EXTREMAL PROBLEMS ON COUNTING COMBINATORIAL STRUCTURES

BY

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DISSERTATION

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Abstract

The fast developing field of extremal combinatorics provides a diverse spectrum of powerful tools with many applications to economics, computer science, and optimization theory. In this thesis, we focus on counting and coloring problems in this field.

The complete balanced bipartite graph on n vertices has $\lfloor n^2/4 \rfloor$ edges. Since all of its subgraphs are triangle-free, the number of (labeled) triangle-free graphs on n vertices is at least $2^{\lfloor n^2/4 \rfloor}$. This was shown to be the correct order of magnitude in a celebrated paper Erdős, Kleitman, and Rothschild from 1976, where the authors furthermore proved that almost all triangle-free graphs are bipartite. In Chapters 2 and 3 we study analogous problems for triangle-free graphs that are maximal with respect to inclusion.

In Chapter 2, we solve the following problem of Paul Erdős: Determine or estimate the number of maximal triangle-free graphs on n vertices. We show that the number of maximal triangle-free graphs is at most $2^{n^2/8+o(n^2)}$, which matches the previously known lower bound. Our proof uses among other tools the Ruzsa-Szemerédi Triangle Removal Lemma and recent results on characterizing of the structure of independent sets in hypergraphs. This is a joint work with József Balogh.

In Chapter 3, we investigate the structure of maximal triangle-free graphs. We prove that almost all maximal triangle-free graphs admit a vertex partition (X, Y) such that $G[X]$ is a perfect matching and Y is an independent set. Our proof uses the Ruzsa-Szemerédi Removal Lemma, the Erdős-Simonovits stability theorem, and recent results of Balogh-Morris-Samotij and Saxton-Thomason on the characterization of the structure of independent sets in hypergraphs. The proof also relies on a new bound on the number of maximal independent sets in triangle-free graphs with many vertex-disjoint P_3's, which is of independent interest. This is a joint work with József Balogh, Hong Liu, and Maryam Sharifzadeh.

In Chapter 4, we seek families in posets with the smallest number of comparable pairs. Given a poset P, a family $\mathcal{F} \subseteq P$ is *centered* if it is obtained by 'taking sets as close to the middle layer as possible'. A poset P is said to have the *centeredness property* if for any M, among all families of size M in P, centered families contain the minimum number of comparable pairs. Kleitman showed that the Boolean lattice $\{0,1\}^n$ has the centeredness property. It was conjectured by Noel, Scott, and Sudakov, and by Balogh and Wagner, that the poset $\{0,1,\ldots,k\}^n$ also has the centeredness property, provided n is sufficiently large compared to k. We show that this conjecture is false for all $k \geq 2$ and investigate the range of M for which it holds. Further, we improve a result of Noel, Scott, and Sudakov by showing that the poset of subspaces of $\mathbb{F}_q^n$ has the centeredness property. Several open problems are also given. This is a joint result with József Balogh and Adam Zsolt Wagner.

In Chapter 5, we consider a graph coloring problem. Kim and Park have found an infinite family of graphs whose squares are not chromatic-choosable. Xuding Zhu asked whether there is some k such that all k-th power graphs are chromatic-choosable. We answer this question in the negative: we show that there is a positive constant c such that for any k there is a family of graphs G with $\chi(G^k)$ unbounded and $\chi_\ell(G^k) \geq c\chi(G^k)\log\chi(G^k)$. We also provide an upper bound, $\chi_\ell(G^k) < \chi(G^k)^3$ for $k > 1$. This is a joint work with Nicholas Kosar, Benjamin Reiniger, and Elyse Yeager.

