NAKAJIMA'S (Q,T)-CHARACTERS AS QUANTUM CLUSTER VARIABLES

BY

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DISSERTATION

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Abstract

Nakajima introduced [36–40] a t-deformation of q-characters, (q,t)-characters for short, and their twisted multiplication through the geometry of quiver varieties. The Nakajima (q,t)-characters of Kirillov-Reshetikhin modules satisfy a t-deformed T-system (see [39]). The T-system is a discrete dynamical system that can be interpreted as a mutation relation in a cluster algebra in two different ways, depending on the choice of direction of evolution. In this thesis, we show that the Nakajima t-deformed T-system of type A_r forms a quantum mutation relation in a quantization of exactly one of the cluster algebra structures attached to the T-system.

There are 2 main parts to our work. The bulk of the work is a combinatorial construction that proves (q,t)-characters of a certain set of Kirillov-Reshetikhin modules t-commute under Nakajima's twisted multiplication. We use a slightly modified version of the tableaux-sum notation for q-characters introduced in [38] and define the notion of a block-tableau, which plays an integral role in the proof. Once t-commutativity is established, the second half of this thesis is concerned with the commutation coefficients of the given set of Kirillov-Reshetikhin modules. In particular, we show that the commutation coefficients are compatible with the cluster algebra exchange matrix and the mutation relations in the language of Berenstein-Zelevinsky [3].

To my family.

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Table of Contents

List of Symbols

$\mathbb{C}$	Complex numbers.
$\mathbb{Z}$	Integers.
$\mathbb{Z}_+$	Positive integers not including 0.
$\mathfrak{g}$	Finite dimensional complex semi-simple Lie algebra of type ADE.
$\mathfrak{sl}_{r+1}$	Finite dimensional complex simple Lie algebra of type A_r.
r	Rank of $\mathfrak{g}$.
I	Index set $\{1, 2, \ldots, r\}$.
$\mathfrak{n}_-$	Negative nilpotent subalgebra of $\mathfrak{g}$.
$\mathfrak{n}_+$	Positive nilpotent subalgebra of $\mathfrak{g}$.
$\mathfrak{h}$	Cartan subalgebra of $\mathfrak{g}$.
$U(\mathfrak{g})$	Universal enveloping algebra of $\mathfrak{g}$.
$U_q(\mathfrak{g})$	Quantum group associated to $\mathfrak{g}$.
$U_q(\widehat{\mathfrak{g}})$	Quantum affine algebra associated to $\mathfrak{g}$.
α_i	Simple root of $\mathfrak{g}$.
Q	Root lattice.
Q_+	Positive root lattice.
ω_i	Fundamental weight of $\mathfrak{g}$.
P	Weight lattice.
$C = (C_{ij})_{i,j \in I}$	Cartan matrix of $\mathfrak{g}$.
W	The Weyl group of $\mathfrak{g}$.
s_i	generators of W.

$\mathrm{Rep}(U_q(\mathfrak{g}))$	The Grothendieck ring of the category of finite-dimensional $U_q(\mathfrak{g})$-modules.
$\mathrm{Rep}(U_q(\widehat{\mathfrak{g}}))$	The Grothendieck ring of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules.
$\mathfrak{S}_s$	Symmetric group on s letters.
χ	Character map.
$\chi_{q,t}$	Nakajima's (q,t)-character map.
q	A formal variable.
$W_k^{(i)}(a)$	Kirillov-Reshetikhin module with spectral parameter $a \in \mathbb{C}^*$.
$\mathbf{P}_{k,a}^{(i)}$	The Drinfeld polynomials defining $W_k^{(i)}(a)$.
$\left\{T_k^{(i)}(a)\right\}$	The variables of the T-system, for $a \in \mathbb{C}^*$.
χ_q	q-character map.
$W_{k,j}^{(i)}$	Kirillov-Reshetikhin module with spectral parameter $a = q^j$.
$\mathbf{P}_{k,j}^{(i)}$	The Drinfeld polynomials defining $W_{k,j}^{(i)}$.
$\left\{T_{k,l}^{(i)}\right\}$	The variables of the T-system, for $a = q^l$.
$\left\{Q_k^{(i)}\right\}$	The variables of the Q-system.
$[n]_q$	The q-number.
δ_{ij}	The delta function.
$\mathcal{Q}$	Quiver.
$\mathcal{Q}_0$	Set of vertices in $\mathcal{Q}$.
$\mathcal{Q}_1$	Set of arrows in $\mathcal{Q}$.
B	Exchange matrix of a cluster algebra.
μ_k	Seed mutation in the direction of k.
$\mathcal{C}$	The fundamental cluster.
$\mathbf{Y}_{k,j}^{(i)}$	The dominant monomial of $\chi_q(W_{k,j}^{(i)})$.
Λ	The commutation matrix of a quantum seed.
$\mathcal{B}_{k,j}^{(i)}$	The set of KR-tableaux.
B_{CT}	The block-tableau associated to a pair of column KR-tableaux (C,T).

Chapter 1

Overview

1.1 A short history

An *R-matrix* is an intertwiner of the co-product of a quantum algebra. An integrable lattice model can be defined using Boltzmann weights that come from the entries of an R-matrix. The partition function of a single row of the lattice (in a lattice model) with a periodic boundary condition is called a *transfer matrix*.

The 6-*vertex model* is an integrable lattice model defined by the R-matrix of the form $V \otimes W \to W \otimes V$, where V and W are 2-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-modules. Bazhanov-Reshetikhin showed in [1] that the transfer matrices of the 6-vertex model obey an interesting functional relation. Later, Bazhanov-Reshetikhin used the generalized RSOS models in [2] and obtained a class of functional relations that are known as the T-system of type A_r, that is of Dynkin type A and rank r.

The existence of the T-system of type A_r is equivalent to the existence of short exact sequences of finite-dimensional modules over the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_{r+1})$ (see [33]). The short exact sequences are as follows:

$$0 \to W_k^{(i+1)}(a) \otimes W_k^{(i-1)}(a) \to W_k^{(i)}(a) \otimes W_k^{(i)}\left(aq^2\right) \to W_{k+1}^{(i)}(a) \otimes W_{k-1}^{(i)}(aq^2) \to 0, \qquad (1.1.1)$$

where $q \in \mathbb{C}^*$ and $W_k^{(i)}(a)$ is the *Kirillov-Reshetikhin module* (KR-module) of type A_r with spectral parameter $a \in \mathbb{C}^*$ (see Definition 2.2.4 and Section 2.3 for examples), $k \in \mathbb{Z}_+$, and $i = 1, 2, \ldots, r$. The T-system of type A_r is a quadratic recursion relation on the variables $\left\{T_k^{(i)}(a)\right\}$, where $T_k^{(i)}(a)$ is the q-*character*, which is a generalization of the usual character over simple Lie algebras to over quantum affine algebras, of the KR-module $W_k^{(i)}(a)$. The T-system is then a character formula induced from the short exact sequences (1.1.1) and is given by:

$$T_k^{(i)}(a)\,T_k^{(i)}\left(aq^2\right) = T_{k+1}^{(i)}(a)T_{k-1}^{(i)}(aq^2) + T_k^{(i+1)}(a)T_k^{(i-1)}(a)\,, \tag{1.1.2}$$

where $T_k^{(0)}(a) = T_k^{(r+1)}(a) = 1$. Without loss of generality, we can always assume $a \in \mathbb{C}^*$ is fixed. Then we need only keep track of the powers of q. Using a change of variables $T_k^{(i)}(aq^j) \to T_{k,k+j}^{(i)}$ and relabeling $l = k + j + 1$, we arrive at another form of the T-system, also known as the *octahedron recurrence*:

$$T_{k,l-1}^{(i)}T_{k,l+1}^{(i)} = T_{k+1,l}^{(i)}T_{k-1,l}^{(i)} + T_{k,l}^{(i-1)}T_{k,l}^{(i+1)}, \tag{1.1.3}$$

for $k \in \mathbb{Z}_+$, $i = 1, 2, \ldots, r$, and with the convention $T_{k,l}^{(0)} = T_{k,l}^{(r+1)} = 1$.

Many important results and conjectures followed this explicit representation theoretical connection. Firstly, the connection of the T-system to a previously known functional relations called the Q-system became evident. More precisely, every $U_q(\widehat{\mathfrak{sl}}_{r+1})$-module can be regarded as $U_q(\mathfrak{sl}_{r+1})$-module. Taking the $U_q(\mathfrak{sl}_{r+1})$-characters of the modules of the short exact sequences (1.1.1), one obtains the Q-system of type A_r given by:

$$(Q_k^{(i)})^2 = Q_{k+1}^{(i)}Q_{k-1}^{(i)} + Q_k^{(i-1)}Q_k^{(i+1)}. \tag{1.1.4}$$

The Q-system (1.1.4) is the $a \to \infty$ limit of the T-system (1.1.2). Kuniba, Nakanishi, and Suzuki used this connection in [33] and derived the general T-system using the Q-systems of all Dynkin types. Similar conjectures of short exact sequences of finite-dimensional modules over the quantum affine algebras of all types followed. The proof of this conjecture was given by Nakajima in [39] for types ADE and by Hernandez in [23] for general types through an extension of Nakajima's deformations to all Dynkin types. The finite-dimensional modules in question are the KR-modules of general type, which were first considered by Kirillov and Reshetikhin in [31] and studied intensively since then with a motivation from the so-called Bethe ansatz (see [4,22,33,34] and the references therein).

In this thesis, we take a closer look at Nakajima's work in [39]. We now give a summary of the results of this thesis.

1.2 Summary of results

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank r, and let $U_q(\widehat{\mathfrak{g}})$ be the corresponding untwisted quantum affine algebra (see Chapter 2 for definitions). Let q be a formal variable. The category of finite-dimensional complex representations of $U_q(\widehat{\mathfrak{g}})$, denoted $\mathrm{Rep}(U_q(\widehat{\mathfrak{g}}))$, has been studied by many authors (see [5–7,18,19, 21,28] and the references therein).

One of the main tools used to study finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules is their q-characters, which are the affine analogs of $U_q(\mathfrak{g})$-characters. The theory of q-characters was introduced by Knight [32] and Frenkel-Reshetikhin [19] for the Yangians and the quantum affine algebras respectively. The combinatorial properties of q-characters were studied by Frenkel-Mukhin in [18] and shown to closely resemble the properties of the classical $U_q(\mathfrak{g})$-characters. Despite the close analogy, little is known about q-characters. In particular, there is no description of the restriction functor to the quantum groups of finite type, no tensor product decomposition rules, and no equivalent of the Weyl character formula. Frenkel-Mukhin gave an algorithm in [18] that can compute the q-character of a module iteratively starting from its highest weight vector for *some* simple objects in $\mathrm{Rep}(U_q(\widehat{\mathfrak{g}}))$. The special class of modules that Frenkel-Mukhin algorithm does produce the correct q-characters includes the KR-modules (see [38]).

One of the key properties concerning the KR-modules is that their characters and q-characters satisfy the Q-system and the T-system respectively. As mentioned before, the former result was proved by Nakajima in [39] for types ADE using the t-analog of q-characters, (q,t)-characters for short, defined geometrically through quiver varieties. The Nakajima (q,t)-character map is a t-deformation of the q-character map that reduces to the q-character when $t = 1$. The (q,t)-characters of KR-modules satisfy a t-deformed T-system with a twisted multiplication on the variables $T^{(i)}_{k,l}$. The deformed T-system is a quadratic recursion relation on non-commutative variables that reduce to the classical T-system when $t = 1$.

Cluster algebras were invented by Fomin-Zelevinsky in [16] as a combinatorial tool to approach total positivity in algebraic groups and canonical bases in quantum groups, which were invented independently and simultaneously by Lusztig in [35] and Kashiwara in [27]. Since then, the theory of cluster algebras have taken a life of its own and many links to areas such as Poisson geometry, integrable systems, higher Teichmuller spaces, algebraic geometry, and representation theory to name a few have emerged (see [30] and references therein).

Cluster algebras are commutative algebras generated by the union of overlapping sets, called *clusters*, of variables $\{\mathbf{x}_0, \mathbf{x}_1, \ldots\}$, where $\mathbf{x}_i = (x_{1:i}, x_{2:i}, \ldots, x_{r:i})$ is a tuple of some rational functions. The generators are related by rational transformations called *mutations*, which are determined by an *exchange matrix* [16]. Since all cluster variables are related to one another via mutations, it suffices to state a single cluster, called the *fundamental cluster*, along with the exchange matrix, in order to define a cluster algebra. More precisely, given an exchange matrix of rank r, we give a set of commutative formal variables $\mathbf{x}_0 = (x_{1:0}, \ldots, x_{r:0})$ as the very first cluster variables and mutate $\mathbf{x}_0$ in all possible directions to obtain the rest of the generators.

When the exchange matrix is invertible, there exists a canonical Poisson structure on the cluster variables [20]. A quantization of this canonical Poisson structure was introduced by Berenstein-Zelevinsky in [3], and is called a *quantum cluster algebra*. Quantum cluster algebras are non-commutative algebras, and as such, their generators are not required to have any commutation relation at all. However, the variables within the same cluster satisfy a t-commutation relation. Quantum cluster algebras play an integral role in this thesis, which we explain next.

Kedem [29] and Di Francesco-Kedem [8] realized Q and T systems as mutation relations in certain cluster algebras. The cluster algebra formulation of the Q-system in [29] was used to obtain a unique quantization of the Q-system in [9,10]. The resulting quantum Q-system was shown to have deep connections with many areas such as the fusion product, defined in [15], of KR-modules [11], the quantum current subalgebra $U_q(\mathfrak{n}_+[u, u^{-1}])$ in $U_q(\widehat{\mathfrak{sl}}_2)$ [13] , where $\mathfrak{n}_+$ is the positive nilpotent subalgebra in $\mathfrak{sl}_2$, and a new set of q-difference operators [12], which are generalizations of the Macdonald raising operators in the limit $t \to \infty$, acting on the characters of KR-modules of type A_r.

We assume the $\left\{T^{(i)}_{k,l}\right\}$ variables to be invertible. Then, the T-system equation can be interpreted as mutation relations in an infinite rank cluster algebra [8,24]. The T-system in (1.1.3) is written such that the direction of mutation is in the direction of the l-parameter. That is, one can obtain $T^{(i)}_{k,l+1}$ using variables with lower value of l only:

$$T^{(i)}_{k,l+1} = (T^{(i)}_{k,l-1})^{-1} \left(T^{(i)}_{k+1,l}T^{(i)}_{k-1,l} + T^{(i-1)}_{k,l}T^{(i+1)}_{k,l}\right).$$

It is possible to re-write the T-system in (1.1.3) such that the direction of mutation is in the k-parameter (see [10]). Because the T-system cluster algebra is an infinite rank cluster algebra, these two choices of directions of mutations define 2 distinct cluster algebra structures. Moreover, since the exchange matrices in both cases are of infinite rank, there are no canonical Poisson structures to these cluster algebras associated with the T-system. Di Francesco and Kedem considered a quantization of the A_1 T-system in [10], which reduces to the quantum Q-system in the $a \to \infty$ limit. This choice corresponds to the quantization of the T-system cluster algebra with direction of mutation in the k-parameter.

In light of the above results, it is natural to ask if the Nakajima deformed T-system forms a quantum mutation in a quantum cluster algebra, and if yes, then which one. The main result of this thesis shows that the Nakajima deformed T-system is a quantum mutation in a quantum cluster algebra when the direction of mutation is in the l-direction only. As a consequence, we show that the Nakajima quantization of the T-system is not compatible with the quantum Q-system. This is because the cluster algebra whose quantization is the Nakajima deformed T-system is a different cluster algebra than the one which reduces to the Q-system cluster algebra in [10], since the former mutates in the l-direction, while the latter mutates in the k-direction.

In order to prove that the deformed T-system forms a quantum mutation, we first define a cluster algebra structure on the classical T-system of type A_r. That is, we explicitly write down the exchange matrix B and the fundamental cluster variables $\mathcal{C}$, which is a set of q-characters of certain KR-modules. Then, we follow the Berenstein-Zelevinsky prescription to show that the non-commutative versions of the fundamental cluster variables, given by the Nakajima (q,t)-characters of the same KR-modules, satisfy the following 3 conditions:

Condition I: The fundamental cluster variables t-commute with respect to the twisted multiplication defined in [39]. That is, for any $x_1, x_2 \in \mathcal{C}$, there exists $\lambda_{12} \in \mathbb{Z}$ such that $x_1 * x_2 = t^{\lambda_{12}} x_2 * x_1$. The matrix $\Lambda = (\lambda_{ij})$ is called the commutation matrix of $\mathcal{C}$.

Condition II: The commutation matrix Λ and the exchange matrix B form a compatible pair (see Definition 3.2.2).

Condition III: The commutation matrix Λ is compatible with the quantum mutation relation, which is given by the Nakajima deformed T-system. (see Section 3.2)

The proofs of Conditions II and III are done in full generality, and in fact, applies to types D and E without any modifications. The proof of Condition I is entirely combinatorial and is specific to type A_r only. As mentioned before, the fundamental cluster variables are given by the (q,t)-characters of certain KR-modules, which are identical to their q-characters in type A_r (see [39]). The Nakajima twisted multiplication is a twisting of the regular multiplication on the monomials of the (q,t)-characters. We use a slightly modified version of the Nakajima crystal basis from [38] for the (q,t)-characters of KR-modules of type A_r and define the notions of the block-tableau and the twisted multiplication of the block-tableau, which play an integral role in the proof of Condition I.

The structure of the remaining text is as follows. In Chapter 2, we review simple Lie algebras, quantum groups, quantum affine algebras of type ADE, and describe q-characters and their properties. In Chapter 3, we describe cluster and quantum cluster algebras, give the formulations of the T and Q-systems as cluster and quantum cluster algebras. We also make Conditions I-III explicit and state the precise conditions that a deformed T-system must satisfy in order to form a valid quantum mutation relation. Chapter 4 contains the proofs of the main results, and Chapter 5 concludes.

Chapter 2

Quantum affine algebras

In this section, we will describe q-characters of finite dimensional modules over quantum affine algebras from ground up. For simplicity of notations, we will provide results and definitions for type A_r since the main result pertains to this type alone. However, everything in this section is directly applicable to types ADE or generalizable to all Dynkin types. When the definition is specific to type A_r, we will specify the type. When the result is applicable to types ADE, we will use general Lie algebra $\mathfrak{g}$ in the notation. We will also provide a complete working example of type A_1 to motivate the need for the q-character theory.

For more detailed exposition on classical Lie algebras and their representations, see [25,26], on quantum affine algebras and their finite-dimensional representations, see [5,6], on q-characters and their combinatorial properties, see [18,19] and the excellent online notes [14].

2.1 Quick primer on $\mathfrak{g}$ and $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be the finite dimensional complex semi-simple Lie algebra of type ADE, $\mathfrak{h}$ its Cartan subalgebra and $r = \mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$ be its rank. Let $\{\alpha_1, \dots, \alpha_r\}$ and $\{\omega_1, \dots, \omega_r\}$ be the sets of simple roots and fundamental weights of $\mathfrak{g}$. Let $I = \{1, 2, \dots, r\}$ be the index set. The Chevalley generators of $\mathfrak{g}$ are $\{e_i, f_i, h_i\}_{i \in I}$ with relations:

$$[h_i, e_j] = C_{ji} e_j, \quad [h_i, f_j] = -C_{ji} f_j, \quad [e_i, f_j] = \delta_{ij} h_j \ ,$$

together with the Serre relations:

$$(\operatorname{ad} e_j)^{1-C_{ij}}(e_i) = (\operatorname{ad} f_j)^{1-C_{ij}}(f_i) = 0\,,$$

where $C = (C_{ij})_{i,j\in I}$ is the Cartan matrix associated to $\mathfrak{g}$ and ad is the adjoint action given by $(\operatorname{ad} x)(y) = [x, y]$ for any $x, y \in \mathfrak{g}$. When $\mathfrak{g}$ is of type A and rank r, the Cartan matrix is an r by r matrix given by:

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}, \tag{2.1.1}$$

Define the q-numbers, q-factorial and q-binomial as follows:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+1} + \cdots + q^{n-1}, \tag{2.1.2}$$

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q, \tag{2.1.3}$$

$$\binom{n}{m}_q := \frac{[n]_q!}{[n-m]_q![m]_q!} \tag{2.1.4}$$

The quantum group $U_q(\mathfrak{g})$ is an associative algebra over $\mathbb{C}(q)$ generated by $E_i, F_i, K_i^{\pm 1}$, $r \in I$, with relations:

$$K_iE_jK_i^{-1} = q^{C_{ji}}E_j; \quad E_iF_j - F_jE_i = \delta_{ij}\tfrac{K_i - K_i^{-1}}{q-q^{-1}}; \quad K_iF_jK_i^{-1} = q^{-C_{ji}}F_j; \tag{2.1.5}$$

for $i, j \in I$, together with the quantum Serre relations:

$$\begin{aligned} \sum_{n=0}^{1-C_{ij}} (-1)^n \binom{1-C_{ij}}{n}_q E_i^n E_j E_i^{1-C_{ij}-n} &= 0, \\ \sum_{n=0}^{1-C_{ij}} (-1)^n \binom{1-C_{ij}}{n}_q F_i^n F_j F_i^{1-C_{ij}-n} &= 0, \end{aligned}$$

where C is the Cartan matrix of $\mathfrak{g}$.

The root data of the quantum group $U_q(\mathfrak{g})$ is the same as that of the simple Lie algebra $\mathfrak{g}$. Therefore, the finite-dimensional $U_q(\mathfrak{g})$-modules are combinatorially identical to those over $\mathfrak{g}$ as well. Let $Q = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ and $P = \bigoplus_{i=1}^r \mathbb{Z}\omega_i$ denote the root and weight lattices of $U_q(\mathfrak{g})$. Let $Q_+ = \bigoplus_{i=1}^r \mathbb{Z}_+\alpha_i$ denote the positive root lattice. Let U_+, U_0, U_- be the subalgebras of $U_q(\mathfrak{g})$ generated by $E_i, K_i^{\pm}$ and F_i respectively.

We now state the main results concerning $U_q(\mathfrak{g})$-modules. For the proofs of each statement, see [25].

Theorem 2.1.1. Let V be an irreducible module over $U_q(\mathfrak{g})$.

1. There exists $\lambda \in P$ and a vector $v_\lambda \in V$ such that $Kv_\lambda = q^{\lambda(K)}v_\lambda$ for all $K \in U_0$, $U_+v_\lambda = 0$, and $V = U_-v_\lambda$. That is, V is a *highest weight module* generated by a *cyclic* vector v_λ. We say that v_λ is the *highest weight vector* of V.

2. V is finite-dimensional if and only if $\lambda(K_i) \in \mathbb{Z}_+$ for all $i \in I$. That is, $\lambda = \sum_{i \in I} n_i\omega_i$ for $n_i \in \mathbb{Z}_+$.

3. V decomposes into a direct sum of finite vector spaces, called the *weight spaces*, of the form:

$$V = \bigoplus_{\alpha \in Q_+} V_{\lambda - \alpha}, \quad \text{where} \quad V_\mu = \left\{ v \in V | Kv = q^{\mu(K)}v \text{ for any } K \in U_0 \right\}$$

4. Given a $U_q(\mathfrak{g})$-module $V = \bigoplus_{\mu \in P} V_\mu$, the *character* of V is the formal sum $\chi(V) = \sum_{\mu \in P} \dim_{\mathbb{C}(q)} V_\mu e^\mu$, which, with the identification $e^{\omega_i} = y_i$, defines an injective ring homomorphism

$$\chi : \mathrm{Rep}(U_q(\mathfrak{g})) \to \mathbb{Z}[y_i^{\pm 1}]_{i \in I} ,$$

where $\mathrm{Rep}(U_q(\mathfrak{g}))$ is the Grothendieck ring of the category of finite-dimensional $U_q(\mathfrak{g})$-modules.

Definition 2.1.2. A monomial $m \in \mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ is called a *dominant* monomial if m consists of positive powers of y_i's only.

By Theorem 2.1.1 (2), a $U_q(\mathfrak{g})$-module is finite-dimensional if and only if its highest weight vector is a positive and integral linear combination of the fundamental weights. Therefore, the highest weight vector of a finite-dimensional module V corresponds to a dominant monomial in $\chi(V)$.

Definition 2.1.3. Let $\mathfrak{g}$ be a Lie algebra of type ADE and let C be the Cartan matrix corresponding to $\mathfrak{g}$. Let $a_i \in \mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ be defined as follows:

$$a_i := y_i^2 \prod_{C_{ji}=-1} y_j^{-1},$$

with the convention $y_j = 1$ if $j \notin I$.

Notice that since y_i corresponds to a vector with weight ω_i, a_i corresponds to a vector with $U_q(\mathfrak{g})$-weight α_i. That is, the monomial $y_i a_j^{-1}$ is a monomial corresponding to a vector with weight $\omega_i - \alpha_j$. Given $V \in \mathrm{Rep}(U_q(\mathfrak{g}))$, the character of V can be written as

$$\chi(V) = m_+ \left(1 + \sum_p M_p\right), \tag{2.1.6}$$

where m_+ is a dominant monomial corresponding to the highest weight vector, and M_p's are monomials in a_i^{-1}'s only. That is, all monomials in $\chi(V)$, except for the dominant one, is of the form $m_+ M_p = m_+ \prod_i a_i^{-v_i}$, $v_i \in \mathbb{Z}_+$, which corresponds to a vector of weight $\lambda - \sum_i v_i \alpha_i$. The statement (2.1.6) is equivalent to Theorem 2.1.1 (3).

