance Boundary Condition |
| MFIE | Magnetic Field Integral Equation |
| MLFMA | Multilevel Fast Multipole Algorithm |
| MoM | Method of Moments |
| MPI | Message Passing Interface |
| MS | Multi-Solver |
| MS-CFIE | Multi-Solver Scheme Based on Combined Field Integral Equation |
| MS-RTC | Multi-Solver Scheme Based on Robin Transmission Condition |
| MVP | Matrix Vector Product |
| PDE | Partial Differential Equation |
| PEC | Perfect Electric Conductor |
| PMC | Perfect Magnetic Conductor |
| RCS | Radar Cross Section |
| RMS | Root Mean Square |
| RTC | Robin Transmission Condition |
| RWG | Rao-Wilton-Glisson |
| SAR | Specific absorption Rate |
| SDIE | Self Dual Integral Equation |
| SIE | Surface Integral Equation |
| SM | Streaming Multiprocessor |
| SP | Streaming Processor |
| 1-D | One-Dimensional |


| 2-D | Two-Dimensional |
| :--- | :--- |
| 3-D | Three-Dimensional |
| 5G | Fifth Generation |

## CHAPTER 1

## INTRODUCTION

Numerical modeling becomes essential for analyzing today's highly complicated engineering electromagnetic (EM) scattering and radiation problems. A variety of numerical methods have been developed in the past to analyze different types of problems [1-6]. If objects are large and impenetrable or homogeneous, the method of moments (MoM) with acceleration of fast algorithms is most efficient. One of the widely used fast algorithms in electromagnetic analysis is the multilevel fast multipole algorithm (MLFMA) because it has a near optimal computational complexity of $O(N \log N)[2,4]$. To further accelerate the efficiency of the MLFMA, parallel computation has been applied to this algorithm to take advantage of computer hardware advancement [7-12]. In this dissertation, a multi-GPU accelerated MLFMA is presented using OpenMP-CUDA (Compute Unified Device Architecture) programming model. If objects contain inhomogeneous and/or anisotropic materials, the finite element method (FEM) is preferred. Although the FEM is able to solve problems with complicated materials, the truncation error introduced by absorbing boundary conditions (ABCs) or perfectly matched layers is undesirable when solving large unbounded problems, especially for applications that require a high accuracy such as modeling and simulation of low observable targets. Thus neither the MoM nor the FEM is both accurate and efficient when the object is electrically large and highly complex. The hybrid finite element-boundary integral (FE-BI) method is one of the most popular methods to deal with such a problem because it is not only powerful for modeling complicated geometries and inhomogeneous materials, but also capable of providing an accurate solution by truncating the computational domain with boundary integral equations (BIEs) [1]. Consequently, the FE-BI method has been widely used in the analysis of large-scale scattering, complex antenna, and deep cavity problems [13-34]. A GPU-accelerated FE-BI method with a mixed testing scheme is proposed in this dissertation
to improve the accuracy and efficiency of numerical solutions. However, if a large portion of an object is modeled by the FEM, the efficiency of the FE-BI method decreases because the poorly conditioned FEM subsystem deteriorates the condition of the entire FE-BI system.

To improve the capability of modeling electrically large and highly complex objects, domain decomposition methods (DDMs) and multi-solver schemes have been developed [33,35-52]. One of the most advanced DDMs, the dualprimal finite element tearing and interconnecting (FETI-DP) method [35,36], is introduced by first tearing the computational domain into non-overlapping subdomains where an incomplete solution of the primary field is evaluated independently, and then constructing the interface problem by applying transmission conditions at the subdomain interfaces. After the solution to the interface problem is obtained, the field inside the subdomains can be calculated independently. For the multi-solver schemes, the domain decompositionbased multi-solver method $[48,49]$ is first proposed for the modeling of electrically large objects by decomposing the object into many subregions and applying the FE-BI(EJ) method [30] to the inhomogeneous subregions and the BIE-based DDMs [44, 45] to perfectly electrical conducting and piecewise homogeneous subregions. However, the approach to constructing system equations with two sets of degrees of freedom (DoFs) defined on the surface of each subregion makes the system matrix expensive to store and solve. Besides, the application of the MLFMA requires a special treatment at the subregion interfaces. Moreover, when a block diagonal preconditioner is applied to accelerate the iterative convergence, one has to solve either the FE-BI(EJ) or BIE subsystems at each global iteration. As a result, even if the iterative solution of the global system converges very quickly, the total amount of computation is still very high.

To design an effective and efficient multi-solver scheme, one must deal with the following major technical challenges:

1) Find robust and accurate individual solvers to model different types of subregion problems,
2) Couple the individual solvers with a unified framework so that all the possible combinations of the solvers can provide a stable and accurate system equation, and
3) Solve the system equation efficiently with fast algorithms and preconditioners.

In this dissertation, two multi-solver schemes are proposed to analyze the scattering from objects with complex materials and structures. One scheme is based on combined field integral equations (CFIEs), denoted as MS-CFIE. In this scheme, the object is decomposed into multiple bodies, and only one set of DoFs is defined on the surface or in the volume of each body. The interactions between the bodies are formulated by the CFIE through their exterior regions. To model the interior region of each body, different methods are used according to different material properties. To be specific, the FEM is used to model inhomogeneous materials, a CFIE with a mixed testing scheme is employed to model homogeneous materials, and the impedance boundary condition (IBC) is applied to model both perfectly and imperfectly conducting materials $[50,53]$. To accelerate the convergence of an iterative solution, the ABC-based preconditioner is applied to the combined system of equations $[24,54]$. Also, the MLFMA with a common tree structure for all bodies is employed to speed up the computation of matrix-vector products and reduce the storage costs $[4,12]$.

The other scheme is based on Robin transmission condition (RTC), denoted as MS-RTC. In this scheme, the entire computational domain consisting of various objects and their background is decomposed into multiple nonoverlapping subdomains. Based on the material and geometry property, the field in each subdomain is formulated by either a PDE (partial differential equation) or a BIE with boundary conditions [55] such as impedance boundary conditions (IBC) [56] and perfectly electrical conducting (PEC) boundary conditions explicitly enforced. The RTC is then employed to couple multiple PDEs and BIEs into a global system equation. To solve each subdomain problem, either the FEM (instead of the FE-BI) or MoM (the method of moments) with one set of DoFs defined in the volume or on the surface of the subdomain is applied. To be more specific, the FEM is applied to model subdomains with inhomogeneous and/or anisotropic materials and the MoM is employed to model subdomains with homogeneous materials.

Different from the MS-CFIE, an impenetrable region such as PEC or IBC region is not considered as a computational subdomain in the MS-RTC. The effect of the impenetrable region is accounted for by applying proper bound-
ary conditions to PDEs or BIEs in the interfaced subdomains. With this it is straightforward to model open surfaces such as metallic structures with zero thickness embedded in or attached to homogeneous materials. Note that the MoM in this scheme is employed to model both the infinite background and the piecewise homogeneous regions in the objects, which is different from the traditional FE-BI method where the MoM is applied only to the truncation boundary. Furthermore, in contrast to the MS-CFIE which couples different bodies by the CFIE, this scheme couples different subdomains by enforcing the field continuity between the subdomains through the Robin transmission condition. Therefore the coupling matrices are sparse so that they are very cheap to compute and store. Moreover, fast algorithms such as the MLFMA [4] can be applied independently to the MoM subdomains and DDMs such as the FETI-DP [36] can be employed directly for the FEM subdomains.

Although the MS methods are highly accurate, flexible, and robust in modeling large and complex objects, the computational cost will be prohibitively high when it is applied to very large EM problems. To further improve the modeling capability, an accelerated MS scheme is developed on distributed computing systems. The parallel strategy for the MS schemes is to parallelize subdomains individually, which is different from the parallelized DDM method where all the subdomains are handled in parallel.

As an application of the MS schemes, this dissertation presents a modeling approach for the simulation of the specific absorption rate (SAR) in a human head at 5 G frequencies. To perform simulation efficiently at such high frequencies, one can simplify the human head model to minimize the computational domain based on the strong skin effect. Then the MS-RTC scheme is employed as a field solver to compute the EM fields. To quantify the EM energy absorbed by human tissue, the SAR is calculated after the fields are computed.

The rest of the dissertation is organized as follows. Chapter 2 presents the GPU-accelerated MLFMA. Chapter 3 proposes a GPU-accelerated FE-BI method with a mixed testing scheme. The MS-CFIE is presented in Chapter 4, which is followed by the discussion of the MS-TRC in Chapter 5. Chapter 6 proposes an accelerated MS scheme on distributed computing systems. The simulation of the specific absorption rate (SAR) in a human head at 5 G frequencies is given in Chapter 7. Finally the conclusion and future work are given in Chapter 8.

## CHAPTER 2

## A MULTILEVEL FAST MULTIPOLE ALGORITHM WITH MULTI-GPU ACCELERATION

### 2.1 Introduction

Fast and accurate computation of electromagnetic scattering from large complex objects is critical to stealth and anti-stealth technologies, radar system design, automatic target recognition, and many other applications. Those applications require full-wave analysis over a large computational domain, which is always computationally intensive and very time-consuming. To accelerate the computation and reduce the memory requirement, the multilevel fast multipole algorithm (MLFMA) has been developed and widely used for electromagnetic scattering analysis due to its $O(\log N)$ computational complexity $[2,4]$.

Even with a near optimal computational complexity, the computational cost of MLFMA is still prohibitively high when it is used for large electromagnetic problems. In practical applications, many of the problems are required to be solved within a very short time. In order to further accelerate the computation, parallel computation has been applied to the traditional MLFMA [7-11] to take advantage of computer hardware advancement. In 2005, a hybrid parallel MLFMA based on the distributed memory system using the message passing interface (MPI) was proposed [8]. The strategy is rather straightforward. For finer levels in MLFMA, the groups at the same level are partitioned into different processors and each processor gets approximately the same number of groups. For coarser levels, the far-field patterns (FFPs) at the same level are partitioned equally among all processors and all groups are replicated for every processor. However, when the number of processors increases, this parallelization strategy is not effective around the transition levels where neither the number of groups nor the number of FFPs is large enough to achieve a good parallel efficiency. To alleviate this prob-
lem, a hierarchical partitioning strategy was later proposed by partitioning the groups and their FFPs simultaneously at each level [10]. More recently, a hybrid MPI-OpenMP implementation was developed to port MLFMA on a hybrid shared/distributed memory architecture to solve problems with over one billion unknowns [11].

All the above-mentioned parallelization strategies are implemented using CPU parallel programming models such as MPI and OpenMP. Recently, the graphics processing unit (GPU), which is basically a many-core computing system, has received more and more attention from computational electromagnetics (CEM) community due to its low price and high computational throughput $[57,58]$. There has been intensive research dedicated to developing a variety of GPU parallelized algorithms. For example, the differential-equation-based methods such as the finite-difference time-domain (FDTD) and the discontinuous Galerkin time-domain (DGTD) methods have been implemented on GPUs [59-61]. The integral-equation-based methods such as the method of moments (MoM) and the time-domain integral equation (TDIE) method have also been accelerated by GPUs [62-64]. Besides, many GPU-incorporated fast algorithms for the efficient evaluation of electromagnetic fields have been proposed [65-70], such as the non-uniform grid interpolation method (NGIM), the box-based adaptive integral method (B-AIM), the fast Fourier transform (FFT)-based method, the multilevel plane-wave time-domain (PWTD) method, and the fast multipole method (FMM). For the GPU-accelerated MLFMA, the CUDA (compute unified device architecture) implementation of a low-frequency MLFMA on a single GPU was proposed with the essential idea of "one thread per observer" [71]. The observer stands for a parent group in the aggregation phase, a child group in the disaggregation phase, and a destination group in the translation phase, respectively. However, that implementation strategy results in a low parallel efficiency when the number of groups decreases at coarse levels.

In order to improve the parallel efficiency and solve large problems, this chapter proposes a multi-GPU accelerated MLFMA, called OpenMP-CUDAMLFMA, which is developed by hybridizing OpenMP and CUDA parallel programming models. In the OpenMP-CUDA-MLFMA, the groups and the FFPs are parallelized hierarchically. For the computation of far-field interactions, a global memory strategy and a pinned memory strategy are proposed for different application situations. This algorithm is shown to have a very
high efficiency when solving large electromagnetic scattering problems.

### 2.2 MLFMA Formulation and Implementation

In order to present the implementation strategy of the OpenMP-CUDAMLFMA clearly, it is necessary to have a brief review of the MLFMA formulation and its numerical implementation.

Consider a three-dimensional (3-D) conducting object illuminated by an incident field $\left(\boldsymbol{E}^{i}, \boldsymbol{H}^{i}\right)$. The electric-field integral equation (EFIE) and the magnetic-field integral equation (MFIE) are given by

$$
\begin{align*}
\eta \mathcal{T}(\boldsymbol{J}) & =-\hat{\boldsymbol{n}} \times \boldsymbol{E}^{i}(\boldsymbol{r}) & \boldsymbol{r} \in S  \tag{2.1}\\
-\frac{1}{2} \boldsymbol{J}+\mathcal{K}(\boldsymbol{J}) & =-\hat{\boldsymbol{n}} \times \boldsymbol{H}^{i}(\boldsymbol{r}) & \boldsymbol{r} \in S \tag{2.2}
\end{align*}
$$

respectively, where $\boldsymbol{J}$ denotes the unknown surface current density and the integral operators $\mathcal{T}$ and $\mathcal{K}$ are defined as

$$
\begin{align*}
& \mathcal{T}(\boldsymbol{J})=i k \hat{\boldsymbol{n}} \times \int_{S}\left(\mathcal{I}+\frac{\nabla \nabla}{k^{2}}\right) \frac{e^{i k R}}{4 \pi R} \cdot \boldsymbol{J}\left(\boldsymbol{r}^{\prime}\right) \mathrm{d} \boldsymbol{r}^{\prime}  \tag{2.3}\\
& \mathcal{K}(\boldsymbol{J})=\hat{\boldsymbol{n}} \times P . V . \int_{S} \nabla \frac{e^{i k R}}{4 \pi R} \times \boldsymbol{J}\left(\boldsymbol{r}^{\prime}\right) \mathrm{d} \boldsymbol{r}^{\prime} \tag{2.4}
\end{align*}
$$

where P.V. stands for the Cauchy principal value integration, $S$ denotes the surface of the conducting object, $k$ and $\eta$ denote the free-space wavenumber and impedance, $\hat{\boldsymbol{n}}$ is the outwardly directed normal unit vector, $\mathcal{I}$ represnts the identity operator, and $R=\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|$ denotes the distance between the field and source points.

Both EFIE and MFIE can be solved for $\boldsymbol{J}$. However, for a given closed surface, both of them will suffer from the problem of interior resonance at certain frequencies when the exterior medium is lossless [72]. To eliminate this problem, we can combine (2.1) and (2.2) together to form the combinedfield integral equation (CFIE) which is given by

$$
\begin{align*}
\alpha \hat{\boldsymbol{n}} & \times \eta \mathcal{T}(\boldsymbol{J})+(1-\alpha) \eta\left[\frac{1}{2} \boldsymbol{J}-\mathcal{K}(\boldsymbol{J})\right]  \tag{2.5}\\
& =-\alpha \hat{\boldsymbol{n}} \times\left[\hat{\boldsymbol{n}} \times \boldsymbol{E}^{i}(\boldsymbol{r})\right]+(1-\alpha) \eta \hat{\boldsymbol{n}} \times \boldsymbol{H}^{i}(\boldsymbol{r}) \quad \boldsymbol{r} \in S
\end{align*}
$$

where $\alpha \in[0,1]$ is the combination parameter.
In order to numerically solve CFIE, the unknown current density can first be expanded as

$$
\begin{equation*}
\boldsymbol{J}=\sum_{n=1}^{N} I_{n} \boldsymbol{f}_{n} \tag{2.6}
\end{equation*}
$$

where $N$ is the number of unknowns, $\boldsymbol{f}_{n}$ denote the vector basis functions, and $I_{n}$ are the expansion coefficients yet to be determined. In this chapter, the curvilinear Rao-Wilton-Glisson (CRWG) functions [73,74] are used as the basis functions. The application of Galerkin's method results in a system of linear equations

$$
\begin{equation*}
\sum_{n=1}^{N} Z_{m n} I_{n}=V_{m} \quad m=1,2, \ldots, N \tag{2.7}
\end{equation*}
$$

in which

$$
\begin{align*}
Z_{m n}= & \alpha \int_{S} \boldsymbol{f}_{m} \cdot\left[\hat{\boldsymbol{n}} \times \eta \mathcal{T}\left(\boldsymbol{f}_{n}\right)\right] \mathrm{d} \boldsymbol{r}  \tag{2.8}\\
& +(1-\alpha) \eta \int_{S} \boldsymbol{f}_{m} \cdot\left[\frac{1}{2} \boldsymbol{f}_{n}-\mathcal{K}\left(\boldsymbol{f}_{n}\right)\right] \mathrm{d} \boldsymbol{r} \\
V_{m}= & \int_{S}\left[\alpha \boldsymbol{E}^{i}+(1-\alpha) \eta \hat{\boldsymbol{n}} \times \boldsymbol{H}^{i}\right] \cdot \boldsymbol{f}_{m} \mathrm{~d} \boldsymbol{r} \tag{2.9}
\end{align*}
$$

For large problems, (2.7) can be solved iteratively, where MLFMA can be employed to speed up the matrix-vector products and reduce the memory requirement. The basic idea of MLFMA is to decompose the computation of matrix-vector products into the near-field and far-field interactions. To achieve such a decomposition, the entire object is first enclosed by a large cubic box, then divided into non-empty subcubes called groups. Each subcube is further subdivided into smaller cubes recursively until the length of non-empty cubes at the finest level is about $0.25 \lambda$ to $0.5 \lambda$. After the decomposition, the system of linear equations can be written as

$$
\begin{equation*}
\boldsymbol{Z}_{\mathrm{near}} \boldsymbol{I}+\boldsymbol{Z}_{\mathrm{far}} \boldsymbol{I}=\boldsymbol{V} \tag{2.10}
\end{equation*}
$$

in which $\boldsymbol{Z}_{\text {near }}$ is a block matrix, and each block represents the interaction between the testing functions in a group at the finest level and the basis functions in the same group or a neighboring group. Subsequently, $\boldsymbol{Z}_{\text {far }}$ is the remaining part of the MoM matrix which represents the interaction
between groups that are well separated [8]. The $\boldsymbol{Z}_{\text {near }}$ can be calculated directly using the standard MoM at the finest level, and the computation of $\boldsymbol{Z}_{\mathrm{far}} \boldsymbol{I}$ can be done in three phases called aggregation, translation, and disaggregation. In the aggregation phase, the fields radiated by the sources $\boldsymbol{f}_{n} I_{n}$ in each group at the finest level are first projected into the spectrum space to obtain the radiation pattern, which is then aggregated to the center of the group at the parent level. This procedure is executed repeatedly until it reaches the coarsest level. It is easy to notice that the number of the groups becomes smaller while the size of the spectrum sampling becomes larger in the aggregation phase. In the disaggregation phase, the receiving patterns at each level come from the two sources: one is the translation of the radiation patterns at the same level; the other is the disaggregation of the receiving pattern from the parent level. Thus, the translation and disaggregation can be executed concurrently as follows $[8,75]$. At the coarsest level, the radiation pattern is first translated to the receiving pattern for each group, which is then distributed to the centers of the groups at the child level. At the same time, the radiation pattern at the child level is translated to the receiving pattern at the same level. Then the total receiving pattern at the child level can be achieved by summing up the above two receiving patterns. After the total receiving pattern is achieved, the next level's disaggregation and translation can be processed. This procedure is executed recursively until it reaches the finest level.

### 2.3 Hybrid OpenMP-CUDA Parallel Programming Model and GPU Architecture

Before the OpenMP-CUDA-MLFMA is presented in detail, the key features of the hybrid OpenMP-CUDA parallel programming model and the GPU/CUDA architecture are reviewed in this section.

The OpenMP programming model is based on the shared memory multicore CPU architecture [76], and CUDA is developed for the shared memory many-core GPUs architecture [58]. A typical OpenMP-CUDA programming model is shown in Fig. 2.1. In general the program consists of one or more phases $[58,76]$. The serial phase of the program is first executed by the master thread on the host (CPU). Then multiple GPUs labeled as devices take over
the work in the parallel phase. Specifically, multiple CPU worker threads are allocated by an OpenMP instruction, and each worker thread manages one device, which is used to execute the data-parallel functions called kernels. When a kernel is launched, a large number of GPU threads are generated to exploit data parallelism. Those threads are organized into a two-dimensional (2-D) grid of blocks, with each block built by 2-D or 3-D array of threads. All of those threads generated by the kernel will carry out the same instructions during the parallel phase. With enough threads in a kernel to execute the same code simultaneously, the latency hiding mechanism [57,58, 71] can be fully utilized to make the parallel computation highly efficient. After every device finishes its parallel computation, the CPU will pick up the runtime and execute the instructions in the serial phase.

It should be emphasized that the significant computational efficiency of GPU results from its specific hierarchical architecture and excellent memory bandwidth. To elaborate the GPU acceleration of MLFMA, an understanding of the hardware architecture of NVIDIA GPUs is necessary. Figure 2.2 shows the architecture of a typical CUDA-capable GPU [57, 58], which is organized into an array of streaming multiprocessors (SMs). Each SM will be assigned with a portion of total thread blocks when a kernel is invoked. Off the chip, all the SMs in one device share a very high bandwidth memory called global memory and a high speed read-only memory called constant memory. The lifetime of variables in the global memory and constant memory is the entire application unless they are freed by the programmer. On the chip, each SM contains a number of streaming processors (SPs) which share control logic, cache, and shared memory. Each SP has its own small number of registers which usually store the private and frequently accessed variables because they can be accessed very quickly.

Usually the size of device memory and on-chip memories are not enough to solve large problems. One remedy is to use multi-GPU, and the other is to use pinned memory. Pinned memory is a special host memory, which is also called page-locked memory. One important property of this memory is that the operating system guarantees pinned memory will never be paged out to disk. Besides, the pinned memory has approximately twice the performance of the standard pageable memory when it is used for transferring data between the host and the device. Moreover, the pinned memory with a mapped memory syntax has a property that the system will automatically overlap
data transfers with kernel execution. However, the transfer speed is restricted by the peripheral component interconnect express (PCIe) transfer speed and the system front-side bus speed [77]. Therefore, the full utilization of the hierarchical memory and reduction of data communications are crucial in GPU computation.

Recognizing that the GPUs are well-suited in dealing with massive data parallelism and weak at executing with logical instructions while the CPUs are optimized for sequential instruction performance, one should implement a CEM code to execute the numerically intensive parts on the GPUs and the sequential parts on the CPU. A well-investigated coordinating strategy can make the GPU-incorporated MLFMA much more efficient than the purely CPU-implemented MLFMA (CPU-MLFMA).

### 2.4 The OpenMP-CUDA-MLFMA Algorithm

The implementation of the OpenMP-CUDA-MLFMA algorithm contains two main parts. One is the calculation of the near-field system matrix $\boldsymbol{Z}_{\text {near }}$, and the other is the evaluation of the far-field interaction $\boldsymbol{Z}_{\text {far }} \boldsymbol{I}$, which includes aggregation, translation, and disaggregation phases. For the near-field interaction $\boldsymbol{Z}_{\text {near }} \boldsymbol{I}$, the CUSPARSE (a set of basic linear algebra subroutines used for handling sparse matrices) is used directly. In this section, the multi-GPU implementation of the near-field system matrix assembly is first detailed. Then we present the parallelization strategy for the calculation of the farfield interaction. Finally, the pinned-memory multi-GPU implementation is presented.

### 2.4.1 Near-Field System Matrix Assembly

In MLFMA, the computationally intensive parts before the iterative solution include the calculation of radiation patterns of the basis functions $\boldsymbol{V}_{\mathrm{s}}$ and receiving patterns of the testing functions $\boldsymbol{V}_{\mathrm{f}}$, the calculation of translator $\boldsymbol{T}$, and the assembly of the near-field system matrix $\boldsymbol{Z}_{\text {near }}$. Here, we focus on the parallelization scheme for the assembly of the near-field system matrix.

Since the order of basis indices in each group at the finest level is sorted so that the indices of basis functions in each group are continuous, the near-
field system matrix $\boldsymbol{Z}_{\text {near }}$ has the pattern shown in Fig. 2.3a. This blocksparse matrix can be separated into two types of block matrices. The solid ones in blue represent block diagonal matrices which come from self-group interactions. The ones marked by red dash lines represent block off-diagonal matrices which come from neighboring-group interactions.

In order to satisfy the requirements for memory coalescing on GPU [57,58], the block matrices are stored in the COO (coordinates list) format which is shown in Fig. 2.3b. The non-zero elements are stored in the array A. The testing stream IA and the basis stream JA contain the information of the testing and basis functions respectively. To implement the matrix assembly on multiple GPUs, the non-zero array A is first separated equally into different devices. An one-dimensional (1-D) grid of threads is allocated for each device to compute a portion of non-zero elements. During the execution, each thread first fetches the data from the testing stream and the basis stream in the global memory, then calculates a non-zero element following the standard steps of MoM, and finally stores the value back to the global memory. The independence of the non-zero elements ensures the efficiency of the hierarchical parallelization regardless of the geometrical shape of the object.

### 2.4.2 Parallelization on Far-Field Interaction

The parallelization strategy for calculating the far-field interaction $\boldsymbol{Z}_{\text {far }} \boldsymbol{I}$ can be implemented by parallelly computing the radiation patterns and receiving patterns of the groups, denoted as $\boldsymbol{S}$ and $\boldsymbol{B}$, in the aggregation, translation, and disaggregation phases. The basic idea is "one thread per spectrum sampling" and "one/several block(s) per group." The hierarchical parallelization by simultaneously partitioning groups and their FFPs ensures a high computational throughput for the GPU calculation.

To be specific, take the aggregation phase for example. The 2-D grids and blocks are allocated for the calculation at each level. Based on our CUDA implementation and the GPU's technical specifications [57], the size of the thread block in the algorithm is set as the size of the spectrum at the finest level for the optimal use of hardware resources. Because the mode number is a function of $k d$ [4], where $d$ denotes the maximum diameter of a group and
$k$ denotes the wavenumber, the spectrum size at the $(L-1)$ th level (finest level) will be $6 \times 12$ if the length of cubes at the finest level is set to $0.3 \lambda$, and the spectrum size is increased by a factor of 4 as it goes to coarser levels. Figure 2.4 shows the thread allocation at the $(L-2)$ th level in the aggregation phase. The size of spectrum at this level is $12 \times 24$, where 12 and 24 are the number of spectrum samplings in the $\theta$ and $\varphi$ directions respectively. Thus 4 blocks should be organized to represent one parent cube, in which one thread is corresponding to one spectrum. At the $(L-3)$ th level the spectrum size is $24 \times 48$, then 16 blocks are assigned to represent one parent cube and so on. In this way, there will be a sufficient number of threads allocated at each level for the parallel computation, which leads to a high parallel efficiency.

There are two strategies to implement this parallelization idea. One is called global memory strategy; the other is called pinned memory strategy. The global memory strategy requires the radiation and receiving patterns to be calculated and stored at all levels on a single GPU. Such strategy avoids data transfer between the host and device during aggregation, which yields a very high computational efficiency. However, the size of the global memory will limit the size of problems that can be solved. The pinned memory strategy calculates the radiation and receiving patterns on multiple GPUs, and stores the results to the pinned memory on the host. The benefit of using pinned memory is that we can solve larger problems because the pinned memory is much larger than the global memory. But the data communications between the host and device become unavoidable. To develop the capability of solving large problems, the pinned memory strategy is presented and discussed in detail.

### 2.4.3 Multi-GPU Implementation Using Pinned Memory

Consider $\boldsymbol{S}_{i-1}$ at level $i-1$ aggregated from $\boldsymbol{S}_{i}$ at level $i$ as shown in Fig. 2.5a. The $\boldsymbol{S}_{i}$ stands for the array lined up with all the groups' radiation patterns at level $i$. To facilitate the computation, $\boldsymbol{S}_{i-1}$ is equally partitioned into different devices by the number of the groups. The thread allocation in each device is determined by the size of spectrum at level $i-1$ and the number of parent groups stored in the device. Each device accesses the data from $\boldsymbol{S}_{i}$ stored in the pinned memory, and calculates a part of $\boldsymbol{S}_{i-1}$.

Then the results from all the devices are stored back to the pinned memory consecutively. Each matrix-vector product using pinned memory has the implicit data transfer between the host and device.

The multi-GPU implementation in the translation phase is similar to that in the aggregation phase. Consider $\boldsymbol{S}_{i}$ translating to $\boldsymbol{B}_{i}$ at the same level $i$ as shown in Fig. 2.5b. The $\boldsymbol{B}_{i}$ stands for the array which consists of all the groups' receiving patterns at level i. $\boldsymbol{S}_{i}$ is equally divided into different devices by the number of groups. The thread allocation in each device is determined by the size of spectrum at level $i$ and the number of groups stored in the device. Similarly, each device accesses the data from $\boldsymbol{S}_{i}$ stored in the pinned memory, and calculates a portion of $\boldsymbol{B}_{i}$. Then the results are stored back to the pinned memory for the use of disaggregation.