To my beloved parents,

Zdenka Petříčková and Libor Petříček

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Table of Contents

List of Symbols

$\mathbb{N}$ .... the set of natural numbers starting with 1

$[n]$ .... $\{1, 2, \ldots, n\}$

$\mathcal{A} - \mathcal{B}$ .... set difference

Graphs

$V(G)$ .... the set of vertices of G

$E(G)$ .... the set of edges of G

$v(G)$ .... the number of vertices in G

$e(G)$ .... the number of edges in G

$E(X, Y)$ .... the set of edges between X and Y

$e(X, Y)$ .... the number of edges between X and Y

$N_G(v)$, $N(v)$ .... the set of neighbors of v in G

$d_G(v)$, $d(v)$ .... the degree of v in G

$\Delta(G)$ .... maximum degree of G

$K_{r,s}$ .... the complete bipartite graph with parts of size r and s

K_{r*s} .... the complete r-partite graph with each part of size s

H-free .... not containing a (not nec. induced) copy of H

$G[X]$ .... graph induced by the vertex set X

$\omega(G)$ .... the clique number of G

$\mathrm{ex}(n, H)$ .... the maximum number of edges in an H-free n-vertex graph

$T(n, r)$ .... complete balanced r-partite graph on n vertices

$t(n, r)$ .... the number of edges in the Turán graph $T(n, r)$

$\mathrm{MIS}(G)$ .... the number of maximal independent sets in G

$L(G)$ .... the line graph of G

$T(G)$ .... the total graph of G

$\chi(G)$ .... the chromatic number of G

$\chi'(G)$ .... the edge-chromatic number of G

$\chi''(G)$ .... the total-chromatic number of G

$\chi_\ell(G)$ .... the list-chromatic number of G

G^k .... the k-th power of a graph G

Posets

$\mathcal{P}(n)$ .... the power set of $[n]$, Boolean lattice

$\text{Comp}(\mathcal{F})$ .... the set of comparable pairs in $\mathcal{F}$

$\text{comp}(\mathcal{F})$ .... the number of comparable pairs in $\mathcal{F}$

$\text{comp}(A, \mathcal{F})$ .... the number of elements comparable with A in $\mathcal{F}$

$\mathcal{L}_r(P)$ .... the set of elements in the r-th layer in P

$\ell_r(P)$ .... the number of elements in the r-th layer in P

$\mathcal{F}_r$ .... the set of elements in the r-th layer that are in $\mathcal{F}$

$\overline{\mathcal{F}_r}$ .... the set of elements in the r-th layer that are NOT in $\mathcal{F}$

$\Sigma_j(P)$ .... the number of elements in the middle j layers in P

A^c .... the complement of A

$\mathbb{F}_q$ .... the field of order q

$\mathcal{V}(q, n)$ .... poset on subspaces of $\mathbb{F}_q^n$ ordered by inclusion

For $P = \{0, 1, \ldots, k\}^n$

$\mathcal{L}_r(n, k)$ .... the set of elements in the r-th layer

$\ell_r(n, k)$ .... the number of elements in the r-th layer

$\Sigma_j(n, k)$ .... the number of elements in the middle j layers

$N_r(A)$ .... elements in the r-th layer comparable with A

A_i .... the i-th coordinate of A

a_i .... the number of coordinates of A equal to i

A^c .... $(k - A_1, \ldots, k - A_n)$

Chapter 1

Introduction

Given a family $\mathcal{F}$ of discrete structures, it is natural to ask the following questions:

- What is the size of $\mathcal{F}$?
- What is the structure of elements in $\mathcal{F}$?
- What are the properties of elements in $\mathcal{F}$?

In this work, we provide answers to several questions of this type for families of graphs and partially ordered sets (posets). In Chapters 2 and 3, we focus on maximal triangle-free graphs, that is, triangle-free graphs in which adding any edge results in a triangle. In Chapter 2, we count the number of maximal triangle-free graphs, and in Chapter 3, we study their typical structure. In Chapter 4, we investigate the structure of families in posets of fixed size that have the smallest number of comparable pairs. In Chapter 5, we show that for every k, the family of k-th powers of graphs is not chromatic-choosable by proving that there exists a graph G whose k-th power is close in structure to a complete multipartite graph.

This chapter is organized as follows: In Section 1.1, we give a brief exposition of fundamental results in extremal graph theory and state our results. In Section 1.2, we discuss essential results in extremal set theory, in particular for partially ordered sets. In Section 1.3, we focus on chromatic graph theory, with emphasis on proper vertex coloring and its list coloring version. The notation and terminology used in this thesis is introduced at the end of this chapter – in Section 1.4.

1.1 Extremal problems in graphs

A systematic study of extremal graph theory was initiated more than 100 years ago by Mantel, who determined the maximum number of edges in a triangle-free graph.

Theorem 1.1.1 (Mantel [56], 1907). *An n-vertex triangle-free graph contains at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.*

This is best possible – consider the complete bipartite graph with the two parts of size $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil$. Furthermore, the complete balanced bipartite graph is the only triangle-free graph that attains this maximum.

A natural question is whether a similar result holds for K_{r+1}-free graphs with $r > 2$ as well. This was answered by Paul Turán in the affirmative.