Let W be the Weyl group of $\mathfrak{g}$, where $\mathfrak{g}$ is of Dynkin type A_r. It is generated by reflections s_i, $i \in I$, that act on the simple roots as follows:

$$s_i(\alpha_i) = -\alpha_i \quad \text{and} \quad s_i(\alpha_j) = \alpha_j + \alpha_i, i \neq j.$$

The action of W can be extended to $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ in a natural way:

$$s_i(a_i) = a_i^{-1} \quad \text{and} \quad s_i(a_j) = a_j a_i.$$

It is a well-known fact that the image of the character map χ consists of W-invariant polynomials in $\mathbb{Z}[y_i^{\pm q}]_{i\in I}$. The subring of invariants of s_i in $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ is equal to:

$$K_i = \mathbb{Z}[y_j^{\pm 1}]_{j\neq i} \otimes \mathbb{Z}[y_i + y_i a_i^{-1}]\,.$$

Then, the image of the character map is given by:

$$\mathrm{Im}\chi \simeq \mathrm{Rep}U_q(\widehat{\mathfrak{g}}) \simeq \bigcap_{i\in I} K_i\,. \tag{2.1.7}$$

The classification of irreducible $U_q(\mathfrak{sl}_2)$-modules is already known and is given by the following theorem:

Theorem 2.1.4. For each $m \in \mathbb{Z}_+$, there exists a unique irreducible $U_q(\mathfrak{sl}_2)$-module V_m of dimension $m+1$, whose U_0-eigenspace decomposition is given by

$$V_m = \bigoplus_{j=0}^{m} (V_m)_{m-2j}$$

where each of the weight spaces is one-dimensional. Moreover, every finite dimensional irreducible $U_q(\mathfrak{sl}_2)$-module has the form V_m for some $m \in \mathbb{Z}_+$.

Example 2.1.5. Let V_m be as in Theorem 2.1.4. Then,

$$\begin{aligned}
\chi(V_0) &= 1 = [1]_y \\
\chi(V_1) &= y^{-1} + y = [2]_y \\
\chi(V_2) &= y^{-2} + 1 + y^2 = [3]_y \\
\chi(V_3) &= y^{-3} + y^{-1} + y + y^3 = [4]_y \\
\chi(V_k) &= [k+1]_y
\end{aligned}$$

2.2 Drinfeld realization of $U_q(\widehat{\mathfrak{g}})$ and finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules

The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is generated by $x^{\pm}_{i,n}$ $(i \in I, n \in \mathbb{Z})$, $h_{i,n}$ $(i \in I, n \in \mathbb{Z}\setminus\{0\})$, $K_i^{\pm 1}$ $(i \in I)$, and central elements $c^{\pm 1/2}$ with the following relations:

$$\begin{gathered}
K_iK_j = K_jK_i\,, \qquad K_ih_{j,n} = h_{j,n}K_i\,, \qquad K_ix^{\pm}_{j,n}K_i^{-1} = q^{\pm C_{ij}}x^{\pm}_{j,n}\,,\\
\left[h_{i,n}, x^{\pm}_{j,m}\right] = \pm[nC_{ij}]_q c^{\mp|n|}x^{\pm}_{j,n+m}\,,\\
x^{\pm}_{i,n+1}x^{\pm}_{j,m} - q^{\pm C_{ij}}x_{j,m} \pm x^{\pm}_{i,n+1} = q^{\pm C_{ij}}x^{\pm}_{i,n}x^{\pm}_{j,m+1} - x^{\pm}_{j,m+1}x^{\pm}_{i,n}\,,\\
[h_{i,n}, h_{j,m}] = \delta_{n,-m}\tfrac{1}{n}\left[nC_{ij}\right]_q \tfrac{c^n - c^{-n}}{q-q^{-1}}\,,\\
\left[x^{+}_{i,n}, x^{-}_{j,m}\right] = \delta_{ij}\tfrac{c^{n-m}\phi^{+}_{i,n+m} - \phi^{-}_{i,n+m}}{q-q^{-1}}\,,\\
\textstyle\sum_{\pi\in\mathfrak{S}_s}\sum_{k=0}^{s}(-1)^k\binom{s}{k}_q x^{\pm}_{i,n_{\pi(1)}}\cdots x^{\pm}_{i,n_{\pi(k)}}x_{j,m}x^{\pm}_{i,n_{\pi(k+1}}\cdots x^{\pm}_{i,n_{\pi(s)}} = 0\,, (i \neq j)\,,
\end{gathered}$$

for $s = 1 - C_{ij}$, and all sequences of integers $n_1, \ldots, n_s$, where $\mathfrak{S}_s$ is the symmetric group on s letters. The elements $\phi^{\pm}_{i,n}$ are defined by:

$$\Phi^{\pm}_i(u) := \sum_{n=0}^{\infty}\phi^{\pm}_{i,\pm n}u^{\pm n} = K_i^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty}h_{i,\pm m}u^{\pm m}\right) \quad \text{and} \quad \phi^{\pm}_{i,\mp n} = 0 \text{ for } n > 0$$

Definition 2.2.1. A $U_q(\mathfrak{g})$-module V is said to be of *type* 1 if it is the direct sum of its weight spaces $V = \bigoplus_{\mu\in P} V_\mu$, where $V_\mu = \left\{v \in V \middle| K_iv = q^{\mu(K_i)}v, i \in I\right\}$. A $U_q(\widehat{\mathfrak{g}})$-module V is said to be of *type* 1 if $c^{\pm 1/2}$ acts as the identity and V, considered as $U_q(\mathfrak{g})$-module, is of type 1.

In this thesis, we consider modules of type 1 only. It is not a restrictive assumption since, according to [7], every finite-dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-module can be obtained from a type 1 module by twisting with an automorphism of $U_q(\widehat{\mathfrak{g}})$.

Let H be the subalgebra in $U_q(\widehat{\mathfrak{g}})$ generated by $c^{\pm 1/2}$ and $\phi^{\pm}_{i,n}$ for all $n \in \mathbb{Z}$ and $i \in I$. Notice that H is not a commutative subalgebra since $h_{i,n}$ and $h_{j,m}$ have a non-trivial commutation relation involving c. However, the action of H on a module of type 1 is commutative since $c^{\pm 1/2}$ acts as the identity. Therefore, the action of H is simultaneously diagonalizable on modules of type 1.

Definition 2.2.2. Let V be a finite dimensional $U_q(\widehat{\mathfrak{g}})$-module of type 1.

1. A vector $v \in V$ is called an *l-highest weight vector* if for any $n \in \mathbb{Z}$ and $i \in I$, we have

$$x_{i,n}^{+} v = 0; \quad \phi_{i,\pm n}^{\pm} v = d_{i,\pm n}^{\pm}\, v \quad \text{for some } d_{i,\pm n}^{\pm} \in \mathbb{C}$$

2. V is called an *l-highest weight module* if there exists an l-highest weight vector v such that $V = U_q(\widehat{\mathfrak{g}})v$. In this case, we call the r-tuple $\left\{\Phi_i^{\pm}(u) := \sum_{n=0}^{\infty} d_{i,\pm n}^{\pm}\, u^{\pm n}\right\}_{i\in I}$ the *l-highest weight* of V.

The classification of finite dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-modules was done by Chari and Pressley, which we state next.

Theorem 2.2.3. (Chari-Pressley, [6,7])

1. Every finite dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-module of type 1 is an l-highest weight module.

2. Let V be a finite dimensional $U_q(\widehat{\mathfrak{g}})$-module with l-highest weight $\Phi_i^{\pm}(u)$, $i \in I$. Then there exists a unique r-tuple of polynomials $\mathbf{P} = (P_1(u), \dots, P_r(u))$, called the Drinfeld polynomials of V, such that $P_i(0) = 1$ and

$$\Phi_i^{\pm}(u) = q^{\deg(P_i)} \frac{P_i(uq^{-1})}{P_i(uq)}$$

   We denote $V := V(\mathbf{P})$.

3. The $U_q(\mathfrak{g})$-highest weight of $V(\mathbf{P})$ is $\lambda = \sum_{i\in I} \deg(P_i)\omega_i$ and it has multiplicity one.

4. For $\mathbf{P}, \mathbf{Q} \in \mathbb{C}[u]^r$, denote $\mathbf{P} \otimes \mathbf{Q} = (P_i Q_i)_{i\in I}$. Then $V(\mathbf{P} \otimes \mathbf{Q})$ is isomorphic to a quotient of the submodule of $V(\mathbf{P}) \otimes V(\mathbf{Q})$ generated by the tensor product of the highest weight vectors.

Definition 2.2.4. Let $i \in I$, $k \in \mathbb{Z}_{\geq 0}$, $a \in \mathbb{C}^*$, let

$$\mathbf{P}_{k,a}^{(i)} = \left((P_{k,a}^{(i)}(u))_\alpha\right)_{\alpha\in I}, \text{ where } (P_{k,a}^{(i)})_\alpha(u) = \begin{cases} \prod_{s=1}^{k}(1 - aq^{2s-2}u) & \text{if } \alpha = i \\ 1 & \text{otherwise} \end{cases} \tag{2.2.1}$$

The irreducible finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module with Drinfeld polynomial $P^{(i)}_{k,a}$ is called the *Kirillov-Reshetikhin module* (KR-module) and denoted $W^{(i)}_{k,a}$. The KR-modules with $k = 1$ are called the *fundamental modules.*

KR-modules were introduced in [31], and then further studied by Kuniba, Nakanishi, and Suzuki in [33], Hatayama, Kuniba, Okado, Takagi, and Yamada in [22] and Chari in [4]. One of the remarkable facts about KR-modules is that their characters and q-characters satisfy the functional relations called the Q-system and the T-system respectively (see Sections 3.3 and 3.5 for definitions of the Q-system and the T-system).

2.3 The Drinfeld polynomials of evaluation modules of $U_q(\widehat{\mathfrak{sl}}_2)$

We now consider the simplest quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, the quantum affine algebra of type A and rank 1. We will introduce an important class of modules, called *evaluation modules*, which exist for type A only. We will derive the l-highest weight of an evaluation module, which will serve as the motivation to introducing the theory of q-characters.

Since the Lie algebra $\mathfrak{sl}_2$ has rank 1, the index set is $I = \{1\}$. For the rest of this section, we drop the index i for notational simplicity.

Proposition 2.3.1. Let $a \in \mathbb{C}^*$. There is a Hopf algebra homomorphism $ev_a : U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$ given by

$$ev_a(c^{\pm 1/2}) = 1 \quad ; \quad ev_a(K^{\pm 1}) = K^{\pm 1}$$
$$ev_a(x_n^+) = q^{-n}a^nK^nE \quad ; \quad ev_a(x_n^-) = q^{-n}a^nFK^n$$

Proof. See [5], Proposition 4.1. □

Definition 2.3.2. For any $U_q(\mathfrak{sl}_2)$-module V, its pull-back via ev_a is called an *evaluation module* of $U_q(\widehat{\mathfrak{sl}}_2)$ and denoted V_a.

Remark 2.3.3. The evaluation module V_a is identical to V as a vector space. In particular, the $U_q(\mathfrak{sl}_2)$-

characters of V_a and V are the same. Recall the Q-system of type A_1 from (1.1.4):

$$Q_{k+1} = \frac{(Q_k)^2 - 1}{Q_{k-1}}, \tag{2.3.1}$$

where we rearranged the terms and dropped the parameter i since it takes exactly one value $i = 1$. We stated in Section 1.1 that the characters of the evaluation modules satisfy the Q-system. We now see that the characters of the evaluation modules of $U_q(\mathfrak{sl}_2)$ are identical to the characters of the underlying highest weight modules, which are described in Theorem 2.1.4. Let us verify the Q-system equation with a few examples.

$$Q_k := \chi(V_{k+1}) = y^k + y^{k-2} + \cdots + y^{-k+2} + y^{-k}$$

Then,

$$\begin{aligned}
\frac{(Q_1)^2 - 1}{Q_0} &= \frac{(y + y^{-1})^2 - 1}{1} = y^2 + 1 + y^{-2} = Q_2, \\
\frac{(Q_2)^2 - 1}{Q_1} &= \frac{\left(y^2 + 1 + y^{-2}\right)^2 - 1}{y + y^{-1}} = \frac{(y^2 + y^{-2})(y + y^{-1})^2}{y + y^{-1}} = y^3 + y + y^{-1} + y^{-3} = Q_3.
\end{aligned}$$

We leave the general form as an exercise.

Proposition 2.3.4. Let V_n be the $(n+1)$-dimensional $U_q(\mathfrak{sl}_2)$-module from Theorem 2.1.4. Then, there exists a basis $v_0, \ldots, v_n$ of V_n such that:

$$Kv_i = q^{n-2i} v_i, \quad Ev_i = [n - i + 1]_q v_{i-1}, \quad Fv_i = [i+1]_q v_{i+1}$$

Proof. It can be seen that v_i's have the correct weights. We directly verify that the coefficients satisfy the relations (2.1.5) as follows:

$$\begin{aligned}
(EF-FE)v_i &= EFv_i - FEv_i = [i+1]_q[n-i]_q\, v_i - [n-i+1]_q[i]_q\, v_i \\
&= \frac{1}{(q-q^{-1})^2}\left((q^{i+1}-q^{-i-1})(q^{n-i}-q^{-n+i}) - (q^{n-i+1}-q^{-n+i-1})(q^{i}-q^{-i})\right) v_i \\
&= \frac{1}{(q-q^{-1})^2}\left(-q^{-n+2i+1} - q^{n-2i-1} + q^{n-2i+1} + q^{-n+2i-1}\right) v_i \\
&= \frac{q^{n-2i}-q^{-n+2i}}{q-q^{-1}} v_i \\
&= [n-2i]_q\, v_i = \frac{K-K^{-1}}{q-q^{-1}} v_i
\end{aligned}$$

By Theorem 2.1.4, there exists a unique irreducible module of the given dimension and the result follows. □

Proposition 2.3.5. Let V_1 be the 2-dimensional $U_q(\mathfrak{sl}_2)$-module from Theorem 2.1.4 with basis vectors $\{v_0, v_1\}$ as in Proposition 2.3.4. Then, the action of $U_q(\widehat{\mathfrak{sl}}_2)$ on the evaluation module $(V_1)_a$ is given by:

$$Kv_0 = qv_0 \quad ; \quad Kv_1 = q^{-1}v_1\, ;$$

$$\phi_k^+\, v_0 = a^k(q-q^{-1})v_0 \quad ; \quad \phi_k^+ v_1 = -a^k(q-q^{-1})v_1\, .$$

Proof. By applying ev_a to the results of Proposition 2.3.4, we can compute:

$$\begin{aligned}
x_k^+ v_i &= ev_a(x_k^+)v_i = q^{-k}a^kK^kEv_i = q^{-k}a^k[2-i]_q q^{k(1-2(i-1))}v_{i-1} = a^k[2-i]_q q^{-2i+2}v_{i-1}\, , \\
x_k^- v_{i-1} &= ev_a(x_k^-)v_{i-1} = q^{-k}a^kFK^kv_{i-1} = q^{-k}a^kq^{k(1-2(i-1))}[i]_qv_i = a^kq^{-2i+2}[i]_qv_i\, .
\end{aligned}$$

Using the relation $\left[x_k^+, x_0^-\right] = \frac{1}{q-q^{-1}}c^k\phi_k^+$, we find that $\phi_k^+ = (q-q^{-1})c^{-k}\left(x_k^+x_0^- - x_0^-x_k^+\right)$. Then,

$$\begin{aligned}
\phi_k^+ v_0 &= (q-q^{-1})c^{-k}x_k^+x_0^-\, v_0 = a^k(q-q^{-1})v_0\, , \\
\phi_k^+ v_1 &= -(q-q^{-1})c^{-k}x_0^-x_k^+v_1 = -a^k(q-q^{-1})v_1\, .
\end{aligned}$$

□

Let us compute the l-weights of v_0 and v_1 in $(V_1)_a$. By definition (see Definition 2.2.2), the l-weights are as follows:

$$\begin{aligned}\Phi^+(u)v_0 &= q+(q-q^{-1})\sum_{k=1}^{\infty}a^ku^k\\ &= q+(q-q^{-1})\frac{au}{1-au}\\ &= q\left(1+\frac{au}{1-au}-\frac{aq^{-2}u}{1-au}\right)\\ &= q\frac{1-aq^{-2}u}{1-au}=q^{\deg P}\frac{P(q^{-1}u)}{P(qu)}\,,\end{aligned}$$

where we denoted:

$$P(u)=1-aq^{-1}u\,. \tag{2.3.2}$$

Notice the resemblance of the result of this computation and the Drinfeld polynomial expression of Theorem 2.2.3(2). In fact, we computed the l-highest weight and the Drinfeld polynomial of $(V_1)_a$.

Similarly, we compute the l-weight of v_1 for later use:

$$\Phi^+(u)v_1 \quad=\quad q^{-1}-(q-q^{-1})\frac{au}{1-au}=q^{-1}\left(1-\frac{aq^2u}{1-au}+\frac{au}{1-au}\right)=q^{-1}\frac{1-aq^2u}{1-au}=q^{-\deg P}\frac{P(q^3u)}{P(qu)}$$

Remark 2.3.6. Notice that the Drinfeld polynomial of $(V_1)_a$ in (2.3.2) is identical to (2.2.1) with $k=1$, i.e the Drinfeld polynomial of the KR-module with $k=1$, also known as the fundamental module. In fact, if we were to compute the Drinfeld polynomial of $(V_k)_a$, we would discover it to be identical to the Drinfeld polynomial of the KR-module $W^{(1)}_{k,a}$ in (2.2.1). Since the evaluation modules are irreducible (because they are irreducible as $U_q(\mathfrak{g})$-modules), they are one and the same as the KR-modules.

Remark 2.3.7. Let $a,a'\in\mathbb{C}^*$ such that $a\neq a'$ and consider $V=V_1$, the 2-dimensional $U_q(\mathfrak{sl}_2)$-module. As $U_q(\mathfrak{sl}_2)$-modules, V_a and $V_{a'}$ are isomorphic since they are both 2-dimensional and there must be exactly one isomorphism class of 2-dimensional modules. However, V_a and $V_{a'}$ are not isomorphic as $U_q(\widehat{\mathfrak{sl}}_2)$-modules since their l-highest weights are different. Indeed, the Drinfeld polynomials of V_a and $V_{a'}$ are $(1-aq^{-1}u)$ and $(1-a'q^{-1}u)$ respectively, which are distinct polynomials when $a\neq a'$. That is, classical character theory does not differentiate between distinct irreducible isomorphism classes of modules over quantum affine algebras. This is the motivation to introducing q-characters.

2.4 q-characters

As seen from Remark 2.3.7, the classical character theory does not differentiate between unique isomorphism classes of irreducible finite-dimensional modules over quantum affine algebras. The theory of q-characters was invented to find the affine analog of the classical character theory. The q-characters were defined by Knight in [32] for the Yangians and by Frenkel and Reshetikhin in [19] for the quantum affine algebras. There are 3 steps in defining q-characters of a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module V. The first step is performing the Jordan decomposition of the action of H on V, where H is the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $c^{\pm 1/2}$ and $\phi^{\pm}_{i,n}$ for $n \in \mathbb{Z}$ and $i \in I$. The second step is identifying polynomials which characterize the l-weights, and finally defining an injective map to some polynomial ring.

Let V be a finite dimensional $U_q(\widehat{\mathfrak{g}})$-module of type 1. Since the action of H on V is commutative, V decomposes into a direct sum of generalized eigenspaces as follows:

$$V = \bigoplus_{\substack{\mathbf{d} = (d^{\pm}_{i,k})_{i \in I, k \in \mathbb{N}}, \\ d^{\pm}_{i,k} \in \mathbb{C}}} V_{\mathbf{d}}, \quad \text{where} \quad V_{\mathbf{d}} = \left\{ v \in V | \text{there exists } p \text{ s.t. } (\phi^{\pm}_{i,k} - d^{\pm}_{i,k})^p v = 0 \text{ for all } i \in I \text{ and } k \in \mathbb{Z}_+ \right\}.$$

The l-weights are described in the following theorem:

Theorem 2.4.1. (Frenkel-Reshetikhin,[19]) Let V be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module, $\mathbf{d} = (d^{\pm}_{i,k})_{i \in I, k \in \mathbb{N}}$ such that $V_{\mathbf{d}} \neq \{0\}$. Then, there exist polynomials $R_i(u)$ and $Q_i(u)$ in $\mathbb{C}[u]$ for every $i \in I$ such that $R_i(0) = Q_i(0) = 1$ and

$$\sum_{k=0}^{\infty} d^{+}_{i,k} u^k = q^{\deg R_i - \deg Q_i} \frac{R_i(q^{-1}u) Q_i(qu)}{R_i(qu) Q_i(q^{-1}u)}, \tag{2.4.1}$$

for any $i \in I$.

Definition 2.4.2. Given $R_i(u)$ and $Q_i(u)$ as in Theorem 2.4.1, denote:

$$R_i(u) = \prod_{j=1}^{k_i}(1 - a_j^{(i)}u) \quad \text{and} \quad Q_i(u) = \prod_{j=1}^{l_i}(1 - b_j^{(i)}u)\,. \tag{2.4.2}$$

Let $\mathbf{d} \in \mathbb{C}^{I\times\mathbb{N}}$ be such that $V_{\mathbf{d}} \neq \{0\}$ with (R, Q) as in (2.4.1) of the form (2.4.2). Define a monomial corresponding to $\mathbf{d}$ in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ as follows:

$$m_{\mathbf{d}} := \prod_{i\in I}\prod_{r=1}^{k_i} Y_{i,a_r^{(i)}} \prod_{s=1}^{l_i} Y_{i,b_s^{(i)}}^{-1}\,. \tag{2.4.3}$$

Given a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module V, the q-character map χ_q is defined as follows:

$$\begin{aligned} \chi_q : \mathrm{Rep}(U_q\widehat{\mathfrak{g}}) &\to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}\,, \\ V &\mapsto \sum_{\mathbf{d}} \dim(V_{\mathbf{d}}) m_{\mathbf{d}}\,, \end{aligned}$$

where $\mathrm{Rep}(U_q\widehat{\mathfrak{g}})$ is the Grothendieck ring of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules and $m_{\mathbf{d}}$ is as in (2.4.3).

Theorem 2.4.3. (Frenkel-Reshetikhin, [19]) The map χ_q is an injective ring homomorphism. Moreover, the diagram below is a commutative diagram.

$$\begin{array}{ccc} \mathrm{Rep}(U_q\widehat{\mathfrak{g}}) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle p} \\ \mathrm{Rep}(U_q\mathfrak{g}) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i\in I} \end{array}$$

where $p(Y_{i,a}) = y_i$ and res is the restriction map.

Example 2.4.4. Let $\mathfrak{g} = \mathfrak{sl}_2$ and let V_1 be the 2-dimensional irreducible $U_q(\mathfrak{g})$-module. Let $(V_1)_a$ be the evaluation module over $U_q(\widehat{\mathfrak{g}})$ considered in Section 2.3. As a vector space, $(V_1)_a$ is isomorphic to $\mathbb{C}v_0 \oplus \mathbb{C}v_1$.

The generalized H-eigenvalues of v_0 and v_1 were computed in the previous section and are given by:

$$\Phi^+(u)v_0 = q^{\deg P}\frac{P(q^{-1}u)}{P(qu)} \quad \text{and} \quad \Phi^+(u)v_1 = q^{-\deg P}\frac{P(q^3u)}{P(qu)}\,,$$

where $P(u) = 1 - aq^{-1}u$. In Theorem 2.4.2 notation, the l-weight of v_0 is given by $R(u) = P(u)$ and $Q(u) = 1$, and the l-weight of v_1 is given by $R(u) = 1$ and $Q(u) = P(q^2u)$. Thus,

$$\chi_q((V_1)_a) = Y_{aq^{-1}} + Y_{aq}^{-1}$$

Remark 2.4.5. Let $a, a' \in \mathbb{C}^*$ such that $a \neq a'$. Then,

$$\chi_q((V_1)_a) = Y_{aq^{-1}} + Y_{aq}^{-1} \neq Y_{a'q^{-1}} + Y_{a'q}^{-1} = \chi_q((V_1)_{a'})\,.$$

We see that the q-character map does differentiate between $(V_1)_a$ and $(V_1)_{a'}$ as promised.

The theory of q-characters was invented to give the affine analog of the usual character theory. Frenkel-Reshetikhin gave 2 conjectures in [19] that are the q-analogs of the well-known properties of the usual character theory. The first conjecture is the q-analog of (2.1.6). That is, the q-character of a finite-dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-module must also be the sum of terms with weights of the form $\lambda - \sum n_i\alpha_i$ for some $n_i \in \mathbb{Z}_+$. The second conjecture is the q-analog of (2.1.7) and gives an explicit description of the image of the q-character map. The proof of both conjectures were given by Frenkel-Mukhin in [18], which we state next.

Definition 2.4.6. Let $a \in \mathbb{C}^*$ and $i \in I$. Define the following monomials:

$$A_{i,a} = Y_{i,aq^{-1}}Y_{i,aq}\prod_{C_{ji}=-1} Y_{j,a}^{-1},$$

with the usual convention that $Y_{i,a} = 1$ if $i \notin I$, and where $C = (C_{ij})$ is the Cartan matrix.

Notice that the monomial $A_{i,a}$ restrict to the monomial a_i in Definition 2.1.3. That is, the monomial $A_{i,a}$ corresponds to a vector with $U_q(\mathfrak{g})$-weight α_i.

Theorem 2.4.7 (Frenkel-Mukhin, [18])**.** Let $V = V(\mathbf{P})$ be a finite dimensional irreducible $U_q(\widehat{\mathfrak{g}})$-module with Drinfeld polynomial $\mathbf{P} = (P_i(u))_{i\in I}$. Then,

$$\chi_q(V) = m_P\left(1 + \prod_p M_p\right),$$

where m_P is the monomial corresponding to $\mathbf{P}$ through the map (2.4.3), aka *highest weight monomial*, and M_p is a monomial in $\mathbb{Z}[A_{i,a}^{-1}]_{i\in I,a\in\mathbb{C}^*}$.