The disaggregation phase of MLFMA is very similar to the aggregation phase. As shown in Fig. 2.5c, the partition strategy, thread allocation rule, and data communication process are all similar to the ones in the aggregation phase.

### 2.5 Numerical Analysis

In this section, a variety of numerical examples are presented to demonstrate the accuracy and efficiency of the OpenMP-CUDA-MLFMA. The CRWG [73,74] functions are used as basis and testing functions to discretize CFIE ( $\alpha=0.5$ ). All the numerical examples are solved by the biconjugate gradient stabilized method (BiCGStab) with a targeted relative residual error of $10^{-3}$. The single-precision floating-point arithmetic is used. The CPU-MLFMA is parallelized and executed by 8 threads on a workstation with a 4 -core Intel Xeon processor W3550 (with a clock speed of 3.06 GHz ). According to our tests, the 8 -threaded computation can achieve around 6.5 times speedup as compared to the single-threaded serial computation. The OpenMP-CUDAMLFMA is executed on a 4-GPU system equipped with 4 Nvidia Tesla C2050 GPUs.

### 2.5.1 Validation of the OpenMP-CUDA-MLFMA

Scattering by a Cone-Sphere With a Gap
A benchmark model, which is a metallic cone-sphere with a gap at the joint, is simulated to validate the OpenMP-CUDA-MLFMA. The object is 0.689m long, oriented in the $z$-direction, and illuminated by a $3-\mathrm{GHz}$ plane wave. Its surface is discretized into 5006 curvilinear triangular patches with 7509 unknowns. The HH-polarized monostatic radar cross section (RCS) in the $x z$-plane is computed, and as can be seen in Fig. 2.6a, a good agreement between the CPU-MLFMA, the OpenMP-CUDA-MLFMA, and the measured data is achieved. Figure 2.6 b shows the real part of the current density induced on the surface of the scatterer. The variation of the current density can easily be observed.

Scattering by a NASA Almond
The next testing benchmark object is a NASA almond with a size of 25.24 $\mathrm{cm} \times 9.75 \mathrm{~cm} \times 3.25 \mathrm{~cm}$. Illuminated by a $9-\mathrm{GHz}$ plane wave, the almond is discretized into 11134 curvilinear triangular patches, resulting in 16701 unknowns. Figure 2.7a shows the HH-polarized monostatic RCS in the $x y$ plane calculated by the CPU-MLFMA and the OpenMP-CUDA-MLFMA. The measured data are used as reference. Both results agree well with the measured data. The real part of the current density induced by the incident wave is shown in Fig. 2.7b.

In both the cone-sphere and almond examples, the OpenMP-CUDA-MLFMA and CPU-MLFMA results are nearly identical to each other, which indicates that there is no loss of accuracy in the GPU computation.

Scattering by a Missile-like Object
First we consider the electromagnetic scattering of a missile-like object which has a 3 -m-long body and 1 -cm-thick wings. The nonuniform mesh is employed to discretize the object into 228158 curvilinear triangular patches, leading to 342237 unknowns. Figure 2.8a shows the HH-polarized bistatic RCS in the $x z$-plane, which demonstrates a good agreement between the
results from the OpenMP-CUDA-MLFMA and the CPU-MLFMA. The real part of the current density induced on the surface of the missile-like object is shown in Fig. 2.8b, in which the wave phenomenon can be observed clearly. The speedup is summarized in Table 2.1, in which "CPU" and "GPU" represent the CPU-MLFMA and the OpenMP-CUDA-MLFMA, respectively. In this example, the same parallelization scheme is applied to the calculation of $\boldsymbol{V}_{\mathrm{s}}$ and $\boldsymbol{V}_{\mathrm{f}}$, the translator factor $\boldsymbol{T}$, and the assembly of $\boldsymbol{Z}_{\text {near }}$, which leads to the same speedups for the pinned memory and global memory cases. For the acceleration of the near-field system matrix assembly, a 60 times speedup is achieved. The BiCGstab solution is parallelized using the pinned memory and the global memory strategies respectively, which leads to 4.2 and 16.1 times speedup for the corresponding strategies. The global memory strategy is faster than the pinned memory strategy because there are no data communications between the host and device when calculating the far-field interaction. However, the global memory has the limited size on GPU so that it cannot solve larger problems. Therefore, our discussion is based on the pinned memory strategy in the following larger examples.

## Scattering by Conducting Spheres

In order to demonstrate the capability and efficiency of the OpenMP-CUDAMLFMA, the scattering from perfect electrically conducting (PEC) spheres with diameters of $4 \lambda, 6 \lambda, 12 \lambda$, and $30 \lambda$ are calculated. The multi-GPU acceleration of the different parts in the MLFMA are investigated as shown in Fig. 2.9a. As can be seen, excellent speedup is achieved in the nearfield system matrix assembly, which increases as the number of unknowns grows. For different numbers of unknowns, the acceleration in the BiCGstab solution remains the same because the data communications between the host and device take the majority of the time, which is determined by the PCIe and front-side bus speed. The total speedup increases a little bit as the number of unknowns increases, because the BiCGstab solution will take a larger portion of the total time. The total speedup achieved is between 10 and 14 times. The HH-polarized bistatic RCS for the $30 \lambda$ sphere with over 1 million unknowns is shown in Fig. 2.9b. The results calculated by the CPUMLFMA and the OpenMP-CUDA-MLFMA agree well with the Mie series solution. The detailed speedup for the $30 \lambda$ sphere is presented in Table 2.2.

The speedup for the near-field system matrix assembly is over 106 times, which is significant. The speedup of the BiCGstab iterative solution is 2.8 times, which is restricted by the data communications between the host and device.

Scattering by an Aircraft
To further illustrate the capability and efficiency of the proposed method, a simplified aircraft is considered. The aircraft, with a length of 12.74 m , a width of 15.06 m , and a height of 2.95 m , is illuminated by plane waves with frequencies of $200 \mathrm{MHz}, 400 \mathrm{MHz}, 780 \mathrm{MHz}$, and 1.5 GHz respectively. The speedup versus the number of unknowns is shown in Fig. 2.10a. In the figure, similar speedups with those for the PEC sphere can be observed. The total speedup is between 5.6 and 8.1 times. Figure 2.10b shows the VVpolarized bistatic RCS calculated by the CPU-MLFMA and the OpenMP-CUDA-MLFMA, respectively. The results are on the top of each other. The real part of the current density induced on the surface of the aircraft is shown in Fig. 2.10c from three different view angles. It is easy to observe the current density variation on the surface of the aircraft. Table 2.3 gives the detailed speedup performance for the computation at 1.5 GHz . The speedup for the near-field system matrix assembly is over 124 times, and the speedup of the BiCGstab iterative solution is 2.9 times. Comparing Table 2.3 with Table 2.2, it is easy to observe that for the problems with a similar number of unknowns, the speedup for each part is similar. In other words, the parallelization scheme is insensitive to the geometrical shape of the object.

### 2.6 Summary

In this chapter, an OpenMP-CUDA based implementation of MLFMA is presented for computing wave scattering problems of 3-D conducting objects on GPU computing systems. For parallelization on a single GPU, a hierarchical parallelization scheme is used by partitioning groups and their FFPs simultaneously. For multi-GPU implementation, a hybrid OpenMP-CUDA parallel programming model is employed. The OpenMP-CUDA-MLFMA is first validated by calculating the monostatic RCS for several benchmark problems.

Larger problems are then solved to demonstrate the capability and efficiency of the proposed algorithm. The near-field system matrix assembly using multi-GPU has an excellent efficiency, which has a speedup independent of the object geometry. For the parallelization of the far-field interaction, by the analysis of the GPU architecture and the numerical results, it is revealed that the global memory strategy is suitable for the fast solution of small problems, and the pinned memory strategy can be employed effectively to accelerate the computation of large problems. The total speedup of the OpenMP-CUDAMLFMA achieved is between 5 and 20 times as compared to the 8-threaded CPU-MLFMA, which can be quite important for practical applications.

### 2.7 Figures and Tables



Figure 2.1: A typical OpenMP-CUDA parallel programming model.


Figure 2.2: A typical CUDA-capable NVIDIA GPU architecture.


Figure 2.3: Near-field system matrix assembly. (a) Pattern of the near-field system matrix. (b) Process of matrix filling.



| Block <br> $(0,2 M)$ | Block <br> $(0,2 M+1)$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Block <br> $(1,2 M)$ | Block <br> $(1,2 M+1)$ |  |  |  |  |  |
| Cube $M$ |  |  |  |  |  |  |

Figure 2.4: Thread allocation for the aggregation phase at the (L-2)th level.


Figure 2.5: Implementation of far-field interaction on multi-GPU. (a) Parallel scheme for aggregation. (b) Parallel scheme for translation. (c) Parallel scheme for disaggregation.


Figure 2.6: Scattering analysis of a cone-sphere with a gap at 3 GHz . The total length of this object is 0.689 m . (a) HH-polarized monostatic RCS in the $x z$-plane. (b) Real part of the current density with the incidence angle $\theta=0^{\circ}$ and $\phi=0^{\circ}$ (in linear scale).


Figure 2.7: Scattering analysis of a NASA almond at 9 GHz . The size of this object is $25.24 \mathrm{~cm} \times 9.75 \mathrm{~cm} \times 3.25 \mathrm{~cm}$. (a) HH-polarized monostatic RCS in the $x y$-plane. (b) Real part of the current density with the incidence angle $\theta=90^{\circ}$ and $\phi=180^{\circ}$ (in linear scale).


Figure 2.8: Scattering analysis of a missile-like object. The length of the body is 3 m , and the thickness of the wing is 1 cm . A 3 GHz plane wave is incident from the angle $\theta=0^{\circ}$ and $\phi=0^{\circ}$. (a) HH-polarized bistatic RCS in the $x z$-plane (step size is $0.25^{\circ}$ ). (b) Real part of the current density induced on the surface of the scatterer (in linear scale).


Figure 2.9: Scattering analysis of the PEC spheres with diameters of $4 \lambda$, $6 \lambda, 12 \lambda$, and $30 \lambda$. (a) 4 -device speedup of the OpenMP-CUDA-MLFMA versus the number of unknowns (the number of unknowns is 18162, 41316, 168333 , and 1063155 respectively). (b) HH-polarized bistatic RCS of the $30 \lambda$ PEC sphere (step size is $0.5^{\circ}$ ).


Figure 2.10: Scattering analysis of the aircraft at frequencies of 200 MHz , $400 \mathrm{MHz}, 780 \mathrm{MHz}$, and 1.5 GHz . (a) 4 -device speedup of the OpenMP-CUDA-MLFMA versus the number of unknowns (the number of unknowns is 20319, 70413, 269859, and 1001946 respectively). (b)
VV-polarized bistatic RCS in the $y z$-plane at 1.5 GHz (step size is $0.25^{\circ}$ ). (c) Real part of the surface current density at 1.5 GHz with the incidence angle $\theta=60^{\circ}$ and $\phi=270^{\circ}$ (in linear scale).

Table 2.1: Speedup of bistatic RCS calculation of a missile-like object at 3 GHz

|  | $\begin{aligned} & \text { CPU } \\ & \text { (sec.) } \end{aligned}$ | GPU (sec.) |  | Speedup |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Pinned | Global | Pinned | Global |
| $\boldsymbol{V}_{\mathrm{s}}$ and $\boldsymbol{V}_{\mathrm{f}}$ | 17 | 7 |  | 2.4 |  |
| $T$ | 4 | 1 |  | 4.0 |  |
| $Z_{\text {near }}$ | 3299 | 55 |  | 60.0 |  |
| BiCGstab | 7569 | 1811 | 469 | 4.2 | 16.1 |
| Total Time | 10889 | 1879 | 539 | 5.8 | 20.2 |

Table 2.2: Speedup of bistatic RCS calculation of a PEC sphere with diameter of $30 \lambda$

|  | CPU (sec.) | GPU (sec.) | Speedup |
| :---: | :---: | :---: | :---: |
| $\boldsymbol{V}_{\mathrm{s}}$ and $\boldsymbol{V}_{\mathrm{f}}$ | 55 | 19 | 2.9 |
| $\boldsymbol{T}$ | 22 | 0.5 | 44.0 |
| $\boldsymbol{Z}_{\text {near }}$ | 4490 | 42 | 106.9 |
| BiCGstab | 933 | 334 | 2.8 |
| Total Time | 5500 | 400 | 13.8 |

Table 2.3: Speedup of bistatic RCS calculation of an aircraft at 1.5 GHz

|  | CPU (sec.) | GPU (sec.) | Speedup |
| :---: | :---: | :---: | :---: |
| $\boldsymbol{V}_{\mathrm{s}}$ and $\boldsymbol{V}_{\mathrm{f}}$ | 52 | 19 | 2.8 |
| $\boldsymbol{T}$ | 44 | 1 | 44.0 |
| $\boldsymbol{Z}_{\text {near }}$ | 3735 | 30 | 124.5 |
| BiCGstab | 1911 | 653 | 2.9 |
| Total Time | 5742 | 705 | 8.1 |

## CHAPTER 3

# A FINITE ELEMENT-BOUNDARY INTEGRAL METHOD WITH GPU ACCELERATION 

### 3.1 Introduction

The hybrid finite element-boundary integral (FE-BI) method has been developed and widely used in the analysis of unbounded electromagnetic scattering and radiation from complicated structures and composite materials. This method not only combines the capability of the finite element method (FEM) in modeling highly complicated geometry and inhomogeneous materials, but also eliminates the truncation error from absorbing boundary conditions (ABCs) and perfectly matched layers by enforcing a rigorous boundary condition through boundary integral equations (BIEs) [1, 72].

A variety of FE-BI formulations have been developed to solve three-dimensional electromagnetic problems in the past $[16,18,19,22,23,26,27,30,33,34]$. As a common approach to constructing an FE-BI formulation, the solution domain of the electromagnetic problem is first divided into an interior and an exterior region. In the interior region, the problem is formulated using the FEM truncated by the Neumann boundary condition, which leads to an underdetermined matrix equation in terms of the unknown volume electric field and surface electric current. In the exterior region, the electric-field integral equation (EFIE), the magnetic-field integral equation (MFIE), or the combined-field integral equation (CFIE) can be used to formulate another underdetermined matrix equation in terms of the unknown surface electric and magnetic currents. These two matrix equations are then coupled together to form a complete set of non-symmetric linear equations to solve for the volume and surface unknowns concurrently. Among different formulations, the FE-BI(CFIE) is widely used in real-life applications and proved to be free of spurious interior resonance. However, the traditional approach of discretizing the FE-BI(CFIE) produces a numerical solution with a poor
accuracy, caused by the improper testing of the surface currents when the tangential electric field is significant on the surface $[1,19]$.

To obtain a more accurate numerical solution, symmetric FE-BI formulations have been developed [26, 27]. For instance, to construct the FE-BI(EJ) formulation, the MFIE is first substituted into the finite element formulation for the interior problem. The EFIE is then used to formulate the exterior problem and coupled with the interior formulation to form a complete set of equations. The numerical solution of such a symmetric FE-BI formulation is found to be very accurate, but the iterative convergence is quite slow due to the mathematical property of the first-kind operators involved in the BI part.

Recently, the Buffa-Christiansen (BC) function [78] has been adopted as the testing function to discretize the integral equations for solving perfectly electrical conducting (PEC) and dielectric problems [79-83]. The numerical accuracy of the second-kind integral equations can be improved by using the BC testing functions, mainly because of the fact that the discretization error from the identity operator is successfully suppressed [80]. To achieve an FE-BI solution with a better numerical accuracy and a faster iterative convergence, a mixed testing scheme together with a special preconditioner are designed in this chapter for the FE-BI(CFIE) formulation. Specifically, the curl-conforming vector functions, the Rao-Wilton-Glisson (RWG) functions [73, 74], and the BC functions are applied as the testing functions to discretize the FEM formulation, the EFIE, and the MFIE, respectively, which are then combined to obtain a complete FE-BI system. To reduce the memory requirement and speed up the matrix-vector products (MVPs), the multilevel fast multipole algorithm (MLFMA) [4] is employed for the BIEs. Furthermore, an efficient preconditioner modified from the ABC-based preconditioner [24] is proposed to accelerate the iterative convergence of the FE-BI(CFIE) formulation.

Although the MLFMA accelerated BIE-related computation has $O\left(N_{\mathrm{s}} \log N_{\mathrm{s}}\right)$ computational and storage complexities, with $N_{\mathrm{s}}$ being the total number of surface unknowns, the cost of the BIE-related computation is still prohibitively high for solving large electromagnetic problems. To alleviate this difficulty, parallel computing techniques have been widely employed to reduce the total computation time. Among these techniques, parallel programming on graphics processing units (GPUs) has received intensive attention because of

GPU's high computational efficiency. To combine the advantages of the fast algorithm and the advancements of computer hardware, a GPU-accelerated MLFMA for simulating scattering from PEC objects has been developed on multi-GPU computing systems [12, 84]. To solve electromagnetic problems with complex structures and composite materials efficiently, a GPUaccelerated FE-BI algorithm with the MLFMA (GPU-FE-BI-MLFMA) is presented in this chapter.

### 3.2 FE-BI Formulation

In this section, the hybrid FE-BI formulation for electromagnetic problems is constructed and discretized into the matrix equation, with a proper choice of testing functions. The MLFMA is then applied to the proposed FE-BI method to accelerate the computation and reduce the memory requirement.

### 3.2.1 Matrix Equation

Consider an arbitrarily shaped inhomogeneous object characterized by relative permittivity and permeability $\left(\epsilon_{\mathrm{r}}, \mu_{\mathrm{r}}\right)$, which is illuminated by an external incident field $\left(\boldsymbol{E}^{\text {inc }}, \boldsymbol{H}^{\text {inc }}\right)$ and excited by an internal source $\boldsymbol{J}_{\text {src }}$. To solve this electromagnetic problem using the FE-BI method, the object is enclosed by an artificial surface $S$, which divides the problem into an interior and an exterior region. The fields in the interior region satisfy the vector wave equation

$$
\begin{equation*}
\nabla \times\left(\frac{1}{\mu_{\mathrm{r}}} \nabla \times \boldsymbol{E}\right)-k_{0}^{2} \epsilon_{\mathrm{r}} \boldsymbol{E}=-\mathrm{j} k_{0} \overline{\boldsymbol{J}}_{\mathrm{src}} \tag{3.1}
\end{equation*}
$$

and the Neumann boundary condition

$$
\begin{equation*}
\hat{\boldsymbol{n}} \times\left(\frac{1}{\mu_{\mathrm{r}}} \nabla \times \boldsymbol{E}\right)=-\mathrm{j} k_{0} \hat{\boldsymbol{n}} \times \overline{\boldsymbol{H}} \tag{3.2}
\end{equation*}
$$

with $\overline{\boldsymbol{H}}=Z_{0} \boldsymbol{H}$ and $\overline{\boldsymbol{J}}_{\text {src }}=Z_{0} \boldsymbol{J}_{\text {src }}$, where $k_{0}$ and $Z_{0}$ denote the free-space wavenumber and impedance, respectively, and $\hat{\boldsymbol{n}}$ represents the outwardly directed unit normal vector. The weak-form of the boundary-value problem
defined by (3.1) and (3.2) can be obtained as

$$
\begin{align*}
\int_{V} & {\left[\frac{1}{\mu_{\mathrm{r}}}\left(\nabla \times \boldsymbol{N}_{i}\right) \cdot(\nabla \times \boldsymbol{E})-k_{0}^{2} \epsilon_{\mathrm{r}} \boldsymbol{N}_{i} \cdot \boldsymbol{E}\right] \mathrm{d} V } \\
& +\mathrm{j} k_{0} \oint_{S} \hat{\boldsymbol{n}} \cdot\left(\boldsymbol{N}_{i} \times \overline{\boldsymbol{H}}\right) \mathrm{d} S=-\mathrm{j} k_{0} \int_{V} \boldsymbol{N}_{i} \cdot \overline{\boldsymbol{J}}_{\text {src }} \mathrm{d} V \tag{3.3}
\end{align*}
$$

where $\boldsymbol{E}$ and $\overline{\boldsymbol{H}}$ are the unknown electric and magnetic fields in the domain $V$ enclosed by $S$, and $\boldsymbol{N}_{i}$ denotes the curl-conforming testing functions such as edge basis functions [1].

For the exterior region, the fields satisfy the EFIE and the MFIE, which are given by

$$
\begin{align*}
\boldsymbol{E} & =\boldsymbol{E}^{\mathrm{inc}}-\mathcal{L}(\overline{\boldsymbol{J}})+\mathcal{K}(\boldsymbol{M})  \tag{3.4}\\
\overline{\boldsymbol{H}} & =\overline{\boldsymbol{H}}^{\mathrm{inc}}-\mathcal{K}(\overline{\boldsymbol{J}})-\mathcal{L}(\boldsymbol{M}) \tag{3.5}
\end{align*}
$$

where $\overline{\boldsymbol{J}}$ and $\boldsymbol{M}$ denote the unknown surface electric and magnetic current densities, respectively; and the integral operators $\mathcal{L}$ and $\mathcal{K}$ are defined as

$$
\begin{align*}
& \mathcal{L}(\boldsymbol{v}) \equiv \mathrm{j} k_{0} \oint_{S}\left(\mathcal{I}+\frac{\nabla \nabla}{k^{2}}\right) \frac{e^{-\mathrm{j} k_{0} R}}{4 \pi R} \cdot \boldsymbol{v}\left(\boldsymbol{r}^{\prime}\right) \mathrm{d} S^{\prime}  \tag{3.6}\\
& \mathcal{K}(\boldsymbol{v}) \equiv \oint_{S} \boldsymbol{v}\left(\boldsymbol{r}^{\prime}\right) \times \nabla \frac{e^{-\mathrm{j} k_{0} R}}{4 \pi R} \mathrm{~d} S^{\prime} \tag{3.7}
\end{align*}
$$

where $\mathcal{I}$ is the identity operator and $R=\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|$. Testing the EFIE with $\boldsymbol{T}_{i}^{\mathrm{E}}$ yields

$$
\begin{align*}
& \oint_{S} \boldsymbol{T}_{i}^{\mathrm{E}} \cdot \boldsymbol{E} \mathrm{~d} S+\oint_{S} \boldsymbol{T}_{i}^{\mathrm{E}} \cdot \mathcal{L}(\overline{\boldsymbol{J}}) \mathrm{d} S \\
& -\oint_{S} \boldsymbol{T}_{i}^{\mathrm{E}} \cdot \mathcal{K}(\boldsymbol{M}) \mathrm{d} S=\oint_{S} \boldsymbol{T}_{i}^{\mathrm{E}} \cdot \boldsymbol{E}^{\text {inc }} \mathrm{d} S \tag{3.8}
\end{align*}
$$

and testing the $\hat{\boldsymbol{n}} \times$ MFIE with $\boldsymbol{T}_{i}^{\mathrm{M}}$ yields

$$
\begin{align*}
& \oint_{S} \boldsymbol{T}_{i}^{\mathrm{M}} \cdot(\hat{\boldsymbol{n}} \times \overline{\boldsymbol{H}}) \mathrm{d} S+\oint_{S} \boldsymbol{T}_{i}^{\mathrm{M}} \cdot[\hat{\boldsymbol{n}} \times \mathcal{K}(\overline{\boldsymbol{J}})] \mathrm{d} S \\
& +\oint_{S} \boldsymbol{T}_{i}^{\mathrm{M}} \cdot[\hat{\boldsymbol{n}} \times \mathcal{L}(\boldsymbol{M})] \mathrm{d} S=\oint_{S} \boldsymbol{T}_{i}^{\mathrm{M}} \cdot\left(\hat{\boldsymbol{n}} \times \overline{\boldsymbol{H}}^{\mathrm{inc}}\right) \mathrm{d} S \tag{3.9}
\end{align*}
$$

To avoid the problem of interior resonance, the CFIE is adopted to formulate the exterior problem, and its mixed testing scheme can be obtained by
summing up (3.8) and (3.9). Hence, the current densities and fields can be globally solved from (3.3) and a combination of (3.8) and (3.9). To couple FE and BI equations on conformal meshes, we can expand the unknown current densities and fields as

$$
\begin{align*}
\overline{\boldsymbol{J}} & =\sum_{j=1}^{N_{\mathrm{S}}} \bar{H}_{j}^{\mathrm{S}} \boldsymbol{T}_{j}  \tag{3.10}\\
\boldsymbol{M} & =-\sum_{j=1}^{N_{\mathrm{S}}} E_{j}^{\mathrm{S}} \boldsymbol{T}_{j}  \tag{3.11}\\
\boldsymbol{E} & =\sum_{j=1}^{N_{\mathrm{S}}} E_{j}^{\mathrm{S}} \boldsymbol{N}_{j}+\sum_{j=1}^{N_{\mathrm{I}}} E_{j}^{\mathrm{I}} \boldsymbol{N}_{j}  \tag{3.12}\\
\overline{\boldsymbol{H}} & =\sum_{j=1}^{N_{\mathrm{S}}} \bar{H}_{j}^{\mathrm{S}} \boldsymbol{N}_{j} \tag{3.13}
\end{align*}
$$

where $\boldsymbol{T}_{j}$ and $\boldsymbol{N}_{j}$ are the RWG and the edge basis function [1], respectively; $N_{\mathrm{S}}$ and $N_{\mathrm{I}}$ denote the number of degrees of freedom (DoFs) on and inside $S$, respectively; $E_{j}^{\mathrm{S}}, E_{j}^{\mathrm{I}}$, and $\bar{H}_{j}^{\mathrm{S}}$ are the expansion coefficients yet to be determined. By substituting (3.10)-(3.13) to (3.3), (3.8) and (3.9), the coupled system equation can be obtained as

$$
\left[\begin{array}{ccc}
K^{\mathrm{II}} & K^{\mathrm{IS}} & 0  \tag{3.14}\\
K^{\mathrm{SI}} & K^{\mathrm{SS}} & B \\
0 & P & Q
\end{array}\right]\left\{\begin{array}{c}
E^{\mathrm{I}} \\
E^{\mathrm{S}} \\
\bar{H}^{\mathrm{S}}
\end{array}\right\}=\left\{\begin{array}{c}
b^{\mathrm{I}} \\
b^{\mathrm{S}} \\
b^{\mathrm{inc}}
\end{array}\right\}
$$

where

$$
\begin{equation*}
K_{i j}^{\mathrm{XY}}=\int_{V}\left[\frac{1}{\mu_{\mathrm{r}}}\left(\nabla \times \boldsymbol{N}_{i}^{\mathrm{X}}\right) \cdot\left(\nabla \times \boldsymbol{N}_{j}^{\mathrm{Y}}\right)-k_{0}^{2} \epsilon_{\mathrm{r}} \boldsymbol{N}_{i}^{\mathrm{X}} \cdot \boldsymbol{N}_{j}^{\mathrm{Y}}\right] \mathrm{d} V \tag{3.15}
\end{equation*}
$$

in which X and Y can be either I or S, and

$$
\begin{align*}
B_{i j}= & \mathrm{j} k_{0} \oint_{S} \hat{\boldsymbol{n}} \cdot\left(\boldsymbol{N}_{i}^{\mathrm{S}} \times \boldsymbol{N}_{j}^{\mathrm{S}}\right) \mathrm{d} S  \tag{3.16}\\
P_{i j}= & \alpha \oint_{S}\left(\hat{\boldsymbol{n}} \times \boldsymbol{T}_{i}^{\mathrm{E}}\right) \cdot\left[\frac{1}{2} \boldsymbol{T}_{j}+\hat{\boldsymbol{n}} \times \tilde{\mathcal{K}}\left(\boldsymbol{T}_{j}\right)\right] \mathrm{d} S \\
& +\beta \oint_{S}\left(\hat{\boldsymbol{n}} \times \boldsymbol{T}_{i}^{\mathrm{M}}\right) \cdot \mathcal{L}\left(\boldsymbol{T}_{j}\right) \mathrm{d} S  \tag{3.17}\\
Q_{i j}= & \beta \oint_{S} \boldsymbol{T}_{i}^{\mathrm{M}} \cdot\left[\frac{1}{2} \boldsymbol{T}_{j}+\hat{\boldsymbol{n}} \times \tilde{\mathcal{K}}\left(\boldsymbol{T}_{j}\right)\right] \mathrm{d} S
\end{align*}
$$

$$
\begin{align*}
& +\alpha \oint_{S} \boldsymbol{T}_{i}^{\mathrm{E}} \cdot \mathcal{L}\left(\boldsymbol{T}_{j}\right) \mathrm{d} S  \tag{3.18}\\
b_{i}^{\mathrm{I}}= & -\mathrm{j} k_{0} \int_{\Omega} \boldsymbol{N}_{i}^{\mathrm{I}} \cdot \overline{\boldsymbol{J}}_{\text {src }} \mathrm{d} V  \tag{3.19}\\
b_{i}^{\mathrm{S}}= & -\mathrm{j} k_{0} \int_{\Omega} \boldsymbol{N}_{i}^{\mathrm{S}} \cdot \overline{\boldsymbol{J}}_{\text {src }} \mathrm{d} V  \tag{3.20}\\
b_{i}^{\mathrm{inc}}= & \alpha \oint_{S} \boldsymbol{T}_{i}^{\mathrm{E}} \cdot \boldsymbol{E}^{\mathrm{inc}} \mathrm{~d} S+\beta \oint_{S} \boldsymbol{T}_{i}^{\mathrm{M}} \cdot\left(\hat{\boldsymbol{n}} \times \overline{\boldsymbol{H}}^{\mathrm{inc}}\right) \mathrm{d} S \tag{3.21}
\end{align*}
$$

where $\tilde{\mathcal{K}}$ is the principal value of the operator $\mathcal{K}$, and $\alpha$ and $\beta$ are the combination factors chosen as $\alpha=\beta=-2 \mathrm{j} k_{0}$. For brevity, we write (3.14) symbolically as

$$
\begin{equation*}
([A]+[G])\{x\}=\{b\} \tag{3.22}
\end{equation*}
$$

where $[A]$ denotes the finite element sparse matrix assembled from (3.15) and (3.16), $[G]$ denotes the boundary integral full matrix assembled from (3.17) and (3.18), $\{x\}$ represents the unknown solution vector, and $\{b\}$ is the known excitation vector given by (3.19)-(3.21).