Theorem 1.1.2 (Turán [76], 1941). *An n-vertex K_{r+1}-free graph contains at most $\left(1 - \frac{1}{r}\right) \left\lfloor \frac{n^2}{2} \right\rfloor$ edges.*

The *Turán graph* $T(n, r)$ is the complete bipartite r-partite graph of order n with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. The *Turán number* $t(n, r)$ is then defined as number of edges in $T(n, r)$. Observe that in the case when r divides n, the Turán graph $T(n, r)$ has exactly $\binom{r}{2} \frac{n^2}{r^2} = (1 - \frac{1}{r}) \frac{n^2}{2}$ edges. Turán's theorem thus states that no K_{r+1}-free graphs on n vertices has more edges than the Turán graph $T(n, r)$. Moreover, as in the triangle-free case, $T(n, r)$ is the only graph that attains the maximum, i.e. every K_{r+1}-free graph with the maximum number of edges must be a complete balanced r-partite graph.

We can further generalize the problem to any graph H. Denote by $\mathrm{ex}(n, H)$ the maximum possible number of edges in an H-free graph on n vertices (so $\mathrm{ex}(n, K_{r+1}) = t(n, r)$). The key parameter here is the chromatic number $\chi(H)$ of H. Since the vertices of H cannot be properly colored with $\chi(H) - 1$ colors, the graph H is not contained in $T(n, \chi(H) - 1)$. Therefore, $\mathrm{ex}(n, H) \geq t(n, \chi(H) - 1)$. It turns out that this is asymptotically best possible.

Theorem 1.1.3 (Erdős-Stone-Simonovits [38], 1966).

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \frac{n^2}{2}.$$

Many interesting problems occur when, instead of counting edges, we attempt to count subgraphs. For fixed graphs T and H, let $\mathrm{ex}(n, T, H)$ be the maximum possible number of copies of T in an H-free graph on n vertices. Note that $\mathrm{ex}(n, K_2, H) = \mathrm{ex}(n, H)$. A recent paper of Alon and Shikhelman [5] presents history of the problem and new results for various choices of T and H.

Perharps the most powerful result in extremal combinatorics is the Szemerédi Regularity Lemma. Before presenting the statement, we need several definitions. For a graph $G = (V, E)$ and two vertex sets $X, Y \subseteq V$, denote by $E(X, Y)$ the set of edges between X and Y, and let $e(X, Y) = |E(X, Y)|$. Furthermore, define the *edge density* of a pair (X, Y) by

$$d(X, Y) := \frac{e(X, Y)}{|X| \cdot |Y|}.$$

Given an $\varepsilon > 0$, a pair (X, Y) is an *ε-regular* if for every $X' \subseteq X$ and every $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$, the edge densities of (X, Y) and (X', Y') differ by at most ε, that is, $|d(X, Y) - d(X', Y')| \leq \varepsilon$.

Theorem 1.1.4 (Szemerédi Regularity Lemma [74, 75], 1975). *For every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an integer M such that every graph $G = (V, E)$ has a vertex partition $V(G) = V_1 \cup \cdots \cup V_t$ with $m \leq t \leq M$ that satisfies the following two conditions:*

- $||V_i| - |V_j|| \leq 1$ *for all $i, j \in [t]$.*
- *All but at most εt^2 pairs (V_i, V_j) are ε-regular.*

The proof of the Szemerédi Regularity Lemma is technical and is based on the "density increment argument". In short, we start with an arbitrary equitable vertex partition, and if this partition is not ε-regular, it can be refined so that a quantity called "the mean square density" increases by a fixed amount. We can then show that the process terminates after a finite number of steps since this quantity is bounded from above.

The standard application of the Szemerédi Regularity Lemma is the Triangle Removal Lemma.

Theorem 1.1.5 (Ruzsa-Szemerédi [68], 1976). *For every $\varepsilon > 0$ there exists $\delta > 0$ such that, every n-vertex graph G with at most δn^3 triangles can be made triangle-free by removing at most εn^2 edges.*

The Triangle Removal Lemma generalizes to larger cliques: Every graph with $o(n^r)$ triangles can be made K_r-free by removing at most $o(n^2)$ edges. This result was first published by Alon, Duke, Lefmann, Rödl, and Yuster [4], and independently by Füredi [41]. For more details on graph removal lemmas, we refer the reader to an excellent survey by Conlon and Fox [29].

Another application of the Szemerédi Regularity Lemma is the celebrated result of Erdős, Kleitman, and Rothschild, which determines the asymptotics for the logarithm of the number of (labeled) K_r-free graphs on n-vertices.