Definition 2.4.8. Let m be a monomial in $\mathbb{Z}[Y_{i,a}^{\pm1}]_{i\in I,a\in\mathbb{C}^*}$. For $i \in I$, we say m is *i-dominant* if m does not contain negative powers of $Y_{i,a}$ for any $a \in \mathbb{C}^*$. If m is i-dominant for all $i \in I$, we say m is *dominant.*

By Theorem 2.4.7, the highest weight monomial is the monomial that corresponds to the Drinfeld polynomial through the map (2.4.3). Therefore, it must be a dominant monomial.

Theorem 2.4.9 (Frenkel-Mukhin, [18])**.** Define

$$\mathcal{K}_i := \mathbb{Z}[Y_{j,a}^{\pm1}]_{j\neq i,a\in\mathbb{C}^*} \otimes \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,bq}^{-1}]_{b\in\mathbb{C}^*},$$

for $i \in I$. Then,

$$\mathrm{Im}\chi_q \simeq \mathrm{Rep}U_q(\widehat{\mathfrak{g}}) \simeq \bigcap_{i\in I}\mathcal{K}_i.$$

Theorem 2.4.9 can be used to obtain the q-character of all $U_q(\widehat{\mathfrak{sl}}_2)$-modules, i.e irreducible finite-dimensional modules of type A_1. Let us demonstrate this claim with an example.

Example 2.4.10. Let $\mathfrak{g} = \mathfrak{sl}_2$, $P(u) = (1 - aq^{-1}u)$, and $V = V(P)$ is the irreducible $U_q(\widehat{\mathfrak{g}})$-module with Drinfeld polynomial $P(u)$. The dominant monomial of $\chi_q(V)$ is $Y_{aq^{-1}}$, where we omitted $i \in \{1\}$. By Theorem 2.4.9, the image of the q-character map must contain $Y_{aq^{-1}} + Y_{aq^{-1}}A_a^{-1} = Y_{aq^{-1}} + Y_{aq}^{-1}$, which is the q-character of V since V is 2-dimensional. Notice that our result is identical to the derivation in Example 2.4.4.

Remark 2.4.11. The q-character of any $U_q(\widehat{\mathfrak{sl}}_2)$-module can be computed using Theorem 2.4.9 as in

Example 2.4.10.

Even though the structure of q-characters are completely analogous to the structure of usual characters, precious little is known about q-characters in generality. In particular, there is no description of the restriction functor to the quantum groups of finite type, no tensor product decomposition rules, and most importantly, there is no equivalent of the Weyl character formula. What *is* known is an algorithm, that can find, in some special cases, the q-character of a module starting with its l-highest weight monomial. The algorithm is a consequence of Theorem 2.4.9 and is known as the *Frenkel-Munkhin algorithm*. The class of special modules whose q-characters can be computed using the Frenkel-Mukhin algorithm includes the fundamental modules [18] and KR-modules [38].

The algorithm is as follows:

1. Start with a dominant monomial m that is the highest weight monomial of the module.

2. If m is i-dominant, define another Laurent polynomial $\phi_i(m)$ as follows:

   (a) If $j \neq i$, replace all $Y_{j,a}$ that appear in m by 1 and $Y_{i,a}$ by Y_a. Denote the resulting monomial in $\mathbb{Z}[Y_a]_{a\in\mathbb{C}^*}$ by $\overline{m}$.

   (b) The monomial $\overline{m}$ can be treated as the dominant monomial of a $U_q(\widehat{\mathfrak{sl}}_2)$-module, which can be computed using Theorem 2.4.9. The q-character with dominant monomial $\overline{m}$ must have the form:

   $$\chi_q(V) = \overline{m}\left(1 + \sum_p \overline{M}_p\right),$$

   where $\overline{M}_p$ are monomials in A_a^{-1}.

   (c) Obtain monomials M_p by replacing A_a^{-1} by $A_{i,a}^{-1}$ in $\overline{M}_p$.

   (d) Define $\phi_i(m) = m(1 + \prod_p M_p)$.

3. Starting from m, perform step 2 in every dominant direction possible and denote the resulting set of monomials $\mathcal{D}_m$. Remove all the dominant monomials other than m, if any, from $\mathcal{D}_m$.

From now on, we will always assume $a = q^j$ for some $q \in \mathbb{C}^*$ that is not a root of unity and $j \in \mathbb{Z}$. We make the following changes in notations accordingly:

$$\mathbf{P}^{(i)}_{k,j} := \mathbf{P}^{(i)}_{k,q^j}; \quad W^{(i)}_{k,j} := W^{(i)}_{k,q^j}; \quad Y_{i,j} := Y_{i,q^j}; \quad A_{i,j} := A_{i,q^j}$$

With this new notation, the Drinfeld polynomial corresponding to the KR-module $W^{(i)}_{k,j}$ defined in (2.2.1) is identified with the dominant monomial $\mathbf{Y}^{(i)}_{k,j}$ through the map (2.4.3) as follows:

$$\mathbf{P}^{(i)}_{k,j} \mapsto \mathbf{Y}^{(i)}_{k,j} := Y_{i,j}Y_{i,j+2}\cdots Y_{i,j+2(k-2)}. \tag{2.4.4}$$

Example 2.4.12. Let $\mathfrak{g} = \mathfrak{sl}_3$ and consider $W^{(1)}_{2,j}$. The dominant monomial is $\mathbf{Y}^{(1)}_{2,j} = Y_{1,j}Y_{1,j+2}$. Recall that $A^{-1}_{1,j} = Y^{-1}_{1,j-1}Y^{-1}_{1,j+1}Y_{2,j}$. The demonstration of the Frenkel-Mukhin algorithm to obtain the q-character of $W^{(1)}_{2,j}$ is shown in Figure 2.1.

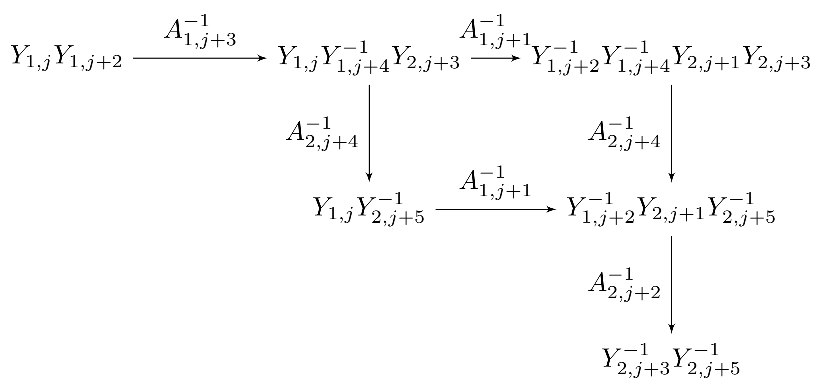


Figure 2.1: The q-character of $W^{(1)}_{2,j}$. An arrow decorated with $A^{-1}_{k,j}$ means the target monomial is obtained by multiplying the source monomial by $A^{-1}_{k,j}$.

Chapter 3

Cluster algebras and quantum cluster algebras

In this section, we will give the definitions of coefficient-free cluster and quantum cluster algebras with examples. For details on cluster algebras, see [16], and for details on quantum cluster algebras, see [3]. We will provide the cluster algebra realization of the Q-system of type A_r introduced in [29], as well as its quantization introduced in [9,10]. We will also provide the definition of the T-system of type A_r and interpret it as a mutation relation in an infinite rank cluster algebra with mutation in the direction of the l-parameter. This interpretation is similar to that in [8], where the direction of mutation is in the k-direction.

3.1 Finite rank cluster algebras

For any $n \in \mathbb{Z}_+$, a cluster algebra of rank n is a commutative ring with unit and no zero-divisors, generated by the union of variables, called *cluster variables*. The cluster variables are related by rational transformations, called *mutations*, determined by an *exchange matrix*, denoted B. For the purposes of this thesis, it suffices to describe coefficient-free cluster algebras.

Let us introduce some useful notations:

$$[x]_+ = \max(0,x) \quad ; \quad \mathrm{sgn}(x) = \begin{cases} -1 & : x<0 \\ 0 & : x=0 \\ 1 & : x>0 \end{cases}$$

We will now describe the coefficient-free cluster algebra of rank n. Let $x_1, \dots, x_n$ be formal variables and let $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$ be an ambient field. Let $\mathbb{T}_n$ be the n-regular tree whose n edges emanating from each vertex receive a different label from the set $\{1, 2, \ldots, n\}$.

Definition 3.1.1. A *cluster pattern* is an assignment $t \mapsto (\mathbf{x}_t, B_t)$ of any vertex $t \in \mathbb{T}_n$ to a *labeled seed* $(\mathbf{x}_t, B_t)$, where:

- the *cluster* tuple $\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t})$ is an n-tuple of elements of $\mathcal{F}$ forming a free generating set,
- the *exchange matrix* $B_t = \left(b_{ij}^{(t)}\right) \in M_{n\times n}(\mathbb{Z})$ is a skew-symmetrizable matrix,
- $t \overset{k}{\longleftrightarrow} t'$ in $\mathbb{T}_n$ if and only if $(\mathbf{x}_t, B_t) \overset{\mu_k}{\longleftrightarrow} (\mathbf{x}_{t'}, B_{t'})$, where μ_k is the *seed mutation in the k-direction* defined as follows:
  - $\mathbf{x}_{t'} = (x_{1;t'}, \ldots, x_{n;t'})$, where $x_{j;t'} = \begin{cases} x_{k;t}^{-1}\left(\prod_{i=1}^n x_{i;t}^{[b_{ik}^{(t)}]_+} + \prod_{i=1}^n x_{i;t}^{[-b_{ik}^{(t)}]_+}\right) & \text{if } j = k \\ x_{j;t} & \text{if } j \neq k \end{cases}$,
  - $B_{t'} = \left(b_{ij}^{(t')}\right)$, where $b_{ij}^{(t')} = \begin{cases} -b_{ij}^{(t)} & \text{if } i = k \text{ or } j = k \\ b_{ij}^{(t)} + \operatorname{sgn}\left(b_{ik}^{(t)}\right)\left[b_{ik}^{(t)} b_{kj}^{(t)}\right]_+ & \text{otherwise} \end{cases}$.

Definition 3.1.2. The *cluster algebra* associated with a cluster pattern $\{(\mathbf{x}_t, B_t) : t \in \mathbb{T}_n\}$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by all cluster variables.

We will consider cluster algebras with skew-symmetric exchange matrices only. In this case, the exchange matrix B and its mutation can be completely described by a quiver and the quiver mutation process.

Definition 3.1.3. A *quiver* $\mathcal{Q}$ is an oriented graph given by a set of vertices $\mathcal{Q}_0$, a set of arrows $\mathcal{Q}_1$, and two maps $s : \mathcal{Q}_1 \to \mathcal{Q}_0$ and $t : \mathcal{Q}_1 \to \mathcal{Q}_0$ taking an arrow to its source and target respectively. A *loop* of a quiver is an arrow whose source and target are the same. A 2*-cycle* of a quiver is a pair of distinct arrows α and β such that $s(\alpha) = t(\beta)$ and $s(\beta) = t(\alpha)$. The *rank* of a quiver is the number of vertices.

Definition 3.1.4. Let $B = (b_{ij}) \in M_{n\times n}(\mathbb{Z})$ be a skew-symmetric matrix. The quiver associated to B, denoted $\mathcal{Q}_B$ has n vertices and b_{ij} arrows from i to j if and only if $b_{ij} > 0$. When the exchange matrix is skew-symmetric, we simplify our notation and denote the labeled seed $(\mathbf{x}, \mathcal{Q}_B) := (\mathbf{x}, B)$.

We will only consider quivers with no loops or 2-cycles. Given $(\mathbf{x}_p, \mathcal{Q}_p) \overset{\mu_k}{\longrightarrow} (\mathbf{x}_{p'}, \mathcal{Q}_{p'})$ for $p \overset{k}{\to} p'$ in $\mathbb{T}_n$, the seed mutation in Definition 3.1.1 can be simplified as follows:

- $\mathbf{x}_{p'} = (x_{1;p'}, \ldots, x_{n;p'})$, where $x_{j;p'} = \begin{cases} x_{k;p}^{-1}\left(\prod_{\substack{\alpha\in(\mathcal{Q}_p)_1 \\ s(\alpha)=k}} x_{t(\alpha);p} + \prod_{\substack{\alpha\in(\mathcal{Q}_p)_1 \\ t(\alpha)=k}} x_{s(\alpha);p}\right) & \text{if } j = k \\ x_{j;p} & \text{if } j \neq k \end{cases}$,

  and $(\mathcal{Q}_p)_1$ is the set of arrows in $\mathcal{Q}_p$.

- $\mathcal{Q}_{p'} = \mu_k(\mathcal{Q}_p)$ is obtained from $\mathcal{Q}_p$ as follows:

  - For each subquiver $i \to k \to j$, add a new arrow $i \to j$.
  - Reverse all arrows with source or target k.
  - Remove the arrows in a maximal set of pairwise disjoint 2-cycles.

It can be easily checked that the quiver mutation is consistent with the seed mutation in Definition 3.1.1.

Example 3.1.5. Let $\mathcal{Q}_B$ be the quiver with $\mathcal{Q}_1 = \{1, 2\}$ with a single arrow $1 \to 2$. Given the seed $((x_1, x_2), \mathcal{Q}_B)$, the mutation in direction 1 gives the following seed:

$$\mathcal{Q}_B : \quad \overset{x_1}{\bullet} \longrightarrow \overset{x_2}{\bullet} \quad \overset{\mu_1}{\Longleftrightarrow} \quad \mathcal{Q}_{B'} : \quad \overset{x_1' = \frac{x_2+1}{x_1}}{\bullet} \longleftarrow \overset{x_2}{\bullet}$$

$$B : \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \overset{\mu_1}{\Longleftrightarrow} \quad B' : \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

That is, the mutated seed in direction 1 is given by:

$$\mu_1((x_1, x_2), \mathcal{Q}_B) = \left(\left(\frac{x_2+1}{x_1}, x_2\right), \mathcal{Q}_{B'}\right).$$

In order to define a cluster algebra, it suffices to give a single seed, called the *initial seed* or the *fundamental cluster*. All other seeds in $\mathbb{T}_n$ can be obtained from the initial seed via iterated mutations. In addition, all other cluster variables are Laurent polynomials in the fundamental cluster variables. This property is called the *Laurent phenomenon*, and is stated in the following theorem:

Theorem 3.1.6 (Fomin-Zelevinsky, [17])**.** The cluster algebra $\mathcal{A}$ defined by an initial seed $(\mathbf{x}, B)$ is contained in the Laurent polynomial ring $\mathbb{Z}[\mathbf{x}^{\pm 1}]$, i.e. every cluster variable is a Laurent polynomial over $\mathbb{Z}$ in the fundamental cluster variables $x_1, \ldots, x_n$.

3.2 Quantum cluster algebras

Cluster algebras, whose exchange matrices are invertible, admit canonical Poisson structures that is consistent with mutations (see [20]). Quantum cluster algebras are the canonical quantizations of this Poisson structure, i.e. non-commutative generalizations of cluster algebras. Therefore, in order to describe quantum cluster algebras, we must give the corresponding non-commutative generalizations of the ambient field $\mathcal{F}$, the initial seed, and seed mutations.

Definition 3.2.1. Let t be a formal variable. The quiver $\mathcal{Q}$ in the quantum case is identical to the non-quantum case. Let $L_{\mathcal{Q}} = \bigoplus_{i \in \mathcal{Q}_0} \mathbb{Z} i$, i.e. the $\mathbb{Z}$-lattice generated by the vertices in $\mathcal{Q}$, and let $\Lambda : L_{\mathcal{Q}} \times L_{\mathcal{Q}} \to \mathbb{Z}$ be a skew-symmetric bilinear form. The *based quantum torus* associated with $(\mathcal{Q}, \Lambda)$, denoted $\mathcal{T} := \mathcal{T}(\mathcal{Q}, \Lambda)$, is the $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-algebra with a distinguished $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-basis $\{X_i | i \in L_{\mathcal{Q}}\}$ such that:

$$X_0 := 1 \quad \text{and} \quad X_{-i} := X_i^{-1}, \tag{3.2.1}$$

and the multiplication defined as:

$$X_i X_j := t^{\frac{1}{2}\Lambda(i,j)} X_{i+j}. \tag{3.2.2}$$

The basis elements satisfy the t-commutation relation:

$$X_i X_j = t^{\Lambda(i,j)} X_j X_i, \tag{3.2.3}$$

It can be derived from (3.2.2) that multiplication in $\mathcal{T}$ is associative:

$$(X_i X_j) X_k = X_i (X_j X_k) = t^{\frac{1}{2}(\Lambda(i,j)+\Lambda(i,k)+\Lambda(j,k))} X_{i+j+k}, \tag{3.2.4}$$

It is well-known that $\mathcal{T}$ forms an Ore domain, that is, it is contained in its skew-field of fractions. We take the skew-field of fractions of $\mathcal{T}$ as the non-commutative ambient field $\mathcal{F}$.

Definition 3.2.2. Let $\mathcal{Q} = \mathcal{Q}_B$ be the quiver associated with a skew-symmetric exchange matrix B, and let $\mathcal{T}$ be a based quantum torus associated with $\mathcal{Q}$ and some skew-symmetric bilinear form Λ. Let $\mathbf{X} = \{X_i | i \in \mathcal{Q}_0\} \subset \mathcal{T}$. The pair $(\mathbf{X}, \mathcal{Q})$ forms a *quantum seed* in $\mathcal{F}$ if the matrix $\Lambda = (\Lambda(i,j))_{i,j\in\mathcal{Q}_0}$ satisfies the following condition:

$$\Lambda B = D,$$

where D is a diagonal matrix with positive entries. In this case, we say (Λ, B) forms a *compatible pair*.

Definition 3.2.3. The *quantum mutation* is given by:

$$X'_k = X_{(-k+\sum_{\alpha\in\mathcal{Q}_1, s(\alpha)=k} i)} + X_{(-k+\sum_{\alpha\in\mathcal{Q}_1, t(\alpha)=k} i)}. \tag{3.2.5}$$

Given a specific quiver, the quantum mutation (3.2.5) can be made explicit using (3.2.2).

Example 3.2.4. We now demonstrate the quantum version of Example 3.1.5. The quivers $\mathcal{Q}_B$ and $\mathcal{Q}_{B'}$ are as in Example 3.1.5. The exchange matrix B is invertible, and, therefore, admits a unique, up to multiplication by a positive diagonal matrix, compatible pair Λ, given by the inverse.

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda = B^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We have a quantum seed $((X_1, X_2), \mathcal{Q}_B)$ with a commutation relation $X_1X_2 = tX_2X_1$. We can explicitly write down the quantum mutation in direction 1 as follows:

$$\begin{aligned} X'_1 &= X_{-1+2} + X_{-1+0} && \text{(by Definition 3.2.3)} \\ &= t^{-\frac{1}{2}\Lambda(1,2)}X_{-1}X_2 + t^{-\frac{1}{2}\Lambda(1,0)}X_{-1}X_0 && \text{(using (3.2.2))} \\ &= t^{-\frac{1}{2}}X_1^{-1}X_2 + X_1^{-1} && \text{(using (3.2.1))} \end{aligned}$$

where $\Lambda(1,2) = \Lambda_{12} = 1$. The mutated seed in direction 1 is given by:

$$\mu_1((X_1, X_2), \mathcal{Q}_B) = \left(\left(t^{-\frac{1}{2}} X_1^{-1} X_2 + X_1^{-1}, X_2\right), \mathcal{Q}_{B'}\right).$$

3.3 The Q-system of type A_r as a finite rank cluster and quantum cluster algebra

The Q-system of type A_r is a recursion relation on commuting variables $\left\{Q_k^{(i)}\right\}$ defined as follows:

$$Q_{k+1}^{(i)} Q_{k-1}^{(i)} = \left(Q_k^{(i)}\right)^2 - Q_k^{(i-1)} Q_k^{(i+1)} \quad \text{for} \quad k \in \mathbb{N}, i \in I\,, \tag{3.3.1}$$

with the convention $Q_k^{(0)} = Q_k^{(r+1)} = 1$.

The Q-system (3.3.1) was formulated as a mutation relation in a rank $2r$ cluster algebra by Kedem in [29]. The variables $Q_k^{(i)}$ were renormalized, denoted $\widehat{Q}_k^{(i)}$, such that the right-hand side of the Q-system (3.3.1) has a positive sign. The renormalized Q-system has a natural interpretation as a mutation relation.

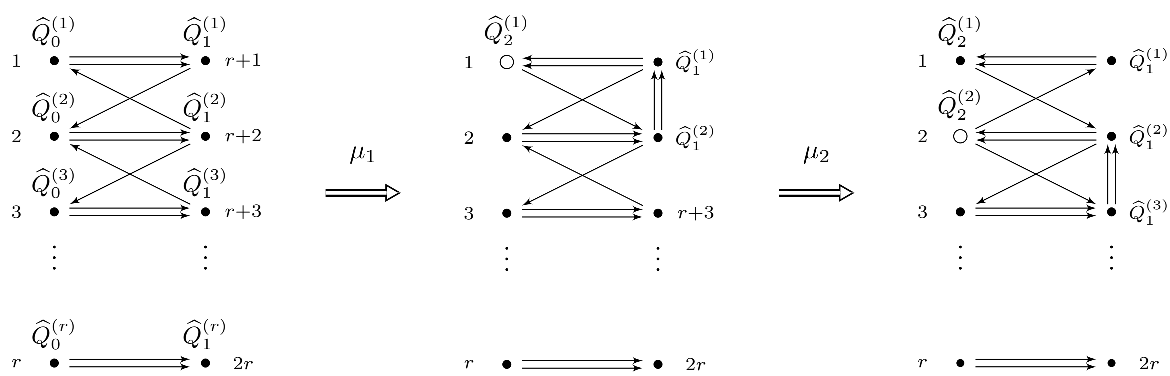


Figure 3.1: Left: the initial seed $(\mathbf{Q}, \Gamma_Q)$, where $\mathbf{Q} = \left\{\widehat{Q}_0^{(i)}, \widehat{Q}_1^{(i)} | i \in I\right\}$ is the fundamental cluster variables and Γ_Q is the underlying quiver. Center: $\mu_1(\mathbf{Q}, \Gamma_Q)$. Right: $\mu_2\mu_1(\mathbf{Q}, \Gamma_Q)$.

The cluster algebra attached to the Q-system is defined by the initial seed $(\mathbf{Q}, \Gamma_Q)$ (see Figure 3.1). Toric mutations (except at the boundaries) are given by the Q-system equations. The exchange matrix associated to the quiver in Figure 3.1, call it B_Q, is invertible, and, therefore, there is a unique quantization (up to scalar) of the initial seed. Di Francesco-Kedem introduced this quantization in [9].

$$B_Q = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} \quad \text{and} \quad \Lambda = B_Q^{-1} = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, \tag{3.3.2}$$

where C is the Cartan matrix of type A_r and $\lambda = -\det(C)C^{-1}$.

By construction, (Λ, B_Q) is a compatible pair. The explicit expression of the quantum mutation in Definition 3.2.3 using Λ in (3.3.2) defines a quantum Q-system as follows:

$$t^{\lambda_{ii}} \widehat{Q}^{(i)}_{k+1} \widehat{Q}^{(i)}_{k-1} = \left(\widehat{Q}^{(i)}_k\right)^2 + \widehat{Q}^{(i-1)}_k \widehat{Q}^{(i+1)}_k \tag{3.3.3}$$

3.4 Infinite rank cluster algebras

Both cluster and quantum cluster algebras were initially defined for finite rank cases. It is possible to generalize the definitions to infinite rank. We let the cluster tuple and the exchange matrix be infinite dimensional. For the mutations to make sense, we assume that for each j, $b_{ij} = 0$ for all but finitely many i. If B is skew-symmetric, this condition is equivalent to saying that an infinite quiver $\mathcal{Q}_B$ has only finitely many incoming or outgoing arrows at all of its vertices. An important distinction between finite rank and infinite rank cluster algebras is that in the finite rank situation, any two clusters can be obtained from each other by a finite sequence of mutations. In an infinite rank cluster algebra, it is no longer the case. That is, it is possible to obtain different cluster algebra structures on the same underlying infinite quiver using different fundamental cluster variables that are not reachable from one to another by a finite sequence of mutations. Another big distinction is that since the exchange matrix is infinite dimensional, it does not admit a unique inverse. That is, there are always infinitely many ways to quantize infinite rank cluster algebras.

3.5 The T-system of type A_r as an infinite rank cluster and quantum cluster algebra

The T-system of type A_r is a recursion relation on commuting variables $\left\{T_{k,l}^{(i)}\right\}$ defined as follows:

$$T_{k,l-1}^{(i)}T_{k,l+1}^{(i)} = T_{k+1,l}^{(i)}T_{k-1,l}^{(i)} + T_{k,l}^{(i-1)}T_{k,l}^{(i+1)} \quad \text{for} \quad k \in \mathbb{Z}_+, i \in I\,, \tag{3.5.1}$$

with the convention $T_{0,l}^{(i)} = T_{k,l}^{(0)} = T_{k,l}^{(r+1)} = 1$. The T-system in (3.5.1) is a discrete dynamical system evolving in the direction of the parameter l, i.e. variables with higher values of l parameter can be obtained from an expression of variables with lower values of l. The T-system can be regarded as a mutation in a cluster algebra with direction of mutation in the l-direction. It is also possible to rearrange the terms such that the evolution is in the k-direction. This rearrangement gives a different cluster algebra structure with a correspondingly different direction of mutation. The latter choice was taken by Di Francesco - Kedem in [8], which is also consistent with the cluster algebra structure attached to the Q-system in [9,10] and described in Section 3.3.

