SOME PROBLEMS IN POLYNOMIAL INTERPOLATION AND TOPOLOGICAL COMPLEXITY

BY

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DISSERTATION

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Abstract

This thesis is comprised of two projects in applied computational mathematics.

In Chapter 1, we discuss the geometry and combinatorics of geometrically characterized sets. These are finite sets of $\binom{n+d}{n}$ points in $\mathbb{R}^d$ which impose independent conditions on polynomials of degree n, and which have Lagrange polynomials of a special form. These sets were introduced by Chung and Yao in a 1977 paper in the SIAM Journal of Numerical Analysis in the context of polynomial interpolation. There are several conjectures on the nature and geometric structure of these sets. We investigate the geometry and combinatorics of GC sets for $d \geq 2$, and prove they are closely related to simplicial complexes for which both Δ and $\Delta^\vee$ are Cohen-Macaulay.

In Chapter 2, we will discuss the motion planning problem in complex hyperplane arrangement complements. The difficulty of constructing a minimally discontinuous motion planning algorithm for a topological space X is measured by an integer invariant of X called topological complexity or $TC(X)$. Yuzvinsky developed a combinatorial criterion for hyperplane arrangement complements which guarantees that their topological complexity is as large as possible. Applying this criterion in the special case when the arrangement is graphic, we simplify the criterion to an inequality on the edge density of the graph which is closely related to the inequality in the arboricity theorem of Nash-Williams.

For Mom and Dad.

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Chapter 1

Simplicial Complexes and the Geometry of Polynomial Interpolation

1.1 Introduction and Background

We will begin our discussion of polynomial interpolation with two simple examples.

Example 1.1.1. Suppose $f(x) \in \mathbb{R}[x]$ is an unknown polynomial with degree at most n. If we know the values that $f(x)$ takes at a set of distinct points $X = \{x_1, \ldots, x_{n+1}\}$ in $\mathbb{R}$, that is

$$f(x_1) = c_1$$
$$f(x_2) = c_2$$
$$\vdots$$
$$f(x_{n+1}) = c_{n+1},$$

is it true that these $n+1$ values of $f(x)$ completely determine $f(x)$? The answer turns out to be an uninteresting "yes". To see this, first observe that for any $1 \leq i \leq n+1$, the degree n polynomial

$$p_i(x) := \frac{\prod\limits_{j \neq i}(x - x_j)}{\prod\limits_{j \neq i}(x_i - x_j)}$$

satisfies $p_i(x_j) = \delta_{ij}$. From this, we can conclude that the evaluation map from the vector space of polynomials of degree at most n to the vector space of $\mathbb{R}$-valued functions on X is an isomorphism. In other words, $f(x)$ is uniquely determined by its values on X, and can be written explicitly as

$$f(x) = \sum_{i=1}^{n+1} c_i p_i(x).$$

It is natural to consider an analogue of the above question for polynomials in more than one variable.

Example 1.1.2. Let $f(x,y) = Ax + By + C$ with A, B and C in $\mathbb{R}$, and let $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$ and $p_3 = (x_3, y_3)$ be three distinct points in $\mathbb{R}^2$. Is it true that the values of $f(x,y)$ on these three points determine $f(x,y)$ completely? The three values $f(p_i) = c_i$ for $i = 1, 2, 3$ determine a linear system in the variables A, B and C:

$$Ax_1 + By_1 + C = c_1$$

$$Ax_2 + By_2 + C = c_2$$

$$Ax_3 + By_3 + C = c_3$$

The coefficients of $f(x,y)$ can be determined from its values at these three points if and only if this system has a unique solution, which is the case exactly when the points p_1, p_2 and p_3 are not collinear. This example illustrates an important distinction between the single variable case and the multivariable case. In the multivariable case, whether or not f is completely determined by its values on X depends on the geometry of the set X.

Our study of polynomial interpolation is focused on finite sets $X \subseteq \mathbb{R}^d$ so that polynomials of degree at most n are completely determined by their values on X. This is codified in the definition below.

Definition 1.1.3. *[dB07] For fixed positive integers d and n, let Π_n denote the vector space of polynomials of degree at most n in the ring $\mathbb{R}[x_1, \ldots, x_d]$. These polynomials can be evaluated as functions on the affine space $\mathbb{R}^d$. For a finite set $X = \{p_1, \ldots, p_N\} \subseteq \mathbb{R}^d$, the evaluation map*

$$ev : \Pi_n \longrightarrow \mathbb{R}^X$$

is a linear map from Π_n to the vector space of $\mathbb{R}-$valued functions on X. We say that X is $n-$correct if any of the following equivalent conditions are satisfied:

(1) The evaluation map is an isomorphism.

(2) $N = \binom{n+d}{n}$ and for any choice of data $\{c_1, \ldots, c_N\} \in \mathbb{R}$, there exists a unique polynomial $f \in \Pi_n$ such that $f(p_i) = c_i$ for $1 \leq i \leq N$.

(3) $N = \binom{n+d}{n}$ and for any $p \in X$ there exists a unique polynomial $Q_p \in \Pi_n$ such that $Q_p(p) = 1$ and $Q_p(q) = 0$ for $q \in X \setminus \{p\}$

In condition (3), the Q_p are called the Lagrange polynomials for X.

While generic sets of $\binom{n+d}{n}$ points are $n-$correct, there are also examples of non-generic sets which are $n-$correct. We will focus on a special class of non-generic $n-$correct sets.

Definition 1.1.4. *With the same notation as above, a finite set X with $|X| = N$ is said to be geometrically characterized, or $GC_{d,n}$ if any of the following equivalent conditions hold.*

(1) X is $n-$correct, and all of the Lagrange polynomials Q_p for X can be written as products

$$Q_p = \prod_{k=1}^{n} l_k$$

of linear polynomials $l_k \in \Pi_1$.

(2) $N = \binom{n+d}{n}$ and for all $p \in X$, we can find n hyperplanes in $\mathbb{R}^d$ so that their union contains $X \setminus \{p\}$ but does not contain p

Note that statement (2) of the above definition naturally extends to finite sets $X \subseteq \mathbb{P}^d_{\mathbb{R}}$, without having to worry about the not-defined "values" of homogeneous polynomials, or choosing affine charts. A finite set of $N = \binom{n+d}{n}$ points in $\mathbb{P}^d_{\mathbb{R}}$ is a $GC_{d,n}$ set if any subset of size $N-1$ is contained in a union of n hyperplanes. As there is no real distinction to be made, we will interchangeably use $GC_{d,n}$ (or simply GC when the dimension and degree are clear) to refer to the property of finite sets in affine or projective space.

It is clear from the geometric form of the definition that $GC_{d,n}$ sets must have fairly large subsets lying on hyperplanes. For example, a $GC_{6,2}$ set would be a set of 28 points in $\mathbb{R}^6$ so that any subset of size 27 is contained in a union of two hyperplanes. This implies immediately that there must exist 14 of the points lying in a single hyperplane. However, in explicit examples there are often hyperplanes containing many more points than the number guaranteed by such simple estimates. We make the following definition.

Definition 1.1.5. *If X is a $GC_{d,n}$ in $\mathbb{R}^d$, a $k-$dimensional plane in $\mathbb{R}^d$ is called maximal if it contains a subset of X of cardinality $\binom{k+n}{n}$.*

In [GM82], Gasca-Maeztu conjectured that in $\mathbb{R}^2$, every $GC_{2,n}$ set contains a line with $n+1$ points of X, which is a maximal hyperplane. In [Bus90], Busch shows the conjecture holds for $n \leq 4$. The last 30 years have seen much additional work on the conjecture; see [dBR90], [CG01], [CGS06], [CG09], [GS00a], [GS00b], [HJZ09], [HR15]. In [CG01] Carnicer-Gasca showed that the Gasca-Maeztu conjecture implies that a $GC_{2,n}$ set in $\mathbb{R}^2$ contains 3 maximal lines. Building on this, in [dB07], de Boor proposed two generalizations of the Gasca-Maeztu conjecture:

Conjecture 1.1.6. *A $GC_{d,n}$ set contains a maximal hyperplane.*

Conjecture 1.1.7. *A $GC_{d,n}$ set contains at least $d+1$ maximal hyperplanes.*

de Boor shows that Conjecture 1.1.7 will require some additional hypothesis: he constructs a $GC_{3,2}$ set which does not have four maximal hyperplanes. Apozyan [Apo11b] used this to construct a $GC_{6,2}$ set with no maximal hyperplane, so Conjecture 1.1.6 fails as stated. On the other hand, [AAK10] shows Conjecture 1.1.6 holds for $GC_{3,2}$ sets. Apozyan-Hakopian conjecture in [Apo11b] that a $GC_{d,n}$ set contains at least $\binom{d+1}{2}$ maximal lines, which is proved for $d = 3, n = 2$ in [Apo11a]. We study $GC_{d,n}$ sets, focusing mainly on the case $d \geq 3$ and $n \geq 2$. Our starting point is work of Sauer-Xu in [SX95] showing that the ideal I_X of a $GC_{d,n}$ set X is minimally generated in degree $n+1$ by $\binom{n+d}{n+1}$ products of linear forms.

The central idea of this chapter is to lift the ideal I_X of polynomials vanishing on X to a monomial ideal: by replacing the generators $\prod l_i$ of I_X with monomials $\prod y_i$ with a new variable y_i for each distinct linear form, we obtain insight into the combinatorial structure of GC sets: the new monomial ideal is squarefree, so corresponds via Stanley-Reisner theory to a simplicial complex Δ. The core of the chapter is Section 1.3.1, where we apply Stanley-Reisner theory to analyze these ideals. Theorem 1.3.12 shows that bi-Cohen-Macaulay squarefree monomial ideals of codimension d and degree $\binom{d+n}{n}$ always specialize to n-correct sets of points.

With the goal of obtaining examples of GC sets, we reverse engineer this process by starting with a bi-Cohen-Macaulay monomial ideal. While specializing yields a n-correct set, the GC condition is quite restrictive: most n-correct sets are not GC. To overcome this obstacle, we introduce an analog of the GC property for monomial ideals. In Theorem 1.3.13, we prove a combinatorial criterion for a component of a monomial ideal to be GC. Example 1.1.8 below illustrates our results in the $d = 2$ case; additional examples appear in Section 1.2.1.

Example 1.1.8. A Chung-Yao *natural lattice* of six points in $\mathbb{R}^2$ consists of the intersection points of four general lines $\{l_1, l_2, l_3, l_4\}$ in the plane.

The ideal of $I_X = \langle l_1l_2l_3, l_1l_2l_4, l_1l_3l_4, l_2l_3l_4\rangle$, so replacing l_i with y_i gives rise to the ideal $I_\Delta = \langle y_1y_2y_3, y_1y_2y_4, y_1y_3y_4, y_2y_3y_4\rangle$. The ideal I_Δ has a decomposition

$$I_\Delta = \langle y_1, y_2\rangle \cap \langle y_1, y_3\rangle \cap \langle y_1, y_4\rangle \cap \langle y_2, y_3\rangle \cap \langle y_2, y_4\rangle \cap \langle y_3, y_4\rangle. \tag{1.1}$$

The results in 1.3.1 show Δ consists of 4 vertices and 6 edges connecting them.

For this example a component $F = \langle y_i, y_j\rangle$ in Equation 1.1 satisfies the monomial version of the GC condition appearing in Definition 1.3.9 if there is a quadratic monomial f such that $f \notin F$ but $f \cdot F \in I_\Delta$.

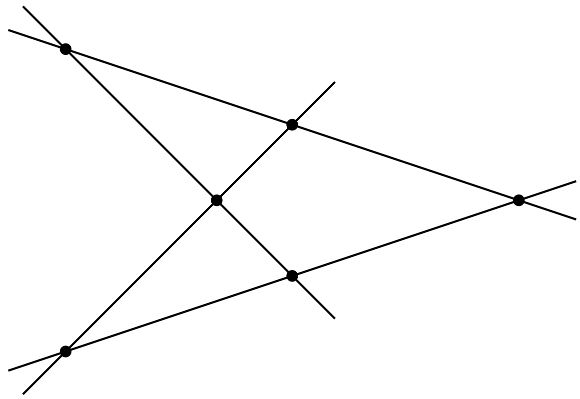


Figure 1.1: Four general lines in the plane, and their intersection points

For example when $F = \langle y_1, y_2 \rangle$, choosing $f = y_3 y_4$ satisfies the condition, and an easy check shows for the other components $\langle y_i, y_j \rangle$ choosing $f = y_k y_l$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$ works. Each of the hyperplanes y_i appears in 3 of the $\langle y_i, y_j \rangle$; the y_i are monomial versions of maximal hyperplanes. Specializing $y_i \mapsto l_i$ preserves these properties, and reproves the well known fact that a Chung-Yao configuration of $n + 2$ lines in the plane is $GC_{2,n}$ and has $n + 2$ maximal hyperplanes.

1.2 The Vanishing Ideal of a $GC_{d,n}$ Set

In this section, we will show that a set of points X having the $GC_{d,n}$ property is very special from an algebraic standpoint. Recall that for any set X, the set of polynomial functions vanishing identically on X is called the vanishing ideal of X and is denoted I_X. It is closed under addition, as well as under multiplication by arbitrary polynomials.

As mentioned above, we will streamline the study of the algebra which arises when considering GC sets by homogenizing the problem. Geometrically, this means we are considering affine space $\mathbb{R}^d$ as a subset of projective space $\mathbb{P}^d_{\mathbb{R}}$. Since $\mathbb{P}^d_{\mathbb{R}}$ may be thought of as $\mathbb{R}^d \bigcup \mathbb{P}^{d-1}_{\mathbb{R}}$, where $\mathbb{P}^{d-1}_{\mathbb{R}}$ is the hyperplane at infinity and $X \subseteq \mathbb{R}^d$, $X \cap \mathbb{P}^{d-1}_{\mathbb{R}} = \emptyset$. The following example makes this explicit.