Theorem 1.1.6 (Erdős-Kleitman-Rothschild [37], 1976). *The number of K_r-free graphs on the vertex set $[n]$ is*

$$2^{\mathrm{ex}(n,K_r)+o(n^2)}, \quad \textit{where} \quad \mathrm{ex}(n,K_r) = \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}.$$

The original proof of Theorem 1.1.6 did not use the Szemerédi Regularity Lemma (which was proved around the same time), but Erdős Frankl, and Rödl [36] later discovered that it follows from it relatively easily: For every K_r-free n-vertex graph G, the n-vertex blow up $B(C_G)$ of the cluster graph C_G for G is K_r-free and omits at most $o(n^2)$ edges of G. Every K_r-free n-vertex graph G can be obtained by

1. choosing a cluster graph C_G, $O(1)$
2. choosing an n-vertex blow-up $B(C_G)$ of the cluster graph C_G, $\leq n^n$
3. deciding which edges of the blow-up $B(C_G)$ are in G, and $\leq 2^{\mathrm{ex}(n,K_r)}$
4. deciding which edges outside of $B(C_G)$ are in G. $\binom{n^2}{o(n^2)} \leq 2^{o(n^2)}$

Together, there are thus at most $2^{\mathrm{ex}(n,K_r)+o(n^2)}$ K_r-free n-vertex graphs.

More recently, there has been a movement to extend classical results of extremal combinatorics to sparse random settings. Conlon and Gowers [30], and independently Schacht [70], developed a technique that allows them to combine an extremal result R with a supersaturation result to obtain a random analogue of R. For example, recall that Turán's theorem states that for every $r \geq 3$:

$$\mathrm{ex}(n, K_r) = \left(1 - \frac{1}{r-1} + o(1)\right) \cdot e(K_n).$$

Let $G(n,p)$ be the classical random graph model, where every edge in an n-vertex graph is chosen independently with probability p. Then the random analogue of Turán theorem states that for every $r \geq 3$, if $p \gg n^{-2/(r+1)}$, then a.a.s.:

$$\mathrm{ex}(G(n,p), K_r) = \left(1 - \frac{1}{r-1} + o(1)\right) \cdot e(G(n,p)).$$

Other applications of the technique used in [30] and [70] include sparse random analogues of Szemerédi's theorem on arithmetic progressions and of Ramsey's theorem.

Inspired by the new development, Balogh, Morris, and Samotij [10], and independently Saxton and Thomason [69], established a powerful counting method called the "container method". Not only this method implies many results of Conlon and Gowers [30], and of Schacht [70], but it also provides their counting counterparts. The main result roughly states that for every "nice" r-uniform hypergraph $\mathcal{H}$, we can find a relatively small family $\mathcal{F}$ of relatively small vertex subsets (called containers), such that every independent set of $\mathcal{H}$ is contained in some member of $\mathcal{F}$. The full result is quite technical and we will omit it here. Instead, we state an important corollary for graphs, obtained by setting $V(\mathcal{H}) = E(K_n)$ and $E(\mathcal{H}) =$'copies of K_r' in the main result for independent sets in a hypergraph $\mathcal{H}$ (Theorem 2.2 in [10], Theorem 3.4 in [69]). Observe that then I is an independent set in $\mathcal{H}$ if and only if the edges in I do not form a copy of a K_r.

Theorem 1.1.7 (Balogh-Morris-Samotij [10], 2015; Saxton-Thomason [69], 2015)**.** *For every $\delta > 0$ there exists a family $\mathcal{F}$ of $2^{O(n^{2-1/(r-1)} \cdot \log n)}$ graphs, each containing at most δn^r copies of K_r, such that every K_r-free graph is contained in some $F \in \mathcal{F}$.*

A similar statement also holds for a general subgraph H, but with the exponent $O(n^{2-1/m_2(H)} \cdot \log n)$ in the term bounding the number of containers, where $m_2(H) := \max \frac{e(H')-1}{v(H')-2}$ taken over all subgraphs H' of H with more than one edge.

Theorem 1.1.7 provides a quick proof of Theorem 1.1.6 (Erdős-Kleitman-Rothschild's theorem). Indeed, every K_r-free graph is in some $F \in \mathcal{F}$, and every such F has $\mathrm{ex}(n, K_r) + o(n^2)$ edges (by a removal lemma for K_r, which has been be proved without the use of Szemerédi Regularity Lemma by Fox [40]). Hence, the number of K_r-free graphs is at most $|\mathcal{F}| \cdot 2^{\mathrm{ex}(n,K_r)+o(n^2)} = 2^{\mathrm{ex}(n,K_r)+o(n^2)}$.

The container method gives new proofs of the sparse analogues of various results, including Szemerédi's theorem, Turán theorem, Erdős-Kleitman-Rothschild's theorem, Erdős-Frankl-Rödl theorem, Erdős-Stone theorem, and Stone-Simonovits theorem. The container method moreover provides new applications for wide spectrum of extremal problems. It was used to prove new results on list coloring hypergraphs [69], counting C_4-free graphs [21], counting metric spaces [21], counting intersecting families of permutations [11], counting maximal sum-free subsets [13], counting r-chains in posets [19, 20], and others.