We now define the cluster algebra formulation of the T-system in a similar way to that in [8]. The parameters i, k, l in (3.5.1) correspond to α, k, j parameters in [8] respectively. Let B be the signed adjacency matrix of the quiver Γ_B in Figure 3.2.

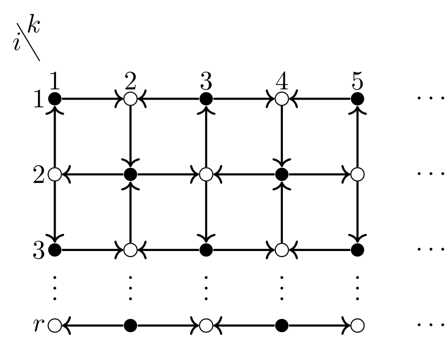


Figure 3.2: The quiver Γ_T, $(i \in I, k \in \mathbb{Z}_+)$

Definition 3.5.1. For $n \in \mathbb{Z}_+$, define $(n)_2 := n \,\mathrm{mod} 2$.

To each vertex (k,i) in Γ_T, we associate a variable $T^{(i)}_{k,(i+k+1)_2}$. The resulting set is:

$$\mathcal{C} := \left\{ T^{(i)}_{k,(i+k+1)_2} \mid i \in I,\ k \in \mathbb{Z}_+ \right\} . \tag{3.5.2}$$

Various boundary conditions can be put on the T-system to obtain different solutions. For example, the solution given by the q-characters of KR-modules corresponds to the boundary condition given by:

$$T^{(i)}_{0,l} = T^{(0)}_{k,l} = T^{(r+1)}_{k,l} = 1 \quad \text{and} \quad T^{(i)}_{1,(i)_2} = \chi_q\left(W^{(i)}_{1,(i)_2}\right).$$

We take $(\mathcal{C}, \Gamma_T)$ to be the initial seed. Then, for any other $l \in \mathbb{Z}$, the value of the variable $T^{(i)}_{k,l}$ can be obtained in terms of the fundamental cluster variables in $\mathcal{C}$ by successive toric mutations, i.e. mutations at vertices with 2 incoming and 2 outgoing arrows only, except at the boundaries. Notice that the choice of the quiver Γ_T is such that each of the mutations are given by the T-system relations in (3.5.1).

Since the exchange matrix B is of infinite rank, there are infinitely many choices of a compatible pair Λ. Consequently, there does not exist a canonical quantization of the T-system cluster algebra.

Suppose we are given non-commutative variables $\left\{T^{(i)}_{k,l}\right\}$ that satisfy a t-deformed T-system with a non-commutative multiplication $*$. Let $(\mathcal{C}, \Gamma_T)$ be as before, only now we substitute $T^{(i)}_{k,l}$ with their non-commutative versions. Then, $(\mathcal{C}, \Gamma_T)$ forms a quantum seed if and only if the following 3 conditions hold:

Condition I: The fundamental cluster variables t-commute. Moreover, the commutation matrix $\Lambda = (\Lambda^{i',k',l'}_{i,k,l})$, defined as:

$$T^{(i)}_{k,l} * T^{(i')}_{k',l'} = t^{\Lambda^{i',k',l'}_{i,k,l}} T^{(i')}_{k',l'} * T^{(i)}_{k,l} \text{ for all } T^{(i')}_{k',l'}, T^{(i)}_{k,l} \in \mathcal{C}, \tag{3.5.3}$$

is an integer-valued matrix.

Condition II: (Λ, B) forms a compatible pair (see Definition 3.2.2).

Condition III: The quantum mutation is given by:

$$T^{(i)}_{k,l-1} * T^{(i)}_{k,l+1} = t^{\frac{1}{2}\Lambda^{i,k-1,l}_{i,k,l-1} + \frac{1}{2}\Lambda^{i,k+1,l}_{i,k,l-1} - \frac{1}{2}\Lambda^{i,k+1,l}_{i,k-1,l}} T^{(i)}_{k-1,l} * T^{(i)}_{k+1,l} + t^{\frac{1}{2}\Lambda^{i-1,k,l}_{i,k,l-1} + \frac{1}{2}\Lambda^{i+1,k,l}_{i,k,l-1} - \frac{1}{2}\Lambda^{i+1,k,l}_{i-1,k,l}} T^{(i-1)}_{k,l} * T^{(i+1)}_{k,l}.$$

This equation is the specialization of Definition 3.2.3 to the quiver Γ_T.

The main result of this thesis concerns a t-deformed T-system introduced by Nakajima in [39]. The deformation comes from Nakajima's geometric construction of certain quiver varieties, and there is no *apriori* reason to suspect that this deformed T-system defines a quantum mutation. We will state Nakajima's deformed T-system and show that conditions I,II,III are satisfied in Chapter 4.

Chapter 4

Nakajima's deformed T-system as quantum cluster variables

Nakajima [36,38,40] introduced a t-analog of the the Grothendieck ring $\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{Rep}U_q(\widehat{\mathfrak{g}})$ and the t-analog of q-characters, the (q,t)-character for short, through the geometry of quiver varieties. The (q,t)-character map is a $\mathbb{Z}[t,t^{-1}]$-linear injective map

$$\chi_{q,t} : \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}} \mathrm{Rep}U_q(\widehat{\mathfrak{g}}) \to \mathbb{Z}[t,t^{-1}] \otimes \mathbb{Z}[Y_{i,j}, Y_{i,j}^{-1}]_{i \in I, j \in \mathbb{Z}} \,,$$

with the property that $\chi_{q,t=1} = \chi_q$. Although $\chi_{q,t}$ is not a ring homomorphism, Nakajima introduced a twisted multiplication on both the source and the target of $\chi_{q,t}$ that makes it into a homomorphism of *non-commutative* rings.

Recall that the main object of this thesis is Nakajima's deformed T-system that (q,t)-characters of KR-modules satisfy. In order to work with the deformed T-system, we must know how to obtain the (q,t)-characters of KR-modules. However, by [36], the (q,t)-characters of the KR-modules of type A are identical to their q-characters. Therefore, we will omit the derivation of the (q,t)-characters and move straight to the description of the twisted multiplication.

4.1 Nakajima's twisted multiplication

Definition 4.1.1. Let $\mathcal{M}$ be the set of monomials in $\mathbb{Z}[Y_{i,j}^{\pm 1}]_{i \in I, j \in \mathbb{Z}}$. Given $m, m' \in \mathcal{M}$, we say m' is a descendant of m if m' is obtained by applying a set of $A_{i,j}^{-1}$ to m, i.e. $m'm^{-1} \in \mathbb{Z}[A_{i,j}^{-1}]_{i \in I, j \in \mathbb{Z}}$.

With this definition, given a q-character of an irreducible module, all its monomials are descendants of the dominant monomial.

Definition 4.1.2. For $m_+, m \in \mathcal{M}$ such that m_+ dominant and m descendant from m_+, define $u_{i,j}(m) \in \mathbb{Z}$ and $v_{i,j}(m, m_+) \in \mathbb{Z}_+$ as follows:

$$m = \prod_{i,j} Y_{i,j}^{u_{i,j}(m)} = m_+ \prod_{i,j} A_{i,j}^{-v_{i,j}(m,m_+)} \ .$$

The following definitions are due to Nakajima [39].

Definition 4.1.3. For $m \in \mathcal{M}$, let $\tilde{u}_{i,j}(m) \in \mathbb{R}$ $(i \in I, j \in \mathbb{Z})$ be a solution of the system

$$u_{i,j}(m) = \tilde{u}_{i,j-1}(m) + \tilde{u}_{i,j+1}(m) - \tilde{u}_{i-1,j}(m) - \tilde{u}_{i+1,j}(m) \ ,$$

such that $\tilde{u}_{i,j}(m) = 0$ for j sufficiently small. As usual, $\tilde{u}_{i,j}(m) = 0$ if $i \notin I$.

Remark 4.1.4. For any monomial $m \in \mathcal{M}$, it can be seen from Definition 4.1.2, there are only finitely many non-zero $u_{i,j}(m)$. The condition $\tilde{u}_{i,j}(m) = 0$ for j sufficiently small ensures there is in fact a unique integral solution to the system, which can be verified through direct computation. However, there can be infinitely many non-zero $\tilde{u}_{i,j}(m)$. The system in Definition 4.1.3 looks cryptic. As we will never have to solve this system explicitly, the reader need only realize that there is a unique solution to the system at this point.

Definition 4.1.5. Let $m_+^1, m_+^2 \in \mathcal{M}$ be dominant monomials and $m^1, m^2 \in \mathcal{M}$ such that m^i is a descendant of m_+^i.

$$\begin{aligned}
\epsilon(m_+^1, m_+^2) &:= -\sum_{i,j} u_{i,j+1}(m_+^1)\tilde{u}_{i,j}(m_+^2) + \sum_{i,j} u_{i,j+1}(m_+^2)\tilde{u}_{i,j}(m_+^1) \ , \\
d(m^1, m_+^1; m^2, m_+^2) &:= \sum_{i,j} v_{i,j+1}(m^1, m_+^1)u_{i,j}(m^2) + u_{i,j+1}(m_+^1)v_{i,j}(m^2, m_+^2) \ , \\
\gamma(m^1, m_+^1; m^2, m_+^2) &:= d(m^1, m_+^1; m^2, m_+^2) - d(m^2, m_+^2; m^1, m_+^1) \ .
\end{aligned}$$

Notice that the above sums are all finite sums as there are only finitely many non-zero values of $u_{i,j}(m)$ and $v_{i,j}(m, m_+)$ for any monomials m and m_+. Also, it follows immediately from the definition that both γ and ϵ are anti-symmetric.

Remark 4.1.6. Given a monomial m in a (q,t)-character, in our application of the γ function, it will always be clear what the dominant monomial of m is. Therefore, we simplify our notation and write

$$\gamma(m^1, m^2) := \gamma(m^1, m^1_+; m^2, m^2_+) \ .$$

Definition 4.1.7. Let V^1, V^2 be finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules and let $\chi^1_{q,t}$ and $\chi^2_{q,t}$ be their (q,t)-characters. Let m^1_+, m^2_+ be the dominant monomials and m^1, m^2 any monomials in $\chi^1_{q,t}$ and $\chi^2_{q,t}$ respectively. The twisted multiplication on monomials is defined as follows:

$$m^1 * m^2 := t^{\gamma(m^1,m^2)+\epsilon(m^1_+,m^2_+)} m^1 m^2 \ ,$$

where m^1m^2 is the usual multiplication of monomials. Multiplication on (q,t)-characters is defined by linearly expanding the twisted multiplication on monomials.

We now state the result on which this work is based.

Theorem 4.1.8 (Nakajima, [39])**.** Let $\chi^{(i)}_{k,j} := \chi_{q,t}(W^{(i)}_{k,j})$, where $W^{(i)}_{k,j}$ is the KR-module with dominant monomial $\mathbf{Y}^{(i)}_{k,j}$ (see (2.4.4)). The following relations hold between the (q,t)-characters:

$$\begin{aligned} t^{-\epsilon(P^{(i)}_{k,j}, P^{(i)}_{k,j+2})} \chi_{q,t}(W^{(i)}_{k,j}) * \chi_{q,t}(W^{(i)}_{k,j+2}) &= t^{-\epsilon(P^{(i)}_{k+1,j}, P^{(i)}_{k-1,j+2})} \chi_{q,t}(W^{(i)}_{k+1,j}) * \chi_{q,t}(W^{(i)}_{k-1,j+2}) \\ &\quad + t^{-1-\epsilon(P^{(i-1)}_{k,j+1}, P^{(i+1)}_{k,j+1})} \chi_{q,t}(W^{(i-1)}_{k,j+1}) * \chi_{q,t}(W^{(i+1)}_{k,j+1}), \end{aligned}$$

with convention $\chi^{(i)}_{k,j} = 1$ if $k=0$ or $i \notin I$.

We also define $*_\gamma$ multiplication as follows:

$$m^1 *_\gamma m^2 := t^{-\epsilon(m^1_+,m^2_+)} m^1 * m^2 = t^{\gamma(m^1,m^2)} m^1 m^2 \ . \tag{4.1.1}$$

Then, the deformed T-system can be written more succinctly with respect to $*_\gamma$ as follows:

$$\chi^{(i)}_{k,j} *_\gamma \chi^{(i)}_{k,j+2} = \chi^{(i)}_{k+1,j} *_\gamma \chi^{(i)}_{k-1,j+2} + t^{-1} \chi^{(i-1)}_{k,j+1} *_\gamma \chi^{(i+1)}_{k,j+1}. \tag{4.1.2}$$

Remark 4.1.9. Equation (4.1.2) looks slightly different from the T-system shown in (3.5.1). However, (4.1.2) can be transformed into a deformation of (3.5.1) by a change of variables $\chi^{(i)}_{k,j} \to T^{(i)}_{k,k+j}$ and relabeling $l = k + j + 1$ with no changes to the coefficients.

Recall the cluster algebra structure attached to the T-system in Section 3.5. We will show that the deformed T-system is a quantum mutation in a quantization of the cluster algebra described in Section 3.5. The exchange matrix of the quantum cluster algebra is the same as before, given by the adjacency matrix B of the quiver Γ_B in Figure 3.2. The fundamental cluster variables are given by the non-commutative versions of (3.5.2). As we will work in $\chi^{(i)}_{k,j}$ variables, the fundamental cluster $\mathcal{C}$ in terms of these variables is as follows:

$$\mathcal{C} = \left\{ \chi^{(i)}_{k,-k+(i+k+1)_2} \mid i \in I,\ k \in \mathbb{Z}_+ \right\} . \tag{4.1.3}$$

Notice that the fundamental cluster (4.1.3) is obtained from (3.5.2) by the change of variables described in Remark 4.1.9.

Remark 4.1.10. Let χ_1 and χ_2 be (q,t)-characters with dominant monomials m^1_+ and m^2_+ respectively. Suppose χ_1 and χ_2 t-commute with respect to $*$. That is, there exists some $\alpha \in \mathbb{Z}$ such that:

$$\chi_1 * \chi_2 = t^{\alpha} \chi_2 * \chi_1 \ , \tag{4.1.4}$$

which is equivalent to:

$$\chi_1 *_\gamma \chi_2 = t^{\alpha - 2\epsilon(m^1_+, m^2_+)} \chi_2 *_\gamma \chi_1 \ . \tag{4.1.5}$$

Notice that $\gamma(m^1_+, m^2_+,) = 0$ as $v_{i,j}(m_+, m_+) = 0$ for all i, j and any dominant monomial m_+. That is, the dominant monomials in $\chi_1 *_\gamma \chi_2$ and $\chi_2 *_\gamma \chi_1$ have coefficient 1. Therefore, (4.1.4) holds if and only if $\alpha = 2\epsilon(m^1_+, m^2_+)$, and $\chi_1 *_\gamma \chi_2 = \chi_2 *_\gamma \chi_1$.

Corollary 4.1.11. The fundamental cluster variables t-commute with respect to $*$, i.e. Condition I holds, if and only if they commute with respect to $*_\gamma$. Moreover, if the fundamental cluster variables t-commute, then the commutation matrix Λ, defined in (3.5.3), is given by

$$\Lambda_{i,j,k}^{i',j',k'} = 2\epsilon(\mathbf{Y}_{k,j}^{(i)}, \mathbf{Y}_{k',j'}^{(i')}) \ ,$$

where $\mathbf{Y}_{k,j}^{(i)}$ is the dominant monomial of $\chi_{q,t}(W_{k,j}^{(i)})$ defined in (2.4.4).

4.2 Proof of Condition I

The main tool we use in this section is the tableaux-sum expression for the q-characters of KR-modules, introduced by Nakajima in [38], and the mapping $m : T \mapsto m_T$ from the space of allowed tableaux, called KR-tableaux in the text and to be defined in Section 4.2.1, to the space of monomials $\mathcal{M}$. Let $\mathcal{B}_{k,j}^{(i)}$ be the set of all KR-tableaux parametrizing the monomials of $\chi_q(W_{k,j}^{(i)})$.

The idea of the proof of Condition I is as follows. We define a division of the set $\mathcal{B}_{k,j}^{(i)} \times \mathcal{B}_{k',j'}^{(i')}$ into 3 disjoint subsets $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}$ and an automorphism σ on this set such that

- σ fixes the elements of $\mathcal{P}_0$ and is an involution between $\mathcal{P}_1$ and $\mathcal{P}_{-1}$
- $m_{(C,T)} = m_{\sigma(C,T)}$, where $m_{(C,T)} := m_C m_T$.
- $\gamma(m_C, m_T) =: \gamma(C,T) = -\gamma(\sigma(C,T))$.

The main result of this section is:

Theorem 4.2.1. Let $(C,T) \in \mathcal{P}_0$, i.e. (C,T) is a fixed point of σ. Then $\gamma(C,T) = 0$.

The proof of Theorem 4.2.1 is given in Section 4.2.5. Let us show how Theorem 4.2.1 implies Condition I.

Corollary 4.2.2. The fundamental cluster variables t-commute with respect to the twisted multiplication $*$ defined in (4.1.7).

Proof. By Corollary 4.1.11, Condition I is equivalent to the statement that elements in $\mathcal{C}$ (see (4.1.3)) commute with respect to $*_\gamma$. That is, given any $\chi_{k,j}^{(i)}, \chi_{k',j'}^{(i')} \in \mathcal{C}$, we want to show

$$\chi_{k,j}^{(i)} *_\gamma \chi_{k',j'}^{(i')} = \chi_{k',j'}^{(i')} *_\gamma \chi_{k,j}^{(i)} \tag{4.2.1}$$

Let's write

$$\chi_{k,j}^{(i)} = \sum_{C \in \mathcal{B}_{k,j}^{(i)}} m_C \text{ and } \chi_{k',j'}^{(i')} = \sum_{T \in \mathcal{B}_{k',j'}^{(i')}} m_T \ ,$$

where $\mathcal{B}_{k,j}^{(i)}$ and $\mathcal{B}_{k',j'}^{(i')}$ are the sets of allowed KR-tableaux. Denote $\mathcal{B} = \mathcal{B}_{k,j}^{(i)} \times \mathcal{B}_{k',j'}^{(i')}$. Then,

$$\begin{aligned}
\chi_{k,j}^{(i)} *_\gamma \chi_{k',j'}^{(i')} &= \sum_{(C,T)\in\mathcal{B}} t^{\gamma(C,T)} m_C m_T \\
&= \underbrace{\sum_{(C,T)\in\mathcal{P}_0\cap\mathcal{B}} t^{\gamma(C,T)} m_{(C,T)}}_{\gamma(C,T)=0 \text{ by Theorem 4.2.1}} + \sum_{(C,T)\in\mathcal{P}_1\cap\mathcal{B}} \left(t^{\gamma(C,T)} m_{(C,T)} + t^{\gamma(\sigma(C,T))} m_{\sigma(C,T)}\right) \\
&= \sum_{(C,T)\in\mathcal{P}_0\cap\mathcal{B}} m_T m_C + \sum_{(C,T)\in\mathcal{P}_1\cap\mathcal{B}} \left(t^{-\gamma(T,C)} m_T m_C + t^{\gamma(T,C)} m_T m_C\right) \\
&= \chi_{k',j'}^{(i')} *_\gamma \chi_{k,j}^{(i)}
\end{aligned}$$

where we used $\gamma(\sigma(C,T)) = -\gamma(C,T) = \gamma(T,C)$ and $m_{(C,T)} = m_C m_T = m_{(T,C)}$. □

4.2.1 Tableaux-sum notation

The tableaux-sum notation we are using is a slight adjustment of the notation introduced in [38]. Let us begin with the tableaux-sum description of the ordinary character $\chi(V)$ for $V \in \mathbf{R}$ as a motivation to the later definitions of the tableaux-sum notation for q-characters.

Let V_λ be the highest weight irreducible $U_q(\mathfrak{sl}_{r+1})$-module with highest weight $\lambda = \sum_{i=1}^r \lambda_i \omega_i$. Then there exists a basis of V_λ parametrized by semi-standard Young tableaux of shape $\Lambda = (\Lambda_1, \ldots, \Lambda_r)$, where $\Lambda_j = \sum_{i=j}^r \lambda_i$. Let $S(\Lambda)$ be the set of all semi-standard Young tableaux on the letters $\{1, 2, \ldots, r+1\}$ of shape Λ. We define a map

$$\begin{aligned}
m: \quad S(\Lambda) &\rightarrow \mathbb{Z}[y_i^{\pm 1}]_{i\in I} \\
T &\mapsto m_T = \textstyle\prod_{i\in I} y_i^{\#T(i)-\#T(i+1)}
\end{aligned} \tag{4.2.2}$$

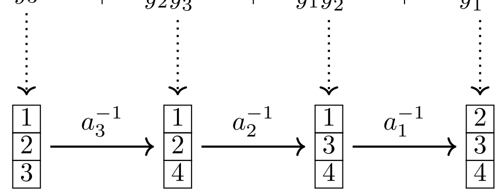
where $\#T(i)$ is the number of times i appears in T. With this mapping, we obtain the tableaux-sum expression for the character of V_λ. More precisely,

$$\chi(V_\lambda) = \sum_{T \in S(\Lambda)} m_T \ .$$

Remark 4.2.3. The $U_q(\mathfrak{g})$-highest weight of the KR-module $W_{k,j}^{(i)}$ is given by $k\omega_i$. Therefore, the tableaux that parametrize the character of KR-modules, considered as $U_q(\mathfrak{g})$-modules, are rectangular of length i and width k.

Example 4.2.4. Consider $\mathfrak{g} = \mathfrak{sl}_4$ and $V = V_{\omega_3}$. Then, $\Lambda = (1, 1, 1)$.

$$\chi(V_{\omega_3}) \quad = \quad y_3 \quad + \quad y_2 y_3^{-1} \quad + \quad y_1 y_2^{-1} \quad + \quad y_1^{-1}$$



Recall that the monomials a_i from Definition 2.1.3, are identified with e^{α_i}. Therefore, multiplying a monomial by a_i^{-1} is equivalent to applying a lowering operator with weight $-\alpha_i$. The action of a_i^{-1} on the tableaux is given by changing a box with i to $i + 1$. □

We now describe the tableaux-sum notation for q-characters. Recall that the restriction of $U_q(\widehat{\mathfrak{g}})$-modules to $U_q(\mathfrak{g})$ corresponds to the map $Y_{i,j} \to y_i$ on the monomials, and the finite weight of $Y_{i,j}$ and y_i are the same, given by ω_i. Therefore, the tableau representation of $Y_{i,j}$ is almost identical to that of y_i, except there are infinitely many $Y_{i,j}$'s corresponding to $j \in \mathbb{Z}$. This infinite property is represented by an extra vertical coordinate added to the usual tableau data.

Definition 4.2.5. 1. Let T be a diagram consisting of a single column of length i equipped with an additional datum $j \in \mathbb{Z}$. To each box in T, we associate an *index* as follows: the top box gets an index $\frac{1-i-j}{2}$ and the indices of the lower boxes increase by one starting from the top box. We call the set of indices of T the *support* of T, denoted Supp(T). Then,

$$\mathrm{Supp(T)} = \left\{ \frac{1-\mathrm{i}-\mathrm{j}}{2}, \ldots, \frac{1+\mathrm{i}-\mathrm{j}}{2} \right\}$$

The diagrams that we will consider in this paper will have i and j such that $(1-i-j)$ is always divisible by 2. That is, $\mathrm{Supp}(T) \subset \mathbb{Z}$. We say T is a *column diagram* of shape (i, j).

2. The *head* of a column diagram T, denoted $\mathrm{head}(T)$, is the index of the first box.

3. The *tail* of a column diagram T, denoted $\mathrm{tail}(T)$, is the index of the last box.

4. The *length* of a column diagram T is given by $i = \mathrm{tail}(T) - \mathrm{head}(T) + 1$. For convenience, we denote it $\mathrm{length}(T)$.

5. A *column tableau* T is a column diagram T of some shape (i, j) decorated with letters $\{1, 2, \ldots, r+1\}$, i.e. we equip T with an arbitrary map

$$\mathrm{Supp}(T) \to \{1, 2, \ldots, r+1\} \ ,$$

where r is the rank of the Lie algebra $\mathfrak{g}$. Equivalently, a column tableau is a column diagram with integers between 1 and $r+1$ filled in each box.

The image of the map at $p \in \mathrm{Supp}(T)$, i.e. the integer in the box with index p, is denoted $T[p]$ and is called the *value* of T at p. If $p < \mathrm{head}(T)$, we set $T[p] = 0$ and if $p > \mathrm{tail}(T)$, we set $T[p] = \infty$. With this redefinition, we can consider T to be defined for all $p \in \mathbb{Z}$ and $\mathrm{Supp}(T)$ is where the value of T is nonzero and finite.

6. A *strip* in T between p_0 and p_1, denoted $T[p_0, p_1]$, is the tableau given by the piece of T between indices p_0 and p_1 with end points included.

7. A *general tableau* is obtained by stacking column tableaux horizontally with the indices of the boxes, defined in (1), aligned.

Let $T = (T_1, \ldots, T_k)$ be a general tableau. Let $\mathrm{Supp}(T) = \cup \mathrm{Supp}(T_l)$. We identify T with a monomial $m_T \in \mathcal{M}$ given by

$$m_T = \prod_{p \in \mathrm{Supp(T)}} \prod_{i=1}^{r} Y_{i,i-2p-1}^{\#(T[p]=i)-\#(T[p+1]=i+1)} \,, \tag{4.2.3}$$

where $\#(T[p] = i)$ is the number of times i appears in T at index p. This is the affine analog of the map (4.2.2).

