SYMPLECTIC TORIC STRATIFIED SPACES WITH ISOLATED SINGULARITIES

BY

SETH WOLBERT

DISSERTATION

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Doctoral Committee:

Professor Susan Tolman, Chair
Professor Eugene Lerman, Director of Research
Professor Rui Loja Fernandes
Professor Ely Kerman

Abstract

Let G be a torus with Lie algebra $\mathfrak{g}$. We provide a classification of two types of toric objects: symplectic toric cones and symplectic toric stratified spaces with isolated singularities. Both types of object are classified via orbital moment map and a second degree cohomology class.

As symplectic toric stratified spaces with isolated singularities are locally modeled on symplectic toric cones, we first focus on classifying symplectic toric cones. We show that symplectic toric cones have a certain type of map $\psi : W \to \mathfrak{g}^*$ (called homogeneous unimodular local embeddings) as orbital moment maps. Conversely, every ψ has a symplectic toric cone for which it is an orbital moment map. We classify the symplectic toric cones with orbital moment map ψ by showing that their isomorphism classes are in bijective correspondence with the first Chern classes $H^2(W; \mathbb{Z}_G)$ of principal G-bundles over W, for $\mathbb{Z}_G$ the integral lattice $\ker(\exp : \mathfrak{g} \to G)$. This generalizes Lerman's classification of compact connected contact toric manifolds.

Symplectic toric stratified spaces with isolated singularities are spaces with neighborhoods of singularities modeled on symplectic cones. We first show their quotients W are space stratified by manifolds with corners and their moment maps are a particular type of map $\psi : W \to \mathfrak{g}^*$ called stratified unimodular local embeddings. Every stratified unimodular local embedding ψ is the orbital moment map of a symplectic toric stratified space. Finally, we show that, for any stratified unimodular local embedding ψ and for W_{reg} the top stratum of W, the isomorphism classes of symplectic toric stratified spaces with isolated singularities with orbital moment map ψ are in bijective correspondence with the cohomology classes $H^2(W_{\mathsf{reg}}; \mathbb{Z}_G) \times \mathcal{C}$, for $\mathcal{C} \subset H^2(W_{\mathsf{reg}}; \mathbb{R})$ a subspace dependent on the topology of W. This generalizes Burns, Guillemin, and Lerman's classification of the compact connected symplectic toric stratified spaces with isolated singularities.

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Chapter 1

Introduction

Symplectic geometry has a recent history of classification results associated to symplectic toric objects. In 1988, Delzant classified compact connected symplectic toric manifolds by the images of their moment maps [9]. Recently, this was extended by Karshon and Lerman to non-compact symplectic toric manifolds [19]. Research in this area has been dominated by two separate pursuits: examining what happens when the symplectic structure is weakened (see origami [7] and folded symplectic manifolds [17], and b-symplectic/log symplectic manifolds [13]/ [12]) and considering weakened versions of manifolds (see symplectic toric orbifolds [23], for instance). The goal of this thesis, the classification of toric symplectic stratified spaces with isolated singularities, follows the latter trend.

The importance of stratified spaces in symplectic geometry arises from the symplectic reduction of Marsden-Weinstein [26] and Meyer [27]. In 1991, Sjamaar and Lerman [32] showed that, in general, symplectic reduction results in a stratified space and furthermore that each strata inherits a symplectic form from the original manifold. In 2005, Burns, Guillemin, and Lerman [6] defined symplectic toric stratified spaces with isolated singularities and classified these in the compact connected case using the images of their moment maps.

The foundation for Delzant's classification are the convexity and connectedness theorems of Atiyah [2] and Guillemin-Sternberg [14]. This is emulated by Burns, Guillemin, and Lerman who use a similar convexity and connectedness theorem for compact symplectic toric stratified spaces with isolated singularities. The issue with the non-compact version of either case is that the image of the moment map no longer needs to be convex and its fibers need not be connected.

Karshon and Lerman's solution to this problem in the case of a symplectic toric manifold (M, ω, μ) is to substitute for the moment map image the orbital moment map as the main classification device: the unique map $\bar{\mu}$ from the quotient of M to the Lie algebra dual through which μ factors. This extra information supplements the loss of connected fibers. As the quotient of M by the torus action needn't be contractible, multiple isomorphism classes may be associated to each orbital moment map and these classes are quantified by cohomology classes of the quotient of M. Our classification will follow this approach.

Fix G a torus and let $\mathfrak{g}$ denote its Lie algebra. A *symplectic toric stratified space with isolated singularities* $(X, \omega, \mu : X \to \mathfrak{g}^*)$ is (roughly) defined as a symplectic toric manifold with isolated singularities whose deleted neighborhoods are modeled on symplectic toric cones. Here, X is the full space, ω is a symplectic form on X_{reg}, the open, dense manifold in X, and μ is a continuous function such that $\mu|_{X_{\mathsf{reg}}}$ is a moment map for the action of G on $(X_{\mathsf{reg}}, \omega)$.

By identifying the orbital moment maps of symplectic toric stratified spaces with isolated singularities as a type of map we call *stratified unimodular local embeddings*, we show that, by grouping together symplectic toric stratified spaces by these orbital moment map types, we make the following classification:

Theorem A. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then for W_{reg} the top stratum of W:

1. The set of symplectic toric stratified isolated singularities with orbital moment map ψ is non-empty.
2. There is a subspace $\mathcal{C} \subset H^2(W_{\mathsf{reg}}, \mathbb{R})$ dependent on the topology of W so that the isomorphism classes of symplectic toric stratified spaces with isolated singularities $(X, \omega, \mu : X \to \mathfrak{g}^*)$ with G-quotient map $\pi : X \to W$ and orbital moment map ψ are in bijective correspondence with the cohomology classes

$$H^2(W_{\mathsf{reg}}; \mathbb{Z}_G) \times \mathcal{C}$$

where $\mathbb{Z}_G$ is the integral lattice of G, the kernel of the exponential map $\exp : \mathfrak{g} \to G$.

Once the relevant language has been established, the subspace $\mathcal{C} \subset H^2(W_{\mathsf{reg}}; \mathbb{R})$ is easily described and may be calculated through the use of relative de Rham cohomology.

On the way to making this classification, we will find it necessary to completely understand symplectic toric cones in this orbital moment map context. Recall that a symplectic toric manifold $(M, \omega, \mu : M \to \mathfrak{g}^*)$ is a *symplectic toric cone* if M has a free and proper action of $\mathbb{R}$ commuting with the action of G and, with respect to any action diffeomorphism $\rho_\lambda : M \to M$ for this $\mathbb{R}$ action (for $\lambda \in \mathbb{R}$), we have $\rho_\lambda^* \omega = e^\lambda \omega$. Additionally, we impose the condition that the moment map μ for M is the *homogeneous moment map* for (M, ω); that is, the moment map μ satisfying $\mu(t \cdot p) = e^t \mu(p)$ for every $t \in \mathbb{R}$ and $p \in M$ (such a moment map for (M, ω) always exists).

As in the case of symplectic toric stratified spaces, the orbital moment maps of symplectic toric cones must take a certain form: that of a *homogeneous unimodular local embedding.* We may group our symplectic toric cones by orbital moment map type to make the classification:

Theorem B. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then:

1. The set of symplectic toric cones with orbital moment map ψ is non-empty.
2. The set of isomorphism classes of symplectic toric cones (M, ω, μ) with G-quotient $\pi : M \to W$ and orbital moment map ψ is in natural bijective correspondence with the cohomology classes $H^2(W; \mathbb{Z}_G)$, where $\mathbb{Z}_G$ is the integral lattice of G, the kernel of the map $\exp : \mathfrak{g} \to G$.

In Lerman's classification of compact connected contact toric manifolds, item 2 of Theorem B is more or less proven in the case where it is known the set of symplectic toric cones with orbital moment map ψ is non-empty. However, entirely new to this thesis is item 1, which tells us that the homogeneous unimodular local embeddings are exactly the orbital moment maps of symplectic toric cones. Confer with the introduction of Chapter 3 for a more detailed discussion.

This thesis is organized as follows: in Chapter 2, we assemble some of the pertinent background required for this paper. In Chapter 3, we present a classification of symplectic toric cones, and in Chapter 4, we present a classification of symplectic toric stratified spaces with isolated singularities. A more complete organizational description will be included in the introduction of each chapter. We also include a three part appendix, dealing with manifolds with corners, stacks, and relative de Rham cohomology.

1.1 Notation and conventions

Manifolds are assumed to be finite dimensional, paracompact, and Hausdorff. For any action of a group K on a manifold M and for any point p of M, K_p will denote the stabilizer of p in K. For an element X of the Lie algebra $\mathfrak{k}$ of K, X_M will always denote the vector field induced by X on M; in other words, the vector field defined at $p \in M$ as:

$$X_M(p) := \left.\frac{d}{dt}\right|_{t=0} (\exp(tX) \cdot p)$$

G will always denote a torus (a compact connected commutative finite dimensional Lie group) and $\mathfrak{g}$ will always denote its Lie algebra. $\mathbb{Z}_G$ will always be used to denote the integral lattice of $\mathfrak{g}$; that is, the lattice $\ker(\exp : \mathfrak{g} \to G)$. The notation $\langle \cdot, \cdot \rangle$ will denote the canonical pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

A Manifold with corners of dimension n is a paracompact Hausdorff space with a maximal atlas of charts diffeomorphic to sectors $\mathbb{R}^k \times \mathbb{R}^{n-k}$; see Appendix A for a full definition of all objects using manifolds with corners. Note: we are expressly *not* thinking of manifolds with corners as stratified spaces with maps of stratified spaces as the only smooth maps. Indeed, our definition of smooth maps of manifolds with corners allow for the image of any open face of the source manifold with corners to be contained in multiple open faces of the target.

For a manifold with corners W, we denote by $\mathring{W}$ the open dense interior of W. There always exists a manifold without corners $\overline{W}$ into which W embeds; in this case, it is said that $\overline{W}$ *contains* W *as a domain.* For a stratified space X, we will always denote by X_{reg} its top dimension open and dense stratum (all stratified spaces we consider will have such a top stratum).

Given two maps $f : M \to N$ and $g : M' \to N$, the symbol $M \times_N M'$ will denote the fiber product of M with M' over N. In the case where we wish to emphasize the maps f and g, we may write $M \times_{f,N,g} M'$.

For any topological space X, $\mathsf{Open}(X)$ will always denote the category of open subsets of X with morphisms inclusions of subsets. The symbols Sets and $\mathsf{Groupoids}$ will denote the categories of sets and (small) groupoids, respectively. By *presheaf of groupoids*, we will always mean a *strict* presheaf of groupoids with domain a (full subcategory of) the category of open subsets on some topological space; in other words, a (1-)functor $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$. To avoid unnecessary generality involving sites and categories fibered in groupoids, we will take *stack* to mean such a presheaf of groupoids satisfying some extra conditions (see Definition B.3).

Chapter 2

Background

In this preliminary chapter, we will discuss the relevant background regarding symplectic toric geometry. Let G be a torus with Lie algebra $\mathfrak{g}$. Recall that a *symplectic toric manifold* is a symplectic manifold (M, ω) together with an effective Hamiltonian action of a torus G with dimension half the dimension of M and a moment map $\mu : M \to \mathfrak{g}^*$. Such an object is denoted by the triple $(M, \omega, \mu : M \to \mathfrak{g}^*)$.

In Section 2.1, we discuss symplectic toric manifolds. Specifically, we will describe a local normal form for symplectic toric manifolds as well as the results of Karshon and Lerman [19] which we will later be adapting to classify symplectic toric cones and symplectic toric stratified spaces with isolated singularities. In Section 2.2, we define and describe symplectic toric cones and contact toric manifolds. This includes a description of the equivalence of categories that links them. We also develop some of the language and results we require later in Chapter 3. Finally, in Section 2.3, we discuss symplectic reduction; specifically, we recall how, in the singular case, symplectic reduction results in a stratified space. This will serve as the motivating model for our definition of a symplectic stratified space.

2.1 Symplectic toric manifolds

In this section, we discuss some pertinent results regarding symplectic toric manifolds. This includes both a discussion of local normal forms of Hamiltonian manifolds as well as a full discussion of the classification of symplectic toric manifolds by Karshon and Lerman.

2.1.1 A local normal form

We recall a local normal form for a neighborhood of orbits of a compact Hamiltonian action with moment map image zero in a symplectic manifold. This was developed in the case of torus actions by Guillemin and Sternberg [15] and later in more general cases by Marle [25]. We use a specific version of this local normal form given by Sjamaar and Lerman in [32].

First, recall the definition of the symplectic slice representation:

Definition 2.1.1. Suppose a Lie group K acts on a symplectic manifold (M, ω) via symplectomorphisms. Suppose additionally that p is a point of M with stabilizer H. Define the vector space V as the quotient

$$V := \frac{T_p(K \cdot p)^\omega}{T_p(K \cdot p) \cap T_p(K \cdot p)^\omega}$$

This vector space with symplectic form ω_V (just the restriction of ω_p to V) and symplectic action of H is called the symplectic slice representation at p.

The action of the stabilizer of a point p in a symplectic manifold (M, ω) is Hamiltonian with respect to a special moment map we will use below:

Proposition 2.1.2. Let (M, ω) be a symplectic manifold with a symplectic action by a Lie group K. Fix a point p with stabilizer H and let (V, ω_V) be its symplectic slice representation. Finally, let $\mathfrak{h}$ be the Lie algebra of H. Then the map $\Phi_V : V \to \mathfrak{h}^*$ defined by

$$\langle X, \Phi_V(v) \rangle := \frac{1}{2}\omega_V(X_V(v), v) \tag{2.1}$$

for every $v \in V$ and every $X \in \mathfrak{h}$ with induced vector field X_V on V is a moment map for the action of H on V.

Proof. First, note that, for any $v \in V$ and any Lie algebra vector $X \in \mathfrak{h}$, $X_V(v) = X \cdot v$, where $X \cdot v$ denotes the infinitesimal action of $\mathfrak{h}$ on V. As this action is represented by an element of $\mathfrak{sp}(V, \omega_V)$ (the Lie algebra to the symplectic group $\mathrm{Sp}(V, \omega_V)$), it follows that, for any w and w' in V, $\omega_V(X \cdot w, w') = -\omega_V(x, X \cdot w')$. Therefore, for any $w \in T_vV$, we have:

$$\begin{aligned}
(d\langle X, \Phi_V \rangle)_v (w) &= \left.\frac{d}{dt}\right|_0 (\langle X, \Phi_V(v + tw) \rangle) \\
&= \left.\frac{d}{dt}\right|_0 \frac{1}{2}\omega_V(X_V(v + tw), v + tw) \\
&= \frac{1}{2}(\omega_V(X \cdot v, w) + \omega_V(X \cdot w, v)) \\
&= \omega_V(X_V(v), w)
\end{aligned} \tag{2.2}$$

Thus, Φ_V is a moment map. $\square$

We may now quote the local normal form we are interested in.

Proposition 2.1.3. Suppose a Lie group K with Lie algebra $\mathfrak{k}$ acts on a symplectic manifold (M, ω). Suppose further this action is Hamiltonian with moment map $\mu : M \to \mathfrak{k}^*$. Let $p \in M$ be a point satisfying $\mu(p) = 0$ with stabilizer H and let $\mathfrak{h} \subset \mathfrak{k}$ be the Lie algebra of $\mathfrak{h}$. Finally, let V be the symplectic slice representation at p. Then:

1. The total space of the vector bundle

$$Y := K \times_H (\mathfrak{h}^o \times V) \to K/H$$

inherits a symplectic form η as a symplectic quotient of the product $T^*K \times V$. (Here, $Y \to K/H$ is the associated bundle built from the principal H-bundle $K \to K/H$ and the H-representation $\mathfrak{h}^o \times V$).

2. Fix a splitting $\mathfrak{g}^* \cong \mathfrak{h}^* \times \mathfrak{h}^o$. Then the action of K on Y given by

$$h \cdot [(k, \xi, v)] := [(h \cdot k, \xi, v)]$$

is Hamiltonian with moment map $J : Y \to \mathfrak{k}^*$ defined by

$$J([(k, \xi, v)]) := \mathrm{Ad}^*(k)(\xi + \Phi_V(v))$$

for $\mathrm{Ad}^*(k)$ the coadjoint action of k on $\mathfrak{k}^*$ and $\Phi_V : V \to \mathfrak{h}^*$ the moment map for the action of H on (V, ω_V) defined in equation (2.6).

3. There are K-invariant neighborhoods U of $K \cdot p$ and U' of the zero section of $Y \to K/H$ and a K-invariant symplectomorphism $\varphi : U \to U'$ such that $\mu = J \circ \varphi$.

Proof. See Proposition 2.5 of [32]. □

In the case where the symplectic manifold (M, ω) is in fact a symplectic toric manifold, Proposition 2.1.3 takes a simpler form:

Lemma 2.1.4 (Lemma B.3, [19])**.** Let G be a torus with Lie algebra $\mathfrak{g}$ and let $(M, \omega, \mu : M \to \mathfrak{g}^*)$ be a symplectic toric manifold. Let p be a point in M with stabilizer K and with symplectic slice representation (V, ω_V).

1. There is an isomorphism $\tau_K : K \to \mathbb{T}^k$ such that (V, ω_V) is isomorphic to $\mathbb{C}^k$ with action of K the composition of the standard $\mathbb{T}^k$ action on $\mathbb{C}^k$ composed with τ_K.

2. Let $\tau : G \to \mathbb{T}^l \times \mathbb{T}^k$ be an isomorphism with $\tau(k) = (1, \tau_K(k))$ for $k \in K$. Then there is a G-invariant neighborhood of the orbit $G \cdot p$ in M and a τ-equivariant open embedding

$$j : U \to T^*\mathbb{T}^l \times \mathbb{C}^k$$

with $j(G \cdot p) = \mathbb{T}^l \times \{0\}$.

3. μ satisfies

$$\mu|_U = \mu(p) + \tau^* \circ \phi \circ j,$$

where $\phi : T^*\mathbb{T}^l \times \mathbb{C}^k \to (\mathbb{R}^l)^* \times (\mathbb{R}^k)^* \to \mathfrak{g}^*$ is the map

$$\phi((q_1, \ldots, q_l, p_1, \ldots, p_l), (z_1, \ldots, z_k)) := \Big((p_1, \ldots, p_l), \sum |z_j|^2 e_j^*\Big)$$

for $\{e_1^*, \ldots, e_k^*\}$ the standard basis of the weight lattice $\mathbb{Z}_G^*$.

Proof. This is essentially Proposition 2.1.3 together with the decomposition of a symplectic vector space into weight spaces as in the Appendix of [23]; see Lemma B.3 of [19]. □

2.1.2 Non-compact symplectic toric manifolds

What follows is a review of the classification of non-compact symplectic toric manifolds by Karshon and Lerman [19]. It is by no means a complete account; the aim is to give a rough outline of their classification. This section also serves as a convenient repository of relevant ideas we will later be citing and adapting.

Fix a torus G with Lie algebra $\mathfrak{g}$. Following Karshon and Lerman, we define unimodular local embeddings as follows.

Definition 2.1.5 (Unimodular cones and unimodular local embeddings)**.** A unimodular cone with vertex $\epsilon \in \mathfrak{g}^*$ is a subset of the form:

$$C = \{\eta \in \mathfrak{g}^* \mid \langle \eta - \epsilon, v_i \rangle \geq 0 \text{ for all } 1 \leq i \leq k\}$$

where $\{v_1, \ldots, v_k\}$ is a basis for an integral lattice of a subtorus of G. This cone is labeled $C_{\{v_1,\ldots,v_k\},\epsilon}$.

For a manifold with corners W, a smooth map $\psi : W \to \mathfrak{g}^*$ is a unimodular local embedding if, for each $w \in W$, there exists neighborhood U of w in W so that $\psi|_U$ is an open embedding of U onto a neighborhood of the vertex of a unimodular cone $C_w := C_{\{v_1,\ldots,v_k\},\psi(w)}$.

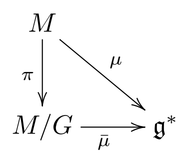
Remark 2.1.6. Note that the unimodular cone $C_{\{v_1,\ldots,v_k\},\epsilon}$ contains the affine subspace

$$A = \{\eta \in \mathfrak{g}^* \mid \langle \eta - \epsilon, v_i \rangle = 0 \text{ for } 1 \leq i \leq k\}$$

Thus, the subspace $A - \epsilon$ is exactly the annihilator of the Lie algebra spanned by $\{v_1, \ldots, v_k\}$.

Rather than looking at moment map images, we will look at orbital moment maps:

Definition 2.1.7. Let $(M, \omega, \mu : M \to \mathfrak{g}^*)$ be a symplectic toric manifold and let $\pi : M \to M/G$ be a G-quotient map. Then the orbital moment map is the unique continuous map $\bar{\mu} : M/G \to \mathfrak{g}^*$ making the diagram



commute.

Orbital moment maps of symplectic toric manifolds are unimodular local embeddings:

Proposition 2.1.8. For a symplectic toric manifold $(M, \omega, \mu : M \to \mathfrak{g}^*)$, the quotient M/G is naturally a manifold with corners. Furthermore, the orbital moment map $\bar{\mu} : M/G \to \mathfrak{g}^*$ is a unimodular local embedding.

Proof. This is a relatively straight forward application of Lemma 2.1.4; see Proposition 1.1 of [19]. □

Given two symplectic toric manifolds (M, ω, μ) and (M', ω', μ'), if there is a G-equivariant symplectomorphism $\varphi : (M, \omega) \to (M', \omega')$, then it is clear that, for G-quotient map $\pi : M' \to M'/G$, $\pi \circ \varphi : M \to M'/G$ is a G-quotient map for M. If φ additionally satisfies $\mu' \circ \varphi = \mu$, they must share an orbital moment map as well. Thus, to understand the collection of all symplectic toric manifolds, it makes sense to group symplectic toric manifolds together by quotient type and orbital moment map.

Definition 2.1.9 (Symplectic toric manifolds over ψ and $\mathsf{STC}_\psi(W)$)**.** Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding. Then a symplectic toric manifold over ψ is a symplectic toric manifold (M, ω, μ) together with a G-quotient map $\pi : M \to W$ such that $\mu = \psi \circ \pi$. This data will be expressed as the triple $(M, \omega, \pi : M \to W)$.

The groupoid of symplectic toric manifolds over ψ, denoted $\mathsf{STM}_\psi(W)$, is the groupoid with

- objects: symplectic toric manifolds over ψ; and

- morphisms: G-equivariant symplectomorphisms

$$f : (M, \omega, \pi : M \to W) \to (M', \omega', \pi' : M' \to W)$$

satisfying $\pi' \circ f = \pi$.

The strategy for actually classifying these spaces is to relate them to a class of simpler objects, namely symplectic toric bundles.

Definition 2.1.10. Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding. Then the category of symplectic toric principal G-bundles over ψ, denoted $\mathsf{STB}_\psi(W)$, is the groupoid with

- objects: pairs $(\pi : P \to W, \omega)$, for $\pi : P \to W$ a principal G-bundle and ω a G-invariant symplectic form with moment map $\psi \circ \pi$; and
- morphisms: G-equivariant symplectomorphisms

$$\varphi : (\pi : P \to W, \omega) \to (\pi' : P' \to W, \omega')$$

for which $\pi' \circ \varphi = \pi$.

Remark 2.1.11. Given a unimodular local embedding $\psi : W \to \mathfrak{g}^*$, for any open subset U of W, $\psi|_U : U \to \mathfrak{g}^*$ is also a unimodular local embedding. Thus we may define

$$\mathsf{STB}_\psi(U) := \mathsf{STB}_{\psi|_U}(U) \quad \text{and} \quad \mathsf{STM}_\psi(U) := \mathsf{STM}_{\psi|_U}(U).$$

These collections of groupoids define presheaves of groupoids

$$\mathsf{STB}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids} \quad \text{and} \quad \mathsf{STM}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids},$$

where $\mathsf{Open}(W)$ denotes the category of open subsets of W with inclusions. Indeed, for any pair of nested open subsets $U \subset V$ of W, we have restriction functors ρ_{VU}, taking a symplectic toric manifold $(M, \omega, \pi : M \to V)$ over $\psi|_V : V \to \mathfrak{g}^*$ to the symplectic toric manifold over $\psi|_U$

$$\rho_{VU}(M, \omega, \pi : M \to V) := (\pi^{-1}(U), \omega|_{\pi^{-1}(U)}, \pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U).$$

And, since any morphism

$$f : (M, \omega, \pi : M \to V) \to (M', \omega', \pi' : M' \to V)$$

in $\mathsf{STM}_\psi(V)$ must satisfy $\pi' \circ f = \pi$, it follows that f restricts to a well-defined morphism

$$\rho_{VU}(f) : \rho_{VU}(M, \omega, \pi : M \to V) \to \rho_{VU}(M', \omega', \pi' : M' \to V).$$

Finally, for $U \subset U' \subset U''$, it is easy to check that $\rho_{U'U} \circ \rho_{U''U'} = \rho_{U''U}$.

A similar argument confirms that STB_ψ is a presheaf as well. To avoid unnecessarily clunky notation, $\rho_{VU}(M, \omega, \pi : M \to V)$ and $\rho_{VU}(f)$ will be denoted as $(M, \omega, \pi : M \to V)|_U$ and $f|_U$, respectively.

To establish the equivalence of the groupoids $\mathsf{STB}_\psi(W)$ and $\mathsf{STM}_\psi(W)$, Karshon and Lerman introduce the functor $c : \mathsf{STB}_\psi(W) \to \mathsf{STM}_\psi(W)$, constructed with the following steps:

1. For every $w \in W$, the orbital moment map ψ picks out a basis $\{v_1^{(w)}, \ldots, v_k^{(w)}\}$ of a subtorus K_w of G. This is the basis of the lie algebra of a subtorus identified by the unimodular cone onto which a neighborhood of w embeds. In turn, this basis defines a symplectic representation $\rho : K_w \to Sp(V, \omega_V)$ on a symplectic vector space (V, ω_V) (namely, the standard symplectic toric representation with weights $\{v_1^*, \ldots, v_k^*\}$).

2. For any principal bundle $\pi : P \to W$, let $\sim$ be the equivalence relation

   $$p \sim p' \text{ if and only if there exists } k \in K_{\pi(p)} \text{ such that } p \cdot k = p'$$

   on P. Then define

   $$c_{\mathsf{Top}}(\pi : P \to W, \omega) := (P/\sim, \bar{\pi} : P/\sim \to W),$$

   for $\bar{\pi}$ the G-quotient of $P/\sim$ descending from π. It follows from the G-equivariance of morphisms of $\mathsf{STB}_\psi(W)$ that morphisms descend to the topological quotients modulo $\sim$. Thus, the relation establishes a functor

   $$c_{\mathsf{Top}} : \mathsf{STB}_\psi(W) \to \text{topological } G \text{ spaces over } W$$

   This functor commutes with restrictions; that is, for every open U in W,

   $$c_{\mathsf{Top}}((P, \omega)|_U) = c_{\mathsf{Top}}(P, \omega)|_U := (\bar{\pi}^{-1}(U), \bar{\pi}). \tag{2.3}$$

3. To define a smooth structure on and "symplectize" these topological quotients, Karshon and Lerman use symplectic cuts, showing for each $w \in W$, there is a neighborhood U_w of w in W (defined independent of P) so that

$$\mathsf{cut}((P,\omega)|_{U_w}) := (P|_{U_w} \times V_w)//_0 K_w$$

is a symplectic toric manifold over $\psi|_{U_w}$. This establishes a functor

$$\mathsf{cut} : \mathsf{STB}_\psi(U_w) \to \mathsf{STM}_\psi(U_w)$$

for each w.

4. For each w and $(P,\omega) \in \mathsf{STB}_\psi(W)$, there is a homeomorphism

$$\alpha_w^P : c_{\mathsf{Top}}((P,\omega)|_{U_w}) \to \mathsf{cut}((P,\omega)|_{U_w})$$

preserving the G-quotients of $c_{\mathsf{Top}}((P,\omega)|_{U_w})$ and $\mathsf{cut}((P,\omega)|_{U_w})$. For any w, w' in W with $U_w \cap U_{w'}$ non-empty, $\alpha_{w'}^P \circ (\alpha_w^p)^{-1}$ is a symplectomorphism. Therefore, $c_{\mathsf{Top}}((P,\omega))$ inherits the structure of a symplectic toric manifold.

5. Finally, for each isomorphism $\varphi : (P,\omega) \to (P',\omega')$, for any $w \in W$, the diagram

$$\begin{array}{ccc} c_{\mathsf{Top}}(P,\omega)|_{U_w} & \xrightarrow{\alpha_w^P} & \mathsf{cut}((P,\omega)|_{U_w}) \\ \downarrow{\scriptstyle c_{\mathsf{Top}}(\varphi)} & & \downarrow{\scriptstyle \mathsf{cut}(\varphi|_{U_w})} \\ c_{\mathsf{Top}}(P',\omega')|_{U_w} & \xrightarrow[\alpha_w^{P'}]{} & \mathsf{cut}((P',\omega')|_{U_w}) \end{array}$$

commutes, and so the morphism $c_{\mathsf{Top}}(\varphi)$ is a symplectomorphism.

As it will be important later, we present below an outline of the process used to "symplectize" the quotient space $c_{\mathsf{Top}}(P,\omega)$. First, an important theorem about extending Marsden-Weinstein and Meyer reduction to a specific scenario involving manifolds with corners is required.

Theorem 2.1.12. Suppose (M,σ) is a symplectic manifold with corners with a proper Hamiltonian action of a Lie group K with moment map $\Phi : M \to \mathfrak{k}^*$ (for $\mathfrak{k}$ the Lie algebra of K). Suppose also that:

- for each $x \in \Phi^{-1}(0)$, the stabilizer K_x of x is trivial;
- Φ admits an extension $\tilde{\Phi}$ to a manifold $\tilde{M}$ containing M as a domain; and

- $\tilde{\Phi}^{-1}(0) = \Phi^{-1}(0)$.

Then $\Phi^{-1}(0)$ is a manifold without corners and the reduction at 0

$$M//_0 K := \Phi^{-1}(0)/K$$

is naturally a symplectic manifold.

Proof. See Theorem 2.23, [19]. □

We now construct $\mathsf{cut}((P,\omega)|_{U_w})$ for a carefully chosen open subset U_w of W.

Construction 2.1.13. Fix a symplectic toric bundle $(\pi : P \to W, \omega)$ over unimodular local embedding $\psi : W \to \mathfrak{g}^*$.

Because ψ is a unimodular local embedding, there exists a unimodular cone $C_{\{v_1,\ldots,v_k\},\psi(w)}$ so that ψ embeds a neighborhood of w in W onto a neighborhood of $\psi(w)$ in the cone. Recall this means $\{v_1, \ldots, v_k\}$ is the basis for the Lie algebra $\mathfrak{k}$ of a subtorus $K_w \subset G$. In turn, these define a symplectic toric representation $\rho_w : K_w \to (\mathbb{C}^k, \omega_{\mathbb{C}^k})$ with symplectic weights $\{v_1^*, \ldots, v_k^*\}$ (for $\omega_{\mathbb{C}^k}$ the standard symplectic form on $\mathbb{C}^k$); ρ_w has moment map

$$\mu_w : \mathbb{C}^k \to \mathfrak{k}^*, \quad (z_1, \ldots, z_k) \mapsto -\sum_{i=1}^{k} |z_i|^2 v_i^*$$

Let $\iota : \mathfrak{k} \to \mathfrak{g}$ be the embedding of $\mathfrak{k}$ into $\mathfrak{g}$ with dual projection $\iota^* : \mathfrak{g}^* \to \mathfrak{k}^*$. Then, since $\psi \circ \pi$ is the moment map for the free action of G on P, $\nu := \iota^* \circ \psi \circ \pi$ is the moment map for the action of K_w on P. Define $\xi_0 := \iota^*(\psi(w))$ and let C'_w be the cone

$$C'_w := \{\xi \in \mathfrak{k}^* \mid \langle \xi - \xi_0, v_i \rangle \geq 0, \, 1 \leq i \leq k\}.$$

Note that we can identify the cone C_w with the product $\mathfrak{k}^o \times C'_w$. Here, $\mathfrak{k}^o$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{g}$ which is embedded in C_w as the affine space $\mathfrak{k}^o + \psi(w)$. This affine space corresponds to the open face of W containing w (near w).

Thus, there exist contractible neighborhoods $\mathcal{U}$ of w in the open face of W containing w and $\mathcal{V}$ of ξ_0 in $\mathfrak{k}^*$ so that, for $\mathcal{V}' := C'_w \cap \mathcal{V}$, a neighborhood U_w is diffeomorphic to $\mathcal{U} \times \mathcal{V}'$. Let $\nu : P|_{U_w} \to \mathcal{V}'$ be the map $\iota^* \circ \psi \circ \pi$. Then ν is a trivializable $\mathcal{U} \times G$ fiber bundle. Thus, $P|_{U_w}$ is contained in a manifold $\tilde{P}$ (diffeomorphic to $\mathcal{V} \times \mathcal{U} \times G$) as a domain and $\nu : P|_{U_w} \to \mathcal{V}'$ admits a smooth extension to a map $\tilde{\nu} : \tilde{P} \to \mathcal{V}$.

Define $\Phi : P|_{U_w} \times V_w \to \mathfrak{k}^*$ by

$$\Phi(p, z) := \nu(p) - \iota^*(\psi(w)) + \mu_w(z).$$

Then Φ is a moment map for the action of K on $P|_{U_w} \times \mathbb{C}^k$ and admits an extension to the map

$$\tilde{\Phi}(p, z) := \tilde{\nu}(p) - \iota^*(\psi(w)) + \mu_w(z)$$

satisfying the conditions of Theorem 2.1.12. Thus, reduction at the zero level set of Φ yields a symplectic manifold (without corners). One may check that $(P|_U \times \mathbb{C}^k)//_0 K_w$ inherits a G-quotient map to U_w $\bar{\pi}$ with respect to which $((P|_U \times \mathbb{C}^k)//_0 K_w, \bar{\pi})$ is a symplectic toric manifold of $\mathsf{STM}_\psi(U_w)$. Define

$$\mathsf{cut}((P, \omega)|_{U_w}) := ((P|_U \times \mathbb{C}^k)//_0 K_w, \bar{\pi}).$$

For $\varphi : (P, \omega) \to (P', \omega')$, the morphism $\varphi \times id_{\mathbb{C}} : P|_{U_w} \times \mathbb{C}^k \to P'|_{U_w} \times \mathbb{C}^k$ descends to a symplectomorphism $\mathsf{cut}(\varphi) : \mathsf{cut}((P, \omega)|_{U_w}) \to \mathsf{cut}((P', \omega')|_{U_w})$.

$\Diamond$

For the purposes of this paper, it will also be important to sketch the construction of the homeomorphisms $\alpha_w^P : c_{\mathsf{Top}}(P, \omega)|_{U_w} \to \mathsf{cut}((P, \omega)|_{U_w})$.

Construction 2.1.14. For each $w \in W$ and U_w defined as in Construction 2.1.13, to define the homeomorphisms $\alpha_w^P : c_{\mathsf{Top}}(P, \omega)|_{U_w} \to \mathsf{cut}((P, \omega)|_{U_w})$, first let $s : \mu_w(\mathbb{C}^k) \to \mathbb{C}^k$ be the continuous section of μ_w defined by

$$s(\eta) := \left(\sqrt{\langle -\eta, v_1 \rangle}, \ldots, \sqrt{\langle -\eta, v_k \rangle}\right)$$

Then one can show that the map $\alpha_w^P : c_{\mathsf{Top}}(P|_{U_w}) \to (P|_{U_w} \times \mathbb{C}^k)//_0 K_w$ defined by

$$[p] \mapsto [p, s(\iota^*(\psi(p)) - \nu(p))]$$

is a well-defined G-equivariant homeomorphism.

$\Diamond$

Remark 2.1.15. For $w \in \mathring{W}$ (the interior of W), we have that $\psi|_{U_w}$ is an open embedding into $\mathfrak{g}^*$ itself (i.e., rather than just an embedding into a cone). This means that K_w is trivial and therefore $\mathsf{cut}((P, \omega)|_{U_w}) =$

$(P|_{U_w}, \omega, \pi)$ as symplectic toric manifolds over ψ.

Remark 2.1.16. Since the collection of functors $c : \mathsf{STB}_\psi(U) \to \mathsf{STM}_\psi(U)$ for each open U in W commute with restriction, it follows that we have a map of presheaves

$$c : \mathsf{STB}_\psi \to \mathsf{STM}_\psi.$$

Remark 2.1.17. In later chapters, we will need to observe that we may shift a unimodular local embedding by any constant in $\mathfrak{g}^*$ and, with respect to this shift, c builds symplectomorphic symplectic toric manifolds. Given a unimodular local embedding $\psi : W \to \mathfrak{g}^*$ and a Lie algebra dual element $\eta \in \mathfrak{g}^*$, let ψ' be the map $\psi'(w) := \psi(w) + \eta$. Then ψ' is also a unimodular local embedding. Let $(\pi : P \to W, \omega)$ be a symplectic toric bundle over ψ. Then clearly $(\pi : P \to W, \omega)$ is a symplectic toric bundle over ψ' as well.

For C_w and C'_w the unimodular cones associated to w, defined relative to ψ and ψ' respectively, it is more or less obvious that $C_w + \eta = C'_w$ and therefore both ψ and ψ' determine the same subtorus K_w. It follows that $c_{\mathsf{Top}}(P, \omega)$ is the same topological G-space when considering (P, ω) as a symplectic toric bundle over ψ or ψ'. Following Construction 2.1.13, it is clear that, since the cutting procedures with respect to ψ or ψ' are performed relative to the cone vertices $\psi(w)$ and $\psi'(w)$, $c_{\mathsf{Top}}(P, \omega)$ is symplectized the same way with respect to either unimodular local embedding.

Therefore, with respect to the identity map on $c_{\mathsf{Top}}(P, \omega)$, the symplectic toric manifolds $c(P, \omega) \in \mathsf{STM}_\psi(W)$ and $c(P, \omega) \in \mathsf{STM}_{\psi'}(W)$ are symplectomorphic. Of course, this symplectomorphism does not preserve the respective moment maps.

In service of classifying the groupoid of symplectic toric bundles over a given unimodular local embedding $\psi : W \to \mathfrak{g}^*$, Karshon and Lerman prove the following lemmas (which we restate as they will become important later in this paper).

Lemma 2.1.18. Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding, let $\pi : P \to W$ be a principal G-bundle, and let $A \in \Omega^1(P, \mathfrak{g})^G$ be a connection 1-form for P. For convenience, define $\mu := \psi \circ \pi$. Then:

- Any closed G-invariant 2-form on P with moment map μ is automatically symplectic; this includes the form $d\langle \mu, A \rangle$.
- The map from closed 2-forms on W to closed 2-forms on P:

$$\beta \mapsto d\langle \mu, A \rangle + \pi^* \beta$$

establishes a bijection between the set of closed 2-forms on W and the set of G-invariant symplectic forms on P with moment map μ.

Proof. See Lemma 3.2, [19]. □

This has an obvious corollary that will be important for us later (though was not explicitly mentioned by Karshon and Lerman).

Corollary 2.1.19. For ψ and P as in the lemma above, let ω be any closed G-invariant 2-form on P with moment map μ. Then the map from closed 2-forms on W to closed 2-forms on P:

$$\beta \mapsto \omega + \pi^*\beta$$

also establishes a bijection between closed 2-forms on W and G-invariant symplectic forms on P with moment map μ.

Lemma 2.1.20. Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding and let $\pi : P \to W$ be a principal G-bundle. For any 1-form γ on W and any G-invariant symplectic form ω on P with moment map μ, there exists a gauge transformation $f : P \to P$ with $f^*(\omega + \pi^*(d\gamma)) = \omega$.

Proof. See Lemma 3.3, [19]. □

Karshon and Lerman then go on to prove that, for every open subset U of W, $c_U : \mathsf{STB}_\psi(U) \to \mathsf{STM}_\psi(U)$ is a fully faithful functor. After observing that the groupoid STM_ψ is locally connected (i.e., all objects are locally isomorphic), they implicitly use the fact that STB_ψ is a stack and STM_ψ is a prestack (see Appendix B) to conclude that c_U is essentially surjective. This yields the following theorem.

Theorem 2.1.21. Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding. Then

$$c : \mathsf{STB}_\psi(W) \to \mathsf{STM}_\psi(W)$$

is an equivalence of categories.

Proof. See Theorem 4.1, [19]. □

Using the tools of Lemma 2.1.18 and Lemma 2.1.20, Karshon and Lerman are able to show that the elements of $\mathsf{STB}_\psi(W)$ are classified by the cohomology classes $H^2(W; \mathbb{Z}_G) \times H^2(W; \mathbb{R})$, (where $\mathbb{Z}_G :=$

ker(exp : $\mathfrak{g} \to G$) is the integral lattice). Thus, using the equivalence of categories c, they conclude the following result.

Theorem 2.1.22. Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding. Then:

1. The groupoid $\mathsf{STM}_\psi(W)$ is non-empty; that is, there exists symplectic toric manifold (M, ω, μ) with G-quotient $\pi : M \to W$ with respect to which ψ is the orbital moment map.

2. $\pi_0(\mathsf{STM}_\psi(W))$, the set of isomorphism classes of $\mathsf{STM}_\psi(W)$, is in bijective correspondence with the cohomology classes:
$$H^2(W, \mathbb{Z}_G \times \mathbb{R}) \cong H^2(W; \mathbb{Z}_G) \times H^2(W; \mathbb{R})$$

Proof. See Theorem 1.3, [19]. □

Since c is in fact an isomorphism of presheaves and it can be shown the identification of symplectic toric bundles with elements of $H^2(W; \mathbb{Z}_G) \times H^2(W; \mathbb{R})$ commutes with restrictions as well, it is fitting to call the elements of $H^2(W; \mathbb{Z}_G) \times H^2(W; \mathbb{R})$ characteristic classes for symplectic toric manifolds over ψ.

2.2 Symplectic cones and contact manifolds

In this section, we describe the relationship between symplectic cones and contact manifolds. Practically all of the results and proofs of this section are rephrased or recontextualized versions of results that can be found in [21] and [22]. We will first discuss general symplectic cones and contact manifolds before introducing a torus action to describe symplectic toric cones and contact toric manifolds. Finally, we will give some local structure results that are easy extensions of known results and will be necessary in the coming chapters.

2.2.1 Preliminary definitions

To start, we recall the definition of a symplectic cone.

Definition 2.2.1. A symplectic cone is any symplectic manifold (M, ω) together with a free and proper $\mathbb{R}$ action
$$\mathbb{R} \times M \longrightarrow M, \;\; (\lambda, p) \mapsto \rho_\lambda(p) := \lambda \cdot p$$
satisfying $\rho_\lambda^* \omega = e^\lambda \omega$ for each $\lambda \in \mathbb{R}$.

A map of symplectic cones is an $\mathbb{R}$-equivariant symplectic map.

The vector field generating the $\mathbb{R}$ action on a symplectic cone is known as the *Liouville* or *expanding* vector field of the cone. It satisfies the following important property.

Proposition 2.2.2. Let (M, ω) be a symplectic cone and let Ξ be the vector field generating the action of $\mathbb{R}$ on M (that is, the vector field Ξ with time t flow corresponding to the action of t on M). Then $L_{\Xi}\omega = \omega$.

Proof. Let φ_t be the time t flow of Ξ. Then $(\varphi_t)^*\omega = e^t\omega$, so by definition

$$L_{\Xi}\omega = \frac{d}{dt}\Big|_{t=0}(e^t\omega) = e^0\omega = \omega$$

□

Rather than defining a symplectic cone as above, one may instead define a symplectic cone as a symplectic manifold (M, ω) with a Liouville vector field Ξ that generates a free and proper action. It is not difficult to show that these two definitions are equivalent.

The symplectic structure on a symplectic cone (M, ω) induces a natural co-oriented contact structure on the quotient $M/\mathbb{R}$. Let us recall the definition of a co-oriented contact structure.

Definition 2.2.3 (Contact forms and contact manifolds)**.** Let B be a $2n+1$ dimensional manifold. Then a contact form is a 1-form α on B such that $(d\alpha)^n \wedge \alpha$ is a volume form on B. Two contact forms α and α' are in the same conformal class of contact forms if there is a function $f \in C^\infty(B)$ such that $e^f\alpha = \alpha'$.

A pair (B, ξ) of manifold B with codimension 1 distribution ξ is a contact manifold if for every $b \in B$ there is an open neighborhood U of b and contact form α_U on U so that $\ker(\alpha_U) = \xi|_U$.

We will only be interested in contact manifolds with contact distributions that are globally the kernel of a contact form.

Definition 2.2.4. A contact manifold (B, ξ) is co-orientable if there exists a contact form α on all of B with $\ker(\alpha) = \xi$. A co-orientable contact manifold (B, ξ) together with a choice of conformal class of contact forms is a co-oriented contact manifold.

Morphisms of co-oriented contact manifolds should preserve both the contact structure as well as the co-orientations of the source and target.