Example 1.2.1. Suppose

$$X = (0,0) \cup (1,0) \cup (0,1) \subseteq \mathbb{R}^2.$$

I_X consists of the intersections of the ideal of the three points, so is

$$\langle x_0, x_1 \rangle \cap \langle x_0 - 1, x_1 \rangle \cap \langle x_0, x_1 - 1 \rangle.$$

If we embed $\mathbb{R}^2 \subseteq \mathbb{P}^2_{\mathbb{R}}$ as the plane with $x_2 = 1$, the points of X become

$(0:0:1) \cup (1:0:1) \cup (0:1:1)$ when written in projective coordinates;

The ideal of homogeneous polynomials in $\mathbb{R}[x_0, x_1, x_2]$ which vanish on X is

$$\langle x_0, x_1 \rangle \cap \langle x_0 - x_2, x_1 \rangle \cap \langle x_0, x_1 - x_2 \rangle = \langle x_0 x_1, x_0(x_0 - x_2), x_1(x_1 - x_2) \rangle.$$

Looking closely, we see that the ideal $\langle x_0 x_1, x_0(x_0 - x_2), x_1(x_1 - x_2) \rangle$ is determinantal, that is, it is generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x_0 - x_2 & x_1 - x_2 \\ -x_1 & 0 \\ 0 & -x_0 \end{bmatrix}$$

This is not an accident: it can be shown that after homogenizing, any $GC_{2,n}$ set is generated by the maximal minors of a $(n+2) \times (n+1)$ matrix of homogeneous linear forms. In Example 1.1.8, I_X is generated by the 3×3 minors of

$$d_2 = \begin{bmatrix} l_4 & 0 & 0 \\ -l_3 & l_3 & 0 \\ 0 & -l_2 & l_2 \\ 0 & 0 & -l_1 \end{bmatrix}$$

However, there is even more structure here: the columns of the matrix d_2 are generators (over the polynomial ring) for the kernel of the matrix

$$d_1 = \begin{bmatrix} l_1 l_2 l_3 & l_1 l_2 l_4 & l_1 l_3 l_4 & l_2 l_3 l_4 \end{bmatrix}.$$

Relations on a matrix with polynomial entries are called *syzygies*. They can be represented by a vector of polynomials, and were systematically studied by Hilbert. For a $GC_{2,n}$ set X, there are three important observations:

- The generators for I_X are products of linear forms.
- The first syzygies of I_X are generated by vectors of linear forms.
- The maximal minors of the syzygy matrix generate I_X.

The second two points are consequences of a famous theorem in commutative algebra, the *Hilbert-Burch* theorem, which describes the behavior of ideals which define sets of points in the projective plane. Most of the remainder of this section is devoted to defining these objects, and to understanding what happens for GC sets in higher dimensions.

By our earlier remarks, we treat X as a set of points in $\mathbb{P}^d_{\mathbb{R}}$; We will use R to denote the ring $\mathbb{R}[x_0, \ldots, x_d]$. For a point $p \in X$, the ideal of functions vanishing at p

$$I_p = \langle l_{p,1}, \ldots, l_{p,d} \rangle$$

can be generated by d linearly independent homogeneous linear forms. We use $\mathcal{Q}$ to denote the ideal $\langle Q_p, p \in X \rangle$, with Q_p as in Definition 1.1.4.

In algebraic geometry, we say that a finite set of points X *imposes independent conditions* on polynomials of degree n if the rank of the evaluation map is equal to $|X|$. So an n-correct set in $\mathbb{R}^d$ is a set of exactly $\binom{d+n}{n}$ points which impose independent conditions in degree n. Let $X \subseteq \mathbb{P}^d$ be a set of $N = \binom{d+n}{n}$ distinct points having property $GC_{d,n}$.

Lemma 1.2.2. *The ideal $\mathcal{Q}$ is of the form $\langle x_0, \ldots, x_d \rangle^n$.*

Proof. **Proof** Since the Q_p are all of degree n, clearly $\mathcal{Q} \subseteq \langle x_0, \ldots, x_d \rangle^n$. The condition that $Q_p(q) = \delta_{pq}$ means that the Q_p are linearly independent; since the dimension of $\langle x_0, \ldots, x_d \rangle^n$ is $\binom{n+d}{d}$, equality holds. this means $GC_{d,n}$ sets are n-correct. □ □

Lemma 1.2.3. *Suppose X has the $GC_{d,n}$ property. Then for each $p \in X$, there are d linearly independent linear forms $l_{p,1}, \ldots, l_{p,d}$ with each $l_{p,j}$ dividing some Q_q, $p \neq q$, such that $l_{p,j}(p) = 0$.*

Proof. The $GC_{d,n}$ property implies that each Q_q with $q \neq p$ has a factor which is a linear form passing through p. Let L be the vector space spanned by all such linear factors, and suppose L has dimension less than d. Changing coordinates, we can suppose $L = \mathrm{span}\{x_0, \ldots, x_m\}$ with $m \leq d-2$. But then

$$\mathcal{Q} = \langle Q_p \rangle + \langle P \rangle,$$

where $\langle P \rangle = \mathcal{Q}_n \cap \langle x_0, \ldots, x_m \rangle$. This is impossible, because by Lemma 1.2.2, $\mathcal{Q} = \langle x_0, \ldots, x_d \rangle^n$ and

$$\{x_{d-1}^n, x_{d-1}^{n-1} x_d, \ldots, x_{d-1} x_d^{n-1}, x_d^n\} \subseteq \mathcal{Q}_n$$

is $n+1$ dimensional and disjoint from the degree n component of the sub-ideal of $\mathcal{Q}$ generated by $\langle P \rangle$, and clearly cannot be spanned by Q_p. □

1.2.1 Minimal free resolutions

The polynomial ring $R = \mathbb{R}[x_0, \dots, x_d]$ is a $\mathbb{Z}-$graded ring: R_i is the vector space of homogeneous polynomials of degree i, and if $r_j \in R_j$ and $r_i \in R_i$ then $r_i \cdot r_j \in R_{i+j}$. As $R_0 = \mathbb{R}$, this means each R_i has the structure of an $R_0 = \mathbb{R}$ vector space, of dimension $\binom{i+d}{i}$, and $R = \oplus_i R_i$. A finitely generated graded R-module N admits a similar decomposition; if $s \in R_p$ and $n \in N_q$ then $s \cdot n \in N_{p+q}$. In particular, each N_q is a $R_0 = \mathbb{R}$-vector space. A graded map of graded modules $M \to N$ preserves the grading, so takes $M_i \to N_i$.

Definition 1.2.4. *For a finitely generated graded R-module N, the Hilbert function is $HF(N,t) = \dim_{\mathbb{R}} N_t$, and the Hilbert series is $HS(N,t) = \sum \dim_{\mathbb{R}} N_q t^q$.*

For $t \gg 0$, the Hilbert function of N is a polynomial in t, called the Hilbert polynomial $HP(N,t)$, of degree at most d ([Sch03], Theorem 2.3.3). For $X \subseteq \mathbb{P}^d$, we define $\operatorname{codim}(I_X)$ as $d - \deg(HP(R/I_X, t))$. The degree of $HP(R/I_X, t)$ is the dimension of X. When X is a set of points in $\mathbb{P}^d$, $I_X = \cap P_i$ with $P_i = \langle l_{i1}, \dots, l_{id} \rangle$ and the codimension of I_X is d.

Definition 1.2.5. *A free resolution for an R-module N is an exact sequence*

$$\mathbb{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0,$$

where the F_i are free R-modules.

If N is graded, then the F_i are also graded, so letting $R(-m)$ denote a rank one free module generated in degree m, we may write $F_i = \oplus_j R(-j)^{a_{i,j}}$. By the Hilbert syzygy theorem [Sch03] a finitely generated, graded R-module N has a free resolution of length at most $d+1$, with all the F_i of finite rank. Since

$$HS(R(-i), t) = \frac{t^i}{(1-t)^{d+1}}$$

$$HP(R(-i), t) = \binom{t+d-i}{d}$$

this means we can read off the Hilbert series, function and polynomial from a free resolution as an alternating sum, which is illustrated in Example 1.2.7.

Definition 1.2.6. *For a finitely generated graded R-module N, a free resolution is called minimal if for each i, $\operatorname{Im}(d_i) \subseteq \mathfrak{m}F_{i-1}$, where $\mathfrak{m} = \langle x_0, \dots, x_d \rangle$. The Castelnuovo-Mumford regularity of N is $\max_{i,j}\{a_{i,j} - i\}$. The projective dimension* $\operatorname{pdim}(N)$ *of N is the length of a minimal free resolution of N.*

Example 1.2.7. For the $R = \mathbb{R}[x_0, x_1, x_2]$ module $R/\langle x_0^2, x_1^2\rangle$, the graded free resolution is

$$0 \longrightarrow R(-4) \xrightarrow{\begin{bmatrix} -x_1^2 \\ x_0^2 \end{bmatrix}} R(-2)^2 \xrightarrow{\begin{bmatrix} x_0^2 & x_1^2 \end{bmatrix}} R \longrightarrow R/\langle x_0^2, x_1^2\rangle \longrightarrow 0,$$

and for I_X of Example 1.1.8, the free resolution is

$$0 \longrightarrow R(-4)^3 \xrightarrow{d_2} R^4(-3) \xrightarrow{d_1} R \longrightarrow R/I \longrightarrow 0,$$

with d_i as in Example 1.2.1; the d_1 map is a 1×4 matrix with cubic entries, giving a map $R^4 \to R^1$. Because we want graded maps, the generators of R^4 must appear in degree 3, which explains the grading shift for the module $R^4(-3)$. So for X the Chung-Yao set of Example 1.1.8, we see that the Hilbert series and Hilbert polynomial are

$$\begin{aligned} HS(R/I_X, t) &= \frac{1 - 4t^3 + 3t^4}{(1-t)^3} \\ HP(R/I_X, t) &= \binom{t+2}{2} - 4\binom{t+2-3}{2} + 3\binom{t+2-4}{2} = 6, \end{aligned}$$

as expected, since the Hilbert polynomial of a $GC_{d,n}$ set X is $|X| = \binom{d+n}{n}$.

While the differentials which appear in a minimal free resolution of N are not unique, the ranks and degrees of the free modules which appear are unique.

Definition 1.2.8. *An ideal $I \subseteq R$ is called Cohen-Macaulay if* $\operatorname{codim}(I) = \operatorname{pdim}(R/I)$.

Example 1.2.9. The two ideals in Example 1.2.7 both have $\operatorname{pdim}(R/I) = 2$; because the ideals define zero dimensional subsets of the plane they are codimension two, so both ideals are Cohen-Macaulay. This is a general phenomenon: the ideal I_X of a set of points $X \subseteq \mathbb{P}^d$ is Cohen-Macaulay, of codimension d.

Definition 1.2.8 is hard to digest, but the Cohen-Macaulay condition has many useful consequences, see Chapter 10 of [Sch03]. The Hilbert-Burch theorem states that a codimension two Cohen-Macaulay ideal $\mathcal{I} = \langle f_1, \ldots, f_m\rangle$ is generated by the maximal minors of an $m \times m-1$ matrix, whose columns are a basis for the syzygies on $\mathcal{I}$. To generalize the Hilbert-Burch theorem to codimension greater than two, we need the *Eagon-Northcott* complex:

Definition 1.2.10. *Let $R^m \simeq F \xrightarrow{\phi} G \simeq R^n$ be a homomorphism of free R-modules, with $m \geq n$. Then ϕ induces a homomorphism*

$$\Lambda^n(F) \xrightarrow{\Lambda\phi} \Lambda^n(G) = R,$$

where the entries of $\Lambda\phi$ are the $n \times n$ minors of ϕ. With suitable conditions (see [Pee11]) on ϕ, the ideal I_ϕ of $n \times n$ minors has a minimal free resolution, in which the free modules are tensor products of exterior and symmetric powers:

$$\cdots \longrightarrow S_2(G^*) \otimes \Lambda^{n+2}(F) \longrightarrow S_1(G^*) \otimes \Lambda^{n+1}(F) \xrightarrow{d_1} \Lambda^n(F) \longrightarrow R \longrightarrow R/I_\phi \longrightarrow 0.$$

This complex is called the Eagon-Northcott complex of ϕ. The key map is d_1: since $\phi^* : G^* \to F^*$, for $\alpha \in G^*$, $\phi^*(\alpha) \in F^*$, and

$$d_1(\alpha \otimes e_1 \wedge \cdots \wedge e_{n+1}) = \sum_{j=1}^{n+1} (-1)^j (\phi^*(\alpha)(e_j)) \cdot e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_{n+1},$$

with higher differentials defined similarly.

1.2.2 The ideal of a $GC_{d,n}$ set is generated by products of linear forms

We start with an algebraic proof of the following key result of Sauer-Xu [SX95], which is a main ingredient in this chapter.

Theorem 1.2.11. *If $X \subseteq \mathbb{R}^d$ is a $GC_{d,n}$ set, then the ideal I_X is generated in degree $n+1$ by $\binom{n+d}{n+1}$ products of linear forms.*

Proof. Let $I_C = \langle Q_p \cdot l_{pj}, p \in X, j \in \{1, \dots, d\} \rangle$, with $l_{p,j}$ as in Lemma 1.2.3. Because X is a set of distinct points in $\mathbb{P}^d$, I_X is Cohen-Macaulay and codimension d. Since $Q_p(q) = \delta_{pq}$, the points of X impose independent conditions (see [Sch03], Chapter 7) on polynomials of degree n, so I_X is generated in degree $> n$. As $\dim_{\mathbb{R}} R_{n+1} = \binom{n+1+d}{d}$ and the $\binom{n+d}{d}$ points impose independent conditions, by Theorem 7.1.8 of [Sch03], I_X is generated by

$$\binom{n+1+d}{d} - \binom{n+d}{d} = \binom{n+d}{n+1}$$

polynomials of degree $n+1$.

By construction, every polynomial in I_C is a product of linear forms of degree $n+1$ and vanishes on X, so $I_C \subseteq I_X$. It suffices to show that the dimension of I_C in degree $n+1$ is $\binom{n+d}{n+1}$. There are relations among the generators of I_C:

$$\sum_{i=1}^{N} Q_i \cdot (\sum_{j=1}^{d} a_{i_j} l_{i_j}) = 0, \tag{1.2}$$

with the $a_{i_j} \in \mathbb{R}$. Such a relation is a *linear syzygy* on $\mathcal{Q} = \langle x_0, \dots, x_d \rangle^n$. By [EK90], $\mathcal{Q}$ has a minimal free resolution of Eagon-Northcott type; in particular, $\mathcal{Q}$ is generated by the $n \times n$ minors of an $(n+d) \times n$

matrix whose entries are the variables of R. As a consequence, all linear syzygies are Eagon-Northcott type syzygies, that is, the image of the leftmost map below:

$$S_1(R^n) \otimes \Lambda^{n+1}(R^{n+d}) \longrightarrow \Lambda^n(R^{n+d}) \longrightarrow \Lambda^n(R^n) = R \longrightarrow R/\mathcal{Q} \longrightarrow 0.$$

So there are $n \cdot \binom{n+d}{n+1}$ minimal linear first syzygies on $\mathcal{Q}$. The minimal value for $dim(I_C)_{n+1}$ is achieved if these syzygies occur in Equation 1.2, so

$$\begin{aligned} \dim(I_C)_{n+1} &\geq d \cdot N - n \cdot \binom{n+d}{n+1} \\ &= d \cdot \binom{n+d}{n} - n \cdot \binom{n+d}{n+1} \\ &= \binom{n+d}{n+1} \\ &= \dim(I_X)_{n+1} \end{aligned}$$

Since $I_C \subseteq I_X$ and both are generated in degree $n+1$, we have $I_C = I_X$. □

An important related result is the next proposition; while the proof is technical the meaning is very concrete: if X is a $GC_{d,n}$ set, then all the matrices in the minimal free resolution have entries of degree at most one: that is, they are matrices of linear forms, just as in the case where $d = 2$.