Maximal triangle-free graphs

Recall that the number of (labeled) copies of triangle-free graphs on n vertices is $2^{n^2/4+o(n^2)}$ by a result of Erdős, Kleitman, and Rothschild (Theorem 1.1.6). Observe that most bipartite graphs are not maximal triangle-free. It is thus natural to ask, how much smaller is the family of maximal triangle-free graphs compared to the family of all triangle-free graphs. Erdős suggested the following problem (as stated in [71]): determine or estimate the number of maximal triangle-free graphs on n vertices. In Chapter 2, we find the asymptotics of the logarithm of this number.

Theorem 1.1.8 (Balogh-Petříčková [17], 2014). *The number of maximal triangle-free graphs with vertex set $[n]$ is at most*

$$2^{\mathrm{ex}(n,K_3)/2+o(n^2)} = 2^{n^2/8+o(n^2)}.$$

Erdős, Kleitman, and Rothschild [37] determined the typical structure of triangle-free graphs, showing that almost all of them are bipartite (i.e., the proportion of n-vertex triangle-free graphs that are not bipartite goes to zero as $n \to \infty$).

Theorem 1.1.9 (Erdős-Kleitman-Rothschild [37], 1976). *Let T_n be the number of triangle-free* *graphs with vertex set* $[n]$*, and* S_n *be the number of bipartite graphs with vertex set* $[n]$*. Then*

$$T_n = S_n \left(1 + O\left(\frac{1}{n}\right)\right).$$

In Chapter 3 we prove the analogue of Theorem 1.1.9 for maximal triangle free-graphs.

Theorem 1.1.10 (Balogh-Liu-Petříčková-Sharifzadeh [12], 2015)**.** *For almost every maximal triangle-free graph* G *on* $[n]$*, there is a vertex partition* $X \cup Y$ *such that* $G[X]$ *is a perfect matching and* Y *is an independent set.*

Furthermore, our proof yields that the number of maximal triangle-free graphs without the desired structure is exponentially smaller than the number of maximal triangle-free graphs: Let $\mathcal{M}_3(n)$ denote the set of all maximal triangle-free graphs on $[n]$, and $\mathcal{G}(n)$ denote the family of graphs from $\mathcal{M}_3(n)$ that admit a vertex partition such that one part induces a perfect matching and the other is an independent set. Then there exists an absolute constant $c > 0$ such that for n sufficiently large, $|\mathcal{M}_3(n) - \mathcal{G}(n)| \leq 2^{-cn}|\mathcal{M}_3(n)|$.

Supersaturation in graphs

The theory of supersaturation is concerned with situations beyond the extremal threshold. In graph theory, the typical such problem is to determine the number of copies of a graph H in a n-vertex graph with $ex(n, H) + t$ edges. Clearly, if $t > 1$, then we are guaranteed at least one copy of H. Somewhat surprisingly, in many cases we are guaranteed many more copies of H. The first such result is by Rademacher.

Theorem 1.1.11 (Rademacher, 1941, unpublished)**.** *Every graph with* $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ *edges contains at least* $\left\lfloor \frac{n}{2} \right\rfloor$ *triangles.*

About two decades later, Erdős proved the following stronger version of Theorem 1.1.11.

Theorem 1.1.12 (Erdős [35], 1962)**.** *There exists a positive constant* c *such that for all* $t < cn/2$*, every graph with* $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ *edges contains at least* $t \cdot \left\lfloor \frac{n}{2} \right\rfloor$ *triangles.*

This was later extended to all $t < n/2$ by Lovász and Simonovits [54]. Note that sharpness examples can be obtained by adding edges to $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$, in particular, to the part with $\lceil \frac{n}{2} \rceil$ vertices in such a way that the new edges do not form a triangle.

Erdős [33] proved the following generalization of Theorem 1.1.12:

Theorem 1.1.13 (Erdős [33], 1962)**.** *For every $r \geq 3$, there exists a positive constant c_r such that for all $t < c_r \cdot n$, every graph with $\mathrm{ex}(n, K_r) + t$ edges contains at least $t \cdot \mathrm{ex}(n, K_r)$ copies of K_r.*

Lovász and Simonovits obtained Theorem 1.1.13 for $t = o(n^2)$, together with a stability result, and furthermore provided a characterization of the extremal configurations. The problem is notoriously difficult for $t = \Omega(n^2)$, but recently, there has been a significant progress. Razborov [66] found an asymptotic solution for the triangle-case, Nikiforov [60] for K_4, and finally, Reiher [67] for general K_r. This concluded the efforts of proving the conjecture of Lovász and Simonovits from the 1970's.