Definition 2.2.5. A map of co-oriented manifolds $\varphi : (B, \xi) \to (B', \xi')$ is a smooth map $\varphi : B \to B'$ so that, for α' a representative of the conformal class for (B', ξ'), $\varphi^*\alpha'$ is in the conformal class co-orienting (B, ξ). A contactomorphism between co-oriented (B, ξ) and (B', ξ') is a diffeomorphism $\varphi : B \to B'$ so that φ and φ^{-1} are both maps of co-oriented contact manifolds.

Given a co-orientable contact manifold (B, ξ), we may produce many examples of symplectic cones (M, ω) with $M/\mathbb{R} \cong B$:

Example 2.2.6. Let (B, ξ) be a co-orientable contact manifold with contact form α. Then the 2-form $d(e^t\alpha)$ is a symplectic form on the manifold $B \times \mathbb{R}$ and, with respect to the action of $\mathbb{R}$ on itself via translation, $(B \times \mathbb{R}, d(e^t\alpha))$ is a symplectic cone.

We will see that, up to an appropriate notion of isomorphism, this is the only example of a symplectic cone. However, it will be useful to have a canonical choice of symplectic cone for each co-oriented contact manifold (rather than one constructed after a choice of contact form).

Given a contact manifold (B, ξ) with $\xi = \ker(\alpha)$, denote by ξ^o the annihilator of ξ in T^*B. Thinking of the 1-form α as a section $\alpha : B \to T^*B$, a (local) contact 1-form for ξ functions as a (local) section trivializing ξ^o. In the case where (B, ξ) is co-orientable, a choice of contact form α on B with $\ker(\alpha) = \xi$ corresponds to a trivialization of all ξ^o. It follows then that $\xi^o \backslash 0$ (ξ^o without its zero section) has two components.

Let ξ^o_+ denote the component of $\xi^o \backslash 0$ selected by the conformal class of contact forms (co-)orienting (B, ξ). Then, for any $b \in B$ and $\eta \in T^*_b B$, the action $t \cdot \eta := e^t \eta$ is free and proper with quotient map the restriction of the natural projection $T^*B \to B$ to ξ^o_+. Additionally, one can show that ξ^o_+ is a symplectic submanifold of T^*B with its canonical symplectic form. This $\mathbb{R}$ action and symplectic form give ξ^o_+ the structure of a symplectic cone.

This canonical association of a symplectic cone to the co-oriented contact manifold (B, ξ) has the following name:

Definition 2.2.7. Let (B, ξ) be a co-oriented contact manifold. Then the symplectic cone ξ^o_+ described above is called the symplectization of (B, ξ).

Just as we are able to produce a symplectic cone from any contact manifold, so too can we produce a co-oriented contact manifold from any symplectic cone.

Proposition 2.2.8. Let (M, ω) be a symplectic cone with Liouville vector field Ξ and let $\pi : M \to B$ be an $\mathbb{R}$-quotient map. Then

1. B inherits a natural co-oriented contact structure ξ from (M, ω).

2. Given any trivialization of M as a principal $\mathbb{R}$-bundle over B

$$\varphi : M \to B \times \mathbb{R},$$

the 1-form $\alpha := \varphi^* \left((\iota_\Xi \omega)|_{B\times\{0\}}\right)$ is a contact form for (B, ξ).

3. Any map of symplectic cones

$$f : (\pi : M \to B, \omega) \to (\pi' : M' \to B, \omega')$$

descends to a co-orientation preserving contact map $\bar{f} : (B, \xi) \to (B', \xi')$.

Proof. Let Ξ be the Liouville vector field for (M, ω) and let $\beta := \iota_\Xi \omega$. Define

$$\xi_b := d\pi_p(\ker(\beta_p)) \text{ for any } p \in \pi^{-1}(b). \tag{2.4}$$

To see this is a well-defined distribution, note that, for any $\lambda \in \mathbb{R}$ with action diffeomorphism $\rho_\lambda : M \to M$,

$$\rho_\lambda^*(\iota_\Xi \omega) = \rho_\lambda^*(\iota_{d\rho_\lambda(\Xi)}\omega) = \iota_\Xi \rho_\lambda^* \omega = e^\lambda \iota_\Xi \omega.$$

Hence for any $p \in M$, $(d\rho_\lambda)_p(\ker(\beta_p)) = \ker(\beta_{\lambda \cdot p})$.

To see this is a co-orientable contact distribution, it is enough to prove item 2. Note that, since $L_\Xi \omega = \omega$, $d\beta = \omega$. Then for any trivialization $\varphi : M \to B \times \mathbb{R}$ of M as a principal $\mathbb{R}$-bundle, it follows that the form $\alpha := (\varphi^* \beta)|_{B\times\{0\}}$ satisfies $\ker(\alpha) = \xi$ and $\varphi^* \omega = d(e^t \alpha)$. It is easy to check then that α is contact.

To see ξ inherits a co-orientation, note that for any $b \in B$ and any $p \in M$ with $\pi(p) = b$, the map

$$(d\pi_p)^T : T_b^* B \to T_p^* M$$

restricts to an isomorphism from ξ_b^o to $\ker(\beta_p)^o = \langle \beta_p \rangle$. Hence, we may choose the co-orientation of ξ defining, for each $b \in B$,

$$(\xi_+^o)_b := \{\eta \in \xi^o \backslash \{0\} \mid \pi^* \eta = \lambda \beta_p \text{ for some } \lambda > 0, p \in \pi^{-1}(b)\} \tag{2.5}$$

Finally, note that, since a map of symplectic cones $f : (M, \omega) \to (M', \omega')$ is $\mathbb{R}$-equivariant, it descends to a smooth map $\bar{f} : B \to B'$ (for (B, ξ) and (B', ξ') the co-oriented contact manifolds associated to the $\mathbb{R}$-quotients $M/\mathbb{R}$ and $M'/\mathbb{R}$). For Ξ, Ξ' the expanding vector fields for (M, ω), (M', ω') respectively, we have that $f_* \Xi = \Xi'$. Thus:

$$f^*(\iota_{\Xi'} \omega') = f^*(\iota_{f_* \Xi} \omega') = \iota_\Xi (f^* \omega') = \iota_\Xi \omega.$$

It follows from the definition of the co-oriented contact distributions ξ and ξ' that $\bar{f}$ is a co-orientation preserving contact map. $\square$

2.2.2 Toric actions on symplectic cones and contact manifolds

To discuss symplectic toric cones, we must first talk about symplectic actions of Lie groups on symplectic cones. We will always be interested in actions of Lie groups commuting with the action of $\mathbb{R}$ on a symplectic cone. In this case, we will be able to show that the action descends to a contact action on the quotient. To prove this, we first must prove the following easy but important lemma:

Lemma 2.2.9. Suppose the action of $\mathbb{R}$ via the flow of a complete vector field X on a manifold M commutes with the action of a Lie group K on M. Then the vector field X is K-invariant.

Proof. Fix a point p in M and let γ be an integral curve of X with $\gamma(0) = p$. The fact that the actions of K and $\mathbb{R}$ on M commute then tells us that, for any $k \in K$, the curve $k \cdot \gamma(t)$ must be the integral curve of X through $k \cdot p$. Thus, we must have $(d\rho_k)_p X(p) = X(k \cdot p)$. $\square$

From here forward, fix G a torus with Lie algebra $\mathfrak{g}$. As a symplectic cone (M, ω) is in particular a symplectic manifold, it makes sense to define a toric action of G on (M, ω). With some additional properties, we will call such a symplectic toric manifold a *symplectic toric cone*:

Definition 2.2.10. For (M, ω) a symplectic cone with an action of the torus G, we call a triple

$$(M, \omega, \mu : M \to \mathfrak{g}^*)$$

a symplectic toric cone if

- the actions of G and $\mathbb{R}$ on M commute;
- the action of G on (M, ω) is a symplectic toric action with moment map μ; and
- $\mu : M \to \mathfrak{g}^*$ is the homogeneous moment map for (M, ω): for every $\lambda \in \mathbb{R}$ and $p \in M$, $\mu(\lambda \cdot p) = e^{\lambda}\mu(p)$.

While the moment map μ of a symplectic toric cone (M, ω, μ) will prove to be an important piece of data, if (M, ω) already has all the other ingredients of a symplectic toric cone, the moment map is redundant:

Proposition 2.2.11. Suppose (M, ω) is a symplectic cone with an effective, symplectic action of a torus G commuting with the action of $\mathbb{R}$ such that $2\dim(G) = \dim(M)$. Then there exists a unique homogeneous moment map $\mu : M \to \mathfrak{g}^*$ with respect to which $(M, \omega, \mu : M \to \mathfrak{g}^*)$ is a symplectic toric cone.

Proof. Let Ξ be the Liouville vector field of the symplectic cone (M, ω). Then by Lemma 2.2.9, Ξ is G-invariant. Therefore, as ω is also G-invariant, the 1-form $\iota_\Xi \omega$ is G-invariant as well.

Now, let X be any vector in $\mathfrak{g}$ with induced vector field X_M on M. Then

$$0 = L_{X_M}(\iota_\Xi \omega) = d(\iota_{X_M}(\iota_\Xi(\omega))) + \iota_{X_M} d(\iota_\Xi \omega) = d(\iota_{X_M}(\iota_\Xi(\omega))) + \iota_{X_M}\omega = d\omega(X_M, \Xi) + \iota_{X_M}\omega.$$

It follows that the equation

$$\langle \mu(p), X \rangle = \omega(X_M(p), \Xi(p)) \tag{2.6}$$

defines a moment map μ for (M, ω).

We may again apply Lemma 2.2.9 to conclude that, for each X in $\mathfrak{g}$, the vector field X_M is invariant with respect to the action of $\mathbb{R}$; indeed, X_M is complete and its flow corresponds to the action of the subgroup $\exp(tX) \subset G$. Thus, for any $\lambda \in \mathbb{R}$ and $p \in M$:

$$\begin{aligned} \omega_{\lambda \cdot p}(X_M(\lambda \cdot p), \Xi(\lambda \cdot p)) &= \omega_{\lambda \cdot p}((d\rho_\lambda)_p(X_M(p)), (d\rho_\lambda)_p(\Xi(p))) \\ &= \rho_\lambda^* \omega_p(X_M(p), \Xi(p)) \\ &= e^\lambda \omega_p(X_M(p), \Xi(p)) \end{aligned}$$

Thus, μ must be homogeneous.

Finally, suppose μ and μ' are two different homogeneous moment maps for (M, ω). Then $\mu - \mu' = \eta$ for some $\eta \in \mathfrak{g}^*$. Since μ and μ' are homogeneous, we must conclude that $\eta = 0$ and so $\mu = \mu'$. □

If an action of a torus on a contact manifold satisfies a certain set of conditions, then the symplectization is a symplectic toric cone.

Example 2.2.12. Suppose that a co-oriented contact manifold (B, ξ) admits an effective action by a torus G with $2\dim(G) + 1 = \dim(B)$ such that the action of each element g preserves ξ and preserves ξ's co-orientation. Then the symplectic action on T^*B induced by the action of G restricts to a symplectic action on ξ_+^o. It is clear that, since the action of G on B is effective, the action of G on ξ_+^o must be effective as well. Thus, by Proposition 2.2.11, ξ_+^o admits a moment map with respect to which it is a symplectic toric cone.

A contact manifold with this type of toric action will be called a *contact toric manifold*:

Definition 2.2.13. A contact toric manifold is a co-oriented contact manifold (B, ξ) with an effective contact action by a torus G with dimension satisfying $2\dim(G) = \dim(B) + 1$.

Example 2.2.14. Suppose a contact toric manifold (B, ξ) has a G-invariant contact form α. Then the manifold $B \times \mathbb{R}$ with form $d(e^t\alpha)$ is a symplectic toric cone. The homogeneous moment map for this symplectic toric cone is of the form $(t, b) \mapsto e^t\mu_\alpha$ for $\mu_\alpha : B \to \mathfrak{g}^*$ the α-moment map which we define below.

Definition 2.2.15. Let (B, ξ) be a co-oriented contact toric manifold with G-invariant contact form α. Then the α-moment map for this action is the unique map $\mu_\alpha : B \to \mathfrak{g}^*$ satisfying

$$\langle \mu_\alpha(b), X \rangle = -\alpha_b(X_B(b)) \tag{2.7}$$

for every $b \in B$, $X \in \mathfrak{g}$ (here, X_B denotes the vector field on B induced by the action of X).

Remark 2.2.16. Note that, if $\alpha' = e^f\alpha$ is another G-invariant contact form for a contact manifold (B, ξ), then $\mu_{\alpha'} = e^f\mu_\alpha$. By fixing a norm $||\cdot||$ on $\mathfrak{g}^*$, one may thereby define *the contact moment map* of a contact toric manifold as the α-moment map $\mu_\alpha : B \to \mathfrak{g}^*$ for which $\mu_\alpha(B) \subset S(\mathfrak{g}^*)$; a unique choice of α for which this is true always exists. Thus, we may define a unique *orbital moment map* $\bar{\mu} : B/G \to \mathfrak{g}^*$ to associated to every contact toric manifold (see [21]).

In Lerman's classification of contact toric manifolds, he proved with a sheaf theoretic argument this very important uniqueness result for contact toric manifolds:

Theorem 2.2.17. Let (B, ξ) be a contact toric manifold with orbital moment map $\bar{\mu} : B/G \to \mathfrak{g}^*$ (see Remark 2.2.16). Then the isomorphism classes of contact toric manifolds with orbital moment map $\bar{\mu}$ is in bijective correspondence with the cohomology classes $H^2(B/G; \mathbb{R})$.

Proof. See Section 5 of [21]. □

Our next goal will be to describe an equivalence of categories between the category of symplectic toric cones and contact toric manifolds; so, we begin by formally describing their respective categories:

Definition 2.2.18. Let STC denote the category of symplectic toric cones: the groupoid with objects symplectic toric cones and morphisms between two symplectic toric cones $(M, \omega, \mu : M \to \mathfrak{g}^*)$ and (M', ω', μ') $(G \times \mathbb{R})$-equivariant symplectomorphisms $f : (M, \omega) \to (M', \omega')$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & M' \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu'} & \\ & \mathfrak{g}^* & \end{array} \tag{2.8}$$

commutes.

Let CTM denote the category of contact toric manifolds: the groupoid with objects contact toric manifolds and morphisms co-orientation preserving G-equivariant contactomorphisms.

Remark 2.2.19. As explained in Proposition 2.2.11, every symplectic toric cone has a unique homogeneous moment map. So, for two symplectic toric cones $(M, \omega, \mu : M \to \mathfrak{g}^*)$ and $(M', \omega', \mu' : M' \to \mathfrak{g}^*)$, a map of symplectic cones $f : M \to M'$ need only be a $(G \times \mathbb{R})$-equivariant symplectomorphism (i.e., diagram 2.8 is superfluous), since μ and $\mu' \circ f$ are both homogeneous moment maps for (M, ω).

As shown in Proposition 2.2.8, the $\mathbb{R}$-quotient of any symplectic toric cone has a natural contact structure. We will now show that the toric structure on a symplectic toric cone descends to a contact toric structure on the quotient.

Proposition 2.2.20. Let $(M, \omega, \mu : M \to \mathfrak{g}^*)$ be a symplectic toric cone. Then for $\mathbb{R}$-quotient $\pi : M \to B$ and ξ the natural co-oriented contact structure of Proposition 2.2.8, there is an action of G on B with respect to which π is G-equivariant and (B, ξ) is a contact toric manifold.

Proof. First, note that $2\dim(G) = \dim(M)$ and $\dim(B) = \dim(M) - 1$; it follows that G has the right dimension relative to the dimension of B. Next, since the actions of $\mathbb{R}$ and G on M commute, the action of G on M descends to the smooth action on B with $g \cdot \pi(p) := \pi(g \cdot p)$ for any $\pi(p)$ in B.

Now, we show that, for any $b \in B$, the isotropy group G_b of b and the isotropy group G_p of any p with $\pi(p) = b$ match. In particular, this implies that, since the action of G on M was effective, the action of G on B is effective as well. So fix $p \in \pi^{-1}(b)$. It is clear from how the action of G on B was defined that $G_p \leq G_b$. Suppose then that $g \in G_b \backslash G_p$. This implies that $g \cdot p = \lambda \cdot p$ for some $\lambda \in \mathbb{R}$. As the actions of G and $\mathbb{R}$ on M commute, this also implies that, for every positive integer n, $g^n \cdot p = (n\lambda) \cdot p$.

Since G is compact, the orbit $G \cdot p$ must be compact as well. Let $\Phi : \mathbb{R} \times M \to M \times M$ be the action map

$$\Phi : \mathbb{R} \times M \longrightarrow M \times M$$
$$(t, p) \mapsto (t \cdot p, p)$$

By definition, the map Φ must be proper. However, by the argument of the previous paragraph, the projection of the preimage $\Phi^{-1}(G \cdot p \times G \cdot p)$ onto the first factor of the product $\mathbb{R} \times M$ cannot be bounded. Thus, such an element $g \in G_b \backslash G_p$ cannot exist and we must conclude that $G_b = G_p$.

It remains now only to show that the action of G on (B, ξ) is contact and co-orientation preserving. First, note that by Lemma 2.2.9, the Liouville vector field Ξ of (M, ω) must be G-invariant. Since ω is also G-invariant, the natural contact distribution on B descending from M (as defined in equation (2.4)) must be G-invariant as well. Since G is connected, the action diffeomorphism of any element of g on a fiber ξ_b of ξ is isotopic to the action of the identity and therefore must be co-orientation preserving. □

As explained in Proposition 2.2.8, a trivialization of a symplectic cone as a principal $\mathbb{R}$-bundle selects a contact form on the quotient. In the case where the symplectic toric cone in question is the symplectization, global sections of this principal $\mathbb{R}$-bundle are contact forms and so finding a G-equivariant trivialization is the same as finding a G-invariant contact form.

Proposition 2.2.21. Let (B, ξ) be a co-oriented contact manifold with an effective action of torus G. Then there exists a G-invariant contact form α with α serving as a section for $\xi^o_+ \to B$.

Rather than proving the above proposition, we now show we can use an averaging argument to obtain a G-equivariant trivialization for *any* symplectic toric cone.

Proposition 2.2.22. Given a symplectic toric cone $(M, \omega, \mu : M \to \mathfrak{g}^*)$ with $\mathbb{R}$-quotient $\pi : M \to B$ and induced co-oriented contact structure ξ on B, there exists a G-equivariant trivialization $\varphi : B \times \mathbb{R} \to M$ of M as a principal $\mathbb{R}$-bundle (where the contact action on B is trivially extended to an action on $B \times \mathbb{R}$).

Proof. To start with, fix $s : B \to M$ a global section of π (this is always possible, as principal $\mathbb{R}$-bundles must always be trivial). Then the map $d : M \times_B M \to \mathbb{R}$ defined by $p \cdot d(p, p') = p'$ is smooth.

Define the map $f : B \times G \to \mathbb{R}$ by:

$$f(b, g) := d(s(g \cdot b), g \cdot s(b))$$

Essentially, f measures the failure of s to be G-equivariant and will be used to adjust s into an equivariant section (f is well defined as π is G-equivariant).

f satisfies the following useful property: for any $b \in B$ and $g, h \in G$,

$$\begin{aligned} f(h \cdot b, g) &= d(s(g \cdot (h \cdot b)), g \cdot s(h \cdot b)) \\ &= d(s(g \cdot (h \cdot b)), g \cdot h \cdot s(b)) + d(g \cdot h \cdot s(b), g \cdot s(h \cdot b)) \\ &= d(s((gh) \cdot b), (gh) \cdot s(b)) + d(h \cdot s(b), s(h \cdot b)) \\ &= f(b, gh) - f(b, h) \end{aligned} \tag{2.9}$$

Note the second line is equivalent to the third as d is G-invariant (with respect to the diagonal action of G on $M \times_B M$); this follows since the actions of G and $\mathbb{R}$ commute.

Now, we average f. Fix a G-invariant measure $d\lambda$ on G with $\int_G d\lambda = 1$. Define $\bar{f} : B \to \mathbb{R}$ by:

$$b \mapsto \int_G f(b, g)\, d\lambda$$

As $\bar{f}$ is the result of integrating a smooth family of functions on G parameterized by B, it is smooth. Calculation (2.9) shows that $\bar{f}(h \cdot b) = \bar{f}(b) - f(b, h)$.

Finally, define a new global section of M $\bar{s} : B \to M$ by $\bar{s}(b) := s(b) \cdot (-\bar{f}(b))$. Then:

$$\begin{aligned}
\bar{s}(h \cdot b) &= s(h \cdot b) \cdot (-\bar{f}(h \cdot b)) \\
&= (s(h \cdot b) \cdot (f(b, h))) \cdot -\bar{f}(b) \\
&= h \cdot s(b) \cdot -\bar{f}(b) \\
&= h \cdot \bar{s}(b)
\end{aligned}$$

The second and third lines are equivalent as, by definition, $f(b, h)$ is the real number whose action takes $s(h \cdot b)$ to $h \cdot s(b)$. So $\bar{s}$ is an equivariant section. It follows that it defines a G-equivariant trivialization of M as a principal $\mathbb{R}$-bundle; i.e., $\varphi : B \times \mathbb{R} \to M$ with $\varphi(b, t) := \bar{s}(b) \cdot t$. ☐

For certain applications, it will be important to know that the image of a homogeneous moment map of a symplectic toric cone cannot contain $0 \in \mathfrak{g}^*$:

Proposition 2.2.23. Let $(M, \omega, \mu : M \to \mathfrak{g}^*)$ be a symplectic toric cone. Then zero is not in the image of μ.

Proof. Let (B, ξ) be the co-oriented contact toric manifold associated to the symplectic toric cone (M, ω, μ) as described above. As shown in Lemma 2.12 of [21], for ξ_+^o the symplectization of (B, ξ), the image of the homogeneous moment map $\nu : \xi_+^o \to \mathfrak{g}^*$ for ξ_+^o does not contain zero.

A choice of G-equivariant trivialization $\varphi : B \times \mathbb{R} \to M$ induces a choice of G-invariant contact form α on (B, ξ) with $\varphi^*\omega = d(e^t\alpha)$. In turn, this contact form α induces a G-equivariant trivialization of the symplectization $\phi : B \times \mathbb{R} \to \xi_+^o$ so that, for η the symplectic form for ξ_+^o, $\phi^*\eta = d(e^t\alpha)$. Thus, $\varphi \circ \phi^{-1}$ yields a $(G \times \mathbb{R})$-equivariant symplectomorphism and hence a map of symplectic toric cones and, since the image of ν does not contain 0, the image of μ cannot contain 0 as well. ☐

The quotient procedure of Proposition 2.2.20 has two important properties. First, it is natural: maps of symplectic toric manifolds descend to maps of contact toric manifolds. Second, it has an "inverse": as shown in Example 2.2.12, the symplectization of a contact toric manifold is a symplectic toric cone with $\mathbb{R}$-quotient B. This is best stated as the following categorical result:

Theorem 2.2.24. There is an equivalence of categories $\Phi : \mathsf{STC} \to \mathsf{CTM}$.

Proof. As explained in Proposition 2.2.20, the $\mathbb{R}$-quotient $B := M/\mathbb{R}$ of a symplectic toric cone (M, ω, μ) inherits a co-oriented contact structure ξ as well as a G action with respect to which (B, ξ) is a contact toric manifold. So define $\Phi(M, \omega, \mu) := (B, \xi)$.

As shown in Proposition 2.2.8, any map of symplectic toric cones $\varphi : (M, \omega, \mu) \to (M', \omega', \mu')$ descends to a map of contact manifolds $\bar{\varphi} : (B, \xi) \to (B' := M'/\mathbb{R}, \xi')$. It follows easily from how $\bar{\varphi}$ was defined that, since φ is G-equivariant, $\bar{\varphi}$ must be G-equivariant as well. As the same can be said about $\overline{\varphi^{-1}} = \bar{\varphi}^{-1}$, we may conclude that $\bar{\phi}$ is indeed a morphism of CTM. Thus, we may define $\Phi(M, \omega, \mu) := (B, \xi)$ and $\Phi(\varphi) := \bar{\varphi}$.

To see Φ is essentially surjective note that, for any contact toric manifold (B, ξ), the symplectization ξ_+^o is a symplectic toric cone (see Example 2.2.12).

Let (M, ω, μ) and (M', ω', μ') be two symplectic toric cones with $\Phi(M, \omega, \mu) = (B, \xi)$ and $\Phi(M', \omega', \mu') = (B', \xi')$. Then by Proposition 2.2.22, there exist of G-equivariant trivializations $\phi : M \to B \times \mathbb{R}$ and $\phi' : M' \to B' \times \mathbb{R}$ of M and M' as principal $\mathbb{R}$-bundles over B and B'. In turn, these induce a choices of G-invariant contact forms α and α' such that $\phi^*(d(e^t\alpha)) = \omega$ and $\phi'^*(d(e^t\alpha')) = \omega'$. As noted in Proposition 2.2.11, there are unique homogeneous moment maps $\nu : B \times \mathbb{R} \to \mathfrak{g}^*$ and $\nu' : B' \times \mathbb{R} \to \mathfrak{g}^*$ such that $(B \times \mathbb{R}, d(e^t\alpha), \nu)$ and $(B' \times \mathbb{R}, d(e^t\alpha'), \nu')$ are symplectic toric cones. Additionally, as noted in Remark 2.2.19, ϕ and ϕ' are maps of symplectic toric cones. Therefore ϕ and ϕ' induce a bijection

$$F : \hom_{\mathsf{STC}}\left((M, \omega, \mu), (M', \omega', \mu')\right) \to \hom_{\mathsf{STC}}\big((B \times \mathbb{R}, d(e^t\alpha), \nu), (B' \times \mathbb{R}, d(e^t\alpha'), \nu')\big)$$
$$f \longmapsto \phi' \circ f \circ \phi^{-1}$$

Thus, to check that Φ is fully faithful, it is enough to check that Φ is bijective on the sets of morphisms of the form

$$\hom_{\mathsf{STC}}\big((B \times \mathbb{R}, d(e^t\alpha), \nu), (B' \times \mathbb{R}, d(e^t\alpha'), \nu')\big)$$

First, fix $f : (B, \xi) \to (B', \xi')$ a co-orientation preserving G-equivariant contactomorphism. Since α and α' lie in the respective conformal classes determined by the co-orientations of (B, ξ) and (B', ξ'), we must have that $f^*\alpha' = e^g\alpha$ for some G-invariant function $g : B \to \mathbb{R}$. It follows that the map

$$\varphi : B \times \mathbb{R} \to B' \times \mathbb{R} \quad \varphi(b,t) := (f(b), t - g(b)) \tag{2.10}$$

is a G-equivariant map of symplectic cones between $(B \times \mathbb{R}, d(e^t\alpha))$ and $(B' \times \mathbb{R}, d(e^t\alpha'))$ and thus a map of symplectic toric cones between $(B \times \mathbb{R}, d(e^t\alpha), \nu)$ and $(B' \times \mathbb{R}, d(e^t\alpha'), \nu')$. Thus, as $\Phi(\varphi) = f$, Φ is full.

To show Φ is faithful, we will show that φ above is *the only* morphism satisfying $\Phi(\varphi) = f$. Suppose a map of symplectic toric cones $\varphi' : (B \times \mathbb{R}, d(e^t\alpha), \nu) \to (B' \times \mathbb{R}, d(e^t\alpha'), \nu')$ satisfies $\Phi(\varphi') = f$. Then there is a smooth map $\tau : B \to \mathbb{R}$ with $\varphi'(b,t) = (f(b), t + \tau(b))$. We calculate:

$$\varphi'^*(d(e^t\alpha')) = d(e^{t+\tau} f^*\alpha') = d(e^{\tau+g}e^t\alpha)$$

Thus, $\varphi'^* d(e^t\alpha') = d(e^t\alpha)$ exactly when $\tau + g = 0$. Therefore the map φ described in equation (2.10) is the unique map of symplectic toric cones with $\Phi(\varphi) = f$. □

2.2.3 Local properties of symplectic toric cones and contact toric manifolds

We now shift our focus to proving a local isomorphism form for symplectic toric cones. This will be largely based on a local normal form proved in Lerman's paper on contact toric manifolds [21] combined with the relationship connecting symplectic toric cones and contact toric manifolds described above. We will also prove that the quotient of a contact toric manifold is a manifold with corners. First, we define symplectic slice representations for contact toric manifolds.

Definition 2.2.25. Let (B, ξ) be a co-oriented contact toric manifold with G-invariant contact form α. Let $\omega := (d\alpha)|_\xi$. Then for any point $x \in B$, the α-symplectic slice representation at x is the G_x-vector space:

$$(V, \omega_V)_\alpha := \left(\frac{(T_x(G \cdot x) \cap \xi_x)^\omega}{T_x(G \cdot x) \cap \xi_x}, \omega|_V \right)$$

Note that another choice of G-invariant contact form $\alpha' = e^f\alpha$ for (B, ξ) defines the same vector space V with symplectic form $d(e^f\omega)$. Thus, the symplectic vector space $(V, \omega_V)_\alpha$ depends on a choice of contact form.

Remark 2.2.26. This matches the definition of "symplectic slice representation" in Definition 3.8 of [21]. We choose to label this with the contact form α defining this symplectic representation both to designate the fact it depends on the choice of contact form α as well as to avoid confusion with the standard symplectic slice representation of a symplectic action on a symplectic manifold (see Definition 2.1.1). We need to relate the two for our local isomorphism result.

Lemma 2.2.27. Suppose $(M, \omega, \mu : M \to \mathfrak{g}^*)$ is a symplectic toric cone with $\mathbb{R}$-quotient $\pi : M \to B$ to the contact toric manifold (B, ξ). Then for each $p \in M$, there is a G-invariant contact form α for (B, ξ) such that the α-symplectic slice representation $(V, \omega_V)_\alpha$ of $\pi(p)$ in (B, ξ) and the symplectic slice representation (W, ω_W) of p in (M, ω) are symplectically isomorphic as representations of $G_{\pi(p)} = G_p$.

Proof. Fix $p \in M$ and define $b := \pi(p)$. As described in the proof of Proposition 2.2.20, we have that $G_p = G_b$. Let $\varphi : B \times \mathbb{R} \to M$ be a G-equivariant trivialization of M as a principal $\mathbb{R}$-bundle such that $\varphi(b, 0) = p$.

Let Ξ be the Liouville vector field associated to (M, ω). Then $\alpha := (\varphi^*(\iota_\Xi \omega))|_{B \times \{0\}}$ is a contact form and $\varphi : (B \times \mathbb{R}, d(e^t \alpha)) \to (M, \omega)$ is a $(G \times \mathbb{R})$-equivariant map symplectomorphism. We have that

$$d\alpha = d(\varphi^*(\iota_\Xi \omega))|_{B \times \{0\}} = (\varphi^* d(\iota_\Xi \omega))|_{B \times \{0\}} = (\varphi^* \omega)|_{B \times \{0\}}$$

So since φ is equivariant, $d\varphi_{(b,0)}$ restricts to an isomorphism between $T_{(b,0)}(G \cdot (b, 0))$ and $T_p(G \cdot p)$. Thus, for any $w \in T_p(G \cdot p) \subset T_p M$, there exists $w' \in T_{(b,0)}(G \cdot (b, 0))$ with $d\varphi_{(b,0)}(w') = w$. For $v \in (T_b(G \cdot b) \cap \xi)^{d\alpha} \subset T_b B$, this means that

$$\omega_p(d\varphi_{(b,0)}(v, 0), w) = \omega_p(d\varphi_{(b,0)}(v, 0), d\varphi_{(b,0)}(w')) = (\varphi^* \omega)_{(b,0)}((v, 0), w') = 0.$$

Thus, $d\varphi_{(b,0)}$ maps $(T_b(G \cdot b) \cap \xi)^{d\alpha} \oplus \{0\} \subset T_b B \times T_0 \mathbb{R}$ to $(T_p(G \cdot p))^\omega$.

We finish by counting dimensions. Suppose M has dimension $2n$. Since the α-moment map ν_α is never zero (see Proposition 2.2.23), it follows we may find a Lie algebra element Y so that $-\alpha_b(Y_B(b)) \neq 0$ (see equation (2.7)). Thus,

$$\dim(T_b(G \cdot b) \cap \xi_b) = (\dim(G) - \dim(G_b)) - 1 = n - \dim(G_b) - 1.$$

So

$$\dim(T_b(G \cdot b) \cap \xi_b)^{d\alpha}) = 2n - 2 - (n - 1 - \dim(G_b)) = n - 1 + \dim(G_b).$$

Since $Y_M(b)$ maps to $T_p(G \cdot p)$, we have that the $n + \dim(G_b)$ dimensional subspace

$$(T_b(G \cdot b) \cap \xi_b)^{d\alpha} \oplus \langle Y_M(b) \rangle \oplus 0 \subset T_b(G \cdot b) \oplus T\mathbb{R}$$

maps injectively to the $n + \dim(G_b)$ dimensional subspace $(T_p(G \cdot p))^\omega$. Thus, $d\varphi_{(b,0)}$ restricts to an isomorphism between $T_b(G \cdot b) \cap \xi_b)^{d\alpha} \oplus \langle Y_M(b) \rangle$ and $(T_p(G \cdot p))^\omega$. It follows that this isomorphism descends to an isomorphism between the quotients $(T_b(G \cdot b) \cap \xi_b)^{d\alpha})/(T_b(G \cdot b) \cap \xi)$ and $(T_p(G \cdot p))^\omega/(T_p(G \cdot p))$. □

We now recall the following local isomorphism result for contact toric manifolds from [21] which we will shortly utilize.

Lemma 2.2.28. Let (B, ξ) and (B', ξ') be two co-oriented contact toric manifolds with G-invariant contact forms $\ker \alpha = \xi$ and $\ker \alpha' = \xi'$. Suppose $x \in B$ and $x' \in B'$ satisfy

- $\mu_\alpha(x) = \lambda \mu_{\alpha'}(x')$ for μ_α, $\mu_{\alpha'}$ the moment maps for α, α' and $\lambda > 0$;
- $G_x = G_{x'}$ (i.e., the isotropy groups are equal); and
- For $(V, \omega)_\alpha$ and $(V', \omega_{V'})_{\alpha'}$ the α, α' symplectic slice representations for x, x', there is an G_x-equivariant linear isomorphism $l : V \to V'$ such that $l^*\omega_{V'} = (d(e^g\alpha)_x)|_V$ for some function $g \in C^\infty(B)$

Then there are G-invariant open neighborhoods U of x and U' of x' and a G-equivariant diffeomorphism $\varphi : U \to U'$ satisfying $\varphi(x) = x'$ and $\varphi^*\alpha' = f\alpha$ for some $f \in C^\infty(U)$.

Proof. See Lemma 3.9, [21]. □

We may extend this to a local isomorphism result for symplectic toric cones.

Proposition 2.2.29. Let $(M, \omega, \mu : M \to \mathfrak{g}^*)$ and $(M', \omega', \mu' : M' \to \mathfrak{g}^*)$ be two symplectic toric cones. Suppose there are two points $p \in M$ and $p' \in M'$ so that:

- $G_p = G_{p'}$;
- The symplectic slice representations (V, ω_V) and $(V', \omega_{V'})$ at p and p' are isomorphic as symplectic $G_p = G_{p'}$ vector spaces; and
- $\mu(p) = \mu'(p')$.

Then there exist $(G \times \mathbb{R})$-invariant neighborhoods U and U' of p and p' respectively and a $(G \times \mathbb{R})$-equivariant symplectomorphism $f : U \to U'$ with $f(p) = p'$ and $\mu' \circ f = \mu|_U$.

Proof. Let (B, ξ) and (B', ξ') be the contact toric bases of (M, ω) and (M', ω'). Denote the $\mathbb{R}$-quotient maps by $\pi : M \to B$ and $\pi' : M' \to B'$ and define $b := \pi(p)$ and $b' := \pi'(p')$. Then by Lemma 2.2.27, there exist G-equivariant trivializations $\varphi : B \times \mathbb{R} \to M$ and $\varphi' : B' \times \mathbb{R} \to M'$ so that $\varphi(b, 0) = p$, $\varphi'(b', 0) = p'$, and, for

$\varphi^*\omega = d(e^t\alpha)$ and ${\varphi'}^*\omega' = d(e^t\alpha')$, the symplectic slice representations (V, ω_V) and $(V', \omega_{V'})$ are isomorphic to the α and α' symplectic slice representations $(W, \omega_W)_\alpha$ and $(W', \omega_{W'})_{\alpha'}$ of b and b', respectively.

By Lemma 2.2.28 there are G-invariant neighborhoods U and U' of b and b' and a G-equivariant co-orientation preserving contactomorphism $\phi : U \to U'$ with $\phi(b) = b'$ and $\phi^*\alpha' = e^g\alpha$, for some $g \in C^\infty(B)$. The map $\tilde{\phi} : U \times \mathbb{R} \to U' \times \mathbb{R}$, defined by $\tilde{\phi}(b, t) := (\phi(b), t - g(b))$ is $(G \times \mathbb{R})$-equivariant and satisfies $\tilde{\phi}^*(d(e^t\alpha')) = d(e^t\alpha)$. Hence, $f := \varphi' \circ \tilde{\phi} \circ \varphi^{-1}$ yields a map of symplectic toric cones between $\pi^{-1}(U)$ and ${\pi'}^{-1}(U')$. Since $\mu|_U$ and $\mu' \circ f$ are both homogeneous moment maps for $\pi^{-1}(U)$, it follows that $\mu|_U = \mu' \circ f$.

Finally, note that

$$f(p) = \varphi'(\tilde{\phi}(\varphi^{-1}(p))) = \varphi'(\tilde{\phi}(b, 0)) = \varphi'(\phi(b), -g(b)) = \varphi'(b', -g(b)).$$

Thus, $f(p) = -g(b) \cdot p'$. But then

$$\mu(p) = \mu'(f(p)) = \mu'(-g(b) \cdot p') = e^{-g(b)}\mu'(p'),$$

so, since $\mu(p) = \mu'(p')$, we must conclude that $g(b) = 0$ (as the image of μ' may not contain 0: see Proposition 2.2.23) and therefore $f(p) = p'$. □

To finish the section, we prove that the quotients of contact toric manifolds are manifolds with corners; this will be vital in Chapter 4. We first quote a local normal form for contact toric manifolds.

Lemma 2.2.30. Let (L, ξ) be a contact toric manifold with G-invariant form α and α-moment map Ψ_α. Given point $p \in L$, denote the α-symplectic slice representation by $G_p \to Sp(V, \omega_V)$ and let $\mathfrak{k} := (\mathbb{R}\Psi_\alpha(p))^o$. Then there exists a G-invariant neighborhood of the orbit of p in L that is G-equivariantly diffeomorphic to a neighborhood of the zero section of the vector bundle $N = G \times_{G_p} ((\mathfrak{g}/\mathfrak{k})^* \oplus V)$

Proof. See Lemma 3.10, [21]. □

Lemma 2.2.31. Let (B, ξ) be a contact toric manifold. Then the quotient B/G is a manifold with corners.

Proof. For each point $p \in B$, the stabilizer G_p is a torus (see Lemma 3.13 of [21]). Let α be a G-invariant contact form for (B, ξ). By Lemma 2.2.30, there exists a subspace $U \subset \mathfrak{g}^*$ so that, for α-symplectic slice representation $G_p \to Sp(V, \omega_V)$, there is a G-invariant neighborhood of $G \cdot p$ in B that is G-equivariantly diffeomorphic to a neighborhood of the zero section of the vector bundle: $N = G \times_{G_p} (U \times V)$ (where U has trivial G_p action).

So to understand what B/G locally looks like, it is enough to understand N/G:

$$\begin{aligned} N/G &= ((G \times U \times V)/G_p)/G \\ &\cong ((G \times U \times V)/G)/G_p \\ &\cong U \times V/G_p \end{aligned}$$

Here, we may reverse the quotients as the actions of G and G_p commute. Since G_p is a torus, we can decompose V into weight spaces (as in the appendix of [23]) and easily see V/G_p is a manifold with corners diffeomorphic to a sector (i.e., a manifold with corners of the form $[0,\infty)^k \times \mathbb{R}^l$). So the G-equivariant diffeomorphism above descends to a manifold with corners chart for $[p] \in B/G$ centered at the origin in $U \times V/G_p$. □

2.3 Symplectic reduction

In this section, we'll very briefly discuss the main motivation for symplectic stratified spaces: symplectic reduction. There are many different definitions and versions of stratified spaces in mathematics. We present an inductive definition of a stratified space that will be most convenient for us (see, for instance, [24]).

First, as stratified spaces are a particular form of partitioned space, we define a partitioned space and maps of partitioned spaces.

Definition 2.3.1 (Partitioned spaces, maps/isomorphisms of partitioned spaces)**.** Let X be a Hausdorff topological space. Then X, together with a partition $X = \bigsqcup_{\alpha \in A} X_\alpha$ is called a partitioned space if each X_α is a manifold (with respect to the subset topology of $X_\alpha \subset X$).

A map of partitioned spaces between partitioned spaces $(X, \{X_\alpha\}_{\alpha\in A})$ and $(Y, \{Y_\alpha\}_{\beta\in B})$ is a continuous map $f : X \to Y$ such that, for every $\alpha \in A$, there is a $\beta \in B$ with

- $f(X_\alpha) \subset Y_\beta$
- $f|_{X_\alpha} : X_\alpha \to Y_\beta$ is smooth.

f is an isomorphism of partitioned spaces if there is another map of partitioned spaces g from $(Y, \{Y_\alpha\}_{\beta\in B})$ to $(X, \{X_\alpha\}_{\alpha\in A})$ so that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Any manifold is, of course, a partitioned space with respect to the trivial partition. We can also build a partitioned space from any other space with a cone construction:

Definition 2.3.2. Let $[-\infty, \infty)$ be the set $\mathbb{R} \sqcup \{-\infty\}$ with topology generated by sets of the form $(-\infty, a) \sqcup \{-\infty\}$ and the open subsets of $\mathbb{R}$. Then for a partitioned space $(X, \{X_\alpha\}_{\alpha \in A})$, the open cone on X, denoted $c(X)$, is the topological quotient

$$c(X) := \frac{X \times [-\infty, \infty)}{X \times \{-\infty\}}$$

with partition $\{\{X_\alpha \times \mathbb{R}\}_{\alpha \in A}, \{-\infty\}\}$.

Stratified spaces are partitioned spaces that are locally cones of smaller partitioned spaces:

Definition 2.3.3. A Hausdorff paracompact partitioned space $(X, \{X_\alpha\}_{\alpha \in A})$ is a stratified space if:

- $\sup_{\alpha \in A} \dim(X_\alpha) < \infty$; and
- For every α and for every $x \in X_\alpha$, there is an open neighborhood U of x in X, an open ball $B \subset X_\alpha$, a stratified space L, called the link of x, and an isomorphism of partitioned spaces

$$\varphi : U \to B \times c(L).$$

Note that the first condition of the above definition is necessary to ensure the recursive second condition terminates.

Remark 2.3.4. Definition 2.3.3 presents a particularly nice form of stratified space known as a locally trivial stratified space. It matches Definition 2.7 of [24] save one important distinction: we *do not* require that the links of a stratum are compact. Indeed, the links of stratified spaces are generally assumed to be compact and any Whitney stratified space has compact links (see Theorem 3.9.3 of [31]).

While it is slightly easier to work with stratified spaces with compact links, we can (and do) prove the classification of symplectic toric manifolds with isolated singularities of Chapter 4 without this assumption. As we will eventually see in Proposition 4.5.13, in high enough dimensions, the classes of stratified spaces with compact links are essentially the same as the classes of manifolds with symplectic toric manifolds.

Examples 2.3.5. Many objects in differential geometry take the form of stratified spaces. For instance:

- Any manifold is a stratified space with the trivial partition is a stratified space; here, each link is just the empty set.
- A manifold with boundary M is a stratified space for partition $M = \mathring{M} \sqcup \partial M$. The link of a point on the boundary is just a one point set.

- More generally, any manifold with corners N is a stratified space. Here, N is partitioned into its interior and (the connected components of) each piece of the boundary with constant codimension. The link of a point in the codimension k boundary of N is the $n-k-1$ dimensional simplex.
- Any manifold with a proper action of a Lie group K is a stratified space with respect to its partition into orbit types (see [10]). Here, for a subgroup H of K, the subset $M_{(H)}$ of points of orbit type H is the subset of points with stabilizer conjugate to H; that is

$$M_{(H)} := \{p \in M \mid K_p = kHk^{-1} \text{ for some } k \in K\} \tag{2.11}$$

We now recall (but do not prove) the construction of symplectic reduction, introduced by Marsden and Weinstein [26] and independently by Meyer [27]:

Theorem 2.3.6. Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Suppose K acts on the symplectic manifold (M, ω) so that $(M, \omega, \mu : M \to \mathfrak{k}^*)$ is a Hamiltonian manifold. Finally, suppose 0 is a regular value for the moment map $\mu : M \to \mathfrak{k}^*$ and that K acts freely on the level set $\mu^{-1}(0)$. Then the manifold $M//_0K := \mu^{-1}(0)$, known as the symplectic reduction or symplectic quotient of (M, ω, μ), inherits a symplectic form ω_0 such that, for K-quotient map $\pi : \mu^{-1}(0) \to M//_0K$, $\pi^*\omega_0 = \omega|_{\mu^{-1}(0)}$.