Compared with the traditional approach to constructing the FE-BI(CFIE) [1,72], the proposed method first discretizes the FE equation, the EFIE, and the MFIE individually, and then combines them into a complete system, which provides freedom to choose proper testing functions for different equations.

### 3.2.2 Choice of the Testing Functions

To obtain an accurate solution from the FE-BI(CFIE) algorithm, the operators contained in the diagonal block $[Q]$ in (3.14) have to be well tested by carefully choosing the testing functions $\boldsymbol{T}_{i}^{\mathrm{E}}$ and $\boldsymbol{T}_{i}^{\mathrm{M}}$.

In general, there are two kinds of vector functions which can be chosen as testing functions: the divergence- and the curl-conforming functions. The typical divergence-conforming functions are the RWG [73,74] and the BC function [78], which are denoted as $\boldsymbol{f}^{\mathrm{r}}$ and $\boldsymbol{f}^{\mathrm{b}}$, respectively. The typical curl-conforming functions can be obtained by rotating the RWG and the BC function with respect to the normal direction, which are denoted as $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\mathrm{r}}$ and $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\mathrm{b}}$, respectively. If we choose $\boldsymbol{T}_{i}^{\mathrm{E}}=\boldsymbol{T}_{i}^{\mathrm{M}}=\boldsymbol{f}_{i}^{\mathrm{r}}$, the resulting FE-BI solution has a poor numerical accuracy, and becomes even worse when it
deals with dielectric objects or objects with a thick coating [1, 19, 72]. One reason for this inaccuracy is the troublesome evaluation of the divergence of curl-conforming basis functions, which results in a contour integral in the third term of $(3.17)[1,80]$. The other reason is that $\boldsymbol{f}^{\mathrm{r}}$ is not a good testing function for the operator $\hat{\boldsymbol{n}} \times \mathcal{K}$ in (3.18) [79-83]. To remove these problems, a simple approach is to set the coefficient $\beta$ to zero, which reduces the FEBI system to the FE-BI(EFIE) with the EFIE tested by $\boldsymbol{f}^{\mathrm{r}}$. Although the FE-BI(EFIE) method can provide an accurate solution, it suffers from the interior resonance corruption $[1,72]$. A better way to overcome this problem is to choose proper testing functions for both the EFIE and the MFIE based on the mathematical properties of the integral operators.

Mathematically, the proper basis and testing functions for an integral operator should be in the domain and the dual of the range of the integral operator, respectively [85]. Because the operators $\hat{\boldsymbol{n}} \times \mathcal{L}$ and $\hat{\boldsymbol{n}} \times \mathcal{K}$ map a space of divergence-conforming functions onto itself, their dual of the range is the space of the curl-conforming functions $[78,86]$. In the finite-element space, if $\boldsymbol{f}^{\mathrm{r}}$ is chosen as the basis function, a good candidate to expand the dual of the range of $\hat{\boldsymbol{n}} \times \mathcal{L}$ and $\hat{\boldsymbol{n}} \times \mathcal{K}$ is $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\mathrm{r}}$ and $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\text {b }}$, respectively $[78,80,82,87]$. Hence, to test the $\mathcal{I}, \tilde{\mathcal{K}}$, and $\mathcal{L}$ operators in (3.18) well, the testing functions should be chosen as $\boldsymbol{T}_{i}^{\mathrm{E}}=\boldsymbol{f}_{i}^{\mathrm{r}}$ and $\boldsymbol{T}_{i}^{\mathrm{M}}=\hat{\boldsymbol{n}} \times \boldsymbol{f}_{i}^{\mathrm{b}}$. As a result, $P_{i j}, Q_{i j}$, and $b_{i}^{\text {inc }}$ can be rewritten as

$$
\begin{align*}
P_{i j}= & -\frac{\alpha}{2} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot\left(\hat{\boldsymbol{n}} \times \boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S+\alpha \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot \tilde{\mathcal{K}}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S \\
& -\beta \oint_{S} \boldsymbol{f}_{i}^{\mathrm{b}} \cdot \mathcal{L}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S  \tag{3.23}\\
Q_{i j}= & -\frac{\beta}{2} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{b}} \cdot\left(\hat{\boldsymbol{n}} \times \boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S+\beta \oint_{S} \boldsymbol{f}_{i}^{\mathrm{b}} \cdot \tilde{\mathcal{K}}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S \\
& +\alpha \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot \mathcal{L}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S  \tag{3.24}\\
b_{i}^{\mathrm{inc}}= & \alpha \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot \boldsymbol{E}^{\mathrm{inc}} \mathrm{~d} S+\beta \oint_{S} \boldsymbol{f}_{i}^{\mathrm{b}} \cdot \overline{\boldsymbol{H}}^{\text {inc }} \mathrm{d} S . \tag{3.25}
\end{align*}
$$

It can be observed from (3.23) that no contour integral exists anymore when the Gauss divergence theorem is applied to its third term [80]. Therefore, with the proposed testing scheme, not only the contour integral in the evaluation of the matrix $[P]$ is avoided, but also the $\mathcal{I}, \tilde{\mathcal{K}}$, and $\mathcal{L}$ operators in the matrix $[Q]$ are all well tested.

### 3.2.3 Application of the MLFMA

To reduce the computational and storage complexities, the MLFMA [4] is applied to evaluate the matrix entries and accelerate the MVPs. The nearfield interactions in the BIE-related matrices are the same as those given in (3.23) and (3.24), and the far-field interactions can be calculated by

$$
\begin{align*}
& P_{i j}=-\frac{\mathrm{j} k_{0}}{4 \pi} \oint \boldsymbol{V}_{f m i}^{P} \cdot \alpha_{m m^{\prime}}\left(\hat{\boldsymbol{k}}, \hat{\boldsymbol{r}}_{m m^{\prime}}\right) \boldsymbol{V}_{s m^{\prime} j} \mathrm{~d}^{2} \hat{\boldsymbol{k}}  \tag{3.26}\\
& Q_{i j}=\frac{\mathrm{j} k_{0}}{4 \pi} \oint \boldsymbol{V}_{f m i}^{Q} \cdot \alpha_{m m^{\prime}}\left(\hat{\boldsymbol{k}}, \hat{\boldsymbol{r}}_{m m^{\prime}}\right) \boldsymbol{V}_{s m^{\prime} j} \mathrm{~d}^{2} \hat{\boldsymbol{k}} \tag{3.27}
\end{align*}
$$

where

$$
\begin{align*}
\boldsymbol{V}_{f m i}^{P}= & \oint_{S} e^{-\mathrm{j} \boldsymbol{k}_{0} \cdot \boldsymbol{r}_{i m}}\left[\alpha \hat{\boldsymbol{k}} \times \boldsymbol{f}_{i}^{\mathrm{r}}\left(\boldsymbol{r}_{i m}\right)\right. \\
& \left.+\beta(\mathcal{I}-\hat{\boldsymbol{k}} \hat{\boldsymbol{k}}) \cdot \boldsymbol{f}_{i}^{\mathrm{b}}\left(\boldsymbol{r}_{i m}\right)\right] \mathrm{d} S  \tag{3.28}\\
\boldsymbol{V}_{f m i}^{Q}= & \oint_{S} e^{-\mathrm{j} \boldsymbol{k}_{0} \cdot \boldsymbol{r}_{i m}}\left[\alpha(\mathcal{I}-\hat{\boldsymbol{k}} \hat{\boldsymbol{k}}) \cdot \boldsymbol{f}_{i}^{\mathrm{r}}\left(\boldsymbol{r}_{i m}\right)\right. \\
& \left.-\beta \hat{\boldsymbol{k}} \times \boldsymbol{f}_{i}^{\mathrm{b}}\left(\boldsymbol{r}_{i m}\right)\right] \mathrm{d} S  \tag{3.29}\\
\boldsymbol{V}_{s m^{\prime} j}= & \oint_{S} e^{\mathrm{j} \boldsymbol{k}_{0} \cdot \boldsymbol{r}_{j m^{\prime}}} \boldsymbol{f}_{j}^{r}\left(\boldsymbol{r}_{j m^{\prime}}\right) \mathrm{d} S^{\prime} \tag{3.30}
\end{align*}
$$

and the translation operator is given by

$$
\begin{align*}
\alpha_{m m^{\prime}}\left(\hat{\boldsymbol{k}}, \hat{\boldsymbol{r}}_{m m^{\prime}}\right)= & \frac{-\mathrm{j} k_{0}}{4 \pi} \sum_{l=0}^{L}(-\mathrm{j})^{l}(2 l+1) h_{l}^{(2)}\left(k_{0} r_{m m^{\prime}}\right) \\
& \cdot P_{l}\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}}_{m m^{\prime}}\right) \tag{3.31}
\end{align*}
$$

where $h_{l}^{(2)}$ denotes the spherical Hankel function of the second kind, $P_{l}$ denotes the Legendre polynomial of degree $l$, and $L$ represents the number of multipole expansion terms.

### 3.3 Efficient ABC-Based Preconditioner

To solve a linear system of equations with a very large number of unknowns, iterative methods are usually employed due to their lower computational and memory costs compared with direct methods. Unfortunately, due to the poor
condition of the FE-BI system matrix, it is impractical to apply an iterative method to solve the matrix equation directly. To achieve a highly convergent solution, an efficient preconditioner is required. For the FE-BI(CFIE) formulation, such an efficient preconditioner can be constructed by replacing the integral equations with their local approximations

$$
\begin{align*}
& \boldsymbol{E} \approx \boldsymbol{E}^{\mathrm{inc}}-\mathcal{L}^{\prime}(\overline{\boldsymbol{J}})+\mathcal{K}^{\prime}(\boldsymbol{M})  \tag{3.32}\\
& \overline{\boldsymbol{H}} \approx \overline{\boldsymbol{H}}^{\mathrm{inc}}-\mathcal{K}^{\prime}(\overline{\boldsymbol{J}})-\mathcal{L}^{\prime}(\boldsymbol{M}) \tag{3.33}
\end{align*}
$$

on the truncation surface, where the operators $\mathcal{L}^{\prime}$ and $\mathcal{K}^{\prime}$ denote only the self-patch interactions. By using the RWG function as both the testing and basis functions, and scaling (3.32) and (3.33) by a factor of $-2 \mathrm{j} k_{0}$, the preconditioner can be obtained as

$$
([A]+[M])=\left[\begin{array}{ccc}
K^{\mathrm{II}} & K^{\mathrm{IS}} & 0  \tag{3.34}\\
K^{\mathrm{SI}} & K^{\mathrm{SS}} & B \\
0 & P^{\prime} & Q^{\prime}
\end{array}\right]
$$

where $[A]$ is the same as that given in (3.22), $[M]$ denotes the sparse matrix obtained from the discretization of the locally approximated integral operators, and

$$
\begin{align*}
P_{i j}^{\prime}= & -2 \mathrm{j} k_{0} \oint_{S}\left(\hat{\boldsymbol{n}} \times \boldsymbol{f}_{i}^{\mathrm{r}}\right) \cdot\left[\frac{1}{2} \boldsymbol{f}_{j}^{\mathrm{r}}+\hat{\boldsymbol{n}} \times \tilde{\mathcal{K}}^{\prime}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right)\right] \mathrm{d} S \\
& -2 \mathrm{j} k_{0} \oint_{S}\left(\hat{\boldsymbol{n}} \times \boldsymbol{f}_{i}^{\mathrm{r}}\right) \cdot \mathcal{L}^{\prime}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S  \tag{3.35}\\
Q_{i j}^{\prime}= & -2 \mathrm{j} k_{0} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot\left[\frac{1}{2} \boldsymbol{f}_{j}^{\mathrm{r}}+\hat{\boldsymbol{n}} \times \tilde{\mathcal{K}}^{\prime}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right)\right] \mathrm{d} S \\
& -2 \mathrm{j} k_{0} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot \mathcal{L}^{\prime}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S \tag{3.36}
\end{align*}
$$

where $\tilde{\mathcal{K}}^{\prime}$ is the principal value of the operator $\mathcal{K}^{\prime}$. From the spectrum analysis of the integral operators, it can be seen that the last two terms in (3.35) and the second term in (3.36) are negligible because their contributions to the spectrum vanish if the testing and basis functions lie in the same plane. Therefore, the $P^{\prime}$ and $Q^{\prime}$ matrices can be further simplified as

$$
\begin{equation*}
P_{i j}^{\prime}=\mathrm{j} k_{0} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot\left(\hat{\boldsymbol{n}} \times \boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S \tag{3.37}
\end{equation*}
$$

$$
\begin{equation*}
Q_{i j}^{\prime}=-\mathrm{j} k_{0} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot \boldsymbol{f}_{j}^{\mathrm{r}} \mathrm{~d} S-2 \mathrm{j} k_{0} \oint_{S} \boldsymbol{f}_{i}^{\mathrm{r}} \cdot \mathcal{L}^{\prime}\left(\boldsymbol{f}_{j}^{\mathrm{r}}\right) \mathrm{d} S \tag{3.38}
\end{equation*}
$$

With all the approximations and simplifications, the proposed preconditioner $([A]+[M])$ eventually becomes a symmetric sparse matrix. This preconditioner can also be obtained by scaling the original ABC-based preconditioner [24] by a factor of $\mathrm{j} k_{0}$ and adding back the locally approximated operator $\mathcal{L}^{\prime}$ to $Q^{\prime}$. It is well known that the spectrum distribution of the $\mathcal{L}$ operator clusters at origin and infinity $[86,88]$. By adding back the $\mathcal{L}^{\prime}$ term, the modified ABC-based preconditioner is closer to the FE-BI system than the original ABC-based preconditioner in terms of the spectrum distribution.

To demonstrate the effectiveness of the modified ABC-based preconditioner, the scattering from a conducting sphere with a dielectric coating $\left(\epsilon_{\mathrm{r}}=4.0-\mathrm{j} 1.0\right)$ is considered as an example. The total diameter of the sphere is 0.6846 m , and the coating thickness is 0.047 m . Illuminated by a $300-\mathrm{MHz}$ plane wave, the object is discretized using curvilinear tetrahedrons with an average mesh size of 0.1 m , which leads to a total of 7,297 unknowns. With the BI surface chosen to be the same as that of the object, the spectrum distributions of the unpreconditioned FE-BI system $([A]+[G])$, the FE-BI system with the original ABC-based preconditioner $([A]+[G])([A]+[L])^{-1}[24]$, and the FE-BI system with the modified ABCbased preconditioner $([A]+[G])([A]+[M])^{-1}$ are presented and compared in Fig. 3.1a. As can be seen from the figure, the FE-BI system with the modified preconditioner has eigenvalues clustered within a smaller region centered around $(1,0)$ in the complex plane. The convergence history of the iterative solution using the biconjugate gradient stabilized method (BiCGSTAB) shown in Fig. 3.1b indicates that the system with the modified ABC-based preconditioner has a better convergence than the one with the original ABCbased preconditioner. From the inset of Fig. 3.1b, it can be seen that the bistatic RCS obtained using the proposed mixed testing scheme is on top of the Mie series solution.

To achieve the same convergence residual as that of the unpreconditioned system, the right preconditioner is adopted in this chapter. By applying $([A]+[M])$ as the right preconditioner on (3.22), the preconditioned system equation can be obtained as

$$
\begin{equation*}
([A]+[G])([A]+[M])^{-1}\{u\}=\{b\} . \tag{3.39}
\end{equation*}
$$

Once the auxiliary unknown vector $\{u\}$ is obtained, the unknown vector $\{x\}$ can be recovered by one more application of the preconditioner matrix $\{x\}=([A]+[M])^{-1}\{u\}$. The application of the preconditioner can be carried out by solving a matrix equation

$$
\begin{equation*}
([A]+[M])\{y\}=\{u\} \tag{3.40}
\end{equation*}
$$

with either a direct or an iterative solver. If a direct solver is used, the LU decomposition is applied to $([A]+[M])$, and the forward and back substitutions are performed to solve the matrix equation. However, for large problems, the LU decomposition will result in a large storage and computational cost. Therefore, an iterative method is preferred for large problems. To differentiate the two iterative processes, the one performed on the preconditioned FE-BI system is called the outer iteration, and the one applied to (3.40) is called the inner iteration. Similar to the FE-BI matrix $([A]+[G])$, the preconditioner $([A]+[M])$ has a poor condition. Therefore, another preconditioner has to be used to accelerate the convergence of the inner iteration. Different preconditioners for the inner iteration, such as the incomplete LU (ILU) decomposition, the approximate inverse (AI), and the Jacobi preconditioners [89], will be discussed in the following sections.

### 3.4 GPU Acceleration

Although the proposed FE-BI-MLFMA algorithm with the ABC-based preconditioner is capable of solving electromagnetic problems accurately and efficiently, the computation can be further accelerated by GPU parallel computing techniques. The GPU-accelerated FE-BI-MLFMA algorithm contains three major parts. The first part is the assembly of the BIE-related matrices. The second part is the iterative solution, which includes the acceleration of the inner and outer iterations. The third part is the evaluation of the scattered fields.

The assembly of the BIE-related matrices can be parallelized on multiple GPUs by hybridizing OpenMP and NVIDIA's CUDA (compute united device architecture) parallel programming models [58]. Since the MLFMA is applied, only the near-field interactions in the matrices $[P]$ and $[Q]$ need
to be assembled. For each matrix, as can be seen in Fig. 3.2, a massive number of GPU threads in each device are assigned to carry out the assembly of a portion of the matrix, and each thread computes one entry in the matrix [12, 84].

To accelerate the iterative solution using GPUs, special care should be taken on the evaluation of the far-field interaction of the BIE-related part, which includes the aggregation, the translation, and the disaggregation phases. These procedures can be accelerated by computing the radiation patterns $\boldsymbol{S}$ and the receiving patterns $\boldsymbol{B}$ on each level in parallel. To achieve a maximum parallel efficiency on all the levels, the implementation strategy of "one thread per spectrum sampling" and "one/several block(s) per group" is adopted to ensure the groups and their far-field patterns partitioned simultaneously $[12,84]$. Another important issue is to develop an inner iteration scheme which can be parallelized efficiently on GPUs. Here, the GPU-accelerated BiCGSTAB with the ILU0, the AI, and the Jacobi preconditioners are employed to solve (3.40) [90-92]. The ILU0 preconditioner requires roughly the same amount of memory as the input matrix $([A]+[M])$, but the forward and back substitutions make it inefficient for parallel processing on GPUs. The AI preconditioner is an incomplete approximation of the inverse of the input matrix based on the minimization of the Frobenius norm. In contrast to the ILU-based preconditioners, the AI preconditioner is applied using MVPs, which makes it very suitable for GPU parallelization. The Jacobi preconditioner is very cheap to generate and apply, but the iterative convergence is slow when the input matrix is ill-conditioned. The comparison of the convergence and efficiency between these three preconditioners will be given in Section 3.5.

Fast evaluation of the scattered fields is critical to the calculation of the RCS and radiation patterns. Since the basis functions can be regarded as the current sources from which the scattered fields are generated, a onedimensional grid of threads is allocated on the GPU, and each thread is related to one basis function. To evaluate the scattered field at a specific observation angle, each GPU thread calculates the corresponding scattered field in parallel, and one CPU thread superposes all the calculated scattered fields in series to avoid write conflicts on the GPU. To further accelerate the scattered field evaluation, the OpenMP parallel technique is employed to generate multiple CPU threads, and each CPU thread manages one GPU
device to calculate a portion of the scattered fields [84].

### 3.5 Numerical Analysis

In this section, the accuracy of the proposed FE-BI algorithm is first investigated. The interior resonance test is then performed using a coated conducting sphere. After that, different approaches of applying the modified ABC-based preconditioner are discussed in detail. Last, the efficiency and capability of the proposed GPU-FE-BI-MLFMA algorithm are demonstrated. All of the computations are performed on a workstation equipped with a 4-core Intel Xeon W3520 CPU (with a memory of 18 GB ) and 2 GeForce GTX 660 GPUs (with a global memory of 2 GB for each).

### 3.5.1 Accuracy Test

In this test, the RCS is calculated to study the numerical accuracy of the proposed FE-BI(CFIE) method, and the system of linear equations is solved by a direct method. The numerical error is measured in terms of the relative root-mean-square (RMS) error of the RCS results [83].

A Conducting Sphere with a Dielectric Coating
A dielectric coated PEC sphere with a total radius of 0.5 m is illuminated by a $300-\mathrm{MHz}$ plane wave. The $0.25-\mathrm{m}$-thick dielectric coating layer has a relative permittivity of $\epsilon_{\mathrm{r}}=4.0-\mathrm{j} 1.0$. Curvilinear tetrahedrons with an average mesh size of 0.05 m are used to discretize the coating layer, which results in 37, 915 FE DoFs and 4, 455 BI DoFs. The bistatic RCS is depicted in Fig. 3.3a, which shows that the results obtained by the FE-BI(EJ) and the proposed FE-BI(CFIE) methods all agree well with the Mie series solution. However, the result obtained from the traditional FE-BI(CFIE) method (the CFIE is tested by the RWG) is not as accurate as the proposed FE-BI(CFIE) method. The corresponding RMS errors are given in Table 3.1, from which it can be seen that the proposed and the FE-BI(EJ) methods have a similar accuracy, and both of them give more accurate results than the traditional FE-BI(CFIE) method.

A Dielectric Sphere
The next testing example is a lossless dielectric sphere with a relative permittivity of $\epsilon_{\mathrm{r}}=2.0$ and a radius of $a=1 \mathrm{~m}$. Illuminated by a $110-\mathrm{MHz}$ plane wave, the object is discretized into curvilinear tetrahedrons with an average size of 0.15 m , which results in 10, 182 FE DoFs and 1, 806 BI DoFs. From the bistatic RCS shown in Fig. 3.3b, it can be seen that the results from the FE-BI(EJ) and the proposed FE-BI(CFIE) methods are all in good agreement with the Mie series solution, and the traditional FE-BI(CFIE) method gives a much less accurate result, which can also be observed from the RMS errors presented in Table 3.2.

## A Dielectric Cuboid

As discussed in Section 3.2, the difference between the proposed FE-BI(CFIE) and the traditional FE-BI(CFIE) is the testing function for the MFIE. To illustrate the reason for the accuracy improvement, we calculate the RCS of a lossless dielectric cuboid using the FE-BI(EFIE) and the FE-BI(MFIE) methods with the same mesh discretization. For the FE-BI(EFIE), $\boldsymbol{f}^{r}$ is applied as the testing function for the EFIE. For the FE-BI(MFIE), $\boldsymbol{f}^{\mathrm{r}}$ and $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\text {b }}$ are employed to test the $\hat{\boldsymbol{n}} \times$ MFIE, respectively. The lossless dielectric cuboid has a size of $1.2 \mathrm{~m} \times 0.5 \mathrm{~m} \times 0.2 \mathrm{~m}$ with $\epsilon_{\mathrm{r}}=2.0$, and is illuminated by a plane wave of 310 MHz . The bistatic RCS is presented in Fig. 3.3c. It can be observed that the result from the FE-BI(MFIE) becomes as accurate as the one from the FE-BI(EFIE) after using $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{b}$ to test the $\hat{\boldsymbol{n}} \times$ MFIE. Therefore, having the MFIE well tested by using the BC function leads to the accuracy improvement for the FE-BI(CFIE) method.

Finally, three observations can be made from the presented examples in this section.

1) The results calculated by the proposed FE-BI(CFIE) method are as accurate as the ones calculated by the $\mathrm{FE}-\mathrm{BI}(\mathrm{EJ})$ method, and much more accurate than the ones calculated by the traditional FE-BI(CFIE) method.
2) The traditional FE-BI(CFIE) method has the problem of inaccuracy when it deals with dielectric objects or objects with a thick coating. In these cases, nontrivial tangential electric and magnetic fields exist simultaneously
on the truncation surface. If the tangential magnetic fields on this surface are not negligible compared with the tangential electric fields, then traditional FE-BI(CFIE) method cannot provide an accurate solution because the $\hat{\boldsymbol{n}} \times$ MFIE cannot be well tested by $\boldsymbol{f}^{\mathrm{r}}$.
3) The FE-BI(MFIE) method with the $\hat{\boldsymbol{n}} \times$ MFIE tested by $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\mathrm{b}}$ is as accurate as the FE-BI(EFIE) method with the EFIE tested by $\boldsymbol{f}^{r}$. Hence, the accuracy of the FE-BI(CFIE) method can be improved by using the mixed testing scheme.

### 3.5.2 Interior Resonance Test

Next, the interior resonance of the FE-BI(CFIE) method with the mixed testing scheme is studied. Theoretically, the FE-BI(CFIE) method has been proved to be free of spurious interior resonance corruption with the aid of the CFIE [93, 94]. As a numerical validation, the condition number of the proposed FE-BI(CFIE) method is calculated over a frequency band, and the accuracy of the solution is studied at the interior resonant frequency. The object considered is a conducting sphere with a dielectric coating $\left(\epsilon_{\mathrm{r}}=4.0\right)$. The total radius of the sphere is 1.0 m , and the coating thickness is 0.05 m . The lowest analytical resonant frequency of a spherical cavity formed by the outside surface of the coated sphere is 130.92 MHz . To capture the numerical resonant frequency, a frequency search is applied using the FE-BI(EFIE), the FE-BI(EJ), and the FE-BI(CFIE) methods under a discretization with an average mesh size of 0.15 m . As shown in Fig. 3.4a, the numerical resonant frequency captured by the $\mathrm{FE}-\mathrm{BI}$ (EFIE) method is 130.96 MHz , and there is no resonant frequency observed for the FE-BI(CFIE) and the FE-BI(EJ) methods. It can also be observed that the FE-BI(CFIE) method with the mixed testing scheme has the smallest condition number among all the methods.

The bistatic RCS is calculated using the FE-BI(EFIE) and the proposed $\mathrm{FE}-\mathrm{BI}(\mathrm{CFIE})$ methods at the numerical resonant frequency. It is evident from Fig. 3.4b that the accuracy of the FE-BI(EFIE) method is significantly compromised because of the spurious interior resonance corruption, while the result of the proposed FE-BI(CFIE) method remains very accurate. At such a frequency, the iterative solution of the proposed FE-BI(CFIE) method
takes only 4 iterations to converge to a targeted relative residual error of $10^{-3}$ by using the modified ABC-based preconditioner. Therefore, it can be concluded that the mixed testing scheme does not deteriorate the immunity of the interior resonance corruption of the FE-BI(CFIE) method.

### 3.5.3 Application of the Modified ABC-Based Preconditioner

To apply the ABC-based preconditioner $([A]+[M])$, one can solve (3.40) using either a direct or an iterative method. For the direct method, $([A]+[M])$ is first decomposed into a lower and an upper triangular matrix, and solved by the forward and back substitutions. For the iterative method, the ILU0, the AI, or the Jacobi preconditioned BiCGSTAB method with a targeted relative residual error of $10^{-3}$ is applied to solve (3.40).

To find the most efficient and effective way of applying the modified ABCbased preconditioner on GPUs, a comparative study is performed using a cylinder with 30 fins as a numerical example. This object has an impedance surface with a normalized surface impedance of $\eta_{\mathrm{r}}=0.5$, and is embedded in a larger dielectric cylinder with $\epsilon_{\mathrm{r}}=2.0-\mathrm{j} 1.0$. Illuminated by a $1.0-\mathrm{GHz}$ plane wave, the whole object is discretized into 379,234 FE DoFs and $32,628 \mathrm{BI}$ DoFs. The average construction and solution times and the average iteration counts for different methods are presented in Table 3.3. Compared with the iterative methods, the direct method is expensive in the construction phase, and cheap in the solution phase. However, the direct method will require a larger storage and a higher computational cost when the problems become large, and the solution is very difficult to parallelize on GPUs. Therefore, we consider only the iterative methods for the GPU calculation. It can be seen from the table that the AI preconditioned iterative method is most efficient in the solution phase, and the Jacobi preconditioned one is cheapest in the construction phase. The ILU0 preconditioned iterative method is not well suited for GPU parallelization because it requires the forward and back substitutions. With the aid of the modified ABC-based preconditioner, the outer iteration converges in 5 steps. As shown in Fig. 3.5, the HHpolarized bistatic RCS calculated by the CPU- and GPU-FE-BI-MLFMA and the discrete body-of-revolution (DBOR) algorithm [95] agree well with each other.

### 3.5.4 Efficiency and Capability

Next, the efficiency and capability of the proposed algorithm are demonstrated by applying all the techniques discussed in this chapter. The BI surface is chosen as the surface of the objects. The preconditioned BiCGSTAB method is applied with a targeted relative residual error of $10^{-3}$.