Remark 4.2.6. If we drop the indices of the tableaux and collapse all the columns until the heads of every column are on the same level, then we obtain Young tableaux, which gives the classical tableaux-sum notation for the highest weight modules. In particular, if we collapse the general tableau, then the map (4.2.3) reduces to the map (4.2.2). Diagramaticaly, we can add a third column to the commutative diagram

$$\begin{array}{ccccc}
\widehat{\mathbf{R}} & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,j}^{\pm 1}]_{i \in I, j \in \mathbb{Z}} & \xleftarrow{(4.2.3)} & \{\text{General Tableaux}\} \\
\downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle collapse} \\
\mathbf{R} & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i \in I} & \xleftarrow{(4.2.2)} & \{\text{Tableaux}\}
\end{array}$$

Definition 4.2.7. Let T be a column tableau of shape (i, j). We define T_{dom} (dominant) to be the column tableau of the same underlying diagram given by

$$T_{dom}[\mathrm{head}(T)] = 1 \quad \text{and} \quad T_{dom}[p+1] = T_{dom}[p] + 1 \,, \quad \text{for } p \in \mathrm{Supp}(T)$$

Remark 4.2.8. Let T be of shape (i, j). Then the map (4.2.3) identifies $m_{T_{dom}} = Y_{i,j}$. In other words, T_{dom} corresponds to the dominant monomial of all column tableaux T of shape (i, j). More generally, the tableaux corresponding to the descendants of any dominant monomial (not necessarily column) will have the same underlying diagram as their dominant monomial, i.e. the lengths and the indices of each column will be the same.

Let us illustrate all these definitions with an example.

Example 4.2.9. Let $r \geq 5$. Let $T = (T_1, T_2)$ be the general tableau consisting of columns of shapes $\{(3, 2), (2, 1)\}$. The first column diagram T_1 is of shape $(3, 2)$, which means the length of T_1 is 3 and

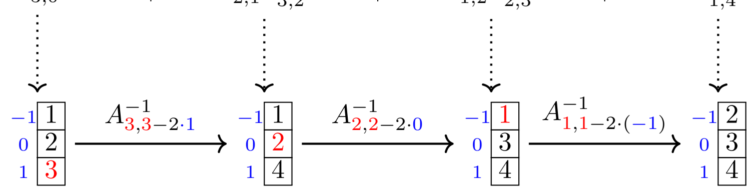
$h(T_1) = \frac{1-i_1-j_1}{2} = -2$. Similarly, the second column diagram T_2 is of length 2 and $h(T_2) = -1$. An example of such general tableau is given below, together with the corresponding collapsed tableau and its dominant tableau.

	diagram of T	tableau T	T_{dom}	$m_{T_{dom}}$
General Tableaux	$\begin{array}{r\|c\|c\|} -2 & \ & \\ -1 & \ & \ \\ 0 & \ & \ \end{array}$	$\begin{array}{r\|c\|c\|} -2 & 1 & \\ -1 & 3 & 4 \\ 0 & 4 & 5 \end{array}$	$\begin{array}{r\|c\|c\|} -2 & 1 & \\ -1 & 2 & 1 \\ 0 & 3 & 2 \end{array}$	$Y_{3,2}Y_{2,1}$
Collapsed Tableaux	$\begin{array}{\|c\|c\|} \ & \ \\ \ & \ \\ \ & \end{array}$	$\begin{array}{\|c\|c\|} 1 & 4 \\ 3 & 5 \\ 4 & \end{array}$	$\begin{array}{\|c\|c\|} 1 & 1 \\ 2 & 2 \\ 3 & \end{array}$	y_3y_2

□

Example 4.2.10. Consider $\mathfrak{g} = \mathfrak{sl}_4$. Let V be the fundamental $U_q(\widehat{\mathfrak{sl}_4})$-module with dominant monomial $Y_{3,0}$. The $U_q(\mathfrak{sl}_4)$-highest weight of this module is ω_3. This is the affine analog of Example 4.2.4. We give the q-character of V below:

$$\chi_q(V) = Y_{3,0} + Y_{2,1}Y_{3,2}^{-1} + Y_{1,2}Y_{2,3}^{-1} + Y_{1,4}^{-1}$$



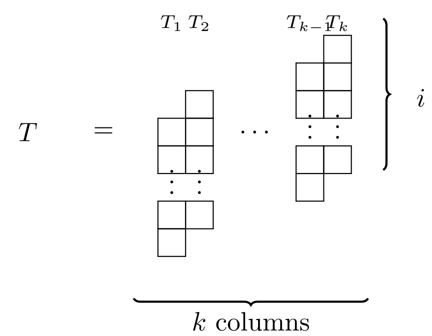
□

Let $T = (T_1, \ldots, T_k)$ be a general tableaux. The monomial m_T in (4.2.3) can also be written as:

$$m_T = m_{T_{dom}} \prod_{l=1}^{k} \prod_{p \in \mathrm{Supp}(T_l)} \prod_{i=(T_l)_{dom}[p]}^{T_l[p]-1} A^{-1}_{i,i-2p}\,. \tag{4.2.4}$$

We have presented so far the modified crystal basis for q-character monomials in [38]. From here on, we define new concepts.

Consider KR-modules and the associated general tableaux. Consider the KR-module with dominant monomial $\mathbf{Y}_{k,j}^{(i)} = Y_{i,j}Y_{i,j+2}\cdots Y_{i,j+2k-2}$. The diagram associated with this dominant monomial and all its descendants consists of k columns stacked as follows:



where each column T_l, $1 \leq l \leq k$, is of length i with appropriate indices determined by $j_l = j + 2l - 2$. We call a diagram of this shape the staircase diagram of shape (i, j, k). Notice that the corresponding collapsed diagram is rectangular, as expected by Remark 4.2.3.

Definition 4.2.11. Define $\mathcal{B}_{k,j}^{(i)}$ to be the set of general tableaux given by staircase diagrams $T = (T_1, \ldots, T_k)$ of shape (i, j, k) along with decorations by $\{1, 2, \ldots, r+1\}$ such that

1. the values of the columns strictly increase from top to bottom, i.e. $T_i[p] < T_i[p+1]$ for all i, p. (see Figure 4.1, left)

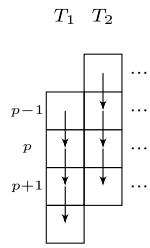
2. the values of the diagonals weakly increase from left to right, i.e. $T_i[p] \leq T_{i+1}[p-1]$ for all i, p. (see Figure 4.1, right)

We call a tableau that belongs to $\mathcal{B}_{k,j}^{(i)}$ for some (i, j, k) a KR-tableau.

Theorem 4.2.12. The q-character of the KR-module $W_{k,j}^{(i)}$ is parametrized by the KR-tableaux in $\mathcal{B}_{k,j}^{(i)}$. That is, we have:

$$\chi_q(W_{k,j}^{(i)}) = \sum_{T \in \mathcal{B}_{k,j}^{(i)}} m_T \ ,$$

where m_T is the monomial associated to the tableau T through the map (4.2.3).



and

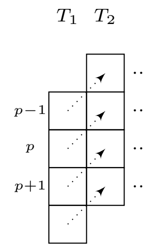


Figure 4.1: Solid and dotted arrows indicate strict and weak inequalities in the direction of the arrows respectively, i.e. $a \to b$ is equivalent to $a < b$, and $a \dashrightarrow b$ is equivalent to $a \leq b$.

Proof. Notice that if we collapse a KR-tableau, we obtain a rectangular semi-standard Young tableau. Since $\mathrm{res}(W^{(i)}_{k,j}) = V_{k\omega_i}$, and $\chi(V_{k\omega_i})$ is given by rectangular semi-standard Young tableaux, the result follows. $\square$

4.2.2 Fundamental cluster diagrams

Recall the fundamental cluster $\mathcal{C}$ defined in (4.1.3). The column diagrams in $\mathcal{C}$ are shown in Figure 4.2(a). Given $\chi^{(i)}_{k,j(k,i)} \in \mathcal{C}$, the corresponding staircase diagram of shape (i, j, k) is constructed as follows: start with the column diagram $\chi^{(i)}_{1,j(1,i)}$ (Figure 4.2(a)), call it the central column and start adding columns of equal length alternatingly to both sides of the central tableau. When i is odd, we start adding on the left and when i is even, we start adding on the right (see Figure 4.2(b) for an example).

(a) Diagrams corresonding to the Fundamental modules ($k = 1$) in $\mathcal{C}$

(b) $i = 3$ and $k = 1, 2, 3, 4$. The central and the last column that is added are colored.

Figure 4.2: Diagrams corresponding to the fundamental cluster variables

4.2.3 Twisted multiplication of tableaux

We will now describe the twisted multiplication $*_\gamma$ from (4.1.1) on the tableaux. Let C, T be KR-tableaux. In order to simplify our notation of Definition 4.1.5, we denote

$$u_{i,j}(C) := u_{i,j}(m_C)\,; \quad v_{i,j}(C) := v_{i,j}(m_C, m_{C_{dom}})\,; \quad \gamma(C,T) := \gamma(m_C, m_T)\,. \tag{4.2.5}$$

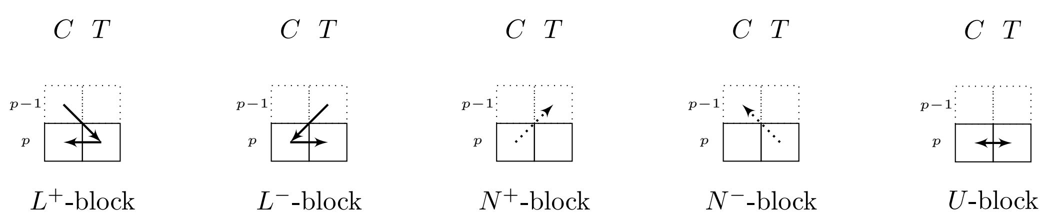
Remark 4.2.13. In order to compute $\gamma(C,T)$ for any general KR-tableaux C and T, it suffices to compute the value of γ between column tableaux only. Indeed, suppose $T = (T_l)$. Then,

$$\begin{aligned} m_T &= \prod_l \left(\prod_{i,j} Y_{i,j}^{u_{i,j}(T_l)} \right) = \prod_{i,j} Y_{i,j}^{\sum_l u_{i,j}(T_l)}\,, \\ m_T &= \prod_l \left(m_{T_{l\,dom}} \prod_{i,j} A_{i,j}^{-v_{i,j}(T_l)} \right) = m_{T_{dom}} \prod_{i,j} A_{i,j}^{-\sum_l v_{i,j}(T_l)}\,. \end{aligned}$$

That is, both $u_{i,j}$ and $v_{i,j}$ are additive. Therefore, for any general tableau $C = (C_k)$, since $\gamma(C,T)$ is a linear expression in $u_{i,j}$ and $v_{i,j}$, we have

$$\gamma(C,T) = \sum_{k,l} \gamma(C_k, T_l)\,.$$

Definition 4.2.14. Let (C,T) be a pair of column KR-tableaux. A *block* in (C,T) at index p is a pair of boxes given by $(C[p], T[p])$. We say there is an $L^{\pm}$-block, an $N^{\pm}$-block, or a U-block at p in (C,T) if the following inequality conditions hold:



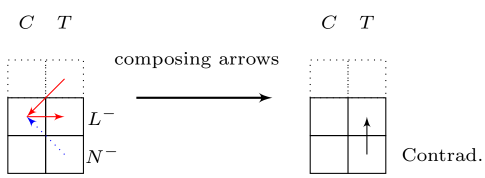
The arrow notation is the same as in Figure 4.1 and two-sided arrow is equivalent to equality.

It is important to stress that a block consists of 2 boxes at index p, even though there may be an inequality requirement coming from boxes with index $p-1$, such as the case of L and N blocks.

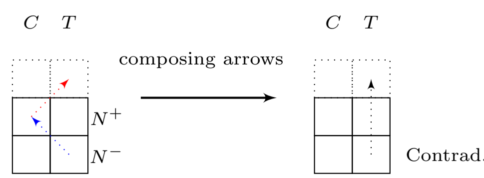
Definition 4.2.15. If there is either an L-block or a U-block at p in (C,T), for convenience we say there is an LU-block at p. Any other combinations are allowed (e.g. L^+UN^--block).

Remark 4.2.16. As we are dealing with KR-tableaux, inequality conditions in Figure 4.1 are always present. Due to this restriction, any block in a pair of KR-tableaux (C,T) is precisely one of the 5 types: $L^\pm, N^\pm$ or U. Moreover, the strictly increasing columns condition puts additional restrictions on the order with which the blocks can appear. For example, (a) an $L^\pm$-block is never followed by an $N^\pm$-block; (b) an $N^\pm$-block is never followed by an $N^\mp$-block, and (c) a U-block is never followed by an N-block. The reason can be seen by simply composing the arrows. For example,

(a)



(b)



Definition 4.2.17. Let (C,T) be a pair of column KR-tableaux. We define for each $p\in\mathbb{Z}$ functions L_p and N_p as follows:

$$L_p(C,T)=\begin{cases}1 & \text{if } C[p-1]<T[p]<C[p]\\ -1 & \text{if } T[p-1]<C[p]<T[p]\\ 0 & \text{otherwise}\end{cases}\quad ;\quad N_p(C,T)=\begin{cases}1 & \text{if } C[p]\le T[p-1]\\ -1 & \text{if } T[p]\le C[p-1]\\ 0 & \text{otherwise}\end{cases}$$

In other words, $L_p(C,T)$ is ±1 if there is an $L^\pm$-block at p and 0 otherwise, and similarly for $N_p(C,T)$. When there is no confusion as to which KR-tableaux we are referring to, we will suppress the dependance on C and T and simply write L_p and N_p.

Definition 4.2.18. Let (C,T) be a pair of column KR-tableaux with $\mathrm{Supp}(C)\cap\mathrm{Supp}(T)=\{h,\dots,t\}$. The block-tableau of (C,T), denoted B_{CT}, is a single column diagram with $\mathrm{Supp}(B_{CT})=\{h,\dots,t,t+1\}$ decorated with letters $\{L^+,L^-,N^+,N^-,U\}$, such that the value of B_{CT} at index p is given by the corresponding block in (C,T) at p.

Example 4.2.19. Block-tableau is a convenient way to describe all the blocks of (C, T) at once. Saying there is an L^+-block at p in (C, T) is equivalent to $B_{CT}[p] = L^+$.

Definition 4.2.20. Let (C, T) be a pair of column tableaux. We say (C, T) is a *fundamental* pair if $\text{head}(C) \geq \text{head}(T)$ and $\text{tail}(C) \leq \text{tail}(T)$. We say (C, T) is *anti-fundamental* if (T, C) is fundamental. We say (C, T) is a *regular* pair if $\text{head}(C) > \text{head}(T)$ and $\text{tail}(C) > \text{tail}(T)$. We say (C, T) is *anti-regular* if (T, C) is regular (see Figure 4.3).

The reason for the name fundamental pair is because of the diagrams in $\mathcal{C}$ corresponding to the fundamental modules, i.e KR-modules with $k = 1$. Every pair of fundamental modules in $\mathcal{C}$ forms a fundamental or an anti-fundamental pair as in Definition 4.2.20 (see Figure 4.2(a)).

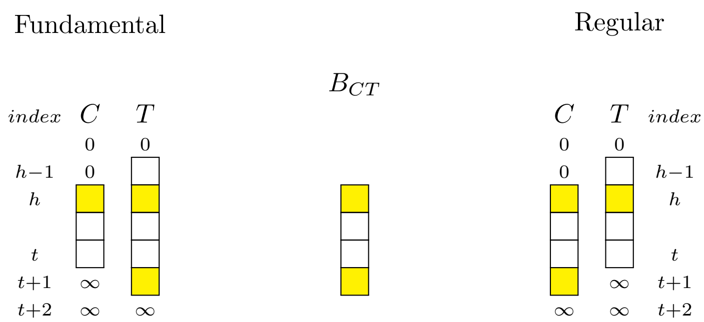


Figure 4.3: Types of pairs of column diagrams

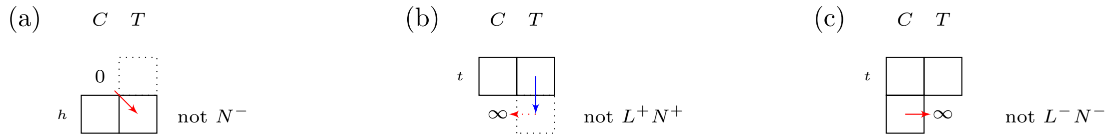
Lemma 4.2.21. Let (C, T) be a pair of column KR-tableaux and suppose (C, T) is either fundamental or regular. Denote $h = \text{head}(C, T)$ and $t = \text{tail}(C, T)$. Then,

(a) $B_{CT}[h]$ is never N^- since $C[h-1] = 0 \not\geq T[p]$.

(b) If (C, T) is fundamental, $B_{CT}[t+1]$ is never L^+ or N^+ since $T[t] < T[t+1] \leq \infty = C[t]$.

(c) If (C, T) is regular, $B_{CT}[t+1]$ is never L^- or N^- since $C[t] < C[t+1] < \infty = T[t]$.

Proof. The statements are clear from the following pictures:



□

Remark 4.2.22. It suffices to compute $\gamma(C,T)$ for (C,T) either fundamental or regular. If (C,T) is of neither type, then (T,C) is either fundamental or regular and $\gamma(C,T) = -\gamma(T,C)$.

The value of $\gamma(C,T)$ depends on the occurrence of L and N blocks in (C,T) encoded in the functions L_p and N_p. More precisely,

Theorem 4.2.23. Let (C,T) be a pair of column KR-tableaux and let $\mathrm{Supp}(C) \cap \mathrm{Supp}(T) := \{h, \dots, t\}$ (see Figure 4.3). Then

$$\gamma(C,T) = \sum_{p=h}^{t} L_p(C,T) + \begin{cases} N_{t+1}(C,T) & \text{if } (C,T) \text{ is (anti-)fundamental} \\ L_{t+1}(C,T) & \text{otherwise} \end{cases} .$$

Remark 4.2.24. In other words, $\gamma(C,T)$ counts the number of L^+-blocks minus the number of L^--blocks in (C,T) with one more contribution from the tail block.

Proof. Let's decompose L_p into its positive L_p^+ and negative L_p^- parts such that

$$L_p = L_p^+ - L_p^-$$

Recall that $\gamma(C,T) = d(C,T) - d(T,C)$. The calculation of $d(C,T)$ is borrowed from [38]. For the reader's convenience, we reproduce it here.

We want to compute (see Definition 4.1.5, Remark 4.1.6, and (4.2.5))

$$d(C,T) = \sum_{i,p} v_{i,i-2p}(C) u_{i,i-2p-1}(T) + \sum_{i,p} u_{i,i-2p}(C_{dom}) v_{i,i-2p-1}(T) \ . \tag{4.2.6}$$

Let $\langle condition \rangle$ be 1 if the *condition* is true and 0 otherwise. By (4.2.4) and Definition 4.1.2, we know

$$v_{i,i-2p}(C) = \langle C_{dom}[p] \leq i \leq C[p]-1 \rangle \ . \tag{4.2.7}$$

By (4.2.3) and Definition 4.1.2, we have

$$u_{i,i-2p-1}(T) = \langle T[p] = i \rangle - \langle T[p+1] = i+1 \rangle \ . \tag{4.2.8}$$

Putting (4.2.7) and (4.2.8) together,

$$\begin{aligned} \sum_i v_{i,i-2p}(C)u_{i,i-2p-1}(T) &= \sum_i \langle C_{dom}[p] \leq i \leq C[p]-1 \rangle \left(\langle T[p] = i \rangle - \langle T[p+1] = i+1 \rangle \right) \\ &= \langle C_{dom}[p] \leq T[p] < C[p] \rangle - \langle C_{dom}[p+1] \leq T[p+1] \leq C[p] \rangle \ , \end{aligned}$$

where we used $C_{dom}[p+1] = C_{dom}[p]+1$. We want to sum up the above value for all p. By shifting the summation index p, we consider the sums of the form

$$-\langle C_{dom}[p] < T[p] \leq C[p-1] \rangle + \langle C_{dom}[p] \leq T[p] < C[p] \rangle = \langle C[p-1] < T[p] < C[p] \rangle = L_p^+ \ .$$

Now, summing over all p, we obtain:

$$\sum_{i,p} v_{i,i-2p}(C)u_{i,i-2p-1}(T) = \sum_{p=h}^{t} L_p^+ - \langle C_{dom}[t] < T[t+1] \leq C[t] \rangle \ , \tag{4.2.9}$$

which gives the first half of $d(C,T)$. The second half of (4.2.6) consists of a single term. That is because $m_{C_{dom}} = Y_{i,j}$, where $i = \text{length}(C) = \text{tail}(C) - \text{head}(C) + 1$ and $j = -\text{tail}(C) - \text{head}(C)$. By setting $j = i - 2p + 1$, we evaluate $p^* = \text{tail}(C)+1$ and $i^* = \text{length}(C)$ are the only values for which $u_{i^*,i^*-2p^*+1}(C_{dom}) = 1$. Therefore,

$$\begin{aligned} \sum_{i,p} u_{i,i-2p+1}(C_{dom})v_{i,i-2p}(T) &= v_{i^*,i^*-2p^*}(T) = \langle T_{dom}[p^*] \leq i^* \leq T[p^*]-1 \rangle \\ &= \langle T_{dom}[\text{tail}(C)+1] \leq \text{length}(C) \leq T[\text{tail}(C)+1]-1 \rangle \ . \end{aligned} \tag{4.2.10}$$

By adding (4.2.9) and (4.2.10), we obtain $d(C,T)$. The value of $d(T,C)$ is obtained by switching C and T, which gives a sum of L_p^- in the equivalent of (4.2.9). Putting everything together, we obtain

$$\gamma(C,T) = d(C,T) - d(T,C) = \sum_{p=n}^{m} (L_p^+ - L_p^-) + \mathrm{Bd}(C,T) \ ,$$

where $Bd(C,T)$ is the boundary term given by

$$\begin{aligned}
\mathrm{Bd}(C,T) \quad = \quad & \langle T_{dom}[t] < C[t+1] \leq T[t] \rangle - \langle C_{dom}[t] < T[t+1] \leq C[t] \rangle \\
& + \langle T_{dom}[\mathrm{tail}(C)+1] \leq \mathrm{length}(C) < T[\mathrm{tail}(C)+1] \rangle \\
& - \langle C_{dom}[\mathrm{tail}(T)+1] \leq \mathrm{length}(T) < C[\mathrm{tail}(T)+1] \rangle \ .
\end{aligned}$$

We simplify the boundary term next. We consider two cases: when (C,T) is fundamental and when (C,T) is regular.

Suppose (C,T) is fundamental. Then $\mathrm{tail}(C) = t$, $\mathrm{tail}(T) \geq t$, and $C[p] = C_{dom}[p] = \infty$ for $p \geq t+1$ (see Figure 4.3). Notice that,

$$C_{dom}[t] = \mathrm{length}(C) \leq \mathrm{length}(T[\mathrm{head}(T), t+1]) = T_{dom}[t+1] \leq T[t+1] \ . \qquad (4.2.11)$$

We compute

$$\begin{aligned}
0 \quad = \quad & \left\langle T_{dom}[t] < \underbrace{C[t+1]}_{\infty} \leq \underbrace{T[t]}_{\neq \infty} \right\rangle = \left\langle \underbrace{T_{dom}[t+1] \leq \mathrm{length}(C)}_{\text{never true by Eq. 4.2.11}} < T[t+1] \right\rangle \\
= \quad & \left\langle \underbrace{C_{dom}[\mathrm{tail}(T)+1]}_{\infty \text{ since } \mathrm{tail}(T) \geq t} \leq \mathrm{length}(T) < C[\mathrm{tail}(T)+1] \right\rangle \ .
\end{aligned}$$

Thus,

$$\mathrm{Bd}(C,T) = \left\langle \underbrace{C_{dom}[t] < T[t+1]}_{\text{always true by Eq. 4.2.11}} \leq C[t] \right\rangle = \langle T[t+1] \leq C[t] \rangle = N_{t+1} \qquad (4.2.12)$$

Suppose (C,T) is regular. Then $\mathrm{tail}(T) = t$, $\mathrm{tail}(C) \geq t$, and $T[p] = T_{dom}[p] = \infty$ for all $p \geq t+1$.

Therefore,

$$\left\langle C_{dom}[t] < \underbrace{T[t+1]}_{\infty} \leq \underbrace{C[t]}_{\neq\infty} \right\rangle = \left\langle \underbrace{T_{dom}[\text{tail}(C)+1]}_{\infty \text{ since tail}(C)\geq t} \leq \text{length}(C) < T[\text{tail}(C)+1] \right\rangle = 0 \ .$$

Notice that

$$C_{dom}[t+1] = t+1-\text{head}(C)+1 < \underbrace{t-\text{head}(C)+1 \leq \text{tail}(T)-\text{head}(T)+1}_{\text{head}(C)>\text{head}(T) \text{ (see Figure 4.3) and tail}(T)=t} = \text{length}(T) \ .$$

Therefore, removing the condition that is always satisfied and substituting $\text{length}(T) = T_{dom}[t]$, we obtain:

$$\begin{aligned} \text{Bd}(C,T) &= \langle T_{dom}[t] < C[t+1] \leq T[t] \rangle - \langle T_{dom}[t] < C[t+1] \rangle \\ &= -\langle T[t] < C[t+1] \rangle = -\langle T[t] < C[t+1] < T[t+1] \rangle = -L^-_{t+1} = L_{t+1} \ , \end{aligned}$$

where we added the condition $C[t+1] < T[t+1] = \infty$, which is always satisfied and, therefore, does not affect the outcome. □

4.2.4 Exchanging boxes, compatibility conditions, and the involution σ

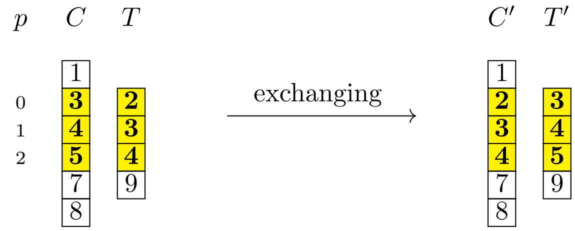
In this section, we define the core concept of exchanging boxes of tableaux, which defines the map σ and the three subsets $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}$ of the set $\mathcal{B}^{(i)}_{k,j} \times \mathcal{B}^{(i')}_{k',j'}$ for any (i,j,k) and (i',j',k').