The conditions of this theorem may be tweaked by changing the μ level set to another regular value and/or considering level sets on which the action of K is only locally free, which may result in a symplectic orbifold. However, in the most general case, one must rely on stratified spaces to describe the structure of the symplectic quotient. Using Proposition 2.1.3 as a jumping off point, Sjamaar and Lerman [32] proved that:

Theorem 2.3.7. Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Suppose K acts on the symplectic manifold (M, ω) so that $(M, \omega, \mu : M \to \mathfrak{k}^*)$ is a Hamiltonian manifold. Finally, for every subgroup $H < K$, let $Z_{(H)}$ denote the intersection $\mu^{-1}(0) \cap M_{(K)}$ (see equation (2.11)). Then:

1. The topological quotient $M//_0K := \mu^{-1}(0)$ is a stratified space with partition $\{Z_{(H)}/K\}_{H<K}$.
2. Each stratum $Z_{(H)}/K$ inherits a symplectic form $\omega_{(H)}$ such that $\pi^*\omega_{(H)} = \omega|_{Z_{(H)}}$, for π the quotient map $\pi : \mu^{-1}(0) \to M//_0K$.
3. For a point $p \in \mu^{-1}(0)$ with stabilizer H and symplectic slice representation (V, ω_V) with moment map $\Phi_V : V \to \mathfrak{h}^*$ (as described in equation (2.6)), there is a neighborhood of $[p]$ in $M//_0K$ isomorphic to

a neighborhood of $\bar{0}$ in $\Phi_V^{-1}(0)/H$. Furthermore, this isomorphism restricts to a symplectomorphism on each stratum.

4. For p, (V, ω_V), and Φ_V as above, the subspace V_H of vectors fixed by H is a symplectic subspace of V. For W the symplectic perpendicular to V_H in V, denote by Φ_W the restriction of the moment map to W. Then $\Phi_V^{-1}(0)/H$ is isomorphic to $\Phi_W^{-1}(0)/H \times V_H$ restricting to a symplectomorphism on each stratum.

Proof. This theorem is a summary of a number of pertinent results of Sjamaar and Lerman in [32]. See also Lerman and Willett [24] for a proof of Item 1 that directly matches our inductive definition of a stratified space. □

Sjamaar and Lerman also showed that the strata of the reduced space inherit a particular local normal form. As the general statement is difficult and unnecessary for the purposes of this paper, we instead now provide the local normal form for symplectic reductions resulting in stratified spaces with isolated singularities:

Proposition 2.3.8. Let (M, ω) be a symplectic manifold on which a compact Lie group K with Lie algebra $\mathfrak{k}$ acts via symplectomorphism. Suppose $\mu : K \to \mathfrak{k}^*$ is a moment map for this action such that K acts freely $\mu^{-1}(0)$ except for an isolated collection of points $\{p_\alpha\}_{\alpha \in A}$ which K fixes. Then for each singularity $[p_\alpha]$ of the quotient $M//_0K$, there is:

- a compact contact manifold L with contact form β;
- a neighborhood U of $[p_\alpha]$ in $M//_0K$; and
- an open embedding $\varphi : U \to c(L)$ with φ mapping $[p_\alpha]$ to the vertex of $c(L)$

so that φ restricts to a symplectomorphism between $U\backslash\{[p_\alpha]\}$ and its image in $L \times \mathbb{R}$ with symplectic form $d(e^t\alpha)$.

To prove this result, we need the concept of contact reduction:

Proposition 2.3.9. Suppose a compact Lie group K acts on a co-orientable contact manifold (M, ξ). Let α be a K-invariant contact form for ξ with α-moment map $\mu_\alpha : M \to \mathfrak{k}^*$ (see Definition 2.2.15). If 0 is a regular value of μ and K acts freely on $\mu^{-1}(0)$, then the quotient $M//_0K := \mu_\alpha^{-1}(0)/K$ has an induced contact structure $\bar{\xi}$ and, for quotient map $\pi : \mu_\alpha^{-1}(0) \to \mu_\alpha^{-1}(0)/K$, there is a contact form $\bar{\alpha}$ for $\bar{\xi}$ satisfying $\pi^*\bar{\alpha} = \alpha|_{\mu^{-1}(0)}$.

Proof. See Theorem 6 of [11]. □

Proof of Proposition 2.3.8. Let (V, ω_V) be the symplectic slice representation of p_α in (M, ω). By assumption, p_α is fixed and so (V, ω_V) is a symplectic K representation. Let $\Phi_V : V \to \mathfrak{k}^*$ be the moment map given in equation (2.1).

By Theorem 2.3.7, it is enough to consider the quotient $\Phi_V^{-1}(0)/K$. As we assume that K acts freely on $\mu^{-1}(0)$ except at the singularities, we have that the linear symplectic action of K on $\Phi_V^{-1}(0)$ must be free except at $0 \in \mathfrak{k}^*$.

Now, fix a K-invariant norm $||\cdot||$ on V and let S be the unit sphere on V defined with respect to this norm. It follows that the action of K on V restricts to an action on S. For R the radial vector field on V, it is not difficult to show that the contraction $\iota_R\omega_V$ restricts to a K-invariant contact form on S. Define $\alpha := \frac{1}{2}\iota_R\omega_V$. Then the map

$$f : S \times \mathbb{R} \to V\backslash\{0\}, \quad (v, t) \mapsto e^{t/2}v$$

satisfies $f^*\omega_V = d(e^t\alpha)$. Since the action of K on S is the restricted action from V and the action on V is linear, it follows that the map f is K-equivariant. As f has smooth inverse

$$v \mapsto (v/||v||, \log(||v||^2)),$$

we therefore have that f is a K-equivariant symplectomorphism.

If $\mu_\alpha : S \to \mathfrak{k}^*$ is the α-moment map of (S, α), then f intertwines Φ_V with the moment map $(v, t) \mapsto e^t\mu_\alpha$ on $(S \times \mathbb{R}, d(e^t\alpha))$. Finally, note that as f is K-equivariant, we must have that K acts freely on $\mu_\alpha^{-1}(0) \subset S$. Calculation (2.2) reveals that, for nonzero $v \in V$, $(d\Phi_V)_v$ is surjective. Thus, for $v \in S \cap \mu_V^{-1}(0)$, it follows that $(d\Phi_V|_S)_v$ is surjective (as the radial direction contributes nothing here) and so 0 is a regular value of $\mu_\alpha^{-1}(0)$.

It is now easy to check that the reduction $(S \times \mathbb{R})//_0K$ is isomorphic to the symplectic cone $(S//_0K \times \mathbb{R}, d(e^t\bar{\alpha}))$ (for $\bar{\alpha}$ the contact form of Proposition 2.3.9). It follows that $f : S \times \mathbb{R} \to V\backslash\{0\}$ descends to an isomorphism

$$\bar{f} : (S//_0K \times \mathbb{R}, d(e^t\bar{\alpha})) \to (\Phi_V^{-1}(0)\backslash\{0\})/K.$$

Thus, $\Phi_V^{-1}(0)/K$ is the space $c(L)$ required in the statement of the proposition. □

Remark 2.3.10. Again, let (M, ω) be a symplectic manifold with Hamiltonian action of a compact Lie group K and moment map $\mu : M \to \mathfrak{k}^*$. Sjamaar and Lerman observe that the symplectic quotient $X := M//_0K$

inherits a ring of "smooth" functions $C^\infty(X)$; namely, the functions on X that descend from smooth K-invariant functions on M. Additionally, $C^\infty(X)$ inherits a Poisson bracket $\{\cdot,\cdot\}_X$ that is coherent with the symplectic structure on each stratum of $M//_0K$ (See Section 3 of [32]).

Sjamaar and Lerman thereby define symplectic stratified spaces as stratified spaces with a symplectic form on each stratum together with a Poisson algebra of smooth functions coherent with the symplectic strata. Burns, Guillemin, and Lerman consequently use the term *singular symplectic space* to describe a stratified space with symplectic forms on each stratum but without a global Poisson algebra of "smooth" functions.

However, we will still refer to stratified spaces with a symplectic structure on each stratum with an extra local conical condition (as in Proposition 2.3.8) as symplectic stratified spaces. The choice of a Poisson algebra of smooth functions on such a stratified space is, in particular, a choice of smooth functions for the stratified space. We view this choice of smooth functions as *an additional* piece of data associated to a symplectic stratified space (rather than a requirement).

Chapter 3

A classification of symplectic toric cones

The goal of this chapter is to classify symplectic toric cones. In Section 3.1, we define and describe *homogeneous unimodular local embeddings*, the orbital moment maps of symplectic toric cones. Since any symplectic toric cone (M, ω, μ) is, in particular, a symplectic toric manifold, it follows, as in [19], that the G-quotient M/G is a manifold with corners and the orbital moment map $\bar{\mu} : M/G \to \mathfrak{g}^*$ is a unimodular local embedding (see Definition 2.1.5). As a consequence of μ being homogeneous, we may conclude that $\bar{\mu}$ satisfies two additional properties:

1. the quotient M/G inherits a free and proper $\mathbb{R}$ action; and

2. with respect to this action, $\bar{\mu}$ is itself homogeneous

A unimodular local embedding $\psi : W \to \mathfrak{g}^*$, we call ψ a homogeneous unimodular local embedding if it satisfies these two additional properties.

As in the case of symplectic toric manifolds, it makes sense to group together symplectic toric cones by orbital moment map: for any homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$, we define *the category of symplectic toric cones over* ψ, denoted $\mathsf{STC}_\psi(W)$, as the groupoid with objects symplectic toric cones admitting a G-quotient map to W for which ψ is the orbital moment map and with morphisms symplectomorphisms preserving these quotients that are both G and $\mathbb{R}$-equivariant. It is important to note that we must prove this groupoid is non-empty.

For any homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$ and for any $\mathbb{R}$-invariant open subset U of W, $\psi|_U$ is a homogeneous unimodular local embedding as well. It follows that, for $\mathsf{Open}_{\mathbb{R}}(W)$ the category of $\mathbb{R}$-invariant open subsets of W, we may form a presheaf of groupoids

$$\mathsf{STC}_\psi : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$$

In Section 3.2, we define *homogeneous symplectic toric bundles over* ψ for any homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$. These are pairs $(\pi : P \to W, \omega)$ for $\pi : P \to W$ a principal G-bundle over the manifold with corners W and ω a G-invariant symplectic form on P with moment map $\psi \circ \pi$. Additionally, P comes with an $\mathbb{R}$ action making $(P, \omega, \psi \circ \pi)$ a symplectic toric cone. Taking a map of homogeneous symplectic toric bundles over ψ to be any isomorphism of principal G-bundles over W that is both a symplectomorphism and $\mathbb{R}$-equivariant, we may then define *the category of homogeneous symplectic toric bundles over* ψ, denoted $\mathsf{HSTB}_\psi(W)$.

As in the case of symplectic toric cones, homogeneous symplectic toric bundles also define a presheaf

$$\mathsf{HSTB}_\psi : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$$

We also describe in this section some of the important properties of homogeneous symplectic toric bundles. Of particular note is Proposition 3.2.6, in which we show that every principal G-bundle $\pi : P \to W$ with an $\mathbb{R}$ action for which π is $\mathbb{R}$-equivariant admits a G-invariant symplectic form ω with respect to which (P, ω) is a homogeneous symplectic toric bundle. In Proposition 3.2.10, we show that any two homogeneous symplectic toric bundles over the same homogeneous unimodular local embedding are isomorphic exactly when they have the same structure as a principal G-bundle with free $\mathbb{R}$ action.

In Section 3.3, we define the map of presheaves $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$. In essence, this is a version of the map c of Karshon and Lerman taking symplectic toric bundles to symplectic toric manifolds that remembers the $\mathbb{R}$ action of a homogeneous symplectic toric bundle. In showing that the category $\mathsf{HSTB}_\psi(W)$ is non-empty for any homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$, this functor allows us to conclude that the groupoid $\mathsf{STC}_\psi(W)$ must be non-empty as well. We show in Theorem 3.3.7 that hc is an isomorphism of presheaves over $\mathsf{Open}_{\mathbb{R}}(W)$. With this in mind, we may focus on identifying the isomorphism classes of homogeneous symplectic toric bundles.

In Section 3.4, we provide characteristic classes for symplectic toric cones. This is done via Proposition 3.4.5, which shows that, for every homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$, the isomorphism classes of $\mathsf{STC}_\psi(W)$ are in bijective correspondence with the isomorphism classes of $BG(W)$ (the groupoid of principal G-bundles over W). We therefore conclude that these bundles admit characteristic classes of the form $H^2(W; \mathbb{Z}_G)$, for $\mathbb{Z}_G$ the integral lattice $\ker(\exp : \mathfrak{g} \to G) \subset \mathfrak{g}$ (this is the content of Proposition 3.4.8). Finally, we are able to use the isomorphism of presheaves hc to conclude:

Theorem B. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then:

1. The set of symplectic toric cones with orbital moment map ψ is non-empty.

2. The set of isomorphism classes of symplectic toric cones (M, ω, μ) with G-quotient $\pi : M \to W$ and orbital moment map ψ is in natural bijective correspondence with the cohomology classes $H^2(W; \mathbb{Z}_G)$, where $\mathbb{Z}_G$ is the integral lattice of G, the kernel of the map $\exp : \mathfrak{g} \to G$.

For all intents and purposes, part of Theorem B is already known. Indeed, in [21], Lerman showed that, given an orbital moment map $\bar{\mu} : B/G \to \mathfrak{g}^*$ of a contact toric manifold (B, ξ), the isomorphism classes of contact toric manifolds with orbital moment map $\bar{\mu}$ are in bijective correspondence with the cohomology classes $H^2(B/G; \mathbb{R})$ (see Theorem 2.2.17). Via the equivalence of categories between symplectic toric cones and contact toric manifolds (see Theorem 2.2.24), it is relatively straight forward to check that, in the case we know that $\psi : W \to \mathfrak{g}^*$ is an orbital moment map of a symplectic toric cone $(B/G \times \mathbb{R}, \omega)$, then Lerman's bijection extends to item 2.

However, a requirement of Lerman's approach was the knowledge a particular map $\bar{\mu} : B/G \to \mathfrak{g}^*$ was an orbital moment map. So a new feature of our classification is the complete characterization of all orbital moment maps of symplectic toric cones as homogeneous unimodular local embeddings (as given by item 1 of Theorem B).

3.1 Homogeneous unimodular local embeddings

In this section, we define homogeneous unimodular local embeddings, the orbital moment maps of symplectic toric cones. We also discuss some of their properties that will be used later in the chapter.

Definition 3.1.1. A unimodular local embedding $\psi : W \to \mathfrak{g}^*$ (see Definition 2.1.5) with a free and proper action of $\mathbb{R}$ on W is a homogeneous unimodular local embedding if $\psi(t \cdot w) = e^t \psi(w)$ for every $t \in \mathbb{R}$ and $w \in W$.

To begin, we will show that the orbital moment map of a symplectic toric cone must be a homogeneous unimodular local embedding. To accomplish this, we will need the following technical lemma:

Lemma 3.1.2. Let H and K be Lie groups and let K be compact. Let X is a Hausdorff topological space on which H and K have commuting actions and let $\pi : X \to X/K$ be the quotient. Then the action of H descends to a continuous action on X/K. Furthermore, this action of H on X/K is proper if and only if the action of H on X is proper.

Proof. That the action of H descends to an action on X/K is simply a consequence of the fact that the actions of H and K on X commute. Then we have the following commutative diagram:

$$\begin{array}{ccc} H \times X & \xrightarrow{\Phi} & X \times X \\ {\scriptstyle (id,\pi)}\downarrow & & \downarrow{\scriptstyle (\pi,\pi)} \\ H \times X/K & \xrightarrow[\bar{\Phi}]{} & X/K \times X/K \end{array}$$

where Φ and $\bar{\Phi}$ are the action maps $\Phi(h,x) := (h \cdot x, x)$ and $\bar{\Phi}(h, \cdot[x]) := (h \cdot [x], [x])$.

First, assume the action of H on X/K is proper. Let C be a compact subset of $X \times X$. We then have that $(\pi,\pi)(C)$ is a compact subset of $X/K \times X/K$. Then, since the action of H on X/K is proper, $\bar{\Phi}^{-1}(\pi(C))$ is a compact subspace of $H \times X/K$. Since π is proper (as shown in Theorem 3.1, pp. 38 of [5]), it follows that $(id,\pi)^{-1}(\bar{\Phi}^{-1}(\pi(C)))$ is a compact subset of $H \times X$.

Finally, note that by the commutativity of the above diagram, $\Phi^{-1}(C) \subset (id,\pi)^{-1}(\bar{\Phi}^{-1}(\pi(C)))$. Then since $X \times X$ is Hausdorff, C is closed and therefore $\Phi^{-1}(C)$ is closed as well. As a closed subset of a compact set, we may conclude that $\Phi^{-1}(C)$ is compact in Hausdorff $H \times X$.

Now, assume the action of H on X is proper. Suppose now C is a compact subset of $X/K \times X/K$. Then there is a compact subset C' of X/K such that $C \subset C' \times C'$. Again, π is proper, so

$$(\pi,\pi)^{-1}(C' \times C') = \pi^{-1}(C') \times \pi^{-1}(C') \subset X \times X$$

is compact. Thus, $\Phi^{-1}(C' \times C')$ is also compact, as is $(id,\pi)(\Phi^{-1}(C' \times C')) = \bar{\Phi}^{-1}(C' \times C')$. As K is compact, the quotient X/K is Hausdorff (again, see Theorem 3.1, pp. 38 of [5]), so the closed subset $\bar{\Phi}^{-1}(C)$ of $\bar{\Phi}^{-1}(C' \times C')$ is compact. □

Proposition 3.1.3. Let $(M, \omega, \mu : M \to \mathfrak{g}^*)$ be a symplectic toric cone. Then for G-quotient $\pi : M \to M/G$ of M, the orbital moment map $\bar{\mu} : M/G \to \mathfrak{g}^*$ is a homogeneous unimodular local embedding.

Proof. As explained by Karshon and Lerman in [19], M/G is a manifold with corners and $\bar{\mu} : M/G \to \mathfrak{g}^*$ is a unimodular local embedding (see Proposition 2.1.8). Note that, since the actions of $\mathbb{R}$ and G on M commute, the action of $\mathbb{R}$ descends to a free smooth action on M/G. By Lemma 3.1.2, this action of $\mathbb{R}$ on M/G must also be proper. Finally, as μ is homogeneous, $\bar{\mu}$ must be homogeneous as well. □

By definition, every point in the domain of a unimodular local embedding ψ is contained in an open neighborhood on which ψ is an open embedding onto its image. If ψ is a homogeneous unimodular local embedding, then we may choose this neighborhood to be $\mathbb{R}$-invariant.

Lemma 3.1.4. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then for every point $w \in W$, there is an open $\mathbb{R}$-invariant neighborhood U of w such that $\psi|_U$ is an open embedding onto its image.

Proof. Fix a norm on $\mathfrak{g}^*$ with unit sphere $S(\mathfrak{g}^*)$. Note first that, since ψ is homogeneous, the image of ψ is closed under the radial action of $\mathbb{R}$ on $\mathfrak{g}^*$. Thus, there exists $w' \in W$ such that $\psi(w') \in S(\mathfrak{g}^*)$ and $t \cdot w = w'$. Let V be an open subset of W containing w' for which $\psi|_V$ is an open embedding onto its image (again, such a subset exists because ψ is in particular a unimodular local embedding; see Definition 2.1.5).

Now note that $V' := \psi^{-1}(S(\mathfrak{g}^*)) \cap V$ is a slice for the action of $\mathbb{R}$ on W; that is, each orbit of the action intersects V' exactly once and $\mathbb{R} \cdot V' = \psi^{-1}(\mathbb{R} \cdot \psi(V'))$ is an open subset of W. Define $U := \mathbb{R} \cdot V'$. As ψ is homogeneous, $\psi|_U$ is injective. Therefore, $\psi|_U$ is an open embedding onto its image. □

We now prove that the radial action of $\mathbb{R}$ on $\mathfrak{g}^*$ uniquely determines an $\mathbb{R}$ action on the domain of a homogeneous unimodular local embedding.

Lemma 3.1.5. Let $\psi : W \to \mathfrak{g}^*$ be a unimodular local embedding. Then there is at most one $\mathbb{R}$ action on W with respect to which ψ is a homogeneous unimodular local embedding.

Proof. Suppose there is an action of $\mathbb{R}$ with respect to which ψ is a homogeneous unimodular local embedding. Then by Lemma 3.1.4, for any $w \in W$, there is an $\mathbb{R}$-invariant open neighborhood U of w such that $\psi|_U$ is an open embedding onto its image. Thus, any $\mathbb{R}$ action on W for which ψ is homogeneous must satisfy $t \cdot w = \psi|_U^{-1}(e^t \psi(w))$ and, since $\psi|_U$ is injective, this uniquely determines $t \cdot w$. □

An interesting consequence of Lemma 3.1.5 is the following proposition which we will use later as a shortcut: if a symplectic toric manifold with an $\mathbb{R}$ action has all the ingredients of a symplectic toric cone without the properness of the free $\mathbb{R}$ action, then the properness comes for free.

Proposition 3.1.6. Let (M, ω, μ) be a symplectic toric manifold. Suppose further that M has a free $\mathbb{R}$ action commuting with the action of G such that

- the orbital moment map $\bar{\mu} : M/G \to \mathfrak{g}^*$ is a homogeneous unimodular local embedding (with respect to the $\mathbb{R}$ action descending from M to M/G); and
- For each $\lambda \in \mathbb{R}$ with action diffeomorphism $\rho_\lambda : M \to M$, $\rho_\lambda^* \omega = e^\lambda \omega$.

Then (M, ω, μ) is a symplectic toric cone.

Proof. Since $\bar{\mu}$ is a homogeneous unimodular local embedding with respect to the action of $\mathbb{R}$ on M/G, the action of $\mathbb{R}$ on W is proper. By Lemma 3.1.5, since ψ is also homogeneous with respect to the action of $\mathbb{R}$ descending from M to M/G, π is $\mathbb{R}$-equivariant. Therefore, by Lemma 3.1.2, the $\mathbb{R}$ action on M is proper and so (M, ω, μ) is a symplectic toric cone. □

We now group symplectic toric cones together by orbital moment map to define the category of symplectic toric cones over a homogeneous unimodular local embedding ψ.

Definition 3.1.7. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then a symplectic toric cone over ψ is a symplectic toric cone (M, ω, μ) together with a G-quotient $\pi : M \to W$ so that $\mu = \psi \circ \pi$. This data is represented by the triple $(M, \omega, \pi : M \to W)$.

Denote by $\mathsf{STC}_\psi(W)$ the category of symplectic toric cones over ψ, the groupoid with

- objects: symplectic toric cones over ψ; and
- morphisms: $(G \times \mathbb{R})$-equivariant symplectomorphisms

$$\varphi : (M, \omega, \pi : M \to W) \to (M', \omega', \pi' : M' \to W)$$

  satisfying $\pi' \circ \varphi = \pi$.

We will soon prove that, for $\psi : W \to \mathfrak{g}^*$ a homogeneous unimodular local embedding, the quotient $\pi : W \to W/\mathbb{R}$ is a principal $\mathbb{R}$-bundle of manifolds with corners (see Definition A.10). As the action of $\mathbb{R}$ on W is free and proper, this is certainly a *topological* principal $\mathbb{R}$-bundle. However, we must take care to prove that $W/\mathbb{R}$ is a manifold with corners and that π admits *smooth* trivialization data. It is likely that one may use a version of the Slice Theorem for proper actions of manifolds with corners to conclude in general there are slices of the $\mathbb{R}$ action (the compact group action case is discussed in [1]). However, in Section 3.3, we will need to use a very particular type of slice for the $\mathbb{R}$ action on W; thus, we build these slices now and as a consequence conclude that $\pi : W \to W/\mathbb{R}$ is a principal $\mathbb{R}$-bundle of manifolds with corners.

Lemma 3.1.8. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Fix a point $w \in W$ and let $C = C_{\{v_1,\dots,v_k\},\psi(w)}$ be the unimodular cone into which ψ embeds near w. Let K be the subtorus with Lie algebra $\mathfrak{k}$ spanned by $\{v_1, \dots, v_k\}$. Finally, let C' be the unimodular cone $C_{\{v_1,\dots,v_k\},0} \subset \mathfrak{k}^*$. Then there is an open $\mathbb{R}$-invariant neighborhood U_w of w such that:

1. $\psi|_{U_w} : U_w \to \mathfrak{g}^*$ is an open embedding onto a neighborhood of $\psi(w)$ in C.

2. There is a contractible open subset $\mathcal{U}$ of the sphere $S(\mathfrak{k}^o)$ and a contractible open neighborhood $\mathcal{V}$ of 0 in $\mathfrak{k}^*$ and a diffeomorphism

$$\varphi : \psi(U_w) \to \mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C').$$

3. φ is $\mathbb{R}$-equivariant, where $\psi(U_w)$ inherits the radial action of $\mathbb{R}$ on $\mathfrak{g}^*$ and we trivially extend the action of $\mathbb{R}$ on itself via translation to the product $\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C')$.

Proof. Let $\iota : \mathfrak{k} \to \mathfrak{g}$ be the inclusion and $\iota^* : \mathfrak{k}^* \to \mathfrak{g}^*$ the dual to this inclusion. We first show that $\iota^*(\psi(w)) = 0$. As noted in Remark 2.1.6, the unimodular cone $C = C_{\{v_1,\ldots,v_k\},\psi(w)}$ contains the affine subspace

$$A = \{\eta \in \mathfrak{g}^* \,|\, \langle \eta - \psi(w), v_i \rangle = 0\}.$$

Since ψ is homogeneous, the image of ψ contains the ray $\{t\psi(w) \,|\, t > 0\}$. In particular, this means that A, as an affine subspace of $\mathfrak{k}$, must contain the origin. It follows A is a linear subspace and, since $\psi(w) \in A$, $A - \psi(w) = A$. As $A - \psi(w) = \mathfrak{k}^o$, we may then conclude that $\psi(w)$ is in $\mathfrak{k}^o$. Thus, $\iota^*(\psi(w)) = 0$.

By choosing a section of ι^* embedding $\mathfrak{k}^*$ into $\mathfrak{g}^*$, we have an isomorphism of vector spaces $\mathfrak{k}^* \times \mathfrak{k}^o \cong \mathfrak{g}^*$ with $\psi(w)$ corresponding to the point $(0, \psi(w))$ in $\mathfrak{k}^* \times \mathfrak{k}^o$. This isomorphism descends to an isomorphism $\phi : C \to C' \times \mathfrak{k}^o$ commuting with scalar multiplication when defined. Since C' is a unimodular cone based at the origin, C' is closed under the radial action of $\mathbb{R}$ on $\mathfrak{k}^*$ and therefore it follows that C is closed under the radial action of $\mathbb{R}$ on $\mathfrak{g}^*$. Thus, the isomorphism $\phi : C \to C' \times \mathfrak{k}^o$ is $\mathbb{R}$-equivariant with respect to the radial action of $\mathbb{R}$.

Now fix a norm on $\mathfrak{k}^o$ with unit sphere $S(\mathfrak{k}^o)$. Then the $\mathbb{R}$-equivariant diffeomorphism between $\mathfrak{k}^o \backslash \{0\}$ and $\mathbb{R} \times S(\mathfrak{k}^o)$ induces an $\mathbb{R}$-equivariant diffeomorphism

$$f : C' \times (\mathfrak{k}^o \backslash \{0\}) \to \mathbb{R} \times S(\mathfrak{k}^o) \times C'.$$

Note that $\mathbb{R}$ acts on $\mathbb{R} \times S(\mathfrak{k}^o) \times C'$ both by translation on the first factor of the product and by radial scaling on the third factor of the product.

Now, the map

$$g : \mathbb{R} \times S(\mathfrak{k}^o) \times C' \to \mathbb{R} \times S(\mathfrak{k}^o) \times C' \qquad (\lambda, p, v) \mapsto (\lambda, p, e^{-\lambda} v)$$

is an $\mathbb{R}$-equivariant diffeomorphism for which the codomain $\mathbb{R} \times S(\mathfrak{k}^o) \times C'$ has $\mathbb{R}$ action $\lambda \cdot (t, p, v) := (\lambda + t, p, v)$.

Thus, the map $g \circ f \circ \phi$ is an $\mathbb{R}$-equivariant diffeomorphism from C to $\mathbb{R} \times S(\mathfrak{k}^o) \times C'$ with the required

$\mathbb{R}$ action. Finally, note that by Lemma 3.1.4, there is an $\mathbb{R}$-invariant open subset U_w on which ψ is an open embedding onto its image. By shrinking U_w as necessary, we may ensure that the $\mathbb{R}$-equivariant $g \circ f \circ \phi$ takes $\psi(U_w)$ to the prescribed neighborhood type. □

We may now use the slices described above to prove the domain of a homogeneous unimodular local embedding is always the total space of a principal $\mathbb{R}$-bundle of manifold with corners.

Proposition 3.1.9. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding and let $q : W \to W/\mathbb{R}$ be an $\mathbb{R}$-quotient map for the free $\mathbb{R}$ action on W. Then q is a principal $\mathbb{R}$-bundle in the category of manifolds with corners.

Proof. From Lemma 3.1.8, it follows that each point $w \in W$ has an $\mathbb{R}$-invariant neighborhood U_w equivariantly diffeomorphic to $\mathbb{R} \times V$, for V an open subset of $[0,\infty)^k \times \mathbb{R}^{\dim(G)-k}$. Thus, $[w] \in W/\mathbb{R}$ has a neighborhood homeomorphic to V. These neighborhoods in the quotient are clearly coherent and give a (possibly non-Hausdorff) manifold with corners structure on $W/\mathbb{R}$.

Since $\psi|_{U_w}$ is an open embedding, and since $(\mathfrak{g}^* \backslash \{0\})/\mathbb{R} \cong S^{\dim(G)-1}$ is Hausdorff, $\psi(v)$ and $\psi(v')$ are separable by $\mathbb{R}$-invariant neighborhoods. Thus, $W/\mathbb{R}$ is Hausdorff.

Finally, note that our slices from Lemma 3.1.8 naturally give us smooth local trivializations of W as a principal $\mathbb{R}$-bundle. □

Like symplectic toric manifolds over a specific unimodular local embedding, symplectic toric cones over a homogeneous unimodular local embedding form a presheaf. Rather than using a site of open subsets over a topological subspace, we instead consider a smaller site.

Definition 3.1.10. Let W be a manifold with corners with a free $\mathbb{R}$ action. Denote by $\mathsf{Open}_{\mathbb{R}}(W)$ the full subcategory of $\mathsf{Open}(W)$ of $\mathbb{R}$-invariant subsets of W. That is, the category with objects $\mathbb{R}$-invariant open subset of W and morphisms inclusions of open subsets.

Proposition 3.1.11. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then

$$U \mapsto \mathsf{STC}_{\psi}(U) := \mathsf{STC}_{\psi|_U}(U)$$

defines a presheaf over $\mathsf{Open}_{\mathbb{R}}(W)$.

Proof. For each $\mathbb{R}$-invariant open subset U of W, $\psi|_U$ is still a homogeneous unimodular local embedding. Thus, the groupoid $\mathsf{STC}_{\psi}(U) := \mathsf{STC}_{\psi|_U}(U)$ is well-defined. For $U \subset V$ $\mathbb{R}$-invariant open subsets of W, we define restriction by

$$(M,\omega,\pi:M\to V)|_U := (\pi^{-1}(U),\omega|_{\pi^{-1}(U)},\pi|_{\pi^{-1}(U)})$$

To see this is still a symplectic cone, note that, since the actions of G and $\mathbb{R}$ on M commute, the action of $\mathbb{R}$ on M descends to an action on V with respect to which π is $\mathbb{R}$-equivariant. It follows from Lemma 3.1.5 that this action of $\mathbb{R}$ matches the action of $\mathbb{R}$ on W (as ψ is homogeneous with respect to both actions). Thus, $\pi^{-1}(U)$ is an $\mathbb{R}$-invariant subset of M and therefore $(M,\omega,\pi:M\to V)|_U$ is a well-defined symplectic toric cone.

Because morphisms of $\mathsf{STC}_\psi(V)$ must cover the identity on V, any morphism restricts to a morphism over U in $\mathsf{STC}_\psi(U)$. It is easy to check that, with these restriction morphisms,

$$\mathsf{STC}_\psi : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}, \quad U \mapsto \mathsf{STC}_\psi(U)$$

is a presheaf. □

3.2 Homogeneous symplectic toric bundles

To classify symplectic toric cones, we use homogeneous symplectic toric bundles together with an analogue of the isomorphism of presheaves $c:\mathsf{STB}_\psi\to\mathsf{STM}_\psi$ from [19] (see Section 2.1.2). In this section, we present a definition and some important properties of homogeneous symplectic toric bundles.

Definition 3.2.1. Let $\psi:W\to\mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then a homogeneous symplectic toric bundle over ψ is a symplectic toric bundle $(\pi:P\to W,\omega)$ (see Definition 2.1.10) together with a free and proper $\mathbb{R}$ action on P so that

- The actions of G and $\mathbb{R}$ on P commute;
- (P,ω) is a symplectic cone with respect to the given $\mathbb{R}$ action; and
- $\psi\circ\pi$ is a *homogeneous* moment map for the action of G on (P,ω).

These bundles are represented by pairs $(\pi:P\to W,\omega)$.

Denote by $\mathsf{HSTB}_\psi(W)$ the category of homogeneous symplectic toric bundles over W. This is the groupoid with objects homogeneous symplectic toric bundles $(\pi:P\to W,\omega)$ and morphisms $\varphi:(\pi:P\to W,\omega)\to(\pi':P'\to W,\omega')$ $\mathbb{R}$-equivariant gauge transformations such that $\varphi^*\omega'=\omega$.

While choosing $\psi \circ \pi$ to be a homogeneous moment map is the "correct" condition to impose from the standpoint of creating a coherent definition, there is a simpler and more useful condition we now provide.

Lemma 3.2.2. Let $\psi : W \to \mathfrak{g}^*$ is a homogeneous unimodular local embedding. Suppose that $(\pi : P \to W, \omega)$ is a symplectic toric bundle with a free and proper $\mathbb{R}$ action, commuting with the action of G, with respect to which (P, ω) is a symplectic cone. Then $(\pi : P \to W, \omega)$ is a homogeneous symplectic toric bundle if and only if π is $\mathbb{R}$-equivariant.

Proof. Since $(\pi : P \to W, \omega)$ is a symplectic toric bundle, $\psi \circ \pi$ is a moment map for the action of G on (P, ω). Thus, all that remains to be shown is that $\psi \circ \pi$ is homogeneous if and only if π is $\mathbb{R}$-equivariant.

If π is $\mathbb{R}$-equivariant, then for any $p \in P$, $\psi(\pi(t \cdot p)) = \psi(t \cdot (\pi(p))) = e^t \psi(\pi(p))$, so $\psi \circ \pi$ is homogeneous. On the other hand, since the actions of $\mathbb{R}$ and G on P commute, the free action of $\mathbb{R}$ on P descends to a free action on W with respect to which π is $\mathbb{R}$-equivariant and with respect to which $\psi(t \cdot w) = e^t \psi(w)$. By Lemma 3.1.5, this implies that this induced $\mathbb{R}$ action matches the $\mathbb{R}$ action on W with respect to which ψ is a homogeneous unimodular local embedding. □

As in the case of symplectic toric cones, the collection of groupoids of homogeneous symplectic toric bundles over restrictions of ψ to $\mathbb{R}$-invariant open subsets of W is a presheaf of groupoids over $\mathsf{Open}_{\mathbb{R}}(W)$.

Proposition 3.2.3. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then the function on $\mathsf{Open}_{\mathbb{R}}(W)$

$$U \mapsto \mathsf{HSTB}_{\psi}(U) := \mathsf{HSTB}_{\psi|_{(U)}}((U))$$

together with the appropriate restriction morphisms defines a presheaf of groupoids

$$\mathsf{HSTB}_{\psi} : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}.$$

As the justification here is more or less the same as that for Proposition 3.1.11, we omit the proof.

Remark 3.2.4. In fact, $\mathsf{HSTB}_{\psi} : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a stack. This fact will be important later, but as the proof is essentially a marginally adjusted retelling of the proof that the presheaf of principal bundles over a site is a stack, we relegate the proof to the appendix (Proposition B.8).

As in the case of symplectic toric bundles, it is not immediately clear that the category $\mathsf{HSTB}_{\psi}(W)$ is non-empty. However, a G-invariant symplectic form ω for any principal G-bundle P over W with appropriate $\mathbb{R}$ action can be built from a choice of connection 1-form on P. Before showing this, we need a technical lemma.

Lemma 3.2.5. Let $\pi : P \to B$ be a principal G-bundle of manifolds with corners. Further, suppose P and B admit free actions of $\mathbb{R}$ with respect to which π is $\mathbb{R}$-equivariant and the $\mathbb{R}$-quotient $q' : B \to B/\mathbb{R}$ is a principal $\mathbb{R}$-bundle of manifolds with corners. Let $q : P \to P/\mathbb{R}$ be an $\mathbb{R}$-quotient. Finally, suppose that the actions of G and $\mathbb{R}$ on P commute.

Then $P/\mathbb{R}$ admits the structure of a manifold with corners. Furthermore, there is a smooth map $\varpi : P/\mathbb{R} \to B/\mathbb{R}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & P/\mathbb{R} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varpi} \\ B & \xrightarrow[q']{} & B/\mathbb{R} \end{array} \tag{3.1}$$

commutes. Finally, the maps $\varpi : P/\mathbb{R} \to B/\mathbb{R}$ and $q : P \to P/\mathbb{R}$ are a principal G-bundle and principal $\mathbb{R}$-bundle of manifolds with corners, respectively.

Proof. We will first work only topologically. The existence of ϖ is a consequence of the universal property of a quotient: as π is $\mathbb{R}$-equivariant and q' is $\mathbb{R}$-invariant, the composition $q' \circ \pi$ must collapse $\mathbb{R}$-orbits. Thus, there exists a unique map $\varpi : P/\mathbb{R} \to B/\mathbb{R}$ making diagram (3.1) commute.

Since the actions of G and $\mathbb{R}$ commute on P, the free action of G on P descends to a free action of G on $P/\mathbb{R}$. Let U be any contractible open subset of $B/\mathbb{R}$ and let $s : U \to B$ be a local section of the principal $\mathbb{R}$-bundle $q' : B \to B/\mathbb{R}$. This induces an $\mathbb{R}$-equivariant homeomorphism $\varphi_s : B|_U \to U \times \mathbb{R}$ with $\varphi_s^{-1}(b, t) := t \cdot s(b)$. It follows that $B|_U$ is contractible, so we may find another local section $s' : B|_U \to P$ of the principal G-bundle $\pi : P \to B$.

Now, let φ_s^i to be the homeomorphism φ_s followed by the projection onto the i^{th} factor of the product $U \times \mathbb{R}$. We may adjust s' to an $\mathbb{R}$-equivariant section t by defining

$$t : B|_U \to P, \;\; b \mapsto \varphi_s^2(b) \cdot s'(s(\varphi_s^1(b))).$$

This yields a $(G \times \mathbb{R})$-equivariant homeomorphism $\varphi_t : P|_{B|_U} \to B|_U \times G$ with $\varphi_t^{-1}(b, g) = g \cdot t(b)$.

Using the notation φ_t^i as above, we have a $(G \times \mathbb{R})$-equivariant homeomorphism

$$\phi : P|_{B|_U} \to U \times \mathbb{R} \times G, \;\; p \mapsto (\varphi_s(\varphi_t^1(p)), \varphi_t^2(p)).$$

Since we have that

$$\varpi \circ q \circ t \circ s = q' \circ \pi \circ t \circ s = id_{B/\mathbb{R}},$$

$q \circ t \circ s$ is a section of ϖ and therefore $q(P|_{B|_U}) = P/\mathbb{R}|_U$. Thus, ϕ descends to a G-equivariant homeomorphism

$$\bar{\phi} : P/\mathbb{R}|_U \to U \times G$$

and we may conclude ϖ is a (topological) principal G-bundle.

Now, note that we may choose smooth sections s and t above. Then φ_s and φ_t must both be diffeomorphisms and so the homeomorphisms $\bar{\phi}$ as above defined for each contractible subset U of $B/\mathbb{R}$ define a smooth manifold with corners structure on $P/\mathbb{R}$. It is clear then that, with this smooth structure in mind, ϖ is smooth and $P/\mathbb{R}$ has smooth trivializations as a principal G-bundle; therefore, $\varpi : P/\mathbb{R} \to B/\mathbb{R}$ is a principal G-bundle of manifolds with corners. $\square$

Now, we build a symplectic form for any principal G-bundle $\pi : P \to W$ with an appropriate $\mathbb{R}$ action.

Proposition 3.2.6. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding and let $\pi : P \to W$ be any principal G-bundle with a free action of $\mathbb{R}$ commuting with the action of G such that π is $\mathbb{R}$-equivariant. Then there exists a connection 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ so that $(\pi : P \to W, d\langle \psi \circ \pi, A\rangle)$ is a homogeneous symplectic toric bundle with respect to this $\mathbb{R}$ action.

Proof. First note that, by Lemma 3.1.2, since the action of $\mathbb{R}$ on W is proper, the action of $\mathbb{R}$ on P must be proper as well.

Next, note that Karshon and Lerman [19] showed that, for any connection 1-form A, $d\langle \psi \circ \pi, A\rangle$ is a G-invariant symplectic form for P with respect to which $\psi \circ \pi$ is a moment map (see Lemma 2.1.18). So it remains to show we can find a particular connection satisfying the additional conditions required of a homogeneous symplectic toric bundle.

Let $Q := P/\mathbb{R}$ and $B := W/\mathbb{R}$ with $\mathbb{R}$-quotient maps $q' : W \to B$ and $q : P \to Q$. From Lemma 3.2.5, we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{q} & Q \\ \pi \downarrow & & \downarrow \varpi \\ W & \xrightarrow[q']{} & B \end{array}$$

where $\varpi : Q \to B$ is a principal G-bundle and $q : P \to Q$ is a principal $\mathbb{R}$-bundle.

So, the bundle $q : P \to Q$ is trivializable: there is a gauge transformation $\phi : P \to Q \times \mathbb{R}$ of principal $\mathbb{R}$-bundles over Q. As q is G-equivariant, it follows that, with respect to the G action on Q extended trivially to $Q \times \mathbb{R}$, ϕ is G-equivariant. Thus, $\pi \circ \phi^{-1} : Q \times \mathbb{R} \to W$ is a principal G-bundle. On the other hand,

$\varpi \times id_{\mathbb{R}} : Q \times \mathbb{R} \to B \times \mathbb{R}$ is of course also a principal G-bundle. As connection 1-forms on principal bundles must only satisfy conditions related to the associated G action on the total space, we may define a connection 1-form for the bundle $\varpi \times id_{\mathbb{R}} : Q \times \mathbb{R} \to B \times \mathbb{R}$ which will also be a connection for the bundle $\pi \circ \phi^{-1} : Q \times \mathbb{R} \to W$.

Let A' be any connection 1-form on $\varpi \times id_{\mathbb{R}} : Q \times \mathbb{R} \to B \times \mathbb{R}$ extended trivially from a connection 1-form on $\varpi : Q \to B$. Define $A := \phi^* A'$. We must show that $d\langle \psi \circ \pi, A \rangle$ satisfies the necessary conditions for a symplectic form of a symplectic cone. Fix a real number λ and let ρ_λ be the diffeomorphism associated to its action on P. Let ρ'_λ be the diffeomorphism associated to the action of λ on $Q \times \mathbb{R}$. Then, as ϕ is $\mathbb{R}$-equivariant, $\phi \circ \rho_\lambda = \rho'_\lambda \circ \phi$. As A' came from a connection on Q, it follows that $\rho'^*_\lambda A' = A'$. Using these facts, we calculate:

$$\begin{aligned} \rho_t^* d\langle \psi \circ \pi, A \rangle &= d\langle \psi \circ \pi \circ \rho_\lambda, \rho_\lambda^*(\phi^* A') \rangle \\ &= d\langle e^\lambda \cdot (\psi \circ \pi), \phi^*({\rho'}^*_\lambda A') \rangle \\ &= d(e^\lambda \langle \psi \circ \pi, \phi^* A' \rangle) \\ &= e^\lambda d\langle \psi \circ \pi, A \rangle \end{aligned}$$

This is exactly the condition $d\langle \psi \circ \pi, A \rangle$ must satisfy so that $(P, d\langle \psi \circ \pi, A \rangle)$ is a symplectic cone. ☐

We will soon show that two homogeneous symplectic toric bundles are isomorphic in HSTB_ψ exactly when there is an $\mathbb{R}$-equivariant gauge transformation between them. To prove this, we need the following lemma.