A Large Coated Sphere
First, the scattering from a large coated sphere with a total diameter of $25 \lambda_{0}$ ( $\lambda_{0}$ being the free-space wavelength) is simulated. The thickness of the coating dielectric is $0.1 \lambda_{0}$, and the relative permittivity of the coating is $\epsilon_{\mathrm{r}}=2.5-\mathrm{j} 0.5$. The object is discretized into curvilinear tetrahedrons with an average mesh size of $0.1 \lambda_{0}$, which results in 2, 697, 315 FE DoFs and 738,099 BI DoFs. From the HH-polarized bistatic RCS shown in Fig. 3.6a, it is evident that the numerical result of the proposed method has a good agreement with the Mie series solution. The corresponding RMS error is $1.47 \%$. The convergence history of the iterative solutions with the ABC-based preconditioners is given in Fig. 3.6b. Obviously, the iterative solution with the modified preconditioner converges faster than the one with the original preconditioner. The corresponding inner iteration with the Jacobi preconditioner takes an average of 180 and 143 iterations to converge, respectively. The total memory cost is about 12.6 GB .

## A Partial Human Body

The radiation of a $1.5-\mathrm{GHz}$ dipole beside a partial human body, as shown in Fig. 3.7a, is considered. The partial human body with a brain and two eyeballs has a total height of 392.3 mm and a total width of 539.6 mm . The relative permittivities $\epsilon_{\mathrm{r}}$ and mesh sizes $h$ of different parts are listed in Table 3.4. The nonuniform mesh is employed to discretize the object into $1,049,558$ FE DoFs and 52, 935 BI DoFs. The power pattern radiated by a dipole located at $x=0 \mathrm{~mm}, y=100 \mathrm{~mm}$, and $z=250 \mathrm{~mm}$ is calculated by the CPU-FE-BI-MLFMA and the GPU-FE-BI-MLFMA, respectively. The computational costs are summarized in Tables 3.5 and 3.6, which show that the total speedup achieved is 16.4 and 18.7 times, respectively, using two differ-
ent preconditioners for the inner iteration. The total memory cost is 7.6 GB with the Jacobi preconditioner and 7.9 GB with the AI preconditioner. The normalized radiation patterns are shown in Fig. 3.8. To demonstrate the advantage of the modified ABC-based preconditioner, the convergence history of the iterative solutions is given in Fig. 3.7b. It is shown that the iterative solution with the modified preconditioner takes far fewer iterations than the one with the original preconditioner. For the inner iteration with the AI preconditioner, the solution of the modified and the original preconditioners take an average of 180 and 167 steps to converge, respectively.

## A Dielectric Coated Missile-Like Object

A missile-like object with a 3 -m-long body and 1 -cm-thick wings is illuminated by a $1.5-\mathrm{GHz}$ plane wave. With a normalized impedance surface $\left(\eta_{\mathrm{r}}=0.5\right)$ and a 11-mm-thick coating layer $\left(\epsilon_{\mathrm{r}}=4.0-\mathrm{j} 1.0\right)$, the object is discretized into curvilinear tetrahedrons with an average mesh size of 10 m m, which results in 915, 833 FE DoFs and 187, 635 BI DoFs. The VV- and HH-polarized bistatic RCS are presented in Fig. 3.9a, which demonstrates a good agreement between the results from the CPU-FE-BI-MLFMA and the GPU-FE-BI-MLFMA. The computational costs are summarized in Tables 3.7 and 3.8 , from which it can be seen that a total speedup of 23.6 and 25.5 times is achieved, respectively. The total memory cost is 7.8 GB with the Jacobi preconditioner and 8.1 GB with the AI preconditioner. The convergence histories of the iterative solutions are presented in Fig. 3.9b, which shows again that the modified ABC-based preconditioner has a much faster convergence. For each outer iteration, the inner solutions of the modified and the original ABC-based preconditioners take an average of 104 and 211 iterations to converge when the AI preconditioner is applied to the inner iteration. Therefore, the modified ABC-based preconditioner is more efficient than the original one.

### 3.6 Summary

In this chapter, a GPU-accelerated FE-BI(CFIE) method was presented for three-dimensional electromagnetic analysis of complicated objects in free
space. First, a mixed testing scheme was applied to improve the accuracy of the numerical solution, so that the proposed FE-BI(CFIE) method can be as accurate as the $\mathrm{FE}-\mathrm{BI}(\mathrm{EJ})$ method, and much more accurate than the traditional FE-BI(CFIE) method. The proposed FE-BI(CFIE) method was also numerically shown to have a better condition than the conventional methods and be free of the interior resonance corruption. To accelerate the iterative convergence, a modified ABC-based preconditioner was proposed. Compared with the original ABC -based preconditioner, this modification not only made the preconditioner symmetric, but also improved the effectiveness significantly. To further speed up the computation, a GPU-accelerated FE-BI algorithm was developed. Extensive numerical experiments were conducted to validate the numerical accuracy and demonstrate the computational efficiency and capability. Through the numerical results, the proposed GPU-accelerated FE-BI(CFIE) method was shown to be accurate, robust, and efficient for practical applications.

### 3.7 Figures and Tables



Figure 3.1: A comparison of the ABC-based preconditioners through a dielectric coated sphere. (a) Spectrum distribution of $([A]+[G])$, $([A]+[G])([A]+[L])^{-1}$, and $([A]+[G])([A]+[M])^{-1}$. (b) Convergence histories of the BiCGTAB solution with a targeted relative error of $10^{-5}$.


Figure 3.2: Assembly of matrices $[P]$ and $[Q]$ on GPU.


Figure 3.3: Bistatic RCS of (a) a dielectric coated PEC sphere, (b) a lossless dielectric sphere, and (c) a lossless dielectric cuboid.


Figure 3.4: Interior resonance test of the proposed FE-BI(CFIE) method. (a) Condition numbers as a function of frequency in a small band around the analytical resonant frequency. (b) Bistatic RCS at the numerical resonant frequency.


Figure 3.5: HH-polarized bistatic RCS of a dielectric coated cylinder with fins at 1.0 GHz .


Figure 3.6: Scattering from a dielectric coated sphere with a total diameter of $25 \lambda_{0}$. (a) HH-polarized bistatic RCS. (b) Convergence histories of the BiCGSTAB method with a targeted relative error of $10^{-3}$.


Figure 3.7: Radiation of a $1.5-\mathrm{GHz}$ dipole beside a partial human body. (a) A partial human body with a brain and two eyeballs. (b) Convergence histories of the BiCGSTAB method with a targeted relative error of $10^{-3}$.


Figure 3.8: Normalized power pattern (in linear scale) of a $1.5-\mathrm{GHz}$ dipole beside a partial human body. (a) In the $y z$-plane. (b) In the $x y$-plane.


Figure 3.9: Scattering from a dielectric coated missile-like object at 1.5 GHz. (a) Bistatic RCS calculated by CPUs and GPUs, respectively. (b) Convergence histories of the BiCGSTAB method with a targeted relative error of $10^{-3}$.

Table 3.1: RMS errors of the bistatic RCS of a dielectric coated PEC sphere

|  | FE-BI(EJ) | FE-BI(CFIE) ${ }^{1}$ | FE-BI(CFIE) $^{2}$ |
| :--- | :---: | :---: | :---: |
| VV-pol. (\%) | 1.50 | 4.52 | 1.43 |
| HH-pol. (\%) | 1.48 | 7.91 | 1.40 |

${ }^{1}$ The CFIE is tested by the RWG.
${ }^{2}$ The mixed testing scheme is applied to the CFIE.

Table 3.2: RMS errors of the bistatic RCS of a lossless dielectric sphere

|  | FE-BI(EJ) | FE-BI(CFIE) ${ }^{1}$ | FE-BI(CFIE) $^{2}$ |
| :--- | :---: | :---: | :---: |
| VV-pol. (\%) | 1.95 | 21.19 | 2.01 |
| HH-pol. (\%) | 1.11 | 25.47 | 1.11 |

${ }^{1}$ The CFIE is tested by the RWG.
${ }^{2}$ The mixed testing scheme is applied to the CFIE.

Table 3.3: Construction and solution costs for different preconditioners

|  | LUD | ILU0 | AI | Jacobi |
| :---: | :---: | :---: | :---: | :---: |
| Const. ${ }^{1}$ (sec.) | 17.2 | 0.49 | 7.8 | 0.039 |
| Sol. ${ }^{1}$ (sec.) | 0.71 | 39.4 | 7.8 | 14.1 |
| Iterations ${ }^{1}$ | N/A | 175 | 65 | 159 |
| Const. $^{2}($ sec. $)$ | N/A | 0.18 | 7.7 | 0.0007 |
| Sol. $^{2}($ sec. $)$ | N/A | 14.2 | 1.3 | 2.1 |
| Iterations $^{2}$ | N/A | 189 | 62 | 164 |

${ }^{1}$ The 4-threaded Intel MKL pardiso routines are used in the LUD, and the 4 -threaded CULA Sparse host routines are used in the ILU0, the AI, and the Jacobi.
${ }^{2}$ The CULA Sparse device routines are used on one GPU.

Table 3.4: Material information and mesh size of a partial human body

|  | Brain | Eyes | Body |
| :---: | :---: | :---: | :---: |
| $\epsilon_{\mathrm{r}}$ | $37.5-\mathrm{j} 9.52$ | $54.0-\mathrm{j} 17.2$ | $15.0-\mathrm{j} 10.0$ |
| $h(\mathrm{~mm})$ | 3.2 | 2.7 | 5.0 |

Table 3.5: Computational costs of the power pattern calculation of a $1.5-\mathrm{GHz}$ dipole beside a partial human body

|  | Assy. | Precon. ${ }^{1}$ | Sol. | RCS | Total |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{CPU}^{2}$ (sec.) | 5746.0 | 0.09 | 2210.8 | 125.9 | 8082.8 |
| $\mathrm{GPU}^{3}$ (sec.) | 170.0 | 0.03 | 319.3 | 2.2 | 491.5 |
| Speedup | 33.8 | 3.0 | 6.9 | 57.2 | 16.4 |

${ }^{1}$ The Jacobi preconditioner is applied.
${ }^{2}$ The code is executed by 8 threads on an Intel Xeon W3520 CPU.
${ }^{3}$ The code is accelerated by 2 GeForce GTX 660 GPUs.

Table 3.6: Computational costs of the power pattern calculation of a $1.5-\mathrm{GHz}$ dipole beside a partial human body

|  | Assy. | Precon. | Sol. | RCS | Total |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{CPU}^{2}$ (sec.) | 5746.0 | 23.7 | 1100.0 | 125.9 | 6995.6 |
| $\mathrm{GPU}^{3}$ (sec.) | 170.0 | 23.0 | 177.0 | 2.2 | 373.2 |
| Speedup | 33.8 | 1.0 | 6.2 | 57.2 | 18.7 |

${ }^{1}$ The AI preconditioner is applied.
2 The code is executed by 8 threads on an Intel Xeon W3520 CPU.
${ }^{3}$ The code is accelerated by 2 GeForce GTX 660 GPUs.

Table 3.7: Computational costs of the bistatic RCS calculation of a coated missile-like object at 1.5 GHz

|  | Assy. | Precon. ${ }^{1}$ | Sol. | RCS | Total |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{CPU}^{2}$ (sec.) | 5868.6 | 0.07 | 920.0 | 218.9 | 7007.6 |
| $\mathrm{GPU}^{3}$ (sec.) | 161.0 | 0.02 | 130.8 | 4.4 | 296.4 |
| Speedup | 36.5 | 3.5 | 7.1 | 49.8 | 23.6 |

${ }^{1}$ The Jacobi preconditioner is applied.
${ }^{2}$ The code is executed by 8 threads on an Intel Xeon W3520 CPU.
${ }^{3}$ The code is accelerated by 2 GeForce GTX 660 GPUs.

Table 3.8: Computational costs of the bistatic RCS calculation of a coated missile-like object at 1.5 GHz

|  | Assy. | Precon. ${ }^{1}$ | Sol. | RCS | Total |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{CPU}^{2}$ (sec.) | 5868.6 | 18.4 | 549.6 | 218.9 | 6655.5 |
| $\mathrm{GPU}^{3}$ (sec.) | 161.0 | 18.8 | 76.7 | 4.4 | 260.9 |
| Speedup | 36.5 | 0.98 | 7.2 | 49.8 | 25.5 |

${ }^{1}$ The AI preconditioner is applied.
${ }^{2}$ The code is executed by 8 threads on an Intel Xeon W3520 CPU.
${ }^{3}$ The code is accelerated by 2 GeForce GTX 660 GPUs.

## CHAPTER 4

## A MULTI-SOLVER SCHEME BASED ON COMBINED FIELD INTEGRAL EQUATIONS

To analyze electrically large and/or highly complex objects, a multi-solver scheme based on combined field integral equations (CFIEs) is proposed in this chapter. In this scheme, an object is decomposed into multiple bodies. The choice of the individual solver for each body is based on the material property and geometry. The FE-BI method is applied to model bodies with complicated materials. The MoM is employed to model homogeneous or conducting bodies. To couple the individual solvers together, the CFIE is applied to the exterior regions of each body.

### 4.1 Formulation

Consider an object immersed in free space with permittivity $\epsilon_{0}$ and permeability $\mu_{0}$ and illuminated by an external incident field ( $\left.\boldsymbol{E}^{\mathrm{inc}}, \overline{\boldsymbol{H}}^{\mathrm{inc}}=Z_{0} \boldsymbol{H}^{\mathrm{inc}}\right)$, where $Z_{0}$ is the free-space impedance, as shown in Fig. 4.1. The object is decomposed into multiple bodies $\Omega_{s}\left(s=1, \ldots, N_{\mathrm{b}}\right)$ according to its material property. The surface of $\Omega_{s}$ is denoted as $S_{s}$, where the equivalent surface electric current $\overline{\boldsymbol{j}}_{s}$ and magnetic current $\boldsymbol{m}_{s}$ are defined. For each body, there is an exterior region denoted as region 0 and an interior region denoted as region $s$. The scattered electric and magnetic fields in region $d(d=0, s)$ generated by the currents $\overline{\boldsymbol{j}}_{s}$ and $\boldsymbol{m}_{s}$ on $S_{s}$ can be written as

$$
\begin{gather*}
\boldsymbol{E}_{d}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)=-\eta_{\mathrm{rd}} \mathcal{L}_{d}\left(\overline{\boldsymbol{j}}_{s} ; S_{s}\right)+\mathcal{K}_{d}\left(\boldsymbol{m}_{s} ; S_{s}\right)  \tag{4.1}\\
\overline{\boldsymbol{H}}_{d}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)=-\eta_{\mathrm{rd}} \mathcal{K}_{d}\left(\overline{\boldsymbol{j}}_{s} ; S_{s}\right)-\mathcal{L}_{d}\left(\boldsymbol{m}_{s} ; S_{s}\right) \tag{4.2}
\end{gather*}
$$

where $\overline{\boldsymbol{j}}_{s}=Z_{0} \hat{\boldsymbol{n}}_{s} \times \boldsymbol{H}_{s}, \boldsymbol{m}_{s}=\boldsymbol{E}_{s} \times \hat{\boldsymbol{n}}_{s}\left(\hat{\boldsymbol{n}}_{s}\right.$ is the unit normal vector pointing to region 0$), \eta_{\mathrm{r} d}=\sqrt{\mu_{\mathrm{r} d} / \epsilon_{\mathrm{r} d}}$, and the integral operators $\mathcal{L}_{d}$ and $\mathcal{K}_{d}$ are defined

$$
\begin{align*}
& \mathcal{L}_{d}\left(\boldsymbol{v} ; S_{s}\right)=\mathrm{j} k_{d} \int_{S_{s}}\left(\mathcal{I}+\frac{\nabla \nabla}{k_{d}^{2}}\right) \frac{\mathrm{e}^{-\mathrm{j} k_{d} R}}{4 \pi R} \cdot \boldsymbol{v}\left(\boldsymbol{r}^{\prime}\right) \mathrm{d} S^{\prime}  \tag{4.3}\\
& \mathcal{K}_{d}\left(\boldsymbol{v} ; S_{s}\right)=\int_{S_{s}} \boldsymbol{v}\left(\boldsymbol{r}^{\prime}\right) \times \nabla \frac{\mathrm{e}^{-\mathrm{j} k_{d} R}}{4 \pi R} \mathrm{~d} S^{\prime} \tag{4.4}
\end{align*}
$$

where $\mathcal{I}$ is the identity operator, $k_{d}$ is the wavenumber in region $d$ of $\Omega_{s}$, and $R=\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|$. Note that in (4.1) and (4.2) a homogeneous dielectric with a relative permittivity $\epsilon_{\mathrm{r} d}$ and permeability $\mu_{\mathrm{r} d}$ is assumed for the interior region $(d=s)$.

In the following subsections, we first present the formulation to model the exterior and interior regions of a body. Then we give the dual formulation. Finally, we discuss the choice of testing functions and solution strategies.

### 4.1.1 EM Modeling in the Exterior Region

All the bodies share the same exterior region, which is free space. Thus we can formulate the electromagnetic fields in region 0 for all the bodies uniformly. The currents $\overline{\boldsymbol{j}}_{s}$ and $\boldsymbol{m}_{s}$ on surface $S_{s}$ satisfy the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE) in region 0 as follows

$$
\begin{align*}
& -\boldsymbol{m}_{s}-\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}_{0}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right) \\
& \quad=\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}^{\mathrm{inc}}+\hat{\boldsymbol{n}}_{s} \times \sum_{q \neq s} \boldsymbol{E}_{0}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{q}, \boldsymbol{m}_{q} ; S_{q}\right)  \tag{4.5}\\
& \eta_{\mathrm{r} 0} \overline{\boldsymbol{j}}_{s}-\hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{0}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right) \\
& \quad=\hat{\boldsymbol{n}}_{s} \times \eta_{\mathrm{r} 0} \overline{\boldsymbol{H}}^{\mathrm{inc}}+\hat{\boldsymbol{n}}_{s} \times \sum_{q \neq s} \overline{\boldsymbol{H}}_{0}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{q}, \boldsymbol{m}_{q} ; S_{q}\right) . \tag{4.6}
\end{align*}
$$

Testing the EFIE with $\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{0}^{\alpha}$ yields

$$
\begin{align*}
& -\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{0}^{\alpha}, \boldsymbol{m}_{s}\right\rangle_{S_{s}}-\left\langle\boldsymbol{T}_{0}^{\alpha}, \boldsymbol{E}_{0}^{\text {sca }}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)\right\rangle_{S_{s}} \\
& \quad=\left\langle\boldsymbol{T}_{0}^{\alpha}, \boldsymbol{E}^{\mathrm{inc}}\right\rangle_{S_{s}}+\sum_{q \neq s}\left\langle\boldsymbol{T}_{0}^{\alpha}, \boldsymbol{E}_{0}^{\text {sca }}\left(\overline{\boldsymbol{j}}_{q}, \boldsymbol{m}_{q} ; S_{q}\right)\right\rangle_{S_{s}} \tag{4.7}
\end{align*}
$$

and testing the MFIE with $\boldsymbol{T}_{0}^{\beta}$ yields

$$
\begin{align*}
& \eta_{\mathrm{r} 0}\left\langle\boldsymbol{T}_{0}^{\beta}, \overline{\boldsymbol{j}}_{s}\right\rangle_{S_{s}}-\left\langle\boldsymbol{T}_{0}^{\beta}, \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{0}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)\right\rangle_{S_{s}} \\
& =\eta_{\mathrm{r} 0}\left\langle\boldsymbol{T}_{0}^{\beta}, \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}^{\mathrm{inc}}\right\rangle_{S_{s}} \\
& \quad+\sum_{q \neq s}\left\langle\boldsymbol{T}_{0}^{\beta}, \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{0}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{q}, \boldsymbol{m}_{q} ; S_{q}\right)\right\rangle_{S_{s}} \tag{4.8}
\end{align*}
$$

where the surface integral $\langle\bullet, \bullet\rangle_{S}$ is defined as

$$
\begin{equation*}
\langle\boldsymbol{u}, \boldsymbol{v}\rangle_{S}=\int_{S} \boldsymbol{u} \cdot \boldsymbol{v}^{\mathrm{T}} \mathrm{~d} S \tag{4.9}
\end{equation*}
$$

in which $\boldsymbol{u}$ and $\boldsymbol{v}$ are column vectors containing vector functions, and the superscript T denotes the transpose of a column vector. The choice of the testing functions $\boldsymbol{T}_{0}^{\alpha}$ and $\boldsymbol{T}_{0}^{\beta}$ will be discussed later in Section 4.1.4.

To obtain the matrix equation which can be solved numerically, the unknown currents are expanded as

$$
\begin{align*}
\overline{\boldsymbol{j}}_{s} & =\boldsymbol{T}_{s}^{\mathrm{T}} h_{s}  \tag{4.10}\\
\boldsymbol{m}_{s} & =-\boldsymbol{T}_{s}^{\mathrm{T}} e_{s} \tag{4.11}
\end{align*}
$$

where $\boldsymbol{T}_{s}$ is a column vector consisting of the basis functions, $e_{s}$ and $h_{s}$ are column vectors consisting of the expansion coefficients yet to be determined. To avoid the problem of spurious interior resonance, the CFIE is used to formulate the exterior problem. Hence, the matrix equation obtained by combining (4.7) and (4.8) with substitution of (4.10) and (4.11) can be written as

$$
\begin{align*}
& {\left[P_{s 0}\right]\left\{e_{s}\right\}+\eta_{\mathrm{r} 0}\left[Q_{s 0}\right]\left\{h_{s}\right\}} \\
& \quad=\left\{b_{s}^{\text {inc }}\right\}-\sum_{q \neq s}\left[C_{s q}^{e}\right]\left\{e_{q}\right\}-\eta_{\mathrm{r} 0} \sum_{q \neq s}\left[C_{s q}^{h}\right]\left\{h_{q}\right\} \tag{4.12}
\end{align*}
$$

where

$$
\begin{align*}
{\left[P_{s 0}\right]=} & \alpha_{0}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{0}^{\alpha}, \frac{1}{2} \boldsymbol{T}_{s}+\hat{\boldsymbol{n}}_{s} \times \tilde{\mathcal{K}}_{0}\left(\boldsymbol{T}_{s} ; S_{s}\right)\right\rangle_{S_{s}} \\
& +\beta_{0}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{0}^{\beta}, \mathcal{L}_{0}\left(\boldsymbol{T}_{s}, S_{s}\right)\right\rangle_{S_{s}}  \tag{4.13}\\
{\left[Q_{s 0}\right]=} & \alpha_{0}\left\langle\boldsymbol{T}_{0}^{\alpha}, \mathcal{L}_{0}\left(\boldsymbol{T}_{s} ; S_{s}\right)\right\rangle_{S_{s}}
\end{align*}
$$

$$
\begin{align*}
& +\beta_{0}\left\langle\boldsymbol{T}_{0}^{\beta}, \frac{1}{2} \boldsymbol{T}_{s}+\hat{\boldsymbol{n}}_{s} \times \tilde{\mathcal{K}}_{0}\left(\boldsymbol{T}_{s} ; S_{s}\right)\right\rangle_{S_{s}}  \tag{4.14}\\
{\left[C_{s q}^{e}\right]=} & \alpha_{0}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{0}^{\alpha}, \frac{1}{2} \boldsymbol{T}_{q}+\hat{\boldsymbol{n}}_{s} \times \tilde{\mathcal{K}}_{0}\left(\boldsymbol{T}_{q} ; S_{q}\right)\right\rangle_{S_{s}} \\
& +\beta_{0}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{0}^{\beta}, \mathcal{L}_{0}\left(\boldsymbol{T}_{q}, S_{q}\right)\right\rangle_{S_{s}}  \tag{4.15}\\
{\left[C_{s q}^{h}\right]=} & \alpha_{0}\left\langle\boldsymbol{T}_{0}^{\alpha}, \mathcal{L}_{0}\left(\boldsymbol{T}_{q} ; S_{q}\right)\right\rangle_{S_{s}} \\
& +\beta_{0}\left\langle\boldsymbol{T}_{0}^{\beta}, \frac{1}{2} \boldsymbol{T}_{q}+\hat{\boldsymbol{n}}_{s} \times \tilde{\mathcal{K}}_{0}\left(\boldsymbol{T}_{q} ; S_{q}\right)\right\rangle_{S_{s}}  \tag{4.16}\\
\left\{b_{s}^{\mathrm{inc}}\right\}= & \alpha_{0}\left\langle\boldsymbol{T}_{0}^{\alpha}, \boldsymbol{E}^{\mathrm{inc}}\right\rangle_{S_{s}}+\beta_{0} \eta_{\mathrm{r} 0}\left\langle\boldsymbol{T}_{0}^{\beta}, \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}^{\mathrm{inc}}\right\rangle_{S_{s}} \tag{4.17}
\end{align*}
$$

where $\tilde{\mathcal{K}}_{0}$ is the principal value of the operator $\mathcal{K}_{0}$, and $\alpha_{0}$ and $\beta_{0}$ are combination factors, which are both chosen as 1 in this chapter.

### 4.1.2 EM Modeling in the Interior Region

To model the electromagnetic fields in the interior region of a body, the most suitable formulation will be considered based on the material property of the body.

## Conducting Bodies

To model an imperfectly conducting body, the IBC is widely used to approximately describe the relation between $\overline{\boldsymbol{j}}_{s}$ and $\boldsymbol{m}_{s}$ on surface $S_{s}$ as

$$
\begin{equation*}
\hat{\boldsymbol{n}}_{s} \times \boldsymbol{m}_{s}-z_{s} \overline{\boldsymbol{j}}_{s}=0 \tag{4.18}
\end{equation*}
$$

where $z_{s}$ is the normalized surface impedance. Testing the IBC equation (4.18) with $\boldsymbol{T}_{s}^{\gamma}$ leads to

$$
\begin{equation*}
\left\langle\boldsymbol{T}_{s}^{\gamma}, \hat{\boldsymbol{n}}_{s} \times \boldsymbol{m}_{s}\right\rangle_{S_{s}}-z_{s}\left\langle\boldsymbol{T}_{s}^{\gamma}, \overline{\boldsymbol{j}}_{s}\right\rangle_{S_{s}}=0 \tag{4.19}
\end{equation*}
$$

where the choice of $\boldsymbol{T}_{s}^{\gamma}$ will be discussed in Section 4.1.4. By substituting (4.10) and (4.11) into (4.19), the matrix equation can be obtained as

$$
\begin{equation*}
\left[U_{s}\right]\left\{e_{s}\right\}-z_{s}\left[V_{s}\right]\left\{h_{s}\right\}=0 \tag{4.20}
\end{equation*}
$$

where

$$
\begin{align*}
& {\left[U_{s}\right]=\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}^{\gamma}, \boldsymbol{T}_{s}\right\rangle_{S_{s}}}  \tag{4.21}\\
& {\left[V_{s}\right]=\left\langle\boldsymbol{T}_{s}^{\gamma}, \boldsymbol{T}_{s}\right\rangle_{S_{s}} .} \tag{4.22}
\end{align*}
$$

When (4.20) is coupled with (4.12), a CFIE(IBC) matrix is obtained. By letting $z_{s}=0$, the CFIE(IBC) can be used to model a perfectly electric conducting (PEC) object.

Homogeneous Dielectric Bodies
If body $s$ is a homogeneous dielectric object, the currents $\overline{\boldsymbol{j}}_{s}$ and $\boldsymbol{m}_{s}$ on surface $S_{s}$ satisfy the EFIE and MFIE in region $s$ given by

$$
\begin{align*}
& -\boldsymbol{m}_{s}+\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}_{s}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)=0  \tag{4.23}\\
& \eta_{\mathrm{rs}} \overline{\boldsymbol{j}}_{s}+\hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{s}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)=0 . \tag{4.24}
\end{align*}
$$

By discretizing (4.23) and (4.24) in a similar manner as given in Section 4.1.1, the matrix equation for the interior region can be obtained as

$$
\begin{equation*}
\left[P_{s}\right]\left\{e_{s}\right\}+\eta_{\mathrm{rs}}\left[Q_{s}\right]\left\{h_{s}\right\}=0 \tag{4.25}
\end{equation*}
$$

where

$$
\begin{align*}
{\left[P_{s}\right]=} & \alpha_{s}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}^{\alpha},-\frac{1}{2} \boldsymbol{T}_{s}+\hat{\boldsymbol{n}}_{s} \times \tilde{\mathcal{K}}_{s}\left(\boldsymbol{T}_{s} ; S_{s}\right)\right\rangle_{S_{s}} \\
& +\beta_{s}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}^{\beta}, \mathcal{L}_{s}\left(\boldsymbol{T}_{s}, S_{s}\right)\right\rangle_{S_{s}}  \tag{4.26}\\
{\left[Q_{s}\right]=} & \alpha_{s} \eta_{\mathrm{r} s}\left\langle\boldsymbol{T}_{s}^{\alpha}, \mathcal{L}_{s}\left(\boldsymbol{T}_{s} ; S_{s}\right)\right\rangle_{S_{s}} \\
& +\beta_{s} \eta_{\mathrm{r} s}\left\langle\boldsymbol{T}_{s}^{\beta},-\frac{1}{2} \boldsymbol{T}_{s}+\hat{\boldsymbol{n}}_{s} \times \tilde{\mathcal{K}}_{s}\left(\boldsymbol{T}_{s} ; S_{s}\right)\right\rangle_{S_{s}} \tag{4.27}
\end{align*}
$$

in which $\tilde{\mathcal{K}}_{s}$ is the principal value of the operator $\mathcal{K}_{s}$, and $\alpha_{s}$ and $\beta_{s}$ are combination factors chosen as $\alpha_{s}=-1$ and $\beta_{s}=1$. Again, the choice of the testing functions $\boldsymbol{T}_{s}^{\alpha}$ and $\boldsymbol{T}_{s}^{\beta}$ will be discussed in Section 4.1.4. A CFIE(Dielectric) matrix can be obtained by coupling (4.12) with (4.25).