Definition 4.2.25. Let (C,T) be a pair of column tableaux. A *strip* in (C,T) is a slice of (C,T) given by $(C[p_0,p_1], T[p_0,p_1])$ for some $p_0 < p_1 \in \text{Supp}(C) \cap \text{Supp}(T)$. An *L-strip* is a strip that starts with an L-block and includes all the N-blocks that follow.

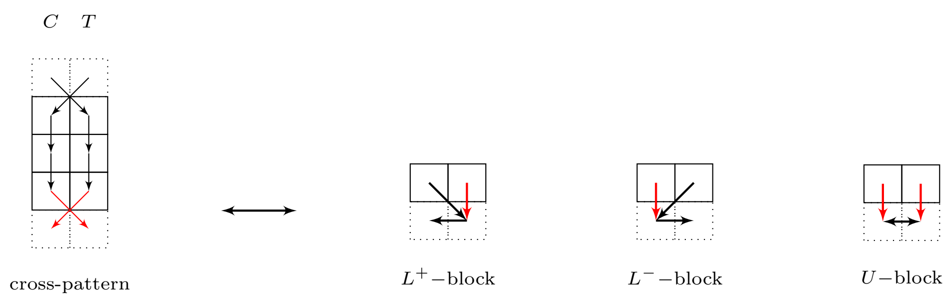
Definition 4.2.26. Let (C,T) be a pair of column KR-tableaux. We say an L-strip in (C,T) is *column-compatible* if it is possible to exchange the boxes in the L-strip and obtain a pair of valid column KR-tableaux. That is, the new tableaux also have strictly increasing columns.

Example 4.2.27. Here is an L^+-strip starting at $p = 0$ and ending at $p = 2$. It is column-compatible because it is possible to exchange $C[0,2]$ and $T[0,2]$ and obtain a new pair of column tableaux (C', T').

Notice that both C' and T' have strictly increasing columns.

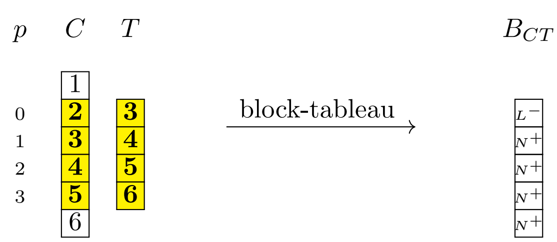


Remark 4.2.28. For a strip to be column-compatible, it must begin and end with an equality condition of a cross-pattern, which occurs at LU-blocks. Indeed,



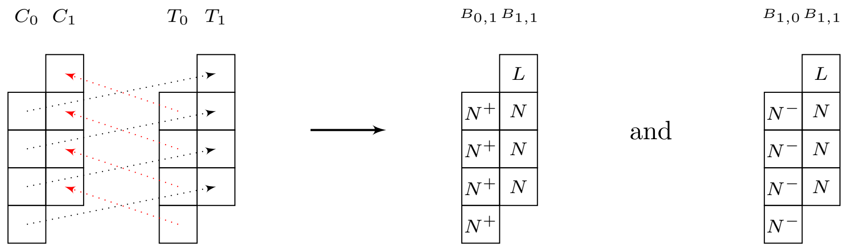
An L-strip can either be followed by an LU-block or by nothing at all. In the first case, the L-strip is always column-compatible. In particular, this means only the very last L-strip is potentially not column-compatible. All others are column-compatible since they are, by definition, at the very least followed by the next L-block.

Example 4.2.29. Here is an L^--strip starting at $p = 0$ and ending at $p = 3$. It is not column-compatible since exchanging $C[0, 3]$ and $T[0, 3]$ will violate the strictly increasing columns condition post-exchange.



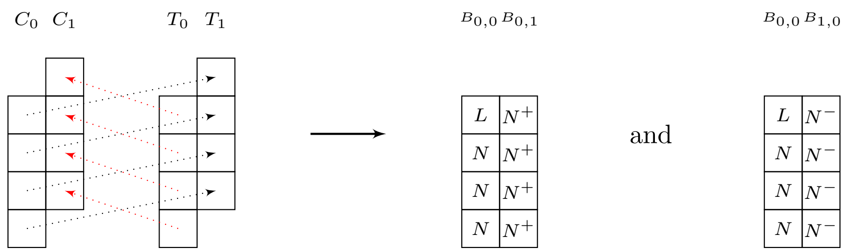
Definition 4.2.30. Let $C = (C_0, C_1)$ and $T = (T_0, T_1)$ be KR-tableaux and let $B_{i,j}$ denote the block-tableau of (C_i, T_j).

Suppose (C_1, T_1) forms an L-strip. We say (C_1, T_1) is *left-compatible* if the weakly increasing diagonals conditions in Definition 4.2.11(2) are not violated when C_1 and T_1 are exchanged. Pictorially, (C_1, T_1) is left-compatible if conditions (lC) (left-compatibility from C) and (lT) (left-compatibility from T) are satisfied:



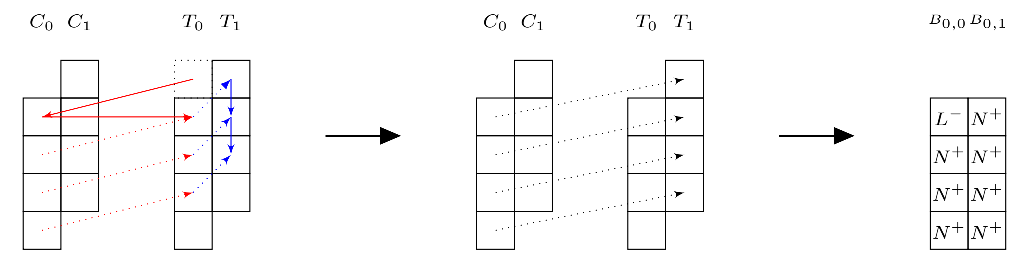
(lC) (lT)

Suppose (C_0, T_0) forms an L-strip. We say L-strip (C_0, T_0) is *right-compatible* if the weakly increasing diagonals condition in Definition 4.2.11(2) is not violated when the boxes in C_0 and T_0 are exchanged. Pictorially, (C_0, T_0) is right-compatible if conditions (rC) (right-compatibility from C) and (rT) (right-compatibility from T) are satisfied:



(rT) (rC)

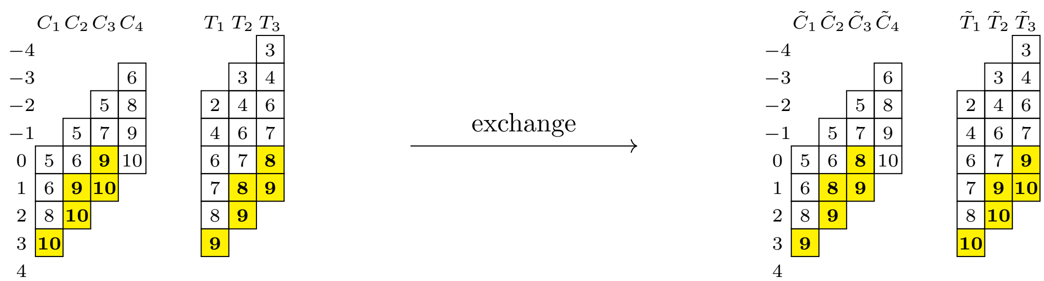
Remark 4.2.31. An L^--strip always satisfies (rT) and (lC) and an L^+-strip always satisfies (rC) and (lT). This can be seen by composing the arrows of the L-strip with the arrows within the KR-tableaux. For example,



By composing the red and blue arrows, we see that the condition (rT) is always satisfied.

Definition 4.2.32. Let (C, T) be a pair of KR-tableaux and let $S = (S_1, \ldots, S_k)$, where S_l is a union of one or more L-strips in (C_l, T_l). We say S is *exchangeable* if it is possible to exchange the boxes in S and still obtain valid KR-tableaux, i.e. the resulting new pair $(\tilde{C}, \tilde{T})$ have strictly increasing columns and weakly increasing diagonals. S is *minimally exchangeable* if removing any nonempty subset of L-strips from S results in a non-exchangeable sequence.

Example 4.2.33. Consider $C = (C_1, C_2, C_3, C_4)$ and $T = (T_1, T_2, T_3)$.

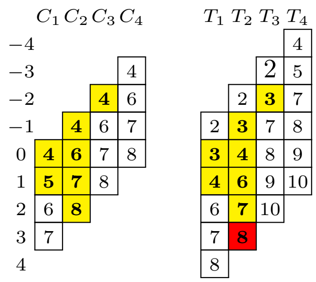


A sequence of L-strips is shown in bold letters above given by $S_1 = (C_1[3], T_1[3])$, $S_2 = (C_2[1,2], T_2[1,2])$, and $S_3 = (C_3[0,1], T_3[0,1])$. Notice that each L-strip is column-compatible, S_1 is left-compatible, S_3 is right-compatible, and S_1, S_2 are not right-compatible. The sequence S is exchangeable since it is possible to exchange the colored boxes and still obtain a pair of valid KR-tableaux. It is also minimal since removing any subset results in a non-exchangeable sequence. For example, exchanging boxes of S_1, S_2 without exchanging the boxes of S_3 will violate the weakly increasing diagonals condition since S_2 is not right-compatible.

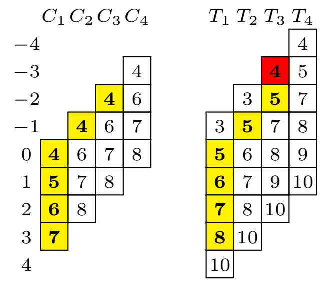
Definition 4.2.34. Given a left-compatible L-strip, we say *it can be completed* to an exchangeable sequence if it is possible to include boxes that, when not exchanged, cause violations of the left or right compatibility conditions and achieve an exchangeable sequence. If, in the process, we end up with an irresolvable contradiction, we say the L-strip *cannot be completed to an exchangeable sequence.*

Let us demonstrate some situations where an L-strip cannot be completed to an exchangeable sequence.

Example 4.2.35. In both cases demonstrated below, there is an L-strip in (C_1, T_1) (bold), which is not right-compatible. We iteratively add all the boxes that cause right or column compatibility violations. In both cases, we run out of boxes to include before resolving all the violations.



(C_2, T_2) is not column-compatible



(C_3, T_3) is not column-compatible

Definition 4.2.36. Let (C, T) be a pair of KR-tableaux such that no L-strip can be completed to an exchangeable sequence. Then we say (C, T) *has no exchangeable sequences.*

We now describe the the map σ. Consider the set $\mathcal{B}^{(i)}_{k,j} \times \mathcal{B}^{(i')}_{k',j'}$ for any (i, j, k) and (i', j', k'). We define

$$\mathcal{P}_0 := \left\{ (C, T) \middle| C \in \mathcal{B}^{(i)}_{k,j}, T \in \mathcal{B}^{(i')}_{k',j'}, \text{ and } (C, T) \text{ has no exchangeable sequences} \right\}$$

Let $(C, T) \in \mathcal{B}^{(i)}_{k,j} \times \mathcal{B}^{(i')}_{k',j'} \backslash \mathcal{P}_0$. By definition, (C, T) has an exchangeable sequence. Suppose $(C', T') \in \mathcal{B}^{(i)}_{k,j} \times \mathcal{B}^{(i')}_{k',j'} \backslash \mathcal{P}_0$ such that (C', T') is obtained from (C, T) by exchanging a minimally exchangeable sequence. Then we assign (C, T) to $\mathcal{P}_1$, $(\tilde{C}, \tilde{T})$ to $\mathcal{P}_{-1}$, and define $\sigma(C, T) = (\tilde{C}, \tilde{T})$. The exact order is not important. All that matters is that we can partition $\mathcal{B}^{(i)}_{k,j} \times \mathcal{B}^{(i')}_{k',j'} \backslash \mathcal{P}_0$ into 2 disjoint subsets. Since the elements in $\mathcal{P}_0$ do not have exchangeable sequences, we define $\sigma(C, T) = (C, T)$ for all $(C, T) \in \mathcal{P}_0$.

4.2.5 Proof of Theorem 4.2.1

Let $C = (C_i)$ and $T = (T_j)$ be a pair of KR-tableaux with no exchangeable sequences. We want to show $\gamma(C,T) = 0$. Recall from Remark 4.2.13 that:

$$\gamma(C,T) = \sum_{i,j} \gamma(C_i, T_j) \ ,$$

and by Lemma 4.2.23:

$$\gamma(C_i,T_j) = \sum_{p=h}^{t} L_p(C_i,T_j) + \begin{cases} N_{t+1}(C_i,T_j) & \text{if } (C_i,T_j) \text{ is (anti-)fundamental} \\ L_{t+1}(C_i,T_j) & \text{otherwise} \end{cases} \ ,$$

where $h = \text{head}(C,T)$ and $t = \text{tail}(C,T)$.

Let us prove first the theorem in case when both C and T are column tableaux. Recall that column diagrams correspond to fundamental modules ($k = 1$) and the fundamental modules in the fundamental cluster $\mathcal{C}$ form (anti-)fundamental pairs (see Section 4.2.2 and Definition 4.2.20).

Lemma 4.2.37. Let (C,T) be a pair of column KR-tableaux of type (anti-)fundamental with no exchangeable sequences, i.e. $(C,T) \in \mathcal{P}_0$. Then $\gamma(C,T) = 0$.

Proof. Suppose (C,T) admits an L-strip. Notice that an L-strip in (C,T) is exchangeable if and only if it is column-compatible since there are no columns to the left or to the right of both C and T. Since (C,T) has no exchangeable sequences, there can be exactly one L-strip. Otherwise, by Remark 4.2.28, the L-strips other than the last one are all column-compatible, and, therefore, exchangeable.

We may assume the L-strip in question is an L^--strip. If not, we simply consider (T,C) instead, where L^+-strips in (C,T) become L^--strips in (T,C). Let p^* be the index of the head of the L-strip. Then,

$$\gamma(C,T) = \underbrace{L_{p^*}(C,T)}_{-1} + N_{t+1}(C,T) \ ,$$

where $t = \text{tail}(C,T)$. Since L_{p^*} is not column-compatible, it is not followed by an LU-block. That is, it must be followed by N-blocks only. By Remark 4.2.16(a,b), an L^--block can only be followed by an N^+, and N^+-blocks cannot be followed by N^--blocks. Therefore, we must have that $N_{t+1}(C,T) = +1$ and $\gamma(C,T) = 0$.

Suppose (C,T) has no L-strips. We can assume (C,T) is a fundamental pair. If not, we simply consider (T,C) instead. If the boundary term is zero, that is $N_{t+1}(C,T) = 0$, then $\gamma(C,T) = 0$ and the result holds. Suppose the boundary term is not zero. By Remark 4.2.21(b), since (C,T) is a fundamental pair, the last block is never N^+. Therefore, $N_{t+1}(C,T) = -1$. By Remark 4.2.16(a,b), an N^--block can only be preceded by L^+ or N^-. However, there are no L-blocks in (C,T). So, there must be only N^--blocks in (C,T). By Remark 4.2.21(a), the very first block cannot be N^-. Contradiction. This concludes the proof. □

Corollary 4.2.38. Non-column-compatible L-strips in (anti-)fundamental pairs do not contribute to γ since their contribution is always canceled out by the boundary term.

Our strategy in proving Theorem 4.2.1 is to find a way to systematically cancel contributions from L_p to γ when the sequence cannot be completed to an exchangeable sequence. In the lemmas that follow, we always have $C = (C_0, C_1)$ and $T = (T_0, T_1)$ a pair of KR-tableaux and (C_0, T_0) has a non-exchangeable L^--strip. The goal is to find a unique non-exchangeable L^+-strip in (C,T) to cancel out the contribution of L^-. Moreover, when the L^- does not contribute to γ (as in Lemma 4.2.37), we want to show that there is no such corresponding L^+. We make an exhaustive list of all the possibilities for L^- to be non-exchangeable and give the lemma that address the situation as a reference in Table 4.1.

(C_0,T_0) type	L^- in (C_0,T_0) is right-compatible	L^- in (C_0,T_0) is column-compatible	Contributes to γ	Lemma
any type	No	Yes	Yes	4.2.41
(anti-)regular	Yes	No	Yes	4.2.43,4.2.44
(anti-)fundamental	No	No	No	4.2.46
(anti-)fundamental	Yes	No	No	4.2.48

Table 4.1: List of non-exchangeable L^--strips

Example 4.2.39. We demonstrate the most common type of cancellation in Figure 4.4. There is L^--block in (C_0, T_0) that is column-compatible, but not right-compatible given by $1 < 2 < 3$ (colored, left), which contributes -1 to γ. It is not right-compatible due to 2 in C_1 since the weakly increasing diagonals condition will be violated post-exchange of the L^--strip. However, there is a non-left-compatible L^+-block in (C_1, T_0) given by $2 < 3 < 4$ (colored, right). The L^+-strip is not left-compatible due to 4 in C_0. That is, we found a non-exchangeable $+1$ contribution to γ to cancel out the previous -1.

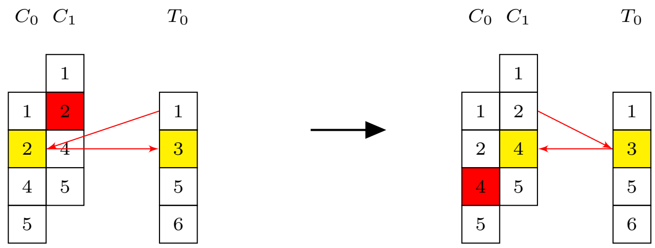


Figure 4.4: Cancellation of non-right-compatible L^--strips with non-left-compatible L^+-strips

Remark 4.2.40. Suppose $C = (C_0, C_1)$ and T_0 are KR-tableaux. Suppose there is an LU-block at p in (C_0, T_0). Then there is an L^+N^--block at $p-1$ in (C_1, T_0) (see Figure 4.5). Indeed, by composing arrows,

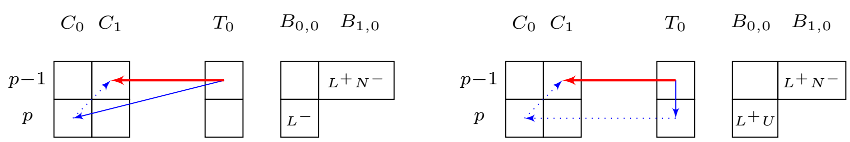


Figure 4.5: Composition of arrows.

we find $T_0[p-1] < C_1[p-1]$, which means there cannot be an L^-U-block in (C_1, T_0) at $p-1$. Also, there cannot be an N^+-block since $T_0[p-2] < T_0[p-1] < C_1[p-1]$. Thus, there can be either L^+ or an N^- block at $p-1$ in (C_1, T_0).

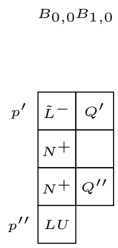
This next lemma is the main cancellation action demonstrated in Example 4.2.39.

Lemma 4.2.41. Let $C = (C_0, C_1)$ and $T = (T_0, T_1)$ be a pair of KR-tableaux. To every right-incompatible L^--strip in (C_0, T_0) that contributes to γ, there exists a unique left-incompatible L^+-strip in (C_1, T_0) that contributes to γ. All other L^+-strips in (C_1, T_0) are left-compatible.

Proof. Without loss of generality, we may assume (C_0, T_0) is either fundamental or regular. If not, we simply consider (T, C) which is of the desired type. As before, denote the block tableaux of (C_i, T_j) by

$B_{i,j}$.

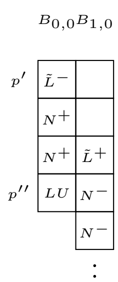
We will consider the case of the very last non-right-compatible L^--strip separately at the end of the proof. Let $\tilde{L}^-$ be not the very last one, and let p' be its index. By Lemma 4.2.31, $\tilde{L}^-$ satisfies the condition (rT), i.e. T_1 does not pose violations. Since $\tilde{L}^-$ is not right-compatible, the condition (rC) must fail. In other words, C_1 must pose a right-compatibility violation. Since $\tilde{L}^-$ is not the last L^-, it is followed by an L-block or a U-block (whichever one comes first). Let p'' be the index of the LU-block that follows $\tilde{L}^-$.



Denote the blocks in (C_1, T_0) at indices p' and $p'' - 1$ by Q' and Q'' respectively. By Remark 4.2.40, Q'' is either L^+ or N^-. If $Q'' = L^+$, we have a candidate. Suppose not, i.e. $Q'' = N^-$. Then, by Remark 4.2.16 it can only be preceded by L^+ or N^-. If we don't allow any L^+ between Q' and Q'', then Condition (rC) is satisfied. Contradiction. Therefore, there must be at least one L^+-block between Q' and Q''.

If there is more than one L^+-block, let $\tilde{L}^+$ be the one with the largest index, i.e. closest to Q''. Then $\tilde{L}^+$ is not left-compatible due to the LU-block in (C_0, T_0). Moreover, all other L^+-strips between Q' and Q'' are left-compatible since (lT) is satisfied (seen from picture) and (lC) is always satisfied for L^+-strips.

It is not clear that the $\tilde{L}^+$ we found contributes to γ. Suppose it does not. There is exactly one situation where an L-block does not contribute to γ. This can happen only if (C_1, T_0) is fundamental and $\tilde{L}^+$ is the last L-block followed by N^-'s. The boundary term of γ is then -1, which cancels out the contribution of $\tilde{L}^+$. Then we have the following picture:



All L^--strips in (C_0, T_0) appearing at any indices $p \geq p''$ satisfy Condition (rC) since $B_{1,0}$ consists of N^--blocks only for all $p \geq p''$, and, therefore, are right-compatible. This means $\tilde{L}^-$ is the very last non-right-compatible L^--strip, which is a contradiction. This concludes the proof for this case.

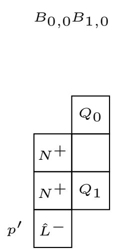
Now let $\tilde{L}^-$ be the very last L^--strip that contributes to γ and is not right-compatible. There are exactly two situations:

1. The situation described above, i.e. $\tilde{L}^-$ is followed by LU, but the corresponding $\tilde{L}^+$ in (C_1, T_0) does not contribute to γ. This happens when (C_1, T_0) is fundamental.

2. $\tilde{L}^-$ is not followed by LU and (C_0, T_0) is regular.

In situation (1), the $\tilde{L}^+$ we found did not contribute to γ and it is the last L^+ in (C_1, T_0). Therefore, we must look for the appropriate L^+ elsewhere in (C_1, T_0). In situation (2), since $\tilde{L}^-$ is the absolute last L^--strip that is not followed by LU, it is followed by N^+-blocks only. This means that any L^+ in (C_1, T_0) adjacent to the strip associated to $\tilde{L}^-$ is left-compatible. Since we are looking for a non-left-compatible L^+ to pair with $\tilde{L}^-$, we must also look elsewhere in (C_1, T_0).

We now consider both situations. Let $\hat{L}^-$ be the very first L^--strip in (C_0, T_0), and let p' be its index. Denote the first block in (C_1, T_0) by Q_0 and the block at index $p' - 1$ in (C_1, T_0) by Q_1.



By Remark 4.2.21, the first block in $B_{1,0}$, i.e. Q_0, is not N^-. By Remark 4.2.40, Q_1 is either L^+ or N^-. The only way to transition from non N^--block to N^--block is through L^+. Therefore, there must be at least one L^+-block between Q_0 and Q_1. Let $\hat{L}^+$ be the L^+ closest to Q_1. Then the $\hat{L}^+$-strip is not left-compatible due to $\hat{L}^-$ in (C_0, T_0) and contributes to γ since it is not the last L-block in (C_1, T_0). All other L^+ above $\hat{L}^+$, if they exist, are left-compatible as seen from the picture. Notice that it is possible to have U-blocks above $\hat{L}^-$. Then we replace $\hat{L}^-$ with the very first U and argue as before.

□

Let us put everything together in Figure 4.6 to emphasize the fact that we have unique pairing of non-right-compatible L^--strips with non-left-compatible L^+-strips.

Corollary 4.2.42. Let $C = (C_{-1}, C_0)$ and $T = (T_{-1}, T_0)$ be a pair of KR-tableaux. To every non-left-compatible L^+-strip in (C_0, T_0) that contributes to γ, there exists a unique non-right-compatible L^--strip in (C_{-1}, T_0) that contributes to γ. All other L^--strips in (C_{-1}, T_0) are right-compatible.