Lemma 3.2.7. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding and let $\pi : P \to W$ be a principal G-bundle with a free $\mathbb{R}$ action commuting with the action of G so that π is $\mathbb{R}$-equivariant. Suppose ω and ω' are two symplectic forms so that $(\pi : P \to W, \omega)$ and $(\pi : P \to W, \omega')$ are both homogeneous symplectic toric bundles. Then the form $\omega - \omega'$ is basic and, for $\omega - \omega' = \pi^*\beta$, β is exact. Furthermore, there is a primitive γ of β satisfying:

$$\rho_\lambda^* \gamma = e^\lambda \gamma$$

for any $\lambda \in \mathbb{R}$ with action diffeomorphism $\rho_\lambda : W \to W$.

Proof. First note that $(\pi : P \to W, \omega)$ and $(\pi : P \to W, \omega')$ are, in particular, symplectic toric bundles. Fix a connection 1-form A for which $(\pi : P \to W, d\langle A, \psi \rangle)$ is a homogeneous symplectic toric bundle (as constructed in Proposition 3.2.6). Then $\omega - d\langle A, \psi \rangle$ and $\omega' - d\langle A, \psi \rangle$ are both basic (see Lemma 2.1.18);

thus, the difference $\omega - \omega'$ is basic as well.

Fix a real number λ. Writing $\tau_\lambda : P \to P$ for the action isomorphism of λ on P, we have by assumption that $\tau_\lambda^* \omega = e^\lambda \omega$ and $\tau_\lambda^* \omega' = e^\lambda \omega'$. So, of course, their difference $\pi^* \beta$ must satisfy this condition as well.

As π is $\mathbb{R}$-equivariant, we have that $\pi \circ \tau_\lambda = \rho_\lambda \circ \pi$. So, we calculate:

$$\begin{aligned} \pi^*(\rho_\lambda^* \beta) &= \tau_\lambda^*(\pi^* \beta) \\ &= e^\lambda \pi^* \beta \\ &= \pi^*(e^\lambda \beta) \end{aligned}$$

Since π is a submersion, it follows that $\rho_\lambda^* \beta = e^\lambda \beta$.

Finally, write Ξ for the vector field on W with flow the action of $\mathbb{R}$. Then β satisfies $L_\Xi \beta = \beta$ meaning, since β is closed, that $\gamma := \iota_\Xi \beta$ is a primitive for β. It is easy to show that γ satisfies $L_\Xi \gamma = \gamma$ as well. It thereby follows that γ satisfies the conditions hypothesized above. □

With this lemma in mind, we may prove the following important lemma (adapted from a lemma of [19]).

Lemma 3.2.8. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding and let $(\pi : P \to W, \omega)$ be a homogeneous symplectic toric bundle. Let γ be a 1-form on W satisfying $\rho_\lambda^* \gamma = e^\lambda \gamma$ for every real λ with action diffeomorphism $\rho_\lambda : W \to W$. Then there is an isomorphism of homogeneous symplectic toric bundles $\varphi : (\pi : P \to W, \omega) \to (\pi : P \to W, \omega + \pi^* d\gamma)$.

Remark 3.2.9. It is clear that, for γ as in the lemma above, the proof of Lemma 3.2.7 may be reversed to conclude that $(\pi : P \to W, \omega + \pi^* d\gamma)$ is indeed a homogeneous symplectic toric bundle over ψ.

proof of Lemma 3.2.8. We will essentially repeat the proof of Lemma 3.3 of [19] with the addition of an $\mathbb{R}$ action; for the convenience of the reader, we will sketch the borrowed details.

To build the map f, we may use Moser's deformation method on the family of symplectic forms:

$$\omega_t = \omega + t\pi^* d\gamma, \;\; t \in [0, 1].$$

Then there is a unique time-dependent vector field X_t on P satisfying:

$$\iota_{X_t} \omega_t = -\pi^* \gamma. \tag{3.2}$$

By showing that X_t is G-invariant and tangent to the compact fibers of π, we may conclude that the time

1 flow of X_t exists and G-equivariant. Therefore, for $\varphi : P \to P$ this time 1 flow, we must have $\pi \circ \varphi = \pi$ and, as is standard in the use of Moser's method (as in [30]), φ satisfies

$$\varphi^*(\omega + \pi^* d\gamma) = \varphi^*(\omega_1) = \varphi^*(\omega_0) = \varphi^*(\omega).$$

It remains to be shown for our case that this gauge transformation is $\mathbb{R}$-equivariant. It is enough to show that the time dependent vector field X_t determined by the family of symplectic forms above is $\mathbb{R}$-invariant. Fix a real number λ and let $\rho_\lambda : P \to P$ be the action diffeomorphism for λ. It is clear, as $\rho_\lambda^* \omega = e^\lambda \omega$ and $\rho_\lambda^*(\pi^*\gamma) = e^\lambda \pi^*\gamma$, that ω_t must satisfy the analogous property.

We calculate:

$$\iota_{(\rho_\lambda)_* X_t}\omega_t = \rho_{-\lambda}^*(\iota_{X_t}(\rho_\lambda^*\omega_t)) = \rho_{-\lambda}^*(\iota_{X_t} e^\lambda \omega_t) = e^\lambda \rho_{-\lambda}^*(-\pi^*\gamma) = e^\lambda e^{-\lambda}(-\pi^*\gamma) = -\pi^*\gamma$$

Because equation (3.2) uniquely determines the vector field X_t, it follows that $(\rho_\lambda)_* X_t = X_t$. Thus, X_t is $\mathbb{R}$-invariant, meaning its time 1 flow φ must be $\mathbb{R}$-equivariant. □

From the previous two lemmas, we may easily prove the following proposition:

Proposition 3.2.10. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then two homogeneous symplectic toric bundles over ψ $(\pi : P \to W, \omega)$ and $(\pi' : P' \to W, \omega')$ are isomorphic as elements of STC_ψ if and only if there exists an $\mathbb{R}$-equivariant isomorphism of principal G-bundles $\varphi : P \to P'$.

Proof. Because an isomorphism in HSTB_ψ is in particular an $\mathbb{R}$-equivariant isomorphism of principal G-bundles, one direction is given by definition.

So suppose there exists an $\mathbb{R}$-equivariant isomorphism of principal G-bundles $\varphi : P \to P'$. Then, by Lemma 3.2.7, it follows that the difference $\varphi^*(\omega') - \omega$ is basic. Furthermore, writing $\varphi^*(\omega') - \omega = \pi^*\beta$, there is a primitive γ of β satisfying $\rho_\lambda^*\gamma = e^\lambda\gamma$, where $\rho_\gamma : W \to W$ is the action diffeomorphism for the action of real number $\mathbb{R}$. Then, by Lemma 3.2.8, we have that there exists an $\mathbb{R}$-equivariant gauge transformation $\phi : P \to P$ satisfying $\phi^*(\omega + d\pi^*\gamma) = \omega$. Therefore, $\varphi \circ \phi$ is an isomorphism of homogeneous symplectic toric bundles. □

3.3 The morphism of presheaves $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$

In this section, we introduce a functor $\mathsf{hc} : \mathsf{HSTB}_\psi(W) \to \mathsf{STC}_\psi(W)$. We then show that hc is an equivalence of categories; in fact, thinking of HSTB_ψ and STC_ψ as presheaves over $\mathsf{Open}_{\mathbb{R}}(W)$, hc is an isomorphism of presheaves. This functor is essentially a homogeneous version of the equivalence of categories $c : \mathsf{STB}_\psi(W) \to \mathsf{STM}_\psi(W)$ of [19] (see Section 2.1.2). We first must verify that, given a homogeneous symplectic toric bundle $(\pi : P \to W, \omega)$ over homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$, the $\mathbb{R}$ action on P descends to an $\mathbb{R}$ action on $c(P, \omega)$ making this into a symplectic cone.

Proposition 3.3.1. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding and let $(\pi : P \to W, \omega)$ be a homogeneous symplectic toric bundle over ψ. Then, regarding (P, ω) simply as a symplectic toric bundle over ψ, the symplectic toric manifold $c(P, \omega)$ inherits an $\mathbb{R}$ action from (P, ω) with respect to which $c(P, \omega)$ is a symplectic toric cone over ψ.

Proof. Recall that $c(P, \omega)$ is built first as a topological G-space $c_{\mathsf{Top}}(P, \omega) := P/\sim$, where $\sim$ is the equivalence relation:

$$p \sim p' \text{ if and only if } k \in K_{\pi(p)} \text{ such that } p \cdot k = p'.$$

This topological quotient is then "symplectized" through the process detailed in Construction 2.1.13.

Now, as the actions of G and $\mathbb{R}$ on P commute, it follows that the action of $\mathbb{R}$ descends to a continuous action on $c_{\mathsf{Top}}(P, \omega)$. To confirm the action is smooth, we will show that we may carefully repeat the cuts giving symplectic structure to $c(P, \omega)$, using $\mathbb{R}$-invariant subsets $U \subset W$ to symplectize $c_{\mathsf{Top}}(P, \omega)$ via the reductions $(P|_U \times \mathbb{C}^k)//_0 K_w$.

Fix an element $w \in W$. Recall that, since ψ is a unimodular local embedding, there is a neighborhood of w diffeomorphic via ψ to a neighborhood of $\psi(w)$ in the unimodular cone $C_w := C_{\{v_1,\ldots,v_k\},\psi(w)}$, where $\{v_1, \ldots, v_k\}$ is a basis for the Lie algebra $\mathfrak{k}$ for a subtorus $K_w \leq G$. As before, let $\iota : \mathfrak{k} \to \mathfrak{g}$ be the natural inclusion with dual $\iota^* : \mathfrak{g}^* \to \mathfrak{k}^*$. As in Construction 2.1.13, define the cone C'_w by

$$C'_w := \{\xi \in \mathfrak{k}^* \mid \langle \xi, v_i \rangle \geq 0,\ 1 \leq i \leq k\}.$$

Then by Lemma 3.1.8, we may find an $\mathbb{R}$-invariant neighborhood U_w of w, a contractible open subset $\mathcal{U}$ of the sphere $S^{\dim(G)-k-1}$, and a contractible open subset $\mathcal{V}$ of the origin in $\mathfrak{k}^*$ so that $\psi|_{U_w}$ is an open embedding and U_w is diffeomorphic to $\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C'_w)$. It is easy to confirm that, with respect to this identification, $(\iota^* \circ \psi)(t, u, v) = e^t v$. Since U_w is contractible, $\pi : P|_{U_w} \to U_w$ is a trivializable principal

G-bundle. So, since the map

$$\mathbb{R} \times \mathcal{U} \times (\mathcal{V} \cap C'_w) \times G \to \mathfrak{k}^* \qquad (t, u, v, g) \mapsto e^t v$$

admits an extension

$$\mathbb{R} \times \mathcal{U} \times \mathcal{V} \times G \to \mathfrak{k}^* \qquad (t, u, v, g) \mapsto e^t v,$$

there is a manifold $\tilde{P}$ (isomorphic to $\mathbb{R} \times \mathcal{U} \times \mathcal{V} \times G$) containing the manifold with corners $P|_{U_w}$ as a domain so that the map $\nu := \iota^* \circ \psi \circ \pi : P|_{U_w} \to \mathfrak{k}^*$ admits an extension to $\tilde{\nu} : \tilde{P} \to \mathfrak{k}^*$.

Now, let $K_w \to (\mathbb{C}^k, \omega_{\mathbb{C}^k})$ be the symplectic representation with weights $\{v_1^*, \dots, v_k^*\}$ (here, $\omega_{\mathbb{C}^k}$ denotes the standard symplectic form on $\mathbb{C}^k$). We fix the moment map

$$\mu_w : \mathbb{C}^k \to \mathfrak{k}^* \qquad \mu_w((z_1, \dots, z_k)) := -\sum_{j=1}^{k} |z_j|^2 v_j$$

for this space. Then the K_w action on $(P|_{U_w} \times \mathbb{C}^k, \omega \oplus \omega_{\mathbb{C}^k})$ has moment map $\Phi(p, z) := \nu(p) + \mu_w(z)$ and this clearly has extension $\tilde{\Phi}(p, z) := \tilde{\nu}(p) + \mu_w(z)$ to the domain $\tilde{P} \times \mathbb{C}^k$ containing $P|_{U_w} \times \mathbb{C}^k$.

The condition $(p, z) \in \tilde{\Phi}^{-1}(0)$ imposes that $\tilde{\nu}(p) = -\mu_w(z)$, meaning the image of $\tilde{\nu}$ must be contained in C'_w. It therefore follows that $\Phi^{-1}(0) = \tilde{\Phi}^{-1}(0)$. Thus, by Theorem 2.1.12, the reduction $(P|_{U_w} \times \mathbb{C}^k)//_0 K_w$ is a symplectic manifold.

As we've proceeded using (essentially) the same method as in Construction 2.1.13, it follows we may use the same form of homeomorphisms (as defined in Construction 2.1.14) to symplectize $c_{\mathsf{Top}}(P, \omega)$. To finish, we need only show that there are compatible smooth $\mathbb{R}$ actions on each $(P|_{U_w} \times \mathbb{C}^k)//_0 K_w$ with respect to which the inherited symplectic form on $c(P, \omega)$ is homogeneous.

So let $\mathbb{R}$ act on $P|_{U_w} \times \mathbb{C}^k$ via the $\mathbb{R}$ action on P restricted to $P|_{U_w}$ and via the "half-radial action" on $\mathbb{C}^k$: the action $t \cdot z := e^{t/2} z$. $\mu_w : \mathbb{C}^k \to \mathfrak{k}^*$ is homogeneous with respect to this action of $\mathbb{R}$ on $\mathbb{C}^k$ and, as $\nu : P|_{U_w} \to \mathfrak{k}^*$ is homogeneous as well, it follows that the action of $\mathbb{R}$ preserves the level set $\Phi^{-1}(0)$. Since the actions of K_w and $\mathbb{R}$ commute, the action of $\mathbb{R}$ descends to a smooth action on $(P|_{U_w} \times \mathbb{C}^k)//_0 K_w$.

It is easy to show that the transition homeomorphisms

$$\alpha_w^P : c_{\mathsf{Top}}(P|_{U_w}) \to (P|_{U_w} \times \mathbb{C}^k)//_0 K_w$$

(again, as outlined in Construction 2.1.14) are $\mathbb{R}$-equivariant, where $c_{\mathsf{Top}}(P|_{U_w})$ and $(P|_{U_w} \times \mathbb{C}^k)//_0 K_w$

inherit the $\mathbb{R}$ actions described above. Thus, the action of $\mathbb{R}$ on $c_{\mathsf{Top}}(P)$ inherited by the commutativity of the action of G and $\mathbb{R}$ on P is in fact a smooth action on the symplectic manifold $c(P, \omega)$.

Finally, to see that the symplectic form η on $c(P, \omega)$ is homogeneous (that is, satisfies $\rho_\lambda^* \eta = e^\lambda \eta$ for the action diffeomorphism ρ_λ defined for each $\lambda \in \mathbb{R}$), recall that, on the open dense interior $\mathring{W}$ of W, the functor c is the identity (see Remark 2.1.15). In other words, for an open subset $U \subset \mathring{W}$, $(P|_U, \omega, \pi : P|_U \to U) = c(P, \omega)|_U$ as symplectic toric manifolds over $\psi|_U$. Thus,

$$\rho_\lambda^*(\eta|_U) = \rho_\lambda^*(\omega|_U) = e^\lambda \omega|_U = e^\lambda \eta|_U$$

As this identity holds on the open dense subset $c(P, \omega)|_{\mathring{W}}$ of $c(P, \omega)$, it follows it must hold over all $c(P, \omega)$. Therefore, the above action of $\mathbb{R}$ on $c(P, \omega)$ renders $c(P, \omega)$ a symplectic toric cone (the properness of the $\mathbb{R}$ action on M is ensured by Proposition 3.1.6). □

Definition 3.3.2. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then denote by

$$\mathsf{hc} : \mathsf{HSTB}_\psi(W) \to \mathsf{STC}_\psi(W)$$

the functor taking a homogeneous symplectic toric bundle (P, ω) to the symplectic manifold $c(P, \omega)$ with $\mathbb{R}$ action inherited from (P, ω), as outlined in Proposition 3.3.1. For a morphism $\varphi : (P, \omega) \to (P', \omega')$, we may take $\mathsf{hc}(\varphi) := c(\varphi)$. It is easy to check that, since φ is $(G \times \mathbb{R})$-equivariant, $\mathsf{hc}(\varphi)$ is $(G \times \mathbb{R})$-equivariant as well. It is also easy to confirm that, as with c, $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$ is a map of presheaves.

To prove that $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$ is an isomorphism of presheaves, we first must prove that hc_U is a fully faithful functor for each $\mathbb{R}$-invariant open subset $U \subset W$. We use the following lemma.

Lemma 3.3.3. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then the forgetful functors $\iota_h : \mathsf{HSTB}_\psi(W) \to \mathsf{STB}_\psi(W)$ and $\iota_c : \mathsf{STC}_\psi(W) \to \mathsf{STM}_\psi(W)$ are faithful.

Proof. ι_h takes a homogeneous symplectic toric bundle to the underlying symplectic toric bundle and ι_c takes a symplectic toric cone to the underlying symplectic toric manifold (in other words, both functors "forget" the $\mathbb{R}$ action on the respective source objects). The morphisms in both source categories are just the morphisms of the target category that happen to be $\mathbb{R}$-equivariant.

It is more or less obvious that, as forgetful functors, ι_h and ι_c are faithful. □

Now, we show that hc is fully faithful.

Lemma 3.3.4. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then for every $\mathbb{R}$-invariant open subset U of W, the functor $\mathsf{hc}_U : \mathsf{HSTB}_\psi(U) \to \mathsf{STC}_\psi(U)$ is fully faithful.

Proof. Note that, as $\psi|_U : U \to \mathfrak{g}^*$ is also a homogeneous unimodular local embedding and the groupoid $\mathsf{HSTB}_\psi(U)$ is, by definition, the groupoid $\mathsf{HSTB}_{\psi|_U}(U)$, we need only worry about the case of $U = W$ as this will generalize to any $\mathbb{R}$-invariant open subset $U \subset W$.

Consider the following diagram

$$\begin{array}{ccc} \mathsf{HSTB}_\psi(W) & \xrightarrow{\iota_h} & \mathsf{STB}_\psi(W) \\ {\scriptstyle \mathsf{hc}}\downarrow & & \downarrow{\scriptstyle c} \\ \mathsf{STC}_\psi(W) & \xrightarrow[\iota_c]{} & \mathsf{STM}_\psi(W) \end{array}$$

where ι_h and ι_c are the faithful functors of Lemma 3.3.3. From the definition of each functor, it follows quite easily that this diagram commutes.

As $c \circ \iota_h = \iota_c \circ \mathsf{hc}$, it is straight forward to check that, as c, ι_h, and ι_c are all faithful, hc must be faithful as well.

To finish, we need to show hc is full. Fix two homogeneous symplectic toric bundles (P, ω) and (P', ω') in $\mathsf{HSTB}_\psi(W)$. Let

$$f : \mathsf{hc}(\pi : P \to W, \omega) \to \mathsf{hc}(\pi' : P' \to W, \omega')$$

be a map of symplectic toric cones. Applying ι_c, we get an $\mathbb{R}$-equivariant map of symplectic toric manifolds

$$\iota_c(f) : \iota_c(\mathsf{hc}(P, \omega)) \to \iota_c(\mathsf{hc}(P', \omega'))$$

which, by the commutativity of the above diagram, is in fact a map

$$\iota_c(f) : c(\iota_h(P, \omega)) \to c(\iota_h(P', \omega')).$$

As c is full, there exists a map of symplectic toric bundles

$$\varphi : \iota_h(P, \omega) \to \iota_h(P', \omega')$$

with $c(\varphi) = \iota_c(f)$.

Now, let $d : P' \times_{\pi', W, \pi'} P' \to G$ be the division map for P': the map defining $d(p, p')$ as the unique element of G such that $p \cdot d(p, p') = p'$ for any $p, p' \in P'$ with $\pi'(p) = \pi'(p')$. This is a smooth map. For each element $t \in \mathbb{R}$, define

$$\tilde{\varphi}_t : P \to G \qquad \tilde{\varphi}_t(p) := d(\varphi(t \cdot p), t \cdot \varphi(p)).$$

By design, this map measures the failure of φ to be equivariant with respect to the action of t.

Again, recall the interior $\mathring{W} \subset W$ is an open dense subset of W and that $c|_{\mathring{W}}$ functions as the identity (see Remark 2.1.15). So, $c(\varphi|_{\mathring{W}}) = \iota_c(f|_{\mathring{W}})$ is $\mathbb{R}$-equivariant and we therefore have that, for every $t \in \mathbb{R}$ and for every $p \in P|_{\mathring{W}}$, $\tilde{\varphi}_t(p) = e$, for e the identity element of G. As $\tilde{\varphi}_t$ is continuous and constant on the open dense subset $P|_{\mathring{W}}$ of P, it follows that $\tilde{\varphi}_t$ must be the constant e on all P for every $t \in \mathbb{R}$. Thus, $\varphi : \iota_h(P, \omega) \to \iota_h(P', \omega')$ must be $\mathbb{R}$-equivariant.

It follows that φ is actually a map of homogeneous symplectic toric bundles $\varphi : (P, \omega) \to (P', \omega')$. Using the commutativity of the above diagram once more, we have that

$$\iota_c(\mathsf{hc}(\varphi)) = c(\iota_h(\varphi)) = \iota_c(f)$$

Since ι_c is faithful, this implies $\mathsf{hc}(\varphi) = f$. Hence hc is full. □

We require two more lemmas before we can use Lemma B.11 to prove that $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$ is an isomorphism of presheaves.

Lemma 3.3.5. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then any two symplectic toric cones over ψ $(M, \omega, \pi : M \to W)$ and $(M', \omega', \pi' : M' \to W)$ are locally isomorphic; explicitly, there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of W by $\mathbb{R}$-invariant open subsets and a collection of isomorphisms

$$\{\varphi_\alpha : (M, \omega, \pi : M \to W)|_{U_\alpha} \to (M', \omega', \pi' : M' \to W)|_{U_\alpha} \in \mathsf{STC}_\psi(U_\alpha) \mid \alpha \in A\}.$$

Proof. As explained in the proof of Lemma B.4 of [19], the symplectic slice representation of M at a point p is determined up to isomorphism by the image of a (sufficiently small) neighborhood of $\pi(p)$ under ψ. Thus, two points $p \in M$ and $p' \in M'$ satisfying $\pi(p) = \pi'(p') \in W$ must have the same stabilizer and isomorphic symplectic slice representations. Therefore, by Proposition 2.2.29, there are $(G \times \mathbb{R})$-invariant neighborhoods U of p and U' of p' and an isomorphism of symplectic toric cones

$$f : (M, \omega, \psi \circ \pi)|_U \to (M', \omega', \psi \circ \pi')|_{U'}$$

satisfying $f(p) = p'$. It follows that f defines an isomorphism of symplectic toric cones over $\psi|_{\pi(U)}$. $\square$

Lemma 3.3.6. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then the presheaf $\mathsf{STC}_\psi : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a prestack (see Definition B.4).

Proof. To show STC_ψ is a prestack, we must show that, for every $\mathbb{R}$-invariant open subset U of W and for any two symplectic toric cones $(M, \omega, \pi : M \to U)$ and $(M', \omega', \pi' : M' \to U)$ in $\mathsf{STC}_\psi(U)$, the presheaf

$$\underline{\mathsf{Hom}}((M, \omega, \pi), (M', \omega', \pi')) : \mathsf{Open}_{\mathbb{R}}(U)^{\mathsf{op}} \to \mathsf{Sets} \quad V \mapsto \mathsf{Hom}_{\mathsf{STC}_\psi}((M, \omega, \pi)|_V, (M', \omega', \pi')|_V)$$

is a sheaf of sets. Clearly every morphism $f : M \to M'$ is uniquely determined by its restrictions to any open cover, so it remains to show that coherent families of local isomorphisms glue to global maps.

So fix an open cover $\{U_\alpha\}_{\alpha \in A}$ of U by $\mathbb{R}$-invariant open subsets. Suppose we have a family of isomorphisms of symplectic toric cones

$$\{f_\alpha : (M, \omega, \pi : M \to U)|_{U_\alpha} \to (M', \omega', \pi' : M' \to U)|_{U_\alpha}\}_{\alpha \in A}$$

that are locally coherent; that is, for α and β with $U_{\alpha\beta} := U_\alpha \cap U_\beta$ non-empty, $f_\alpha|_{U_{\alpha\beta}} = f_\beta|_{U_{\alpha\beta}}$. Then clearly there is a unique smooth map $f : M \to M'$ such that $f|_{\pi^{-1}(U_\alpha)} = f_\alpha$ for every α.

As checking a map is symplectic may be done locally and each f_α is a symplectomorphism, it follows that f must be a symplectic map. By design, $\pi^{-1}(U_\alpha) \subset M$ and ${\pi'}^{-1}(U_\alpha) \subset M'$ are both $(G \times \mathbb{R})$-invariant and f_α is equivariant for every α. Therefore, f must be $(G \times \mathbb{R})$-equivariant. It is clear that, applying the same logic as above, the collection of maps

$$\{f_\alpha^{-1} : (M', \omega', \pi' : M' \to U)|_{U_\alpha} \to (M, \omega, \pi : M \to U)|_{U_\alpha}\}_{\alpha \in A}$$

glue together to a $(G \times \mathbb{R})$-equivariant symplectic map

$$f^{-1} : (M', \omega', \pi' : M' \to U) \to (M, \omega, \pi : M \to U)$$

and that this is indeed the inverse to f. Thus, the $\{f_\alpha\}_{\alpha \in A}$ glue to a unique isomorphism.

It follows that $\underline{\mathsf{Hom}}((M, \omega, \pi), (M', \omega', \pi')) : \mathsf{Open}^{\mathsf{op}}(U) \to \mathsf{Sets}$ is a sheaf of sets and therefore STC_ψ is a prestack. $\square$

We may now put together the proof of the following theorem.

Theorem 3.3.7. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$ is an isomorphism of presheaves. In particular, this means the groupoids $\mathsf{HSTB}_\psi(W)$ and $\mathsf{STC}_\psi(W)$ are equivalent.

Proof. We have from Lemma 3.3.6 that STC_ψ is a prestack and from Proposition B.8 that HSTB_ψ is a stack. To see that $\mathsf{HSTB}_\psi(U)$ is non-empty for every open $\mathbb{R}$-invariant subset U of W, note that the trivial principal G-bundle $U \times G \to U$ has a connection 1-form A with respect to which $(U \times G, d\langle A, \psi\rangle)$ is a homogeneous symplectic toric bundle. From Lemma 3.3.5, we have that, for any open $\mathbb{R}$-invariant subset U of W, any two elements in the groupoid $\mathsf{STC}_\psi(U)$ are locally isomorphic; in other words, STC_ψ is transitive. Finally, from Lemma 3.3.4, we have that $\mathsf{hc}_U : \mathsf{HSTB}_\psi(U) \to \mathsf{STC}_\psi(U)$ is fully faithful for each U.

Thus, HSTB_ψ, STC_ψ, and hc satisfy all the hypotheses of Lemma B.11 (also, see Remark B.12) and so we may conclude that hc is an isomorphism of presheaves. □

3.4 Characteristic classes for symplectic toric cones

In this section, we first give characteristic classes for homogeneous symplectic toric bundles over any homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$. Via the isomorphism of presheaves hc, these classes then yield characteristic classes for symplectic toric cones taking values in the cohomology group $H^2(W; \mathbb{Z}_G)$.

First, we set some notation.

Notation 3.4.1. Given two categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $F : \mathcal{C} \to \mathcal{D}$, denote by the symbols $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$ the collections of isomorphism classes of $\mathcal{C}$ and $\mathcal{D}$ respectively and denote by $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ the function $\pi_0 F([c]) := [F(c)]$ for each class $[c] \in \pi_0\mathcal{C}$. Note $\pi_0 F$ is well-defined as F is a functor.

Remark 3.4.2. For X a topological space, suppose $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a presheaf of groupoids. Then there is a sheaf of sets $\pi_0\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Sets}$ with $(\pi_0\mathcal{F})(U) := \pi_0(\mathcal{F}(U))$ for every open subset U of X. For $U \subset V$ nested open subsets of X, the restriction functor $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ for $\mathcal{F}$ descends to the function $\pi_0\rho_{VU} : \pi_0\mathcal{F}(V) \to \pi_0\mathcal{F}(U)$. It is easy to check these functions satisfy the necessary requirements of restriction functions for $\pi_0\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Sets}$.

Now we relate HSTB_ψ to an easier presheaf of groupoids to classify.

Definition 3.4.3. For W a manifold with corners with a free $\mathbb{R}$ action, let $\mathsf{BG}_\mathbb{R} : \mathsf{Open}_\mathbb{R}(\mathsf{W})^{\mathsf{op}} \to \mathsf{Groupoids}$ be the presheaf of groupoids so that, for every $\mathbb{R}$-invariant open subset U of W, $\mathsf{BG}_\mathbb{R}(\mathsf{U})$ is the groupoid of principal G-bundles over U with morphisms isomorphisms of principal G-bundles.

We need the following theorem, well-known in the case of topological principal bundles (see, for instance [8]).

Theorem 3.4.4. Let $\pi : E \to B$ be a principal G-bundle of manifolds with corners and, for a manifold with corners X, let $f_0 : X \to B$ and $f_1 : X \to B$ be two smoothly homotopic maps. Then the pullbacks $f_0^*(E)$ and $f_1^*(E)$ are isomorphic as principal G-bundles.

Proof. First, assume π is a bundle of manifolds and X is also a manifold. Let $H : X \times [0,1] \to B$ be the hypothesized homotopy with $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$. Then the flow of the horizontal lift of the vector field $\frac{d}{dt}$ on $X \times [0,1]$ with respect to any connection 1-form on E induces an isomorphism of principal G-bundles $f_0^*E \cong f_1^*E$. Note this flow exists and is equivariant as it matches the parallel transport of the curves $t \mapsto H(x,t)$ for each $x \in X$.

To conclude the same result for manifolds with corners, it is enough to show parallel transport is well-defined in this case. The horizontal lift of $\frac{d}{dt}$ still makes sense, but we must be able to show that this lift has a flow. The standard existence argument for parallel transport over a manifold with corners M applies to this case with the possible exception of a path that intersects the boundary of M. But, for any curve $\gamma : [a,b] \to M$, and any s with $a < s < b$ and $\gamma(s) \in \partial M$ (i.e., the boundary of M), $\gamma(s)$ must be tangent to ∂M. It follows the that there is no obstruction to the existence of the required flow. ☐

Proposition 3.4.5. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then there is a map of presheaves $R : \mathsf{HSTB}_\psi \to \mathsf{BG}_\mathbb{R}$ so that $\pi_0 R : \pi_0\mathsf{HSTB}_\psi \to \pi_0\mathsf{BG}_\mathbb{R}$ is an isomorphism.

Proof. For every $\mathbb{R}$-invariant open subset U of W, let $R_U : \mathsf{HSTB}_\psi(U) \to \mathsf{BG}_\mathbb{R}(\mathsf{U})$ be the forgetful functor: for every homogeneous symplectic toric bundle $(\pi : P \to U, \omega)$, let $R(\pi : P \to U, \omega) := \pi : P \to U$. Since any map of homogeneous symplectic toric bundles φ is, in particular, an isomorphism of principal G-bundles, it makes sense to define $R(\varphi) := \varphi$. It is clear that R commutes with restrictions and therefore $R : \mathsf{HSTB}_\psi \to \mathsf{BG}_\mathbb{R}$ is a map of presheaves over $\mathsf{Open}_\mathbb{R}(W)$.

It remains to be shown that, for every $\mathbb{R}$-invariant open subset U of W,

$$(\pi_0 R)_U : \pi_0\mathsf{HSTB}_\psi(U) \to \pi_0\mathsf{BG}_\mathbb{R}(\mathsf{U})$$

is a bijection. Since $\mathsf{BG}_\mathbb{R}(\mathsf{U})$ is a groupoid, it is enough to show that R_U is essentially surjective and, for any two homogeneous symplectic toric bundles $(\pi : P \to U, \omega)$ and $(\pi' : P' \to U, \omega')$, $R(P,\omega)$ and $R(P',\omega')$ are isomorphic only if (P,ω) and (P',ω') are isomorphic.

To begin, fix a principal G-bundle $\pi : P \to U$. By Proposition 3.1.9, the $\mathbb{R}$-quotient $q : U \to U/\mathbb{R}$ is a principal $\mathbb{R}$-bundle. Then there exists a slice Σ for the $\mathbb{R}$ action on U (the image of a global section of $q : U \to U/\mathbb{R}$). With respect to this slice, U is equivariantly isomorphic to $U/\mathbb{R} \times \mathbb{R}$, so there is a homotopy $H : [0,1] \times U \to U$ between the identity map on U and the contraction of U onto a slice of the $\mathbb{R}$ action. By Theorem 3.4.4, this induces an isomorphism of principal G-bundles between $P|_\Sigma \times \mathbb{R} \to U$ and P. It follows that P inherits a free $\mathbb{R}$ action with respect to which π is equivariant. By Proposition 3.2.6, there is a connection 1-form A on P with respect to which $(\pi : P \to U, d\langle \psi \circ \pi, A\rangle)$ is a homogeneous unimodular local embedding with $R_U(\pi : P \to U, d\langle \psi \circ \pi, A\rangle) = \pi : P \to U$.

Now, suppose $(\pi : P \to U, \omega)$ and $(\pi' : P' \to U, \omega')$ are two homogeneous symplectic toric bundles and that $\varphi : P \to P'$ is an isomorphism. By Lemma 3.2.2, π and π' are $\mathbb{R}$-equivariant. Then for any p in P,

$$\pi'(\varphi(t \cdot p)) = \pi(t \cdot p) = t \cdot \pi(p) = t \cdot \pi'(\varphi(p)) = \pi'(t \cdot \varphi(p))$$

So, while φ needn't be $\mathbb{R}$-equivariant, $\varphi(t \cdot p)$ and $t \cdot \varphi(p)$ must lie in the same fiber of π'.

As before, let $d : P' \times_U P' \to G$ be the division map: the smooth map taking each pair (p, p') with $\pi'(p) = \pi'(p')$ to the unique element of G satisfying $p \cdot d(p, p') = p'$. As above, let Σ be a slice for the action of $\mathbb{R}$ on U. Then there is an $\mathbb{R}$-equivariant diffeomorphism $\phi : P \to P|_\Sigma \times \mathbb{R}$ and for $\phi^1 : P \to P|_\Sigma$ and $\phi^2 : P \to \mathbb{R}$ the corresponding projections, we may define the isomorphism

$$\tilde{\varphi} : P \to P' \quad p \mapsto \varphi(p) \cdot d(\varphi(p), \phi^2(p) \cdot \varphi(\phi^1(p)))$$

Then, since $\phi^1(t \cdot p) = \phi^1(p)$ for any p in P and t in $\mathbb{R}$, we have

$$\tilde{\varphi}(t \cdot p) = \varphi(t \cdot p) \cdot d(\varphi(t \cdot p), \phi^2(t \cdot p) \cdot \varphi(\phi^1(t \cdot p))) = \phi^2(t \cdot p) \cdot \varphi(\phi^1(p)) = t \cdot \tilde{\varphi}(p)$$

Thus, $\tilde{\varphi}$ is $\mathbb{R}$-equivariant and therefore a $(G \times \mathbb{R})$-equivariant isomorphism. So, by Proposition 3.2.10, the two homogeneous symplectic toric bundles $(\pi : P \to U, \omega)$ and $(\pi' : P' \to U, \omega')$ are isomorphic. □

Before we can finish, we need the following well-known theorem.

Theorem 3.4.6. Let M be a manifold with corners and let $\mathsf{BG(M)}$ be the category of principal G-bundles over M with morphisms isomorphisms of principal G-bundles. For G our torus and $\mathbb{Z}_G$ the integral lattice of $\mathfrak{g}$ (that is, the kernel of $\exp : \mathfrak{g} \to G$), the function:

$$\mathsf{c}_1 : \pi_0\mathsf{BG(M)} \to \mathsf{H}^2(\mathsf{M}; \mathbb{Z}_\mathsf{G})$$

with $\mathsf{c}_1([P]) := \mathsf{c}_1(P)$ the first Chern class of P is a bijection.

Remark 3.4.7. We may extend the bijection in Theorem 3.4.6 to an isomorphism of presheaves of sets. Since $\mathsf{BG} : \mathsf{Open(M)}^{\mathsf{op}} \to \mathsf{Groupoids}$ is a presheaf of groupoids (in fact, a stack; see Example B.7), as explained in Remark 3.4.2, $\pi_0\mathsf{BG} : \mathsf{Open(M)}^{\mathsf{op}} \to \mathsf{Sets}$ is a presheaf of sets. Since the first Chern class c_1 is a characteristic class, it commutes with restrictions, and so we may think of the collection of bijections

$$\mathsf{c}_1 : \pi_0\mathsf{BG(U)} \to \mathsf{H}^2(\mathsf{U}; \mathbb{Z}_\mathsf{G})$$

as an isomorphism of presheaves of sets.

Now we may classify homogeneous symplectic toric bundles over a homogeneous unimodular local embedding $\psi : W \to \mathfrak{g}^*$.

Proposition 3.4.8. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then, for $H^2(\cdot; \mathbb{Z}_G) : \mathsf{Open}_\mathbb{R}(W)^{\mathsf{op}} \to \mathsf{Sets}$ the presheaf of sets $U \mapsto H^2(U; \mathbb{Z}_G)$, there is an isomorphism of presheaves:

$$\mathsf{ch} : \pi_0\mathsf{HSTB}_\psi \to H^2(\cdot; \mathbb{R}).$$

Proof. Recall we have isomorphisms of presheaves $R : \pi_0\mathsf{HSTB}_\psi \to \pi_0\mathsf{BG}_\mathbb{R}$ of Proposition 3.4.5 and $\mathsf{c}_1 : \pi_0\mathsf{BG} \to \mathsf{H}^2(\cdot; \mathbb{Z}_\mathsf{G})$ from Remark 3.4.7. Therefore, the composition $\mathsf{ch} := \mathsf{c}_1 \circ R$ is an isomorphism of presheaves. □

We may now prove our first main classification which we restate for the convenience of the reader.

Theorem B. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then:

1. The set of symplectic toric cones with orbital moment map ψ is non-empty.
2. The set of isomorphism classes of symplectic toric cones (M, ω, μ) with G-quotient $\pi : M \to W$ and orbital moment map ψ is in natural bijective correspondence with the cohomology classes $H^2(W; \mathbb{Z}_G)$, where $\mathbb{Z}_G$ is the integral lattice of G, the kernel of the map $\exp : \mathfrak{g} \to G$.

Proof. To see the first item holds, note that, by Proposition 3.2.6, there is a connection 1-form A on the trivial principal G-bundle $W \times G \to W$ over W such that $(W \times G \to W, d\langle \psi, A \rangle)$ is a homogeneous symplectic toric bundle over ψ.

For the second item, this bijective correspondence arises from composition of the isomorphisms of presheaves $(\pi_0 \mathsf{hc})^{-1} : \mathsf{STC}_\psi \to \mathsf{HSTB}_\psi$ and $\mathsf{c}_1 : \mathsf{HSTB}_\psi \to H^2(\cdot; \mathbb{Z}_G)$. $\square$

We now define our characteristic class for symplectic toric cones:

Definition 3.4.9. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then the homogeneous Chern class of a symplectic toric cone $(M, \omega, \pi : M \to W)$ over ψ is the cohomology class:

$$\mathsf{h\text{-}ch}((M, \omega, \pi)) := \mathsf{c}_1((\pi_0 \mathsf{hc})^{-1}(M, \omega, \pi)) \in H^2(W; \mathbb{R})$$

To conclude this chapter, we remark on calculating characteristic classes for symplectic toric cones:

Remark 3.4.10. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding and let $(M, \omega, \pi : M \to W)$ be a symplectic toric cone over ψ. Then, as shown in Lemma A.12, the inclusion $\iota : \mathring{W} \to W$ is part of a homotopy equivalence (for $\mathring{W}$ the interior of W). Therefore, restriction to $\mathring{W}$ on forms induces an isomorphism in cohomology:

$$\iota^* : H^2(W; \mathbb{R}) \to H^2(\mathring{W}, \mathbb{R}).$$

As $(\pi_0 \mathsf{hc})^{-1} : \mathsf{STC}_\psi \to \mathsf{HSTB}_\psi$ and $\mathsf{ch} : \mathsf{HSTB}_\psi \to H^2(\cdot; \mathbb{Z}_G)$ are isomorphisms of presheaves, it follows that

$$\mathsf{h\text{-}ch}(M, \omega, \pi)|_{\mathring{W}} = \mathsf{h\text{-}ch}\left((M, \omega, \pi)|_{\mathring{W}}\right)$$

and, since $(M, \omega, \pi)|_{\mathring{W}}$ has a free torus action, it follows that $(M, \omega, \pi)|_{\mathring{W}}$ is just the homogeneous symplectic toric bundle over $\psi|_{\mathring{W}}$ $(\pi : M|_{\mathring{W}} \to \mathring{W}, \omega)$. Thus,

$$\mathsf{h\text{-}ch}((M, \omega, \pi)|_{\mathring{W}}) = \mathsf{c}_1(\pi : M|_{\mathring{W}} \to \mathring{W}) \in H^2(\mathring{W}; \mathbb{Z}_G)$$

and so $\mathsf{h\text{-}ch}(M, \omega, \pi) = (\iota^*)^{-1}(\mathsf{c}_1(\pi : M|_{\mathring{W}} \to \mathring{W}))$.

Chapter 4

A classification of symplectic toric stratified spaces with isolated singularities

The goal of this chapter is to describe and classify symplectic toric stratified spaces with isolated singularities. To begin, we describe in Section 4.1 *singular symplectic toric cones*: these are symplectic toric cones with an added point at infinity. These spaces are important, as they will serve as a model for symplectic toric stratified spaces with isolated singularities.

In Section 4.2, We define and describe symplectic toric stratified spaces with isolated singularities. These are (roughly) stratified spaces with torus actions locally modeled on singular symplectic toric cones (see Definition 4.2.1). We will see that, for such a space $(X, \omega, \mu : X \to \mathfrak{g}^*)$, the topological quotient X/G of any symplectic toric stratified space with isolated singularities $(X, \omega, \mu : X \to \mathfrak{g}^*)$ inherits the structure of a *cornered stratified space with isolated singularities*; essentially, a stratified space for which the stratum are allowed to be manifolds with corners. Furthermore, the moment map $\mu : X \to \mathfrak{g}^*$ descends to a *stratified unimodular local embedding* $\bar{\mu} : X/G \to \mathfrak{g}^*$ (see Definition 4.2.3). This is the continuous extension of a unimodular local embedding to a cornered stratified space with an additional local property near each singularity.

As any isomorphic symplectic toric stratified spaces must have the same orbit space and orbital moment map, we group the spaces with orbital moment map the stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$ together into a groupoid $\mathsf{STSS}_\psi(W)$, the *groupoid of symplectic toric stratified spaces over* ψ. The morphisms of this groupoid are exactly the G-equivariant isomorphisms (i.e., strata preserving homeomorphisms descending to symplectomorphisms on the open dense strata) preserving ψ. As in the case of symplectic toric cones, we may form a presheaf with these groupoids:

$$\mathsf{STSS}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$$

In Section 4.3, we define *conical symplectic toric G-bundles over* ψ. These are principal G-bundles $\pi : P \to W_{\mathsf{reg}}$ over the open dense stratum W_{reg} of W with a G-invariant symplectic form for which $\psi \circ \pi$ is a moment map satisfying a special "conical" condition (see Definition 4.3.1). Together with G-equivariant symplectomorphisms, these form a groupoid $\mathsf{CSTB}_\psi(W)$, the *groupoid of conical symplectic toric G-bundles over ψ*. As in the case of homogeneous symplectic toric bundles, the collection of groupoids $\mathsf{CSTB}_{\psi|_U}(U)$ forms a presheaf of groupoids

$$\mathsf{HSTB}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$$

In Section 4.4, we build a map of presheaves $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$ adapted from the equivalence of categories c presented by Karshon and Lerman. As in the case of c and hc of Chapter 3, we are also able to show that $\tilde{\mathsf{c}}$ is an isomorphism of presheaves (Theorem 4.4.18).

In Section 3.4, we describe a classification for conical symplectic toric bundles and, by extension, a classification of symplectic toric stratified spaces. We show in Proposition 4.5.6 that isomorphism classes of conical symplectic toric bundles are determined both by their structure as principal G-bundles and by a so-called horizontal class. This mirrors the case of Karshon and Lerman where horizontal classes took the form of cohomology classes on the base manifold with corners W. In the case of conical symplectic toric bundles, however, there are restrictions on the allowed cohomology classes; namely classes of forms called (for the purposes of this paper) *good forms*: forms on the open dense manifold piece W_{reg} of a cornered stratified space W that are exact in deleted neighborhoods of each singularity of W (see Definition 4.5.1). We denote the subspace of all classes of good forms by $\mathcal{C} \subset H^2(W_{\mathsf{reg}}; \mathbb{R})$. We may then finally use this result and the isomorphism $\tilde{\mathsf{c}}$ to prove our main classification theorem (Theorem A): the bijection between the isomorphism classes of $\mathsf{STSS}_\psi(W)$ and the cohomology classes $H^2(W_{\mathsf{reg}}; \mathbb{Z}_G) \times \mathcal{C}$.