Analogous problems of determining the number of forbidden configurations in structures slightly denser than the extremal threshold are called *Erdős-Rademacher-type problems*, and we will see some other examples in the next section.

1.2 Extremal problems in posets

A *partially ordered set* P (or *poset*) is a set P together with a binary relation $\leq$ over P that is reflexive, antisymmetric, and transitive. Two elements x and y in P form a *comparable pair* (or are *comparable*) if $x \leq y$ or $y \leq x$. An r-chain is a subset of size r where every two elements are comparable, that is, whose elements can be ordered as $x^1 \leq x^2 \leq \cdots \leq x^r$. An *antichain* is a subset with no comparable pairs.

An important example of a poset is the Boolean lattice $\mathcal{P}(n)$, which is the family of subsets of $[n]$ ordered by set-inclusion. Alternatively, the poset $\mathcal{P}(n)$ can be viewed as the set $\{0,1\}^n$ with the ordering $A \leq B$ if $A_i \leq B_i$ for every $i \in [n]$, where A_i and B_i are the i-th coordinates of the sets A and B, respectively.

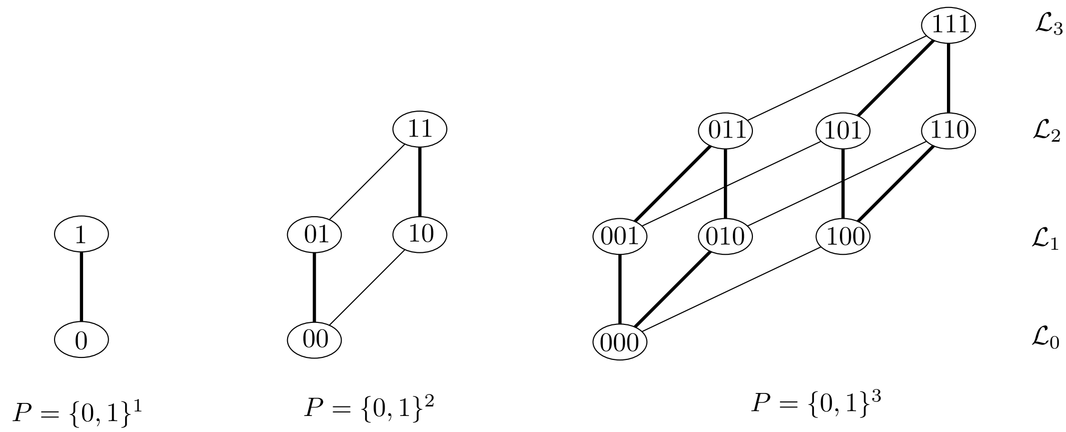


Figure 1.1: The Hasse diagram of the poset $\mathcal{P}(n) = \{0,1,2\}^n$ for $n = 1$, $n = 2$, and $n = 3$.

A fundamental result in extremal combinatorics is the Sperner's theorem, which states that there are no larger antichains in $\mathcal{P}(n)$ than the set of elements in the middle layer (or one of the two middle layers).

Theorem 1.2.1 (Sperner [72], 1928)**.** *The largest antichain in the poset* $\mathcal{P}(n)$ *has size* $\binom{n}{\lfloor n/2 \rfloor}$.

Sperner's theorem was later generalized by Lubell [55] (1966), Yamamoto [80] (1954), and Meshalkin [57] (1963). Their independent discoveries are commonly known as the LYM inequality. This inequality is however also a special case of a powerful theorem proved by Bollobás [22] in 1965 (when he was still an undergraduate student!), and should be therefore called the BLYM inequality.

Theorem 1.2.2 (BLYM inequality, [22, 55, 80, 57])**.** *For an antichain* $\mathcal{F} \in \{0,1\}^n$

$$\sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} \leq 1.$$

Erdős extended Sperner's theorem to longer chains. Also in this case, the natural choice of families closest to the middle layer was shown to be optimal.

Theorem 1.2.3 (Erdős [34], 1945)**.** *The maximum size of a family in* $\{0,1\}^n$ *that does not contain an* $(r+1)$*-chain is*

$$\Sigma_r(\mathcal{P}(n)) := \sum_{i=\lceil (n-r+1)/2 \rceil}^{\lfloor (n+r-1)/2 \rfloor} \binom{n}{i}.$$

Supersaturation in posets

At the end of Section 1.1 we discussed Erdős-Rademacher-type problems in graphs. Here we consider a similar question for posets: How many r-chains are we guaranteed in a family of a poset P which is larger than the extremal threshold? Erdős and Katona (see [31]) proposed the following conjecture for $r = 2$ and the Boolean lattice $\mathcal{P}(n)$.