Proposition 1.2.12. *The minimal free resolution of I_X has the same graded free modules as an Eagon-Northcott resolution of a generic $(n+d) \times (n+1)$ matrix.*

Proof. By Theorem 7.1.8 of [Sch03], the Castelnuovo-Mumford regularity of I_X is the smallest i such that $H^1(\mathcal{I}_X(i-1)) = 0$; because the points impose independent conditions and $H^1(\mathcal{I}_X(i-1))$ is the cokernel of the evaluation map on polynomials of degree $i-1$, the $GC_{d,n}$ property means X is $n+1$ regular. Therefore the minimal free resolution of I_X has the form

$$0 \to R^{a_d}(-d-n) \to R^{a_{d-1}}(-d-n+1) \to \cdots \to R^{a_1}(-n-1) \to R \to R/I_X \to 0,$$

so every differential is a matrix of linear forms. Since the points impose independent conditions, comparing to the Hilbert series yields the result. □

Definition 1.2.13. *We call an ideal I determinantal if I is generated by the $r \times r$ minors of a $m \times r$ matrix, with $m \geq r \geq 2$.*

Example 1.2.14. For a set of points $X \subseteq \mathbb{P}^2$, the Hilbert-Burch theorem [Pee11] shows that I_X is determinantal, with $m = n+2, r = n+1$. This fails in higher dimension: the ideal for ten general points in $\mathbb{P}^3$

has a minimal free resolution of the form

$$0 \longrightarrow R(-5)^6 \longrightarrow R(-4)^{15} \longrightarrow R(-3)^{10} \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

So I_X has 10 cubic generators, 15 linear first syzygies, and 6 linear second syzygies. However, it is not determinantal [Gor07]. By Proposition 1.2.12 the graded free modules are the same as those of a $GC_{3,2}$ set; by Theorem 1.3.12 I_X is determinantal if X is Chung-Yao. Question: are $GC_{d,n}$ sets always determinantal?

1.3 Bi-Cohen-Macaulay simplicial complexes

By Theorem 1.2.11, the ideal I_X can be generated by products of linear forms, and our strategy is to relate I_X to a monomial ideal. Because the forms appearing in any generator F of I_X are distinct, the monomial ideal is actually squarefree. Such ideals are related to the combinatorics of simplicial complexes.

1.3.1 Simplicial complexes and Stanley-Reisner ring

Definition 1.3.1. *[Sch03] A simplicial complex on a vertex set V is a collection of subsets Δ of V, such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. If $|\sigma| = i+1$ then σ is called an $i-$face.*

Let $f_i(\Delta)$ be the number of i-faces of Δ, and $\dim(\Delta) = \max\{i \mid f_i(\Delta) \neq 0\}$. If $\dim(\Delta) = n-1$, let $f_\Delta(t) = \sum_{i=0}^{n} f_{i-1} t^{n-i}$, with $f_{-1} = 1$ for the empty face. The reverse ordered list of coefficients of $f_\Delta(t)$ is the f-vector $f(\Delta)$ of Δ.

Definition 1.3.2. *The Alexander dual $\Delta^\vee$ of Δ is the simplicial complex*

$$\Delta^\vee = \{\tau^\vee \mid \tau \notin \Delta\}, \text{ where } \tau^\vee \text{ denotes the complement } V \setminus \tau.$$

Definition 1.3.3. *Let Δ be a simplicial complex on vertices $\{y_1, \dots, y_n\}$. The Stanley-Reisner ideal I_Δ is*

$$I_\Delta = \langle y_{i_1} \cdots y_{i_j} \mid \{y_{i_1}, \dots, y_{i_j}\} \text{ is not a face of } \Delta \rangle \subseteq S = \mathbb{R}[y_1, \dots y_n],$$

and the Stanley-Reisner ring is $\mathbb{R}[y_1, \dots y_n]/I_\Delta$.

The Stanley-Reisner ideal $I_{\Delta^\vee}$ of $\Delta^\vee$ is obtained from the primary decomposition of I_Δ: for each primary component P_i in the primary decomposition, take the product of the terms in the component. So if

$$I_\Delta = \bigcap_j P_j \text{ with } P_j = \langle y_{j_1}, \ldots, y_{j_d} \rangle,$$

then the minimal generators of $I_{\Delta^\vee}$ are of the form $y_{j_1} \ldots y_{j_d}$.

Definition 1.3.4. *The $j-1$ skeleton of a $i-1$ simplex has as maximal faces all j tuples on a set of i vertices. Denote this complex by $\Delta(i,j)$. The Stanley-Reisner ideal $I_{\Delta(i,j)}$ is generated by all square-free monomials of degree $j+1$ in i variables.*

Example 1.3.5. Figure 1.2 shows $\Delta(4,2)$.

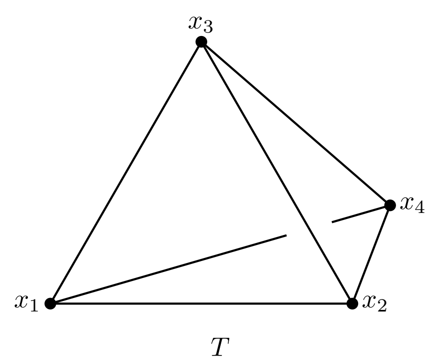


Figure 1.2: The 1−skeleton of a 3−simplex

Δ consists of 4 vertices and 6 edges, so $\Delta = \{\emptyset, \{x_i\}, \{x_i, x_j\} \mid 1 \leq i \leq 4 \text{ and } i < j \leq 4\}$ and $f(\Delta) = (1,4,6)$. Every maximal non-face of Δ is a triangle, so $I_\Delta = \langle x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \rangle$. The complements of the four triangles are the four vertices, so $\Delta^\vee = \Delta(4,1)$, the four vertices. Specializing $x_i \mapsto l_i$ yields the Chung-Yao set of Example 1.1.8.

Definition 1.3.6. *A regular sequence on S/I is a sequence $\{f_1, f_2, \ldots, f_k\} \subseteq S = \mathbb{R}[y_1, \ldots, y_n]$ such that each f_i is not a zero divisor on $S' = S/(I, f_1, \ldots, f_{i-1})$; alternatively, the map $S' \xrightarrow{\cdot f_i} S'$ is injective. The depth of S/I is the length of a maximal regular sequence. It is a theorem [Sch03] that the Cohen-Macaulay condition is equivalent to* $\operatorname{depth}(S/I) = n - \operatorname{codim}(I)$.

1.3.2 The simplicial complex of a GC set

We return to the study of I_X. Let $S = \mathbb{R}[y_1, \ldots, y_m]$, with a variable for each distinct (ignore scaling) linear form which is a factor of one of the $\prod l_i$ which generate I_X, and let $\phi : S \to R$ via $y_i \mapsto l_i$. The kernel L of ϕ is an ideal generated by $m - d - 1$ linear forms. Let I' be the ideal in S obtained by substituting y_i for l_i

in I_X, so ϕ induces a surjective map $\psi : S \xrightarrow{\phi} R \xrightarrow{\pi} R/I_X$. Since

$$\frac{S}{I'+L} \simeq \frac{S/L}{(I'+L)/L} \simeq \frac{\phi(S)}{\phi(I')} \simeq \frac{R}{I_X} = R/\cap_{i=1}^{|X|} \langle l_{i_1}, \ldots, l_{i_d}\rangle,$$

$I' + L = \ker(\psi)$. Let $J = \cap_{i=1}^{|X|}\langle y_{i_1}, \ldots, y_{i_d}\rangle$. If J has $\binom{n+d}{n+1}$ generators in degree $n+1$ then $I' + L = J + L$ with J a codimension d squarefree monomial ideal. Since $S/(J+L) \simeq R/I_X$, $J+L$ is of codimension $m-1$ and depth one, the $m-d-1$ linear forms of L are a regular sequence on S/J; because $\operatorname{depth}(S/(J+L)) = 1$, we can find an additional nonzero divisor on $S/(J+L)$. Thus S/J has depth $m-d$ so is Cohen-Macaulay.

Definition 1.3.7. *For a $GC_{d,n}$ set X with defining ideal I_X, write $J_{\Delta(X)}$ for the squarefree monomial ideal J appearing above, with $\Delta(X)$ the simplicial complex.*

Theorem 1.3.8. *If $I' = J$ then the ideal $J_{\Delta(X)}$ is bi-Cohen-Macaulay: both $J_{\Delta(X)}$ and the Alexander dual $J_{\Delta(X)^\vee}$ are Cohen-Macaulay.*

Proof. The Eagon-Reiner theorem [ER98] states that a Stanley-Reisner ideal I_Δ is Cohen-Macaulay iff the Alexander dual ideal $I_{\Delta^\vee}$ has a minimal free resolution where all the matrices representing the maps have only linear forms as entries. The remarks above show that if $I' = J_{\Delta(X)}$ then the ideal is Cohen-Macaulay and by Proposition 1.2.12 has a linear minimal free resolution, so the result follows. □

Even in algebraic geometry, bi-Cohen-Macaulay simplicial complexes are esoteric objects. In [FyV05], Fløystad-Vatne note that if Δ is a simplicial complex on m vertices, then the face vectors of Δ and $\Delta^\vee$ satisfy the relation

$$f_i(\Delta^\vee) + f_{m-i-2}(\Delta) = \binom{m}{i+1}. \tag{1.3}$$

Since $J_{\Delta(X)^\vee}$ has $\binom{n+d}{d}$ generators in degree d, letting $i^* = m-i-2$ we have

i	0	1	2	$\cdots$	$n-1$	n	$\cdots$	$m-d-1$	$\cdots$	$m-3$	$m-2$
$\binom{m}{i+1}$	m	$\binom{m}{2}$	$\binom{m}{3}$	$\cdots$	$\binom{m}{n}$	$\binom{m}{n+1}$	$\cdots$	$\binom{m}{d}$	$\cdots$	$\binom{m}{2}$	m
$f_i(\Delta)$	m	$\binom{m}{2}$	$\binom{m}{3}$	$\cdots$	$\binom{m}{n}$	$\binom{m}{n+1}-\binom{n+d}{n+1}$	$\cdots$	$\binom{n+d}{d}$	$\cdots$	0	0
$f_{i^*}(\Delta^\vee)$	0	0	0	$\cdots$	0	$\binom{n+d}{n+1}$	$\cdots$	$\binom{m}{d}-\binom{n+d}{d}$	$\cdots$	$\binom{m}{2}$	m

Proposition 3.1 of [FyV05] gives a complete characterization of the f-vectors that are possible if Δ is bi-Cohen-Macaulay: any such f-vector is of the form

$$(1+t)^i \cdot \left(1 + mt + \binom{m}{2}t^2 + \cdots + \binom{m}{k}t^k\right). \tag{1.4}$$

The key definition of this chapter is a version of the GC property for monomial ideals:

Definition 1.3.9. *Let I_Δ be a squarefree bi-Cohen-Macaulay monomial ideal of codimension d and degree $\binom{n+d}{d}$. A primary component P of I_Δ is* monomial GC *if there is a degree n monomial f with $f \in I_\Delta : P$ and $f \notin P$. If every primary component P of I_Δ is monomial GC, then I_Δ is a* monomial $GC_{d,n}$ *ideal. $V(y_i)$ is a* maximal monomial hyperplane *if $V(y_i)$ contains $\binom{n+d-1}{d-1}$ components of $V(I_\Delta)$.*

1.3.3 The simplicial complex of a Chung-Yao set

In certain cases, the $GC_{d,n}$ property is a consequence of combinatorics: it is inherited from a monomial $GC_{d,n}$ ideal. Suppose there is no overlap between the nonzero entries of $f(\Delta^\vee)$ and $f(\Delta)$:

$$f_i(\Delta^\vee) \cdot f_{m-i-2}(\Delta) = 0 \text{ for all } i.$$

As $d \geq 3$ and $n \geq 2$, the assumption above implies that

$$\binom{m}{d} - \binom{n+d}{d} = 0, \text{ so } m = n+d$$

Lemma 1.3.10. *If $f_j(\Delta^\vee) \cdot f_{m-j-2}(\Delta) = 0$ for all j, then $J_\Delta = I_{\Delta(d+n,n)}$*

Proof. By our observation above, $f_j(\Delta^\vee) \cdot f_{m-j-2}(\Delta) = 0$ for all j implies that $m = n+d$. Therefore $i = 0$ in Equation 1.4, so $\Delta = \Delta(m,n)$, with $m = d+n$. By Theorem 1.2.11 J_Δ has $\binom{m}{n+1} = \binom{n+d}{n+1}$ generators, which is exactly the number of squarefree monomials of degree $n+1$ on $n+d$ vertices, hence $J_\Delta = I_{\Delta(d+n,n)}$. □

In Lemma 2.8 of [Gor07], Gorla shows that the ideal $I_{\Delta(d+n,n)}$ is determinantal, and has an Eagon-Northcott resolution. The construction is as follows: take an $(n+d) \times (n+1)$ matrix M of constants, with no minor vanishing. Let M' be the result of multiplying the i^{th} column of M by the variable y_i. Then

$$I_{n+1}(M') = I_{\Delta(d+n,n)}$$

The primary decomposition of $I_{\Delta(d+n,n)}$ is straightforward. Because $\Delta(d+n,n)$ consists of all n tuples on a ground set of size $n+d$,

$$I_{\Delta(d+n,n)} = \bigcap_{1 \leq i_1 < i_2 < \cdots < i_d \leq n+d} \langle y_{i_1}, \ldots, y_{i_d} \rangle.$$

For any of the coordinate hyperplanes y_i, it is clear that there are $\binom{n+d-1}{d-1}$ terms in the primary decomposition which contain the fixed linear form y_i. For each component in the primary decomposition, $V(\langle y_{i_1}, \ldots, y_{i_d} \rangle)$ is a codimension d linear subspace, and the count above shows that every coordinate hyperplane contains $\binom{n+d-1}{d-1}$ such components of $V(I_{\Delta(d+n,n)})$.