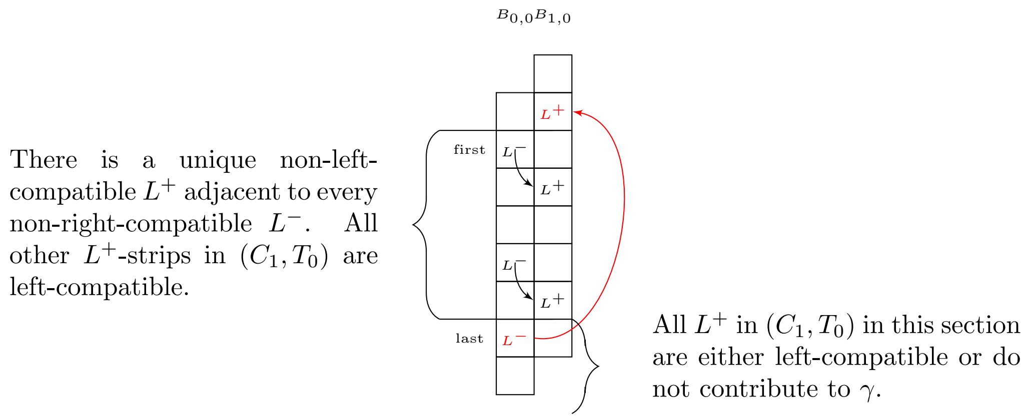


Figure 4.6: Unique pairings of non-exchangeable L^- and L^+.

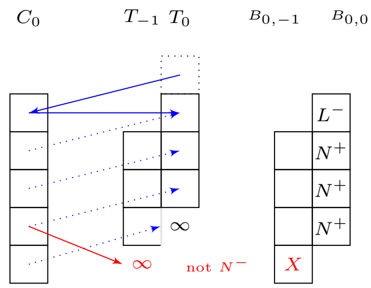
Proof. Left-compatibility and right-compatibility are, in fact, identical requirements with different points of references. This is evident in the underlying inequalities being the same (see Definition 4.2.30). We simply start with non-left-compatible L^+-strips in (C_0, T_0) and assign the L^--blocks in (C_{-1}, T_0) that pair with the L^+-blocks by Lemma 4.2.41. □

Notice that in the proof of Lemma 4.2.41, when considering the last L^--strip with (C_0, T_0) regular (case two), we did not use the fact that L^- is not right-compatible. Moreover, this situation includes the case when L^- is not column-compatible.

Corollary 4.2.43. Suppose (C_0, T_0) is regular and there is a non-column-compatible L^--strip. If there is a column in C to the right of C_0, call it C_1, then there is a unique non-left-compatible L^+ in (C_1, T_0). □

Lemma 4.2.44. Suppose (C_0, T_0) is regular and there is a non-column-compatible L^-. If there is a column in T to the left of T_0, call it T_{-1}, then L^- is never *left*-compatible.

Proof. We want to show that Condition (lT) is not satisfied. We have the following picture:



□

Lemma 4.2.45. Let $C = (C_i)$ and $T = (T_j)$ be KR-tableaux from $\mathcal{C}$. Suppose (C_i, T_j) is regular. Then at least one of C_{i+1} or T_{j-1} must exist. In other words, either there is a column to the right of C_i or to the left of T_j or both.

Proof. Since the central columns of C and T form a fundamental pair (see Section 4.2.2), the central column of T must be to the left of T_j or the central column of C must be to the right of C_i. □

When looking for L^+ to cancel out the contributions of non-exchangeable L^--blocks, we need only address left-compatible L^--blocks. The reason is that non-left-compatible L^--blocks would have already been paired with a non-right-compatible L^+-blocks previously. Lemma 4.2.41 addresses most of the situations when we have a non-exchangeable L^- that contributes to γ. Lemma 4.2.45 shows that there are exactly two other situations when L^- contributes to γ and needs a pair. Lemmas 4.2.43 and 4.2.44 address each of those situations respectively. This concludes the unique pairing when we have a non-exchangeable L^- that contributes to γ.

Next, we address the case when we have a non-exchangeable L^- that does not contribute to γ. We must ensure that there is no corresponding L^+, which would create an imbalance. There are two possibilities as listed in Table 4.1.

Lemma 4.2.46. Suppose (C_0, T_0) is fundamental. Then every L^--strip in (C_0, T_0) is column-compatible, i.e. contributes to γ.

Proof. By Remark 4.2.21(b), the last block in $B_{0,0}$ is not N^+, and, therefore, the boundary term of $\gamma(C_0, T_0)$ is zero. □

Remark 4.2.47. Let $C = (C_0, C_1)$ and $T = (T_0)$ be KR-tableaux. If (C_0, T_0) has an L^+UN^--block at p, then (C_1, T_0) has an N^--block at p (see Figure 4.7). Indeed, by composing arrows, we find $T_0[p] \le C_1[p-1]$,

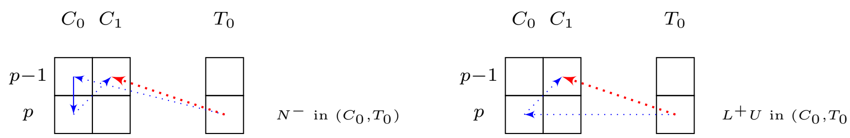
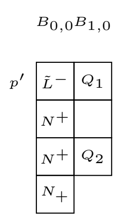


Figure 4.7: Composition of arrows.

which means there is an N^--block at p in (C_1, T_0).

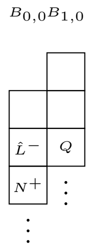
Lemma 4.2.48. Suppose (C_0, T_0) is anti-fundamental and there is a non-right-compatible and non-column-compatible L^--strip. Then there does not exist a corresponding non-exchangeable L^+-strip in (C_1, T_0).

Proof. Let $\tilde{L}^-$ be the L^--strip in question and let p' be its index. Since L^- is not column-compatible, it must be the last L-strip in (C_0, T_0) and the last block in (C_0, T_0) must be N^+. In other words, the boundary term of $\gamma(C_0, T_0)$ is $+1$, which cancels out the contribution of $\tilde{L}^-$. Since the contribution of $\tilde{L}^-$ is already canceled out, we want to show that there is no non-exchangeable L^+ in (C_1, T_0) that is paired with $\tilde{L}^-$. It suffices to show that all possible L^+'s that can be paired with $\tilde{L}^-$ are, in fact, left-compatible. Notice that there must be C_1 in order for $\tilde{L}^-$ to fail the condition (rC).



Let Q_1 and Q_2 be the index p' and the last block in (C_1, T_0). If there are any L^+-blocks between Q_1 and Q_2, then they are all left-compatible. Indeed, (lC) is satisfied (seen from the picture) and (lT) is always satisfied for L^+-strips.

Let $\hat{L}^-$ be the very first L^--strip in (C_0, T_0).



Since (C_0, T_0) is anti-fundamental, by Remark 4.2.21, the first block in $B_{0,0}$ is not N^+. Since $\hat{L}^-$ is the first L^--strip in (C_0, T_0), the first block in $B_{0,0}$ is not an L^--block either. So, it can be either N^-, U or L^+. By Remark 4.2.16, since N^+ cannot follow an L^+N^-U-block, there are no N^+-blocks above $\hat{L}^-$. By Remark 4.2.47, the adjacent blocks in (C_1, T_0), i.e. the blocks above Q, are all N^-. In other words, there are no L^+-blocks above Q. This concludes the proof.

□

We are now ready to put everything together.

Proof of Theorem 4.2.1. If there are no L-blocks anywhere in (C, T), the statement is trivially true. Let's assume there is at least one L-block in (C, T), and without loss of generality, we may assume it is L^-. Otherwise we consider (T, C) instead. Moreover, we can assume the L^- is left-compatible. Indeed, if there is a non-left-compatible L^- in (C_i, T_j), there is a non-left-compatible L^+ in (T_j, C_i). By Corollary 4.2.42, there is a non-right-compatible L^- in (T_{j-1}, C_i). If this L^- is again non-left-compatible, we go through the same chain of arguments. We continue inductively and eventually, since the process must end when C or T runs out of columns, we are guaranteed to find a left-compatible L^-.

Consider a left-compatible L^--strip in (C_i, T_0). If there are no such L^--strips in (C, T_0), which happens if there are no L^--blocks in (C, T_0), we remove T_0 from T and consider (C, T_1). The list of possibilities for L^- are listed in Table 4.1.

1. Suppose the L^--block in question contributes to γ. If it is non-right-compatible, there must be C_{i+1}. This is because all L^--strips satisfy the condition (rT). In order for the L^- to be non-right-compatible, there must be C_{i+1} that pose violations. By Lemma 4.2.41, there exists a unique non-left-compatible L^+ in (C_{i+1}, T_j). If (C_i, T_0) is regular and there exists C_{i+1}, then Corollary 4.2.42 is used. If there is no C_{i+1}, by Lemma 4.2.45 there must be T_{-1} and by Lemma 4.2.44, the L^- is not left-compatible, which is a contradiction.

2. Suppose the L^--block does not contribute to γ. Then (C_i, T_0) is (anti-)fundamental. If (C_i, T_0) is fundamental, then every L^- contributes to γ and we apply the previous analysis. If (C_i, T_0) is anti-fundamental and L^- is right-compatible, there is no need to pair it with anything since L^- does not contribute to γ and pose no violations with C_{i+1}. If (C_i, T_0) is anti-fundamental and L^- is not right-compatible, Lemma 4.2.46 shows there is no corresponding non-left-compatible L^+'s.

We now remove T_0 from T and consider $\gamma(C, T_1)$. Since T_0 is removed, all non-left-compatible L^--strips become left-compatible. However, restrictions on L^+-strips are not changed since left-compatibility comes from C itself. By the exact same argument, all negative non-exchangeable contributions in $\gamma(C, T_1)$ are canceled by positive non-exchangeable contributions. We continue this argument for all T_i. This proves all the negative non-exchangeable terms in $\gamma(C, T)$ are uniquely canceled out by non-right-compatible terms in $\gamma(C, T)$.

Next, we consider (T, C), where all left-compatible L^+-strips in (C, T) become left-compatible L^--strips in (T, C) and apply the same argument. □

4.3 Proofs of Conditions II and III

The previous section showed that the condition I holds. That is, the (q, t)-characters in the fundamental cluster t-commute with each other. Then, by Corollary 4.1.11, the commutation matrix Λ of the fundamental cluster $\mathcal{C}$ is given by

$$\Lambda_{i,k}^{i',k'} = 2\epsilon(\mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2}, \mathbf{Y}^{(i')}_{k',-k'+(i'+k'+1)_2}) \ ,$$

where $\mathbf{Y}^{(i)}_{k,j(i,k)}$ and $\mathbf{Y}^{(i')}_{k',j(i',k')}$ are dominant monomials of KR-modules in the fundamental cluster $\mathcal{C}$.

We now show condition II holds.

Theorem 4.3.1. Let B be the infinite matrix associated to the quiver Γ_B in Figure 3.2 (on page 31). Then,

$$\Lambda B = D \,,$$

where D is a diagonal matrix with positive entries. In other words, (Λ, B) is a compatible pair (as in [3], Section 3).

We will provide some definitions and lemmas first.

Definition 4.3.2. We fix the following notations for convenience:

1. Given $k \in \mathbb{Z}$, the s-number is defined as:

$$\begin{aligned} [0]_s &:= 0\,, \\ [k]_s &:= \frac{s^k - s^{-k}}{s - s^{-1}} = s^{k-1} + s^{k-3} + \cdots + s^{-k+3} + s^{-k+1}\,. \end{aligned}$$

2. Denote $\mathcal{F} := \mathbb{Z}[[s]][s^{-1}]$. Then, multiplication operator is well-defined in $\mathcal{F}$.

3. Given $f(s) \in \mathcal{F}$, we define $[f]_0$ to be the constant term in $f(s)$, e.g. $[s^{-1} + 3 + s]_0 = 3$.

Definition 4.3.3. Let m be a monomial in $\mathbb{Z}[Y_{i,j}^{\pm 1}]$. We define the following generating series:

$$u_i(m)(s) := \sum_{j \in \mathbb{Z}} u_{i,j}(m) s^j \quad ; \quad u(m)(s) := \sum_{i=1}^{r} e_i \otimes u_i(m)(s)\ ,$$

where $u_{i,j}(m)$ is the exponent of $Y_{i,j}$ in m as defined in Definition 4.1.2 and $e_i \in \mathbb{Z}^r$ is the vector with 1 in the ith position and 0's everywhere else.

Lemma 4.3.4. Let $\mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2}$ be the dominant monomial of $\chi^{(i)}_{k,-k+(i+k+1)_2} \in \mathcal{C}$. Then,

$$u(\mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2})(s) \quad = \quad e_i \otimes s^{-1+(i+k+1)_2}\ [k]_s\,.$$

Proof. Denote $j := (i+k+1)_2$. Modules in $\mathcal{C}$ have dominant monomials of the form:

$$\mathbf{Y}^{(i)}_{k,j} = Y_{i,-k+j} Y_{i,-k+j+2} \cdots Y_{i,-k+j+2(k-1)}\,. \tag{4.3.1}$$

Then, using Definition 4.3.3, we directly compute as follows:

$$\begin{aligned} u_i(\mathbf{Y}^{(i)}_{k,j})(s) &= \sum_{p\in\mathbb{Z}} u_{i,p}(\mathbf{Y}^{(i)}_{k,j})s^p \\ &= s^{-k+j} + s^{-k+j+2} + \cdots + s^{j+k-2} \\ &= s^{j-1}\left(s^{-k+1} + s^{-k+3} + \cdots + s^{k-1}\right) \\ &= s^{j-1}[k]_s\,, \end{aligned}$$

where $u_{i,p}\left(\mathbf{Y}^{(i)}_{k,j}\right)$ is the power of $Y_{i,p}$ in $\mathbf{Y}^{(i)}_{k,j}$, which is either 1 or 0 as seen from (4.3.1). Notice that $u_{i',p}(\mathbf{Y}^{(i)}_{k,j}) = 0$ if $i' \neq i$. Therefore, $u_{i'}(\mathbf{Y}^{(i)}_{k,j})(s) = 0$ and we have:

$$u(\mathbf{Y}^{(i)}_{k,j}) = e_i \otimes s^{j-1}[k]_s\,.$$

□

Definition 4.3.5. Let m be a monomial in $\mathbb{Z}[Y^{\pm 1}_{i,j}]$. We define the following generating series:

$$\tilde{u}_i(m)(s) = \sum_{j\in\mathbb{Z}} \tilde{u}_{i,j}(m)s^j \quad \text{and} \quad \tilde{u}(m)(s) = \sum_{i=1}^{r} e_i \otimes \tilde{u}_i(m)(s)\,,$$

where $\tilde{u}_{i,j}(m)$ are the solutions of the system given in Definition 4.1.3.

Definition 4.3.6. Let $M \in \mathrm{Mat}_{r\times r}(\mathbb{Z})$ and $g \in \mathcal{F}$. We define an action of $M \otimes g$ on the space $\mathbb{Z}^r \times \mathcal{F}$ as follows:

$$(M \otimes g)(v \otimes f) = (Mv) \otimes (gf) \quad \text{for any} \quad v \in \mathbb{Z}^r, f \in \mathcal{F},$$

where Mv is the matrix multiplication and gf is the usual multiplication.

Recall the system of equations in Definition 4.1.3:

$$u_{i,j}(m) = \tilde{u}_{i,j-1}(m) + \tilde{u}_{i,j+1}(m) - \tilde{u}_{i-1,j}(m) - \tilde{u}_{i+1,j}(m)\,,$$

defined for any monomial m and any $i \in I, j \in \mathbb{Z}$. Let $A = C - 2I$, where C is the Cartan matrix of $\mathfrak{sl}_{r+1}$. We rewrite this system for $m = \mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2}$ as follows:

$$u^{i,k}(s) = (1 \otimes s + 1 \otimes s^{-1} + A \otimes 1)\tilde{u}^{i,k}(s)\,, \tag{4.3.2}$$

where the action is as in Definition 4.3.6.

Definition 4.3.7. Denote the operator

$$K = (1 \otimes s^{-1})(1 \otimes 1 + A \otimes s + 1 \otimes s^2)\,. \tag{4.3.3}$$

Define operator D as a formal power series in s, expanded at 0, given by:

$$D = (1 \otimes 1 + A \otimes s + 1 \otimes s^2)^{-1}(1 \otimes s)\,. \tag{4.3.4}$$

Then we have $DK = KD = 1 \otimes 1$ and $Du^{i,k}(s) = \tilde{u}^{i,k}(s)$.

Remark 4.3.8. In Definition 4.1.3, we require $\tilde{u}_{i,j}(m) = 0$ for j sufficiently small. This condition is equivalent to expanding the formal inverse of the power series (4.3.3) at 0, which is the choice we made in (4.3.4).

Remark 4.3.9. It is easy to see that K commutes with $1 \otimes s^n$ for any n. Since D is the inverse of K, we have:

$$D(1 \otimes s^n) = D(1 \otimes s^n)KD = DK(1 \otimes s^n)D = (1 \otimes s^n)D\,.$$

That is, the operator D commutes with $1 \otimes s^n$ for $n \in \mathbb{Z}$.

Definition 4.3.10. Let $v, w \in \mathbb{Z}^r$ and $f, g \in \mathcal{F}$. We define the following inner product on $\mathbb{R}^r \otimes \mathcal{F}$ as follows:

$$(v \otimes f) \cdot (w \otimes g) = \langle v, w \rangle \left[f\left(s^{-1}\right) g\left(s\right) \right]_0\,, \tag{4.3.5}$$

where $\langle v, w \rangle$ is the usual inner product on $\mathbb{R}^r$ and $[f(s)]_0$ is the constant term in $f(s)$ as defined in Definition

4.3.2.

Remark 4.3.11. Notice that the inner product in Definition 4.3.10 is symmetric. That is,

$$(v \otimes f) \cdot (w \otimes g) = (w \otimes g) \cdot (v \otimes f)\,.$$

Definition 4.3.12. Given $M \otimes h \in \mathrm{Mat}_{r\times r}(\mathbb{Z}) \times \mathcal{F}$, we define the *transpose* of $M \otimes h$, denoted $(M \otimes h)^t$, by the following condition:

$$(v \otimes f) \cdot M \otimes h\,(w \otimes g) = (M \otimes h)^t\,(v \otimes f) \cdot (w \otimes g)\ ,$$

for any $v, w \in \mathbb{Z}^r$ and $f, g \in \mathcal{F}$. An operator $M \otimes h$ is *symmetric* if $M \otimes h = (M \otimes h)^t$.

Lemma 4.3.13.

$$(1 \otimes s)^t = 1 \otimes s^{-1}; \quad K^t = K; \quad D^t = D\,.$$

Proof. It is easy to see that K is symmetric. Since $D^{-1} = K$, D is also symmetric. The remaining result is shown by direct computation:

$$\begin{aligned}(v \otimes f(s)) \cdot (1 \otimes s)\,(w \otimes g(s)) &= \langle v, w\rangle \left[f(s^{-1})(sg(s))\right]_0 \\ &= \langle v, w\rangle \left[(s^{-1})^{-1} f(s^{-1}) g(s)\right]_0 \\ &= (1 \otimes s^{-1})(v \otimes f(s)) \cdot (w \otimes g(s))\,,\end{aligned}$$

for any $v, w \in \mathbb{Z}^r$ and $f, g \in \mathcal{F}$. □

Lemma 4.3.14. Let p, p' be dominant monomials in $\mathcal{M}$. Then,

$$\epsilon\,(p, p') = (1 \otimes s - 1 \otimes s^{-1}) D u\,(p)\,(s) \cdot u\,(p')\,(s)\,,$$

where ϵ is given in Definition 4.1.5.

Proof. By definition, we have

$$u(p)(s) = \sum_i e_i \otimes \sum_j u_{i,j}(p)s^j \quad \text{and} \quad \tilde{u}(p')(s) = \sum_{i'} e_{i'} \otimes \sum_{j'=1}^{r} \tilde{u}_{i',j'}(p')s^{j'} .$$

Then,

$$\begin{aligned} u(p)(s) \cdot (1 \otimes s)\tilde{u}(p')(s) &= \sum_{i,i'=1}^{r} \langle e_i, e_{i'} \rangle \left[\sum_{j,j'} u_{i,j}(p)s^{-j} \tilde{u}_{i',j'}(p')s^{j'+1} \right]_0 \\ &= \sum_j u_{i,j}(p)\tilde{u}_{i,j-1}(p') . \end{aligned}$$

Therefore, using Deifnition 4.1.5, we have:

$$\begin{aligned} \epsilon(p,p') &= -\sum_{i,j} u_{i,j}(p)\tilde{u}_{i,j-1}(p') + \sum_j u_{i,j}(p')\tilde{u}_{i,j-1}(p) \\ &= -u(p)(s) \cdot (1 \otimes s)\tilde{u}(p')(s) + u(p')(s) \cdot (1 \otimes s)\tilde{u}(p)(s) \\ &= -u(p)(s) \cdot (1 \otimes s)Du(p')(s) + u(p')(s) \cdot (1 \otimes s)Du^(p)(s) \\ &= -D^t(1 \otimes s)^t u(p)(s) \cdot u(p')(s) + (1 \otimes s)Du(p)(s) \cdot u(p')(s) \\ &= (1 \otimes s - 1 \otimes s^{-1})Du(p)(s) \cdot u(p')(s) . \end{aligned}$$

□

Remark 4.3.15. As a sanity check, let us verify that the expression we found for ϵ is also anti-symmetric.

$$\begin{aligned} \epsilon(p',p) &= (1 \otimes s - 1 \otimes s^{-1})Du(p')(s) \cdot u(p)(s) \\ &= u(p)(s) \cdot (1 \otimes s - 1 \otimes s^{-1})Du(p')(s) \\ &= D^t(1 \otimes s^{-1} - 1 \otimes s)u(p)(s) \cdot u(p')(s) = -\epsilon(p,p') . \end{aligned}$$

Definition 4.3.16. When u and $\tilde{u}$ generating functions of Definitions 4.3.3 and 4.3.5 are applied to the dominant monomial of a module in the fundamental cluster $\mathcal{C}$, i.e. monomial of the form $\mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2}$, we make the following simplifying notation:

$$u(\mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2})(s) := u^{i,k}(s) \quad \text{and} \quad \tilde{u}(\mathbf{Y}^{(i)}_{k,-k+(i+k+1)_2})(s) := \tilde{u}^{i,k}(s)\,.$$

Remark 4.3.17. We emphasize that the superscript $u^{i,k}(s)$ in Definition 4.3.16 indicates the monomial $Y^{(i)}_{k,-k+(i+k+1)_2}$. In contrast, the subscript $u_{i,j}(m)$ indicates the exponent of $Y_{i,j}$ in m.

Lemma 4.3.18. The following equation holds:

$$u^{i,k-1}(s) + u^{i,k+1}(s) - u^{i-1,k}(s) - u^{i+1,k}(s) = 1 \otimes s^{-1+(i+k)_2} K e_i \otimes [k]_s\,,$$

for any $i \in I$ and $k \geq 1$.

Proof. We have:

$$\begin{aligned}
[k-1]_s + [k+1]_s &= \left(s^{k-2} + \cdots + s^{-k+2}\right) + \left(s^k + s^{k-2} + \cdots + s^{-k+2} + s^{-k}\right) \\
&= \left(s^{k-2} + \cdots + s^{-k+2} + s^{-k}\right) + \left(s^k + s^{k-2} + \cdots + s^{-k+2}\right) \\
&= s^{-1}\left(s^{k-1} + \cdots + s^{-k+1}\right) + s\left(s^{k-1} + \cdots + s^{-k+1}\right) \\
&= (s^{-1} + s)[k]_s\,.
\end{aligned}$$

Notice that the above equation holds when $k = 1$ as well:

$$[0]_s + [2]_s = [2]_s = s + s^{-1} = \left(s + s^{-1}\right)[1]_s\,.$$

By Lemma 4.3.4, we have $u^{i,k}(s) = e_i \otimes s^{-1+(i+k+1)_2}\ [k]_s$. Then,

$$\begin{aligned}
u^{i,k-1}(s) \quad + \quad & u^{i,k+1}(s) - u^{i-1,k}(s) - u^{i+1,k}(s) = \\
= \quad & e_i \otimes s^{-1+(i+k)_2}[k-1]_s + e_i \otimes s^{-1+(i+k)_2}[k+1]_s + (-e_{i-1} - e_{i+1}) \otimes s^{-1+(i+k)_2}[k]_s \\
= \quad & 1 \otimes s^{-1+(i+k)_2}\left(e_i \otimes \left(s + s^{-1}\right)[k]_s + (-e_{i-1} - e_{i+1}) \otimes [k]_s\right) \\
= \quad & 1 \otimes s^{-1+(i+k)_2} K e_i \otimes [k]_s\,.
\end{aligned}$$

Notice that when $i = 1$ or $i = r$, the above equation still holds since the matrix A incorporates the boundary values. $\square$

We are now ready to prove the theorem stated at the beginning of the section.

Proof of Theorem 4.3.1. We want to compute:

$$\begin{aligned} \frac{1}{2}(\Lambda B)_{i,k}^{i',k'} &= \frac{1}{2}\sum_{p,n} \Lambda_{i,k}^{p,n} B_{p,n}^{i',k'} \\ &= \sum_{p,n} (1 \otimes s - 1 \otimes s^{-1}) D u^{i,k}(s) \cdot u^{p,n}(s) B_{p,n}^{i',k'} . \end{aligned} \tag{4.3.6}$$

The goal is to show that the value of (4.3.6) is 1 if $(i, k) = (i', k')$ and 0 otherwise.