Conjecture 1.2.4 (Erdős-Katona). *In $\mathcal{P}(n)$, every family of size $\binom{n}{\lfloor n/2 \rfloor} + t$ contains at least $t \cdot \lceil \frac{n+1}{2} \rceil$ comparable pairs.*

This conjecture was resolved by Kleitman [49] in 1966, when he showed that the minimum number of comparable pairs occurs for a family of sets of sizes as close to $n/2$ as possible. He introduced the compression method, which allows for replacing sets far from the middle level with elements closer to the middle level, without increasing the number of comparable pairs.

Theorem 1.2.5 (Kleitman [49], 1966). *Conjecture 1.2.4 is true. Moreover, the minimum is attained by some family of sets as close to the middle level as possible.*

Given a poset P, we say that a family $\mathcal{F} \subseteq P$ is *centered* if it is obtained by 'taking sets as close to the middle layer as possible'. A family $\mathcal{F} \subseteq P$ of size M is called *M-optimal* if it contains the smallest number of comparable pairs, among families of size M. A poset P is said to have the *centeredness property* if for every $M \leq |P|$ there exists an M-optimal centered family.

Theorem 1.2.5 thus states that the Boolean lattice has the centeredness property, i.e., that for every $M \leq 2^n$ there exists a family $\mathcal{F}$ of size M such that for every family $\mathcal{G}$ of size M:

1. $\mathcal{G}$ has at least as many comparable pairs as $\mathcal{F}$ (that is, $\mathcal{F}$ is M-optimal), and
2. $\sum_{F \in \mathcal{F}} ||F| - \frac{n}{2}| \leq \sum_{G \in \mathcal{G}} ||G| - \frac{n}{2}|$ (that is, $\mathcal{F}$ is centered).

Kleitman suggested that an analogous result might hold for chains of any length r.

Conjecture 1.2.6 (Kleitman [49]). *In $\mathcal{P}(n)$, the number of r-chains is minimized by a centered family.*

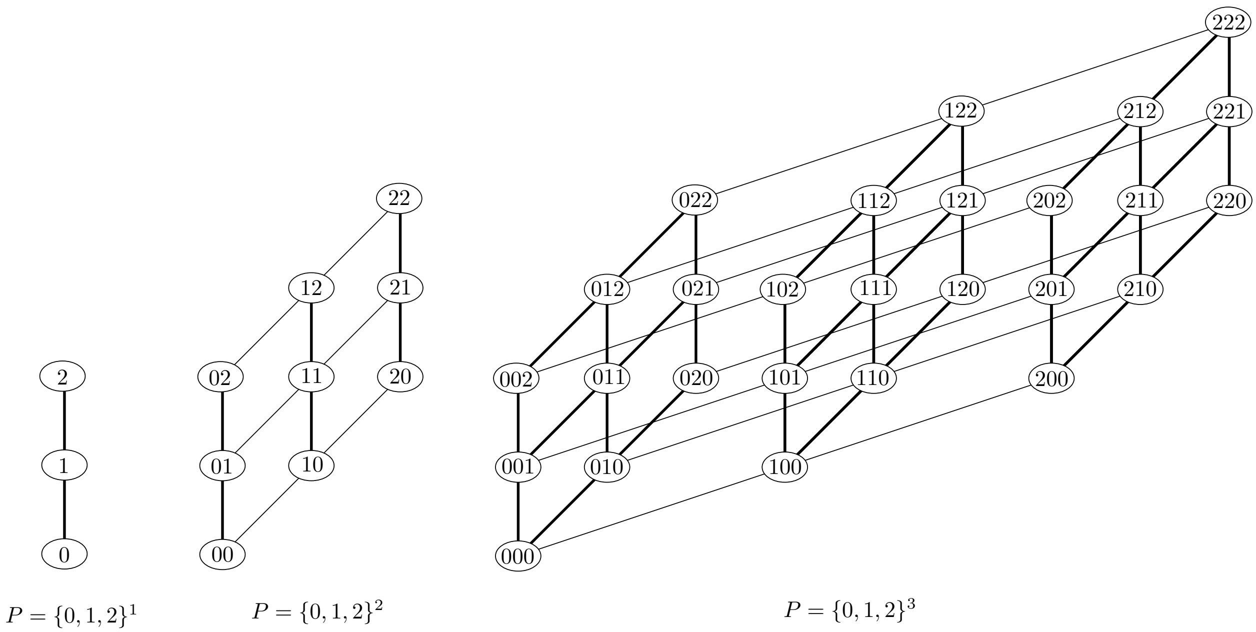


Figure 1.2: The Hasse diagram of the poset $\{0,1,2\}^n$ for $n=1$, $n=2$, and $n=3$.