Theorem 1.3.11. *If I_Δ is a squarefree bi-Cohen-Macaulay monomial ideal of codimension d and degree $\binom{n+d}{d}$, then a specialization by a regular sequence $\phi : y_i \mapsto l_i$ yields a n-correct set. If in addition I_Δ is a monomial $GC_{d,n}$ ideal, then the specialization is also a $GC_{d,n}$ set. If I_Δ has a maximal hyperplane, so does $\phi(I_\Delta)$.*

Proof. As I_Δ is Cohen-Macaulay, specialization by a regular sequence preserves the primary decomposition, hence the $GC_{d,n}$ and maximal hyperplane properties. The fact that the specialization is n-correct follows because specializing by a regular sequence preserves the minimal free resolution, and Proposition 1.2.12. $\square$

Continuing with the example where $m = d + n$, for m generic linear forms $l_i \in R$,

$$\mathbb{R}[y_1, \ldots, y_m] \xrightarrow{\phi} \mathbb{R}[x_1, \ldots, x_d], \; y_i \mapsto l_i$$

yields the $GC_{d,n}$ sets of [CY77], which contain $n + d$ maximal hyperplanes. The argument above shows that they also have additional algebraic structure:

Theorem 1.3.12. *If X is a $GC_{d,n}$ set of Chung-Yao type, then I_X is determinantal.*

1.3.4 Constructing GC sets from I_Δ

One way to construct $GC_{d,n}$ sets is to start with a squarefree bi-Cohen-Macaulay monomial ideal of codimension d and degree $\binom{n+d}{d}$, which is not a $GC_{d,n}$ monomial ideal, but which has many GC components. Any specialization will preserve the GC properties; if I_Δ has a maximal monomial hyperplane, specialization also preserves it. The next theorem is crucial: it gives a necessary and sufficient combinatorial condition for a primary component to be monomial GC:

Theorem 1.3.13. *Let I_Δ be a squarefree bi-Cohen-Macaulay monomial ideal of degree $\binom{n+d}{d}$ and codimension d:*

$$I_\Delta = \bigcap_{i=1}^{\binom{n+d}{d}} P_i, \; \text{with } P_i = \langle x_{i_1}, \ldots, x_{i_d} \rangle \subseteq k[x_1, \ldots, x_m]$$

A primary component $P_i = \langle x_{i_1}, \ldots, x_{i_d} \rangle$ is monomial GC iff there is $\tau \in \Delta_{n-1}$ such that for all $j \in \{1, \ldots, d\}$, $\overline{\tau v_{i_j}} \notin \Delta_n$, where $\overline{\tau v_{i_j}}$ is the join of τ with v_{i_j}.

Proof. From Definition 1.3.9, a primary component P of I_Δ is monomial GC if there is a degree n monomial (necessarily squarefree) f with $f \in I_\Delta : P_i$ and $f \notin P_i$. As I_Δ is generated in degree $n+1$, Δ contains the $n-1$ skeleton $\Delta(m, n)$; in particular, f corresponds to a face $\tau \in \Delta_{n-1}$. But $f \in I_\Delta : P_i$ iff $f \cdot x_{i_k} \in I_\Delta$ for all

$k \in \{1, \ldots, d\}$ iff for all $j \in \{1, \ldots, d\}$, $\overline{\tau v_{i_j}} \notin \Delta$. Finally, the monomial f is in P_i iff for some $j \in \{1, \ldots, d\}$, $x_{i_j} \mid f$, which would imply there is a non-squarefree monomial generator of I_Δ, a contradiction. $\square$

1.4 The Case Where $\Delta(X)$ is Two Dimensional

The previous sections shows that when X is a Chung-Yao $GC_{d,n}$ set, then the primary decomposition of $J_{\Delta(X)} = I_{\Delta(d+n,n)}$ is, by itself, enough to prove that X is GC and has $d+n$ maximal hyperplanes: combinatorics of $J_{\Delta(X)}$ governs both conditions. We give a detailed analysis of the case when $\Delta(X)$ is two dimensional. By equation 1.3, there are four numerical possibilities for $f(\Delta)$ in this situation:

i	f_{-1}	f_0	f_1	f_2
0	1	m	$\binom{m}{2}$	$\binom{m}{3}$
1	1	m	$\binom{m}{2}$	$\binom{m-1}{2}$
2	1	m	$2m-3$	$m-2$
3	1	3	3	1

The case $i = 3$ is impossible (there are only 3 hyperplanes), and the case $i = 2$ is impossible because $J_{\Delta(X)}$ would have minimal generators in degree two, and the assumption that $n \geq 2$ implies that $J_{\Delta(X)}$ is generated in degree three or higher.

In the case that $i = 0$, the numerics force the simplicial complex $\Delta(X)$ to be the two skeleton of an $m-1$ simplex, so $\Delta(X) = \Delta(m, 3)$, and we are in the setting of a Chung-Yao set with $n = 3$. Since $m = n + d = 3 + d$, X is a Chung-Yao $GC_{m-3,3}$ set, with $d + 3$ maximal hyperplanes.

So the only interesting case is when $i = 1$, that is, the case when $f(\Delta) = (1, m, \binom{m}{2}, \binom{m-1}{2})$. This will be our standing assumption for the remained of this section. Because $f_2 < \binom{m}{3}$, we know that $J_{\Delta(X)}$ has generators in degree three, so we must have that $n = 2$. Combining Theorem 1.3.8 and equation 1.3 shows that $f_j(\Delta^\vee) \cdot f_{m-j-2}(\Delta)$ is nonzero only for $j = 3$, hence

$$\binom{m}{d} - \binom{n+d}{d} = \binom{n+d}{d-1},$$

and so $m = n + d + 1$. We also obtain strong constraints from Theorem 1.3.9:

Corollary 1.4.1. *Let I_Δ be a bi-Cohen-Macaulay ideal of degree $\binom{n+d}{d}$ and codimension d, with $\dim(\Delta) = 2$ and $i = 1$. Then a primary component $P_i = \langle x_{i_1}, \ldots, x_{i_{m-3}} \rangle$ is monomial GC if and only if there is an edge $\tau = \overline{v_j v_k}$ with $j, k \notin \{i_1, \ldots, i_{m-3}\}$ such that there is a unique triangle $\overline{v_i v_j v_k}$ in the $2-$skeleton of Δ which contains τ.*

Proof. The first part of the corollary is nothing but a restatement of Theorem 1.3.9 in the case when $n = 2$. Because $dim(\Delta) = 2$ and $i = 1$, we are in the setting where $d = m - 3$, so P_i is monomial GC if and only if $\overline{v_i v_j v_k} \notin \Delta$ for all $i \in \{i_1, \ldots, i_{m-3}\}$, hence the triangle is unique.. ☐

Remark 1.4.2. Notice that the conditions of Corollary 1.4.1 mean that the monomial GC components of I_Δ correspond to edges on the boundary $\delta(\Delta)$.

Remark 1.4.3. One way to build a simplicial complex with the desired f vector is to take the cone cK_{m-1} over a complete graph on $m - 1$ vertices. Such a complex is indeed bi-Cohen Macaulay, but because the variable corresponding to the cone point does not appear among any of the generators of $J_{\Delta(cK_{m-1})}$, this situation never arises in the context of $GC_{d,n}$ sets.

1.4.1 Topological Constraints on Δ

Following [Mun84], define a simplicial surface Δ as a triangulation of a two-dimensional (compact and possibly non-oriented) manifold S. A fundamental invariant of a simplicial surface is its Euler characteristic $\chi(\Delta)$, defined as $f_0(\Delta) - f_1(\Delta) + f_2(\Delta)$. It is a non-trivial fact that $\chi(\Delta)$ depends only on S, i.e. it is independent of the choices made in the triangulation. In the case when $i = 1$, we have

$$\chi(\Delta) = m - \binom{m}{2} + \binom{m-1}{2} = 1.$$

Proposition 1.4.4. *Let I_Δ be the Stanley-Reisner ideal of a simplicial surface Δ with $\chi(\Delta) = 1$. Then none of the primary components of I_Δ are monomial GC.*

Proof. First of all, if Δ is a triangulation of an orientable surface, then its Euler characteristic is even. So we need only consider compact non-orientable surfaces, which are connected sums of copies of $\mathbb{RP}^2$. Since $\mathbb{RP}^2$ has Euler characteristic 1, and since Euler characteristic of connected sums satisfies

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2,$$

it follows that the only compact surface of Euler characteristic 1 is $\mathbb{RP}^2$ itself. So we need only show that the Stanley-Reisner ideal for any triangulation of $\mathbb{RP}^2$ has no monomial GC components. This follows immediately from Remark 1.4.2. ☐

Example 1.4.5. For the triangulation of $\mathbb{RP}^2$ shown in Figure 1.3, the ideal $J_{\Delta(\mathbb{RP}^2)}$ is generated by

$$y_0y_1y_2, y_0y_1y_4, y_0y_2y_5, y_0y_3y_4, y_0y_3y_5, y_1y_2y_3, y_1y_3y_5, y_1y_4y_5, y_2y_3y_4, y_2y_4y_5,$$

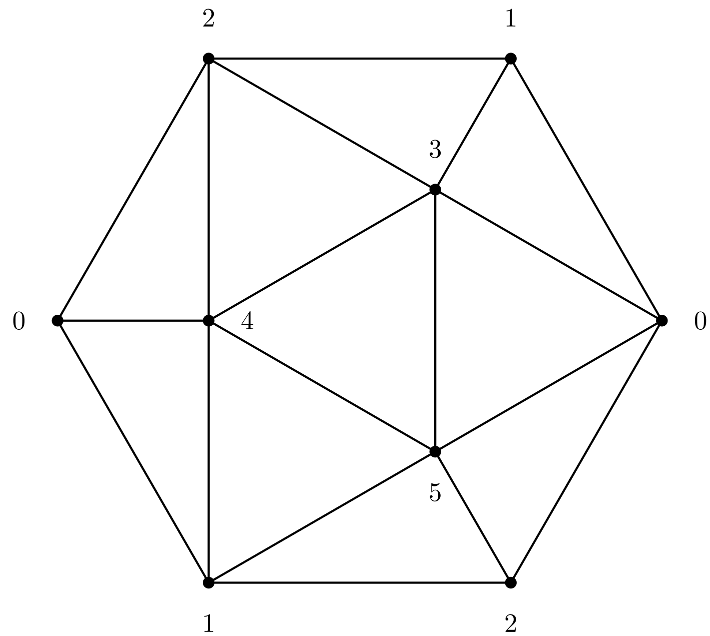


Figure 1.3: A Triangulation of $\mathbb{RP}^2$ with six vertices

and the primary decomposition of $J_{\Delta(\mathbb{RP}^2)}$ is

$$\langle y_0, y_1, y_2 \rangle \cap \langle y_0, y_1, y_4 \rangle \cap \langle y_0, y_2, y_5 \rangle \cap \langle y_0, y_3, y_4 \rangle \cap \langle y_0, y_3, y_5 \rangle \cap$$

$$\langle y_1, y_2, y_3 \rangle \cap \langle y_1, y_3, y_5 \rangle \cap \langle y_1, y_4, y_5 \rangle \cap \langle y_2, y_3, y_4 \rangle \cap \langle y_2, y_4, y_5 \rangle.$$

Notice that $J_\Delta = J_{\Delta^\vee}$; Δ is self-dual, and we can see directly that none of the ten primary components are monomial GC. For example,

$$J_{\Delta(\mathbb{RP}^2)} : \langle y_0, y_1, y_2 \rangle = J_{\Delta(\mathbb{RP}^2)} + \langle y_3 y_4 y_5 \rangle.$$

In particular, there is no quadric in $J_{\Delta(\mathbb{RP}^2)} : \langle y_0, y_1, y_2 \rangle$.

1.4.2 When Δ is not a Simplicial Surface

When Δ is not a simplicial surface, there are many possible combinatorial types. We continue to assume that $dim(\Delta) = 2$ and $i = 1$, hence by the above, $n = 2$ and $m = n + d + 1$. Let

$$E = \{(\epsilon, \tau) | \epsilon \subset \tau\}$$

be the set of pairs where ϵ is an edge contained in the triangle τ. Every triangle has three distinct edges, and there are $\binom{m-1}{2} = \binom{d+2}{2}$ triangles, so $|E| = 3 \cdot f_2 = \frac{3}{2}(d^2 + 3d + 2)$. We write T_i for the number of edges contained in i distinct triangles, so T_1 is the number of edges contained in exactly one triangle; By Corollary 1.4.1, these are the edges which contribute monomial GC components. Since Δ is bi-Cohen-Macaulay, there are no isolated edges, so

$$\sum_{i=1}^{m-2} i \cdot T_i = 3 \cdot f_2$$

Theorem 1.4.6. *For $J_{\Delta(X)}$ bi-Cohen-Macaulay of degree $\binom{n+d}{d}$ and codimension d with $\dim(\Delta) = 2$, the only monomial $GC_{d,n}$ ideal is of Chung-Yao type with $i = 0$.*

Proof. We need every component of I_Δ to be monomial GC, which by Corollary 1.4.1 means that every one of the $\binom{d+n}{d} = \binom{d+2}{2}$ primary components is a triangle with an edge contained in no other triangle. We will call such an edge a *singleton*. Since $i = 1$, this condition means that $T_1 = \binom{d+2}{d}$, so a monomial $GC_{d,n}$ ideal must satisfy

$$\sum_{i=1}^{m-2} i \cdot T_i = 2 \cdot f_2 = d^2 + 3d + 2. \tag{1.5}$$

Since Δ has $\binom{m}{2} = \binom{d+3}{2}$ edges, of which $\binom{d+2}{2}$ are singletons, there are $d+2$ edges remaining. Equation 1.5 holds only if $T_i = 0$ for $i \in \{2, \ldots, d\}$, and $T_{d+1} = d+2$. We will call the edges in T_{d+2} *star* edges. Since Δ has $d+3$ vertices, any star edge $\overline{v_i v_j}$ forms a triangle with each of the other $d+1$ vertices in $\Delta_0 \setminus \{i, j\}$. Every edge of Δ is either a singleton or star edge.

Suppose two star edges do not share a common vertex, and without loss of generality call these edges $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$. The star property means that the triangles $\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$ are all in Δ; since all edges are either singleton or star, this means that all edges $\overline{v_i v_j}$ for $1 \leq i < j \leq 4$ are all star edges. Now for any vertex v_5, the star property implies that $\{v_i, v_j, v_5\}$ are all triangles in Δ for any $1 \leq i < j \leq 4$, which means every edge $\overline{v_i v_5}$, $1 \leq i \leq 4$ is a star edge. Iterating this process shows that Δ is the two-skeleton of a simplex on $m = d+3$ vertices, and this has too many triangles, a contradiction.

So if there are two star edges, they must share a vertex. In fact, if there are more than two star edges, they must all share a *common* vertex. If $\overline{v_1 v_2}, \overline{v_2 v_3}, \overline{v_3 v_4}$ are star edges, we simply apply the above argument to $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$. If $\overline{v_1 v_2}, \overline{v_2 v_3}, \overline{v_3 v_1}$ are star edges that form a triangle, then any vertex v_4 forms a triangle with all three star edges, and as above $\overline{v_i v_4}$ would be star edges for $1 \leq i \leq 3$, yielding a contradiction.

Hence, all star edges share a common vertex, which is therefore a cone vertex. Since we need $\binom{d+2}{2}$ singleton edges, this forces Δ to be the cone over the complete graph K_{d+2}. While this is bi-Cohen-Macaulay, as we noted above it does not arise in the $GC_{d,n}$ context. So there are no monomial $GC_{d,n}$ ideals with $i = 1$. $\square$

The above theorem means that in the setting $dim(\Delta) = 2$ and $i = 1$, we cannot obtain $GC_{d,n}$ sets directly from monomial $GC_{d,n}$ ideals as we did for Chung-Yao sets when $i = 0$. Example 1.6.2 suggests an alternative approach to constructing $GC_{d,n}$ sets: Start with a bi-Cohen-Macaulay monomial ideal that has the correct degree and codimension, and which has many monomial GC components. Proposition 1.4.8 gives an infinite family of such simplicial complexes.