Inhomogeneous Bodies
To model an inhomogeneous body with complicated structures and materials, the FEM can be applied to the interior region. The resulting matrix equation can be written as

$$
\begin{equation*}
\left[K_{s}\right]\left\{E_{s}\right\}+\left[B_{s}\right]\left\{h_{s}\right\}=0 \tag{4.28}
\end{equation*}
$$

By coupling (4.28) with (4.12), the FE-BI(CFIE) matrix is obtained. The expressions for $\left[K_{s}\right]$ and $\left[B_{s}\right]$ can be found in [54].

### 4.1.3 Dual Formulation

Based on the formulation above, the matrix equation for body $s$ can be written uniformly as

$$
\begin{equation*}
\left[A_{s}\right][T]_{s}\{x\}+\sum_{q \neq s}\left[C_{s q}\right][T]_{q}\{x\}=[T]_{s}\{b\} \tag{4.29}
\end{equation*}
$$

where $\left[A_{s}\right]$ denotes the MoM matrix for body $s$ as

$$
\left[A_{s}\right]=\left[\begin{array}{cc}
U_{s} & -z_{s} V_{s}  \tag{4.30}\\
P_{s 0} & \eta_{\mathrm{r} 0} Q_{s 0}
\end{array}\right] \quad \text { or } \quad\left[\begin{array}{cc}
P_{s} & \eta_{\mathrm{rs}} Q_{s} \\
P_{s 0} & \eta_{\mathrm{r} 0} Q_{s 0}
\end{array}\right]
$$

or the FE-BI matrix for body $s$ as

$$
\left[A_{s}\right]=\left[\begin{array}{cc}
K_{s} & B_{s}  \tag{4.31}\\
P_{s 0} & \eta_{\mathrm{r} 0} Q_{s 0}
\end{array}\right]
$$

and $\left[C_{s q}\right]$ is the coupling matrix between bodies $s$ and $q$, which can be expressed as

$$
\left[C_{s q}\right]=\left[\begin{array}{cc}
0 & 0  \tag{4.32}\\
C_{s q}^{e} & \eta_{\mathrm{r} 0} C_{s q}^{h}
\end{array}\right]
$$

Furthermore, $[T]_{S}$ is a Boolean matrix to extract the DOFs defined in body $s$, and $\{x\}$ denotes the unknown vector of the global system. The FE-BI unknown vector in body $s$ is $[T]_{s}\{x\}=\left\{E_{s}, h_{s}\right\}^{\mathrm{T}}$ and the MoM unknown vector in body $s$ is $[T]_{s}\{x\}=\left\{e_{s}, h_{s}\right\}^{T}$. Finally, $\{b\}$ represents the excitation vector of the global system, and the excitation vector for body $s$ is given by $[T]_{s}\{b\}$.

By applying the simple substitutions $e_{s} \rightarrow h_{s}, h_{s} \rightarrow-e_{s}, \boldsymbol{E}^{\text {inc }} \rightarrow \overline{\boldsymbol{H}}^{\text {inc }}$, $\overline{\boldsymbol{H}}^{\mathrm{inc}} \rightarrow-\boldsymbol{E}^{\mathrm{inc}}, \eta_{\mathrm{r} d} \rightarrow 1 / \eta_{\mathrm{r} d}, z_{s} \rightarrow y_{s}$, the dual equation of (4.29) can be obtained as

$$
\begin{equation*}
\left[A_{s}^{\mathrm{d}}\right][T]_{s}\left\{x^{\mathrm{d}}\right\}+\sum_{q \neq s}\left[C_{s q}^{\mathrm{d}}\right][T]_{q}\left\{x^{\mathrm{d}}\right\}=[T]_{s}\left\{b^{\mathrm{d}}\right\} \tag{4.33}
\end{equation*}
$$

which can be used to solve scattering from a multibody system with perfectly magnetic conducting (PMC) bodies by letting $y_{s}=0$.

Solving the global matrix obtained by summing up (4.29), (4.33), or their combinations over all the bodies, the fields in the bodies and the currents on their surfaces can be computed. As can be seen, the proposed multi-solver scheme integrates three solvers: the CFIE(IBC) for conducting bodies, the CFIE(Dielectric) for homogeneous dielectric bodies, and the FE-BI(CFIE) for inhomogeneous bodies. With the unified CFIE coupling scheme for the exterior regions, the electromagnetic problem of a complex object can be solved by applying the most suitable equations to the interior regions of the decomposed bodies. When the proposed multi-solver scheme is applied to a composite object with multiple conducting and dielectric junctions, there is no need to assign special junction basis functions or apply special testing procedures at the junctions. Before we conclude this section, it is worth mentioning that the CFIE(Dielectric) requires less memory than the one published in [45]. To model a dielectric body using the MoM, if the number of DOFs used in the CFIE(Dielectric) is $N_{1}$, then the memory cost is $N_{1}^{2}$. In contrast, the total number of DOFs referred in [45] is $N_{1}+N_{0}$, which results in a memory cost of $N_{1}^{2}+2 N_{0}^{2}$ (referring to (52) in [45]), where $N_{1}$ and $N_{0}$ are the numbers of the interior and exterior DOFs, respectively. When a fast algorithm such as the MLFMA is applied to accelerate the computation, the memory saving is scaled with $N_{0} \log N_{0}$.

### 4.1.4 Choice of Testing Functions

To achieve an accurate solution of (4.29) or (4.33), it has to test the operators in the matrices $\left[P_{s}\right],\left[U_{s}\right],\left[Q_{s 0}\right]$, and $\left[C_{s q}^{h}\right]$ well by applying proper testing functions. In the discrete space, if the Rao-Wilton-Glisson (RWG) function [73,74] is chosen as the basis function, the testing functions should be chosen as $\boldsymbol{T}_{0}^{\alpha}=\boldsymbol{f}_{s}^{\mathrm{r}}$ and $\boldsymbol{T}_{0}^{\beta}=\hat{\boldsymbol{n}}_{s} \times \boldsymbol{f}_{s}^{\mathrm{b}}$ in order to test the $\mathcal{I}, \tilde{\mathcal{K}}$, and $\mathcal{L}$ operators in (4.14) and (4.16) well, where $\boldsymbol{f}_{s}^{\mathrm{r}}$ and $\boldsymbol{f}_{s}^{\mathrm{b}}$ denote column vectors consisting of
the RWG and the Buffa-Christiansen functions [78], respectively. To test the $\mathcal{I}, \tilde{\mathcal{K}}$, and $\mathcal{L}$ operators in (4.26) well, the testing functions should be chosen as $\boldsymbol{T}_{s}^{\alpha}=\boldsymbol{f}_{s}^{\mathrm{b}}$ and $\boldsymbol{T}_{s}^{\beta}=\boldsymbol{f}_{s}^{\mathrm{r}} \times \hat{\boldsymbol{n}}_{s}$. To test the $\mathcal{I}$ operator in (4.21) well, the testing function can be chosen as $\boldsymbol{T}_{s}^{\gamma}=\boldsymbol{f}_{s}^{\mathrm{b}}$. The detailed discussions on the choices of basis and testing functions can be found in [54] and [80, 83, 96].

### 4.1.5 Iterative Solution

Iterative methods are preferred for the solution of a multi-solver system with a very large number of unknowns, because they have lower computational complexities and memory requirements compared to direct methods. To accelerate the computation and reduce the memory requirement, the MLFMA is applied to the MoM, BI, and coupling matrices in the proposed multi-solver scheme. The application of the MLFMA with the mixed testing scheme can be referred to [54]. In the proposed multi-solver scheme, we adopt the socalled common MLFMA tree strategy, in which all the bodies share the same octree structure that is determined by the largest wavenumber among all the bodies. Although this strategy is not optimal when the bodies have high contrast materials, it can avoid a special treatment in the calculation of the near-field interactions on body interfaces as given in [44]. To accelerate the computation, other fast algorithms or other implementations of fast algorithms can be employed easily in the proposed framework without changing the multi-solver formulation.

To further accelerate the iterative solution, a preconditioner based on ABCs is adopted $[24,54]$. The preconditioner can be obtained by substituting the global operator $\mathcal{L}$ with a locally approximate operator $\mathcal{L}^{\prime}$, replacing $\hat{\boldsymbol{n}} \times \boldsymbol{f}^{\mathrm{b}}$ with $\boldsymbol{f}^{\mathrm{r}}$, and neglecting all the interactions that contain the global operator $\tilde{\mathcal{K}}$ in (4.29) and (4.33). By doing so, the spectrum distribution of the obtained ABC-based preconditioner is similar to that of the multi-solver system. The application of the preconditioner can be carried out by solving the preconditioning matrix directly or iteratively as detailed in [54].

### 4.2 Numerical Analyses of Individual Solvers

To obtain a robust and accurate multi-solver algorithm, it is essential to investigate the conditions and accuracy of the individual solvers. The detailed discussion on the FE-BI(CFIE) can be found in [54]; here only the properties of the CFIE(IBC) and CFIE(Dielectric) are discussed. In this chapter, the CFIE(IBC) and the CFIE(Dielectric) are obtained from a combination of the dual formulations in (4.29) and (4.33), so that the electric and magnetic fields are well tested in both the interior and exterior regions by applying the mixed testing scheme. All of the computations in this chapter are performed on a cluster node equipped with a 20-core Intel Xeon E5-2680 v2 CPU and a $250-\mathrm{GB}$ memory.

### 4.2.1 Condition of the Individual Solvers

The CFIE-based methods have been proven theoretically to be free of spurious interior resonance corruption [93, 94, 97] and verified numerically in published papers $[54,98,99]$. In this section, we study the condition with respect to the material property and mesh density.

## CFIE(IBC)

A conducting sphere with a radius of 1 m is considered. The surface of the sphere is discretized into curvilinear triangles with a mesh size of 0.15 m . The condition number of the system matrix obtained from the discretization of the CFIE(IBC) at 130 MHz is shown in Fig. 4.2, in which the condition number is displayed as a two-dimensional function of $z=R+\mathrm{j} X$. The real part of the normalized surface impedance, $R$, varies from $10^{-5}$ to 1.0 , and is shown as the horizontal axis. The imaginary part of the normalized surface impedance, $X$, also varies from $10^{-5}$ to 1.0 , and is shown as the vertical axis. It is found from the figure that the condition numbers for all normalized surface impedance $z$ are smaller than 81 . At the PEC limit, the condition numbers stay around 20 .

To study the condition with respect to the mesh density, the same conducting sphere with a normalized surface impedance $z=0.1+\mathrm{j} 0.1$ is considered. The condition number of the system matrix obtained from the discretization
of the CFIE(IBC) is calculated and plotted in Fig. 4.3. From the result, it can be observed clearly that the CFIE(IBC) has a very small condition number, which increases linearly with respect to the mesh density. This can be explained through the analysis of the CFIE(IBC) formulation. The diagonal blocks of the system matrix consist of the well tested $\mathcal{I}, \mathcal{L}$, and $\tilde{\mathcal{K}}$ operators, and the off-diagonal blocks have eigenvalues clustered at zero. Hence the entire matrix behaves like a matrix consisting of well tested $\mathcal{I}$ and $\mathcal{L}$ operators. The detailed discussion can be found in [53].

## CFIE(Dielectric)

The object considered is a homogeneous dielectric cube with a size of 0.3 m $\times 0.3 \mathrm{~m} \times 0.3 \mathrm{~m}$. The surface of the cube is discretized into triangles with a mesh density of 10 segments per dielectric wavelength corresponding to 300 MHz. The PMCHWT, N-Müller, and the CFIE(Dielectric) are applied to solve the scattering from the cube. As can be seen in Fig. 4.4, among three solvers, the CFIE(Dielectric) is most stable as the relative permittivity varies from 4 to 36 . The increase in the condition numbers around $\epsilon_{\mathrm{r}}=24$ is due to a physical resonance of the dielectric cube [99].

To investigate the condition versus mesh density, the same dielectric cube with $\epsilon_{\mathrm{r}}=4$ is considered. The condition number of the system matrix obtained from the discretization of the CFIE(Dielectric) is calculated and given in Fig. 4.5. Again, the result shows that the condition number of the CFIE(Dielectric) increases linearly with respect to mesh density. The analysis on the blocks of the CFIE(Dielectric) matrix indicates that the system matrix behaves like a matrix consisting of well tested $\mathcal{I}$ and $\mathcal{L}$ operators.

### 4.2.2 Accuracy of the Individual Solvers

The first testing example is a conducting sphere with a radius of 1 m and a normalized surface impedance $z=0.5$. The sphere is discretized into curvilinear triangles with an average size of 0.1 m and illuminated by a $300-\mathrm{MHz}$ plane wave. The bistatic radar cross section (RCS) results calculated by the CFIE(IBC) and SDIE [53] are compared with the Mie series solutions. The relative root-mean-square (RMS) errors [83] are given in Table 4.1, from
which it can be seen that the proposed CFIE(IBC) and the SDIE have a similar accuracy.

To show the accuracy of the CFIE(IBC) at limit cases, we set $z_{s}=0$ in (4.29) and $y_{s}=0$ in (4.33) for the same sphere. The bistatic RCS of the corresponding PEC sphere is identical to that of a PMC sphere with a switched polarization. All results agree very well with the Mie series solutions. The RMS errors for the PEC-VV and the PMC-HH cases are $0.22 \%$, and the ones for the PEC-HH and the PMC-VV cases are $0.10 \%$.

The second testing example is a lossless dielectric sphere with a relative permittivity of $\epsilon_{\mathrm{r}}=2$ and a radius of 1 m . Illuminated by a $110-\mathrm{MHz}$ plane wave, the surface of the sphere is discretized into curvilinear triangles with an average size of 0.15 m . The bistatic RCS is calculated and compared with the Mie series solutions. The RMS errors of the VV- and HH-polarized cases are $0.21 \%$ and $0.48 \%$, respectively, which demonstrate the accuracy of the CFIE(Dielectric) method with the mixed testing scheme.

### 4.3 Numerical Examples of the Multi-Solver Scheme

The numerical analyses in Section 4.2 showed that the individual solvers have good conditions and a high accuracy. In this section, the accuracy, flexibility, and capability of the proposed multi-solver scheme will be demonstrated through several numerical examples. The global systems in the following examples are solved by the biconjugate gradient stabilized (BiCGSTAB) method with a targeted relative residual of $10^{-3}$ unless indicated otherwise. To accelerate the computation and reduce the memory requirement, the MLFMA is employed for all the BI- and SIE-related computation.

### 4.3.1 A Dielectric Sphere

A homogeneous dielectric sphere with $\epsilon_{\mathrm{r}}=2$ is illuminated by a $300-\mathrm{MHz}$ plane wave. As shown in Fig. 4.6, the sphere is decomposed into two bodies, and each body is discretized separately with a mesh size of 0.1 m . The CFIE(Dielectric) is applied to model body I. To model body II, either the CFIE(Dielectric) (case 1) or the FE-BI(CFIE) method (case 2) is applied. The global system is solved iteratively using the ABC-based precondition-
er, which converges in 10 iterations in case 1 and 8 iterations in case 2, respectively. The calculated bistatic RCS of the two cases and the Mie series solutions are given in Fig. 4.6, which demonstrates the accuracy of the proposed multi-solver sheme.

Next, the same dielectric sphere is equally partitioned into eight bodies as shown in Fig. 4.7a, and illuminated by a $131-\mathrm{MHz}$ plane wave. Each body is discretized separately with a mesh size of 0.15 m , and modeled by the CFIE(Dielectric). With the application of the ABC-based preconditioner, the solution takes 13 iterations to converge, and the surface current distribution is shown in Fig. 4.7b. For comparison, the dielectric sphere is simulated by the CFIE(Dielectric) without partition, and the current distribution is given in Fig. 4.7c. As can be seen, a good agreement is achieved between the two simulations.

### 4.3.2 A Composite Sphere

A composite sphere is illuminated by a $131-\mathrm{MHz}$ plane wave from the $z$ direction. As shown in Fig. 4.8, the sphere with a radius of 1 m is made of two hemispheres. One is a dielectric with $\epsilon_{\mathrm{r}}=2$ and the other is made of PEC. The sphere is decomposed into two bodies according to the material property, and each body is discretized separately with a mesh size of 0.15 m . The CFIE(IBC) is used to model the PEC body. To model the dielectric body, either the CFIE(Dielectric) (case 1) or the FE-BI(CFIE) (case 2) is employed. The global system is solved iteratively with the application of the ABC-based preconditioner, which converges in 11 iterations in both cases. The bistatic RCS and the electric current distribution of the two cases are presented in Figs. 4.8 and 4.9, respectively, from which a good agreement can be observed.

### 4.3.3 A Composite Cylinder

A composite cylinder shown in Fig. 4.10a is partitioned into three bodies. Body I is a 0.1-m-thick dielectric coating layer with $\epsilon_{\mathrm{r}}=1.5$, body II is a conducting cylinder with $z=0.1$, and body III is a dielectric cylinder with $\epsilon_{\mathrm{r}}=2$. The conductor is modeled by the CFIE(IBC), and the two
dielectric materials are modeled by the CFIE(Dielectric). The surface of each body is discretized into curvilinear triangles with an average size of 0.05 m , which results in 36,672 DOFs in total. Illuminated by a $500-\mathrm{MHz}$ plane wave from the $z$ direction, the bistatic RCS of the composite cylinder is calculated by solving the multi-solver system, as shown in Fig. 4.10b. The iterative solution takes 11 steps to converge with the application of the ABCbased preconditioner. The ABC-based preconditioner is solved iteratively with the approximate inverse (AI) preconditioner, which takes an average of 14 iterations to converge to a relative error of $10^{-3}$. The RCS results are compared with the FE-BI(CFIE) solutions [54], and a good agreement is observed for both VV- and HH-polarized cases. The current distributions on three bodies are shown in Fig. 4.11. As can be seen, the current pattern on the surface of the inner cylinders is consistent with the one on the inner surface of the coating layer.

### 4.3.4 A Multilayer Dielectric Sphere

The scattering from a multilayer dielectric sphere illuminated by a $300-\mathrm{MHz}$ plane wave is calculated to show the capability in modeling high-contrast dielectric materials. The partially homogeneous sphere is composed of 6 dielectric layers with different relative permittivities, which from inside out are $16,4-\mathrm{j} 0.03,15,5-\mathrm{j} 0.02,14$, and $6-\mathrm{j} 0.01$, respectively. The corresponding radius of each layer is $0.1 \mathrm{~m}, 0.2 \mathrm{~m}, 0.3 \mathrm{~m}, 0.4 \mathrm{~m}, 0.5 \mathrm{~m}$, and 0.6 m . The following two cases are considered: 1) All the dielectric layers are modeled by the CFIE(Dielectric); 2) Layers 1-3 are modeled by the FE-BI(CFIE) and layers 4-6 are modeled by the CFIE(Dielectric).

In the two cases, the sphere is meshed into triangular or tetrahedral elements with an average size of 0.025 m , which results in 141,468 and 189,636 DOFs in total, respectively. The global systems are solved iteratively with the applicaiton of the ABC-based preconditioner, which take 141 and 148 iterations to converge to a relative error of $10^{-3}$ for the two cases. The total computation times for the two cases are 5.8 and 5.0 hours, respectively. The bistatic RCS of the dielectric sphere is given in Fig. 4.12. The RMS errors are $3.4 \% ~(\mathrm{VV})$ and $2.8 \% ~(\mathrm{HH})$ for case 1 , and $3.5 \% ~(\mathrm{VV})$ and $3.1 \% ~(\mathrm{HH})$ for case 2. The current distributions on the surface of each layer are plotted
in Fig. 4.13, from which it can be found that the current densities between surfaces have a large difference.

### 4.3.5 Two Missile-Like Objects

To demonstrate the capability of the proposed multi-solver scheme, two missile-like objects with complex materials and structures are considered. Missile I is a 3 -m-long object with six 1 -cm-thick fins. Missile II is a 4.7m -long object with four thrusts. Based on the material properties, missiles I and II are partitioned into ten bodies as shown in Fig. 4.14a and seven bodies as shown in Fig. 4.14b, respectively, and the details of the bodies are presented as follows.

Body I-A: a 4-cm-thick radome layer with $\epsilon_{\mathrm{r}}=4$.
Body I-B: a 6-cm-thick radome layer with $\epsilon_{\mathrm{r}}=3$.
Body I-C: a circular patch antenna sitting on a radome layer with $\epsilon_{\mathrm{r}}=$ 1.5-j0.008.

Body I-D: a missile body with a conducting surface of $z=0.1$ and a 0.01-m-thick coating layer of $\epsilon_{\mathrm{r}}=2-\mathrm{j}$.

Bodies I-E: six conducting fins with $z=3+\mathrm{j} 0.3$.
Body II-A: a dielectric nose with $\epsilon_{\mathrm{r}}=3-\mathrm{j} 0.003$.
Body II-B: an inner part of the nose, which consists of a dielectric layer with $\epsilon_{\mathrm{r}}=2-\mathrm{j} 0.1$ and an enclosed PEC object.

Body II-C: a conducting missile body with $z=0.5$.
Bodies II-D: four conducting thrusts with $z=2+\mathrm{j} 0.1$.

Illuminated by a $5-\mathrm{GHz}$ plane wave from the $z$ direction, the scattering from the two missile-like objects is solved by the multi-solver algorithm at the moment when missile I is going to intercept missile II. The individual solvers and DOFs for each body are shown in Table 4.2. Curvilinear tetrahedral meshes are employed for the bodies with the FE-BI(CFIE) solver, and curvilinear
triangular meshes are employed for the bodies with the CFIE(Dielectric) and CFIE(IBC) solvers.

The global system with a total of $5,956,148$ unknowns is solved with the application of the ABC-based preconditioner. The solution takes 26 iterations to converge to a relative error of $5 \times 10^{-3}$. The total computation time is 64 hours ( $23 \%$ for assembly and $77 \%$ for solution), and the peak memory consumption is 160 GB . The current distribution on the surfaces of the two missiles is shown in Fig. 4.15, from which the current variation can be observed clearly. The current magnitude on the IBC area is much smaller than that in the dielectric area. Figure 4.16 provides the current distribution at the interface between the head and the body of missile I, from which it can be seen that the current pattern at the interface of the two parts are consistent with each other. The VV- and HH-polarized bistatic RCS results are calculated in the $x z$-plane and $y z$-plane, which are presented in Fig. 4.17.

### 4.4 Summary

In this chapter, a CFIE-based multi-solver algorithm was presented for electromagnetic modeling of electrically large and highly complex objects. To analyze such a problem, the object was decomposed into multiple bodies according to its material property and geometry. For the interior region of each body, different equations were used to model different materials. To couple these equations in different bodies, the CFIE was applied to uniformly model the exterior region of each body. As a result, the proposed multi-solver scheme which consists of the CFIE(IBC), CFIE(Dielectric), and FE-BI(CFIE) is capable to model electromagnetic problems with conducting, homogeneous, and inhomogeneous materials. To obtain accurate numerical results, a mixed testing scheme was applied to discretize the multi-solver system into matrix equations. In the proposed algorithm, the ABC-based preconditioner was employed to speed up the iterative convergence and the MLFMA with a common tree structure was employed to compute the MoM, BI, and coupling matrices. The numerical experiments were conducted to first show the numerical accuracy and condition of the individual solvers, and then to demonstrate the accuracy, flexibility, and capability of the multisolver algorithm. Through the numerical examples, the proposed CFIE-based
multi-solver scheme was shown to be powerful for electromagnetic modeling of highly complex objects.

### 4.5 Figures and Tables



Figure 4.1: Arbitrarily shaped object decomposed into two regions.


Figure 4.2: Condition number of the system matrix obtained from the discretization of the CFIE(IBC) for an IBC sphere at 130 MHz . The condition number is displayed as a two-dimensional function of $R_{\mathrm{s}}$ and $X_{\mathrm{s}}$, which are the real and imaginary parts of the normalized surface impedance, respectively, $z_{\mathrm{s}}=R_{\mathrm{s}}+\mathrm{j} X_{\mathrm{s}}$.


Figure 4.3: Condition number of the system matrix obtained from the discretization of the CFIE(IBC) versus the mesh density. The object is an IBC sphere with a radius of 1.0 m and a normalized surface impedance $z=0.1+\mathrm{j} 0.1$.


Figure 4.4: Condition number of the system matrix obtained from the discretization of the CFIE(Dielectric) for a dielectric cube at 300 MHz . The PMCHWT and N-Müller formulations for dielectric problems are given for comparison.


Figure 4.5: Condition number of the system matrix obtained from the discretization of the CFIE(Dielectric) versus the mesh density. The object is a dielectric cube with a size of $0.3 \mathrm{~m} \times 0.3 \mathrm{~m} \times 0.3 \mathrm{~m}$ and $\epsilon_{\mathrm{r}}=4$.


Figure 4.6: Bistatic RCS of the dielectric sphere at 300 MHz . Case 1: both bodies I and II are modeled by the CFIE(Dielectric). Case 2: body I is modeled by the CFIE(Dielectric) and body II is modeled by the FE-BI(CFIE).


Figure 4.7: Current distribution on the surface of a dielectric sphere with $\epsilon_{\mathrm{r}}=2$ at 131 MHz . (a) The sphere is partitioned into eight bodies. (b) Simulation of eight bodies with each modeled by the CFIE(Dielectric). (c) Simulation of a sphere without partition using CFIE(Dielectric).


Figure 4.8: Bistatic RCS of the composite sphere at 131 MHz . Case 1: the dielectric and PEC bodies are modeled by the CFIE(Dielectric) and CFIE(IBC), respectively. Case 2: the dielectric body is modeled by the FE-BI(CFIE), the PEC body is modeled by the CFIE(IBC).


Figure 4.9: Current distribution on the surface of the PEC hemisphere in (a) case 1 and (b) case 2 .


Figure 4.10: (a) A composite cylinder. Bodies I and III are two different dielectric materials, and body II is a conductor. (b) Bistatic RCS of the composite cylinder at 500 MHz .


Figure 4.11: (a) Current distribution on body I. (b) Current distribution on bodies II and III.


Figure 4.12: Bistatic RCS of the 6-layer sphere with a high dielectric contrast at 300 MHz .


Figure 4.13: Current distribution on the layer with (a) $\epsilon_{\mathrm{r}}=16$, (b)
$\epsilon_{\mathrm{r}}=4-\mathrm{j} 0.03$, (c) $\epsilon_{\mathrm{r}}=15$, (d) $\epsilon_{\mathrm{r}}=5-\mathrm{j} 0.02$, (e) $\epsilon_{\mathrm{r}}=14$, and (f) $\epsilon_{\mathrm{r}}=6-\mathrm{j} 0.01$.


Figure 4.14: Decomposition of the missiles-like objects. (a) Missile I partitioned into ten bodies. (b) Missile II partitioned into seven bodies.


Figure 4.15: Current distribution on the surfaces of the two missiles illuminated by a $5-\mathrm{GHz}$ plane wave from the $z$ direction. The currents around the heads and thrusters are enlarged for a better illustration.


Figure 4.16: Current distribution at the interface between the body and the head of missile I. (a) Body part. (b) Head part. The slight difference between the convex surface in (a) and concave surface in (b) is from the angle of plot.


Figure 4.17: Bistatic RCS of the two missiles at 5 GHz in the (a) $x z$-plane and (b) $y z$-plane.

Table 4.1: RMS errors of the bistatic RCS of an IBC sphere at 300 MHz

|  | SDIE | CFIE(IBC) |
| :---: | :---: | :---: |
| VV-pol. (\%) | 1.98 | 1.70 |
| HH-pol. (\%) | 1.58 | 0.75 |

Table 4.2: Individual solvers and DoFs for missile-like objects at 5 GHz

| Body(ies) | Solver | DoFs |
| :---: | :---: | :---: |
| I-A | CFIE(Dielectric) | 511,188 |
| I-B | CFIE(Dielectric) | 241,110 |
| II-A | CFIE(Dielectric) | 366,138 |
| I-E | CFIE(IBC) | 330,732 |
| II-C | CFIE(IBC) | 901,956 |
| II-D | CFIE(IBC) | $1,078,248$ |
| I-C\&D, II-B | FE-BI(CFIE) | $2,526,776^{1}$ |
| 1 <br> Total number of DoFs in bodies I-C, I-D, and II-B, <br> in which a single FE-BI(CFIE) solver is applied. |  |  |

## CHAPTER 5

## A MULTI-SOLVER SCHEME BASED ON ROBIN TRANSMISSION CONDITIONS

In the last chapter, we discussed the multi-solver scheme based on combined field integral equation (MS-CFIE) [50, 51], where the entire object is decomposed into multiple bodies based on its material property, with the inhomogeneous bodies modeled by the CFIE-based finite element-boundary integral (FE-BI) method [54] and the conducting and homogeneous bodies modeled by CFIE-based boundary integral equation (BIE) methods. The different solvers are uniformly coupled by the CFIE applied to the exterior region of the bodies.