We finish the section by showing that the subspace $\mathcal{C}$ can be identified as the image of the relative de Rham cohomology group $H^2\left(W_{\mathsf{reg}}, \overline{W}\right)$ (as from Bott and Tu, [4]) under a natural inclusion map for an appropriately chosen subset $\overline{W}$ of W_{reg}. This group and image can be computed using the long exact sequence associated to a pair. We also provide some examples of applications of our classification. This includes a confirmation that our results agrees with the classification of compact connected symplectic toric stratified spaces with isolated singularities by Burns, Guillemin, and Lerman [6] as well as a discussion of what happens when one of their technical conditions (the *Reeb type* condition) is dropped.

4.1 Singular symplectic toric cones

Symplectic toric stratified spaces with isolated singularities are symplectic toric manifolds except on a discrete set of isolated singularities fixed by the torus G. These singularities have neighborhoods modeled on cones, both in the topological sense and in the symplectic sense. This mirrors a particular case of the symplectic stratified space structure symplectic quotients must take (see Section 2.3). We will make this idea precise with a series of definitions.

Definition 4.1.1. Let L be a manifold (possibly with corners). Then the open cone on L, denoted $c(L)$, is the topological space

$$c(L) := \frac{L \times [-\infty, \infty)}{L \times \{-\infty\}.}$$

Here, $[-\infty, \infty)$ is the topological space given by compactifying $\mathbb{R}$ at one end and is homeomorphic to $[0, \infty)$ (or any half closed interval, for that matter). We denote by $*$ the vertex of the cone (i.e., the image of $\{-\infty\} \times L$ in $c(L)$ under the quotient).

Note that this is just the cone of Definition 2.3.2 in the case where the link is a manifold. While it is a bit awkward to take the topological quotient at $-\infty$ as above (as opposed to, say, at the end of the ray $[0, \infty)$), it fits the convention for symplectic cones nicely and is more convenient in the long run.

What follows is a definition for stratified spaces with isolated singularities. Note that this is just Definition 2.3.3 specialized to the isolated singularity case.

Definition 4.1.2. A stratified space with isolated singularities is a Hausdorff paracompact topological space X with a partition $X = X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I} \{x_\alpha\})$ such that X_{reg} is a manifold and for each x_α, there exists a neighborhood U_α of x_α in X, a manifold L_α, and an embedding $\varphi_\alpha : U_\alpha \to c(L_\alpha)$ such that

- $\varphi_\alpha(x_\alpha) = *$ (i.e., φ_α maps x_α to the vertex of $c(L_\alpha)$); and
- φ_α restricts to a diffeomorphism between $X_{\mathsf{reg}} \cap U_\alpha$ and its image in $c(L_\alpha) \backslash \{*\} \cong L \times \mathbb{R}$

Formally, this data will be denoted by the pair $(X, X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I} \{x_\alpha\}))$ though informally, the partition may be suppressed.

We will call a choice of link L_α, neighborhood U_α, and embedding $\varphi_\alpha : U_\alpha \to c(L_\alpha)$ a local structure datum for x_α.

To model both symplectic toric stratified spaces and their quotients, we also define stratified spaces with isolated singularities that are modeled on cones of manifolds with corners (we distinguish the latter case with the name *cornered stratified spaces*).

Definition 4.1.3. A cornered stratified space with isolated singularities is a stratified space for which the links of the singularities are manifolds with corners.

It is not difficult to see that the top dimensional stratum of a (cornered) stratified space must be an open dense manifold (with corners). We set some notation for this piece of the stratified space.

Notation 4.1.4. The top dimensional open dense stratum of a (cornered) stratified space with isolated singularities X will always be designated X_{reg}. For any open subset U of X, U inherits the structure of a (cornered) stratified space naturally as a subset of X. It is easy to see that, with respect to this inherited structure, $U_{\mathsf{reg}} = X_{\mathsf{reg}} \cap U$.

We must also define maps and isomorphisms of stratified spaces with isolated singularities. It is not difficult to see that these are just maps of partitioned spaces (see Definition 2.3.1) tailored to the case of isolated singularities.

Definition 4.1.5. A map of stratified spaces with isolated singularities is a continuous map

$$f : \Big(X, X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in A}\{x_\alpha\})\Big) \longrightarrow \Big(X', X'_{\mathsf{reg}} \bigsqcup (\sqcup_{\beta \in B}\{x'_\alpha\})\Big)$$

so that $f(X_{\mathsf{reg}}) \subset X'_{\mathsf{reg}}$ and for every $\alpha \in A$, $f(x_\alpha) = x'_\beta$ for some $\beta \in B$.

Such a map is an isomorphism of stratified spaces with isolated singularities if it is a homeomorphism restricting to a diffeomorphism on the top stratum (it follows that f^{-1} is a map of stratified spaces since f is bijective and a map of stratified spaces).

As a particular important example of a stratified space with isolated singularities note that, for any manifold L, its open cone $c(L)$ is a stratified space with one isolated singularity. If L is a manifold with corners, then $c(L)$ is a cornered stratified space with one isolated singularity.

From any symplectic cone, we can build a stratified space by adding a point at $-\infty$. To build an actual stratified space (as opposed to just a partitioned space), the neighborhoods of this added points must contain neighborhoods of $-\infty$:

Definition 4.1.6. Let M be a manifold with a free and proper $\mathbb{R}$ action. Then a neighborhood of $-\infty$ is an open subset $U \subset M$ that:

- For each real $\lambda \leq 0$, $\lambda \cdot U \subset U$; and
- U intersects each $\mathbb{R}$ orbit non-trivially.

We now define singular symplectic cones. These will be cones in the topological sense (i.e., the sense of Definition 4.1.1) with a *symplectic* cone structure on the top stratum.

Definition 4.1.7. A singular symplectic cone (with corners) is a (cornered) stratified space with one isolated singularity $X = X_{\mathsf{reg}} \sqcup \{x_0\}$ together with a symplectic form $\omega \in \Omega^2(X_{\mathsf{reg}})$ such that

- $(X_{\mathsf{reg}}, \omega)$ is a symplectic cone; and
- every neighborhood U of the vertex x_0 of X contains a neighborhood of $-\infty$ of $(X_{\mathsf{reg}}, \omega)$.

We saw in Section 2.2 that every symplectic toric cone is isomorphic to a cone $(B \times \mathbb{R}, d(e^t\alpha))$, for B a contact manifold with contact form α. For a singular symplectic cone, we now show this isomorphism on the top stratum extends to an isomorphism to the cone $c(B)$.

Proposition 4.1.8. Let $(X = X_{\mathsf{reg}} \sqcup \{x_0\}, X_{\mathsf{reg}})$ be a singular symplectic cone. Then, for $\mathbb{R}$-quotient $B := X_{\mathsf{reg}}/\mathbb{R}$, every trivialization $\varphi : X_{\mathsf{reg}} \to B \times \mathbb{R}$ of X_{reg} as a principal $\mathbb{R}$-bundle admits an extension to an isomorphism of stratified spaces $\tilde{\varphi} : X \to c(L)$.

Proof. Let $\tilde{\varphi} : X \to c(L)$ be the extension of φ taking x_0 in X to $*$ in $c(L)$. Since $\tilde{\varphi}$ is $\mathbb{R}$-invariant, it takes neighborhoods of $-\infty$ in $X \backslash \{0\}$ to neighborhoods if $-\infty$ in $c(L)\backslash\{0\}$. So $\tilde{\varphi}$ is a homeomorphism away from x_0, thus to conclude that $\tilde{\varphi}$ is an open map, it is enough to show that every open subset of the vertex $*$ in $c(L)$ contains a neighborhood of $-\infty$ in $L \times \mathbb{R}$.

To see this, let

$$q : L \times [-\infty, \infty) \to c(L) := ((L \times [-\infty, \infty)) \,/\, (L \times \{-\infty\})$$

be the topological quotient map. Then a set U containing $*$ in $c(L)$ is open if and only if $V := q^{-1}(U)$ is open in $L \times [-\infty, \infty)$. Note that V must contain $L \times \{-\infty\} \subset L \times [-\infty, \infty)$. Therefore, for every $l \in L$, it must also contain a set of the form $V_l \times (-\infty, \epsilon)$ for V_l an open neighborhood of l in L and for some $\epsilon \in \mathbb{R}$. A union of a cover of these open subsets for each $l \in L$ builds a neighborhood of $-\infty$ in $L \times \mathbb{R} \subset V \subset L \times [-\infty, \infty)$. This descends via q to a neighborhood of $-\infty$ in $L \times \mathbb{R} \subset c(L)$.

Thus, every neighborhood of $*$ in $c(L)$ contains a neighborhood of $-\infty$ in $L \times \mathbb{R}$ and so $\tilde{\varphi}$ is open. A similar argument shows that $\tilde{\varphi}^{-1}$ is open as well and therefore $\tilde{\varphi}$ is a homeomorphism. □

Proposition 4.1.8 implicitly demonstrates that, by choosing a trivialization of a symplectic cone as a principal $\mathbb{R}$-bundle, we may extend this symplectic cone to a singular symplectic cone. In Section 4.4, it will be important that we can make this extension in a coordinate free manner (i.e., without first choosing a trivialization). As we show below, this is always possible.

Proposition 4.1.9. Any symplectic cone (M, ω) naturally extends to a singular symplectic cone. That is, there exists a stratified space $\tilde{M}$ such that $\tilde{M}_{\mathsf{reg}} = M$ and $(\tilde{M}, \omega)$ is a singular symplectic cone.

Proof of Proposition 4.1.9. Define the topological space $\tilde{M}$ as follows: as a set, it is simply the disjoint union $M \sqcup \{*\}$, for the point $*$ representing our (soon to be) cone point. $\tilde{M}$ is then given the topology generated by sets of the form:

1. U, an open subset of M
2. $V \sqcup \{*\}$, where $V \subset M$ is a neighborhood of $-\infty$

More succinctly, we topologize the set $\tilde{M}$ by specifying that all open subsets of M closed under negative translation (the neighborhoods of $-\infty$) are in fact open neighborhoods of the singular point $*$. As in the proof of Proposition 4.1.8, any trivialization of $\varphi : M \to M/\mathbb{R} \times \mathbb{R}$ takes neighborhoods of $-\infty$ of M to neighborhoods of $-\infty$ of $M/\mathbb{R} \times \mathbb{R}$ and so extends to a homeomorphism $\tilde{\varphi} : \tilde{M} \to c(M/\mathbb{R})$. Thus, $\tilde{M}$ is a stratified space with one singularity and it is clear then that, by definition, $(\tilde{M}, \omega)$ is a singular symplectic toric cone. □

Now, we add a torus action and define singular symplectic toric cones. From here forward, fix a torus G with Lie algebra $\mathfrak{g}$.

Definition 4.1.10. A singular symplectic toric cone is a singular symplectic cone $X = X_{\mathsf{reg}} \sqcup \{x_0\}$ with form $\omega \in \Omega^2(X_{\mathsf{reg}})$, an action of torus G, and continuous map $\mu : X \to \mathfrak{g}^*$, such that

- G fixes the point x_0;
- the action of G restricts to a smooth action on X_{reg}; and
- $(X_{\mathsf{reg}}, \omega)$ is a symplectic toric cone with respect to the action of G and $\mu|_{X_{\mathsf{reg}}}$ is the homogeneous moment map of $(X_{\mathsf{reg}}, \omega)$.

We denote this data as the triple (X, ω, μ).

An isomorphism of singular symplectic toric cones is a G-equivariant isomorphism of stratified spaces that restricts to an isomorphism of symplectic toric cones on the top strata of the domain and codomain.

Remark 4.1.11. Let $(X = X_{\mathsf{reg}} \sqcup \{x_0\}, \omega, \mu : X \to \mathfrak{g}^*)$ be a singular symplectic toric cone. As $\mu|_{X_{\mathsf{reg}}}$ is homogeneous and μ is continuous, it follows that $\mu(x_0) = 0$.

As in the case of symplectic cones, symplectic toric cones admit trivialization independent extensions to singular symplectic toric cones.

Proposition 4.1.12. Every symplectic toric cone $(M, \omega, \mu : M \to \mathfrak{g}^*)$ naturally extends to a singular symplectic toric cone. That is, there is a stratified space $\tilde{M}$ and an extension $\tilde{\mu} : \tilde{M} \to \mathfrak{g}^*$ of μ such that $(\tilde{M}, \omega, \tilde{\mu} : \tilde{M} \to \mathfrak{g}^*)$ is a singular symplectic toric cone.

To show this, we will need the following lemma that will also be useful in constructing the functor $\tilde{\mathfrak{c}}$ in Section 4.4 and in a homotopy argument in Section 4.5.

Lemma 4.1.13. Let L be a manifold (with corners). Then any neighborhood of $-\infty$ U in $L \times \mathbb{R}$ contains a neighborhood of $-\infty$ diffeomorphic to $L \times \mathbb{R}$. If a compact Lie group K acts on L, then U contains a K-invariant neighborhood of $-\infty$ diffeomorphic to $L \times \mathbb{R}$ via a K-equivariant diffeomorphism.

Proof. Let L be a manifold (with corners). We will first show that, for any neighborhood of $-\infty$ U in $L \times \mathbb{R}$, there is a smooth function $f : L \to \mathbb{R}$ such that $(p, \lambda) \in U$ for any $\lambda \leq f(p)$.

Note that we may choose a locally finite open cover $\{U_i\}_{i \in I}$ of L such that the closure $\overline{U_i}$ of each open subset U_i is compact. It follows that each U_i has a non-trivial intersection with only finitely many elements of $\{U_i\}_{i \in I}$. Since $\overline{U_i}$ is compact, there is a real number ϵ_i such that $(p, t) \in U$ for every $p \in U_i$ and $t \leq \epsilon_i$. For each i, define:

$$\tau_i := \min\{\epsilon_j \mid U_i \cap U_j \neq \emptyset\}$$

Now, let $\{\rho_i\}_{i \in I}$ be a partition of unity subordinate to $\{U_i\}_{i \in I}$. Then $f(p) := \sum_{i \in I} \tau_i \rho_i(p)$ is the required smooth function.

Now, let F be the diffeomorphism

$$F : L \times \mathbb{R} \to L \times \mathbb{R}, \quad F(p, t) := (p, t - f(p)). \tag{4.1}$$

Therefore, $F^{-1}(L \times (-\infty, 0))$ is a neighborhood of $-\infty$ and F restricts to a diffeomorphism

$$F^{-1}(L \times (-\infty, 0)) \to L \times (-\infty, 0) \cong L \times \mathbb{R}.$$

We repeat the process above in the presence of the action of a compact Lie group K. To begin, we first show that, if K acts on L, then there is a K-invariant smooth function $f : L \to \mathbb{R}$ such that $(p, \lambda) \in U$ for any $\lambda \leq f(p)$.

Let $\{U_i\}_{i \in I}$ be an open cover of L chosen as above. For each i, $V_i := K \cdot U_i$ is an open G-invariant subset contained in the compact subset $K \cdot \overline{U_i}$. Then for a possibly smaller indexing set $J \subset I$, we have that

$\{V_j\}_{j\in J}$ is an open cover of L by K-invariant subsets such that each V_j has a nontrivial intersection with only finitely many other V_k in $\{V_j\}_{j\in J}$.

Now, let $\{\psi_j\}_{j\in J}$ be a partition of unity subordinate to $\{V_j\}_{j\in J}$. Let $d\lambda$ be a K-invariant metric on K with $\int_K d\lambda = 1$. Then we average ψ_j to a K-invariant function

$$\bar{\psi}_j(p) := \int_K \psi_j(k\cdot p)\, d\lambda.$$

Since V_j is K-invariant, $\mathrm{supp}(\tilde{\psi}_j) \subset K\cdot \mathrm{supp}(\psi_j) \subset V_j$. Therefore, since for every $p \in L$,

$$\sum_{j\in J} \tilde{\psi}_j(p) = \sum_{j\in J} \left(\int_K \psi_j(k\cdot p)\, d\lambda \right) = \int_K \left(\sum_{j\in J} \psi_j(k\cdot p) \right) d\lambda = 1,$$

$\{\tilde{\psi}_j\}_{j\in J}$ is a K-invariant partition of unity subordinate to $\{V_j\}_{j\in J}$.

As above, for each j there is a δ_j such that, for every $p \in V_j$ and $t \le \delta_j$, $(p,t) \in U$. Defining

$$\sigma_j := \min\{\delta_j \mid V_j \cap V_k \neq \emptyset\},$$

we have that $f(p) := \sum_{j\in J} \sigma_j \tilde{\psi}_j(p)$ is the required smooth, K-invariant function.

Thus, for F as in equation (4.1) defined with respect to the K-invariant function f is K-equivariant. Therefore, $F^{-1}(L\times(-\infty,0))$ is a K-invariant neighborhood of $-\infty$ diffeomorphic via F to $L\times(-\infty,0) \cong L\times\mathbb{R}$. □

Proof of Proposition 4.1.12. By Proposition 4.1.9, (M,ω) extends to a singular symplectic cone $(\tilde{M},\omega)$. To show that the action of G extends to a continuous action on $\tilde{M}$ fixing the singularity of $\tilde{M}$, it is enough to show that every neighborhood of $-\infty$ of M contains a G-*invariant* neighborhood of $-\infty$.

Let $B := M/\mathbb{R}$. By Proposition 2.2.22, there is a G-equivariant trivialization of M as a principal $\mathbb{R}$-bundle $\varphi : M \to B\times\mathbb{R}$. As discussed in the proof of Proposition 4.1.8, φ induces a bijection between the neighborhoods of $-\infty$ of M and the neighborhoods of $-\infty$ of $B\times\mathbb{R}$. Thus, since φ is G-equivariant, it is enough to build a G-invariant neighborhood of $-\infty$ inside every neighborhood of $-\infty$ in $B\times\mathbb{R}$; this is exactly the content of Lemma 4.1.13.

So the action of G on M extends to an action on $\tilde{M}$ fixing the singularity. Finally, as explained in Remark 4.1.11, since μ is continuous, it extends by 0 to a continuous map on $\tilde{M}$. □

As we will be interested in the orbital moment maps of spaces locally modeled on singular symplectic toric cones, we now prove that the quotient of a singular symplectic toric cone must be a cornered stratified space.

Lemma 4.1.14. Let $(X, \omega, \mu : X \to \mathfrak{g}^*)$ be a singular symplectic toric cone. Then for $B := X_{\mathsf{reg}}/\mathbb{R}$, X/G is a cornered stratified space with link B/G.

Proof. As explained in Proposition 2.2.20, B has a natural contact structure ξ and the action of G on X_{reg} descends to a contact toric action on (B, ξ). By Proposition 4.1.8, any trivialization $\phi : X_{\mathsf{reg}} \to B \times \mathbb{R}$ extends to a homeomorphism $\tilde{\varphi} : X \to c(B)$. Furthermore, by Proposition 2.2.22, we may choose φ to be a G-equivariant trivialization.

As the actions of G and $\mathbb{R}$ commute and $\tilde{\varphi}$ is G-equivariant, $\tilde{\varphi}$ descends to a homeomorphism $\bar{\varphi} : X/G \to c(B/G)$. By Lemma 2.2.31, B/G is a manifold with corners and therefore $\tilde{\varphi}$ gives local trivialization data for the singularity x_0 of X as a cornered stratified space. □

To finish this section, we prove that any isomorphism of symplectic toric cones extends to an isomorphism between their extensions as singular symplectic toric cones.

Lemma 4.1.15. Let $(X, \omega, \mu : X \to \mathfrak{g}^*)$ and $(X', \omega', \mu' : X' \to \mathfrak{g}^*)$ be two singular symplectic toric cones for which there is an isomorphism of symplectic toric cones

$$f : (X_{\mathsf{reg}}, \omega, \mu|_{X_{\mathsf{reg}}}) \to ({X'}_{\mathsf{reg}}, \omega', \mu'|_{{X'}_{\mathsf{reg}}}).$$

Then f extends to an isomorphism of singular symplectic toric cones.

Proof. Since f is $(G \times \mathbb{R})$-equivariant, f must take G-invariant neighborhoods of $-\infty$ in $(X_{\mathsf{reg}}, \omega)$ to G-invariant neighborhoods of $-\infty$ in $({X'}_{\mathsf{reg}}, \omega')$. f^{-1} satisfies the same property and, as in the proofs above, we may conclude that f and f^{-1} extend to maps on the respective symplectic toric cones. □

4.2 Symplectic toric stratified spaces with isolated singularities

After all the work of the previous section, we are finally ready to give a definition of symplectic toric stratified spaces with isolated singularities. Recall we've fixed a torus G with lie algebra $\mathfrak{g}$.

Definition 4.2.1. A symplectic toric stratified space with isolated singularities is a stratified space with isolated singularities $((X, X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I} \{x_\alpha\}))$ (see Definition 4.1.2) with a symplectic form $\omega \in \Omega^2(X_{\mathsf{reg}})$, a continuous map $\mu : X \to \mathfrak{g}^*$, and an action of torus G such that G fixes each x_α and restricts to a smooth, toric action on $(X_{\mathsf{reg}}, \omega)$ with moment map $\mu|_{X_{\mathsf{reg}}} : X_{\mathsf{reg}} \to \mathfrak{g}^*$. Furthermore, for each x_α, we require that there exist a G-invariant neighborhood U of x_α in X, a toric singular symplectic cone $(C, \omega, \nu : C \to \mathfrak{g}^*)$, a G-invariant neighborhood V of the vertex of C, and a G-equivariant homeomorphism $\varphi : U \to V$ such that

- $\varphi(x_\alpha) = *$ (for $*$ the vertex of C);
- φ restricts to a symplectomorphism between U_{reg} and V_{reg}; and
- $\mu|_U = \nu \circ \varphi + \mu(x_\alpha)$.

Such an object is denoted by the triple (X, ω, μ) (with the partition of X left implicit).

Remark 4.2.2. We still call μ a moment map for the full stratified space. Indeed, one may think of μ as a map of stratified spaces, serving as a trivial moment map to each zero dimensional symplectic manifold $\{x_\alpha\}$.

As explained in Proposition 4.1.12, any symplectic toric cone with homogeneous moment map can be extended to a singular symplectic cone; these serve for now as our only example of a symplectic toric stratified space with isolated singularities.

Analogous to [19] and Chapter 3, symplectic toric stratified spaces are grouped together together by orbital moment map. The orbital moment maps of symplectic toric stratified spaces are *stratified unimodular local embeddings*:

Definition 4.2.3. Let $(W, W_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I}\{w_\alpha\}))$ be a cornered stratified space with isolated singularities. Then a continuous map $\psi : W \to \mathfrak{g}^*$ is a stratified unimodular local embedding if

- $\psi|_{W_{\mathsf{reg}}}$ is a unimodular local embedding; and
- For each α, there exists a local trivialization datum $\varphi_\alpha : U_\alpha \to c(L_\alpha)$ for w_α in W and a homogeneous unimodular local embedding (see Definition 3.1.1) $\phi_\alpha : L_\alpha \times \mathbb{R} \to \mathfrak{g}^*$ such that $\psi|_{U_{\alpha\,\mathsf{reg}}} = \phi_\alpha \circ \varphi_\alpha + \psi(w_\alpha)$, where ϕ_α is homogeneous with respect to the action by translation on $L_\alpha \times \mathbb{R}$.

We will call the piece of local trivialization datum $\varphi_\alpha : U_\alpha \to c(L_\alpha)$ as above a homogeneous local trivialization datum.

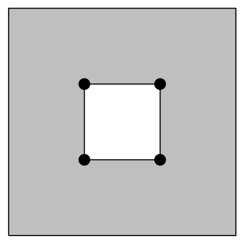
Any unimodular local embedding $\psi : W \to \mathfrak{g}^*$ is a stratified unimodular local embedding for which W is a cornered stratified space with respect to the trivial partition $W_{\mathsf{reg}} = W$. A couple more exotic examples are given below:

Example 4.2.4. Let Δ be the octahedron in $\mathbb{R}^3$; that is, the convex hull of the 6 points

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

Then Δ is a cornered stratified space with singularities the vertices of Δ. The inclusion of Δ into $\mathbb{R}^3$ (identified with the Lie algebra dual of the three torus) is a stratified unimodular local embedding.

Example 4.2.5. The inclusion of the region



into $\mathbb{R}^2$ (identified with the Lie algebra dual of the two torus) is a stratified unimodular local embedding. Here, the singularities of this cornered stratified space are the black vertices. More generally, every covering map of the above region is a stratified unimodular local embedding.

Proposition 4.2.6. Suppose $(X, X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I} \{x_\alpha\}))$ is a stratified space with isolated singularities and that $(X, \omega, \mu : X \to \mathfrak{g}^*)$ is a symplectic toric stratified space with isolated singularities. Then X/G is a cornered stratified space with isolated singularities and, for quotient map $\pi : X \to X/G$, the unique map $\bar{\mu} : X/G \to \mathfrak{g}^*$ satisfying $\bar{\mu} \circ \pi = \mu$ is a stratified unimodular local embedding.

Proof. By Proposition 2.1.8, as $(X_{\mathsf{reg}}, \omega, \mu|_{X_{\mathsf{reg}}})$ is a symplectic toric manifold, we have that X_{reg}/G is a manifold with corners and $\bar{\mu}|_{X_{\mathsf{reg}}/G}$ is a unimodular local embedding. It remains to show that ψ factors through (a translation of) a homogeneous unimodular local embedding near each singularity $[x_\alpha]$.

We have already by definition that, for any singular point x_α, there is a G-invariant neighborhood U_α together with a G-equivariant embedding $\varphi_\alpha : U_\alpha \to C_\alpha$ of U_α onto a neighborhood of the cone point of a singular symplectic toric cone $(C_\alpha, \omega_\alpha, \nu_\alpha : C_\alpha \to \mathfrak{g}^*)$ so that $\mu|_{U_\alpha} = \nu_\alpha \circ \varphi_\alpha + \mu(x_\alpha)$. Thus, φ_α descends to an embedding of $\bar{\varphi}_\alpha : U_\alpha/G \to C_\alpha/G$. From Lemma 4.1.14, we have that C_α/G is isomorphic to $c(L_\alpha)$ for a compact manifold with corners L_α. Therefore, $\bar{\varphi}_\alpha$ is a local structure datum for X.

It is easy to check via the universal properties of quotients that $\bar{\mu}|_{U_\alpha/G} = \bar{\nu}_\alpha \circ \bar{\varphi}_\alpha + \mu(x_\alpha)$, for $\bar{\nu}_\alpha : C_\alpha/G \to \mathfrak{g}^*$ defined in the same fashion as $\bar{\mu}$. As shown in Proposition 3.1.3, $\bar{\nu}_\alpha|_{C_\alpha \setminus \{*\}}$ is a homogeneous unimodular local embedding with respect to the action of $\mathbb{R}$ on C_α/G descending from the action on C_α. It is clear that, with respect to the identification $C_\alpha/G \cong L_\alpha \times \mathbb{R} \subset C(L_\alpha)$, this $\mathbb{R}$ action corresponds to the action by translation on $L_\alpha \times \mathbb{R}$. Therefore, $\bar{\varphi}_\alpha$ is our required local structure datum. ☐

Now, we show that the homogeneous data associated to a stratified unimodular local embedding ψ near a singularity w_α of W is unique up to isomorphism:

Proposition 4.2.7. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then for each singularity w_0 of W, if $\phi : U \to C(L)$ and $\phi' : U' \to C(L')$ are two pieces of homogeneous local trivialization data with associated homogeneous unimodular local embeddings $\varphi : L \times \mathbb{R} \to \mathfrak{g}^*$ and $\varphi' : L' \times \mathbb{R} \to \mathfrak{g}^*$ (see Definition 4.2.3), then there is an isomorphism $f : C(L) \to C(L')$ satisfying $\varphi' \circ f|_{L\times\mathbb{R}} = \varphi$.

Proof. Without loss of generality, assume that $\psi(w_0) = 0$. First, note that φ is uniquely determined by $\psi|_U$. Indeed, ϕ maps U onto a neighborhood of the vertex $*$ of $C(L)$ and U_{reg} onto a neighborhood of $-\infty$ of $L \times \mathbb{R}$. Then, since $\varphi \circ \phi|_{U_{\mathsf{reg}}} = \psi$ and φ is homogeneous, $\varphi(w)$ is uniquely determined by $\varphi(w')$ for $w = e^t w'$ and $w' \in \phi(U_{\mathsf{reg}})$. Similarly, φ' is uniquely determined by $\psi|_{U'}$.

Let $g := \phi' \circ \phi^{-1}$. We will show that g extends to an $\mathbb{R}$-equivariant diffeomorphism $f : L \times \mathbb{R} \to L' \times \mathbb{R}$. For any $w \in \phi(U_{\mathsf{reg}} \cap U'_{\mathsf{reg}})$ and $t \in \mathbb{R}$ for which $t \cdot w \in \phi(U_{\mathsf{reg}} \cap U'_{\mathsf{reg}})$ as well, we have that

$$\varphi'(g(t \cdot w))) = \varphi(t \cdot w) = e^t\varphi(w) = e^t\psi(\phi^{-1}(w)) = e^t\varphi'(g(w)) = \varphi'(t \cdot g(w)).$$

By Lemma 3.1.4, we can pick an $\mathbb{R}$-invariant neighborhood of $g(w)$ on which φ' is an open embedding. It follows that, where defined, g is $\mathbb{R}$-equivariant. As $\phi(U_{\mathsf{reg}} \cap U'_{\mathsf{reg}})$ contains a slice Σ of the $\mathbb{R}$-action on $L \times \mathbb{R}$, we therefore may conclude that

$$f : L \times \mathbb{R} \to L' \times \mathbb{R},\ g(t \cdot p) := t \cdot g(p) \text{ for } p \in \Sigma$$

is a well-defined extension of g satisfying $\varphi' \circ f = \varphi$. ☐

We can now define our main category of interest.

Definition 4.2.8 (Symplectic toric stratified spaces over ψ and $\mathsf{STSS}_\psi(W)$)**.** Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding for W a cornered stratified space with isolated singularities. A symplectic toric stratified space over ψ is a symplectic toric stratified space $(X, \omega, \mu : X \to \mathfrak{g}^*)$ together with a G-quotient map $\pi : X \to W$ so that $\psi \circ \pi = \mu$.

Any symplectic toric stratified space over ψ will be denoted by the triple $(X, \omega, \pi : X \to W)$ (again, the partition of X is left as implicit).

A map of symplectic toric stratified spaces over ψ between $(X, \omega, \pi : X \to W)$ and $(X', \omega', \pi' : X' \to W)$ is a G-equivariant isomorphism of stratified spaces $\varphi : X \to X'$ that restricts to a symplectomorphism between $(X_{\mathsf{reg}}, \omega)$ and $(X'_{\mathsf{reg}}, \omega')$ and satisfies $\pi' \circ \varphi = \pi$.

The category $\mathsf{STSS}_\psi(W)$ of symplectic toric stratified spaces over ψ is the groupoid with objects and morphisms as described above.

Remark 4.2.9. Note that for any open subset $U \subset W$, $\psi|_U$ is also a stratified unimodular local embedding. Therefore, it makes sense to define the presheaf

$$\mathsf{STSS}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids} \quad U \mapsto \mathsf{STSS}_{\psi|_U}(U).$$

Here, for $V \subset U$ open subsets, restriction is defined as

$$(X, \omega, \pi : X \to U)|_V := (\pi^{-1}(V), \omega|_{\pi^{-1}(V)}, \mu|_{\pi^{-1}(V)}).$$

Since morphisms in each groupoid $\mathsf{STSS}_\psi(U)$ must preserve G-quotients, the restriction of a morphism $f : (X, \omega, \pi : X \to U) \to (X', \omega', \pi' : X' \to U)$ descends to a unique morphism in $\mathsf{STSS}_\psi(V)$, giving a well-defined choice for $f|_V$. It is easy to check that these restriction maps satisfy the necessary requirements for a presheaf.

Implicit here is the fact that, for U not containing singularities, the condition that an object of $\mathsf{STSS}_\psi(U)$ must be modeled on certain neighborhoods of singular symplectic toric cones is empty. Hence, on U the presheaves STSS_ψ and $\mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}$ (see Definition 2.1.9) are equal.

Given a stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, the restriction functor

$$\rho_{W W_{\mathsf{reg}}} : \mathsf{STSS}_\psi(W) \to \mathsf{STSS}_\psi(W_{\mathsf{reg}}) = \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}}),$$

which from here forward we will denote res, will be important in Section 4.4 when showing the functor $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$ is fully faithful. We provide a specific definition of res now for completeness and for later citation:

Definition 4.2.10. Given a stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, let

$$\mathrm{res} : \mathsf{STSS}_\psi(W) \to \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

be the functor taking a symplectic toric stratified space $(X, \omega, \mu : X \to \mathfrak{g}^*)$ to the symplectic toric manifold $(X_{\mathsf{reg}}, \omega, \mu|_{X_{\mathsf{reg}}})$ and a morphism of symplectic toric stratified spaces $\varphi : (X, \omega, \mu) \to (X', \omega', \mu')$ to the restriction $\varphi|_{X_{\mathsf{reg}}} : (X_{\mathsf{reg}}, \omega, \mu|_{X_{\mathsf{reg}}}) \to (X'_{\mathsf{reg}}, \omega', \mu'|_{X'_{\mathsf{reg}}})$ (which is, by definition, a moment map preserving symplectomorphism).

Lemma 4.2.11. For any stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, the functor

$$\mathrm{res} : \mathsf{STSS}_{\psi}(W) \to \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

is fully faithful.

Proof. Let's start by showing res is faithful. Given any two symplectic toric stratified spaces over ψ $(X, \omega, \pi : X \to W)$ and $(X', \omega', \pi' : X' \to W)$, for each singularity x_α of X, it must be the case that any map φ of symplectic toric stratified spaces over ψ must satisfy $\varphi(x_\alpha) = \pi'^{-1}(\pi(x_\alpha))$. Since the singularities of both X and X' are in bijection with those of W via their respective quotient maps, there is no question as to where φ must send each singularity.

For convenience, write $x'_\alpha := \pi'^{-1}(\pi(x_\alpha))$. Then, by the logic above, every φ must satisfy $\varphi(x_\alpha) = x'_\alpha$, so *if* a map of symplectic toric manifolds $\phi : (X_{\mathsf{reg}}, \omega, \mu|_{X_{\mathsf{reg}}}) \to (X'_{\mathsf{reg}}, \omega', \mu'|_{X'_{\mathsf{reg}}})$ admits an extension to a map of symplectic toric stratified spaces, this extension must be unique. Thus, res is faithful.

To show it is full, it is enough to show the unique extension of ϕ described above is continuous. This follows much as in the proof of Proposition 4.1.12. As a neighborhood of each singularity of X' is symplectomorphic to a neighborhood of $-\infty$ of a symplectic cone, we have seen in the previously cited proposition that every neighborhood of the singularity may be written as the union of a G-invariant neighborhood of $-\infty$ U and some open subset V of X'_{reg}. Then $\phi^{-1}(U \cup V) = \phi^{-1}(U) \cup \phi^{-1}(V)$. $\phi^{-1}(V)$ is an open subset of X_{reg} which we will now ignore.

It remains to show $\phi^{-1}(U)$ is a (deleted) open neighborhood of x_α in X. As ϕ is G-equivariant and U is G-invariant, $\phi^{-1}(U)$ is also G-invariant. Thus, $\phi^{-1}(U) = \pi^{-1}(\pi'(U))$. So, since $\pi'(U \sqcup \{x'_\alpha\})$ is an open neighborhood of $\pi'(x'_\alpha)$ in W, it follows $\phi^{-1}(U \sqcup \{x'_\alpha\})$ is an open neighborhood of x_α in X. Thus, the extension of ϕ is continuous. By the same logic, we may also show that the extension of ϕ^{-1} also continuous and therefore ϕ extends to a homeomorphism between X and X' covering the identity on W. So res is full. $\square$

Remark 4.2.12. It is important to note that res is in general *not* essentially surjective. Indeed, for $\psi : W \to \mathfrak{g}^*$ a stratified unimodular local embedding, a symplectic toric manifold over $\psi|_{W_{\mathsf{reg}}}$ $(M, \omega, \pi : \psi|_{W_{\mathsf{reg}}})$

only comes from a symplectic toric stratified space over ψ when near each singularity, M is equivariantly symplectomorphic to a neighborhood of $-\infty$ in a symplectic toric cone. See Example 4.5.14 for a specific example of a stratified unimodular local embedding for which res is not essentially surjective.

To finish this section, we describe how symplectic toric stratified spaces may be "pulled back" over certain open embeddings of cornered stratified spaces. This will be important later in Section 4.4 when constructing the functor $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$.

Lemma 4.2.13. Let $\psi : W \to \mathfrak{g}^*$ and $\psi' : W' \to \mathfrak{g}^*$ be two stratified unimodular local embeddings and let $\varphi : W' \to W$ be an open embedding of cornered stratified spaces with $\psi \circ \varphi = \psi'$. Then for any $(X, \omega, \pi : X \to W) \in \mathsf{STSS}_\psi(W)$, there exists a stratified symplectic toric space $(X', \omega', \pi' : X' \to W')$ and a G-equivariant isomorphism of symplectic stratified spaces $\tilde{\varphi} : (X', \omega') \to (X, \omega)|_{\varphi(W')}$ with $\pi \circ \tilde{\varphi} = \varphi \circ \pi'$. We denote $(X', \omega', \pi' : X' \to W')$ by $\varphi^*(X, \omega, \pi : X \to W)$.

Proof. Let $\varphi(W') := U$. As a stratified symplectic G-space, we simply take $(X', \omega') = (X, \omega)|_U$. Since φ is an open embedding, we have a homeomorphism $\varphi^{-1} : U \to W'$ and so the map $\pi' := \varphi^{-1}|_U \circ \pi : X' \to W'$ is a G-quotient for X'. Since $(X, \omega)|_U$ has moment map

$$\psi \circ \pi = \psi \circ \varphi \circ \varphi^{-1} \circ \pi = \psi' \circ \pi',$$

it follows that $(X', \omega', \pi' : X' \to W')$ is a symplectic toric stratified space over $\psi' : W' \to \mathfrak{g}^*$. By design, $\tilde{\varphi}$ is simply the embedding of $X' = \pi^{-1}(U)$ into X. ☐

Remark 4.2.14. Let $\psi : W \to \mathfrak{g}^*$ and $\psi' : W' \to \mathfrak{g}^*$ be two stratified unimodular local embeddings and let $\varphi : W' \to W$ be an open embedding of cornered stratified spaces with $\psi \circ \varphi = \psi'$. Also let $(X, \omega, \pi : X \to W)$ and $(X', \omega', \pi' : X' \to W)$ be two symplectic toric stratified spaces in $\mathsf{STSS}_\psi(W)$ with pullbacks $\varphi^*(X, \omega, \pi : X \to W)$ and $\varphi^*(X', \omega', \pi' : X' \to W)$ and with isomorphisms

$$\tilde{\varphi} : \varphi^*(X, \omega, \pi : X \to W) \to (X, \omega, \pi : X \to W)|_{\psi(W')}$$

and

$$\tilde{\varphi}' : \varphi^*(X', \omega', \pi' : X' \to W) \to (X', \omega', \pi' : X' \to W)|_{\psi(W')}$$

of Lemma 4.2.13. Then any isomorphism

$$f : (X, \omega, \pi : X \to W) \to (X', \omega', \pi' : X' \to W)$$

(also in $\mathsf{STSS}_\psi(W)$) pulls back via φ to an isomorphism in $\mathsf{STSS}_{\psi'}(W)$. Explicitly, f induces the isomorphism

$$\varphi^* f := \tilde{\varphi}'^{-1} \circ f|_{\varphi(W')} \circ \tilde{\varphi} : \varphi^*(X, \omega, \pi : X \to W) \to \varphi^*(X', \omega', \pi' : X' \to W).$$

It is easy to check that $\varphi^* f$ must be an isomorphism in $\mathsf{STSS}_{\psi'}(W')$.

4.3 Conical symplectic toric bundles

Analogous to [19] and Chapter 3, we define a category $\mathsf{CSTB}_\psi(W)$ of principal G-bundles associated to a stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$ with some special properties. These will be called conical symplectic toric bundles; they are symplectic toric bundles locally modeled on homogeneous symplectic toric bundles over deleted open neighborhoods of the singularity of the base cornered stratified space. In Section 4.4, we will show that there is an equivalence of categories $\tilde{\mathsf{c}}$ between $\mathsf{CSTB}_\psi(W)$ and $\mathsf{STSS}_\psi(W)$ and in Section 4.5, we classify conical symplectic toric bundles and therefore, via $\tilde{\mathsf{c}}$, classify symplectic toric stratified spaces with isolated singularities.

Definition 4.3.1 (Conical symplectic toric bundles and $\mathsf{CSTB}_\psi(W)$)**.** Let $(W, W_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I} \{w_\alpha\}))$ be a cornered stratified space with isolated singularities and let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then a conical symplectic toric principal G-bundle is a symplectic toric bundle $(\pi : P \to W_{\mathsf{reg}}, \omega)$ over $\psi|_{W_{\mathsf{reg}}}$ (see Definition 2.1.10) satisfying the following local condition: for each singularity w_α of W, there exists

- a neighborhood U of w_α in W with a homogeneous local trivialization datum $\varphi : U \to c(L)$ and homogeneous unimodular local embedding $\phi : L \times \mathbb{R} \to \mathfrak{g}^*$ (see Definition 4.2.3);
- a homogeneous symplectic toric bundle $(\varpi : Q \to L \times \mathbb{R}, \eta)$; and
- for $V := \varphi(U)$, there is a G-equivariant symplectomorphism $\tilde{\varphi} : (P, \omega)|_{U_{\mathsf{reg}}} \to (Q, \eta)|_{V_{\mathsf{reg}}}$ so that the diagram

$$\begin{array}{ccc} P|_{U_{\mathsf{reg}}} & \xrightarrow{\tilde{\varphi}} & Q|_{V_{\mathsf{reg}}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \varpi} \\ U & \xrightarrow[\varphi]{} & c(L) \end{array}$$

  commutes.

The category $\mathsf{CSTB}_\psi(W)$ of conical symplectic toric bundles over ψ is the groupoid with objects as described above and morphisms G-equivariant symplectomorphisms covering the identity on W_{reg}.

We now remark that, given a conical symplectic toric bundle (P, ω) over a stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, *every* piece of homogeneous local trivialization data for ψ has an associated piece of data defined on P. Further, we show that we may pick a particularly nice homogeneous symplectic toric bundle to compare P to:

Remark 4.3.2. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $(\pi : P \to W_{\mathsf{reg}}, \omega)$ be a conical symplectic toric bundle. Fix a singularity w_α and let $\varphi : U \to c(L)$, $\phi : L \times \mathbb{R} \to \mathfrak{g}^*$, $(\varpi : Q \to L \times \mathbb{R}, \eta)$, and $\tilde{\varphi} : (P, \omega)|_{U_{\mathsf{reg}}} \to (Q, \eta)|_{V_{\mathsf{reg}}}$ be as in Definition 4.3.1. Given another choice of homogeneous local trivialization data $\varphi' : U' \to C(L')$ and homogeneous unimodular local embedding $\phi' : L' \times \mathbb{R} \to \mathfrak{g}^*$ near the singularity w_α, by Proposition 4.2.7, there is an isomorphism $f : L' \times \mathbb{R} \to L \times \mathbb{R}$ such that $\phi \circ f = \phi'$.

Then $\varpi' : f^*(Q, \eta) \to L' \times \mathbb{R}$ is a homogeneous symplectic toric bundle over ϕ' and for $\tilde{f} : f^*(Q, \eta) \to (Q, \eta)$ the natural isomorphism of principal G-bundles covering f, we have $\varpi' \circ (\tilde{f}^{-1} \circ \tilde{\varphi}) = (f \circ \varphi) \circ \pi$. Thus, $\varphi' : U' \to C(L')$, $\phi' : L' \times \mathbb{R} \to \mathfrak{g}^*$, $f^*(Q, \eta)$, and $\tilde{f}^{-1} \circ \tilde{\varphi}$ also give (P, ω) a homogeneous structure near w_α.

Remark 4.3.3. As in the case of STSS_ψ (see Remark 4.2.9), we have a presheaf of groupoids

$$\mathsf{CSTB}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids},\ U \mapsto \mathsf{CSTB}_{\psi|_U}(U).$$

Open subsets U of W not containing singularities renders the extra conditions of Definition 4.3.1 empty and here $\mathsf{CSTB}_\psi(U)$ and $\mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(U)$ (the category of symplectic toric bundles over U, see Definition 2.1.10) are equal.

In fact, CSTB_ψ is a stack over W. As the proof of this is more or less just a retelling of the proof that the presheaf of principal bundles over a topological space is a stack, we relegate this proof to Appendix B (see Proposition B.9).