Definition 1.4.7. *Let cK_{m-1} denote the two dimension simplicial complex obtained by taking the cone over a complete graph on $m-1$ vertices, with cone vertex v_m. For any even $k \leq m$, let $cK_{m-1}(k)$ be the simplicial complex obtained by replacing triangles $[1,2,m],[3,4,m],\ldots,[k-1,k,m]$ with triangles $[1,2,4],[3,4,6],\ldots,[k-1,k,k+2]$, where if $k = m-1$ we replace $k+2$ by 2, and if $k = m-2$, replace $k+2$ by 1.*

Proposition 1.4.8. *The complex $cK_{m-2}(k)$ is bi-Cohen-Macaulay of degree $\binom{n+d}{d}$, codimension d, and has $\binom{m-1}{2} - 2k$ monomial GC components.*

Proof. Write $I(0)$ for the ideal of squarefree cubics in $\{x_1,\ldots,x_{m-1}\}$; it is the ideal of the one skeleton of an $m-2$ simplex, so is bi-Cohen-Macaulay by [Gor07]. Let $I(k)$ be the ideal obtained by deleting monomials $x_1x_2x_4, x_3x_4x_6, \ldots, x_{k-1}x_kx_{k+2}$ from $I(0)$. Since $I(k-2) = I(k) + x_{k-1}x_kx_{k+2}$, there is a short exact sequence

$$0 \longrightarrow S(-3)/I(k) : x_{k-1}x_kx_{k+2} \longrightarrow S/I(k) \longrightarrow S/I(k-2) \longrightarrow 0 \qquad (1.6)$$

and an easy exercise shows that for k in the designated range,

$$I(k) : x_{k-1}x_kx_{k+2} = \{x_1,\ldots,x_{m-1}\} \setminus \{x_{k-1},x_k,x_{k+2}\}.$$

A mapping cone [Sch03] construction shows that the ideals $I(k)$ are all bi-Cohen-Macaulay. Now we run the process in reverse: starting from $I(m-1)$ when $m-1$ is even, and from $I(m-2)$ when $m-1$ is odd, add the monomial $x_{k-1}x_kx_{k+2}$ at the k^{th} step. Arguing as for $x_{k-1}x_kx_{k+2}$ shows that $I(k) : x_{k-1}x_kx_m = \{x_1,\ldots,x_{m-1}\} \setminus \{x_{k-1},x_k,x_{k+2}\}$, and the mapping cone construction gives the desired result. The key is that since $I(k) : x_{k-1}x_kx_{k+2}$ and $I(k) : x_{k-1}x_kx_m$ are ideals of variables, their minimal free resolutions are linear Koszul complexes.

The fact that $cK_{m-1}(k)$ has the correct degree and codimension is immediate: there are $\binom{m-1}{2} = \binom{n+d}{2} = \binom{d+2}{2}$ maximal components which are two dimensional in $\mathbb{P}^{d+2}$, so of codimension d. That the primary decomposition has $\binom{m-1}{2} - 2k$ monomial GC components follows from the construction: each time we exchange a triangle with a cone vertex for a triangle without the cone vertex, two singleton edges are lost. Now apply Corollary 1.4.1. $\square$

Example 1.4.9. The cone cK_6 over K_6 has 15 maximal triangles:

$$[1,2,7],[1,3,7],[2,3,7],[1,4,7],[2,4,7],[3,4,7],[1,5,7],[2,5,7],$$

$$[3,5,7],[4,5,7],[1,6,7],[2,6,7],[3,6,7],[4,6,7],[5,6,7].$$

1. To obtain $cK_6(1)$, exchange $[1,2,7]$ for $[1,2,4]$.
2. To obtain $cK_6(2)$, exchange $[1,2,7],[3,4,7]$ for $[1,2,4],[3,4,6]$.
3. To obtain $cK_6(3)$, exchange $[1,2,7],[3,4,7],[5,6,7]$ for $[1,2,4],[3,4,6],[5,6,2]$.

These give bi-Cohen-Macaulay simplicial complexes with 15 primary components, which have, respectively, 13, 11 and 9 monomial GC components. A computation shows that specializing with a regular sequence $y_i \mapsto l_i$ with $l_i \in k[x_0,\ldots,x_4]$ yields $GC_{4,2}$ sets for all three of these complexes.

1.5 Specialization and the Segre Map

By Theorem 1.2.11 and Theorem 1.3.8, a $GC_{d,n}$ set can arise as a specialization of the Stanley-Reisner ideal of a bi-Cohen-Macaulay simplicial complex Δ of degree $\binom{n+d}{d}$ and codimension d. If Δ has many monomial GC components, then these components will still be GC after specialization, So the question raised by the previous section is how to specialize the non GC components to obtain a $GC_{d,n}$ set. It turns out that there is a natural connection to classical geometry, via the Segre map. The strategy is as follows: Suppose I_Δ is a bi-Cohen-Macaulay ideal of degree $\binom{n+d}{d}$ and codimension d, and P_i is a primary component of I_Δ which is not monomial GC. After specializing by a regular sequence, we obtain a set X of $\binom{n+d}{d}$ points in $\mathbb{P}^d$. For any $p \in X$, there is a unique degree n polynomial f vanishing on $X \setminus \{p\}$ but not at p. To decide whether X is a $GC_{d,n}$ set, we just need to determine whether f is a product of linear forms.

We give a short review of the Segre map. For a detailed treatment see [Lan12]. The set of polynomials $f \in S = Sym(V^*)_n$ such that $f = \prod_{i=1}^{n} l_i$ with l_i a linear form is parametrized by a projection of the *Segre variety*. Let V be a $d+1$ dimensional vector space, and identify a linear form with a point in projective space $\mathbb{P}^d = \mathbb{P}(V)$ via

$$l = \sum_{i=0}^{d} a_i x_i \longleftrightarrow (a_0 : \ldots : a_d) \in \mathbb{P}^d.$$

With this identification, the Segre map $\sigma_{d,\ldots,d}$ takes n copies of $\mathbb{P}^d$ to $\mathbb{P}^{(d+1)^n-1}$ by taking all $(d+1)^n$ products of coordinates; write $\Sigma_{d,\ldots,d}$ for the image of this map. $\Sigma_{d,\ldots,d}$ is an algebraic variety, defined by polynomial equations. A coordinate free description of the Segre map is

$$\mathbb{P}(V^*) \times \mathbb{P}(V^*) \times \ldots \times \mathbb{P}(V^*) \longrightarrow \mathbb{P}((V^*)^{\otimes n}),$$

where

$$(v_1, v_2, \ldots, v_n) \mapsto v_1 \otimes v_2 \otimes \ldots \otimes v_n.$$

Example 1.5.1. The Segre map $\sigma_{2,2}$ is the map

$$\mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8$$

defined by

$$(a_0 : a_1 : a_2), (b_0 : b_1 : b_2) \mapsto (a_0b_0 : a_0b_1 : a_0b_2 : a_1b_0 : a_1b_1 : a_1b_2 : a_2b_0 : a_2b_1 : a_2b_2).$$

If $y_{i,j}$ is the coordinate on $\mathbb{P}^8$ associated to the image of a_ib_j in the above map, then the defining ideal of $\Sigma_{2,2}$ is given by the 2×2 minors of the matrix

$$\begin{bmatrix} y_{0,0} & y_{0,1} & y_{0,2} \\ y_{1,0} & y_{1,1} & y_{1,2} \\ y_{2,0} & y_{2,1} & y_{2,2} \end{bmatrix}$$

Identifying $l_1 = a_0x_0 + a_1x_1 + a_2x_2$ and $l_2 = b_0x_0 + b_1x_1 + b_2x_2$, we see that the coefficient of x_1x_2 will be $a_1b_2 + a_2b_1$, which is where the projection comes into the picture: rather than mapping to $\mathbb{P}(V^*)$, the map should be to $\mathbb{P}(Sym_n(V^*))$.

Specializing $I_\Delta \subseteq S$ by a regular sequence means defining a map

$$S = k[y_1, \ldots, y_m] \xrightarrow{\phi} R = k[x_0, \ldots, x_d] = Sym(W^*)$$

defined by sending y_i to the linear form l_i.

Then I_Δ specializes to a $GC_{d,n}$ set exactly when in the primary decomposition

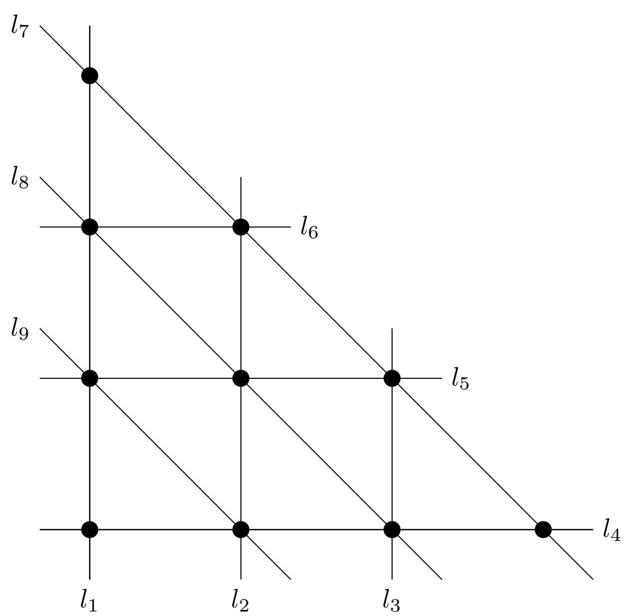
$$I_\Delta = \bigcap_{j=1}^{\binom{n+d}{n}} P_j,$$

for each j, the image $\phi(I_{\Delta:P_j})$ contains some $f \in R_n$ that is a product of n linear forms.

1.6 Examples and Macaulay2

To demonstrate the utility of the computer algebra system Macaulay2, we give here a detailed example which illustrates many of the results in this chapter.

Example 1.6.1. Consider the set $X \subseteq \mathbb{R}^2$, shown below, consisting of integer points on or inside the triangle with vertices $(0,0)$, $(3,0)$ and $(0,3)$.



The projective coordinates of these 10 points are the rows of the following matrix.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

And we can use Macaulay2 to study this set of points and its associated combinatorial objects. We'll start with a naive presentation of the ideal I_X.

```
i1 : m = matrix {{0, 0, 1}, {1, 0, 1}, {2, 0, 1}, {3, 0, 1}, {0, 1, 1},
{1, 1, 1}, {2, 1, 1}, {0, 2, 1}, {1, 2, 1}, {0, 3, 1}}

o1 = | 0 0 1 |
     | 1 0 1 |
     | 2 0 1 |
     | 3 0 1 |
     | 0 1 1 |
     | 1 1 1 |
     | 2 1 1 |
     | 0 2 1 |
     | 1 2 1 |
     | 0 3 1 |

               10        3
o1 : Matrix ZZ   <--- ZZ
```

The following script will compute a list of the primary components of the ideal I_X

```
i2 : mtiHelp = (v,r) -> (
         minors(2,(matrix v | matrix r))
         )

o2 = mtiHelp

o2 : FunctionClosure

i3 :
     matrixToPrimaryComponentList = m -> (
         n := (numColumns m) - 1;
         R := ZZ/31991[x_0..x_n];
         v := transpose vars R;
         N := (numRows m) - 1;
         pdcomp := apply(toList (0..N), i -> mtiHelp(v,transpose m^{i}));
         return apply(pdcomp, I -> ideal mingens I);
         )

o3 = matrixToPrimaryComponentList

o3 : FunctionClosure
```

And applying it to our matrix `m` yields

```
i4 : matrixToPrimaryComponentList m

o4 = {ideal (x , x ), ideal (x , x  - x ), ideal (x , x  - 2x ), ideal (x , x  - 3x ), ideal (x  -
              1   0            1   0    2             1   0     2             1   0     2            1
     ------------------------------------------------------------------------------------------------
     x , x ), ideal (x  - x , x  - x ), ideal (x  - x , x  - 2x ), ideal (x  - 2x , x ), ideal (x  -
      2   0           1    2   0    2            1    2   0     2            1     2   0            1
     ------------------------------------------------------------------------------------------------
     2x , x  - x ), ideal (x  - 3x , x )}
       2   0    2           1     2   0

o4 : List
```

We can find the generators of I_X, and factor them into their component linear forms, via

```
i5 : IX = fold(intersect, oo)

             4     3        2 2      3     3        2          2   2 2    2          2         2
o5 = ideal (x  - 6x x  + 11x x  - 6x x , x x  - 3x x x  + 2x x x , x x  - x x x  - x x x  + x x x ,
             1     1 2      1 2     1 2   0 1     0 1 2     0 1 2   0 1    0 1 2    0 1 2    0 1 2
     ------------------------------------------------------------------------------------------------
      3       2             2   4     3        2 2       3
     x x  - 3x x x  + 2x x x , x  - 6x x  + 11x x  - 6x x )
      0 1     0 1 2     0 1 2   0     0 2      0 2     0 2

               ZZ
o5 : Ideal of -----[x , x , x ]
              31991  0   1   2

i6 : apply(toList(0..4), i -> factor(IX_i))

o6 = {(15995x  + x )(10664x  - x )(x )(x  - x )(-6), (15995x  + x )(x )(x  - x )(x )(-2), (x )(x  -
             1    2        1    2   1   1    2              1    2   1   1    2   0         1   1
     ------------------------------------------------------------------------------------------------
      x )(x )(x  - x ), (x )(15995x  + x )(x )(x  - x )(-2), (15995x  + x )(10664x  - x )(x )(x  -
       2   0   0    2     1        0    2   0   0    2              0    2        0    2   0   0
     ------------------------------------------------------------------------------------------------
      x )(-6)}
       2

o6 : List
```

This tells us that the ideal I_X can be written as

$$
\begin{aligned}
I_X = \quad & \langle (x_1 - 2x_2)(x_1 - 3x_2)(x_1)(x_1 - x_2) \\
& (x_1 - 3x_2)(x_1)(x_1 - x_2)(x_0) \\
& (x_1)(x_1 - x_2)(x_0)(x_0 - x_2) \\
& (x_1)(x_0 - 2x_2)(x_0)(x_0 - x_2) \\
& (x_1 - 2x_2)(x_0 - 3x_2)(x_0)(x_0 - x_2) \rangle
\end{aligned}
$$

Unfortunately, this is not the presentation of the ideal that we want. We can see that several of the linear forms which appear among the generators above correspond to lines that only contain one point in X, for example, $x_1 - 3x_2$, which only contains the point $(0 : 3 : 1)$.