In this chapter, we propose a new multi-solver scheme based on Robin transmission conditions (MS-RTC) for large-scale electromagnetic modeling and simulation. The entire computational domain consisting of the object and its background is first partitioned into multiple non-overlapping subdomains with each modeled by either a PDE (partial differential equation) or a BIE. These equations are then coupled into a multi-solver system by applying the Robin transmission condition at the subdomain interfaces.

### 5.1 Formulation

This section presents the detailed formulation of the MS-RTC scheme. The modeling of general objects by coupling PDEs with BIEs through Robin transmission conditions is first introduced, which is followed by the FEM and MoM analysis for the PDEs and BIEs, respectively. Finally, the approach to solving the MS-RTC system is discussed.

### 5.1.1 Modeling of General Objects

Consider an arbitrarily shaped object immersed in free space with permittivity $\epsilon_{0}$ and permeability $\mu_{0}$. As shown in Fig. 5.1, the object consists of multiple regions which can have different material properties. According to the material property of the object, the entire computational domain $\Omega \subseteq \mathbb{R}^{3}$ can be decomposed into non-overlapping subdomains $\Omega_{s}\left(s=0,1, \ldots, N_{\mathrm{s}}\right)$ with $\Omega_{0}$ denoting the free-space subdomain. The surface of each subdomain is denoted as $S_{s}$ and the portion that interfaces with its neighboring subdomains is denoted as $\Gamma_{s}$. The impenetrable regions which do not belong to the computational domain are represented by $\mathbb{R}^{3} \backslash \Omega$.

If subdomain $s$ contains inhomogeneous and/or anisotropic materials ( $\bar{\epsilon}_{\mathrm{r}}$, $\overline{\boldsymbol{\mu}}_{\mathrm{r}}$ ), a partial differential equation with proper boundary conditions [1] can be applied to model the subdomain, which can be written as

$$
\begin{array}{rlrl}
\nabla \times\left(\overline{\boldsymbol{\mu}}_{\mathrm{r}}^{-1} \cdot \nabla \times \boldsymbol{E}_{s}\right)-k_{0}^{2} \overline{\boldsymbol{\epsilon}}_{\mathrm{r}} \cdot \boldsymbol{E}_{s} & =-\mathrm{j} k_{0} \overline{\boldsymbol{J}}_{\mathrm{src}} & & \text { in } \Omega_{s} \\
\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}_{s} & =0 & & \text { on } \Gamma_{\mathrm{PEC}} \\
\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}_{s} \times \hat{\boldsymbol{n}}_{s} & =z_{\mathrm{s}} \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{s} & & \text { on } \Gamma_{\mathrm{IBC}} \\
\hat{\boldsymbol{n}}_{s} \times\left(\overline{\boldsymbol{\mu}}_{\mathrm{r}}^{-1} \cdot \nabla \times \boldsymbol{E}_{s}\right) & =-\mathrm{j} k_{0} \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{s} & \text { on } \Gamma_{s} \tag{5.4}
\end{array}
$$

where $k_{0}, Z_{0}$, and $z_{\mathrm{s}}$ denote the free-space wavenumber, free-space impedance, and normalized surface impedance, respectively. Furthermore, $\overline{\boldsymbol{H}}_{s}=Z_{0} \boldsymbol{H}_{s}$, $\overline{\boldsymbol{J}}_{\text {src }}=Z_{0} \boldsymbol{J}_{\text {src }}$, and $\hat{\boldsymbol{n}}_{s}$ represents the unit vector normal to $S_{s}\left(=\Gamma_{\mathrm{PEC}} \cup \Gamma_{\mathrm{IBC}} \cup\right.$ $\left.\Gamma_{s}\right)$ and pointing toward the interior of $\Omega_{s}$.

If the subdomain is a large homogeneous object described as $\left(\epsilon_{\mathrm{r}}, \mu_{\mathrm{r}}\right)$, a surface integral equation is employed to model the field in the subdomain. The scattered electric and magnetic fields in subdomain $s$ generated by the surface electric current $\overline{\boldsymbol{j}}_{s}=\hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{s}$ and magnetic current $\boldsymbol{m}_{s}=\boldsymbol{E}_{s} \times \hat{\boldsymbol{n}}_{s}$ on $S_{s}$ can be written as

$$
\begin{align*}
& \boldsymbol{E}_{s}^{\text {sca }}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)=-\eta_{\mathrm{r}} \mathcal{L}_{s}\left(\overline{\boldsymbol{j}}_{s} ; S_{s}\right)+\mathcal{K}_{s}\left(\boldsymbol{m}_{s} ; S_{s}\right)  \tag{5.5}\\
& \overline{\boldsymbol{H}}_{s}^{\text {sca }}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right)=-\eta_{\mathrm{r}} \mathcal{K}_{s}\left(\overline{\boldsymbol{j}}_{s} ; S_{s}\right)-\mathcal{L}_{s}\left(\boldsymbol{m}_{s} ; S_{s}\right) \tag{5.6}
\end{align*}
$$

where $\eta_{\mathrm{r}}=\sqrt{\mu_{\mathrm{r}} / \epsilon_{\mathrm{r}}}$ and the integral operators $\mathcal{L}_{s}$ and $\mathcal{K}_{s}[2]$ are defined as

$$
\begin{equation*}
\mathcal{L}_{s}\left(\boldsymbol{v} ; S_{s}\right)=\mathrm{j} k \int_{S_{s}}\left(\mathcal{I}+\frac{\nabla \nabla}{k^{2}}\right) \frac{\mathrm{e}^{-\mathrm{j} k R}}{4 \pi R} \cdot \boldsymbol{v}\left(\boldsymbol{r}^{\prime}\right) \mathrm{d} S^{\prime} \tag{5.7}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{K}_{s}\left(\boldsymbol{v} ; S_{s}\right)=\int_{S_{s}} \boldsymbol{v}\left(\boldsymbol{r}^{\prime}\right) \times \nabla \frac{\mathrm{e}^{-\mathrm{j} k R}}{4 \pi R} \mathrm{~d} S^{\prime} \tag{5.8}
\end{equation*}
$$

in which $\mathcal{I}$ is the identity operator, $k$ is the wavenumber in subdomain $s$, and $R=\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|$. The electromagnetic fields satisfy the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE) [2] given by

$$
\begin{align*}
-\boldsymbol{m}_{s}-\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}_{s}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right) & =\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}^{\mathrm{inc}} \quad \text { on } S_{s}  \tag{5.9}\\
\eta_{\mathrm{r}} \overline{\boldsymbol{j}}_{s}-\hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}_{s}^{\mathrm{sca}}\left(\overline{\boldsymbol{j}}_{s}, \boldsymbol{m}_{s} ; S_{s}\right) & =\eta_{\mathrm{r}} \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{H}}^{\mathrm{inc}} \text { on } S_{s} \tag{5.10}
\end{align*}
$$

and the IBC [56] given by

$$
\begin{equation*}
\boldsymbol{m}_{s}+z_{s} \hat{\boldsymbol{n}}_{s} \times \overline{\boldsymbol{j}}_{s}=0 \quad \text { on } \Gamma_{\mathrm{IBC}} \tag{5.11}
\end{equation*}
$$

where $\boldsymbol{E}^{\mathrm{inc}}$ and $\overline{\boldsymbol{H}}^{\mathrm{inc}}$ denote the incident fields excited by the sources in $\Omega_{s}$ with $\mathbb{R}^{3}$ filled with the same medium as $\Omega_{s}$. By letting $z_{\mathrm{s}}=0$ in (5.11), the PEC boundary condition can be obtained. By default, the free-space subdomain $\Omega_{0}$ is modeled by a boundary integral equation unless indicated otherwise. It is noted that although the subscripts of the material-related variables are omitted for brevity in the preceding formulation, the material in each subdomain can be different.

To couple all the subdomains together, the Robin transmission condition [1] is applied to the subdomain interfaces, which is written as

$$
\begin{equation*}
\overline{\boldsymbol{j}}_{s}-\boldsymbol{e}_{s}=-\overline{\boldsymbol{j}}_{q}-\boldsymbol{e}_{q} \quad \text { on } \Gamma_{s}^{q} \tag{5.12}
\end{equation*}
$$

where $\boldsymbol{e}_{s}=\hat{\boldsymbol{n}}_{s} \times \boldsymbol{E}_{s} \times \hat{\boldsymbol{n}}_{s}$, and $\Gamma_{s}^{q}$ denotes the interface between subdomains $s$ and $q$. The MS-RTC global system can be constructed by coupling (5.1)(5.4) or (5.9)-(5.11) with (5.12) over all the subdomains. By solving the global system numerically, the electromagnetic fields in $\Omega_{s}$ or the equivalent currents on $S_{s}$ can be obtained. Before proceeding to the discussion of the numerical methods, it is convenient to define the bilinear form $\mathcal{B}(\bullet, \bullet)_{\Omega}$, the volume integral $\langle\bullet, \bullet\rangle_{\Omega}$, and the surface integral $\langle\bullet, \bullet\rangle_{S}$ as

$$
\begin{equation*}
\mathcal{B}(\boldsymbol{a}, \boldsymbol{b})_{\Omega}=\int_{\Omega}\left[(\nabla \times \boldsymbol{a}) \cdot \overline{\boldsymbol{\mu}}_{\mathrm{r}}^{-1} \cdot(\nabla \times \boldsymbol{b})^{\mathrm{T}}-k_{0}^{2} \boldsymbol{a} \cdot \overline{\boldsymbol{\epsilon}}_{\mathrm{r}} \cdot \boldsymbol{b}^{\mathrm{T}}\right] \mathrm{d} V \tag{5.13}
\end{equation*}
$$

$$
\begin{align*}
& \langle\boldsymbol{a}, \boldsymbol{b}\rangle_{\Omega}=\int_{\Omega} \boldsymbol{a} \cdot \boldsymbol{b}^{\mathrm{T}} \mathrm{~d} V  \tag{5.14}\\
& \langle\boldsymbol{a}, \boldsymbol{b}\rangle_{S}=\int_{S} \boldsymbol{a} \cdot \boldsymbol{b}^{\mathrm{T}} \mathrm{~d} S \tag{5.15}
\end{align*}
$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are column vectors containing vector functions, and the superscript T denotes the transpose of a column vector.

### 5.1.2 FEM Analysis in 3-D Subdomains

For the subdomains modeled by PDEs, the FEM can be applied to solve for the fields numerically. The weak form of the boundary-value problem (BVP) defined by (5.1)-(5.4) [1] can be obtained as

$$
\begin{align*}
& \mathcal{B}\left(\boldsymbol{N}_{s}, \boldsymbol{E}_{s}\right)_{\Omega_{s}}+\frac{\mathrm{j} k_{0}}{z_{\mathrm{s}}}\left\langle\boldsymbol{N}_{s}, \boldsymbol{E}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}} \\
&+\mathrm{j} k_{0}\left\langle\boldsymbol{N}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{\Gamma_{s}}=-\mathrm{j} k_{0}\left\langle\boldsymbol{N}_{s}, \overline{\boldsymbol{J}}_{\mathrm{src}}\right\rangle_{\Omega_{s}} \tag{5.16}
\end{align*}
$$

where $\boldsymbol{N}_{s}$ is a column vector containing the curl-conforming vector functions in $\Omega_{s}$. In this chapter, $\boldsymbol{N}_{s}$ is chosen as a column vector consisting of the edge-based vector functions [1]. Testing the transmission condition (5.12) by $\boldsymbol{N}_{s}$ with a scaling factor of $-\mathrm{j} k_{0} / 2$ and adding it to (5.16), the weak form of the BVP with the TRC can be rewritten as

$$
\begin{align*}
& \mathcal{B}\left(\boldsymbol{N}_{s}, \boldsymbol{E}_{s}\right)_{\Omega_{s}}+\frac{\mathrm{j} k_{0}}{z_{\mathrm{s}}}\left\langle\boldsymbol{N}_{s}, \boldsymbol{E}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}} \\
& \quad+\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{N}_{s}, \boldsymbol{e}_{s}\right\rangle_{\Gamma_{s}}+\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{N}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{\Gamma_{s}} \\
& =\frac{\mathrm{j} k_{0}}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\boldsymbol{N}_{s}, \boldsymbol{e}_{q}\right\rangle_{\Gamma_{s}^{q}}+\frac{\mathrm{j} k_{0}}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\boldsymbol{N}_{s}, \overline{\boldsymbol{j}}_{q}\right\rangle_{\Gamma_{s}^{q}} \\
& \quad-\mathrm{j} k_{0}\left\langle\boldsymbol{N}_{s}, \overline{\boldsymbol{J}}_{\mathrm{src}}\right\rangle_{\Omega_{s}} \tag{5.17}
\end{align*}
$$

where $\mathcal{N}(s)$ denotes the neighbors of subdomain $s$. Testing (5.12) again with $\hat{\boldsymbol{n}}_{s} \times \boldsymbol{N}_{s}$ defined on $\Gamma_{s}$ and scaling it with a factor of $\mathrm{j} k_{0} / 2$ yields the discretized RTC as

$$
\begin{aligned}
& -\frac{\mathrm{j} k_{0}}{2}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{N}_{s}, \boldsymbol{e}_{s}\right\rangle_{\Gamma_{s}}+\frac{\mathrm{j} k_{0}}{2}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{N}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{\Gamma_{s}} \\
& \quad=-\frac{\mathrm{j} k_{0}}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{N}_{s}, \boldsymbol{e}_{q}\right\rangle_{\Gamma_{s}^{q}}
\end{aligned}
$$

$$
\begin{equation*}
-\frac{\mathrm{j} k_{0}}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{N}_{s}, \overline{\boldsymbol{j}}_{q}\right\rangle_{\Gamma_{s}^{q}} . \tag{5.18}
\end{equation*}
$$

By coupling (5.17) with (5.18), a subdomain-level FEM system can be obtained [1].

To discretize the FEM subsystem, the fields and currents are expanded as

$$
\begin{align*}
\overline{\boldsymbol{j}}_{s} & =\boldsymbol{f}_{s}^{\mathrm{rT}} h_{s}  \tag{5.19}\\
\boldsymbol{e}_{s} & =\boldsymbol{N}_{s}^{\mathrm{T}} e_{s}  \tag{5.20}\\
\boldsymbol{E}_{s} & =\boldsymbol{N}_{s}^{\mathrm{T}} E_{s}^{\mathrm{T}}+\boldsymbol{N}_{s}^{\mathrm{T}} e_{s} \tag{5.21}
\end{align*}
$$

where $\boldsymbol{f}_{s}^{\mathrm{r}}=\hat{\boldsymbol{n}}_{s} \times \boldsymbol{N}_{s}$ denotes a column vector consisting of the Rao-WiltonGlisson (RWG) functions. Furthermore, $h_{s}, e_{s}$, and $E_{s}^{\mathrm{I}}$ are unknown vectors containing the expansion coefficients to be determined in the FEM subdomain $s$. By substituting (5.19)-(5.21) into (5.17) and (5.18), the FEM subdomain equation can be obtained as

$$
\begin{equation*}
\left[F_{s}\right][T]_{s}\{x\}+\sum_{q \in \mathcal{N}(s)}\left[C_{s q}\right][T]_{q}\{x\}=[T]_{s}\{b\} \tag{5.22}
\end{equation*}
$$

where $[T]_{s}$ is a Boolean matrix to extract the DoFs defined in subdomain $s$, $\{x\}$ denotes the unknown vector of the global system, and the FEM unknown vector in subdomain $s$ is given by $[T]_{s}\{x\}=\left\{E_{s}^{\mathrm{I}}, e_{s}, h_{s}\right\}^{\mathrm{T}}$. Furthermore, $\{b\}$ represents the excitation vector of the global system, and the subdomain excitation vector is given by $[T]_{s}\{b\}=\mathrm{j} k_{0}\left\langle\boldsymbol{N}_{s}, \overline{\boldsymbol{J}}_{\text {src }}\right\rangle_{\Omega_{s}}$. Finally, the subdomain matrix and the coupling matrix can be written as

$$
\begin{align*}
{\left[F_{s}\right] } & =\left[\begin{array}{ccc}
K_{s}^{\mathrm{II}} & K_{s}^{\mathrm{IS}} & 0 \\
K_{s}^{\mathrm{SI}} & K_{s}^{\mathrm{SS}}+U_{s} & V_{s} \\
0 & W_{s} & -U_{s}
\end{array}\right]  \tag{5.23}\\
{\left[C_{s q}\right] } & =\left[\begin{array}{cc}
-U_{s q} & -V_{s q} \\
-W_{s q} & -T_{s q}
\end{array}\right] \tag{5.24}
\end{align*}
$$

where

$$
\begin{equation*}
\left[K_{s}^{\mathrm{XY}}\right]=\mathcal{B}\left(\boldsymbol{N}_{s}^{\mathrm{X}}, \boldsymbol{N}_{s}^{\mathrm{Y}}\right)_{\Omega_{s}}+\frac{\mathrm{j} k_{0}}{z_{\mathrm{s}}}\left\langle\boldsymbol{N}_{s}, \boldsymbol{N}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}} \tag{5.25}
\end{equation*}
$$

with X and Y being either I or S , and

$$
\begin{align*}
& {\left[U_{s q}\right]=\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{N}_{s}, \boldsymbol{N}_{q}\right\rangle_{\Gamma_{s}^{q}}}  \tag{5.26}\\
& {\left[V_{s q}\right]=\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{N}_{s}, \boldsymbol{f}_{q}^{\mathrm{r}}\right\rangle_{\Gamma_{s}^{q}}}  \tag{5.27}\\
& {\left[W_{s q}\right]=\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \boldsymbol{N}_{q}\right\rangle_{\Gamma_{s}^{q}}}  \tag{5.28}\\
& {\left[T_{s q}\right]=\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \boldsymbol{f}_{q}^{\mathrm{r}}\right\rangle_{\Gamma_{s}^{q}}} \tag{5.29}
\end{align*}
$$

and $\left[U_{s}\right],\left[V_{s}\right],\left[W_{s}\right]$, and $\left[T_{s}\right]$ can be calculated by (5.26)-(5.29) with $q=s$. From these expressions, it is clear that $\left[F_{s}\right]=\left[F_{s}\right]^{\mathrm{T}}$ and $\left[C_{s q}\right]=\left[C_{q s}\right]^{\mathrm{T}}$.

### 5.1.3 MoM Analysis on 3-D Subdomain Surfaces

For the subdomains modeled by BIEs, the MoM can be applied to solve for the equivalent surface currents numerically. Testing (5.9) and (5.10) with the rotated divergence-conforming vector functions $\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}$ defined on the surface $S_{s}$ yields

$$
\begin{align*}
&-\left\langle\boldsymbol{T}_{s}, \tilde{\mathcal{K}}_{s}\left(\boldsymbol{m}_{s}\right)\right\rangle_{S_{s}}+ \eta_{\mathrm{r}}\left\langle\boldsymbol{T}_{s}, \mathcal{L}_{s}\left(\overline{\boldsymbol{j}}_{s}\right)\right\rangle_{S_{s}} \\
&+\frac{1}{2}\left\langle\boldsymbol{T}_{s}, \boldsymbol{e}_{s}\right\rangle_{S_{s}}=\left\langle\boldsymbol{T}_{s}, \boldsymbol{E}^{\mathrm{inc}}\right\rangle_{S_{s}}  \tag{5.30}\\
&\left\langle\boldsymbol{T}_{s}, \mathcal{L}_{s}\left(\boldsymbol{m}_{s}\right)\right\rangle_{S_{s}}+\eta_{\mathrm{r}}\left\langle\boldsymbol{T}_{s}, \tilde{\mathcal{K}}_{s}\left(\overline{\boldsymbol{j}}_{s}\right)\right\rangle_{S_{s}} \\
&+\frac{\eta_{\mathrm{r}}}{2}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{S_{s}}=\eta_{\mathrm{r}}\left\langle\boldsymbol{T}_{s}, \overline{\boldsymbol{H}}^{\mathrm{inc}}\right\rangle_{S_{s}} \tag{5.31}
\end{align*}
$$

where $\boldsymbol{e}_{s}=\hat{\boldsymbol{n}}_{s} \times \boldsymbol{m}_{s}$ and $\tilde{\mathcal{K}}_{s}$ is the principal value of the operator $\mathcal{K}_{s}$. To apply the boundary condition on $\Gamma_{\mathrm{IBC}}$, Equation (5.11) is tested by $\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}$ and scaled with $1 / \sqrt{z_{\mathrm{s}}}$ before being added to (5.30). To apply the transmission condition on $\Gamma_{s}$, Equation (5.12) is substituted into the third term of (5.30), which is finally rewritten as

$$
\begin{align*}
& -\left\langle\boldsymbol{T}_{s}, \tilde{\mathcal{K}}_{s}\left(\boldsymbol{m}_{s}\right)\right\rangle_{S_{s}}+\eta_{\mathrm{r}}\left\langle\boldsymbol{T}_{s}, \mathcal{L}_{s}\left(\overline{\boldsymbol{j}}_{s}\right)\right\rangle_{S_{s}}+\frac{1}{2}\left\langle\boldsymbol{T}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{\Gamma_{s}} \\
& +\left(\frac{1}{\sqrt{z_{\mathrm{s}}}}-\frac{1}{2}\right)\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \boldsymbol{m}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}}+\sqrt{z_{\mathrm{s}}}\left\langle\boldsymbol{T}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}} \\
= & -\frac{1}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\boldsymbol{T}_{s}, \boldsymbol{e}_{q}\right\rangle_{\Gamma_{s}^{q}}-\frac{1}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\boldsymbol{T}_{s}, \overline{\boldsymbol{j}}_{q}\right\rangle_{\Gamma_{s}^{q}}+\left\langle\boldsymbol{T}_{s}, \boldsymbol{E}^{\mathrm{inc}}\right\rangle_{S_{s}} . \tag{5.32}
\end{align*}
$$

A similar treatment to (5.31) yields

$$
\begin{align*}
& -\left\langle\boldsymbol{T}_{s}, \mathcal{L}_{s}\left(\boldsymbol{m}_{s}\right)\right\rangle_{S_{s}}-\eta_{\mathrm{r}}\left\langle\boldsymbol{T}_{s}, \tilde{\mathcal{K}}_{s}\left(\overline{\boldsymbol{j}}_{s}\right)\right\rangle_{S_{s}}-\frac{\eta_{\mathrm{r}}}{2}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \boldsymbol{e}_{s}\right\rangle_{\Gamma_{s}} \\
& -\frac{1}{\sqrt{z_{\mathrm{s}}}}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \boldsymbol{e}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}}-\left(\frac{\eta_{\mathrm{r}}}{2}-\sqrt{z_{\mathrm{s}}}\right)\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \overline{\boldsymbol{j}}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}} \\
= & -\frac{\eta_{\mathrm{r}}}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \boldsymbol{e}_{q}\right\rangle_{\Gamma_{s}^{q}}-\frac{\eta_{\mathrm{r}}}{2} \sum_{q \in \mathcal{N}(s)}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{T}_{s}, \overline{\boldsymbol{j}}_{q}\right\rangle_{\Gamma_{s}^{q}} \\
& -\eta_{\mathrm{r}}\left\langle\boldsymbol{T}_{s}, \overline{\boldsymbol{H}}^{\mathrm{inc}}\right\rangle_{S_{s}} . \tag{5.33}
\end{align*}
$$

In the above equations, the testing column vector $\boldsymbol{T}_{s}$ can be set to either $\boldsymbol{f}_{s}^{\mathrm{r}}$ or $\boldsymbol{f}_{s}^{\mathrm{b}}$ (a column vector containing the Buffa-Christiansen (BC) functions [80, 83, 96, 100]). Summing up (5.30) with $\boldsymbol{T}_{s}=\boldsymbol{f}_{s}^{\mathrm{b}}$ and (5.33) with $\boldsymbol{T}_{s}=\boldsymbol{f}_{s}^{\mathrm{r}}$ forms an equation denoted as CFIE(1). Summing up (5.31) with $\boldsymbol{T}_{s}=\boldsymbol{f}_{s}^{\mathrm{b}}$ and (5.32) with $\boldsymbol{T}_{s}=\boldsymbol{f}_{s}^{\mathrm{r}}$ forms another equation denoted as CFIE(2). The subdomain-level MoM system can be obtained by coupling CFIE(1) with CFIE(2).

To discretize the MoM subsystem, the fields and currents are expanded as

$$
\begin{align*}
\overline{\boldsymbol{j}}_{s} & =\boldsymbol{f}_{s}^{\mathrm{rT}} h_{s}  \tag{5.34}\\
\boldsymbol{e}_{s} & =\boldsymbol{N}_{s}^{\mathrm{T}} e_{s}  \tag{5.35}\\
\boldsymbol{m}_{s} & =-\boldsymbol{f}_{s}^{\mathrm{rT}} e_{s} \tag{5.36}
\end{align*}
$$

where $e_{s}$ and $h_{s}$ are the unknown vectors containing the expansion coefficients to be determined in the MoM subdomain $s$. By substituting (5.34)-(5.36) into the CFIE(1) and CFIE(2), the MoM subdomain equation can be obtained as

$$
\begin{equation*}
\left[M_{s}\right][T]_{s}\{x\}+\sum_{q \in \mathcal{N}(s)}\left[C_{s q}\right][T]_{q}\{x\}=[T]_{s}\{b\} \tag{5.37}
\end{equation*}
$$

where the MoM unknown vector in subdomain $s$ is $[T]_{s}\{x\}=\left\{e_{s}, h_{s}\right\}^{\mathrm{T}}$, and the subdomain excitation vector can be represented by $[T]_{s}\{b\}=\left\{b_{s}^{e}, b_{s}^{h}\right\}^{\mathrm{T}}$, in which

$$
\begin{align*}
& \left\{b_{s}^{e}\right\}=-\mathrm{j} k_{0} \eta_{\mathrm{r}}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \overline{\boldsymbol{H}}^{\mathrm{inc}}\right\rangle_{S_{s}}+\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{b}}, \boldsymbol{E}^{\mathrm{inc}}\right\rangle_{S_{s}}  \tag{5.38}\\
& \left\{b_{s}^{h}\right\}=\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \boldsymbol{E}^{\mathrm{inc}}\right\rangle_{S_{s}}+\mathrm{j} k_{0} \eta_{\mathrm{r}}\left\langle\boldsymbol{f}_{s}^{\mathrm{b}}, \overline{\boldsymbol{H}}^{\mathrm{inc}}\right\rangle_{S_{s}} . \tag{5.39}
\end{align*}
$$

Finally, the subdomain matrix and the coupling matrix can be written as

$$
\begin{align*}
& {\left[M_{s}\right]=\left[\begin{array}{cc}
Q_{s}+\eta_{\mathrm{r}} U_{s}+\frac{2}{\sqrt{z_{\mathrm{s}}}} R_{s} & -\eta_{\mathrm{r}} P_{s}+\left(\eta_{\mathrm{r}}-2 \sqrt{z_{\mathrm{s}}}\right) S_{s} \\
P_{s}+\left(\frac{2}{\sqrt{z_{\mathrm{s}}}}-1\right) S_{s} & \eta_{\mathrm{r}} Q_{s}+U_{s}+2 \sqrt{z_{\mathrm{s}}} R_{s}
\end{array}\right]}  \tag{5.40}\\
& {\left[C_{s q}\right]=\left[\begin{array}{cc}
-\eta_{\mathrm{r}} U_{s q} & -\eta_{\mathrm{r}} V_{s q} \\
W_{s q} & T_{s q}
\end{array}\right]} \tag{5.41}
\end{align*}
$$

where

$$
\begin{align*}
{\left[P_{s}\right]=} & \mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \tilde{\mathcal{K}}_{s}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}-\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{b}}, \mathcal{L}_{s}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}  \tag{5.42}\\
{\left[Q_{s}\right]=} & \mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \mathcal{L}_{s}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}+\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{b}}, \tilde{\mathcal{K}}_{s}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}} \\
& +\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{f}_{s}^{\mathrm{b}}, \boldsymbol{N}_{s}\right\rangle_{S_{s}}  \tag{5.43}\\
{\left[R_{s}\right]=} & \frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{N}_{s}, \boldsymbol{N}_{s}\right\rangle_{\Gamma_{\mathrm{IBC}}}  \tag{5.44}\\
{\left[S_{s}\right]=} & \frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{N}_{s}, \boldsymbol{f}_{s}^{\mathrm{r}}\right\rangle_{\Gamma_{\mathrm{IBC}}} \tag{5.45}
\end{align*}
$$

and the other matrices are the same as those defined in Section 5.1.2. When the proposed MS-RTC is applied to a composite object with multiple conducting and dielectric junctions, there is no need to assign special junction basis functions or apply special testing procedures at junctions.

Before we conclude the discussion of the multi-solver formulation, it is worth pointing out several special cases: 1) If only one FEM subdomain and the background subdomain $\Omega_{0}$ exist, the MS-RTC is reduced to an FE-BI method with two sets of DoFs on the BI surface, denoted as FE-BI2(CFIE), which permits the application of non-conformal meshes and basis functions with different orders on the inner and outer surfaces of the geometrical boundary. 2) If the MoM is applied to all the subdomains, the MS-RTC is reduced to a CFIE-based BIE method with the capability of modeling piecewise homogeneous objects with PEC and IBC boundary conditions. This method is a good candidate to model metallic patches with zero thickness attached to homogeneous materials. 3) If the MS-RTC is used to model an impenetrable object, only the background subdomain $\Omega_{0}$ is necessary. By applying the IBC boundary condition to the BIE in subdomain $\Omega_{0}$, it can model nonuniformly coated objects easily.