As in the case of symplectic toric stratified spaces (see Lemma 4.2.13), conical symplectic toric bundles may be pulled back over open embeddings of conical stratified spaces that preserve the respective stratified unimodular local embeddings.

Lemma 4.3.4. Let $\psi : W \to \mathfrak{g}^*$ and $\psi' : W' \to \mathfrak{g}^*$ be two stratified unimodular local embeddings and let $\varphi : W' \to W$ be an open embedding of cornered stratified spaces with $\psi' \circ \varphi = \psi$. Then, for $(\pi : P \to$

$W_{\mathsf{reg}}, \omega) \in \mathsf{CSTB}_\psi(W)$, there exists a conical symplectic toric bundle $(\pi' : P' \to W', \omega')$ and a G-equivariant symplectomorphism

$$\tilde{\varphi} : (\pi' : P' \to W'_{\mathsf{reg}}, \omega') \to (\pi : P \to W_{\mathsf{reg}}, \omega)|_{\varphi(W')}$$

with $\pi \circ \tilde{\varphi} = \varphi \circ \pi'$. We denote (P', ω') as $\varphi^*(P, \omega)$.

Proof. As a symplectic G-space, let $(P', \omega') = (P, \omega)|_{\varphi(W')}$. Since $\varphi : W' \to W$ is an open embedding, there is an inverse homeomorphism $\varphi^{-1} : \varphi(W') \to W'$. Let $\pi' : P' \to W'$ be the map $\varphi^{-1} \circ \pi$. Then clearly $(\pi' : P' \to W'_{\mathsf{reg}}, \omega')$ is a conical symplectic toric bundle over ψ' and the embedding from $P' = P|_{\varphi(W')}$ to P satisfies the requirements for $\tilde{\varphi}$. $\square$

Pullbacks allow us to more efficiently describe conical symplectic toric bundles in a way that will be useful in constructing the functor $\tilde{\mathsf{c}}$.

Lemma 4.3.5. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then a symplectic toric bundle $(\pi : P \to W_{\mathsf{reg}})$ is a conical symplectic toric bundle exactly when, for each singularity w_α of W, there exists

- an open neighborhood U of w_α;
- a homogeneous local trivialization datum $\varphi : U \to c(L)$ with homogeneous unimodular local embedding $\phi : L \times \mathbb{R} \to \mathfrak{g}^*$ satisfying $\psi|_{U_{\mathsf{reg}}} = \phi \circ \varphi + \psi(w)$; and
- a homogeneous symplectic toric bundle $(\varpi : Q \to L \times \mathbb{R}, \eta) \in \mathsf{HSTB}_\phi(L \times \mathbb{R})$

so that, thinking of (Q, η) as a symplectic toric bundle over $\varphi + \psi(w_\alpha)$ (see Remark 2.1.17), $\varphi^*(Q, \eta)$ and $(P, \omega)|_U$ are isomorphic in $\mathsf{CSTB}(U)$.

Proof. This is easily confirmed from the definition of a conical symplectic toric bundle (see Definition 4.3.1) and the description of a pullback of a conical symplectic toric bundle (see Lemma 4.3.4). $\square$

We now discuss an important fully faithful functor ι that will be vital both for defining $\tilde{\mathsf{c}}$ and showing that $\tilde{\mathsf{c}}$ is fully faithful.

Definition 4.3.6. Given a stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, let

$$\iota : \mathsf{CSTB}_\psi(W) \to \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

denote the forgetful functor. Explicitly, as an object $(\pi : P \to W_{\mathsf{reg}}, \omega)$ in $\mathsf{CSTB}_\psi(W)$ is in particular a symplectic toric G-bundle over W_{reg}, we may define

$$\iota(\pi : P \to W_{\mathsf{reg}}, \omega) := (\pi : P \to W_{\mathsf{reg}}, \omega) \in \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

and, as the morphisms of $\mathsf{CSTB}_\psi(W)$ are symplectic isomorphisms of principal G-bundles, any morphism in $\mathsf{CSTB}_\psi(W)$ descends to a morphism $\iota(\varphi)$ in $\mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$.

Lemma 4.3.7. For any stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, $\iota : \mathsf{CSTB}_\psi(W) \to \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$ is fully faithful.

Proof. As explained in Remark 4.3.3, $\mathsf{CSTB}_\psi(W_{\mathsf{reg}}) = \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$. This is because both are exactly the groupoid of symplectic toric bundles over the unimodular local embedding $\psi|_{W_{\mathsf{reg}}} : W_{\mathsf{reg}} \to \mathfrak{g}^*$. With this identification in mind, ι is just the restriction functor in the presheaf CSTB_ψ taking elements of $\mathsf{CSTB}_\psi(W)$ to $\mathsf{CSTB}_\psi(W_{\mathsf{reg}})$. That ι is fully faithful is more or less obvious from its definition. □

It is not clear that $\mathsf{CSTB}_\psi(W)$ is non-empty for a general choice of $\psi : W \to \mathfrak{g}^*$. Analogous again to [19] and Chapter 3, we can use a connection 1-form on any principal bundle over W_{reg} to create a symplectic form with respect to which the corresponding bundle is conical symplectic toric. We will need the following fact about connections on principal bundles.

Lemma 4.3.8. Let $\pi : P \to B$ be a principal K-bundle with connection 1-form A, where K is a commutative Lie group. Suppose also that η is any element of K's Lie algebra dual $\mathfrak{k}^*$ and $f : P \to \mathbb{R}$ is any K-invariant function on P. Then $d\langle \eta, fA\rangle$ is a basic 2-form.

Proof. As K is commutative, note that dA is exactly the curvature of A. Furthermore, it is a standard fact (see, for instance, Proposition 6.39, pp. 266, [29]) that dA is:

1. Horizontal: $dA|_{\ker \pi} = 0$

2. Equivariant: $\rho_k^* dA = \mathrm{Ad}_{k^{-1}} \circ dA$ for any $k \in K$.

Again, as K is commutative, $\mathrm{Ad}_k = id_{\mathfrak{k}}$ for any $k \in K$, so the second condition actually implies that dA is K-invariant. It follows that dA is basic (again, this is standard; see Lemma 6.44, pp. 275, [29]).

Now, since $d(fA) = df \wedge A + f dA$ and f is K-invariant, $d(fA)|_{\ker \pi} = 0$. Also,

$$\rho_k^* d(fA) = d(\rho_k^* f \rho_k^* A) = d(fA)$$

for every $k \in K$. Therefore, the form $d\langle \eta, fA\rangle = \langle \eta, d(fA)\rangle$ is basic. $\square$

Proposition 4.3.9. For $\psi : W \to \mathfrak{g}^*$ a stratified unimodular local embedding and any principal bundle $\pi : P \to W_{\mathsf{reg}}$, there exists an exact G-invariant symplectic form ω on P so that $(\pi : P \to W_{\mathsf{reg}}, \omega)$ is a conical symplectic toric bundle over ψ.

Proof. Recall from Definition 4.2.3, for each singularity w_α in W, there exists an open subset U_α of w_α in W, local trivialization datum $\varphi_\alpha : U_\alpha \to c(L_\alpha)$, and a homogeneous unimodular local embedding $\phi_\alpha : L_\alpha \times \mathbb{R} \to \mathfrak{g}^*$ such that $\psi|_{U_\alpha} = \phi_\alpha \circ \varphi_\alpha + \psi(w_\alpha)$. Fix such a piece of data for each α. As the singularities of W are isolated, we may also assume that the U_α are pairwise disjoint.

Complete the set $\{U_\alpha\}_{\alpha \in I}$ to an open cover of all W with an open set $U_0 \subset W_{\mathsf{reg}}$ so that, for each α, there is an open neighborhood of w_α $U'_\alpha \subset U_\alpha$ with $U'_\alpha \cap U_0$ empty. Note that such a subset U_0 exists as, for each α, the subset $U_\alpha \backslash \{w_\alpha\}$ is diffeomorphic to a neighborhood of $-\infty$ which itself contains a neighborhood of $-\infty$ V diffeomorphic to $L_\alpha \times \mathbb{R}$ (see Lemma 4.1.13). Then pick U'_α corresponding via this diffeomorphism to $L \times (-\infty, 0)$ and $U_0 \cap V$ corresponding to $L \times (1, \infty)$.

Now, for each α, we will build a G-invariant symplectic form for $P|_{(U_\alpha)_{\mathsf{reg}}}$ and show these forms may be patched together to a form with the properties we desire. Fix $\alpha \in I$. Our key tool will be Proposition 3.2.6. Without loss of generality, via Lemma 4.1.13, we may assume that $\varphi_\alpha : U_\alpha \to c(L_\alpha)$ is the inclusion $\iota : L_\alpha \times (-\infty, 0) \sqcup \{*\} \to c(L)$. Fix $\tau < 0$ and let $B := L_\alpha \times \{\tau\}$. Then since the actions of G and $\mathbb{R}$ on Q commute, $\varpi : Q \to B$ is a principal G-bundle as well. Note that the retraction $r : L_\alpha \times (-\infty, 0) \to L_\alpha \times (-\infty, 0)$ of $L_\alpha \times (-\infty, 0)$ onto B is homotopy equivalent to the identity on $L_\alpha \times (-\infty, 0)$.

Therefore, by Theorem 3.4.4, the bundles $r^*(P|_{L_\alpha \times (-\infty,0)})$ and $P|_{L_\alpha \times (-\infty,0)}$ are isomorphic via an isomorphism of principal G-bundles $f : P|_{L_\alpha \times (-\infty,0)} \to r^*(P|_{L_\alpha \times (-\infty,0)})$. Since we may represent the pullback $r^*(P|_{L_\alpha \times (-\infty,0)})$ by the bundle $(\varpi \times id) : Q \times (-\infty, 0) \to L_\alpha \times (-\infty, 0)$ (with an implicit identification of B with L_α), we will take the image of f to be this bundle.

Now, note that we may extend $Q \times (-\infty, 0)$ to the principal G-bundle $(\varpi \times id) : L_\alpha \times \mathbb{R} \to B \times \mathbb{R}$ and that $\varpi \times id$ is clearly $\mathbb{R}$-equivariant. Therefore, by Proposition 3.2.6, there exists a connection 1-form A' on $Q \times \mathbb{R}$ so that $(Q \times \mathbb{R}, d\langle \phi_\alpha \circ (\varpi \times id), A'\rangle)$ is a homogeneous symplectic toric bundle over ϕ_α.

Write $A_\alpha := f^*(A'|_{Q\times(-\infty,0)})$. Then, for $\iota' : Q\times(-\infty,0)\to Q\times\mathbb{R}$ the inclusion, we have:

$$
\begin{aligned}
f^*\iota'^*(d\langle\phi_\alpha\circ(\varpi\times id), A'\rangle) &= f^*(d\langle\phi_\alpha\circ\iota\circ(\varpi\times id), A'|_{Q\times(-\infty,0)}\rangle) \\
&= d\langle\phi_\alpha\circ\iota\circ\pi|_{(U_\alpha)_{\mathsf{reg}}}, A_\alpha\rangle \\
&= d\langle\psi\circ\pi|_{(U_\alpha)_{\mathsf{reg}}} - \psi(w_\alpha), A_\alpha\rangle
\end{aligned}
$$

To finish, let $\{A_\alpha\}_{\alpha\in I}$ be the collection of connection 1-forms on each $P|_{(U_\alpha)_{\mathsf{reg}}}$ selected as above and let A_0 be any connection 1-form for U_0 (for U_0 as in the beginning of this proof). Let $\{\{\rho_\alpha\}_{\alpha\in I},\rho_0\}$ be a partition of unity subordinate to the open cover $\{\{(U_\alpha)_{\mathsf{reg}}\}_{\alpha\in I}, U_0\}$ of W_{reg}. Then for $\rho'_\alpha := \rho_\alpha\circ\pi$, $\{\{\rho'_\alpha\}_{\alpha\in I},\rho_0\}$ is a G-invariant partition of unity subordinate to the cover $\{\{P|_{(U_\alpha)_{\mathsf{reg}}}\}_{\alpha\in I}, P|_{U_0}\}$ of P. The form $A := \rho'_0 A_0 + \sum_\alpha \rho'_\alpha A_\alpha$ is a connection 1-form for P.

By Lemma 4.3.8, $-d\langle\psi(w_\alpha),\rho'_\alpha A_\alpha\rangle$ is basic for each α. Define γ_α as the form on $(U_\alpha)_{\mathsf{reg}}$ such that

$$
\pi|^*_{(U_\alpha)_{\mathsf{reg}}}\gamma_\alpha := -d\langle\psi(w_\alpha),\rho'_\alpha A_\alpha\rangle.
$$

Note that $\operatorname{supp}(\gamma_\alpha)\subset U_\alpha$. Therefore, we may extend each γ_α by 0 to all W_{reg} and define $\gamma := \sum_{\alpha\in I}\gamma_\alpha$. Then $\pi^*\gamma$ is exact. By Lemma 2.1.18, the form $\omega := d\langle\psi\circ\pi, A\rangle + \pi^*\gamma$ is a G-invariant symplectic form on P with moment map $\psi\circ\pi$. It follows by our work above that $(\pi : P\to W_{\mathsf{reg}},\omega)$ is a conical symplectic toric bundle over ψ. □

Remark 4.3.10. Let $\pi : P\to W_{\mathsf{reg}}$ be a symplectic toric bundle and let $\varphi : P\to P$ be a gauge transformation. Then for open cover $\{\{U_\alpha\}_{\alpha\in I}, U_0\}$ and subordinate partition of unity $\{\{\rho_\alpha\}_{\alpha\in I},\rho_0\}$ as in Proposition 4.3.9, for $\rho'_\alpha := \rho_\alpha\circ\pi$, we have that $\rho'_\alpha\circ\varphi = \rho'_\alpha$.

So for η defined as in Proposition 4.3.9 with respect connection 1-forms $\{\{A_\alpha\}_{\alpha\in I}, A_0\}$ on $\{\{P|_{U_\alpha}\}_{\alpha\in I}, P|_{U_0}\}$ patching to the form A, we have that

$$
\varphi^*\eta - \eta = \sum_{\alpha\in I}\left(d\langle\psi\circ\pi,\varphi^*A - A\rangle - d\langle\psi(w_\alpha),\rho'_\alpha(A-\varphi^*A)\rangle\right)
$$

is exact with a basic primitive.

Now, we prove the result that will help us to characterize conical symplectic toric bundles (i.e., among all symplectic toric bundles):

Proposition 4.3.11. Let $\psi : W\to\mathfrak{g}^*$ be a stratified unimodular local embedding and let $\pi : P\to W_{\mathsf{reg}}$

be a principal G-bundle. If ω and ω' are any two symplectic forms for which (P,ω) and (P,ω') are conical symplectic toric bundles, then there is a open neighborhood U of each singularity of w such that $(\omega-\omega')|_{U_{\mathsf{reg}}} = d\pi^*\gamma$ for γ a 1-form on U_{reg}.

Proof. Fix a singularity w_0 of W. As discussed in Remark 4.3.2, every homogeneous local trivialization datum $\varphi : U \to c(L)$ with homogeneous unimodular local embedding $\phi : L \times \mathbb{R} \to \mathfrak{g}^*$ for W near w_0 has an associated homogeneous symplectic toric bundle serving as a model for P near w_0. That is, there are local models $(\varpi : Q \to L\times\mathbb{R}, \eta), (\varpi' : Q' \to L\times\mathbb{R}, \eta') \in \mathsf{HSTB}_\phi(L\times\mathbb{R})$ and G-invariant symplectic isomorphisms $f : (P,\omega)|_U \to (Q,\eta)|_{\varphi(U_{\mathsf{reg}})}$ and $f' : (P,\omega')|_U \to (Q',\eta')|_{\varphi(U_{\mathsf{reg}})}$ satisfying $\varpi \circ f = \varphi \circ \pi$ and $\varpi' \circ f' = \varphi \circ \pi$.

Then $\varpi \circ f = \varpi' \circ f'$, so $g := f' \circ f^{-1} : Q|_{\varphi(U_{\mathsf{reg}})} \to Q'|_{\varphi(U_{\mathsf{reg}})}$ is an isomorphism of principal G-bundles. We first show we may extend g to an isomorphism of principal G-bundles between all of Q and Q'. As $\varphi(U_{\mathsf{reg}})$ is a neighborhood of $-\infty$ in $L \times \mathbb{R}$, there is a slice $\Sigma \subset \varphi(U)$ for the $\mathbb{R}$-action on $L \times \mathbb{R}$. By Lemma 3.2.2, ϖ and ϖ' are $\mathbb{R}$-equivariant, so $Q|_\Sigma$ and $Q'|_\Sigma$ are slices for the actions of $\mathbb{R}$ on Q and Q' and $g(Q|_\Sigma) = Q'_\Sigma$. For each $p \in \Sigma$, if $t \cdot p \in Q|_\Sigma$, then

$$\varpi'(t \cdot g(p)) = t \cdot \varpi'(g(p)) = t \cdot \varpi(p) = \varpi(t \cdot p) = \varpi'(g(t \cdot p)).$$

Let $d : Q' \times_{L\times\mathbb{R}} Q' \to G$ be the division map defined by $q' \cdot d(q', q) = q$. Then for $p \in \Sigma$ and t as above, by definition $g(t \cdot p) = (t \cdot g(p)) \cdot d(t \cdot g(p), g(t \cdot p))$.

Let ρ be a non-negative function for which $\rho \equiv 1$ on V a neighborhood of $-\infty$ and satisfies $\operatorname{supp}(\rho) \subset \varphi(U)$. Define the map F as

$$F : Q \to Q', \quad F(t \cdot p) := t \cdot g(p) \cdot d(\rho(\varpi(p))t \cdot g(p), g(\rho(\varpi(p))t \cdot p)) \text{ for each } p \in Q|_\Sigma$$

In words, this is the map that over V corresponds to g and over $(L \times \mathbb{R})\backslash\operatorname{supp}(\rho)$ corresponds to the map $t \cdot p \mapsto t \cdot g(p)$. It is easy to check F is an isomorphism of principal G-bundles.

Finally, note that, for connection 1-forms A and A' chosen on Q and Q' as in Proposition 3.2.6, $(Q, d\langle \phi \circ \varpi, A\rangle)$ and $(Q', d\langle \phi \circ \varpi', A'\rangle)$ are homogeneous symplectic toric bundles. By Lemma 3.2.7, there exist 1-forms

γ, γ' on $L \times \mathbb{R}$ so that $\eta = d\langle \phi \circ \varpi, A\rangle - \varpi^* d\gamma$ and $\eta' = d\langle \phi \circ \varpi', A'\rangle - \varpi'^* d\gamma'$. Thus:

$$\begin{aligned}(\omega' - \omega)|_{\varphi^{-1}(V)} &= f'^*\eta'|_V - f^*\eta|_V \\ &= f^*((F^*\eta')|_V - \eta|_V) \\ &= f^*(F^*(d\langle \phi \circ \varpi', A'\rangle - \varpi'^* d\gamma') - d\langle \phi \circ \varpi, A\rangle + \varpi^* d\gamma) \\ &= f^*(d\langle \phi \circ \varpi, F^*A'\rangle - \varpi^* d\gamma - d\langle \phi \circ \varpi, A\rangle + \varpi^* d\gamma)\end{aligned}$$

Then, since $\langle \phi \circ \varpi, F^*A' - A\rangle$ is basic, the result follows. □

4.4 The morphism of presheaves $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$

In this section, we will define the functor $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$ for any stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$. We will also show that $\tilde{\mathsf{c}}$ is an isomorphism of presheaves. This will allow us to classify symplectic toric stratified spaces with orbital moment map $\psi : W \to \mathfrak{g}^*$ by classifying conical symplectic toric bundles.

4.4.1 Constructing $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$

Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. For each $(\pi : P \to W_{\mathsf{reg}}, \omega) \in \mathsf{CSTB}_\psi(W)$, we build the symplectic toric stratified space $\tilde{\mathsf{c}}((P, \omega))$ by adding a singularity to (P, ω) for each singularity of W and applying the fiber quotient construction of Karshon and Lerman to P while "remembering" the singularities we've added.

Formally, we build the functor $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$ in the following steps.

Step 1: We first define the functor $\tilde{\mathsf{c}}_{\mathsf{top}} : \mathsf{CSTB}_\psi(W) \to G\text{-}\mathsf{Top}(W)$, taking a conical symplectic toric bundle $(\pi : P \to W_{\mathsf{reg}}, \omega)$ in $\mathsf{CSTB}_\psi(W)$ to a topological G-spaces with quotient map to W:

$$\tilde{\mathsf{c}}_{\mathsf{top}}(\pi : P \to W_{\mathsf{reg}}, \omega) = (\bar{P}, \bar{\pi} : \bar{P} \to W).$$

$\tilde{\mathsf{c}}_{\mathsf{top}}$ also takes morphisms of conical symplectic toric bundles to maps of topological G-spaces over W: G-equivariant homeomorphisms $f : (X, \varpi : X \to W) \to (X', \varpi' : X' \to W)$ satisfying $\varpi = f \circ \varpi'$.

Step 2: Now, we define $\tilde{\mathsf{c}}((\pi : P \to W_{\mathsf{reg}}, \omega))$ as a tuple

$$\left(X, X_{\mathsf{reg}} \bigsqcup \left(\sqcup_{\alpha \in A}\{x_\alpha\}\right), \omega, \bar{\pi} : X \to W\right),$$

where $(X, \bar{\pi}) = \tilde{\mathsf{c}}_{\mathsf{top}}(\pi : P \to W_{\mathsf{reg}}, \omega)$ and $X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in A}\{x_\alpha\})$ is a partition of X such that

$$(X_{\mathsf{reg}}, \omega, \bar{\pi}|_{X_{\mathsf{reg}}} : X_{\mathsf{reg}} \to W_{\mathsf{reg}})$$

is a symplectic toric manifold over $\psi|_{W_{\mathsf{reg}}} : W_{\mathsf{reg}} \to \mathfrak{g}^*$ (we will call this type of object a *partitioned symplectic toric space over* ψ). Note that this does not necessarily imply that $\tilde{\mathsf{c}}((\pi : P \to W_{\mathsf{reg}}, \omega))$ is a symplectic toric stratified space over ψ as we must also check that it has a singular symplectic toric cone structure near each singularity.

We also show that, for any morphism of conical symplectic toric principal bundles φ, the morphism $\tilde{\mathsf{c}}(\varphi) := \tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)$ is in fact a map of partitioned spaces (see Definition 2.3.1) that restricts to a symplectomorphism on the top strata of the source and target partitioned symplectic toric spaces. Thus, when the source and target of $\tilde{\mathsf{c}}(\varphi)$ are symplectic toric stratified spaces over ψ. $\tilde{\mathsf{c}}(\varphi)$ is an isomorphism in $\mathsf{STSS}_\psi(W)$.

Step 3: Next, we show that $\tilde{\mathsf{c}}$ commutes with pullbacks. That is, suppose $\varphi : W' \to W$ is an open embedding, $\psi : W \to \mathfrak{g}^*$ and $\psi' : W' \to \mathfrak{g}^*$ are stratified unimodular local embeddings for which $\psi \circ \varphi = \psi'$, (P, ω) is a conical symplectic toric bundle over ψ, and $\tilde{\mathsf{c}}(P, \omega)$ is a symplectic toric stratified space. Then $\tilde{\mathsf{c}}(\varphi^*(P, \omega)) = \varphi^*\tilde{\mathsf{c}}(P, \omega)$ (see Lemma 4.2.13 and Lemma 4.3.4 for the definitions of these pullbacks). In particular, as pullbacks include (but are not limited to) restrictions, this means $\tilde{\mathsf{c}}$ commutes with restrictions.

Step 4: Now, we prove that $\tilde{\mathsf{c}}(P, \omega)$ is a symplectic toric stratified space in a very specific case of stratified unimodular local embedding: suppose $\psi : c(B) \to \mathfrak{g}^*$ is a stratified unimodular local embedding for which $\psi|_{B \times \mathbb{R}}$ is a *homogeneous* unimodular local embedding. We use our work from Chapter 3 to show that, if a conical symplectic toric bundle (P, ω) over ψ is in fact a homogeneous symplectic toric bundle over $\psi|_{B \times \mathbb{R}}$, then $\tilde{\mathsf{c}}(P, \omega)$ is a symplectic toric stratified space over ψ.

Step 5: Finally, as conical symplectic toric bundles are locally modeled on homogeneous symplectic toric bundles, Step 4 allows us to show $\tilde{\mathsf{c}}(\pi : P \to W_{\mathsf{reg}}, \omega)$ is actually a symplectic toric stratified space over ψ.

We now flesh out the details of each step.

Step 1: To start, we formally define topological G-spaces over a space W.

Definition 4.4.1. Let W be a cornered stratified space. Then a topological G-space over W is a topological space X with a continuous action of G and a G-quotient map $\pi : X \to W$. A map of topological G-spaces over W between $(X, \pi : X \to W)$ and $(X', \pi' : X' \to W)$ is a G-equivariant homeomorphism $\varphi : X \to X'$ such that $\pi' \circ \varphi = \pi$.

Denote by $G\text{-}\mathsf{Top}(W)$ the category of topological G-spaces: the category with objects topological G-spaces over W and morphisms maps of topological G-spaces.

Now, for each conical symplectic toric bundle (P, ω), we construct $\tilde{\mathsf{c}}_{\mathsf{top}}(P, \omega)$, a topological G-space over W.

Construction 4.4.2. [Building $\tilde{\mathsf{c}}_{\mathsf{top}}$ for objects] Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and fix $(\pi : P \to W_{\mathsf{reg}}, \omega)$ a conical symplectic toric bundle in $\mathsf{CSTB}_\psi(W)$. Suppose W has partition $W = W_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in A} \{w_\alpha\})$.

We first construct the cornered stratified space $\tilde{P}$ as follows: as a set, $\tilde{P} := P \bigsqcup (\sqcup_{\alpha \in A}\{p_\alpha\})$ for a set of points $\{p_\alpha\}$ in bijection with the singularities of W.

We then give $\tilde{P}$ the topology generated by

1. open subsets U of P and
2. sets of the form $P|_{V_{\mathsf{reg}}} \sqcup \{p_\alpha\}$, for V an open neighborhood of w_α in W.

Recall that, for each $w \in W_{\mathsf{reg}}$, as $\psi|_{W_{\mathsf{reg}}}$ is a unimodular local embedding, there exists a subtorus $K_w < G$ determined by the image of ψ in $\mathfrak{g}^*$ (see Section 2.1.2). Then let $\sim$ be the equivalence relation on $\tilde{P}$ defined on $P \subset \tilde{P}$ by

$$p \sim p' \text{ if and only if there exists } k \in K_{\pi(p)} \text{ such that } p \cdot k = p'$$

and extended to $\tilde{P}$ so that the added singularities p_α occupy their own equivalence classes.

Since every open neighborhood of a singularity p_α of $\tilde{P}$ contains a G-invariant open neighborhood (namely, a subset of the form $P|_{U_{\mathsf{reg}}}$ for U an open neighborhood of w_α in W), the action of G on P extends to an action of G on $\tilde{P}$ fixing each singularity p_α. It follows that the G-quotient $\pi : P \to W_{\mathsf{reg}}$ extends to a G-quotient $\tilde{\pi} : \tilde{P} \to W$ with $\tilde{\pi}(p_\alpha) = w_\alpha$ for every α.

Now, let $q : \tilde{P} \to \tilde{P}/\sim$ be the topological quotient map. Then, since $\sim$ only identifies elements of the same G-orbit, $\tilde{P}/\sim$ inherits a G action with respect to which q is equivariant; namely, the action $g \cdot [p] := [g \cdot p]$. Via the universal property of q as a quotient map, there exists a unique map $\overline{\pi} : \tilde{P}/\sim \to W$ such that the diagram

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{\pi}} & W \\ {\scriptstyle q}\downarrow & \nearrow_{\overline{\pi}} & \\ \tilde{P}/\sim & & \end{array}$$

commutes. Furthermore, it is clear from how we've defined the G action on $\tilde{P}/\sim$ that $\overline{\pi}$ is the G-quotient map for this action. ◊

Next, we show that maps of conical symplectic toric bundles descend to maps of topological G-spaces over W.

Construction 4.4.3. [Building $\tilde{\mathsf{c}}_{\mathsf{top}}$ for morphisms] Let $\varphi : (P, \omega) \to (P', \omega')$ be a map of conical symplectic toric bundles and let $\tilde{P} := P \bigsqcup (\sqcup_{\alpha \in A}\{p_\alpha\})$ and $\tilde{P}' := P' \bigsqcup (\sqcup_{\alpha \in A}\{p'_\alpha\})$ be the topological spaces of Construction 4.4.2. Let $\tilde{\mathsf{c}}_{\mathsf{top}}(P, \omega) = (\tilde{P}/\sim, \overline{\pi})$ and $\tilde{\mathsf{c}}_{\mathsf{top}}(P', \omega') = (\tilde{P}'/\sim, \overline{\pi}')$.

First note that, for U an open subset of any singularity w_α in W, φ is a map of principal G-bundles and so takes the open subset $P|_{U_{\mathsf{reg}}}$ in P to the open subset $P'|_{U_{\mathsf{reg}}}$ in P'. As every open subset of p_α in $\tilde{P}$ and of p'_α in $\tilde{P}'$ contains a neighborhood of the form $P|_{U_{\mathsf{reg}}}$ or $P'|_{U_{\mathsf{reg}}}$, it follows that φ extends to a homeomorphism $\tilde{\varphi} : \tilde{P} \to \tilde{P}'$ with $\tilde{\varphi}(p_\alpha) = p'_\alpha$ for each α. As $\pi' \circ \varphi = \pi$, we also have that $\tilde{\pi} = \tilde{\pi}' \circ \tilde{\varphi}$, for $\tilde{\pi}$ and $\tilde{\pi}'$ the extensions of π and π' to $\tilde{P}$ and $\tilde{P}'$.

Denote by $q : \tilde{P} \to \tilde{P}/\sim$ and $q' : \tilde{P}' \to \tilde{P}'/\sim$ the topological quotient maps of $\tilde{P}$ and $\tilde{P}'$. Since $\tilde{\varphi} : \tilde{P} \to \tilde{P}'$ is a G-equivariant homeomorphism, it descends to a homeomorphism

$$\tilde{\mathsf{c}}_{\mathsf{top}}(\varphi) : \tilde{P}/\sim \to \tilde{P}'/\sim \qquad \tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)([p]) := [\varphi(p)] \tag{4.2}$$

satisfying $q' \circ \tilde{\varphi} = \tilde{\mathsf{c}}_{\mathsf{top}}(\varphi) \circ q$.

For $\overline{\pi} : \tilde{P}/\sim \to W$ and $\overline{\pi}' : \tilde{P}'/\sim \to W$ the G-quotient maps associated to $\tilde{\mathsf{c}}_{\mathsf{top}}(P, \omega)$ and $\tilde{\mathsf{c}}_{\mathsf{top}}(P', \omega')$, we have from Construction 4.4.2 that $\overline{\pi} \circ q = \tilde{\pi}$ and $\overline{\pi}' \circ q' = \tilde{\pi}'$. So, since $\tilde{\pi} = \tilde{\pi}' \circ \tilde{\varphi}$, we have

$$\overline{\pi}' \circ \tilde{\mathsf{c}}_{\mathsf{top}}(\varphi) \circ q = \overline{\pi}' \circ q' \circ \tilde{\varphi} = \tilde{\pi}' \circ \tilde{\varphi} = \tilde{\pi} = \overline{\pi} \circ q.$$

As q is an epimorphism, it follows that $\overline{\pi}' \circ \tilde{\mathsf{c}}_{\mathsf{top}}(\varphi) = \overline{\pi}$. Therefore, $\tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)$ is a map of topological G-spaces over W. $\Diamond$

We summarize Construction 4.4.2 and Construction 4.4.3 with the following definition:

Definition 4.4.4. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then denote by $\tilde{\mathsf{c}}_{\mathsf{top}}$ the functor

$$\tilde{\mathsf{c}}_{\mathsf{top}} : \mathsf{CSTB}_\psi(W) \to G\text{-}\mathsf{Top}(W)$$

with $\tilde{\mathsf{c}}_{\mathsf{top}}(P,\omega) := (\tilde{P}/\sim, \overline{\pi} : \tilde{P} \to W)$ (for $\tilde{P}/\sim$ and $\overline{\pi}$ as in Construction 4.4.2) and, for φ an isomorphism in $\mathsf{CSTB}_\psi(W)$, $\tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)$ the map defined in equation (4.2).

Step 2: We now begin to define $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$. First, we show that $\tilde{\mathsf{c}}$ maps to the larger category of partitioned symplectic toric spaces over ψ; these are essentially symplectic toric stratified spaces over ψ without the extra condition on the neighborhoods of the singularities. We formally define these spaces and their category below:

Definition 4.4.5. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then a partitioned symplectic toric stratified space over ψ is a partitioned space $X = X_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in A}\{x_\alpha\})$ (see Definition 2.3.1) with a symplectic form $\omega \in \Omega^2(X_{\mathsf{reg}})$, a continuous action by the torus G restricting to a smooth toric action on X_{reg}, and a quotient map $\pi : X \to W$ for which $\psi \circ \pi|_{X_{\mathsf{reg}}} : X_{\mathsf{reg}} \to \mathfrak{g}^*$ is a moment map for $(X_{\mathsf{reg}}, \omega)$. Denote this object as the triple $(X, \omega, \pi : X \to W)$ (with the partition generally left as implicit).

A map of partitioned symplectic toric stratified spaces over ψ between (X, ω, π) and (X', ω', π') is a G-equivariant isomorphism of partitioned spaces $f : X \to X'$ (again, see Definition 2.3.1) such that $f|^*_{X_{\mathsf{reg}}}\omega' = \omega$ and $\pi' \circ f = \pi$.

Denote by $\mathsf{Part}_\psi(W)$ the category of partitioned symplectic toric stratified spaces over ψ with objects and morphisms as described above.

Now, from each conical symplectic toric bundle over ψ, we construct a partitioned symplectic toric stratified space over ψ:

Construction 4.4.6. [Building $\tilde{\mathsf{c}}$ for objects] Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $(\pi : P \to W_{\mathsf{reg}}, \omega)$ be a conical symplectic toric bundle. Let $\tilde{P}$ and $\sim$ be the extension of P and equivalence relation as described in Construction 4.4.2. Note that the subset $P \subset \tilde{P}$ descends to the open dense subset $P/\sim \subset \tilde{P}$. Then $P/\sim$ is exactly the topological space $c_{\mathsf{Top}}(\iota(P,\omega))$, for c_{Top} as defined in equation (2.3). Recall $\iota : \mathsf{CSTB}_\psi(W) \to \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$ is the restriction from W to W_{reg} in the presheaf

CSTB_ψ together with the identification $\mathsf{CSTB}_\psi(W_{\mathsf{reg}}) = \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$; see Remark 4.3.3 and Definition 4.3.6.

For

$$c : \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}}) \to \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

the isomorphism of Karshon and Lerman [19] (see Section 2.1.2), we have that $c(\iota(P,\omega))$ is a symplectic toric manifold over $\psi|_{W_{\mathsf{reg}}}$ homeomorphic to the topological space $c_{\mathsf{Top}}(\iota(P,\omega))$. So $\tilde{P}/\sim$ has partition into manifolds $c_{\mathsf{Top}}(\iota(P,\omega))\coprod(\sqcup_{\alpha\in A}\{[p_\alpha]\})$.

Denote by $c(\iota(P,\omega)) = (M, \bar{\omega}, \varpi : M \to W_{\mathsf{reg}})$. We now define:

$$\tilde{\mathsf{c}}((\pi : P \to W_{\mathsf{reg}}, \omega)) = (\tilde{P}/\sim, \bar{\omega}, \overline{\pi} : \tilde{P}/\sim \to W).$$

This is a partitioned symplectic toric space over ψ as the quotient map $\varpi : M \to W_{\mathsf{reg}}$ is the G-quotient map for the topological space $(P/\sim)_{\mathsf{reg}} = c_{\mathsf{Top}}(\iota(P,\omega)) \subset \tilde{P}/\sim$ which in turn is the restriction of $\overline{\pi} : \tilde{P}/\sim \to W$. Thus, $\psi \circ \overline{\pi}|_{(\tilde{P}/\sim)_{\mathsf{reg}}} = \psi \circ \varpi$ is a moment map for $((\tilde{P}/\sim)_{\mathsf{reg}}, \bar{\omega})$. $\diamondsuit$

Remark 4.4.7. Let $\varphi : (P,\omega) \to (P',\omega')$ be a map of conical symplectic toric bundles over ψ. Then since our definition for $\tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)$ matches $c_{\mathsf{Top}}(\iota(\varphi))$ (again, see Section 2.1.2), it follows that the morphism $\tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)$ restricts to an isomorphism of symplectic toric manifolds between the top strata of $\tilde{\mathsf{c}}(P,\omega)$ and $\tilde{\mathsf{c}}(P',\omega')$. Thus, $\tilde{\mathsf{c}}_{\mathsf{top}}(\varphi)$ induces an isomorphism of symplectic toric partitioned spaces over ψ.

To finish this step, we formally define the functor $\tilde{\mathsf{c}}$:

Definition 4.4.8. Denote by $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{Part}_\psi(W)$ the functor which takes a conical symplectic toric bundle over ψ (P,ω) to $\tilde{\mathsf{c}}(P,\omega)$ (as defined in Construction 4.4.6) and the isomorphism φ of $\mathsf{CSTB}_\psi(W)$ to the homeomorphism $\tilde{\mathsf{c}}(\varphi)$ (which, as explained Remark 4.4.7, is in fact a morphism of $\mathsf{Part}_\psi(W)$).

Step 3: In this step, we show that $\tilde{\mathsf{c}}$ commutes with restrictions and, more generally commutes with pullbacks (as defined in Lemmas 4.2.13 and 4.3.4). This will be important in Step 5.

Lemma 4.4.9. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $(\pi : P \to W_{\mathsf{reg}}, \omega)$ be a conical symplectic toric manifold over ψ. Then for any open subset of U in W, if $\tilde{\mathsf{c}}(P,\omega)$ is a symplectic toric stratified space, then $\tilde{\mathsf{c}}(P,\omega)|_U = \tilde{\mathsf{c}}((P,\omega)|_U)$.

Proof. Fix open U in W. We will trace step by step through Construction 4.4.2 and Construction 4.4.6 to confirm that the resulting partitioned symplectic toric space $\tilde{\mathsf{c}}((P,\omega)|_U)$ is exactly the symplectic toric stratified space $\tilde{\mathsf{c}}(P,\omega)|_U$.

First, note that the total space of $(P,\omega)|_U$ is $P|_{U_{\mathsf{reg}}}$. Recall from Construction 4.4.6 that

$$\tilde{\mathsf{c}}(P,\omega) := \left(\tilde{P}/\sim, \bar{\omega}, \overline{\pi} : \tilde{P}/\sim \to W\right).$$

Also recall from Construction 4.4.2 that $\tilde{\pi} : \tilde{P} \to W$ is the G-quotient of the extension $\tilde{P}$ to a cornered stratified space over W. When starting to build $\tilde{\mathsf{c}}((P,\omega)|_U)$, we extend $(P,\omega)|_U$ to the cornered stratified space $\widetilde{P|_{U_{\mathsf{reg}}}}$. It is more or less clear that this cornered stratified space may be identified with the open subset $\tilde{\pi}^{-1}(U)$ of $\tilde{P}$.

Now, let $q : \tilde{P} \to \tilde{P}/\sim$ the topological quotient map. Since $\tilde{\pi}^{-1}(U)$ is G-invariant, the set $q(\tilde{\pi}^{-1}(U))$ is an open subset of $\tilde{P}/\sim$. As the subtorus K_w defining the equivalence relation $\sim$ on each fiber P_w is determined by the restriction of $\psi : W \to \mathfrak{g}^*$ to any open neighborhood V of w (see Lemma 2.4 of [19]), it follows that $\psi|_{U_{\mathsf{reg}}}$ and ψ induce the same equivalence relation on $\widetilde{P|_{U_{\mathsf{reg}}}}$ and so the quotient $\widetilde{P|_{U_{\mathsf{reg}}}}/\sim$ may be identified with $q(\tilde{\pi}^{-1}(U))$.

By definition (again, see Construction 4.4.2), $\overline{\pi} \circ q = \tilde{\pi}$. Thus, as $\tilde{\pi}^{-1}(U)$ is G-invariant,

$$\overline{\pi}(q(\tilde{\pi}^{-1}(U))) = (\overline{\pi} \circ q)(\tilde{\pi}^{-1}(U)) = \tilde{\pi}(\tilde{\pi}^{-1}(U)) = U.$$

Since $q(\tilde{\pi}^{-1}(U))$ is G-invariant and $\overline{\pi}$ is the G-quotient of $\tilde{P}$, $\overline{\pi}(q(\tilde{\pi}^{-1}(U))) = U$ implies that $q(\tilde{\pi}^{-1}(U)) = (\overline{\pi})^{-1}(U)$. Therefore, $\tilde{\mathsf{c}}_{\mathsf{top}}((P,\omega)|_U)$ is the topological G-space $\left((\overline{\pi})^{-1}(U), \overline{\pi}|_{(\overline{\pi})^{-1}(U)}\right)$ and so $\tilde{\mathsf{c}}_{\mathsf{top}}((P,\omega)|_U) = \tilde{\mathsf{c}}_{\mathsf{top}}((P,\omega))|_U$ as topological G-spaces over U.

Finally, recall we have the fully faithful functor $\iota : \mathsf{CSTB}_{\psi}(W) \to \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$ described in Definition 4.3.6. It is easy to confirm that, since this is just a forgetful functor (after the identification of the groupoids $\mathsf{CSTB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$ and $\mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$) that $\iota((P,\omega)|_U) = \iota(P,\omega)|_{U_{\mathsf{reg}}}$. Therefore, since c is a map of presheaves, we have

$$c(\iota((P,\omega)|_{U_{\mathsf{reg}}})) = c(\iota(P,\omega)|_{U_{\mathsf{reg}}}) = c(\iota(P,\omega))|_{U_{\mathsf{reg}}}.$$

Thus, the symplectic structure on the top strata of $\tilde{\mathsf{c}}((P,\omega)|_U)$ and $\tilde{\mathsf{c}}(P,\omega)|_U$ match and so $\tilde{\mathsf{c}}((P,\omega)|_U)$ and $\tilde{\mathsf{c}}(P,\omega)|_U$ are the same partitioned symplectic toric space; hence, under the assumption that $\tilde{\mathsf{c}}(P,\omega)$ is a symplectic toric stratified space, both partitioned symplectic toric spaces are in fact symplectic toric stratified spaces. □

Lemma 4.4.10. Suppose $\varphi : W' \to W$ is an open embedding of cornered stratified spaces and $\psi : W \to \mathfrak{g}^*$ and $\psi' : W' \to \mathfrak{g}^*$ are stratified unimodular local embeddings with $\psi \circ \varphi = \varphi'$. Then for any conical symplectic toric bundle $(\pi : P \to W_{\mathsf{reg}}, \omega)$ for which $\tilde{\mathsf{c}}(P,\omega)$ is a symplectic toric stratified space over ψ, $\tilde{\mathsf{c}}(\varphi^*(P,\omega)) = \varphi^*\tilde{\mathsf{c}}(P,\omega)$.

Proof. Write $\tilde{\mathsf{c}}(P,\omega) = (\tilde{P}/\sim, \bar{\omega}, \overline{\pi} : \tilde{P}/\sim \to W)$. Recall that pullbacks are constructed, for $U := \varphi(W') \subset W$, as:

$$\varphi^*(\pi : P \to W_{\mathsf{reg}}, \omega) := (\varphi^{-1} \circ \pi : P|_{U_{\mathsf{reg}}} \to W'_{\mathsf{reg}}, \omega)$$

and

$$\varphi^*\tilde{\mathsf{c}}(P,\omega) := ((\overline{\pi})^{-1}(U), \bar{\omega}, \varphi^{-1} \circ \overline{\pi} : (\overline{\pi})^{-1}(U) \to W'_{\mathsf{reg}}).$$

As shown in the proof of Lemma 4.4.9, $\overline{\pi}^{-1}(U) = \widetilde{P|_{U_{\mathsf{reg}}}}/\sim$. Thus,

$$\tilde{\mathsf{c}}(\varphi^*(P,\omega)) = ((\overline{\pi})^{-1}(U), \bar{\omega}, \overline{\varphi^{-1} \circ \pi} : (\overline{\pi})^{-1}(U) \to W')$$

for $\overline{\varphi^{-1} \circ \pi} : (\overline{\pi})^{-1}(U) \to W'$ the quotient map defined as in Construction 4.4.2. So the only thing to show is that $\overline{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \overline{\pi}$.

Recall that, when constructing $\overline{\varphi^{-1} \circ \pi}$, we first build $\widetilde{\varphi^{-1} \circ \pi} : \tilde{P} \to W'$, the extension of $\varphi^{-1} \circ \pi$ to $\tilde{P}$. It is clear that, for $\tilde{\pi} : \tilde{P} \to W$ the extension of π to $\tilde{P}$, that $\widetilde{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \tilde{\pi}$.