We should be using a generating set like the Q stuff tells us to.

$$
\begin{aligned}
l_1 =&\quad x_0\\
l_2 =&\quad x_0 - x_2\\
l_3 =&\quad x_0 - 2x_2\\
l_4 =&\quad x_1\\
l_5 =&\quad x_1 - x_2\\
l_6 =&\quad x_1 - 2x_2\\
l_7 =&\quad x_0 + x_1 - 3x_2\\
l_8 =&\quad x_0 + x_1 - 2x_2\\
l_9 =&\quad x_0 + x_1 - x_2
\end{aligned}
$$

(Notice that we cannot avoid having some repeated generators)

$$
\begin{aligned}
I_X = \quad &\langle l_1l_7l_8l_9, \quad l_4l_7l_8l_9, \quad l_1l_2l_7l_8, \quad l_1l_4l_7l_8, \quad l_1l_2l_4l_7,\\
&l_1l_2l_7l_8, \quad l_1l_2l_3l_4, \quad l_1l_2l_3l_7, \quad l_4l_5l_7l_8, \quad l_4l_7l_8l_9,\\
&l_1l_2l_4l_7, \quad l_1l_4l_5l_7, \quad l_1l_2l_3l_4, \quad l_1l_2l_4l_5, \quad l_4l_5l_7l_8,\\
&l_4l_5l_6l_7, \quad l_1l_4l_5l_6, \quad l_1l_4l_5l_7, \quad l_1l_4l_5l_6, \quad l_4l_5l_6l_7\rangle
\end{aligned}
$$

Now let S be a polynomial ring in the variables $T_1, \ldots, T_9$ and let I_Δ be the monomialization of this presentation of I_X.

$$
\begin{aligned}
I_\Delta = \quad &\langle T_1T_7T_8T_9, \quad T_4T_7T_8T_9, \quad T_1T_2T_7T_8, \quad T_1T_4T_7T_8, \quad T_1T_2T_4T_7,\\
&T_1T_2T_7T_8, \quad T_1T_2T_3T_4, \quad T_1T_2T_3T_7, \quad T_4T_5T_7T_8, \quad T_4T_7T_8T_9,\\
&T_1T_2T_4T_7, \quad T_1T_4T_5T_7, \quad T_1T_2T_3T_4, \quad T_1T_2T_4T_5, \quad T_4T_5T_7T_8,\\
&T_4T_5T_6T_7, \quad T_1T_4T_5T_6, \quad T_1T_4T_5T_7, \quad T_1T_4T_5T_6, \quad T_4T_5T_6T_7\rangle
\end{aligned}
$$

Or, in Macaulay2

```
i7 = loadPackage("SimplicialComplexes")

o7 = SimplicialComplexes

o7 : Package

i8 : S = ZZ/31991[T_1..T_9]

o8 = S

i9 : IDelta = monomialIdeal (T_1*T_7*T_8*T_9,T_4*T_7*T_8*T_9,T_1*T_2*T_7*T_8,T_1*T_4*T_7*T_8,
      T_1*T_2*T_4*T_7,T_1*T_2*T_7*T_8,T_1*T_2*T_3*T_4,T_1*T_2*T_3*T_7,
      T_4*T_5*T_7*T_8,T_4*T_7*T_8*T_9,T_1*T_2*T_4*T_7,T_1*T_4*T_5*T_7,
      T_1*T_2*T_3*T_4,T_1*T_2*T_4*T_5,T_4*T_5*T_7*T_8,T_4*T_5*T_6*T_7,
      T_1*T_4*T_5*T_6,T_1*T_4*T_5*T_7,T_1*T_4*T_5*T_6,T_4*T_5*T_6*T_7)
```

```
o9 = monomialIdeal (T T T T , T T T T , T T T T , T T T T , T T T T , T T T T , T T T T , T T T T ,
                    1 2 3 4   1 2 4 5   1 4 5 6   1 2 3 7   1 2 4 7   1 4 5 7   4 5 6 7   1 2 7 8
  -----------------------------------------------------------------------------------------------
  T T T T , T T T T , T T T T , T T T T )
   1 4 7 8   4 5 7 8   1 7 8 9   4 7 8 9

o9 : MonomialIdeal of S

i10 : fVector simplicialComplex IDelta

o10 = HashTable{-1 => 1 }
                0 => 9
                1 => 36
                2 => 84
                3 => 114
                4 => 84
                5 => 29
                6 => 3
o10 : HashTable
```

So this monomial ideal corresponds to a simplicial complex with f-vector $(-1, 9, 36, 84, 114, 84, 29, 3)$. Naturally, if we specialize the variables back to their corresponding linear forms, we will arrive at the same $GC_{3,2}$ set that we started with, which is not very interesting. The next example is the punchline of the results in this chapter.

Example 1.6.2. Consider the monomial ideal J_Δ generated by

$$\langle y_1y_5y_6, y_2y_6y_7, y_3y_7y_8, y_4y_5y_8, y_1y_5y_7, y_2y_6y_8, y_5y_6y_7, y_5y_6y_8, y_5y_7y_8, y_6y_7y_8\rangle$$

is the Stanley-Reisner ideal for a simplicial complex Δ on 8 vertices, with

$$f(\Delta) = (1, 8, 28, 46, 35, 10) = f(\Delta^\vee).$$

A computation shows that 6 of the 10 components are monomial GC; and that Δ has four maximal monomial hyperplanes: $\{y_5, y_6, y_7, y_8\}$. The ideal I_Δ is codimension three, and specializing yields a $GC_{3,2}$ set, which is a one-lattice.

Chapter 2

Graphic Arrangements with Maximal Topological Complexity

2.1 Background and Motivation

2.1.1 Topological Complexity

The topological complexity, denoted $TC(X)$, of a space X is an integer which measures the extent to which any motion planning algorithm for X must be discontinuous. $TC(X)$ is a special case of the Schwarz genus introduced in [Švar58]. Specifically, $TC(X)$ is the Schwarz genus of the path fibration

$$\pi : X^I \to X \times X \quad \text{given by} \quad \pi(\gamma) \mapsto (\gamma(0), \gamma(1)).$$

The exact value of the topological complexity of a space can be difficult to compute, but its value is of interest for spaces like configuration spaces and their generalizations; spaces for which explicit motion planning algorithms are desired for practical applications. Examples of such spaces include the space of configurations of a mechanical system, or the space of configurations of a multi-body system in a 2 or 3 dimensional space. Higher topological complexities, denoted $TC_s(X)$ for $s > 2$, were defined in [Rud10] to address algorithms for planning more complicated motions. Recent work has been done on computing the topological complexity of hyperplane arrangement complements and other combinatorially determined spaces, for example [CP08],[GG16],[GGM16],[GGY15],[Yuz07] and [Yuz14]. In this chapter we focus on a particular class of hyperplane arrangements called graphic arrangements.

In recent work in [Yuz14], Yuzvinsky gives a combinatorial condition on a complex hyperplane arrangement $\mathcal{A}$ which guarantees that the topological complexity of the arrangement complement is maximized. An arrangement satisfying the condition defined in [Yuz14] is called *large.* In the case of graphic arrangements, we show that this condition is equivalent to a strengthened version of the inequality in a theorem of Nash-Williams which guarantees that the edges of a graph can be decomposed into two acyclic subgraphs. Our main result is the following.

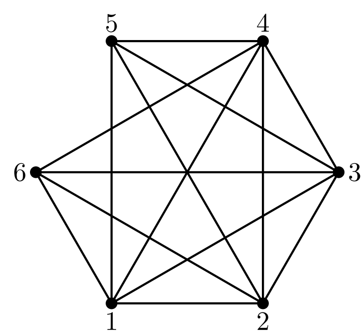
Theorem 2.1.1. *Let $G = (V, E)$ be a graph with $|V| = r + 1$ and no isolated vertices, and let $\mathcal{A}_G$ be its* *associated graphic arrangement. Then $\mathcal{A}_G$ is large if and only if G contains a spanning subgraph H having $2r-1$ edges and satisfying that for every nonempty, non-singleton subset $U \subseteq V$ we have*

$$|E_H(U)| < 2(|U|-1).$$

Here $E_H(U)$ denotes the set of edges of the subgraph of H induced by U. In particular, if such an H exists, then the higher topological complexity TC_s of the complement of $\mathcal{A}_G$ is equal to $sr-1$.

Before proceeding, we give a quick illustrative example.

Example 2.1.2. *Let $\mathcal{A}$ be the arrangement attained by removing one hyperplane from the arrangement A_5. In other words, $\mathcal{A}$ is the arrangement in $\mathbb{C}^6$ consisting of the hyperplanes defined by the 14 equations $\{x_1-x_2, x_1-x_3, x_1-x_4, x_1-x_5, x_1-x_6, x_2-x_3, x_2-x_4, x_2-x_5, x_2-x_6, x_3-x_4, x_3-x_5, x_3-x_6, x_4-x_5, x_4-x_6\}$. This arrangement is the graphic arrangement associated to the graph attained by deleting one edge from the complete graph K_6.*

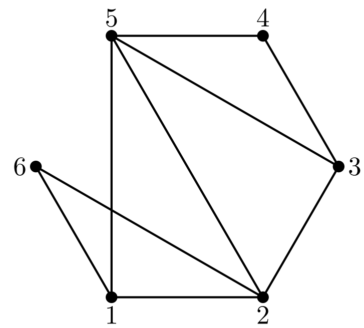


It is shown in [Yuz14] that the topological complexity of the complement of the A_5 arrangement is $5s-1$. The topological complexity $TC_s(\mathcal{A})$ of the complement of the arrangement described above is at most $5s-1$, but it may be lower. If we can find a full rank subarrangement $\mathcal{A}'$ with $TC_s(\mathcal{A}') = 5s-1$, then we can conclude that $TC_s(\mathcal{A}) = 5s-1$ as well. Such a subarrangement would correspond to a subgraph H which satisfies the theorem above. Let $\mathcal{A}'$ be the subarrangement determined by the subgraph H shown below.

It is easy to confirm that for any non-empty, non-singleton subset U of the vertex set of H, the subgraph induced by U has strictly fewer than $2(|U|-1)$ edges. Applying the theorem above lets us conclude that $TC_s(\mathcal{A}) = TC_s(\mathcal{A}') = 5s-1$.

2.1.2 Hyperplane Arrangements

We begin by establishing the terminology and results we will need for our discussion of hyperplane arrangements. Additional background and details can be found in [OT92].



Definition 2.1.3. *A* hyperplane arrangement $\mathcal{A}$ *is a finite set* $\{H_1, \ldots, H_n\}$ *of codimension* 1 *linear subspaces of a complex affine space* $\mathbb{C}^r$*.*

An arrangement $\mathcal{A}$ is called *central* if the intersection $H_1 \cap H_2 \cap \ldots \cap H_n$ is nonempty, and *essential* if the intersection contains exactly one point. When we refer to the *combinatorics* of the arrangement, we mean the partially ordered set of all intersections of subsets of $\mathcal{A}$, ordered by reverse inclusion. This is called the *intersection lattice* of $\mathcal{A}$. When we refer to the *topology* of the arrangement, we mean the topology of its complement

$$M_{\mathcal{A}} = \mathbb{C}^r \setminus \bigcup_{i=1}^{n} H_i.$$

A subset $\{H_{i_1}, \ldots, H_{i_t}\}$ of t hyperplanes of $\mathcal{A}$ is called *independent* if the intersection $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_t}$ has codimension t, and is called *dependent* otherwise. It turns out that the cohomology of $M_{\mathcal{A}}$ is determined by the combinatorial data of the arrangement.

Theorem 2.1.4 (Orlik, Solomon [OS80])**.** *Let* $E_{\mathcal{A}}$ *be the exterior algebra with generators* $\{e_1, \ldots, e_n\}$ *in natural correspondence with the hyperplanes in* $\mathcal{A}$*, and let* $I_{\mathcal{A}}$ *be the ideal in* $E_{\mathcal{A}}$ *given by*

$$\langle \sum_{j=1}^{t} (-1)^j e_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_t} \mid \{H_{i_1}, \ldots, H_{i_t}\} \text{ is a dependent set in } \mathcal{A} \rangle.$$

$I_{\mathcal{A}}$ *is called the Orlik-Solomon ideal of* $\mathcal{A}$*, and the* Orlik-Solomon algebra *of* $\mathcal{A}$*, denoted* $A(\mathcal{A})$*, is the quotient of* $E_{\mathcal{A}}$ *by* $I_{\mathcal{A}}$*. The Orlik-Solomon algebra is isomorphic to the cohomology of the complement:* $H^*(M_{\mathcal{A}}, \mathbb{C}) \cong A(\mathcal{A})$*.*

Any linear ordering $\preccurlyeq$ of the hyperplanes in $\mathcal{A}$ determines a basis for $A(\mathcal{A})$, called the *no-broken-circuit basis.* A *circuit* in $\mathcal{A}$ is a dependent set of hyperplanes which is minimal with respect to containment, and a *broken circuit* is a circuit with its minimal (with respect to $\preccurlyeq$) hyperplane removed. A subset of $\mathcal{A}$ is called *no-broken-circuit* or **nbc** if it does not contain a broken circuit.

The monomials in $E_{\mathcal{A}}$ are naturally identified with subsets of $\mathcal{A}$. For any choice of linear ordering $\preccurlyeq$, the images of the **nbc** monomials in $A(\mathcal{A})$ form a $\mathbb{C}$-basis for the Orlik-Solomon algebra, called the **nbc** basis for that ordering.

2.1.3 Topological Complexity and Motion Planning

Let X be a topological space and suppose we are interested in the motion planning problem for X [FY04]: given any two points a and b in X we would like a path $\gamma : I \to X$ starting at a and ending at b, i.e. with $\gamma(0) = a$ and $\gamma(1) = b$. We would like this assignment of paths to be a continuous function of a and b, but this is only possible when X is a contractible space. So instead we seek a collection of local assignments of paths to pairs. If we let $\pi : X^I \to X \times X$ be the *path fibration* of X, defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$, we are led to the following definition, which first appeared in [Far03].

Definition 2.1.5. *A* motion planning algorithm *for X, or simply a* motion planner *is a finite open cover $\{U_0, \ldots, U_n\}$ of $X \times X$ together with a map $s_i : U_i \to X^I$ satisfying that $\pi \circ s_i = id_{U_i}$. The open sets U_i are called the* local domains *of the motion planner.*

If a pair (a, b) is in one local domain, while a nearby pair (a', b') is in a different local domain, then it is possible that the motion planner will assign very different paths to these pairs, even if the pairs themselves are very close. For this reason, in practical applications it is desirable to have a motion planner with as few local domains as possible. The extent to which this goal can be achieved is measured by the topological complexity of X.

Definition 2.1.6. *The* topological complexity *of X, denoted by $TC(X)$, is the smallest integer n such that there exists a motion planner for X with $n+1$ local domains $\{U_0, \ldots, U_n\}$.*

Note that we are using the reduced version of topological complexity; a space for which a motion planner exists with a single local domain will have topological complexity 0 using this definition. Higher topological complexity, generalizing the notions given above, was defined in [Rud10]. In the above definitions, replace $X \times X$ with the s-fold product of X, and replace the path fibration with

$$\pi_s : X^I \to \underbrace{X \times X \times \ldots \times X}_{s \text{ times}}$$

defined by evaluation at s points in I:

$$\pi_s(\gamma) := (\gamma(0), \gamma(\frac{1}{s-1}), \gamma(\frac{2}{s-1}), \ldots, \gamma(\frac{s-2}{s-1}), \gamma(1))$$

Higher topological complexity $TC_s(X)$ is one less than the number of open subsets needed to cover the base $X \times X \times \ldots \times X$ so that on each open subset, π_s admits a continuous section. When $s = 2$, this recovers the definition of TC given above. Before we proceed, we state a few useful properties of TC_s which we will use in later sections.