### 5.1.4 Iterative Solution

Assembling the FEM subdomain equation (5.22) or the MoM subdomain equation (5.37) for all the subdomains forms a global system matrix $[A]\{x\}=$ $\{b\}$, which can be solved by either a direct or an iterative method. To solve the MS-RTC system with a very large number of unknowns, an iterative method with fast algorithms and preconditioners is preferred.

To reduce the computational complexity and memory costs, the MLFMA with the mixed testing scheme [54] is applied to evaluate the matrix entries and accelerate the computation of matrix-vector products. The near-field interactions in the block matrices $\left[P_{s}\right]$ and $\left[Q_{s}\right]$ are the same as those given in (5.42) and (5.43). The calculation of the far-field interactions in each subdomain can be found in [54]. It is noted that because the Robin transmission condition is applied to couple different subdomains, the MLFMA tree structures in subdomains are completely independent so that it is flexible to choose the size of the groups and the number of multipole expansion terms based on the material properties for different subdomains, and there is no special treatment needed at the interfaces between the subdomains.

To achieve a highly convergent solution of the proposed scheme, it is required to apply an efficient preconditioner to the global system. Since two individual methods are involved in the proposed MS-RTC, a global preconditioner can be constructed by considering both the FEM and MoM subsystems. The preconditioner for the FEM subsystem is the same as (5.22). To construct a preconditioner for the MoM subsystem, the integral operators $\mathcal{L}_{s}$ and $\tilde{\mathcal{K}}_{s}$ in (5.30)-(5.33) are first approximated by local operators $\mathcal{L}_{s}^{\prime}$ and $\tilde{\mathcal{K}}_{s}^{\prime}$ that only take into account the self-patch interactions, and then discretized by Galerkin's method with $\boldsymbol{T}_{s}=\boldsymbol{f}_{s}^{\mathrm{r}}$. The MoM subsystem preconditioner can then be written as

$$
\begin{equation*}
\left[M_{s}^{\prime}\right][T]_{s}\{x\}+\sum_{q \in \mathcal{N}(s)}\left[C_{s q}\right][T]_{q}\{x\}=[T]_{s}\{b\} \tag{5.46}
\end{equation*}
$$

where $\left[M_{s}^{\prime}\right]$ has the same block matrices as the ones in $\left[M_{s}\right]$ except that $\left[P_{s}\right]$ and $\left[Q_{s}\right]$ are replaced with $\left[P_{s}^{\prime}\right]$ and $\left[Q_{s}^{\prime}\right]$ expressed as

$$
\begin{align*}
& {\left[P_{s}^{\prime}\right]=\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \tilde{\mathcal{K}}_{s}^{\prime}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}+\mathrm{j} k_{0}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{f}_{s}^{\mathrm{r}}, \mathcal{L}_{s}^{\prime}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}}  \tag{5.47}\\
& {\left[Q_{s}^{\prime}\right]=\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \mathcal{L}_{s}^{\prime}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}-\mathrm{j} k_{0}\left\langle\hat{\boldsymbol{n}}_{s} \times \boldsymbol{f}_{s}^{\mathrm{r}}, \tilde{\mathcal{K}}_{s}^{\prime}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}}
\end{align*}
$$

$$
\begin{equation*}
+\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \boldsymbol{f}_{s}^{\mathrm{r}}\right\rangle_{S_{s}} . \tag{5.48}
\end{equation*}
$$

Through the spectrum analysis for the local operators $\mathcal{L}_{s}^{\prime}$ and $\tilde{\mathcal{K}}_{s}^{\prime}$, it is easy to find that all the terms in the right-hand side of (5.47) and the second term in the right-hand side of (5.48) are negligible because their contributions to the spectrum vanish when the testing and basis functions lie in the same plane. Therefore, the matrices $P_{s}^{\prime}$ and $Q_{s}^{\prime}$ can be simplified as

$$
\begin{align*}
& {\left[P_{s}^{\prime}\right]=0}  \tag{5.49}\\
& {\left[Q_{s}^{\prime}\right]=\mathrm{j} k_{0}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \mathcal{L}_{s}^{\prime}\left(\boldsymbol{f}_{s}^{\mathrm{r}}\right)\right\rangle_{S_{s}}+\frac{\mathrm{j} k_{0}}{2}\left\langle\boldsymbol{f}_{s}^{\mathrm{r}}, \boldsymbol{f}_{s}^{\mathrm{r}}\right\rangle_{S_{s}} .} \tag{5.50}
\end{align*}
$$

The preconditioning matrix for the MS-RTC system can be formed by assembling (5.22) or (5.46) for all the subdomains. This matrix can also be obtained by approximating the CFIEs in the MS-RTC system with absorbing boundary conditions (ABCs) [54], thus the resulting preconditioner is called the extended ABC-based preconditioner. To show the effectiveness of this preconditioner, a cuboid illuminated by a plane wave at 131 MHz is considered. As shown in Fig. 5.2a, the cuboid is $1.0 \mathrm{~m} \times 2.0 \mathrm{~m} \times 0.5 \mathrm{~m}$ in size and consists of two subdomains. The first subdomain has a lossless dielectric material with $\epsilon_{\mathrm{r}}=2$, which is discretized into tetrahedrons with an average size of 0.15 m . The second one is an IBC subdomain with $z_{\mathrm{s}}=0.5$, whose surface is discretized into triangles with an average size of 0.15 m . The resulting number of unknowns is 4,972 . The spectrum distributions of the unpreconditioned MS-RTC system, the ABC-based preconditioner, and the preconditioned MS-RTC system are presented in Fig. 5.2b. As can be seen from the figure, the unpreconditioned system and the preconditioner have similar spectrum distributions, and the preconditioned system has eigenvalues clustered within a small region centered around $(1,0)$ in the complex plane. To apply the ABC-based preconditioner, one can solve the preconditioning matrix either directly or iteratively [54].

The MS-RTC algorithm is summarized as follows. The entire object with its background is first partitioned into multiple non-overlapping subdomains. To model the FEM and MoM subdomains, equations (5.22) and (5.37) are applied, respectively, and then coupled to form a global system. Finally, the global equation is solved iteratively with the extended ABC-based pre-
conditioner which is constructed by assembling (5.22) or (5.46) for all the subdomains.

### 5.2 Numerical Examples

In this section, several examples are presented to demonstrate the accuracy, versatility, and capability of the proposed MS-RTC scheme. The differences between the proposed MS-RTC and MS-CFIE presented in [51] are emphasized again through the numerical examples.

### 5.2.1 A Dielectric Sphere

To test the condition and accuracy of the MS-RTC with respect to partition, a lossless dielectric sphere with a relative permittivity of $\epsilon_{\mathrm{r}}=2$ and a radius of 1 m is illuminated by a $131-\mathrm{MHz}$ plane wave. The following four cases are considered:

The entire sphere is modeled by the FEM.
The entire sphere is modeled by the MoM.
The sphere is equally partitioned into two subdomains with each modeled by the MoM.

The sphere is equally partitioned into eight subdomains with each modeled by the MoM.

With a mesh size of 0.15 m in each case, the system matrices are solved directly. The bistatic radar cross section (RCS) is calculated and compared with the Mie series solutions, respectively. The condition number of the systems and the relative root-mean-square (RMS) errors [83] are given in Table 5.1. It can be seen that the results from four cases are all in good agreement with the Mie series solution. Thus the accuracy can be guaranteed when objects are partitioned into subdomains. It is also shown from cases 2 to 4 that the condition numbers of the system matrices have only small changes when the MoM is applied to all the subdomains. Thus the condition of the proposed scheme is very stable with respect to the number of partitions.

Figure 5.3 demonstrates that the electric currents at the interface of the two subdomains in case 3 are identical to each other. Also the electric currents at the surface of eight subdomains in case 4 are continuous and the current pattern is consistent with that on the surface of subdomain $\Omega_{0}$ as shown in Fig. 5.4.

Case 4 of this problem is also solved by the MS-CFIE with a total number of 9,384 unknowns, in which the biconjugate gradient stabilized (BiCGSTAB) method with the ABC-based preconditioner takes 13 iterations to converge to a relative residual error of $10^{-3}$ [51], while the MS-RTC takes 11 iterations to reach the same relative residual error. The memory costs with the application of the MLFMA are 516.1 MB and 811.0 MB for the MS-RTC and MS-CFIE, respectively. Both the MS-RTC and MS-CFIE can provide highly accurate and convergent solutions, but the modeling approaches are different. In the MS-CFIE, the sphere is equally divided into eight bodies with each modeled by the CFIE-based BIE method. The coupling between bodies is based on global interactions. In contrast, the MS-RTC partitions the sphere with its background into nine subdomains. The coupling between subdomains is based on local interactions. Thus the memory cost in the MS-RTC is smaller even though it has more DoFs than the MS-CFIE.

### 5.2.2 A Conducting Sphere with a Dielectric Coating

To show the two special cases mentioned in Section 5.1.3, a dielectric coated conducting sphere with a total radius of 0.5 m is illuminated by a $300-\mathrm{MHz}$ plane wave. The $0.25-\mathrm{m}$-thick dielectric coating layer has a relative permittivity of $\epsilon_{\mathrm{r}}=4-\mathrm{j}$. The following two cases are considered:

The coating layer is modeled by the FEM.
The coating layer is modeled by the MoM.
The coated sphere is meshed into tetrahedral or triangular elements, both with an average size of 0.05 m . The resulting system matrix is solved by the generalized minimal residual (GMRES) method [101] without restart or any preconditioner. The calculated bistatic RCS and the Mie series solutions are given in Fig. 5.5a. The RMS errors for case 1 are $1.4 \%$ and $1.3 \%$ for the VV- and HH-polarized cases, respectively, and the ones for case 2 are $0.7 \%$
and $0.4 \%$ for the VV- and HH-polarized cases, respectively. The convergence history of the scheme in the two cases and the FE-BI(EJ) method published in [30] are compared and shown in Fig. 5.5b. It is evident that the MoM (case 2) has the best convergence, and the FE-BI2(CFIE) method (case 1) has a better convergence than the $\mathrm{FE}-\mathrm{BI}(\mathrm{EJ})$ method in [30]. It is noted that the material property of the conducting sphere is modeled by the PEC boundary condition to the PDE (case 1) or the BIE (case 2) in the coating layer.

### 5.2.3 A Mixed IBC/PEC Cylinder

A mixed IBC/PEC cylinder is considered to demonstrate the capability of modeling nonuniformly coated objects. The cylinder has a length of $10 \lambda_{0}$ and a diameter of $1 \lambda_{0}$, where $\lambda_{0}$ is the free-space wavelength. Half of the cylinder is coated with $z_{\mathrm{s}}=2+\mathrm{j} 0.1$, and the other half is a perfect electric conductor. The system is solved by the GMRES method without restart or any preconditioner. The current distribution on the surface of the cylinder is given in Fig. 5.6a, from which the current variation along the cylinder can be easily observed. The bistatic RCS is calculated by the MS-RTC and the self dual integral equation (SDIE) method [53], respectively. The RCS results solved by the two methods are shown in Fig. 5.6b. With the solution from the SDIE method used as the reference, the RMS differences between the two solutions are $2.5 \%$ and $0.7 \%$ for the VV- and HH-polarized cases, respectively. The convergence history of the MS-RTC and the SDIE method is shown in Fig. 5.6c, which indicates that the proposed scheme for IBC problems has a better convergence than the SDIE method.

### 5.2.4 Two Separate Spheres

To demonstrate the versatility in assigning different methods to different subdomains, an example of two separate spheres is studied. As shown in Fig. 5.7, the radii of the two spheres are both $1 \lambda_{0}$. The subdomains I and II have a relative permittivity of $\epsilon_{\mathrm{r}}=1.5$, while subdomain III has a relative permittivity of $\epsilon_{\mathrm{r}}=2$. The IBC hemisphere has a normalized surface impedance $z_{\mathrm{s}}=0.5$. The following three cases are considered:

Subdomains I, II, and III are modeled by the MoM. The mesh size is $0.1 \lambda_{0}$, which results in 41,460 DoFs.

Subdomains I and II are modeled by the FEM, and subdomain III is modeled by the MoM. The mesh size is $0.1 \lambda_{0}$, which results in 73,257 DoFs.

Subdomains I, II, and III are modeled by the FEM. The exterior surfaces of subdomains I and III are meshed with an average size of $0.1 \lambda_{0}$. The volumes are meshed with an average size of $0.06 \lambda_{0}$. The number of DoFs is 102,134.

The BiCGSTAB method [102] with the extended ABC-based preconditioner and the MLFMA acceleration is applied to solve the systems in all the three cases. The numbers of iterations with a targeted relative residual error of $10^{-3}$ are 9,7 , and 6 , respectively. The ABC-based preconditioners can be solved easily by a direct or an iterative method. The bistatic RCS results in the three cases are in good agreement with each other as can be seen in Fig. 5.7.

To compare the MS-RTC and MS-CFIE in the modeling of this example, the scattering problem is solved by the MS-CFIE by assigning the FE-BI solver to subdomains I and II and the CFIE-based solvers to subdomain III and the IBC region. Different from the MS-RTC, the IBC region is modeled by an IBC solver. The RCS result shown in Fig. 5.7 agrees well with the those from the MS-RTC. With the result from the MS-CFIE used as the reference, the RMS differences for cases 1,2 , and 3 are $2.7 \%, 1.6 \%$, and $2.9 \%$, respectively. The detailed comparison between the MS-RTC in case 2 and the MS-CFIE is presented in Table 5.2. It shows that the two schemes have a similar memory usage although the MS-RTC has more DoFs than the MS-CFIE. Also, the MS-RTC takes fewer iterations and computation time than the MS-CFIE to solve the global system.

### 5.2.5 An Aircraft with a Launched Missile

To show the capability of the proposed MS-RTC, an aircraft with a launched missile is simulated at 2 GHz . As can be seen in Fig. 5.8a, the aircraft, with a length of 12.74 m , a width of 15.06 m , and a height of 2.95 m , consists of a nose, a cockpit, and a body. The aircraft nose is made of a two-layer
radome ( $\epsilon_{\mathrm{r} 1}=3-\mathrm{j} 0.03$ and $\epsilon_{\mathrm{r} 2}=2-\mathrm{j} 0.002$ ) and a conducting array, which is shown in Fig. 5.8b. The surface of the cockpit is coated with $z_{\mathrm{s}}=2+\mathrm{j} 0.2$. The surface of the aircraft body is coated with $z_{\mathrm{s}}=2+\mathrm{j} 0.2$ (in grey) and $z_{\mathrm{s}}=0.4+\mathrm{j} 0.04$ (in green). The missile with a head and a body is 4.17 m in length and 0.254 m in diameter. The missile head is made of a threelayer radome $\left(\epsilon_{\mathrm{r} 1}=3-\mathrm{j} 0.03, \epsilon_{\mathrm{r} 2}=4-\mathrm{j} 0.06\right.$, and $\left.\epsilon_{\mathrm{r} 3}=2-\mathrm{j} 0.002\right)$ and a patch antenna array, which is shown in Fig. 5.8c. The surface of the missile body is coated with $z_{\mathrm{s}}=2+\mathrm{j} 0.2$ (in grey) and $z_{\mathrm{s}}=0.4+\mathrm{j} 0.04$ (in green). To solve the scattering using the MS-RTC, the object is first decomposed into subdomains based on the material properties, and then assigned with a suitable method with proper boundary conditions. The details are given as follows.

Subdomain I: missile head consisting of layers 1 to 3 modeled by the FEM. Its volume is discretized into curvilinear tetrahedrons with an average size of 6.5 mm , which results in 776,341 DoFs.

Subdomain II: layer 1 in the aircraft nose modeled by the FEM. Its volume is discretized into curvilinear tetrahedrons with an average size of 6.5 mm , which results in $3,601,665$ DoFs.

Subdomain III: layer 2 in the aircraft nose modeled by the MoM. Its surface is discretized into curvilinear triangles with an average size of 10 mm , which results in 359,502 DoFs.

Subdomain IV: the default subdomain which is the free space. This subdomain is modeled by the MoM, and its surface is discretized into curvilinear triangles with an average size of 15 mm , which results in 4,818,132 DoFs.

The BiCGSTAB method with the extended ABC-based preconditioner is applied to solve the MS-RTC system with a total of 9,555,640 unknowns. The solver takes 16 and 14 iterations to converge to a relative residual error of $10^{-3}$ for the VV- and HH-polarized cases, respectively. The ABC-based preconditioner itself is also solved iteratively using the BiCGSTAB method with an approximate inverse preconditioner [92,103], which takes an average of 118 iterations to converge to $10^{-3}$. The application of the ABC-based preconditioner takes $19 \%$ of the total solution time. The equivalent electric current distributed over the entire surfaces of the aircraft and the missile is
shown in Fig. 5.9. It is easy to observe the current variation from the figures. More detailed views are provided in Fig. 5.10, from which the continuity of the current distribution across the nose and body and the consistency of the current patterns at the interfaces between subdomains II and III can be observed. For the FEM subdomains I and II, the electric field distributions are shown in Fig. 5.11. The VV- and HH-polarized bistatic RCS results are calculated in both the $x z$-plane and $y z$-plane, which are presented in Fig. 5.12.

As can be seen from this example, the patch arrays were easily modeled by applying the IBC or PEC boundary conditions to the BIE in layer 2 of the aircraft nose and the PDE in layer 3 of the missile head. However, when the MS-CFIE is used to solve this scattering problem, one has to cut out a region that encloses the patch arrays and model the region using the FE-BI method.

### 5.3 Discussion

From the theoretical investigation and numerical demonstration in the preceding sections, it is clear that the MS-RTC is able to perform an EM analysis on large-scale and complex objects. The features and advantages of the proposed scheme are summarized as follows.

1) To model individual subdomains, either the FEM or the MoM is chosen to model the interior region of each subdomain. An impenetrable region such as PEC or IBC region is not considered as a computational domain. The material property of the impenetrable region is enforced by applying the corresponding boundary conditions to the PDEs or BIEs in the computational domain. Therefore, it is easy to deal with subdomains with complicated boundary conditions. In contrast, in the methods presented in $[46,49]$, and $[51]$, impenetrable regions have to be modeled by surface integral equations. Consequently, problems such as those involving a dielectric subdomain with a partially conducting boundary can only be modeled by the FE-BI solvers.
2) The inter-subdomain coupling strategy is different from the ones applied in [44-46, 49], and [51]. To couple multiple subdomain equations in this
work, the Robin transmission condition is applied to avoid global interactions between the DoFs at the subdomain interfaces. The resulting coupling matrices are sparse. Also, the application of the Robin transmission condition provides the flexibility to use non-conformal meshes and different orders of basis functions in different subdomains, and the capability to apply FEM-based DDMs to FEM subdomains.
3) The MLFMA data structures for each subdomain are completely independent so that the size of the cubes and the number of multipole expansion terms are very flexible to choose based on the material property of each subdomain, and there is no special treatment needed at the interface between subdomains. This is different from the strategy used in [44] where a buffer region is required when near-field interactions are calculated between the subdomains, and the strategy used in [51] where a common MLFMA tree is applied.
4) The ABC-based preconditioner is extended and applied to precondition the proposed MS-RTC systems. The application of the ABC-based preconditioner can be carried out by solving a purely sparse preconditioning matrix either directly or iteratively, whereas the block diagonal preconditioner used in [44-46] and [49] consists of partly sparse and partly dense FE-BI and fully populated BIE subdomain matrices.
5) When the proposed MS-RTC is applied to a composite object with multiple subdomain junctions, there is no special junction basis function or special testing procedure needed.

Although the MS-RTC scheme has the aforementioned desirable properties, it is worth pointing out that the MS-RTC and MS-CFIE [51] schemes are parallel and complementary to each other, and their application and performance depend on specific problems. The MS-RTC scheme is more suited for modeling objects with complicated materials and open structures, whereas the MS-CFIE scheme can alleviate the difficulties of modeling extremely large objects by decomposing the object into multiple bodies with each modeled by a suitable solver.

### 5.4 Summary

This chapter presented a flexible MS-RTC scheme for numerical simulation of large-scale complex EM problems. In this scheme, the object and its surrounding background were decomposed into multiple non-overlapping subdomains, each of which was solved by either the FEM or MoM. The fields in subdomains were connected through the Robin transmission condition. Both the RWG and BC functions were applied to test the MS-RTC system in order to obtain a good accuracy of the solution. Furthermore, the MLFMA was applied to accelerate the computation and reduce the memory costs in each subdomain independently, and the ABC-based preconditioner was extended and employed to accelerate the global iterative convergence. Three special cases in the MS-RTC were discussed and applied to solve scattering problems. The differences between the MS-RTC and MS-CFIE were elaborated in the chapter. Numerical examples clearly demonstrated that the MS-RTC is a highly accurate, efficient, flexible, and robust numerical simulation tool that is powerful and capable for EM analysis of electrically large and highly complicated objects.

### 5.5 Figures and Tables



Figure 5.1: Arbitrarily shaped object with multiple subdomains.


Figure 5.2: (a) Geometry of a cuboid with two subdomains. (b) Spectrum distributions of the unpreconditioned MS-RTC system, the ABC-based preconditioner, and the preconditioned MS-RTC system.


Figure 5.3: Current distribution at the interface in case 3. (a) The first half of the sphere. (b) The second half of the sphere.


Figure 5.4: Current distribution in case 4. (a) A sphere is partitioned into eight subdomains. (b) Current distribution on the surface of subdomain $\Omega_{0}$. (c) Current distribution on the surfaces of the eight subdomains $\Omega_{1}$ to $\Omega_{8}$.


Figure 5.5: (a) Bistatic RCS of the coated sphere at 300 MHz calculated by different methods. (b) Convergence history of the GMRES method without restart or any preconditioner for the MS-RTC in the two cases and the FE-BI(EJ) method.


Figure 5.6: (a) Current distribution on the IBC/PEC cylinder. (b) Bistatic RCS of the IBC/PEC cylinder calculated by the MS-RTC and the SDIE method. (c) Convergence history of the GMRES method without restart or any preconditioner for the MS-RTC and the SDIE method.


Figure 5.7: Bistatic RCS of the two separate spheres calculated by different methods.


Figure 5.8: (a) An aircraft with a launched missile. (b) Aircraft nose which consists of a two-layer radome and a patch antenna array. (c) Missile head which consists of a three-layer radome and a patch antenna array.


Figure 5.9: Equivalent electric current distribution on the entire surface of the aircraft with a launched missile. (a) Top view. (b) Bottom view.


Figure 5.10: Detail of the equivalent electric current distribution on the aircraft nose. (a) The exterior surface with the area across the nose and body enlarged, (b) layer 1, and (c) layer 2.


Figure 5.11: Electric field distribution in (a) layer 1 of the aircraft nose and (b) the missile head.


Figure 5.12: Bistatic RCS of an aircraft with a launched missile at 2 GHz in the (a) $x z$-plane and (b) $y z$-plane.

Table 5.1: Condition and accuracy of different cases for modeling a dielectric sphere at 131 MHz

| Case | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: |
| Unknowns | 15,600 | 7,224 | 9,816 | 12,936 |
| Cond. Num. | 921 | 24 | 26 | 28 |
| RMS Err. (\%) | 1.0 | 0.4 | 0.96 | 1.4 |

Table 5.2: Comparison of the MS-RTC and MS-CFIE in solving the scattering of two separate spheres

|  | MS-RTC | MS-CFIE |
| :---: | :---: | :---: |
| Unknowns | 73,257 | 58,310 |
| Iterations | 7 | 13 |
| Memory $(\mathrm{GB})$ | 1.02 | 0.99 |
| Solution $(\mathrm{s})^{1}$ | 555.8 | 973.9 |

${ }^{1}$ Multi-solver systems are solved on a workstation with a 4-core Intel Xeon W3520 CPU.

## CHAPTER 6

## A PARALLELIZED MULTI-SOLVER SCHEME AND ITS APPLICATIONS

### 6.1 Introduction

Electromagnetic (EM) modeling and simulation of electrically large and highly complex objects are critical for applications related to radar scattering and antenna radiation. To solve such a problem, the hybrid finite element and boundary integral method is widely used. This method is not only capable of modeling complicated geometries and inhomogeneous materials but also able to eliminate the truncation error by enforcing a rigorous boundary condition through boundary integral equations [1].

Recently, a multi-solver algorithm based on Robin transmission conditions (MS-RTCs) has been proposed to significantly improve the modeling capability [55]. In this algorithm, an arbitrarily shaped object with its background is decomposed into multiple non-overlapping subdomains. The finite element method (FEM) is used to model subdomains with inhomogeneous and/or anisotropic materials, and the method of moments (MoM) is applied to model homogeneous subdomains. To couple different subdomains, Robin transmission conditions are employed to enforce the field continuity between neighboring subdomains. Assembling all the subdomain matrices and coupling matrices yields a global system matrix, which is solved iteratively with preconditioners and fast algorithms. Specifically, a preconditioner based on absorbing boundary conditions (ABCs) is applied to accelerate the convergence of the global solution. The multilevel fast multipole algorithm (MLFMA) is employed to accelerate the computation of MoM subdomains. Although the MS-RTC method is highly accurate, flexible, and robust in modeling large and complicated objects, the computational cost will be prohibitively high when it is used for very large EM simulations. To further improve the efficiency of the MS-RTC method, an accelerated MS-RTC al-
gorithm is developed on distributed computing systems in this chapter. First, a modeling strategy using the MS-RTC method is presented so that readers can easily follow the guideline to decompose objects into subdomains, assign suitable solvers, and apply the MS-RTC algorithm to solve real EM problems. Based on the modeling strategy, the parallelization strategy is proposed, which is followed by numerical examples.

### 6.2 Modeling Strategy Using the MS-RTC Scheme

To model a complex object using the MS-RTC scheme, the first step is to study the material properties in the different parts of the object. As shown in Fig. 6.1, the object is decomposed into three parts according to the material properties of each part. If the part is inhomogeneous, the FEM is applied to model it. If the part is a impenetrable material, the MoM can be employed to model. If the part is a homogeneous material, either the FEM or MoM can be used to model this part. It is preferred to use the MoM if this part is large. For the default region which is the free space, the MoM is applied. On the surface of each part, the boundary condition can be PEC, IBC, or RTC. Therefore, it is straightforward to model open surfaces such as metallic structures with zero thickness embedded in or attached to homogeneous materials. As can be seen, the subdomain sizes are not similar, and the solvers applied to subdomains have different computational complexities.

### 6.3 Parallel Algorithm

### 6.3.1 Overall Strategy

The parallelization strategy for the MS-RTC algorithm is to parallelize individual subdomains, which is different from conventional parallelized domain decomposition methods (DDMs), where an object is usually partitioned into a large number of subdomains in order to achieve a good parallel efficiency and all the subdomains are computed with the same solver. For the MS-RTC algorithm, an object is partitioned into a small number of subdomains based on the material properties and geometries. For example, it is preferable to
apply a single MoM solver to a large homogeneous dielectric region without partitioning this region further into smaller subdomains to avoid additional unknowns generated on the subdomain interfaces. Therefore, the number of subdomains in the MS-RTC algorithm is typically much smaller than that in the DDMs. Since the subdomains in the MS-RTC algorithm are modeled by different solvers which have different computational complexities, it is difficult to control and balance the computation workload among processors. To tackle this difficulty, it is preferred to parallelize subdomains individually in order to obtain a good parallel efficiency. Such a strategy can take advantage of the well-developed parallelized algorithms and apply them directly to each subdomain.

### 6.3.2 Assembly of System Matrix

In the MS-RTC algorithm, the individual solvers are either the FEM or the MoM. Hence, the parallelization of the system matrix assembly includes the parallelization of the assembly of the FEM subdomain matrices, the MoM subdomain matrices, and the coupling matrices. Among these matrices, the assembly of the MoM subdomain matrices is most time-consuming. Hence, the parallelized MLFMA based on the message passing interface (MPI) techniques is applied to accelerate the assembly of MoM matrices. As shown in Fig. 6.2, in each MPI process, multiple OpenMP threads are allocated to further accelerate the computation. The hybridization of the MPI and OpenMP parallel programming models can reduce the memory duplication and communication overhead in the MPI applications. The assembly of the sparse FEM and coupling matrices is very fast, hence they are parallelized using OpenMP technique in this work. The pseudocode for system matrix assembly is given in Algorithm 1. The detailed expression for the matrices $F_{s}, M_{s}$, and $C_{s q}$ can be found in Chapter 5.