Next, for $q : \tilde{P} \to \tilde{P}/\sim$ the topological quotient map, $\overline{\varphi^{-1} \circ \pi} : \tilde{P}/\sim \to W'$ is by definition the unique map for which $\overline{\varphi^{-1} \circ \pi} \circ q = \widetilde{\varphi^{-1} \circ \pi}$. But since $\overline{\pi} \circ q = \tilde{\pi}$, we have:

$$\widetilde{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \tilde{\pi} = \varphi^{-1} \circ \overline{\pi} \circ q$$

Therefore, by uniqueness from the universal property of a quotient, $\overline{\varphi^{-1} \circ \pi} = \varphi^{-1} \circ \overline{\pi}$. ☐

Step 4: We now prove that $\tilde{\mathsf{c}}(P,\omega)$ is a symplectic toric stratified space (indeed, a singular symplectic toric cone) for a particular case of stratified unimodular local embedding.

Proposition 4.4.11. Let B be a manifold with corners and let $\psi : c(B) \to \mathfrak{g}^*$ be a stratified unimodular local embedding such that $\psi|_{B\times\mathbb{R}}$ is a homogeneous unimodular local embedding with respect to the action of translation on the second factor (see Definition 3.1.1). Suppose that $(\pi : P \to B \times \mathbb{R}, \omega)$ is a homogeneous symplectic toric bundle in $\mathsf{HSTB}_{\psi|_{B\times\mathbb{R}}}(B \times \mathbb{R})$. Then (P,ω) is a conical symplectic toric bundle over ψ and $\tilde{\mathsf{c}}(P,\omega)$ is a singular symplectic toric cone over ψ.

Proof. First, it is clear from the definition of a conical symplectic toric bundle that (P,ω), the homogeneous symplectic toric bundle over $\psi|_{B\times\mathbb{R}}$, is a conical symplectic toric bundle over ψ. As in the steps above, we have $\tilde{\mathsf{c}}(P,\omega) = (\tilde{P}/\sim, \bar{\omega}, \overline{\pi} : \tilde{P}/\sim \to B)$.

Let $\tilde{P}$ and $\tilde{\pi} : \tilde{P} \to W$ correspond to the of P and π as in Construction 4.4.2. Then it is apparent from how both spaces are defined that $(\tilde{P}, \omega, \psi \circ \tilde{\pi})$ is exactly the extension of the symplectic toric cone $(P, \omega, \psi \circ \pi)$ to a singular symplectic toric cone as given in Proposition 4.1.12.

Since $P \subset \tilde{P}$ is a homogeneous symplectic toric bundle over $\psi|_{B\times\mathbb{R}}$, we may apply hc to (P, ω). Recall that

$$\mathsf{hc} : \mathsf{HSTB}_{\psi|_{B\times\mathbb{R}}}(B \times \mathbb{R}) \to \mathsf{STC}_{\psi|_{B\times\mathbb{R}}}(B \times \mathbb{R})$$

is just the functor

$$c : \mathsf{STB}_{\psi|_{B\times\mathbb{R}}}(B \times \mathbb{R}) \to \mathsf{STM}_{\psi|_{B\times\mathbb{R}}}(B \times \mathbb{R})$$

that remembers the action of $\mathbb{R}$ on a homogeneous symplectic toric bundle (see Definition 3.3.2). Thus, as the top stratum of $\tilde{\mathsf{c}}(P,\omega)$ is the symplectic manifold $c(P,\omega)$, we have that the top stratum inherits the structure of the symplectic toric cone over $\psi|_{B\times\mathbb{R}}$ $\mathsf{hc}(P,\omega)$.

To finish, let $L := P/\mathbb{R}$ and let $\varphi : P \to L \times \mathbb{R}$ be a G-equivariant trivialization of P as a principal $\mathbb{R}$-bundle (by Proposition 2.2.22, such a trivialization exists). Then by Proposition 4.1.8, this extends to a homeomorphism $\tilde{\varphi} : \tilde{P} \to c(L)$. Note that, since $\psi_{B\times\mathbb{R}}$ is homogeneous, the subtori $K_{(b,t)}$ and $K_{(b,0)}$ induced by $\psi_{B\times\mathbb{R}}$ are equal for any $(b,t) \in B \times \mathbb{R}$. So for $L' := (B \times \{0\})/\sim$ (where $\sim$ is the equivalence relation defining $\tilde{\mathsf{c}}_{\mathsf{top}}$), since the actions of G and $\mathbb{R}$ on $L \times \mathbb{R}$ commute, $\tilde{\mathsf{c}}_{\mathsf{top}}(L \times \mathbb{R})$ is isomorphic to $c(L')$ as topological G-spaces.

Now, note that $\tilde{\varphi} : \tilde{P} \to c(L)$ descends to a $(G \times \mathbb{R})$-equivariant homeomorphism $\phi : \tilde{P}/\sim \to c(L')$. As ϕ restricts to a $(G\times\mathbb{R})$-equivariant homeomorphism between $\mathsf{hc}(P,\omega) \subset \tilde{P}/\sim$ and $L' \times \mathbb{R}$, we may conclude that L' is in fact diffeomorphic to the manifold $\mathsf{hc}(P,\omega)/\mathbb{R}$. Therefore, ϕ gives local trivialization data for $[*]$ in $\tilde{P}$ and so $\tilde{P}/\sim$ is a stratified space with one singularity with respect to the partition $P/\sim \sqcup \{[*]\}$.

Thus, $\tilde{\mathsf{c}}(P,\omega)$ is a singular symplectic toric cone over ψ (i.e., a singular symplectic toric cone with moment map $\psi \circ \overline{\pi}$). □

Step 5: To finish, we show that $\tilde{\mathsf{c}}$ is in fact a functor into the category of symplectic toric stratified spaces.

Proposition 4.4.12. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $(\pi : P \to W_{\mathsf{reg}}, \omega)$ be a conical symplectic toric bundle. Then $\tilde{\mathsf{c}}(P,\omega)$ is a symplectic toric stratified space over ψ. Therefore, $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{Part}_\psi(W)$ in fact defines a functor $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$.

Proof. We need to show that the partitioned symplectic toric space over ψ $\tilde{\mathsf{c}}(\pi : P \to W_{\mathsf{reg}}, \omega)$ defined in Construction 4.4.6 is in fact a symplectic toric stratified space. To do this, we must show that

1. $\tilde{\mathsf{c}}(\pi : P \to W_{\mathsf{reg}}, \omega)$ is in fact a stratified space with isolated singularities; and

2. Each singularity of $\tilde{\mathsf{c}}(\pi : P \to W_{\mathsf{reg}}, \omega)$ has a neighborhood isomorphic to the neighborhood of the singularity in a singular symplectic toric cone (see Definition 4.2.1).

Via Lemma 4.1.13, we have that any neighborhood of the singularity of any singular symplectic toric cone is isomorphic to $c(L)$ for some manifold with corners L. Therefore, if we can show the second condition above holds, the first must hold as well.

So fix a singularity p_α of $\tilde{P}$ lying over w_α in W. Then by Lemma 4.3.5, there exists an open neighborhood U of w_α, a homogeneous local trivialization datum $\varphi : U \to c(L)$ with homogeneous unimodular local embedding $\phi : L \times \mathbb{R} \to \mathfrak{g}^*$ such that $\psi|_{U_{\mathsf{reg}}} = \phi \circ \varphi + \psi(w)$, a homogeneous symplectic toric bundle $(\varpi : Q \to L \times \mathbb{R}, \eta) \in \mathsf{HSTB}_\phi(L \times \mathbb{R})$, and a G-equivariant isomorphism $f : (P, \omega)|_U \to \varphi^*(Q, \eta)$ in $\mathsf{CSTB}_\psi(U)$.

So since $\tilde{\mathsf{c}}(\varphi^*(Q, \eta)) = \varphi^*\tilde{\mathsf{c}}(Q, \eta)$ (by Lemma 4.4.10) and $\tilde{\mathsf{c}}((P, \omega)|_U) = \tilde{\mathsf{c}}(P, \omega)|_U$ (by Lemma 4.4.9), we have an isomorphism

$$\tilde{\mathsf{c}}(f) : \tilde{\mathsf{c}}(P, \omega)|_U \to \varphi^*\tilde{\mathsf{c}}(Q, \eta).$$

Thus, we have an isomorphism from $\tilde{\mathsf{c}}(P, \omega)|_U$ to the neighborhood of the singularity $\varphi^*\tilde{\mathsf{c}}(Q, \eta)$ of the singular symplectic toric cone $\tilde{\mathsf{c}}(Q, \eta)$. □

Remark 4.4.13. As in the case of symplectic toric bundles and symplectic toric manifolds, we may also make the following observation about shifting moment maps. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding, let $\eta \in \mathfrak{g}^*$, and let $\psi' : W \to \mathfrak{g}^*$ be the map $\psi'(w) := \psi(w) + \eta$. Then it is clear that ψ' is also a stratified unimodular local embedding.

Write

$$\tilde{\mathsf{c}}_\psi : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W) \text{ and } \tilde{\mathsf{c}}_{\psi'} : \mathsf{CSTB}_{\psi'}(W) \to \mathsf{STSS}_{\psi'}(W)$$

for the functor $\tilde{\mathsf{c}}$ applied with respect to ψ and ψ', respectively. As we are simply shifting the moment map, it follows easily that, for any conical symplectic toric bundle over ψ $(\pi : P \to W, \omega)$, this bundle must be a conical symplectic toric bundle over ψ' as well.

It is easy then to check that, from how $\tilde{\mathsf{c}}_\psi$ and $\tilde{\mathsf{c}}_{\psi'}$ were defined, $\tilde{\mathsf{c}}_\psi(\pi : P \to W, \omega)$ and $\tilde{\mathsf{c}}_{\psi'}(\pi : P \to W, \omega)$ are the same as symplectic stratified G-spaces; that is, they are the same topological space with the same partition, same G action, and the same symplectic form on the top stratum. It follows there is a G-equivariant isomorphism of stratified spaces

$$f : \tilde{\mathsf{c}}_\psi(\pi : P \to W, \omega) \to \tilde{\mathsf{c}}_{\psi'}(\pi : P \to W, \omega)$$

restricting to a symplectomorphism on the top strata and preserving the respective G-quotients to W. Of course, for moment maps μ and μ' for $\tilde{\mathsf{c}}_\psi(\pi : P \to W, \omega)$ and $\tilde{\mathsf{c}}_{\psi'}(\pi : P \to W, \omega)$, we have $\mu' \circ f = \mu + \eta$.

4.4.2 Showing $\tilde{\mathsf{c}}$ is an isomorphism of presheaves

We now show that $\tilde{\mathsf{c}}$ is an isomorphism of presheaves. Rather than directly showing $\tilde{\mathsf{c}}$ is fully faithful, we show that $\tilde{\mathsf{c}}$ fits into a diagram of fully faithful functors. Recall that, for stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, we have the functors

$$\iota : \mathsf{CSTB}_\psi(W) \to \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

(see Definition 4.3.6) and

$$\mathrm{res} : \mathsf{STSS}_\psi(W) \to \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$$

(see Definition 4.2.10), which both may be thought of as presheaf restrictions from W to W_{reg} followed by identifications $\mathsf{CSTB}_\psi(W_{\mathsf{reg}}) = \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$ and $\mathsf{STSS}_\psi(W_{\mathsf{reg}}) = \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$, respectively.

Proposition 4.4.14. For any stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, the diagram:

$$\begin{array}{ccc} \mathsf{CSTB}_\psi(W) & \xrightarrow{\ \iota\ } & \mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}}) \\ {\scriptstyle \tilde{\mathsf{c}}}\downarrow & & \downarrow{\scriptstyle c} \\ \mathsf{STSS}_\psi(W) & \xrightarrow[\mathrm{res}]{} & \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}}) \end{array}$$

commutes.

Proof. This is more or less obvious from the details of the construction of $\tilde{\mathsf{c}}$. □

Lemma 4.4.15. For any stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$ is fully faithful.

Proof. Since ι and c are both fully faithful (see Lemma 4.3.7), it follows $c \circ \iota$ is as well. Hence, by Proposition 4.4.14, $\mathrm{res} \circ \tilde{\mathsf{c}}$ is also fully faithful. It is easy to check that, since res is fully faithful (see Lemma 4.2.11), it must follow that $\tilde{\mathsf{c}}$ is fully faithful. □

Now, we show that any element of STSS_ψ is locally isomorphic to elements of the image of $\tilde{\mathsf{c}}$.

Lemma 4.4.16. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $(X, \omega, \pi : X \to W)$ be a symplectic toric stratified space over ψ. Then for each $w \in W$, there is an open neighborhood U_w of w and a conical symplectic toric bundle $(\varpi : P \to U_w, \eta)$ in $\mathsf{CSTB}_\psi(U_w)$ such that $\tilde{\mathsf{c}}(P, \eta)$ is isomorphic to $(X, \omega, \pi : X \to W)|_{U_w}$.

Proof. In the case where $w \in W_{\mathsf{reg}}$, this is done simply by choosing a contractible open neighborhood of w small enough so that $U_w \subset W_{\mathsf{reg}}$. Here, $\mathsf{STSS}_\psi(U_w) = \mathsf{STM}|_{\psi|_{W_{\mathsf{reg}}}}(U_w)$ and, since U_w is contractible, all elements of $\mathsf{STM}|_{\psi|_{W_{\mathsf{reg}}}}(U_w)$ are isomorphic. Thus, the image of any element of $\mathsf{CSTB}_\psi(U_w)$ is isomorphic to the restriction our original symplectic toric stratified space. By Proposition 4.3.9, $\mathsf{CSTB}_\psi(U_w)$ is non-empty, so we may find such a bundle.

So consider the case where w is a singularity of W. Then let U_w be any neighborhood of w in W for which there exists a singular symplectic toric cone $(C, \omega', \nu : C \to \mathfrak{g}^*)$ with neighborhood V of the vertex $*$ of C, and a map $\varphi : \pi^{-1}(U_w) \to V$ satisfying the conditions described in Definition 4.2.1. Then for orbital moment map $\bar{\nu} : C/G \to \mathfrak{g}^*$, $\bar{\nu}|_{(C/G)_{\mathsf{reg}}}$ is a homogeneous unimodular local embedding.

Let $q : C \to C/G$ be the G-quotient map. As φ is G-equivariant, it descends to an open embedding of cornered stratified spaces $\bar{\varphi} : U_w \to (C/G)_{\mathsf{reg}}$ satisfying $q \circ \varphi = \bar{\varphi} \circ \pi$. By assumption, we have that $\psi \circ \pi = \nu \circ \varphi + \psi(w)$ and therefore

$$\psi \circ \pi = \nu \circ \varphi + \psi(w) = \bar{\nu} \circ q \circ \varphi + \psi(w) = \bar{\nu} \circ \bar{\varphi} \circ \pi + \psi(w).$$

Since π is an epimorphism, $\bar{\nu} \circ \bar{\varphi} + \psi(w) = \psi$.

As discussed in Remark 4.4.13, $(C, \omega', q : C \to C/G)$ is also a symplectic toric stratified space over the stratified unimodular local embedding $\psi' := \bar{\nu} + \psi(w)$. So, since $\psi = \psi' \circ \bar{\varphi}$, it follows we may pull (C, ω', q) back to the symplectic toric stratified space over ψ $\bar{\varphi}^*(C, \omega', q)$ and there is a G-equivariant isomorphism of symplectic stratified spaces

$$\phi : \bar{\varphi}^*(C, \omega', q) \to (C, \omega', q)|_{U_w} = (V, \omega'|_V, \nu|_V)$$

where $\bar{\varphi}^*(C, \omega', q) = (C, \omega')$ as a symplectic stratified G-space and has quotient map $q' := \bar{\varphi}^{-1} \circ q : C \to U_w$ (see Lemma 4.2.13). So

$$q' \circ \varphi = \bar{\varphi}^{-1} \circ q \circ \varphi = \bar{\varphi}^{-1} \circ \bar{\varphi} \circ \pi = \pi$$

and so

$$\tilde{\varphi} := \phi^{-1} \circ \varphi|_{U_w} : (X, \omega, \pi : X \to W)|_{U_w} \to \bar{\varphi}^*(C, \omega', q : C \to C/G)$$

is an isomorphism in $\mathsf{STSS}_\psi(U_w)$.

On the other hand, since

$$\mathsf{hc} : \mathsf{HSTB}_{\bar{\nu}|_{C_{\mathsf{reg}}/G}}(C_{\mathsf{reg}}/G) \to \mathsf{STC}_{\bar{\nu}|_{C_{\mathsf{reg}}/G}}(C_{\mathsf{reg}}/G)$$

is an equivalence of categories (see Theorem 3.3.7), there exists a homogeneous symplectic toric bundle $(\pi : P \to C_{\mathsf{reg}}/G, \eta)$ and an isomorphism of symplectic toric cones over $\bar{\nu}|_{C_{\mathsf{reg}}/G}$

$$f : (C_{\mathsf{reg}}, \omega', q|_{C_{\mathsf{reg}}} : C_{\mathsf{reg}} \to C_{\mathsf{reg}}/G) \to \mathsf{hc}(\pi : P \to C_{\mathsf{reg}}/G, \eta)$$

We may view $(\pi : P \to C_{\mathsf{reg}}/G, \eta)$ as a conical symplectic toric bundle over $\bar{\nu} : C/G \to \mathfrak{g}^*$ and, by Proposition 4.4.11, $\tilde{\mathsf{c}}(\pi : P \to C_{\mathsf{reg}}/G, \eta)$ is a singular symplectic toric cone with top stratum the symplectic toric cone $\mathsf{hc}(\pi : P \to C_{\mathsf{reg}}/G, \eta)$. By Lemma 4.1.15, the isomorphism f extends to a unique isomorphism of symplectic toric stratified spaces over $\bar{\nu}$

$$\tilde{f} : (C, \omega', q : C \to C/G) \to \tilde{\mathsf{c}}(\pi : P \to C_{\mathsf{reg}}/G, \eta).$$

Again, as in Remark 4.4.13, we may shift the moment map for $(C, \omega', q : C \to C/G)$ and $\tilde{\mathsf{c}}(\pi : P \to C_{\mathsf{reg}}/G, \eta)$ by $\psi(w)$ to obtain an isomorphism

$$\tilde{f}' : (C, \omega', q : C \to C/G) \to \tilde{\mathsf{c}}(\pi : P \to C_{\mathsf{reg}}/G, \eta)$$

in $\mathsf{STSS}_{\bar{\nu}+\psi(w)}(C/G)$; here we also "shift" $(\pi : P \to C_{\mathsf{reg}}/G, \eta)$ to a conical symplectic toric bundle over $\bar{\nu} + \psi(w)$.

Finally, as discussed in Remark 4.2.14, isomorphisms pullback as well. So, the isomorphism $\tilde{f}'$ pulls back to an isomorphism

$$\bar{\varphi}^*\tilde{f} : \bar{\varphi}^*(C, \omega', q : C \to C/G) \to \bar{\varphi}^*\tilde{\mathsf{c}}(\pi : P \to C_{\mathsf{reg}}/G, \eta)$$

in $\mathsf{STSS}_\psi(U_w)$. By Lemma 4.4.10, $\bar{\varphi}^*\tilde{\mathsf{c}}(\pi : P \to C_{\mathsf{reg}}/G, \eta) = \tilde{\mathsf{c}}(\bar{\varphi}^*(\pi : P \to C_{\mathsf{reg}}/G, \eta))$, so we have an isomorphism

$$(\tilde{\varphi})^{-1} \circ (\bar{\varphi}^*\tilde{f})^{-1} : (X, \omega, \pi : X \to W)|_{U_w} \to \tilde{\mathsf{c}}(\bar{\varphi}^*(\pi : P \to C_{\mathsf{reg}}/G, \eta))$$

in $\mathsf{STSS}_\psi(U_w)$.

□

We need one more lemma before we can finish.

Lemma 4.4.17. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then the presheaf of groupoids $\mathsf{STSS}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a prestack (see Definition B.4).

Proof. Let $U \subset W$ be any open subset and let $(X, \omega, \pi : X \to U)$ and $(X', \omega', \pi' : X' \to U)$ be any two symplectic toric stratified spaces over $\psi|_U$. As a map of symplectic toric stratified spaces $f : (X, \omega, \pi : X \to U) \to (X', \omega', \pi' : X' \to U)$ is, in particular, a homeomorphism $f : X \to X'$, it is clear that f is uniquely determined by its restrictions to any open cover of X.

So, suppose that $\{U_\alpha\}_{\alpha \in A}$ is any open cover of U. Suppose also that we have a family of isomorphisms

$$\{f_\alpha : (X, \omega, \pi : X \to U)|_{U_\alpha} \to (X', \omega', \pi' : X' \to U)|_{U_\alpha}\}_{\alpha \in A}$$

so that, for every α and β with $U_\alpha \cap U_\beta$ non-empty, $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$. Then clearly the f_α's glue together to a homeomorphism $f : X \to X'$. Since $f_\alpha|_{X_{\mathsf{reg}} \cap U_\alpha}$ is a symplectomorphism for each α, it follows that $f|_{X_{\mathsf{reg}}}$ must be a symplectomorphism as well. Finally, as each f_α preserves the quotients $\pi|_{U_\alpha}$ and $\pi'|_{U_\alpha}$, the glued together map f must preserve these quotients as well. Therefore, this $f : X \to X'$ in fact defines an isomorphism $f : (X, \omega, \pi) \to (X', \omega', \pi')$.

Thus, the presheaf of sets $\underline{\mathsf{Hom}}_{\mathsf{STSS}_\psi}(X, X')$ defined by $U \mapsto \mathsf{Hom}_{\mathsf{STSS}_\psi}(X|_U, X'|_U)$ (again, as in Definition B.4) is a sheaf. □

We may now prove the following theorem.

Theorem 4.4.18. For any stratified unimodular local embedding $\psi : W \to \mathfrak{g}^*$, the map of presheaves $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$ is an isomorphism of presheaves.

Proof. We've shown in Lemma 4.4.15 that $\tilde{\mathsf{c}}_U : \mathsf{CSTB}_\psi(U) \to \mathsf{STSS}_\psi(U)$ is fully faithful for every U and in Lemma 4.4.17 that STSS_ψ is a prestack. In Proposition B.9, we showed that CSTB_ψ is a stack. Therefore, with Lemma 4.4.16, we have that $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$ satisfies the hypotheses of Lemma B.11 and is an isomorphism of presheaves. □

4.5 A classification of symplectic toric stratified spaces over ψ

In this section, we describe a set of cohomology classes that will help identify the isomorphism classes of $\mathsf{CSTB}_\psi(W)$ and hence, via our equivalence of categories $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi(W) \to \mathsf{STSS}_\psi(W)$, the isomorphism classes of $\mathsf{STSS}_\psi(W)$. This is done by classifying conical symplectic toric bundles over a unimodular local embedding $\psi : W \to \mathfrak{g}^*$. We use the same general method as in [19], distinguishing elements of $\mathsf{CSTB}_\psi(W)$ by their isomorphism of principal G-bundle together with a piece of "horizontal data"; these horizontal data take values in a subspace $\mathcal{C}$ of $H^2(W_{\mathsf{reg}}; \mathbb{R})$. We will also discuss how to calculate $\mathcal{C}$ and apply this classification to some examples.

4.5.1 Establishing an isomorphism between $\mathsf{CSTB}_\psi(W)$ and $H^2(W_{\mathsf{reg}}; \mathbb{R}) \times \mathcal{C}$

We first focus on horizontal data; this will allow us to distinguish conical symplectic toric bundles that are isomorphic as principal G-bundles over W_{reg}. These horizontal data for a conical symplectic toric bundle will take the following form.

Definition 4.5.1. Let $(W, W_{\mathsf{reg}} \coprod (\sqcup_{\alpha \in A} \{w_\alpha\}))$ be a cornered stratified space. Then a form $\beta \in \Omega^2(W_{\mathsf{reg}})$ is a good form on W (for the purposes of this paper) if there exist a neighborhood U of w_α for each $\alpha \in A$ such that $\beta|_{U_{\mathsf{reg}}}$ is exact.

Notation 4.5.2. It follows directly from the definition above that, given a closed good form β on a cornered stratified space W, $\beta + \eta$ is also closed for any exact form $\eta \in \Omega^2(W_{\mathsf{reg}})$. We denote by $\mathcal{C} \subset H^2(W_{\mathsf{reg}})$ the subspace of all de Rham classes of good forms.

We set the following notation which we will need to establish part of our bijection:

Notation 4.5.3. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $\pi : P \to W_{\mathsf{reg}}$ be a principal G-bundle. Then we will denote by $[(P, \cdot)]$ the subset of isomorphism classes

$$\{[(\pi : P \to W_{\mathsf{reg}}, \omega)] \in \pi_0 \mathsf{CSTB}_\psi(W)\}$$

of $\pi_0\mathsf{CSTB}_\psi(W)$. In other words, these are the classes in $\pi_0\mathsf{CSTB}_\psi(W)$ containing conical symplectic toric bundles with principal bundle $\pi : P \to W_{\mathsf{reg}}$ and any valid G-invariant symplectic form ω on P.

Now, we build a bijection between $[(P, \cdot)]$ and $\mathcal{C}$.

Lemma 4.5.4. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding, let $\pi : P \to W_{\mathsf{reg}}$ be a principal G-bundle, and define $\mu := \psi \circ \pi$. Then a choice of 2-form η built as in Proposition 4.3.9 induces a bijection $\mathsf{c}_\eta : [(P, \cdot)] \to \mathcal{C}$.

Proof. By Proposition 4.3.9, we have that there exists a G-invariant symplectic form η on P with respect to which (P, η) is a conical symplectic toric bundle over ψ. By Corollary 2.1.19 we have that, for any other G-invariant symplectic form ω on P with moment map μ, $\omega - \eta$ is a basic closed form. Then let f be the function

$$f : \{G\text{-invariant symplectic forms on } P \text{ with moment map } \mu\} \longrightarrow \{\text{Closed 2-forms on } W_{\mathsf{reg}}\} \qquad (4.3)$$

for which $f(\omega)$ is the 2-form satisfying $\eta - \omega = \pi^* f(\omega)$. This is well-defined as π is a submersion.

Let $\varphi : (P, \omega) \to (P, \omega')$ be an isomorphism in $\mathsf{CSTB}_\psi(W)$. By definition, we also have that $\varphi^*(\omega') = \omega$ and $\pi \circ \varphi = \pi$. Then we have

$$\pi^* f(\omega') = \varphi^*(\pi^* f(\omega')) = \varphi^*\eta - \varphi^*(\omega') = \varphi^*\eta - \omega = \varphi^*\eta - (\eta - \pi^* f(\omega))$$

As $\varphi^*\eta - \eta$ is exact with a basic primitive (see Remark 4.3.10), it follows that $[f(\omega)] = [f(\omega')] \in H^2(W_{\mathsf{reg}}; \mathbb{R})$.

Note that, given any conical symplectic toric bundle (Q, β) such that $[Q, \beta] \in [(P, \cdot)]$ there is, by assumption, a map of principal G-bundles $\phi : P \to Q$ for which $(P, \phi^*\beta) \in \mathsf{CSTB}_\psi(W)$. Given another map of principal G-bundles $\phi' : P \to Q$ for which $(P, \phi'^*\beta) \in \mathsf{CSTB}_\psi(W)$, $\phi' \circ \phi^{-1}$ is an isomorphism in $\mathsf{CSTB}_\psi(W)$ between $(P, \phi^*\beta)$ and $(P, \phi'^*\beta)$. Thus, by the discussion in the previous paragraph, f takes any two representatives of the same class in $[(P, \cdot)]$ to cohomologous forms.

By Proposition 4.3.11, for each singularity w_α of W, there is a neighborhood U_α for which $(\omega - \eta)|_{P|_{U_\alpha}}$ is basic with an exact primitive. It follows that $[f(\omega)] \in \mathcal{C}$. Therefore, f descends to a well-defined function

$$\mathsf{c}_\eta : [(P, \cdot)] \to \mathcal{C}, \quad \text{for} \quad \mathsf{c}_\eta([P, \omega]) := [f(\omega)].$$

We now show that c_η is injective. For ω and ω' G-invariant symplectic forms on P for which $(P, \omega), (P, \omega') \in \mathsf{CSTB}_\psi(W)$, if $\omega - \omega'$ is exact, then Lemma 2.1.20 tells us that $\iota(P, \omega)$ and $\iota(P, \omega')$ are isomorphic in $\mathsf{STB}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$. Recall that ι (see Definition 4.3.6) is fully faithful (see Lemma 4.3.7); therefore, if $\iota(P, \omega)$ and $\iota(P, \omega')$ are isomorphic, then (P, ω) and (P, ω') must be isomorphic as well.

Finally, to show c_η is surjective, suppose $[\beta]$ is any class in $\mathcal{C}$ and fix a good form $\beta \in \Omega^2(W_{\mathsf{reg}})$ to represent this class. Then, by definition, for each singularity w_α of W, there is an open subset U_α of w_α and a form γ_α on $(U_\alpha)_{\mathsf{reg}}$ with $\beta|_{(U_\alpha)_{\mathsf{reg}}} = d\gamma_\alpha$. Let $U_0 \subset W_{\mathsf{reg}}$ be an open subset so that:

- $\{U_0, \{U_\alpha\}_{\alpha\in A}\}$ is an open cover of W
- For each α, there is an open neighborhood $V_\alpha \subset U_\alpha$ of w_α with $V_\alpha \cap U_0 = \emptyset$

(as noted in the proof of Proposition 4.3.9, this is possible via Lemma 4.1.13).

Let $\{\rho_0, \{\rho_\alpha\}_{\alpha\in A}\}$ be a partition of unity for this open cover. Define $\gamma := \sum_\alpha \rho_\alpha\gamma_\alpha$. Then, for $\beta' := \beta - d\gamma$, we have $[\beta'] = [\beta]$. Since $\beta'|_{(V_\alpha)_{\mathsf{reg}}} = 0$ for every α, it follows that $(P, \eta + \pi^*\beta)$ is an element of $\mathsf{CSTB}_\psi(W)$. Therefore, we have that $\mathsf{c}_\eta(P, \eta + \pi^*\beta') = [\beta]$ and so c_η must be surjective. □

Remark 4.5.5. In [19], Karshon and Lerman proved that, for unimodular local embedding $\psi : W \to \mathfrak{g}^*$ there is a characteristic class $\mathsf{c}_{\mathsf{hor}} : \pi_0\mathsf{STB}_\psi(W) \to H^2(W;\mathbb{R})$ such that, for any choice of connection 1-form A on $\pi : P \to W$ and ω a G-invariant symplectic form with moment map $\psi\circ\pi$, $\mathsf{c}_{\mathsf{hor}}(P,\omega)$ is the unique class $[\alpha] \in H^2(W;\mathbb{R})$ with representative α satisfying $\pi^*\alpha + d\langle\psi\circ\pi, A\rangle = \omega$.

As shown in Lemma 4.3.7, there is a fully faithful functor $\iota : \mathsf{CSTB}_\psi(W) \to \mathsf{STB}_\psi(W)$. Thus, $\mathsf{c}_{\mathsf{hor}} \circ \pi_0\iota : \pi_0\mathsf{CSTB}_\psi(W) \to H^2(W;\mathbb{R})$ is an injective function. While it is tempting to work with $\mathsf{c}_{\mathsf{hor}} \circ \pi_0\iota$ as a characteristic class for conical symplectic toric bundles, one may show that, for $\pi : P \to W_{\mathsf{reg}}$ a principal G-bundle not admitting a flat connection 1-form, $(\mathsf{c}_{\mathsf{hor}} \circ \pi_0\iota)([(P,\cdot)])$ is only an *affine* subspace of $H^2(W_{\mathsf{reg}};\mathbb{R})$. Furthermore, for another principal bundle $\varpi : Q \to W_{\mathsf{reg}}$ non-isomorphic to Q, $(\mathsf{c}_{\mathsf{hor}} \circ \pi_0\iota)([(P,\cdot)])$ and $(\mathsf{c}_{\mathsf{hor}} \circ \pi_0\iota)([(Q,\cdot)])$ needn't be the same affine subspace.

By instead using our approach for defining c_η as in Lemma 4.5.4, we can ensure $\mathcal{C}$ will be the image under c_η of every $[(P,\cdot)]$. Unfortunately, the function c_η is very much dependent on a choice of η.

We now classify conical symplectic toric bundles over any stratified unimodular local embedding.

Proposition 4.5.6. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then there is a bijection

$$(\mathsf{c}_1, \mathsf{s}\text{-}\mathsf{c}_{\mathsf{hor}}) : \pi_0\mathsf{CSTB}_\psi \to H^2(W_{\mathsf{reg}};\mathbb{Z}_G) \times \mathcal{C}$$

(recall $\mathbb{Z}_G := \ker(\exp : \mathfrak{g} \to G)$).

Proof. We have essentially the same ingredients as the similar Proposition 5.1 of [19] had for symplectic toric bundles, so we approach the proof in the same way.

For each isomorphism class of principal G-bundle over W_{reg}, pick a representative $\pi : R \to W_{\mathsf{reg}}$ and, for each other $\varpi : Q \to W_{\mathsf{reg}}$ isomorphic to R, fix an isomorphism $\phi_Q : Q \to R$. Define $\eta_R := \eta$ for η as chosen in Proposition 4.3.9. For any Q isomorphic to R, define $\eta_Q := \phi_Q^*\eta_R$. Then, via the discussion in Remark 4.3.10, we have that, for any isomorphism of principal G-bundles $\varphi : Q \to Q'$, $\varphi^*\eta_{Q'} - \eta_Q$ is exact with a basic primitive.

For $(P, \omega) \in \mathsf{CSTB}_\psi(W)$, define

$$\mathsf{s\text{-}c_{hor}} : \pi_0\mathsf{CSTB}_\psi(W) \to \mathcal{C}, \quad \mathsf{s\text{-}c_{hor}}([P,\omega]) := \mathsf{c}_{\eta_\mathsf{P}}([P,\omega]) \tag{4.4}$$

To see this is well defined, suppose $\varphi : (P', \omega') \to (P, \omega)$ is any isomorphism of conical symplectic toric bundles. Then

$$\omega' - \eta_{P'} = \varphi^*\omega - \eta_{P'} = (\varphi^*\omega - \varphi^*\eta_P) + (\varphi^*\eta_P - \eta_{P'}) = \varphi^*(\omega - \eta_P) + \varphi^*\eta_P - \eta_{P'}$$

It follows as in the proof of Lemma 4.5.4 that, since $\varphi^*\eta_P - \eta_{P'}$ are exact with a basic primitive, η_P and $\varphi^*\eta_{P'}$ induce the same cohomology class for $[P, \omega]$ and $[P', \omega']$.

Now, we define

$$(\mathsf{c}_1, \mathsf{s\text{-}c_{hor}})([P,\omega]) := (\mathsf{c}_1(P), \mathsf{s\text{-}c_{hor}}([P,\omega]))$$

We first show that $(\mathsf{c}_1, \mathsf{s\text{-}c_{hor}})$ is injective. Suppose $(\pi : P \to W_{\mathsf{reg}}, \omega)$ and $(\pi' : P' \to W_{\mathsf{reg}}, \omega')$ are two elements of $\mathsf{CSTB}_\psi(W)$ so that $(\mathsf{c}_1, \mathsf{s\text{-}c_{hor}})(P, \omega) = (\mathsf{c}_1, \mathsf{s\text{-}c_{hor}})(P', \omega')$. In particular, this means there is an isomorphism of principal bundles $\varphi : P' \to P$ since $\mathsf{c}_1(P) = \mathsf{c}_1(P')$. As shown in the proof of Lemma 4.5.4, it follows that, since $\mathsf{s\text{-}c_{hor}}(\omega) = \mathsf{s\text{-}c_{hor}}(\omega')$, $[P, \omega] = [P', \omega']$.

Finally, to see $(\mathsf{c}_1, \mathsf{s\text{-}c_{hor}})$ is surjective, fix a pair $(c, [\beta]) \in H^2(W_{\mathsf{reg}}; \mathbb{R}) \times \mathcal{C}$. Then there exists a principal G-bundle $\pi : P \to W_{\mathsf{reg}}$ with $\mathsf{c}_1(P) = c$. As shown in Lemma 4.5.4, $\mathsf{c}_{\mathsf{P}_\eta} : [(P, \cdot)] \to \mathcal{C}$ is surjective and there exists β' with $[\beta] = [\beta']$ for which $(\pi : P \to W_{\mathsf{reg}}, \eta_P + \pi^*\beta')$ is a conical symplectic toric bundle.

□

Now, we may easily prove the following classification.

Theorem A. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then for W_{reg} the top stratum of W:

1. The set of symplectic toric stratified isolated singularities with orbital moment map ψ is non-empty.

2. There is a subspace $\mathcal{C} \subset H^2(W_{\mathsf{reg}}, \mathbb{R})$ dependent on the topology of W so that the isomorphism classes of symplectic toric stratified spaces with isolated singularities $(X, \omega, \mu : X \to \mathfrak{g}^*)$ with G-quotient map

$\pi : X \to W$ and orbital moment map ψ are in bijective correspondence with the cohomology classes

$$H^2(W_{\mathsf{reg}}; \mathbb{Z}_G) \times \mathcal{C}$$

where $\mathbb{Z}_G$ is the integral lattice of G, the kernel of the exponential map $\exp : \mathfrak{g} \to G$.

Proof. For the first part of the theorem, note that by Proposition 4.3.9, there is a conical symplectic toric bundle (P, ω) over ψ. Thus, $\tilde{\mathsf{c}}(P, \omega)$ is a symplectic toric stratified space over ψ.

For the second part, we have a bijection

$$(\pi_0 \tilde{\mathsf{c}})^{-1} : \pi_0 \mathsf{STSS}_\psi(W) \to \pi_0 \mathsf{CSTB}_\psi(W)$$

(which is indeed a bijection since $\tilde{\mathsf{c}}$ is an equivalence of categories, by Theorem 4.4.18) and

$$(\mathsf{c}_1, \mathsf{s\text{-}c_{hor}}) : \pi_0 \mathsf{CSTB}_\psi(W) \to H^2(W_{\mathsf{reg}}; \mathbb{Z}_G) \times \mathcal{C}$$

(which is a bijection by Proposition 4.5.6). Thus, we have a bijection

$$(\mathsf{c}_1, \mathsf{s\text{-}c_{hor}}) \circ (\pi_0 \tilde{\mathsf{c}})^{-1} : \mathsf{STSS}_\psi(W) \to H^2(W_{\mathsf{reg}}; \mathbb{Z}_G) \times \mathcal{C}$$

□

Definition 4.5.7. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding and let $\mathsf{s\text{-}c_{hor}} : \mathsf{CSTB}_\psi(W) \to \mathcal{C}$ be a choice of horizontal class as defined in equation (4.4) (as remarked before, this is generally dependent on a choice of coherent forms $\{\eta_P\}$ chosen as in the proof of Proposition 4.5.6). Then for a symplectic toric stratified space over ψ $(X, \omega, \pi : X \to W)$, define its the stratified Chern class as the class

$$\mathsf{s\text{-}ch}(X, \omega, \pi : X \to W) := \mathsf{c}_1((\pi_0 \tilde{\mathsf{c}})^{-1}(X, \omega, \pi : X \to W)) \in H^2(W_{\mathsf{reg}}; \mathbb{Z}_G)$$

and define its stratified horizontal class as the class

$$\mathsf{s\text{-}c_{hor}}(X, \omega, \pi : X \to W) := \mathsf{c_{hor}}((\pi_0 \tilde{\mathsf{c}})^{-1}(X, \omega, \pi : X \to W)) \in \mathcal{C}$$

Remark 4.5.8. It is not difficult to check that the classes s-ch is exactly just the first Chern class of Karshon and Lerman of the symplectic toric manifold $(X_{\mathsf{reg}}, \omega, \pi|_{X_{\mathsf{reg}}})$ (see Definition 5.3 of [19]). We add the adjective *stratified* simply to remember the fact that these classes came from a symplectic toric stratified space.

4.5.2 Calculating $\mathcal{C}$ and some examples

Now we've identified cohomology classes quantifying symplectic toric stratified spaces with isolated singularities, we discuss how to actually calculate what the subspace $\mathcal{C}$ actually looks like for a general cornered stratified space. For this, we turn to relative de Rham cohomology, as presented by Bott and Tu [4]. A full description is given in Appendix C. We can identify a subset $\overline{W}$ of W_{reg} so that the image of $H^2\left(W_{\mathsf{reg}}, \overline{W}\right)$ into $H^2(W_{\mathsf{reg}}; \mathbb{R})$ via the long exact sequence of relative de Rham cohomology (see equation (C.1)) is exactly $\mathcal{C}$.

We identify this subset $\overline{W}$ with the following lemma.

Lemma 4.5.9. For any cornered stratified space $(W, W_{\mathsf{reg}} \bigsqcup (\sqcup_{\alpha \in I}\{w_\alpha\}))$, there exists a subset $\overline{W}$ of W_{reg} so that, if β is any good form on W (see Definition 4.5.1), then β is exact on all of $\overline{W}$. More precisely, we can identify a subset $\overline{W}$ of W_{reg} so that forms exact in (deleted) neighborhoods of singularities of W must be exact on $\overline{W}$.

Proof. Fix a singularity w_α of W. Then a neighborhood U_α of w_α in W is isomorphic to $c(L)$ for some manifold with corners L, meaning $(U_\alpha)_{\mathsf{reg}}$ is a deleted neighborhood of w_α in W_{reg} diffeomorphic to $L \times \mathbb{R}$. Now, suppose β is any good form on W. Then there exists $V \subset (U_\alpha)_{\mathsf{reg}}$ on which β is exact and so that $V \sqcup \{w_\alpha\}$ is a neighborhood of w_α in W. It follows that V is diffeomorphic to a neighborhood of $-\infty$ in $L \times \mathbb{R}$.

As in the proof of Lemma 4.1.13, there is a diffeomorphism $F : (U_\alpha)_{\mathsf{reg}} \to L \times \mathbb{R}$ for which $F^{-1}(L \times (-\infty, 0))$ is a neighborhood of $-\infty$ contained in V. In particular, $F^{-1}(L \times \{-1\})$ is contained in V and so $\beta|_{F^{-1}(L \times \{-1\})}$ is exact. As the retraction of $L \times \mathbb{R}$ onto $L \times \{-1\}$ is a homotopy equivalence, we may conclude that β is exact on all of $(U_\alpha)_{\mathsf{reg}}$.

Thus, any good form beta must be exact on the open subset $\overline{W} := \sqcup_{\alpha \in A} (U_\alpha)_{\mathsf{reg}}$. □

So, in light of the previous lemma, we may identify $\mathcal{C}$ as those elements of $H^2(W; \mathbb{R})$ that are exact on $\overline{W}$. Now that we have one standard neighborhood on which all good forms are exact, we may proceed more easily.

Proposition 4.5.10. Let W be any cornered stratified space and let $f : \overline{W} \to W_{\mathsf{reg}}$ the inclusion of the open subspace $\overline{W}$ of W_{reg} identified above. Then, for $H^2\left(W_{\mathsf{reg}}, \overline{W}\right)$ the relative de Rham cohomology group (see Appendix C), $\mathcal{C} \subset H^2(W_{\mathsf{reg}};\mathbb{R})$ is exactly the image of $H^2\left(W_{\mathsf{reg}},\overline{W}\right)$ under the map $\tilde{\pi}([(\alpha,\beta)]) := [\alpha]$ of the long exact sequence:

$$\ldots \xrightarrow{f^*} H^1\left(\overline{W};\mathbb{R}\right) \xrightarrow{\tilde{\iota}} H^2\left(W_{\mathsf{reg}},\overline{W}\right) \xrightarrow{\tilde{\pi}} H^2(W_{\mathsf{reg}};\mathbb{R}) \xrightarrow{f^*} H^2\left(\overline{W};\mathbb{R}\right) \longrightarrow \ldots.$$

Here, $\tilde{\iota} : H^1\left(\overline{W};\mathbb{R}\right) \to H^2\left(W_{\mathsf{reg}},\overline{W}\right)$ is the map $\tilde{\iota}([\beta]) = [(0,\beta)]$.

Proof. Note that any class $[\alpha,\beta] \in H^2\left(W_{\mathsf{reg}},\overline{W}\right)$ must satisfy $d\alpha = 0$ and $\alpha|_{W_{\mathsf{reg}}} = d\beta$. So $\tilde{\pi}([\alpha,\beta]) = [\alpha]$ and is exact on W_{reg}. Thus, $[\alpha] \in \mathcal{C}$.

On the other hand, if $[\gamma]$ is a class of good forms in $\mathcal{C}$, then Lemma 4.5.9 tells us that γ is exact on $\overline{W}$. So there exists $\delta \in \Omega^1\left(\overline{W}\right)$ such that $d\delta = \gamma|_{\overline{W}}$. Thus $[(\gamma,\delta)] \in H^2\left(W_{\mathsf{reg}},\overline{W}\right)$ and $\tilde{\pi}([(\gamma,\delta)]) = [\gamma]$. ☐

We now discuss some applications of the above classification. First, we perform a quick check to confirm our classification matches that of Karshon and Lerman in the case of symplectic toric manifolds:

Example 4.5.11. Any unimodular local embedding $\psi : W \to \mathfrak{g}^*$ is a stratified unimodular local embedding in the case where W has no singularities. Then in this case, the condition for good forms on $W_{\mathsf{reg}} = W$ is empty and so $\mathcal{C} = H^2(W;\mathbb{R})$. Thus, we have recovered Karshon and Lerman's classification (see Theorem 2.1.22).