Proposition 2.1.7. *[[Yuz14]]*

1. *$TC_s(X)$ is an invariant of the homotopy type of X [Far03].*

2. *If X has homotopy dimension r, $TC_s(X) \leq sr$.*

3. *If X is the complement of an arrangement of hyperplanes in $\mathbb{C}^r$, then $TC_s(X) \leq sr - 1$.*

4. *There is a lower bound for $TC_s(X)$, given by the higher zero-divisors-cup-length:*

$$zcl_s(X) \leq TC_s(X).$$

This lower bound $\mathrm{zcl}_s(X)$ is called the s^{th} zero-divisors-cup-length of X. It is computed using the multiplication structure of $H^*(X;\mathbb{C})$. More precisely, if we let K_s denote the kernel of the cup product map

$$\underbrace{H^*(X;\mathbb{C}) \otimes H^*(X;\mathbb{C}) \otimes \ldots \otimes H^*(X;\mathbb{C})}_{s \text{ times}} \to H^*(X;\mathbb{C}),$$

then the s^{th} zero-divisors cup length of X, denoted $\mathrm{zcl}_s(X)$, is the largest integer z so that the ideal K_s^z, i.e. the product $K_s \cdot K_s \cdot \ldots \cdot K_s$ with z factors, is not the zero ideal.

2.2 Specializing a Result of Yuzvinsky

2.2.1 Large Arrangements

For general spaces, it can be very difficult to compute the exact value of $TC_s(X)$. The main technique for proving that a space has a certain topological complexity is to explicitly construct a motion planner for it having a number of local domains greater by one than the cohomological lower bound. This is extremely difficult for complicated spaces. Of course, if the cohomological lower bound is equal to the dimensional upper bound, then the value of $TC_s(X)$ is also equal to that upper bound.

In what follows, $\mathcal{A}$ will denote an essential arrangement of complex hyperplanes in $\mathbb{C}^r$. By abuse of notation, we will interchangeably use $\mathcal{A}$ to refer to the hyperplane arrangement or its complement. In

[Yuz14], Yuzvinsky gives a combinatorial condition on $\mathcal{A}$ which guarantees that the cohomological lower bound and the dimensional upper bound for $TC_s(\mathcal{A})$ are equal. We recall the relevant definitions here.

Definition 2.2.1. *A pair (B, C) of subsets of $\mathcal{A}$ is called a* basic pair *if there exists a linear order $\preccurlyeq$ on the set of hyperplanes such that the following conditions are met:*

1. *B and C are disjoint.*
2. *B is maximal **nbc** (meaning that $|B| = r$) for the order $\preccurlyeq$.*
3. *C is **nbc** for the order $\preccurlyeq$.*

An arrangement $\mathcal{A}$ is called large *if it admits a basic pair with $|C| = r - 1$.*

Our result for graphic arrangements is proven by leveraging the result in [Yuz14] which states that large arrangements have maximal topological complexity.

Proposition 2.2.2. *[[Yuz14]] If $\mathcal{A}$ contains a basic pair with $|C| = r - 1$ then $zcl_s(X) = sr - 1$ and hence $TC_s(X) = sr - 1$.*

2.2.2 Graphic Arrangements

The condition in Proposition 2.2.2, when satisfied, gives the topological complexity of any essential hyperplane arrangement. We are interested in applying it to the case of graphic arrangements, a class of arrangements which has overlap with reflection arrangements and Coxeter arrangements, and which has very well-behaved combinatorics.

Definition 2.2.3. *Let $G = (V, E)$ be a finite graph with vertices $\{v_1, \ldots, v_{r+1}\}$. Consider a complex affine space $\mathbb{C}^{r+1}$ with coordinates $\{x_1, \ldots, x_{r+1}\}$, and for any edge $(v_i, v_j) \in E$ let H_{ij} denote the hyperplane in $\mathbb{C}^{r+1}$ defined by $x_i - x_j = 0$. The* graphic arrangement *associated to G is given by $\{H_{ij} | (v_i, v_j) \in E\}$.*

If G is a connected graph, then the intersection of all of the H_e is 1-dimensional. By projecting to the orthogonal complement of this subspace, we see that the complement of $\mathcal{A}$ is homotopy equivalent to the complement of an essential arrangement in $\mathbb{C}^r$ with the same combinatorics as $\mathcal{A}_G$. Since TC_s is a homotopy invariant, the topological complexity of this essential arrangement will be the same as the topological complexity of $\mathcal{A}_G$.

2.3 When is a Graphic Arrangement Large?

The topological complexity $TC_s(\mathcal{A}_G)$ is at most $sr-1$, with equality guaranteed when $\mathcal{A}_G$ is a large arrangement. So we would like a combinatorial condition on G which is equivalent to the condition that $\mathcal{A}_G$ is large. To find such a condition, we should first formulate Definition 2.3 for the graphic arrangement case.

A set of hyperplanes in $\mathcal{A}_G$ is independent if and only if the corresponding subset S of the edge set of G is acyclic. Furthermore, for a given linear order $\preccurlyeq$ on E, an independent set of hyperplanes in $\mathcal{A}_G$ in **nbc** if and only if the corresponding subset S of the edge set of G is acyclic and for any path P in S and any edge e not in S such that $\{e\} \cup P$ is a cycle, e is not the minimal element of the cycle $\{e\} \cup P$ with respect to $\preccurlyeq$. For brevity, we will use **nbc** to refer to subsets of the edge set of G which correspond to **nbc** subsets of $\mathcal{A}_G$.

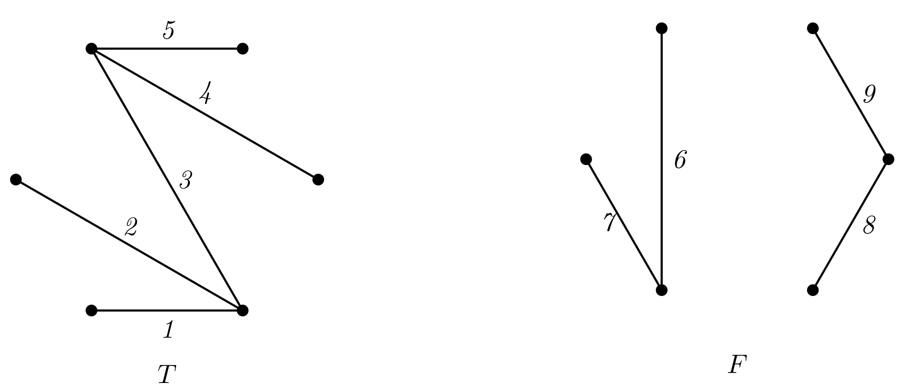
Using this, we can rephrase the condition that $\mathcal{A}_G$ is large in terms of the underlying graph as follows

Proposition 2.3.1. *When G is a connected graph, there exists a basic pair (B, C) in $\mathcal{A}_G$ if and only if there exists a pair (T, F) of disjoint subsets of E, and a linear ordering $\preccurlyeq$ on E, satisfying the following.*

1. *T is a spanning tree for G.*
2. *F is a disjoint union of at least two trees.*
3. *If P is a path in T and e is an edge in F such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to $\preccurlyeq$.*
4. *If P is a path in F and e is an edge in T such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to $\preccurlyeq$.*

It is immediate that $\mathcal{A}_G$ is a large arrangement if and only if there exists a pair (T, F) and an order $\preccurlyeq$ as above with $|F| = r-1$. In this situation, it must be the case that F is a disjoint union of exactly two trees, which together form a spanning subgraph of G.

Example 2.3.2. *Revisiting example 2.6.1, if we decompose the edge set of the subgraph G' into subsets T and F and choose an ordering on the edges as shown below.*



This partition (T, F) of the edge set and ordering of the edges corresponds to a basic pair (B, C) for the graphic arrangement $\mathcal{A}'$ with $|C| = r - 1 = 4$. The existence of this pair is what guarantees that $TC_s(\mathcal{A}') = 5s - 1$. It is worth noting that it will not generally be true that the linear ordering is such that all edges in T are less than all edges in F.

Proposition 2.3.1 is a reformulation of Definition for the graphic arrangement case. In the above example, we simply found the decomposition and ordering by hand. But using tools from graph theory, we can obtain a simpler condition on the graph G which is equivalent to the conditions in Proposition 2.3.1 being satisfied.

2.3.1 Arboricity and a theorem of Nash-Williams

We've seen that questions about whether a graphic arrangement is large are closely related to questions about decomposing finite graph into acyclic subsets, and this is a well-understood subject. For a finite graph $G = (V, E)$, there is a smallest integer k such that E can be written as a disjoint union of k acyclic subsets. This number is called the *arboricity* of G and is tightly connected to the density of edges among the vertex-induced subgraphs of G. In particular, a theorem of Nash-Williams gives us the following.

Theorem 2.3.3 (Nash-Williams [NW64]). *Let $G = (V, E)$ be a finite graph. For any subset $U \subseteq V$, we will use $E(U)$ to denote the edge set of the subgraph induced by U. The edge set of G can be partitioned into k forests if and only if for all nonempty $U \subseteq V$ we have*

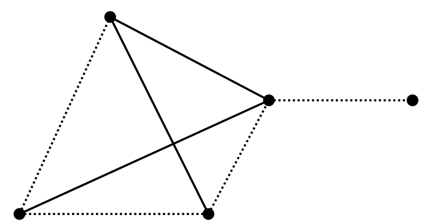
$$|E(U)| \leq k(|U| - 1).$$

This inequality is a condition on the density of the edges of G among the vertex-induced subgraphs of G. If the edge set of G is a union of a small number of acyclic subsets, then no vertex-induced subgraph of G can have a high density of edges, and in fact the reverse is true.

Applying this theorem with $k = 2$ to a graph with $r + 1$ vertices and $2r - 1$ edges, we see that when

$$|E(U)| \leq 2(|U| - 1)$$

for all $U \subseteq V$, not only will the edge set of G be partitioned into two forests, but in fact one of the two forests will be a spanning tree and the other will be a disjoint union of exactly two trees. This additional structure is nothing but numerics; it can't be the case that both of the forests have fewer than r edges, since G has a total of $2r - 1$ edges. And neither forest can have more than r edges, since such a subgraph would have a cycle. So the only possibility is that one of the forests has exactly r edges and is a spanning tree, and the other has $r-1$ edges and is a forest composed of exactly two trees. These two subsets are naturally the pair (T, F) which form a candidate for a basic pair for the arrangement $\mathcal{A}_G$. But we don't yet know whether there exists the necessary linear order $\preccurlyeq$ on the edge set. For an example of why G having arboricity 2 is not sufficient to guarantee that $\mathcal{A}_G$ is large, consider the complete graph K_4 with one additional vertex and pendant edge shown below. For all subsets of its vertex set, it satisfies the Nash-Williams inequality, so it can be partitioned into a spanning tree T (dotted edges) and a forest F composed of exactly two trees (solid edges). Note that one of the trees in F is just an isolated vertex.



However, T and F cannot form a basic pair. To see this, let $\preccurlyeq$ be any linear ordering on the edges of this graph. If we restrict our attention to the K_4 subgraph, we see that every dotted edge can be completed to a cycle by a path of solid edges. And every solid edge can be completed to a cycle by a path of dotted edges. This means that no matter which edge in the K_4 is minimal with respect to $\preccurlyeq$, it will force one of either T or F to fail to be **nbc**.

If we let U be the vertex set of the K_4, we see that it satisfied the Nash-Williams inequality with equality, i.e. $6 = |E(U)| = 2(|U|-1) = 2(4-1)$. In what follows, we will show that such subsets are the only obstruction to the existence of a linear ordering $\preccurlyeq$ with the needed properties.

2.4 Result

Our main result is that strengthening the inequality in the Nash-Williams theorem guarantees the existence of the desired linear order. We only need to change the non-strict inequality to a strict inequality to guarantees that $\mathcal{A}_G$ is a large arrangement.

Theorem 2.4.1. *Let $G = (V, E)$ be a graph with $|V| = r+1$ and no isolated vertices, and let $\mathcal{A}_G$ be its associated graphic arrangement. Then $\mathcal{A}_G$ is large if and only if G contains a spanning subgraph H having $2r-1$ edges and satisfying that for every nonempty, non-singleton $U \subseteq V$ we have*

$$|E_H(U)| < 2(|U|-1),$$

where $E_H(U)$ denotes the set of edges of the subgraph of H induced by U. In particular, if this inequality is *satisfied, then the higher topological complexity TC_s of the complement of $\mathcal{A}_G$ is equal to $sr-1$.*

In order to prove this Theorem, we will make use of the following technical lemma. This is a translation of Theorem 4.1 from [Yuz14] into the language of graphs.

Lemma 2.4.2. *Let $H=(V,E)$ be a graph with $|E_H(U)| < 2(|U|-1)$ for all nonempty, non-singleton $U \subseteq V$, and suppose that E can be written as a union of disjoint subsets T and F, where T is a spanning tree and F is a proper forest. Then there exists a linear ordering $\preccurlyeq$ of E so that*

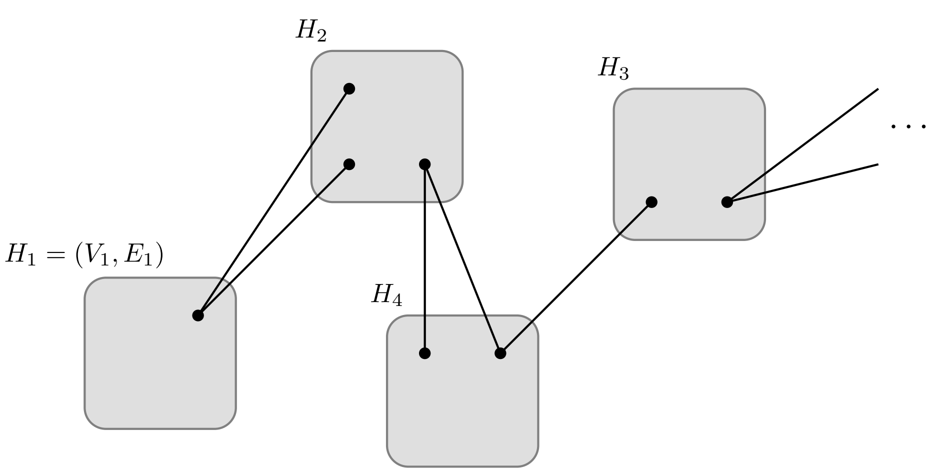
1. *If P is a path in T and e is an edge in F such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to $\preccurlyeq$.*
2. *If P is a path in F and e is an edge in T such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to $\preccurlyeq$.*

Proof. We will prove this lemma by induction on $|V|$. It is vacuously true when $|V|=1$.