### 6.3.3 Solution of Global System

To solve the global system iteratively, matrix-vector products (MVPs) have to be computed at each iteration. The MPI-based MLFMA is employed to accelerate the MVPs in the MoM subdomains. Because the Robin transmis-

```
Algorithm 1 Assembly of the MS-RTC System Matrix
    function SystemAssembly \((F, M, C) \quad \triangleright\) Where \(F\) - FEM subdomain
    matrices, \(M\) - MoM subdomain matrices, \(C\) - coupling matrices
        \(F=0, M=0, C=0\)
        for \(s=0\) to \(N_{s}\) do
            if \(s\) is the FEM subdomain then
                    Compute \(F_{s} \triangleright F_{s}\) is the FEM subdomain matrix which is
    computed using OpenMP programming model
            else \(\triangleright s\) is the MoM subdomain
            Compute \(M_{s} \triangleright M_{s}\) is the MoM subdomain matrix which can
    be computed using the MPI-based MLFMA
            for \(q \neq s\) do
                Compute \(C_{s q} \quad \triangleright C_{s q}\) is the coupling matrix between
    subdomain \(s\) and \(q\)
```

sion condition is used to couple the subdomains, the MLFMA tree structures in different subdomains are completely independent so that the group sizes and the number of multipole expansion terms are chosen based only on the subdomain's material property [55]. The parallelization approach adopted in this work is based on a hybrid strategy with an introduction of a transition level, which has been shown to yield a satisfactory parallel efficiency [104]. For the finer levels below the transition level, the far field patterns (FFPs) of a group are replicated in each processor while the groups are partitioned equally among all processors; for the coarser levels above the transition level, the groups are replicated in every processor, and the FFPs are partitioned equally among all processors. The MVPs in the FEM subdomains are accelerated using the OpenMP parallel technique. All the subdomains are parallelized and computed individually. The pseudocode for system matrix-vector products is given in Algorithm 2.

To summarize the overall procedure, it is worth to present the flowchart of the MS-RTC scheme. As can be seen in Fig. 6.3, when the program starts, either the FEM or the MoM solver is assigned to each subdomain until all the subdomains are associated with a solver. Once a subdomain is assigned with a solver, the subdomain system will be computed. After all the subdomain systems are computed, a global system will be formed to solve. Within a subdomain, the computation is parallelized using the MPI or OpenMP programming model.

```
Algorithm 2 Matrix-vector products for the MS-RTC system
    function SYSTEMMVPs \((M, F, C, v, r) \quad \triangleright\) Where \(M\) -
    MoM subdomain matrices, \(F\) - FEM subdomain matrices, \(C\) - coupling
    matrices, \(v\) - input vector, \(r\) - output vector
        \(r=0\)
        for \(s=0\) to \(N_{s}\) do
            if \(s\) is the FEM subdomain then
                \(r_{s}=r_{s}+F_{s} v_{s} \quad \triangleright F_{s} v_{s}\) is parallelized using OpenMP
    programming model
            else \(\quad \triangleright s\) is the MoM subdomain
                \(r_{s}=r_{s}+M_{s} v_{s} \triangleright M_{s} v_{s}\) is accelerated by the MPI-based
    MLFMA
            for \(q \neq s\) do
                \(r_{s}=r_{s}+C_{s q} v_{q}\)
```


### 6.4 Numerical Examples

In this section, the scattering from an aircraft with a launched missile is first computed to validate the implementation of the parallelized MS-RTC method. Then several large and complex examples are given to show parallel efficiency, and the modeling and computational capability of the proposed method. Due to the limited memory for each core and a long time for serial computation, the parallel efficiency with respect to the baseline execution time $T_{q}$ on $q$ cores is defined as

$$
\begin{equation*}
\text { Efficiency }=\frac{q T_{q}}{p T_{p}} \times 100 \% \tag{6.1}
\end{equation*}
$$

where $T_{p}$ is the execution time using $p$ cores.
The computations are carried out on CISCO Arcetri cluster, with each node quipped with up to $282.80-\mathrm{GHz}$ Intel Xeon E5-2680 processors. Intel MKL Pardiso solver based on distributed computing systems is applied to solve the preconditioner.

### 6.4.1 An Aircraft with a Launched Missile

To validate the implementation of the parallelized MS-RTC method, an aircraft with a launched missile as presented in Chapter 6 is simulated at 2

GHz. The OpenMP-based MS-RTC and MPI-based MS-RTC methods with the same decomposition and solver assignment strategy are applied to solve this problem. The OpenMP-based MS-RTC method is performed on a single node with 20 OpenMP threads, and BiCGSTAB method takes 179 min to converge to a relative residual error of $10^{-3}$. The MPI-based MS-RTC method uses 150 MPI processes with 1 OpenMP thread for each process. The iterative solution takes 36 min to converge to the same residual error. The speedup is about 5 times. The radar cross section of the object is calculated and shown in Fig. 6.4. As can be seen, the two results are in good agreement with each other. The relative root-mean-square difference between two results is $0.11 \%$. The difference is caused by the different parallel implementations of the MS-RTC algorithm.

### 6.4.2 A Missile-like Object

Next, a missile-like object with a $3-\mathrm{m}$-long body having a surface impedance of $z_{\mathrm{s}}=0.4+\mathrm{j} 0.04$ and $1-\mathrm{cm}$-thick fins having a surface impedance of $z_{\mathrm{s}}=3+$ j0.3 is illuminated by a $8-\mathrm{GHz}$ plane wave. As shown in Fig. 6.5, the radome layer with $\epsilon_{\mathrm{r}}=3$ is modeled by the MoM (495,252 DoFs); the enclosed patch antenna sitting on a dielectric layer with $\epsilon_{\mathrm{r}}=1.5-\mathrm{j} 0.008$ is modeled by the FEM $(413,966 \mathrm{DoFs})$; and the surrounding free space is modeled by the MoM (3,185,076 DoFs). The preconditioned BiCGSTAB method is applied to solve the MS-RTC system with 4,094,294 DoFs in total, which takes 17 iterations to converge to a targeted relative residual error of $10^{-3}$. The current distribution is shown in Fig. 6.6, and the parallel efficiency of the two MoM solvers is given in Fig. 6.7. The computation is performed using $16,32,48$, and 64 MPI processes and 1 OpenMP thread for each process, respectively, with the baseline computation of 16 processes. In the 32- and 64process cases, since there are few nonempty cubes in the radome subdomain, the communication and computation overheads are comparable. Therefore, the MVP in this subdomain has a low parallel efficiency. The total parallel efficiency is $73.3 \%$.

### 6.4.3 Human Head with a Cell Phone

To demonstrate the capability of the parallelized MS-RTC method, large and complex examples are given in the following sections. As shown in Fig. 6.8, the first example is the human head excited by a dipole in the cell phone at 60 GHz . The head has a size of $257 \mathrm{~mm} \times 222 \mathrm{~mm} \times 182 \mathrm{~mm}$ with a relative permittivity of $\epsilon_{\mathrm{r}}=8.0-\mathrm{j} 10.8$. It is modeled by the MoM method, and its surface is discretized into curvilinear triangles with an average size of 0.3 mm , which results in $10,885,284$ DoFs. The phone is a dielectric coated IBC object with $\epsilon_{\mathrm{r}}=3.0-\mathrm{j} 0.03$ and $z_{\mathrm{s}}=0.4+\mathrm{j} 0.04$, which is modeled by the FEM method. Its volume is discretized into curvilinear tetrahedrons with an average size of 0.3 mm , which results in $8,497,137$ DoFs. The free space is modeled by the MoM method, and its surface is discretized curvilinear triangles with an average size of 0.3 mm , which results in $12,161,664$ DoFs.

The BiCGSTAB method is applied to solve the MS-RTC system with a total of $31,544,085$ unknowns, which takes 16 iterations to converge to a relative residual error of $10^{-2}$. The preconditioner used in the iterative solution consists of the FEM subdomain matrices and the diagonal entries of the MoM subdomain matrices. The detailed computation times and parallel efficiency are given in Table 6.1. The baseline computation is using 20 MPI processes and 10 OpenMP threads for each process. For the assembly, the parallel efficiency is between $70 \%$ and $80 \%$. The total parallel efficiency for 900 cores is $52.8 \%$. The equivalent electric current distribution on the human head and electric field distribution in the cell phone are shown in Fig. 6.9. It can be seen that the energy is concentrated around the ear at such a high frequency.

### 6.4.4 Three Aircraft

Next, three aircraft as shown in Fig. 6.10 are simulated at 3 GHz . Each aircraft has a length of 12.74 m , a width of 15.06 m , and a height of 2.95 m . Aircraft I with its launched missile is modeled as a composite object. The detailed material information can be referred to Chapter 5. Aircraft II and III are modeled as a coated object with $z_{\mathrm{s}}=0.4+\mathrm{j} 0.04$. To solve the scattering using the parallelized MS-RTC method, the objects are first decomposed into subdomains based on the material properties, and then
assigned with a proper numerical method with boundary conditions. The details are given as follows.

1) Subdomain I: missile head consisting of layers 1 to 3 modeled by the FEM. Its volume is discretized into curvilinear tetrahedrons with an average size of 6.5 mm , which results in 1,535,183 DoFs.
2) Subdomain II: layer 1 in the aircraft nose modeled by the FEM. Its volume is discretized into curvilinear tetrahedrons with an average size of 6.5 mm , which results in 7,353,049 DoFs.
3) Subdomain III: layer 2 in the aircraft nose modeled by the MoM. Its surface is discretized into curvilinear triangles with an average size of 10 mm , which results in 615,540 DoFs.
4) Subdomain IV: the default subdomain which is the free space. This subdomain is modeled by the MoM , and its surface is discretized into curvilinear triangles with an average size of 15 mm , which results in 29,795,982 DoFs.

The BiCGSTAB method with the extended ABC-based preconditioner is applied to solve the MS-RTC system with a total of 39,299,754 unknowns. The solver takes 16 iterations to converge to a relative residual error of $10^{-3}$. The equivalent electric current distributed over the entire surfaces of the aircraft and the missile is shown in Fig. 6.11. It is easy to observe the current variation from the figures. More detailed views are provided in Fig. 6.12, from which the continuity of the current distribution across the nose and body and the consistency of the current patterns at the interfaces between subdomains II and III can be observed. The detailed computation times and parallel efficiency are given in Table 6.2. The baseline computation is using 20 MPI processes and 10 OpenMP threads for each process. For the assembly, the parallel efficiency is between $74 \%$ and $89 \%$. The total parallel efficiency for 900 cores is $49.5 \%$.

### 6.4.5 An Aircraft Carrier

Lastly, the scattering from an aircraft carrier at 300 MHz is considered. As shown in Fig. 6.13, an aircraft carrier has a length of 322 m , a width of 75
m , and a height of 35 m . The surfaces of the carrier body and control tower are coated with $z_{\mathrm{s}}=0.4+\mathrm{j} 0.4$ and $z_{\mathrm{s}}=2+\mathrm{j} 0.2$, respectively. There are three aircraft sitting on the carrier, and one aircraft taking off from the carrier. The aircraft with $\epsilon_{\mathrm{r}}=2$ is $12.74-\mathrm{m}$ long, $15.06-\mathrm{m}$ wide, and $2.95-\mathrm{m}$ high. The three-layer radome $\left(\epsilon_{\mathrm{r}}=1,2\right.$, and 4 , respectively from inside out) on the top of the control tower has a diameter of 4 m . The detailed modeling strategy for different objects is given as follows.

1) Subdomain I: the three-layer radome is modeled by the FEM. Its volume is discretized into curvilinear tetrahedrons with a varying size from 0.03 to 0.06 m , which results in $4,746,143$ DoFs.
2) Subdomains II-V: four dielectric aircraft are modeled by the MoM. Their surfaces are discretized into curvilinear triangles with an average size of 0.07 m , which results in 798,168 DoFs in total.
3) Subdomain VI: the default subdomain which is the free space. This subdomain is modeled by the MoM, and its surface is discretized into curvilinear triangles with an average size of 0.1 m , which results in 43,360,746 DoFs.

The BiCGSTAB method is applied to solve the MS-RTC system with a total of $48,968,921$ unknowns, which takes 21 iterations to converge to a relative residual error of $10^{-2}$. To reduce the computation and memory cost of the preconditioner, the preconditioner used for the iterative solution is created by combining the FEM subdomain matrices and the diagonal entries in the MoM subdomain matrices. The equivalent electric current distributed over the entire surfaces of the aircraft and the aircraft carrier is shown in Fig. 6.14. It is easy to observe the current variation from the figures. The electric field distribution in the radome is provided in Fig. 6.15. The wave pattern is obvious and the maximum value is located around the monopole antenna. The detailed computation times and parallel efficiency are given in Table 6.3. The baseline computation is using 30 MPI processes and 10 OpenMP threads for each process. For the assembly, the parallel efficiency is between $66 \%$ and $80 \%$. The total parallel efficiency for 900 cores is $46.8 \%$.

### 6.5 Summary

An accelerated multi-solver (MS) method was developed on distributed computing systems to simulate the scattering from large and complex objects. In this method, the targeted object with its background was decomposed into multiple subdomains which were modeled by either the finite element method or the method of moments. The parallelization strategy for the MS method was to parallelize different subdomains individually, which was different from the parallelized domain decomposition methods, where the subdomains were handled in parallel. The multilevel fast multipole algorithm was parallelized to enable computation on many processors. Numerical examples were given to show the parallel efficiency and modeling capability of the proposed strategy.

Through the numerical examples, the factors affecting the parallel efficiency and workload balance can be analyzed. It can be seen that the models in the examples consist of different materials and complex structures and geometries. To solve the problems accurately, the meshes are required to resolve the detailed structures and material properties, which results in nonuniform meshes for the overall models. Also, due to the complex geometry of the models, the number of nonempty cubes at the same level in the MLFMA varies significantly, which decreases the parallel efficiency.

### 6.6 Figures and Tables



Figure 6.1: Modeling strategy using the MS-RTC scheme.


Figure 6.2: Hybrid MPI-OpenMP parallel programming model.


Figure 6.3: Flowchart for the MS-RTC scheme.


Figure 6.4: Bistatic RCS of an aircraft with a launched missile at 2 GHz .


Figure 6.5: Decomposition of the missile-like object.


Figure 6.6: Equivalent electric current distribution on the entire surface of the missile.


Figure 6.7: Parallel efficiency of the two MoM solvers for a missile-like object. The baseline computation is using 16 processes.


Figure 6.8: Human head with a cell phone.


Figure 6.9: (a) Equivalent electric current distribution on the human head. (b) Electric field distribution in the cell phone.


Figure 6.10: Three aircraft with a launched missile.


Figure 6.11: Equivalent electric current distribution on the surfaces of the aircraft. (a) Top view (log scale). (b) Bottom view (linear scale).


Figure 6.12: Detail of the equivalent electric current distribution on aircraft I. (a) The exterior surface with the area across the nose and body enlarged, (b) layer 1, and (c) layer 2.


Figure 6.13: Aircraft carrier model.


Figure 6.14: Equivalent electric current distribution on an aircraft carrier. (a) Top view (linear scale). (b) Side view (log scale).


Figure 6.15: Electric field distribution in the radome.

Table 6.1: Parallel efficiency for the simulation of the human head with a phone at 60 GHz , with a total number of DoFs being 31,544,085.

| Number of Processes | 20 | 40 | 62 | 90 |
| :---: | :---: | :---: | :---: | :---: |
| Number of Cores | 200 | 400 | 620 | 900 |
| Assembly (min.) | 186.7 | 127.4 | 84.2 | 51.4 |
| Precond. Prep. (min.) | 13.9 | 8.5 | 6.8 | 5.7 |
| MV (min.) | 123.2 | 88.8 | 75.9 | 66.4 |
| Precond. Appl. (min.) | 29.3 | 28.6 | 25.8 | 25.2 |
| Total (min.) | 353.1 | 253.3 | 192.7 | 148.7 |
| Parallel Efficiency (\%) | baseline | 69.7 | 59.1 | 52.8 |

Table 6.2: Parallel efficiency for the simulation of the three aircraft at 3 GHz , with a total number of DoFs being 39,299,754.

| Number of Processes | 20 | 40 | 62 | 90 |
| :---: | :---: | :---: | :---: | :---: |
| Number of Cores | 200 | 400 | 620 | 900 |
| Assembly (min.) | 219.6 | 123.2 | 87.2 | 65.4 |
| Precond. Prep. (min.) | 18.7 | 15.9 | 12.4 | 11.0 |
| MV (min.) | 98.8 | 92.1 | 64.6 | 50.6 |
| Precond. Appl. (min.) | 37.8 | 42.7 | 33.5 | 41.3 |
| Total (min.) | 374.9 | 273.9 | 197.7 | 168.3 |
| Parallel Efficiency (\%) | baseline | 68.4 | 61.2 | 49.5 |

Table 6.3: Parallel efficiency for the simulation of the aircraft carrier at 300 MHz , with a total number of DoFs being 48,968,921.

| Number of Processes | 30 | 62 | 75 | 90 |
| :---: | :---: | :---: | :---: | :---: |
| Number of Cores | 300 | 620 | 750 | 900 |
| Assembly (min.) | 84.0 | 54.0 | 50.8 | 40.8 |
| Precond. Prep. (min.) | 29.8 | 17.4 | 15.5 | 14.3 |
| MV (min.) | 112.1 | 102.5 | 90.5 | 92.0 |
| Precond. Appl. (min.) | 66.8 | 62.1 | 62.1 | 61.5 |
| Total (min.) | 292.7 | 236.0 | 218.9 | 208.6 |
| Parallel Efficiency (\%) | baseline | 60.0 | 53.5 | 46.8 |

## CHAPTER 7

# ELECTROMAGNETIC SIMULATION OF SPECIFIC ABSORPTION RATE AT 5G FREQUENCIES WITH A MULTI-SOLVER METHOD 

### 7.1 Introduction

As the ownership of mobile devices such as smartphones and pads increases rapidly, the potential human health hazards caused by the electromagnetic (EM) radiation from electronic devices have received intensive attention. When the EM energy absorbed by human tissues exceeds a certain limit, there can be a harmful impact on the nervous system, blood and immune system, and others. Therefore, the assessment of the radiation impact on human health becomes important in the electronic design [105]. To quantify the EM energy absorbed by human tissues, specific absorption rate (SAR) is widely used.

Numerical simulation is an effective way to assess the SAR during the electronic design process. To evaluate SAR values, a full-wave EM analysis is performed first, and then followed by the SAR calculation. For the full-wave analysis, the methods based on partial differential equations such as the finite difference and the finite element methods are usually used because of their flexibility in modeling inhomogeneous materials and complicated geometries. However, due to the complexity of media and structures of tissues and organs in the human body, the efficiency of the numerical simulation is decreased when the simulation is performed at very high frequencies such as those for 5G applications. To perform simulation efficiently, one can first simplify the model to minimize the computational domain at very high frequencies based on the strong skin effect, then apply the multi-solver method based on the Robin transmission conditions (MS-RTC) [55] to maximize the capability of modeling electrically large and complex objects. In this chapter, such a modeling approach which includes the numerical method and the simplification criteria is discussed and justified, and the SAR results calculated by the

MS-RTC on the simplified human model are provided.

### 7.2 Modeling

### 7.2.1 Problem Description

The problem to be addressed here is the SAR evaluation in a human head with a dipole excitation at 60 GHz . Shown in Fig. 7.1a, the human head with a brain and two eyeballs has a total height of 260.5 mm and a total width of 150.0 mm . The relative permittivities and conductivities of different body tissues can be found in [106]. To evaluate the SAR distribution in the human head, EM fields are first computed by a numerical method, and then the SAR distribution is calculated using

$$
\begin{equation*}
\mathrm{SAR}=\frac{\sigma|E|^{2}}{\rho} \tag{7.1}
\end{equation*}
$$

where $|E|$ is the electric field strength at a given frequency and location, $\sigma$ is the local conductivity inside in the human model, and $\rho$ is the local density of the head tissue [107].

### 7.2.2 EM Field Computation

The EM modeling of a human head at a very high frequency is difficult because the discrete representation of a human head usually results in a tremendous number of degrees of freedom (DoFs) to be simulated. Therefore, an accurate and efficient numerical solver is required to simulate the EM fields in the human head. In this chapter, the MS-RTC method is employed to model the human head. Specifically, the human head model with its background is partitioned into multiple non-overlapping subdomains. The finite element method is applied to model the subdomains with inhomogeneous materials and the moment method is employed to model the subdomains with homogeneous materials. To couple different solvers, the Robin transmission condition is applied to enforce the field continuity between different subdomains. To accelerate the computation, the multilevel fast multipole algorithm (MLFMA) and a preconditioner based on an absorbing boundary
condition are applied. For a detailed discussion on the MS-RTC method, the reader is referred to [55].

To demonstrate the EM modeling of the human head using the MS-RTC method, a homogeneous human head with a relative permittivity of $\epsilon_{\mathrm{r}}=8.0-$ j 10.8 is considered. The phone is a two-layer dielectric coated PEC object with $\epsilon_{\mathrm{r}}=1.0$ and 4.0, respectively from inside out. The head and the cell phone are excited by a dipole in a cell phone at 3 GHz . As shown in Fig. 7.2a, the human head is decomposed into two portions. The portion close to the cell phone is modeled by the FEM, and the other is modeled by the MoM. The cell phone is modeled by the FEM. The preconditioned BiCGSTAB method with the MLFMA acceleration is applied to solve the combined system with 176,887 unknowns, which takes 10 iterations to converge to a targeted relative residual error of $10^{-3}$. The surface current distribution is plotted in Fig. 7.2b, which shows that the energy is concentrated around the ear.

### 7.2.3 Simplified Human Head Model

When the SAR distribution is evaluated at a very high frequency, it is essential to minimize the computational domain in order to reduce the computational cost. To reduce the computational domain, the human head model can be simplified with the consideration of the strong skin effect at very high frequencies. Since dry skin has a relative permittivity of $\epsilon_{\mathrm{r}}=7.9$ and a conductivity of $\sigma=36.4 \mathrm{~S} / \mathrm{m}$ at 60 GHz [106], the corresponding skin depth is 0.386 mm . Therefore, the field strength will decay to $0.67 \%$ after 5 skin depths ( 1.9 mm ) and $0.091 \%$ after 7 skin depths ( 2.7 mm ). Consequently, the human head model given in Fig. 7.1a can be simplified to a human head shell, which contains only the human skin. To further reduce the computational domain, only the lit region is retained in the lateral direction, which is shown in Fig. 7.1b. The mass density used in the simulation is $1020 \mathrm{~kg} / \mathrm{m}^{3}$.

### 7.3 Numerical Results

To justify the proposed simplification, a spherical body tissue with a radius of 5 mm is simulated using a dipole excitation at 60 GHz . To reduce the computational domain, a portion of the interior tissue is hollowed out, which
results in a spherical shell with a thickness of 1.5 mm . Further, only the lit region of the spherical shell is retained and modeled by the MS-RTC method. The SAR distribution in the original spherical tissues and simplified model are shown in Figs. 7.3a and 7.3b, respectively. As can be seen, there is very little effect on the SAR distribution when the model is carefully simplified according to the proposed approach at such a high frequency. As shown in Fig. 7.3c, the SAR values along the $y$-axis agree well between the two results. The relative difference between the two maximum values is $1.5 \%$.

To evaluate the SAR distribution in the human head at 60 GHz , a dipole is placed close to the ear to radiate EM fields. To compute the fields, the multisolver method is applied to the simplified model with an average thickness of 3.0 mm . Modeled by the finite element method, the simplified model is discretized into tetrahedrons with an average size of 0.15 mm , which results in $21,342,571 \mathrm{DoFs}$. The surrounding free space is modeled by the moment method with its surface discretized into curvilinear triangles with an average size of 0.2 mm , which results in 4,964,682 DoFs. The preconditioned BiCGSTAB method with the MLFMA acceleration is applied to solve the system equation with $26,307,253$ unknowns in total, which takes 11 iterations to converge to a targeted relative residual error of $10^{-3}$. The preconditioner is also solved iteratively, which takes an average of 20 iterations to converge to $10^{-3}$.

The SAR distribution in the simplified model is shown in Fig. 7.4. As can be seen, most of the energy absorbed by the human head is concentrated around the ear. It can also be seen from Fig. 7.4b that the energy can only penetrate into the skin by a very short distance due to the very strong skin effect. Based on the observation of the simulation results, the simplified human head model is sufficient to represent the original problem for the SAR evaluation.

### 7.4 Summary

An electromagnetic modeling approach was presented for the simulation of the specific absorption rate (SAR) in a human head at 60 GHz . Based on the strong skin effect, the human head model was simplified to reduce the computation cost. The multi-solver method based on the Robin transmission
condition was employed to compute electromagnetic fields in the simplified human head. The SAR was calculated after the electromagnetic fields were computed. Numerical examples showed that the multi-solver method was very efficient in solving electromagnetic fields in the human head and the simplified human head model can be used in the SAR simulation with an acceptable accuracy.

### 7.5 Figures



Figure 7.1: (a) Human head model with a brain and two eyeballs. (b) Simplified human head model.


Figure 7.2: A homogeneous human head excited by an excitation of a dipole in a cell phone. (a) Model. (b) Current distribution.


Figure 7.3: SAR distribution in a spherical model. (a) Original model. (b) Simplified model. (c) SAR values along the $y$-axis, which is shown as the white lines marked in (a) and (b).

(a)

(b)

Figure 7.4: SAR distribution in the simplified human head model. (a) On the skin. (b) In a cutting plane.

## CHAPTER 8

## CONCLUSION AND FUTURE WORK

In this thesis, multi-solver frameworks were studied for fast and accurate numerical analysis of electromagnetic (EM) scattering and radiation problems. To develop multi-solver schemes for modeling of electrically large and complex objects, a variety of advanced numerical algorithms and parallel computing techniques were investigated and applied.

First, a GPU-accelerated multilevel fast multiple algorithm (MLFMA) was developed on multiple GPU computing systems to improve the efficiency of the traditional MLFMA by taking advantage of GPU hardware advancement. The linear systems can be solved efficiently with the multi-GPU accelerated MLFMA, which paves a way to solve very large EM problems using boundary-integral-equation-based methods. Compared with the 8threaded CPU-based MLFMA, the OpenMP-CUDA-MLFMA method can achieve from 5 to 20 total speedup ratios.

Second, to model large and complex objects efficiently and accurately, a finite element-boundary integral (FE-BI) method was proposed. For the accuracy improvement, a mixed testing scheme, in which the Rao-Wilton-Glisson and the Buffa-Christiansen functions were both applied as testing functions, was presented. For the efficiency improvement, an efficient absorbing boundary condition (ABC)-based preconditioner was proposed to accelerate the convergence of the iterative solution, and the GPU-accelerated MLFMA was applied to speed up the iterative solution. Compared with the 8-threaded CPU-based algorithm, the GPU-accelerated FE-BI-MLFMA algorithm can achieve a total speedup of up to 25.5 times.

Third, a multi-solver scheme based on combined field integral equation (MS-CFIE) was proposed to solve large and complex electromagnetic problems. In this algorithm, an object was partitioned into multiple bodies based on its material property and geometry. The FE-BI method was applied to model bodies with complicated materials, and the method of moments (MoM) was
applied to model bodies with homogeneous or conducting materials. Numerical examples showed that the proposed multi-solver scheme was accurate and flexible to solve scattering problems of electrically large and complex objects.

Fourth, a multi-solver scheme based on Robin transmission condition (MSRTC) was presented. In this scheme, the entire computational domain consisting of the object and its background was decomposed into multiple nonoverlapping subdomains. For the subdomains with inhomogeneous materials and complex geometries, the FEM was applied. For the homogeneous or impenetrable subdomains, the MoM was applied. To couple different subdomains, the Robin transmission condition was employed at the subdomain interfaces. This scheme was demonstrated to have a good accuracy, versatility, and capability through a variety of numerical examples.

Fifth, to improve the modeling capability of multi-solver schemes, a parallelized MS-RTC scheme was proposed on distributed computing systems. The parallelization strategy for the MS method was to parallelize different subdomains individually, which as different from the parallelized domain decomposition methods, where the subdomains were handled in parallel. The assembly of system matrices was parallelized by hybridizing MPI and OpenMP. The iterative solution was accelerated by MPI-based MLFMA on many computing processors. Numerical examples were given to show the parallel efficiency of the proposed strategy and capability of the proposed method.

Finally, as an application of the MS schemes, the simulation of the specific absorption rate (SAR) in a human head at 5 G frequencies was performed. In order to simulate such a problem, the human head model was simplified based on the strong skin effect at such a high frequency. The MS-RTC method was employed to compute electromagnetic fields. Numerical examples showed that the multi-solver method was very efficient in solving electromagnetic fields in the human head and the simplified human head model can be used in the SAR simulation with an acceptable accuracy.

The proposed MS schemes provide very general frameworks where different numerical methods are coupled to solve complex problems. While many aspects of the MS schemes have been studied and discussed in this dissertation, there are still a few research directions that can be pursued in the future to further improve the modeling capability.

1) Domain decomposition methods can be easily incorporated into individual solvers in the MS-RTC method. For example, the dual-primal finite element tearing and interconnecting method can be applied to the FEM subdomains.
2) Non-conformal mesh can be used at the interfaces of subdomains. Different mesh densities and order of basis functions can be applied to different subdomains so that the mesh generation becomes easier and more flexible.
3) The ABC-based preconditioner was proved to be very effective for preconditioning the MS systems. In this dissertation, the MPI-based direct solver and OpenMP-based iterative solver are used. Parallelized iterative solvers on distributed computing systems are worth studying in the future. Other effective preconditioners for the proposed MS schemes are also open to discussion.
4) The MS schemes can be extended to EM problems at low frequencies. To avoid the low-frequency breakdown problem, tree-co-tree splitting and loop-star decomposition techniques can be applied to the FEM subdomains and the MoM subdomains, respectively.

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