Given (i', k'), the only non-zero terms in B are given as follows:

$$B_{i'-1,k'}^{i',k'} = B_{i'+1,k'}^{i',k'} = (-1)^{i'+k'} \quad \text{and} \quad B_{i',k'-1}^{i',k'} = B_{i',k'+1}^{i',k'} = (-1)^{i'+k'+1} ,$$

and $B_{n,p}^{i',k'} = 0$ if $n \notin I$ or $p < 0$ (see Figure 3.2 on page 31). Then,

$$\begin{aligned} \frac{1}{2}(\Lambda B)_{i,k}^{i',k'} &= (1 \otimes s - 1 \otimes s^{-1}) D u^{i,k}(s) \cdot \\ &\qquad (-1)^{i'+k'+1} \left(u^{i',k'-1}(s) + u^{i',k'+1}(s) - u^{i'-1,k'}(s) - u^{i'+1,k'}(s) \right) \\ &= (-1)^{i'+k'+1} (1 \otimes s - 1 \otimes s^{-1}) D e_i \otimes s^{-1+(i+k+1)_2} [k]_s \cdot (1 \otimes s^{-1+(i'+k')_2}) K e_{i'} \otimes [k']_s \\ &= (-1)^{i'+k'+1} (1 \otimes s^{(i+k+1)_2 - (i'+k')_2}) (1 \otimes s - 1 \otimes s^{-1}) e_i \otimes [k]_s \cdot e_{i'} \otimes [k']_s \\ &= (-1)^{i'+k'+1} \langle e_i, e_{i'} \rangle \left[(s^{-1} - s)(s^{(i'+k')_2 - (i+k+1)_2} [k]_{s^{-1}} [k']_s \right]_0 \\ &= (-1)^{i+k'+1} \delta_{ii'} \left[s^{\delta} \frac{1}{s - s^{-1}} \left(s^{k'-k} - s^{-k-k'} - s^{k+k'} + s^{k'-k} \right) \right]_0 , \end{aligned} \tag{4.3.7}$$

where $\delta := (i + k')_2 - (i + k + 1)_2$. Notice that δ can only be $+1, 0$ or -1.

It is clear that if $i \neq i'$, the value of (4.3.7) is zero. Suppose $k \neq k'$, and without loss of generality, let's assume $k' > k$. We want to show that the constant term part of (4.3.7) vanishes in this case. Let $b := k' - k > 0$ and $a := -k' - k < 0$. Notice that a and b have the same parity.

If $\delta = 1$, the constant term expression of (4.3.8) is as follows:

$$\begin{aligned} s \quad \times \quad & \tfrac{1}{s(1-s^{-2})}\left((s^b + s^{-b}) - (s^a + s^{-a})\right) = \\ = \quad & (s^b + s^{-b})(1 + s^{-2} + s^{-4} + \cdots) - (s^a + s^{-a})(1 + s^{-2} + s^{-4} + \cdots) \end{aligned} \tag{4.3.8}$$

Notice that $-b - 2n, a - 2m < 0$ for any $n, m \geq 0$, and therefore, $s^{-b}s^{-2n}$ and $s^a s^{-2m}$ are not constants for any $n, m \geq 0$. If a and b are odd, it is clear that there is no constant term in (4.3.8). If a and b are even, there exists some $n > 0$ such that $s^{b-2n} = 1$. Since $a < 0$, there must exist some $m > 0$ such that $s^{-a-2m} = 1$. Therefore, the constant term in (4.3.8) vanishes.

If $\delta = -1$, we write

$$s^{-1} \times \frac{1}{s^{-1}(s^2 - 1)}\left((s^b + s^{-b}) - (s^a + s^{-a})\right) ,$$

and use the same argument.

If $\delta = 0$, the constant term part of (4.3.7) can be written as:

$$\frac{1}{(1 - s^{-2})}\left((s^{b-1} + s^{-b-1}) - (s^{a-1} + s^{-a-1})\right) ,$$

and we use the same argument. Therefore, the value of (4.3.7) is always zero if $k \neq k'$.

Suppose $k = k'$. If $(i + k)_2 = 0$, we have $\delta = -1$ and (4.3.7) can be written as:

$$\begin{aligned} \frac{1}{2}(\Lambda B)^{i,k}_{i,k} &= -\left[s^{-1}\frac{1}{-s^{-1}(1 - s^2)}\left(2 - s^{2k} - s^{-2k}\right)\right]_0 \\ &= \left[\left(2 - s^{2k} - s^{-2k}\right)(1 + s^2 + s^4 + \cdots + s^{2k} + \cdots)\right]_0 = 1\,. \end{aligned}$$

If $(i + k)_2 = 1$, we have $\delta = 1$ and (4.3.7) can be written as:

$$\begin{aligned} \frac{1}{2}(\Lambda B)^{i,k}_{i,k} &= \left[s\frac{1}{s(1 - s^{-2})}\left(2 - s^{2k} - s^{-2k}\right)\right]_0 \\ &= \left[\left(2 - s^{2k} - s^{-2k}\right)(1 + s^{-2} + s^{-4} + \cdots + s^{-2k} + \cdots)\right]_0 = 1\,. \end{aligned}$$

This concludes the proof. □

We are now ready to prove the final condition.

Theorem 4.3.19. The quantum mutation is given by:

$$T^{(i)}_{k,l-1} * T^{(i)}_{k,l+1} = t^{\frac{1}{2}\Lambda^{i,k-1,l}_{i,k,l-1}+\frac{1}{2}\Lambda^{i,k+1,l}_{i,k,l-1}-\frac{1}{2}\Lambda^{i,k+1,l}_{i,k-1,l}} T^{(i)}_{k-1,l} * T^{(i)}_{k+1,l} + t^{\frac{1}{2}\Lambda^{i-1,k,l}_{i,k,l-1}+\frac{1}{2}\Lambda^{i+1,k,l}_{i,k,l-1}-\frac{1}{2}\Lambda^{i+1,k,l}_{i-1,k,l}} T^{(i-1)}_{k,l} * T^{(i+1)}_{k,l}.$$

Theorem 4.3.19 is stated in terms of $T^{(i)}_{k,l}$ variables, while Nakajima's t-deformed T-system is written in terms of $\chi^{(i)}_{k,j}$ variables, which is achieved by a change of variables as described in Remark 4.1.9. Also recall that Λ is expressed in terms of the ϵ function (see Corollary 4.1.11). We now restate Theorem 4.3.19 in terms of $\chi^{(i)}_{k,j}$ variables and ϵ expressions for Λ.

Theorem 4.3.20. The following equation holds:

$$\begin{array}{c} \chi^{(i)}_{k,j} * \chi^{(i)}_{k,j+2} = t^{\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k-1,j+2})+\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k+1,j})-\epsilon(\mathbf{Y}^{(i)}_{k+1,j+2},\mathbf{Y}^{(i)}_{k-1,j})} \chi^{(i)}_{k+1,j} * \chi^{(i)}_{k-1,j+2} \\ +t^{\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i-1)}_{k,j+1})+\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i+1)}_{k,j+1})-\epsilon(\mathbf{Y}^{(i-1)}_{k,j+1},\mathbf{Y}^{(i+1)}_{k,j+1})} \chi^{(i-1)}_{k,j+1} * \chi^{(i+1)}_{k,j+1} \end{array}. \tag{4.3.9}$$

Proof. Recall Nakajima's t-deformed T-system (see Theorem 4.1.8):

$$t^{-\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k,j+2})} \chi^{(i)}_{k,j} * \chi^{(i)}_{k,j+2} = t^{-\epsilon(\mathbf{Y}^{(i)}_{k+1,j+2},\mathbf{Y}^{(i)}_{k-1,j})} \chi^{(i)}_{k+1,j} * \chi^{(i)}_{k-1,j+2} + t^{-1-\epsilon(\mathbf{Y}^{(i-1)}_{k,j+1},\mathbf{Y}^{(i+1)}_{k,j+1})} \chi^{(i-1)}_{k,j+1} * \chi^{(i+1)}_{k,j+1}.$$

We will show that the t-deformed T-system is equivalent to (4.3.9). It suffices to show:

$$\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k-1,j+2}) + \epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k+1,j}) = \epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k,j+2}), \tag{4.3.10}$$

$$\epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i-1)}_{k,j+1}) + \epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i+1)}_{k,j+1}) = -1 + \epsilon(\mathbf{Y}^{(i)}_{k,j},\mathbf{Y}^{(i)}_{k,j+2}). \tag{4.3.11}$$

By an abuse of notation, let us denote $\mathbf{Y}^{(i)}_{k,j} := u(\mathbf{Y}^{(i)}_{k,j})(s) = e_i \otimes (s^j + s^{j+2} + \cdots + s^{j+2(k-1)})$ (see (2.4.4) and Definition 4.3.3).

$$\begin{aligned} \mathbf{Y}^{(i)}_{k-1,j+2} + \mathbf{Y}^{(i)}_{k+1,j} &= e_i \otimes \Big((s^{j+2} + \cdots + s^{j+2+2(k-2)}) + (s^j + s^{j+2} + \cdots + s^{j+2k})\Big) \\ &= e_i \otimes \big((s^{j+2} + \cdots + s^{j+2k-2} + s^{j+2k}) + (s^j + s^{j+2} + \cdots + s^{j+2k-2})\big) \\ &= \mathbf{Y}^{(i)}_{k,j+2} + \mathbf{Y}^{(i)}_{k,j}\,. \end{aligned}$$

Then, using Definition 4.3.14, we compute:

$$\begin{aligned}
\epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k-1,j+2}) + \epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k+1,j}) &= (1 \otimes s - 1 \otimes s^{-1}) D\mathbf{Y}^{(i)}_{k,j} \cdot \left(\mathbf{Y}^{(i)}_{k-1,j+2} + \mathbf{Y}^{(i)}_{k+1,j}\right) \\
&= (1 \otimes s - 1 \otimes s^{-1}) D\mathbf{Y}^{(i)}_{k,j} \cdot \left(\mathbf{Y}^{(i)}_{k,j+2} + \mathbf{Y}^{(i)}_{k,j}\right) \\
&= \epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k,j+2}) + \underbrace{\epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k,j})}_{0},
\end{aligned}$$

where $\epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k,j}) = 0$ due to anti-commutativity of ϵ. This proves (4.3.10).

Notice that

$$(1 \otimes s + 1 \otimes s^{-1})\mathbf{Y}^{(i)}_{k,j+1} = \mathbf{Y}^{(i)}_{k,j+2} + \mathbf{Y}^{(i)}_{k,j}\,.$$

Then,

$$\begin{aligned}
\mathbf{Y}^{(i-1)}_{k,j+1} + \mathbf{Y}^{(i+1)}_{k,j+1} &= \mathbf{Y}^{(i-1)}_{k,j+1} + \mathbf{Y}^{(i+1)}_{k,j+1} - (1 \otimes s + 1 \otimes s^{-1})\mathbf{Y}^{(i)}_{k,j+1} + (\mathbf{Y}^{(i)}_{k,j+2} + \mathbf{Y}^{(i)}_{k,j}) \\
&= -(1 \otimes s + 1 \otimes s^{-1} + A \otimes 1)\mathbf{Y}^{(i)}_{k,j+1} + \mathbf{Y}^{(i)}_{k,j+2} + \mathbf{Y}^{(i)}_{k,j} \\
&= -K\mathbf{Y}^{(\alpha)}_{k,j+1} + \mathbf{Y}^{(\alpha)}_{k,j} + \mathbf{Y}^{(\alpha)}_{k,j+2}\,.
\end{aligned}$$

Next,

$$\begin{aligned}
\epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i-1)}_{k,j+1}) + \epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i+1)}_{k,j+1}) &= (1 \otimes s - 1 \otimes s^{-1}) D(\mathbf{Y}^{(i)}_{k,j}) \cdot \left(\mathbf{Y}^{(i-1)}_{k,j+1} + \mathbf{Y}^{(i+1)}_{k,j+1}\right) \\
&= (1 \otimes s - 1 \otimes s^{-1}) D(\mathbf{Y}^{(i)}_{k,j}) \cdot \left(-K(\mathbf{Y}^{(i)}_{k,j+1}) + \mathbf{Y}^{(i)}_{k,j} + \mathbf{Y}^{(i)}_{k,j+2}\right) \\
&= -(1 \otimes s - 1 \otimes s^{-1})\mathbf{Y}^{(i)}_{k,j} \cdot \mathbf{Y}^{(i)}_{k,j+1} + \underbrace{\epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k,j})}_{0} + \epsilon(\mathbf{Y}^{(i)}_{k,j}, \mathbf{Y}^{(i)}_{k,j+2}),
\end{aligned}$$

where

$$\begin{aligned}(1\otimes s-1\otimes s^{-1})\mathbf{Y}^{(i)}_{k,j}\cdot\mathbf{Y}^{(i)}_{k,j+1} &= \left[(s^{-1}-s)(s^{-j}+\cdots+s^{-j-2k+2})(s^{j+1}+\cdots+s^{j+2k-1})\right]_0 \\ &= \left[(s^{-1}-s)s^{-j-k+1}[k]_s s^{j+k}[k]_s\right]_0 \\ &= \left[-s\frac{1}{s-s^{-1}}(s^{2k}-2+s^{-2k})\right]_0 = 1\end{aligned}$$

This concludes the proof of (4.3.11). □

4.3.1 Explicit proof of Condition II for type A_1

The commutation matrix Λ can be computed explicitly and Theorem 4.3.1 can be verified through direct computation. We will work out the explicit description of Λ and the direction verification of Theorem 4.3.1 for the case of type A_1 now.

Definition 4.3.21. Given a matrix $M=(m_{i,j})_{i\in I,j\in J}$ for some index sets I and J (possibly infinite), we can write M as a generating series $M(z_1,z_2)=\sum_{i\in I,j\in J} m_{i,j}z_1^i z_2^j$, where z_1,z_2 are indeterminates keeping track of the indices of the matrix.

The quiver associated to the T-system of type A_1 is as follows:

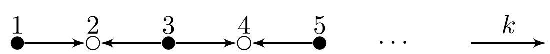


Figure 4.8: The quiver Γ_T, ($k\in\mathbb{Z}_+$)

The signed adjacency matrix of the quiver Γ_T in Figure 4.8 is as follows:

$$B = \left(\begin{array}{c|cccccccccc}
k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ \hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
4 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots \\
5 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\
6 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & \cdots \\
7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & \cdots \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

which can be written as:

$$B(z_1, z_2) = z_1 z_2 \frac{z_2 - z_1}{1 + z_1 z_2}.$$

Recall that to each vertex k, we associate the KR-module $W_{k,l=(k)_2-k}$ (see Equation (3.5.2)), and the commutation matrix Λ is given by:

$$\Lambda(k, k') = 2\epsilon(\mathbf{Y}_{k,(k)_2-k}, \mathbf{Y}_{k',(k')_2-k'}),$$

where $\mathbf{Y}_{k,j} = Y_j Y_{j+2} \cdots Y_{j+2k-2}$ (see Equation (2.4.4)), where we dropped the index i in $Y_{i,j}$ since i can take exactly one value, and ϵ is from Definition 4.1.5. More precisely, we have:

$$\epsilon(\mathbf{Y}_{k,(k)_2-k}, \mathbf{Y}_{k',(k')_2-k'}) = -\sum_{j=0}^{k-1} \tilde{u}_{(k)_2-k+2j-1}(\mathbf{Y}_{k',(k')_2-k'}) + \sum_{j=0}^{k'-1} \tilde{u}_{(k')_2-k'+2j-1}(\mathbf{Y}_{k,(k)_2-k}),$$

where $\tilde{u}_j(m) \in \mathbb{R}$ $(j \in \mathbb{Z})$ is the unique solution of the system:

$$u_j(m) = \tilde{u}_{j-1}(m) + \tilde{u}_{j+1}(m) \ ,$$

such that $\tilde{u}_j(m) = 0$ for j sufficiently small (see Definition 4.1.3). Again, we dropped the dependance on i and set $u_{i,j}(m) = u_j(m)$.

Example 4.3.22. Consider $m = \mathbf{Y}_{1,0} = Y_0$. Then $\tilde{u}_1(m) = 1$ and $\tilde{u}_j(m) = 0$ for all $j \geq 1$.

$$\begin{array}{rcl} & \vdots & \\ u_4(m) = 0 & = & \tilde{u}_5(m) + \tilde{u}_7(m) = 0 + 0 \\ u_2(m) = 0 & = & \tilde{u}_1(m) + \tilde{u}_3(m) = 1 + 0 \\ u_0(m) = 1 & = & \tilde{u}_{-1}(m) + \tilde{u}_1(m) = 0 + 1 \\ u_{-2}(m) = 0 & = & \tilde{u}_{-3}(m) + \tilde{u}_{-1}(m) = 0 + 0 \\ & \vdots & \end{array}$$

Here, we must have $\tilde{u}_{-1}(m) = 0$. Otherwise, $\tilde{u}_{-2j-1}(u) = 0$ for all $j \geq 0$, which contradicts the condition $\tilde{u}_j(m) = 0$ for j sufficiently small.

Example 4.3.23. When $m = \mathbf{Y}_{2,-2} = Y_{-2}Y_0$, we can compute $\tilde{u}_{-1}(m) = 1$ and $\tilde{u}_j(m) = 0$ for all $j \neq -1$. Then,

$$\epsilon(\mathbf{Y}_{1,0}, \mathbf{Y}_{2,-2}) \quad = \quad -\tilde{u}_{-1}(Y_{-2}Y_0) + (\tilde{u}_{-3}(Y_0) + \tilde{u}_{-1}(Y_0)) = -1 + 0 = -1.$$

Example 4.3.24. When $m = \mathbf{Y}_{4,-4} = Y_{-4}Y_{-2}Y_0Y_2$, we can compute $\tilde{u}_{-3}(m) = \tilde{u}_1(m) = 1$ and $\tilde{u}_j(m) = 0$ for all $j \neq -3, 1$. Then,

$$\epsilon(\mathbf{Y}_{1,0}, \mathbf{Y}_{4,-4}) \quad = \quad -\tilde{u}_{-1}(Y_{-4}Y_{-2}Y_0Y_2) + (\tilde{u}_{-5}(Y_0) + \tilde{u}_{-3}(Y_0) + \tilde{u}_{-1}(Y_0) + \tilde{u}_1(Y_0)) = -0 + 1 = 1.$$

Similarly, the commutation matrix Λ can be computed and is given by:

$$\Lambda = \left(\begin{array}{c|cccccccccc}
k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\
\hline
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & \cdots \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
3 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & \cdots \\
4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
5 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & \cdots \\
6 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\
8 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & \cdots \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

which can be written as:

$$\Lambda(z_1, z_2) \quad = \quad \frac{z_1 z_2 (z_1 - z_2)}{(1 + z_1 z_2)(1 + z_1^2)(1 + z_2^2)}.$$

Then,

$$\begin{aligned}
\frac{1}{2}\Lambda B(z_1, z_2) &= \mathrm{Res}_w \left(\frac{1}{w}\Lambda(z_1, w) B(\frac{1}{w}, z_2) \right) \\
&= \mathrm{Res}_w \left(\frac{1}{w} \frac{z_1 w (z_1 - w)}{(1 + w z_1)(1 + z_1^2)(1 + w^2)} \cdot \frac{1}{w} z_2 \frac{z_2 - \frac{1}{w}}{1 + \frac{z_2}{w}} \right) \\
&= \mathrm{Res}_w \left(\frac{1}{w} \frac{z_1 z_2 (z_1 - w)(w z_2 - 1)}{(1 + w z_1)(1 + z_1^2)(1 + w^2)(w + z_2)} \right) \\
&= -\frac{z_1^2 z_2}{(1 + z_1^2) z_2} + \frac{z_1 z_2 (z_1 + z_2)(-z_2^2 - 1)}{(-z_2)(1 - z_1 z_2)(1 + z_1^2)(1 + z_2^2)} \\
&= -\frac{z_1^2}{1 + z_1^2} + \frac{z_1 (z_1 + z_2)}{(1 - z_1 z_2)(1 + z_1^2)} \\
&= \frac{-z_1^2 + z_1^3 z_2 + z_1^2 + z_1 z_2}{(1 - z_1 z_2)(1 + z_1^2)} = \frac{z_1 z_2}{1 - z_1 z_2} = I(z_1, z_2)
\end{aligned}$$

4.4 Evolution in k-direction

We showed that Nakajima's deformed T-system forms a quantum cluster algebra with evolution in l-direction in $T_{k,l}^{(i)}$ variables (equivalently in j-direction in $\chi_{k,j}^{(i)}$ variables). We now show that the same deformed T-system is not a quantum cluster algebra with evolution in k-direction. In particular, this shows that the quantum T-system is not compatible with the quantum Q-system considered in [9].

By re-writing Nakajima's t-deformed T-system of Theorem 4.1.8 so that the evolution is in k-direction, we obtain:

$$\chi_{k+1,j}^{(i)} *_\gamma \chi_{k-1,j+2}^{(i)} = \chi_{k,j}^{(i)} *_\gamma \chi_{k,j+2}^{(i)} - t^{-1} \chi_{k,j+1}^{(i-1)} *_\gamma \chi_{k,j+1}^{(i+1)} .$$

Notice that the negative sign on the right-hand side is not compatible with cluster algebra interpretation, where the right-hand side expression must have exactly 2 positive contributions. However, it is possible to renormalize $\chi_{k,j}^{(i)}$'s such that the negative sign becomes positive (see [8]). We call the resulting variables $\widehat{\chi}_{k,j}^{(i)}$. The ensuing T-system of type A_1 is as follows:

$$\widehat{\chi}_{k+1,j} *_\gamma \widehat{\chi}_{k-1,j+2} = \widehat{\chi}_{k,j} *_\gamma \widehat{\chi}_{k,j+2} + t^{-1}, \tag{4.4.1}$$

where we dropped the parameter i since it can have only one value.

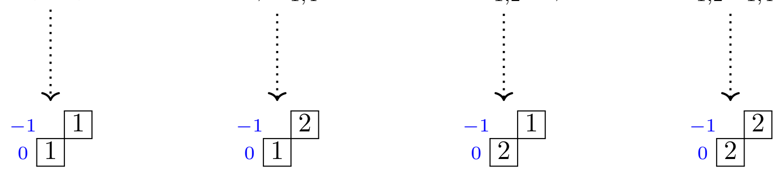
The variables on the right-hand side of (4.4.1) must belong to the same cluster, and therefore, must t-commute if (4.4.1) does form a quantum mutation. We will give a simple counter example to this condition, which shows that (4.4.1) is not a quantum cluster algebra.

Example 4.4.1. Let $\mathfrak{g} = \mathfrak{sl}_2$. We consider $\widehat{\chi}_{1,0}$ and $\widehat{\chi}_{1,2}$, which are both on the right-hand side of (4.4.1). The values of these variables are given as follows:

$$\widehat{\chi}_{1,0} = \chi_{q,t}(W_{1,0}^{(1)}) = \underset{{}_0\boxed{1}}{Y_{1,0}} + \underset{{}_0\boxed{2}}{Y_{1,2}^{-1}} \quad ; \quad \widehat{\chi}_{1,2} = \chi_{q,t}(W_{1,2}^{(1)}) = \underset{{}_{-1}\boxed{1}}{Y_{1,2}} + \underset{{}_{-1}\boxed{2}}{Y_{1,4}^{-1}}$$

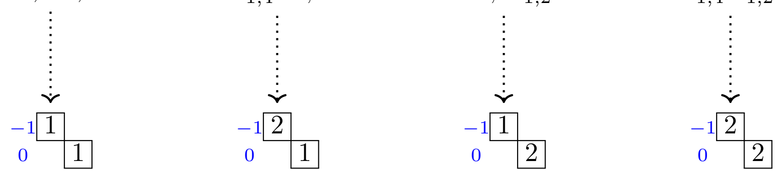
Their twisted product can be computed using Theorem 4.2.23, and is given by:

$$\widehat{\chi}_{1,0} *_\gamma \widehat{\chi}_{1,2} \quad = \quad Y_{1,0}Y_{1,2} \quad + \quad Y_{1,0}Y_{1,4}^{-1} \quad + \quad t^{-1}\, Y_{1,2}^{-1}Y_{1,2} \quad + \quad Y_{1,2}^{-1}Y_{1,4}^{-1}$$



Notice that the third term on the right-hand side forms a pair of column tableaux of regular type with non-zero boundary term in γ. On the other hand,

$$\widehat{\chi}_{1,2} *_\gamma \widehat{\chi}_{1,0} \quad = \quad Y_{1,2}Y_{1,0} \quad + \quad Y_{1,4}^{-1}Y_{1,0} \quad + \quad t\, Y_{1,2}Y_{1,2}^{-1} \quad + \quad Y_{1,4}^{-1}Y_{1,2}^{-1}$$



We see that $\widehat{\chi}_{1,0}$ and $\widehat{\chi}_{1,2}$ do not t-commute.

Chapter 5

Conclusion

The Nakajima (q, t)-characters of KR-modules satisfy a deformed T-system, introduced in [39], which is a t-deformed discrete dynamical system with 3 independent parameters: i, k, j. In this thesis, we showed that this t-deformed T-system forms a quantum mutation in a quantization of the T-system cluster algebra when the direction of mutation is in the l-parameter. We also showed that, when the direction of mutation is taken to be in the k-direction, the fundamental cluster variables do not t-commute with respect to the twisted multiplication of the deformed T-system. In particular, this means that the Nakajima quantum T-system does not restrict to the quantum Q-system considered in [8–10].

This result pertains to type A only. It is an open question whether the same holds for other types, and in particular to type D. The proofs of Conditions II and III are applicable to type D with minimal adjustments. The hard part is the proof of Condition I, the t-commutativity of the fundamental cluster variables. The proof of Condition I is entirely combinatorial and requires the knowledge of the (q, t)-characters of all the KR-modules in the fundamental cluster.

There are noticeable differences between types A and D. For example, unlike the case in type A, the KR-modules of type D are reducible as $U_q(\mathfrak{g})$-modules. Also, the (q, t)-characters of the KR-modules are not identical to their q-characters. Nakajima's tableaux-sum notation exists for type D as well (see [38]). However, it is combinatorially different from the type A case, and, therefore, all the combinatorial structures introduced as part of the proof of Condition I will need to be translated to the combinatorics of the type D.

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