If $|V|>1$, then the disjoint trees in F partition V into disjoint subsets $V_1,\ldots,V_k$ with $k \geq 2$. Let E_i denote $E_H(V_i)$. Since T is a spanning tree, there must be at least one edge connecting a vertex in V_i to a vertex in V_j for some $i \neq j$. We let E_0 denote the set of all such edges, so that

$$E_0 = E \setminus \bigcup_{i=1}^{k} E_i,$$

and E is the disjoint union $E_0 \cup E_1 \cup \ldots \cup E_k$. We will denote by H_i the graph (V_i, E_i). In this way, H can be seen as a disjoint union of at least 2 vertex-induced subgraphs $H_1,\ldots,H_k$, represented here as gray boxes, connected by the edges of E_0.



We will construct the linear ordering $\preccurlyeq$ on E by giving an order on each E_i, then concatenating these orderings so that when $i<j$, all edges in E_i are less that all edges in E_j.

First, let e be an edge in E_0 and let P be any path in F. Since e connects two disjoint subtrees of F, $\{e\} \cup P$ does not form a cycle. For this reason, the edges of E_0 can be chosen to be minimal among the edges of E without introducing any broken circuits. The ordering of E_0 itself can be chosen arbitrarily.

Now consider E_i for $i \geq 1$. The graph H_i satisfies that $|E_{H_i}(U)| < 2(|U| - 1)$ for all nonempty, non-singleton $U \subseteq V_i$. $F \cap E_i$ is a spanning tree for H_i by design, and $T \cap E_i$ is a forest. We see immediately that $T \cap E_i$ must be a proper forest, because if it were a spanning tree, then we would have $|E_{H_i}(V_i)| = 2(|V_i| - 1)$, a contradiction. Since H_i has strictly fewer vertices than H, we know by induction that there is a linear ordering $\preccurlyeq$ on E_i so that the following two conditions are met.

1. If P is a path in $F \cap E_i$ and e is an edge in $T \cap E_i$ such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to $\preccurlyeq$.

2. If P is a path in $T \cap E_i$ and e is an edge in $F \cap E_i$ such that $P \cup \{e\}$ forms a cycle, then e is not the minimal element of that cycle with respect to $\preccurlyeq$.

Let $\preccurlyeq$ be the linear ordering of E defined by concatenating the arbitrary ordering of E_0 with the orderings of the E_i as described above. All that remains is to verify that the ordering $\preccurlyeq$ satisfies the necessary conditions.

Let P be a path in T and suppose that e is an edge in F so that $P \cup \{e\}$ is a cycle. If $P \cap E_0$ is nonempty, then e cannot be the minimal element of that cycle by construction. If $P \cap E_0$ is empty, then P is contained in H_i for some $1 \leq i \leq k$, so e must also be an edge in H_i, so by induction e is not the minimal element of the cycle.

Similarly, let P be a path in F and let e be an edge in T so that $P \cup \{e\}$ is a cycle. P must be a path in H_i for some $1 \leq i \leq k$, which means e must be an edge in H_i and so is not the minimal edge of the cycle by induction. □

With the above lemma in place, we now proceed to the proof of the theorem.

Proof. Since any arrangement which contains a large subarrangement is itself large [Yuz14], it is enough to show that the graphic arrangement $\mathcal{A}_H$ is a large arrangement.

$\mathcal{A}_H$ is large if and only if it contains a basic pair (B, C), which is equivalent to the existence of a pair (T, F) and a linear order $\preccurlyeq$ as described in Proposition 5.2.

Suppose $|E(U)| < 2(|U| - 1)$ for all $U \subseteq V$. By Nash-Williams, we know that the edge set of H can be written as a disjoint union of two forests T and F. As mentioned above, we can assume that T is a spanning tree and $|F| = r - 1$, and so F is a disjoint union of exactly two trees. Since F is a proper forest, the above lemma guarantees the existence of a linear order $\preccurlyeq$ on E so that (T, F) forms a basic pair, so $\mathcal{A}_H$ is a large arrangement and $TC_s(\mathcal{A}_H) = sr - 1$.

For the reverse implication, Let G be a graph satisfying that for each spanning subgraph H with $2r - 1$ edges, there is at least one nonempty, non-singleton subset $U \subseteq V$ for which the above strict inequality does not hold. We will show that the edges of H cannot form a basic pair, and so G cannot be large. If H is such a subgraph and U satisfies $|E_H(U)| > 2(|U| - 1)$, then by Nash-Williams the edges of H can't be decomposed into two acyclic subsets and so cannot correspond to a basic pair in $\mathcal{A}_G$. Next suppose H is a subgraph so that $|E_H(U)| \leq 2(|U| - 1)$ for all nonempty, non-singleton subsets U, but with at least one subset U satisfying $|E_H(U)| = 2(|U| - 1)$. We will show that there cannot exist a linear ordering on the edges of H satisfying the needed conditions. Let H' be the subgraph of H induced by U. By Nash-Williams, the edge set of H' will decompose into a disjoint union of two forests. Since $|H'| = 2(|U| - 1)$, both of these forests must be spanning trees of H', call them T_1 and T_2. For an arbitrary edge e in T_1, there exists a path in T_2 which is completed to a cycle by e. So e cannot be the minimal edge of H'. By symmetry, we can make the same argument about an arbitrary edge in T_2. So no linear ordering of the edges of H' can satisfy the conditions of Proposition 5.2, and so the edges of H cannot form a basic pair for $\mathcal{A}_G$. □

2.5 Computations with Macaulay2

In this section, we show some code snippets in Macaulay2 which compute the bounds TC_s for hyperplane arrangements. Making use of the existing hyperplane arrangements package, we give below scripts to get the Orlik-Solomon algebra $A(\mathcal{A})$ of an arrangement, to compute the kernel of the s-fold multiplication map from $A(\mathcal{A})^{\otimes s} \to A(\mathcal{A})$, and to find the cup-length lower bound and dimensional upper bound for TC_s of a given arrangement.

```
    needsPackage("HyperplaneArrangements")

del = (R,m) -> diff(sum gens R,m);

os = (A,v) ->         (
    n := numColumns matrix A;
    E := QQ[v_1..v_n,SkewCommutative => true];
    L := circuits A;
    C := apply(L, c -> (apply(c, t -> t+1)));
    genlist := apply(C, c -> del(E,product(apply(c,i -> (gens E)_(i-1)))));
    I := ideal mingens ideal genlist;
    if #genlist == 0 then return E;
    return E/I
    )

tensorPower = (s,A) -> (
    algList := apply(toList (1..s), i -> A);
    return fold(algList, (R,S) -> R**S);
    )

multker = (s,A) -> (
    n := numgens A;
    T := tensorPower(s,A);
```

```
    l := gens T;
    kergens := flatten(apply(toList (0..(s-2)),
                       i -> apply(toList ((n*i)..((n*i)+n-1)),
                            t -> l_(t) - l_(t+n))));
    return ideal  kergens;
    )
nil = J -> (
    i := 0;
    nthpower := J;
    while (not nthpower == 0) list (i = i+1; nthpower = ideal mingens (nthpower*J););
    return i;
    )

zcl = (s,X) -> (
    A := os(X,e);
    K := multker(s,A);
    return nil K;
    )

ub = (s,X) -> (
    numplanes := numColumns matrix X;
    ambientdim := (numgens ring X);
    codimen := ambientdim - (rank flat(X,toList (0..(numplanes-1))));
    return s*(ambientdim - codimen) - 1;
    )

TC = (s,X) -> (
    return [zcl(s,X),ub(s,X)];
    )
```

The result of `TC(s,X)` for an integer $s \geq 2$ and an arrangement X is a pair `[a,b]`, where a is the zcl_s lower bound and b is the dimensional upper bound for $TC_s(X)$. We will explore a small non-graphic example, as well as the type A arrangement A_3 and A_4, which are graphic arrangements corresponding to the complete graphs K_4 and K_5.

```
i1 : X = arrangement(transpose matrix{{1,0,0},{1,1,0},{0,1,0},{0,1,1},{0,0,1}})

o1 = {x , x  + x , x , x  + x , x }
       1   1    2   2   2    3   3

o1 : Hyperplane Arrangement

i2 : TC(2,X)

o2 = [5, 5]

o2 : Array

i3 : TC(3,X)

o3 = [8, 8]

o3 : Array

i4 : A3 = typeA(3)

o4 = {x  - x , x  - x , x  - x , x  - x , x  - x , x  - x }
       1    2   1    3   1    4   2    3   2    4   3    4

o4 : Hyperplane Arrangement

i5 : TC(2,A3)

o5 = [5, 5]

o5 : Array

i6 : TC(3,A3)

o6 = [8, 8]

o6 : Array
```

```
i7 : A4 = typeA(4)

o7 = {x  - x , x  - x , x  - x , x  - x , x  - x , x  - x , x  - x , x  - x , x  - x , x  - x }
       1    2   1    3   1    4   1    5   2    3   2    4   2    5   3    4   3    5   4    5

o7 : Hyperplane Arrangement

i8 : TC(2,A4)

o8 = [7, 7]

o8 : Array
```

This confirms a few special cases of what was already known from results in this chapter, and which also follows from earlier work of Yuzvinsky, that A_n are large arrangements, so have $TC_s(A_n) = sn - 1$

However, if we compute the bounds on the topological complexity of the arrangement corresponding to a 4-cycle, or the arrangement from the non-large example above, we see the following.

```
i9 : C = arrangement(transpose matrix{{1,-1,0,0},{0,1,-1,0},{0,0,1,-1},{1,0,0,-1}})

o9 = {x  - x , x  - x , x  - x , x  - x }
       1    2   2    3   3    4   1    4

o9 : Hyperplane Arrangement

i10 : TC(2,C)

o10 = [4, 5]

o10 : Array

i11 : Y = arrangement(transpose matrix{{1,-1,0,0,0},{1,0,0,-1,0},
      {1,0,0,0,-1},{0,1,-1,0,0},{0,1,0,-1,0},{0,1,0,0,-1},{0,0,0,1,-1}})

o11 = {x  - x , x  - x , x  - x , x  - x , x  - x , x  - x , x  - x }
        1    2   1    4   1    5   2    3   2    4   2    5   4    5

o11 : Hyperplane Arrangement

i33 : TC(2,Y)

o33 = [6, 7]

o33 : Array
```

This gives two examples of graphic arrangements which are demonstrably not large, the first because its graph does not have enough edges, the second because it does not satisfy the inequality. In both cases we see that the zcl_s lower bound does not meet the dimensional upper bound, and so we cannot determine the TC_s of these arrangements by these methods alone.

2.6 Examples

We close with some examples of graphs that determine large graphic arrangements.

Example 2.6.1. *Let $\mathcal{A}$ be the graphic arrangement associated to the complete tripartite graph $K_{r-1,1,1}$. The arrangement $\mathcal{A}$ is large and $TC_s(\mathcal{A}) = sr - 1$.*

Proof. Let v_1 and v_2 be the vertices of the singleton parts, and let U be a nonempty, non-singleton subset of the vertex set. If neither v_1 nor v_2 are in U, then $|E(U)| = 0 < 2(|U| - 1)$. If exactly one of the v_1 or v_2 is in U, then $|E(U)| = |U| - 1 < 2(|U|) - 1$. If both v_1 and v_2 are in U, then $|E(U)| = 2(|U| - 2) < 2(|U| - 1)$. So $\mathcal{A}_G$ is large. □

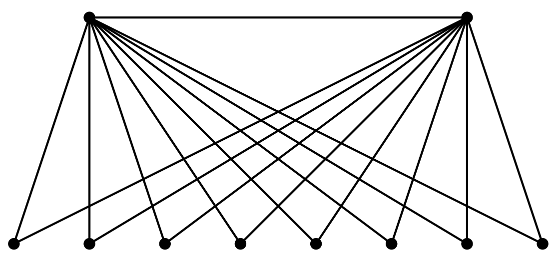


Figure 2.1: The arrangement $\mathcal{A}_G$ associated to the complete tripartite graph $G = K_{8,1,1}$ has $TC_s(\mathcal{A}_G) = 9s - 1$.

Example 2.6.2. *Let G be the wheel graph W_{r+1}, that is the graph join of the singleton graph with an r-cycle. The graphic arrangement $\mathcal{A}_G$ is large and $TC(\mathcal{A}_G) = sr - 1$.*

Proof. Let v denote the central vertex of G and let H be the subgraph obtained by deleting one edge which is not incident to v. Let U be a nonempty, non-singleton subset of the vertex set. If v is not in U, then $|E_H(U)| \leq |U| - 1 < 2(|U| - 1)$. If v is in U, then $|E_H(U)| \leq (|U| - 2) + (|U| - 1) = 2|U| - 3 < 2(|U| - 1)$. So the edges of H form a basic pair in $\mathcal{A}_G$. □

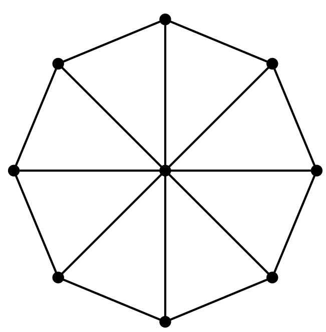


Figure 2.2: The arrangement $\mathcal{A}_G$ associated to the wheel graph $G = W_9$ has $TC_s(\mathcal{A}_G) = 8s - 1$.

Example 2.6.3. *Let G be any graph with r vertices for which $\mathcal{A}_G$ is large, and let v_1 and v_2 be two distinct vertices in G. Let G' be the graph formed by adding a new vertex v' to G which is adjacent only to v_1 and v_2. Then $\mathcal{A}_{G'}$ is large and $TC_s(\mathcal{A}_{G'}) = sr - 1$.*

Proof. Let H be the spanning subgraph of G satisfying theorem 7.1 and let H' be the spanning subgraph of G' formed by adding v' and both its incident edges to H. Let U be a subset of the vertex set of G'. If v' is not in U, then by assumption $|E_{H'}(U)| < 2(|U|-1)$. If v' is in U, then by assumption $|E_{H'}(U \setminus \{v'\})| < 2(|U|-2)$, and so $|E_{H'}(U)| = |E_{H'}(U \setminus \{v'\})| + 2 < 2(|U|-2) + 2 = 2(|U|-1)$. So the edges of H' will form a basic pair for G' and hence $\mathcal{A}_{G'}$ is large. □

Since the graphic arrangement associated to K_3 is easily seen to be large, and since both $K_{r-1,1,1}$ and wheel graphs with a deleted edge can be built by iteratively applying the above construction to K_3, the above example gives a second proof that the above graphs determined large graphic arrangements.

Example 2.6.4. *Let G be the graph obtained by inserting a diagonal edge into each square of an $n \times 1$ grid graph $P_n \times P_1$ as shown below. Since G can be built by iteratively applying the construction from example 8.3 to K_3, $\mathcal{A}_G$ is large and hence $TC_s(\mathcal{A}_G) = s(2n-1) - 1$.*

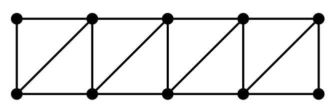


Figure 2.3: Because this 10-vertex graph G can be built by the iterative construction in example 8.3, the associated arrangement has $TC_s(\mathcal{A}_G) = 9s - 1$.

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