As a more interesting example, note we may build a moment map image by deleting a point from the octahedron:

Example 4.5.12. Let Δ be the octahedron in $\mathbb{R}^3$; that is, the convex hull of

$$\{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}$$

(as in Example 4.2.4). Let $\Delta_0 := \Delta\backslash\{0\}$. Then the inclusion $\iota : \Delta_0 \to \mathbb{R}^3$ of Δ_0 into $\mathbb{R}^3$, identified with the Lie algebra dual of the 3-torus, is a stratified unimodular local embedding with singularities the vertices of Δ. Then $H^2((\Delta_0)_{\mathsf{reg}};\mathbb{R}) \cong \mathbb{R}$ and, as the link of each singularity of Δ_0 is convex, it follows that, for $\overline{\Delta_0}$ the open subset of Lemma 4.5.9,

$$H^1(\overline{\Delta_0};\mathbb{R}) \cong H^2(\overline{\Delta_0};\mathbb{R}) \cong 0.$$

Thus, the map of Proposition 4.5.10

$$\tilde{\pi} : H^2(\Delta_0,\overline{\Delta_0}) \longrightarrow H^2((\Delta_0)_{\mathsf{reg}};\mathbb{R}) \cong \mathbb{R}$$

is an isomorphism. Therefore, the symplectic toric stratified spaces with orbital moment map ι are in bijective correspondence with the cohomology classes $H^2((\Delta_0)_{\mathsf{reg}}; \mathbb{R}) \cong \mathbb{R}$.

Recall that, for any stratified unimodular local embedding ψ, there is a fully faithful functor

$$\text{res} : \mathsf{STSS}_\psi(W) \to \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}}).$$

This induces an injection

$$\pi_0\text{res} : \pi_0\mathsf{STSS}_\psi(W) \to \pi_0\mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}}).$$

We can use our classification to prove that, for symplectic toric stratified spaces of dimension 8 or higher with compact links, $\pi_0$res is in fact an isomorphism:

Proposition 4.5.13. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular embedding such that the link of each singularity w_α of W is compact and suppose that $\dim(\mathfrak{g}^*) \geq 4$. Then every symplectic toric manifold with orbital moment map $\psi|_{W_{\mathsf{reg}}} : W_{\mathsf{reg}} \to \mathfrak{g}^*$ is isomorphic to $(X, \omega, \pi)|_{W_{\mathsf{reg}}}$ for some symplectic toric stratified space over ψ (X, ω, π).

Proof. Let (X, ω, π) be any symplectic toric stratified space over ψ. Then every singularity of X has a neighborhood isomorphic to a neighborhood of $-\infty$ in a symplectic toric cone $(C, \eta, \mu : C \to \mathfrak{g}^*)$. Furthermore, by assumption, the link $C/\mathbb{R}$ of C is compact and so, for each connected component C_i of C, the moment cone $\mu(C_i) \sqcup \{0\} \subset \mathfrak{g}^*$ is a convex rational polyhedral cone and the fibers of $\mu|C_i$ are connected (see [20]). It follows that each connected component of the link of w_α in W $C_i/(G \times \mathbb{R})$ is convex or diffeomorphic to a sphere S^d for $d = \dim(\mathfrak{g}^*) - 1$. Therefore

$$H^1(\overline{W}; \mathbb{R}) \cong H^2(\overline{W}; \mathbb{R}) \cong 0.$$

So Proposition 4.5.10 tells us that *every* class in $H^2(W_{\mathsf{reg}}; \mathbb{R})$ is in $\mathcal{C}$.

Now, let (M, β, ϖ) be a symplectic toric manifold over $\psi|_{W_{\mathsf{reg}}}$. Then there is a symplectic toric manifold $(\pi : P \to W_{\mathsf{reg}}, \eta)$ for which $c(P, \eta) \cong (M, \beta, \pi)$. Let η_P be a form on P chosen as in Proposition 4.3.9 such that (P, η_P) is a conical symplectic toric bundle. Note that, by Corollary 2.1.19, $\eta_P - \eta = \pi^*\gamma$ for some closed 2-form γ on W_{reg}. Then as discussed in the previous paragraph, $[\gamma] \in \mathcal{C}$.

So, by Lemma 4.5.4, there is a choice of representative γ' with $[\gamma'] = [\gamma]$ for which $(P, \eta_P - \pi^*\gamma')$ is a conical symplectic toric bundle. Then

$$\eta = \eta_P - \pi^*\gamma = (\eta_P - \pi^*\gamma') + \pi^*(\gamma' - \gamma)$$

and so, by Lemma 2.1.20, there is an isomorphism of symplectic toric bundles $\varphi : (P, \eta) \to (P, \eta_P - \pi^*\gamma')$.

Finally, let $\mathrm{res} : \mathsf{STSS}_\psi(W) \to \mathsf{STM}_{\psi|_{W_{\mathsf{reg}}}}(W_{\mathsf{reg}})$ be the restriction functor (see Definition 4.2.10) and let $\iota : \mathsf{CSTB}_\psi(W) \to \mathsf{STB}_\psi(W)$ be the forgetful functor (see Definition 4.3.6). By Proposition 4.4.14,

$$\tilde{\mathsf{c}}(P, \eta_P - \pi^*\gamma')|_{W_{\mathsf{reg}}} = \mathrm{res}(\tilde{\mathsf{c}}(P, \eta_P - \pi^*\gamma')) = c(\iota(P, \eta_P - \pi^*\gamma'))$$

so φ descends to an isomorphism $c(P, \eta) \cong \tilde{\mathsf{c}}(P, \eta_P - \pi^*\gamma')|_{W_{\mathsf{reg}}}$.

□

$\pi_0\mathrm{res}$ in general is *not* an isomorphism:

Example 4.5.14. Let $\mathfrak{g}$ be the Lie algebra of the 3-torus $\mathbb{T}^3$ and let W be the cornered stratified space with total space $\mathfrak{g}^*$ and partition $W = \mathfrak{g}^*\backslash\{0\} \sqcup 0$. Let $\psi : W \to \mathfrak{g}^*$ be the identity homeomorphism. Then $W_{\mathsf{reg}} = \overline{W}$ is homotopy equivalent to S^2. Thus, the restriction morphism $H^2(W_{\mathsf{reg}}; \mathbb{R}) \to H^2(\overline{W}; \mathbb{R})$ is an isomorphism and so $\mathcal{C} = 0$.

So for every principal bundle over $\mathfrak{g}^*\backslash\{0\}$, there is exactly one isomorphism class of corresponding symplectic toric stratified space while the symplectic toric manifold structures are in bijective correspondence with $H^2(W_{\mathsf{reg}}; \mathbb{R}) \cong \mathbb{R}$. In the particular case where we start with the trivial principal $\mathbb{T}^3$-bundle, the symplectic toric manifold $T^*\mathbb{T}^3\backslash\{0\}$ (the cotangent bundle minus the zero section) is the unique symplectic toric manifold (up to isomorphism) that is isomorphic to the restriction of a symplectic toric stratified space over ψ. Indeed, it is the restriction of the topological quotient of $T^*\mathbb{T}^3$ modulo its zero section.

As noted in the introduction, our classification generalizes that of Burns, Guillemin, and Lerman [6] in the compact connected case. We also generalize their result in the following manner: the authors of [6] assume that their symplectic toric stratified spaces are of *Reeb type*. That is, they only model symplectic toric stratified spaces with neighborhoods of $-\infty$ in symplectic toric cones $(M, \omega, \mu : M \to \mathfrak{g}^*)$ such that there is a Lie algebra element X so that $\langle \mu(p), X \rangle > 0$ for all $p \in M$. With this condition, they prove that:

Theorem 4.5.15. For (X, ω, μ) a compact connected symplectic toric stratified space with isolated singularities:

1. $\mu(X)$ is a rational polytope, simple except possibly at its vertices;

2. The fibers of μ are G-orbits; and

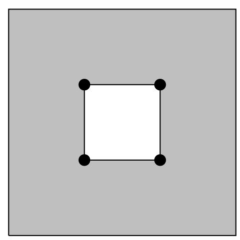
3. (X, ω, μ) is uniquely determined up to isomorphism by $\mu(X)$.

Proof. See Theorem 1, [6]. □

As a compact connected symplectic toric stratified space with isolated singularities (X, ω, μ) with links of Reeb type must have connected fibers, we must have that $X/G \cong \mu(X)$. Since the image $\mu(X)$ is convex, it follows that $H^2((X/G)_{\mathsf{reg}}; \mathbb{R}) \cong 0$ and so our classification agrees with Theorem 4.5.15.

The following example illustrates dropping the Reeb type condition:

Example 4.5.16. As in Example 4.2.5, the inclusion of the region



into $\mathbb{R}^2$ is a stratified unimodular local embedding, as is every covering map of this region (where the black dots represent the images of singularities). Since the 2^{nd} degree cohomology of this region is 0, for each covering map ψ, there is a unique isomorphism class of symplectic toric stratified spaces with orbital moment map ψ.

In particular, this means every finite sheeted connected cover ψ of the above moment map image corresponds to a unique compact connected symplectic toric stratified space with isolated singularities, so we've lost unique characterization via moment map image. Interestingly enough, however, the singularities of these symplectic toric stratified spaces have links diffeomorphic to S^3 and so the associated stratified spaces are homeomorphic to 4-manifolds.

Appendix A

Manifolds with corners

In his appendix, we briefly recall some facts about manifolds with corners. Our primary source is [18]. We will also use some results in [28].

First, we remind the reader that, for $C \subset \mathbb{R}^m$ and $D \subset \mathbb{R}^n$, a map $f : C \to D$ is *smooth* if there are open subsets $C \subset U \subset \mathbb{R}^m$ and $D \subset V \subset \mathbb{R}^n$ such that f admits a smooth extension $\tilde{f} : U \to V$. A manifold with corners of dimension n is essentially a manifold for which the domains of charts are allowed to be open subsets of sectors of $[0, \infty)^k \times \mathbb{R}^{n-k}$. More formally:

Definition A.1 (Atlases of manifolds with corners, manifolds with corners, smooth maps)**.** Let M be a Hausdorff, second-countable, topological space. Denote by R^n_k the sector $[0, \infty)^k \times \mathbb{R}^{n-k}$. Then an n-dimensional chart with corners for M is a homeomorphism $\phi : U \to V$ for U an open subset of $\mathbb{R}^n_k$ and V an open subset of M. Two charts $\phi_i : U_i \to V_i$, $i = 1, 2$ are compatible if the maps

$$\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(V_1 \cap V_2) \to \phi_2(V_1 \cap V_2) \quad \text{and} \quad \phi_1^{-1} \circ \phi_2 : \phi_2^{-1}(V_1 \cap V_2) \to \phi_1(V_1 \cap V_2)$$

are smooth (in the sense described in the above paragraph).

A collection $\{\phi_\alpha : U \to V\}_{\alpha \in A}$ of n-dimensional charts for M is called an atlas for M if $\{V_\alpha\}_{\alpha \in A}$ is a cover of M. An atlas is maximal if it is not properly contained in any other atlas. Finally, we call the pair $(M, \{\phi_\alpha : U \to V\}_{\alpha \in A})$ of a space M with a maximal atlas an n-dimensional manifold with corners. As is standard, we will here forward leave the maximal atlas as an implicit piece of the data.

The set of all points $p \in M$ for which there is a chart with domain $\mathbb{R}^n$ is called the interior of M and is denoted $\mathring{M}$. The set $M \backslash \mathring{M}$ is called the boundary of M and is denoted ∂M.

A continuous map $f : M \to N$ between manifolds with corners $(M, \{\phi_\alpha : U \to V\}_{\alpha \in A})$ and $(N, \{\varphi_\beta : U' \to V'\}_{\beta \in B})$ is smooth if for any $p \in M$ and any two charts $\phi : U \to V$ and $\varphi : U' \to V'$ with $p \in V$ and $f(p) \in V'$, the map $\varphi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(V') \to U'$ is smooth.

An action of a Lie group on a manifold with corners is just a smooth group action on a manifold with corners:

Definition A.2. Given a manifold with corners M and a Lie group K, an action of K on M is a group action of K on M for which the associated map $\rho : K \times M \to M$ is smooth.

We must also define the tangent/cotangent spaces of a manifold with corners; these are again standard extensions of the definitions for smooth manifolds.

Definition A.3. Let M be an n-dimensional manifold with corners. Then for any point $p \in M$, the tangent space at p, denoted by T_pM, is the n-dimensional vector space:

$$\{v : C^\infty(M) \to \mathbb{R} \mid v(fg) = v(f)g(p) + f(p)v(g)\}.$$

The cotangent space at p, denoted by T_p^*M, is the dual vector space to T_pM.

As discussed by Michor, the collection of tangent and cotangent spaces on a manifold with corners naturally inherit a manifold with corners structure.

Lemma A.4. The tangent and cotangent bundles $TM := \bigsqcup_{p\in M} T_pM$ and $T^*M := \bigsqcup_{p\in M} T_p^*M$ are naturally manifolds with corners with respect to which the projections to M are smooth. Similarly, for any positive integer k, $\Lambda^k T^*M := \bigsqcup_{p\in M} \Lambda^k T_p^*M$ is a manifold with corners.

Proof. See p.19, [28]. □

Definition A.5. A k-form on M is a section $\alpha : M \to \Lambda^k T^*M$ and the collection of all k-forms on M is denoted $\Omega^k(M)$.

Remark A.6. Given an n dimensional manifold with corners M and an l-form $\alpha \in \Omega^l(M)$, near boundary points on M, α locally extends to a differential form on a manifold without boundary. Thus, as the exterior derivative may be defined locally, it makes sense to extend d to forms on manifolds with corners.

Additionally, the concept of non-degeneracy is local and thus also extends to manifolds with corners. Thus, we may define symplectic forms for manifolds with corners. It also makes sense to talk about moment maps and Hamiltonian actions on a symplectic manifold with corners.

For a torus G, we need to know what it means to be a principal G-bundle in the realm of manifolds with corners:

Definition A.7. Given a manifold with corners M and a Lie group K, a principal K-bundle over M is a topological principal K-bundle $\pi : P \to M$ for which P is a manifold with corners, the action of K is smooth, π is smooth, and there exist *smooth* local trivialization data for $\pi : P \to M$.

Remark A.8. It is easy to show that, for $\pi : P \to M$ a principal K-bundle of manifolds with corners, then π is a submersion.

Remark A.9. It is relatively straightforward to check that there is a partition of unity subordinate to any cover $\{U_\alpha\}$ of a manifold with corners M. Indeed, it is enough to note that, for any open subset U of the sector $\mathbb{R}^n_k \subset \mathbb{R}^n$ and compact subset $K \subset U$, there is an open subset $\tilde{U}$ of $\mathbb{R}^n$ with $U = \tilde{U} \cap \mathbb{R}^n_k$. As K must also be compact in $\mathbb{R}^n$, there exists a smooth function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $\rho|_K = 1$ and $\mathrm{Supp}(\rho) \subset \tilde{U}$. It follows then that $\bar{\rho} := \rho|_U : U \to \mathbb{R}$ is a smooth function with $\bar{\rho}|_K = 1$ and $\mathrm{Supp}(\bar{\rho}) \subset U$. From here, it is clear that one may proceed as in the case of manifolds without corners.

Definition A.10. For M a manifold with corners and $\tilde{M}$ a manifold without boundary of the same dimension, if there is an embedding $\iota : M \to \tilde{M}$, then we say that $\tilde{M}$ *contains* M *as a domain.*

Every manifold with corners is contained as a domain in a manifold without boundary.

Lemma A.11 (Lemma, [28], p. 21)**.** For M an n-dimensional manifold with corners, there exists an n-dimensional manifold without boundary $\tilde{M}$ and an embedding $\iota : M \to \tilde{M}$.

As we will shortly use the map constructed in this lemma, we sketch its proof:

Proof. First, on any sector $\mathbb{R}^n_k$, there is a strictly inward pointing vector field; that is, a vector field $(X_1, \ldots, X_k, Y_1, \ldots Y_{n-k})$ such that, on the boundary, each X_i is non-negative not every Y_j is zero. Now, let $\{U_\alpha\}_{\alpha \in A}$ be a cover of ∂M by chart codomains. Using a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in A} \sqcup \{\mathring{M}\}$, we may patch together the strictly inward pointing vector fields on each U_α together with the zero vector field on $\mathring{M}$ to a vector field X on M.

By shrinking X as necessary, we may insure that X has a time 1 flow $\phi : M \to M$. Since the flow of X takes ∂M into the interior of M, it follows that $\phi(M) \subset \mathring{M}$ and so ϕ is the required embedding of M into the manifold without boundary $\mathring{M}$. □

Lemma A.12. Let M be an n-dimensional manifold with corners. Then the inclusion map $\iota : \mathring{M} \to M$ is part of a homotopy equivalence.

Proof. Let X be the vector field of the proof of Lemma A.11 and let ϕ_t be its time t flow. Then for $\psi_t := (\phi_t)|_{\mathring{M}}$, $\psi_t\left(\mathring{M}\right) \subset \mathring{M}$ for all t and so ψ_t defines a smooth family of maps on $\mathring{M}$. ϕ_t yields a homotopy between $\iota \circ \phi_1$ and id_M and ψ_t yields a homotopy between $\mathrm{id}_{\mathring{M}}$ and $\phi_1 \circ \iota$. Therefore, ι and ϕ_1 define a homotopy equivalence between M and $\mathring{M}$. □

Finally, we give a brief explanation for why de Rham cohomology is invariant under homotopy for manifolds with corners. As explained in Chapter 1, Section 4 of Bott and Tu [4], as $\mathbb{R}^n$ yields a local model for any manifold without corners M, one may use the Poincaré Lemma to conclude that de Rham cohomology for manifolds without boundary is homotopy invariant. Thus, using this same logic, we may conclude that, if a version of the Poincaré Lemma holds for sectors of $\mathbb{R}^n$, de Rham cohomology for manifolds with corners is homotopy invariant as well. That is, it is enough to show the following:

Lemma A.13. Let $\mathbb{R}^n_k$ be a sector of $\mathbb{R}^n$. Then for $\pi : \mathbb{R}^n_k \times \mathbb{R} \to \mathbb{R}^n_k$ the natural projection and $s : \mathbb{R}^n_k \to \mathbb{R}^n_k \times \mathbb{R}$ the zero section, π and s are a homotopy equivalence.

Proof. Again, we refer to Chapter 1, Section 4 of Bott and Tu [4]. Note that their proof of Proposition 4.1 (the current lemma in the case where $k = 0$) requires the construction of a homotopy operator K requiring fiber integration. Given a smooth function $f(x,t) \in C^\infty(\mathbb{R}^n_k \times \mathbb{R})$, note that f extends to a smooth function $\tilde{f}$ on an open subset $U \subset \mathbb{R}^{n+1}$ containing $\mathbb{R}^n_k \times \mathbb{R}$. Using a partition of unity argument to extend $\tilde{f}$ by zero away from $\mathbb{R}^n_k \times \mathbb{R}$, we may in fact extend f by a function $\tilde{f} \in C^\infty(\mathbb{R}^{n+1})$. Thus, the fiber integral

$$\int_0^t f(x,\tau)d\tau = \int_0^t \tilde{f}(x,\tau)d\tau$$

is smooth on $\mathbb{R}^n_k \times \mathbb{R}$. It follows then that the homotopy operator K makes sense for de Rham forms on $\mathbb{R}^n_k \times \mathbb{R}$ and satisfies the Fundamental Theorem of Calculus; therefore, we may use the proof of Proposition 4.1 of Chapter 1 in [4] to conclude our lemma. □

Appendix B

Stacks

In this appendix, we will provide some results on stacks. We also prove that our presheaves of groupoids HSTB_ψ and CSTB_ψ are stacks and prove the major technical lemma we require in proving that $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$ and $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$ are isomorphisms of presheaves of groupoids.

For simplicity's sake, we will be using a less general definition that would perhaps be more accurately named a "sheaf of groupoids". Since the stacks we are interested in are, in fact, arising from presheaves of groupoids, this will be ideal in our case (rather than using lax presheaves or categories fibered in groupoids). Additionally, for our purposes we need only worry about defining stacks over categories of open subsets of a topological space (or full subcategories of these categories), as in the case of traditional sheaves of sets. A few good sources for a more complete story on stacks are [33] (which is focused more on stacks in algebraic geometry), [3] (which is focused on using stacks in differential geometry), and [16] (which discusses stacks over manifolds and over topological spaces).

Fix a topological space X. For $\{U_\alpha\}_{\alpha\in A}$ an open cover of X, we write $U_{\alpha\beta} := U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$. First, we need some preliminaries.

Definition B.1. $\mathsf{Open}(X)$ is the category of open sets of X: the objects of X are open subsets $U \subset X$ and the morphisms are inclusions of open subsets $\iota : U \to V$. Write $\mathsf{Open}(X)^{\mathsf{op}}$ for the opposite category of $\mathsf{Open}(X)$.

Now we may define the category of descent data for a presheaf of groupoids:

Definition B.2 (The category of descent data)**.** Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X and let $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ be a presheaf of groupoids. Then $\{U_\alpha\}_{\alpha\in A}$ descent data for $\mathcal{F}$ are pairs of tuples

$$(\{\xi_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha\in A}, \{\varphi_{\alpha\beta} : \xi_\alpha|_{U_{\alpha\beta}} \to \xi_\beta|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A})$$

such that the morphisms $\{\varphi_{\alpha\beta} : \xi_\alpha|_{U_{\alpha\beta}} \to \xi_\beta|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A}$ (known as transition morphisms) satisfy the cocycle condition: for every non-empty triple intersection $U_{\alpha\beta\gamma}$, we have that $\varphi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} \circ \varphi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = \varphi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}$.

A morphism of descent data

$$\{\eta_\alpha\}_{\alpha\in A} : (\{\xi_\alpha\}_{\alpha\in A}, \{\varphi_{\alpha\beta}\}_{\alpha,\beta\in A}) \to (\{\xi'_\alpha\}_{\alpha\in A}, \{\varphi'_{\alpha\beta}\}_{\alpha,\beta\in A})$$

is a collection of morphisms $\{\eta_\alpha : \xi_\alpha \to \xi'_\alpha\}_{\alpha\in A}$ so that the diagram

$$\begin{array}{ccc} \xi_\alpha & \xrightarrow{\eta_\alpha} & \xi'_\alpha \\ {\scriptstyle \varphi_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \varphi'_{\alpha\beta}} \\ \xi_\beta & \xrightarrow[\eta_\beta]{} & \xi'_\beta \end{array} \tag{B.1}$$

commutes for every α and β with $U_{\alpha\beta}$ non-empty.

Write $\mathcal{D}_\mathcal{F}(\{U_\alpha\}_{\alpha\in A})$ for the descent category: the category of $\{U_\alpha\}_{\alpha\in A}$ descent data for $\mathcal{F}$ with morphisms of descent data.

We may now formally define stacks:

Definition B.3. Let $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ be a presheaf of groupoids. For an open cover $\{U_\alpha\}_{\alpha\in A}$ of X, define the restriction functor $\Phi : \mathcal{F}(X) \to \mathcal{D}_F(\{U_\alpha\})$ as the functor taking an object $\xi \in \mathcal{F}(X)$ to the descent data:

$$(\{\xi|_{U_\alpha}\}_{\alpha\in A}, \{id : (\xi|_{U_\alpha})|_{U_{\alpha\beta}} \to (\xi|_{U_\beta})|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A})$$

and a morphism $\varphi : \xi \to \xi'$ to the morphism of descent data $\{\varphi|_{U_\alpha} : \xi|_{U_\alpha} \to \xi'|_{U_\alpha}\}_{\alpha\in A}$.

$\mathcal{F}$ is a stack if, for every open subset U of X and for every open cover $\{U_\alpha\}_{\alpha\in A}$, the restriction morphism $\Phi : \mathcal{F}_U(U) \to \mathcal{D}_{\mathcal{F}_U}(\{U_\alpha\}_{\alpha\in A})$ is an equivalence of groupoids.

For $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ a presheaf and U any open subset of X, note that, for $V \subset U$ an open subset, the restriction morphism from U to V is a map of groupoids. Thus, for any two objects ξ and ξ' in $\mathcal{F}(U)$, we have a map of sets

$$\mathsf{Hom}_{\mathcal{F}(U)}(\xi, \xi') \to \mathsf{Hom}_{\mathcal{F}(V)}(\xi|_V, \xi'|_V)$$

It is easy to check that this corresponds to a presheaf of sets we will write as $\underline{\mathsf{Hom}}(\xi, \xi') : \mathsf{Open}(U)^{\mathsf{op}} \to \mathsf{Sets}$.

Definition B.4. A presheaf of groupoids $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a prestack if for every open subset $U \subset X$ and any two ξ and ξ' in $\mathcal{F}(U)$, the presheaf $\underline{\mathsf{Hom}}(\xi, \xi') : \mathsf{Open}(U)^{\mathsf{op}} \to \mathsf{Sets}$ is a sheaf of sets.

Remark B.5. It is more or less clear that, for W a manifold with an $\mathbb{R}$ action, we may just as easily define a stack over the category $\mathsf{Open}_\mathbb{R}(W)$, as defined in Definition 3.1.10. Indeed, a presheaf of groupoids over this category is again just a functor $\mathcal{F} : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$ and we may replace the open covers of $\mathsf{Open}(W)$ as in Definition B.3 with open covers of elements of $\mathsf{Open}_{\mathbb{R}}(W)$ by $\mathbb{R}$-invariant subsets.

Remark B.6. It is easy to check that a presheaf of groupoids $\mathcal{F} : \mathsf{Open}(U)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a prestack if and only if for every open subset $U \subset X$ and for any open cover $\{U_\alpha\}_{\alpha\in A}$ of U the restriction functor $\Phi : \mathcal{F} \to \mathcal{D}_{\mathcal{F}_U}(\{U_\alpha\})$ is fully faithful.

Note also that all the presheaves of groupoids we consider in this paper are clearly prestacks. Since the groupoids of these presheaves consist of spaces with extra information (bundles/manifolds/stratified spaces with symplectic forms) and the morphisms are maps of these spaces, it is clear that, in these cases, collections of maps between two objects on local restrictions that are coherent will glue to a unique map.

Example B.7. Let X be a manifold with corners and let $\mathsf{BG} : \mathsf{Open(X)}^{\mathsf{op}} \to \mathsf{Groupoids}$ be the presheaf of principal G-bundles over X: for each open U, the groupoid $\mathsf{BG(U)}$ is that with objects principal G-bundles over U and with morphisms isomorphisms of principal G-bundles (G-equivariant maps of bundles covering the identity on U). The restriction morphisms $\mathsf{BG(V)} \to \mathsf{BG(U)}$ for $U \subset V$ open subsets of X are simply just the morphisms taking principal bundles $\pi : P \to V$ to $\pi : P|_U \to U$.

The proof that BG is a stack comes in two easy parts. Let U be an open subset of X with open cover $\{U_\alpha\}_{\alpha\in A}$ and let $\Phi : \mathsf{BG(U)} \to \mathcal{D}_{\mathsf{BG}}(\{\mathsf{U}_\alpha\}_{\alpha\in\mathsf{A}})$ be the restriction functor. Let $\pi : P \to U$ and $\pi' : P' \to U$ be two principal G-bundles. Then it is clear that $\underline{\mathsf{Hom}}(P, P')$ is a sheaf of sets. Thus, Φ is fully faithful (see Remark B.6).

To show Φ is essentially surjective, let

$$(\{\pi_\alpha : P_\alpha \to U_\alpha\}_{\alpha\in A}, \{\varphi_{\alpha\beta} : P_\alpha \to P_\beta\}_{\alpha,\beta\in A})$$

be a piece of decent data. Then the construction

$$P := \left(\bigsqcup_{\alpha\in A} P_\alpha\right) \Big/ \sim \tag{B.2}$$

where $\sim$ is the equivalence relation generated by

$$p \sim q \text{ if } p \in P_\alpha|_{U_{\alpha\beta}},\ q \in P_\beta|_{U_{\alpha\beta}},\ \text{and } \varphi_{\alpha\beta}(p) = q$$

yields the total space of a principal G-bundle with quotient map $\pi : P \to U$ for $\pi([p]) = \pi_\alpha(p)$, where $p \in P_\alpha$. It is clear then that $\Phi(P)$ is isomorphic to our original descent data.

Proposition B.8. Let $\psi : W \to \mathfrak{g}^*$ be a homogeneous unimodular local embedding. Then $\mathsf{HSTB}_\psi : \mathsf{Open}_{\mathbb{R}}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a stack.

Proof. Recall HSTB_ψ is a presheaf of groupoids over W with an $\mathbb{R}$-invariant open subset $U \subset W$ corresponding to the groupoid $\mathsf{HSTB}_{\psi|_U}(U)$. Fix an open cover of U by $\mathbb{R}$-invariant subsets $\{U_\alpha\}_{\alpha \in A}$. Then we must show $\Phi : \mathsf{HSTB}_\psi(U) \to \mathcal{D}_{\mathsf{HSTB}_\psi}(\{U_\alpha\})$ is an equivalence of categories.

Let $(\pi : P \to U, \omega)$ and $(\pi' : P' \to U, \omega')$ be any two homogeneous symplectic toric bundles in $\mathsf{HSTB}_\psi(U)$. Then as morphisms in HSTB_ψ must be $(G \times \mathbb{R})$-equivariant, it follows that any family of morphisms $\varphi_\alpha : P|_{U_\alpha} \to P'|_{U_\alpha}$ in HSTB_ψ that successfully patch together to a map of principal G-bundles must also patch together to an $\mathbb{R}$-equivariant diffeomorphisms. As the $\varphi_{\alpha\beta}$ are symplectomorphisms, it also follows the patched together map is a symplectomorphism as well. Thus,

$$\underline{\mathsf{Hom}}((\pi : P \to U, \omega), (\pi' : P' \to U, \omega'))$$

must be a sheaf and therefore Φ is fully faithful (see Remark B.6).

The case of principal G-bundles provides a guide for showing Φ is essentially surjective. Let

$$(\{(\pi_\alpha : P_\alpha \to U_\alpha, \omega_\alpha)\}_{\alpha \in A}, \{\varphi_{\alpha\beta} : (P_\alpha, \omega_\alpha)|_{U_{\alpha\beta}} \to (P_\beta, \omega_\beta)|_{U_{\alpha\beta}}\}_{\alpha,\beta \in A}) \tag{B.3}$$

be a piece of descent data. As in the case of principal G-bundles, let $\pi : P \to U$ be the bundle built as in equation (B.2). Since the transition maps $\varphi_{\alpha\beta}$ are $\mathbb{R}$-equivariant, it follows that the actions of $\mathbb{R}$ on each P_α patch together to give a free and proper action on P.

As the transition maps $\varphi_{\alpha\beta}$ must also be symplectomorphisms, it is clear that the symplectic forms from each piece must patch together. Finally, since the condition $\rho_\lambda^* \omega = e^\lambda \omega$ for $\rho_\lambda : P \to P$ the action diffeomorphism for real λ is local, it follows that, since each ω_α satisfies this property, ω must satisfy this property as well. Finally, note that for glued quotient map $\pi : P \to U$, $\psi \circ \pi$ is locally and hence globally a moment map for (P, ω). So descent data correctly patches together to an element $(\pi : P \to U, \omega)$ with $\Phi(\pi : P \to U, \omega)$ isomorphic to the descent data (B.3).

Thus, HSTB_ψ is a stack. □

Proposition B.9. Let $\psi : W \to \mathfrak{g}^*$ be a stratified unimodular local embedding. Then $\mathsf{CSTB}_\psi : \mathsf{Open}(W)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a stack.

Proof. Fix an open subset U in W with open cover $\{U_\alpha\}_{\alpha \in A}$. We must show $\Phi : \mathsf{CSTB}_\psi(U) \to \mathcal{D}_{\mathsf{CSTB}_\psi}(\{U_\alpha\})$

is an equivalence of categories.

Let $(\pi : P \to U_{\mathsf{reg}}, \omega)$ and $(\pi' : P' \to U_{\mathsf{reg}}, \omega')$ be any two conical symplectic toric bundles in $\mathsf{CSTB}_\psi(U)$. Then since maps of conical symplectic toric bundles between $(\pi : P \to U_{\mathsf{reg}}, \omega)$ and $(\pi' : P' \to U_{\mathsf{reg}}, \omega')$ are just isomorphisms of principal G-bundles that are also symplectomorphisms, it is easy to check as in the case of HSTB_ψ that

$$\underline{\mathsf{Hom}}((\pi : P \to U, \omega), (\pi' : P' \to U, \omega'))$$

must be a sheaf and therefore Φ is fully faithful (see Remark B.6).

Again, to show Φ is an essentially surjective functor, we use BG as a model. Let

$$(\{(\pi_\alpha : P_\alpha \to U_{\alpha\mathsf{reg}}, \omega_\alpha)\}_{\alpha\in A}, \{\varphi_{\alpha\beta} : (P_\alpha, \omega_\alpha)|_{U_{\alpha\beta}} \to (P_\beta, \omega_\beta)|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A})$$

be a piece of descent data. Let $\pi : P \to U$ be the principal G-bundle built from the bundles of the above descent data as in equation (B.2). As in the case of homogeneous symplectic toric bundles above, we may conclude that the forms ω_α glue together to a symplectic form ω on all P and that $\psi \circ \pi$ is a moment map for this form.

Finally, for each singularity w of W, and for any element U_α of the cover $\{U_\alpha\}_{\alpha\in A}$ containing w, by definition there must be an open subset $V \subset U_w$ containing w so that $(P|_{U_\alpha}, \omega_\alpha)|_V$ is isomorphic to a neighborhood of $-\infty$ in a symplectic cone. It then follows that $(P, \omega)|_V$ is also isomorphic to a neighborhood of $-\infty$ in a symplectic cone. Therefore, $(\pi : P \to W_{\mathsf{reg}}, \omega)$ is an element of $\mathsf{CSTB}_\psi(U)$. It is clear then that $\Phi(\pi : P \to W_{\mathsf{reg}}, \omega)$ is isomorphic to the above descent data. Thus Φ is essentially surjective and CSTB_ψ is a stack. □

A special class of presheaves we are interested in are transitive stacks.

Definition B.10. A presheaf of groupoids $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ is called transitive if, for every open subset $U \subset X$,any two objects ξ and ξ' in $\mathcal{F}(U)$ are locally isomorphic; that is, there exists a cover $\{U_\alpha\}_{\alpha\in A}$ such that the restrictions $\xi|_{U_\alpha}$ and $\xi'|_{U_\alpha}$ are isomorphic for each α.

The payoff for working with stacks in our case will be the following technical lemma. This is a generalized version of the proof presented in [19] that, for $\psi : W \to \mathfrak{g}^*$ a unimodular local embedding, the functor $c|_U : \mathsf{STB}_\psi(U) \to \mathsf{STM}_\psi(U)$ is essentially surjective on each open subset $U \subset W$.

Lemma B.11. Let X be a topological space. Suppose $\mathcal{F} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a stack and that $\mathcal{G} : \mathsf{Open}(X)^{\mathsf{op}} \to \mathsf{Groupoids}$ is a prestack. If a map of presheaves $\Psi : \mathcal{F} \to \mathcal{G}$ satisfies

1. $\Psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful

2. For each $x \in U$ and each $\xi \in \mathcal{G}(U)$, there is an open subset $V \subset U$ and an element $\eta \in \mathcal{F}(V)$ such that $\Psi(\eta)$ is isomorphic to $\xi|_V$

for each open subset U X, then Ψ is an isomorphism of presheaves. In particular, $\mathcal{G}$ must be a stack.

Remark B.12. Note that, in the case of the map of presheaves $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$ over $\mathsf{Open}_\mathbb{R}(W)$, this lemma still works (if we are sure to use open $\mathbb{R}$-invariant subsets and covers by open $\mathbb{R}$-invariant subsets of W). Additionally, if $\mathcal{G}$ is a transitive prestack and, for every open U, $\mathcal{F}(U)$ is non-empty, any map of presheaves satisfies condition (2) of the above lemma; in fact, elements of $\mathcal{G}(U)$ are always locally isomorphic to any elements of the image of $\mathcal{F}(U)$. This will also be the case with $\mathsf{hc} : \mathsf{HSTB}_\psi \to \mathsf{STC}_\psi$. However, we must also have this slightly more general version of the lemma to apply to the case of $\tilde{\mathsf{c}} : \mathsf{CSTB}_\psi \to \mathsf{STSS}_\psi$, where STSS_ψ in general *is not* a transitive prestack.

Proof of Lemma B.11. Fix an open subset U of X. To show Ψ is an isomorphism of presheaves, it is enough to show that $\Psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of groupoids for each U. By hypothesis, we have already that $\Psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful, so it remains to show that it is essentially surjective.

Fix an element $\xi \in \mathcal{G}(U)$. Then by hypothesis there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of U, elements $\{\eta_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha\in A}$, and a family of isomorphisms $\{\varphi_\alpha : \Psi(\eta_\alpha) \to \xi|_{U_\alpha}\}_{\alpha\in A}$. Then since $\Psi_{U_{\alpha\beta}}$ is full for every $U_{\alpha\beta}$ and since $(\Psi(\eta_\alpha))|_{U_{\alpha\beta}} = \Psi(\eta_\alpha|_{U_{\alpha\beta}})$, there exist morphisms

$$\phi_{\alpha\beta} : \eta_\alpha|_{U_{\alpha\beta}} \to \eta_\beta|_{U_{\alpha\beta}}$$

for every α and β with $U_{\alpha\beta}$ non empty such that $\Psi(\phi_{\alpha\beta}) = \varphi_\beta^{-1}\varphi_\alpha$.

For any α, β, and γ so that $U_{\alpha\beta\gamma}$ is non-empty, as $\Psi_{U_{\alpha\beta\gamma}}$ is faithful, it follows that $\phi_{\beta\gamma}|_{U_{\alpha\beta\gamma}} \circ \phi_{\alpha\beta}|_{U_{\alpha\beta\gamma}} = \phi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}$. Thus, the family of isomorphisms $\{\phi_{\alpha\beta}\}_{\alpha,\beta\in A}$ satisfies the cocycle condition.

Therefore, the pair of subsets $\{\{\eta_\alpha\}_{\alpha\in A}, \{\phi_{\alpha\beta}\}_{\alpha,\beta\in A}\}$ is a piece of descent data for $\mathcal{F}$ with respect to the cover $\{U_\alpha\}_{\alpha\in A}$. As $\mathcal{F}$ is a stack, the restriction functor $\Phi : \mathcal{F}(U) \to \mathcal{D}_\mathcal{F}(\{U_\alpha\}_{\alpha\in A})$ from $\mathcal{F}(U)$ to the descent category is essentially surjective. Thus, there exists an element η in $\mathcal{F}(U)$ and an isomorphism of descent data:

$$\{\rho_\alpha\}_{\alpha\in A} : \Phi(\eta) \to \{\{\eta_\alpha\}_{\alpha\in A}, \{\phi_{\alpha\beta} : \eta_\alpha|_{U_{\alpha\beta}} \to \eta_\beta|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A}\}$$

Finally, we have the commutative diagram

$$\begin{array}{ccccc}
\Psi(\eta)|_{U_{\alpha\beta}} & \xrightarrow{\Psi(\rho_\alpha)|_{U_{\alpha\beta}}} & \Psi(\eta_\alpha)|_{U_{\alpha\beta}} & \xrightarrow{\varphi_\alpha} & \xi|_{U_{\alpha\beta}} \\
\Vert & & \downarrow{\scriptstyle \Psi(\phi_{\alpha\beta})} & & \Vert \\
\Psi(\eta)|_{U_{\alpha\beta}} & \xrightarrow[\Psi(\rho_\beta)|_{U_{\alpha\beta}}]{} & \Psi(\eta_\beta)|_{U_{\alpha\beta}} & \xrightarrow[\varphi_\beta]{} & \xi|_{U_{\alpha\beta}}
\end{array}$$

for every α and β with $U_{\alpha\beta}$ non-empty. This commutes as the left square is exactly the image under Ψ of the diagram (B.1) corresponding to the isomorphism of descent data $\{\rho_\alpha\}_{\alpha\in A}$ while the right hand side commutes by definition of $\phi_{\alpha\beta}$.

For each α, let $f_\alpha : \Psi(\eta)|_{U_\alpha} \to \xi|_{U_\alpha}$ be the composition $f_\alpha := \varphi_\alpha \circ \Psi(\rho_\alpha)|_{U_\alpha}$. Then the above diagram demonstrates that, on the intersections $U_{\alpha\beta}$, the f_α's are coherent. As $\mathcal{G}$ is a prestack, $\underline{\mathsf{Hom}}(\Psi(\eta), \xi)$ is a sheaf and the isomorphisms $\{f_\alpha\}_{\alpha\in A}$ glue to an isomorphism $f : \Psi(\eta) \to \xi$.

Thus, $\Psi_U : \mathcal{F} \to \mathcal{G}$ is essentially surjective for every U and, by hypothesis, is fully faithful. Therefore, Ψ is an isomorphism of presheaves. □

Appendix C

Relative de Rham cohomology

In this section, we review relative de Rham cohomology, as presented by Bott and Tu [4]. While their treatment uses manifolds without boundary, it should be more or less clear that, as we are not using any special properties of manifolds beyond the existence of the de Rham complex, everything generalizes to manifolds with corners.

Definition C.1. Let M and N be two manifolds with corners and $f : M \to N$ a smooth map. Then the relative de Rham complex $\Omega^{\cdot}(f)$ is the cochain complex with $\Omega^p(f) := \Omega^p(N) \oplus \Omega^{p-1}(M)$ and differential $d_f((\alpha, \beta)) := (d\alpha, f^*\alpha - d\beta)$ (here, we take $\Omega^k(f) = 0$ for $k < 0$ and $\Omega^0(f) = C^\infty(N)$). Denote by $H^{\cdot}(f)$ the cohomology of this cochain complex.

In the case where f is the inclusion of a submanifold M into N, we use the notation $\Omega^{\cdot}(N, M)$ and $H^{\cdot}(N, M)$ for the relative cochain complex and relative cohomology associated to f, respectively.

Proposition C.2. Let $f : M \to N$ be a map of manifolds with corners. Then there is a long exact sequence:

$$\dots \xrightarrow{f^*} H^p(M) \xrightarrow{\tilde{\iota}} H^{p+1}(f) \xrightarrow{\tilde{\pi}} H^{p+1}(N) \xrightarrow{f^*} H^{p+1}(M) \longrightarrow \dots \tag{C.1}$$

where $\tilde{\iota} : H^p(M) \to H^{p+1}(f)$ is the map $\tilde{\iota}([\beta]) := [(0, \beta)]$ and $\tilde{\pi} : H^p(f) \to H^p(N)$ is the map $\tilde{\pi}([(\alpha, \beta)]) := [\alpha]$.

Proof. Let $\tilde{\Omega}^{\cdot}(M)$ be the cochain complex with $\tilde{\Omega}^k(M) := \Omega^{k-1}(M)$ and with differentials $-d$ (for d the normal exterior differential on forms). Then clearly the collection of inclusions $\iota_p : \tilde{\Omega}^p(M) \to \Omega^p(f)$ with $i(\beta) := (0, \beta)$ for each p defines a map of cochain complexes $\iota : \tilde{\Omega}^{\cdot}(M) \to \Omega^{\cdot}(f)$.

On the other hand, let $\pi_p : \Omega^p(f) \to \Omega^p(N)$ be the collection of projections $\pi_p(\alpha, \beta) := \alpha$ for each p. Then we also have a chain map $\pi : \Omega^{\cdot}(f) \to \Omega^{\cdot}(N)$. ι and π give rise to a short exact sequence of chain complexes:

$$0 \longrightarrow \tilde{\Omega}^{\cdot}(M) \xrightarrow{\iota} \Omega(f)^{\cdot} \xrightarrow{\pi} \Omega^{\cdot}(M) \longrightarrow 0$$

Therefore, we have a long exact sequence of cohomology groups:

$$\dots \longrightarrow \tilde{H}^p(M) \xrightarrow{\tilde{\imath}} H^p(f) \xrightarrow{\tilde{\pi}} H^p(N) \longrightarrow \tilde{H}^{p+1}(M) \xrightarrow{\tilde{\imath}} \dots \tag{C.2}$$

for $\tilde{H}^{\cdot}(M)$ the cohomology of $\tilde{\Omega}^{\cdot}(M)$.

Note now that it is more or less obvious that $\tilde{H}^p(M) = H^{p-1}(M)$ as vector spaces. To see that f^* is the connecting homomorphism for long exact sequence (C.2), note that, for $\gamma \in \Omega^p(N)$ a closed form, $d_f(\gamma, 0) = (d\gamma, f^*\gamma) = (0, f^*\gamma)$. Therefore, with the identification $\tilde{H}^p(M) = H^{p-1}(M)$, equation (C.2) becomes equation (C.1). □

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