Some partial results were recently obtained by Das, Gan, and Sudakov [31]. They confirmed that Kleitman's conjecture is true for families of size at most $\Sigma_r(\mathcal{P}(n))$, and for families of size at most $\Sigma_{r+1}(\mathcal{P}(n))$ with $r \leq n-6$. In both cases, they provide a corresponding stability result, i.e., they show that if the number of r-chains in a family is close to the minimum, then the family must be close in structure to the extremal example.

Using the 'container method' discussed in Section 1.1, Balogh and Wagner [20] were able to prove an asymptotic version of Kleitman's conjecture.

Theorem 1.2.7 (Balogh and Wagner [20], 2017+)**.** *For every r and $\varepsilon > 0$ there exists $n_0 = n_0(k, \varepsilon)$ such that if $n \geq n_0$ and $M \leq (1-\varepsilon)2^n$, then among the families of size M, the number of r-chains is minimized by a centered family.*

Another direction for extending Theorem 1.2.5 is to count the minimum number of comparable pairs of families in other posets. In Chapter 4, we study such problems for two posets that generalize the Boolean lattice:

- The poset on $\{0, 1, \ldots, k\}^n$ with $(A_1, \ldots, A_n) \leq (B_1, \ldots, B_n)$ iff $A_i \leq B_i$ for all $i \in [n]$.

- The poset of subspaces of $\mathbb{F}_q^n$ (where q is a prime power) ordered by inclusion.

Recently, Noel, Scott, and Sudakov [62], and Balogh and Wagner [20] conjectured that for every k there exists an n_0 such that if $n \geq n_0$ then the poset $\{0, 1, \ldots, k\}^n$ has the centeredness property. In Chapter 4 we show that this conjecture does not hold.

Theorem 1.2.8 (Balogh-Petříčková-Wagner, 2017+)**.** *The poset $\{0, 1, \ldots, k\}^n$ does not have the centeredness property for any $k \geq 2$.*

We prove several other more technical results for the poset $\{0, 1, 2\}^n$, and also the following theorem for the poset of subspaces of $\mathbb{F}_q^n$, which generalizes a recent result of Noel, Scott, and Sudakov [62].

Theorem 1.2.9 (Balogh-Petříčková-Wagner, 2017+)**.** *For any prime power q, the poset of subspaces of $\mathbb{F}_q^n$ has the centeredness property.*

1.3 Chromatic graph theory

In general, a *proper vertex k-coloring* is a function $c : V(G) \to [k]$ such that if $uv \in E(G)$, then $c(u) \neq c(v)$. The least k such that there exist a proper vertex k-coloring of G is called the *(vertex) chromatic number of* G, and is denoted $\chi(G)$. Coloring the vertices of G greedily shows that every graph can be colored with at most $\Delta(G)+1$ colors. The most fundamental result in vertex coloring, Brooks' theorem, states that $\Delta(G)$ colors are sufficient for most graphs G:

Theorem 1.3.1 (Brooks [25], 1941)**.** *Let G be a connected graph. If G is not a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$. Otherwise $\chi(G) = \Delta(G) + 1$.*

The next well-studied type of graph coloring is proper edge coloring, where we color edges instead of vertices and require every pair of adjacent edges to receive different colors. Every edge coloring of G can be transformed into a vertex coloring by considering the line graph $L(G)$ of G. Indeed, two edges e and f of G are adjacent in G if and only if the corresponding vertices e and f of $L(G)$ are adjacent in $L(G)$. As in the case of proper vertex coloring, a proper edge coloring is a special case of an H-free edge coloring, where H is now a path of length two. A graph is *k-edge-colorable*

TYPE OF COLORING	standard	list version
vertex coloring	χ	χ_ℓ
edge coloring	χ'	χ'_ℓ
total coloring	χ''	χ''_ℓ

Table 1.1: Basic types of graph colorings and the coresponding parameters.

if there exists a proper edge k-coloring of G. The least k such that G is k-edge-colorable is called the *edge chromatic number of* G, and denoted $\chi'(G)$. The obvious lower bound on $\chi'(G)$ is $\Delta(G)$. Somewhat surprisingly, the upper bound on $\chi'(G)$ differs from the lower bound only by 1, as states the celebrated Vizing's theorem.

Theorem 1.3.2 (Vizing [78], 1964). *For any graph G either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

A graph G is said to be of *class 1* if $\chi'(G) = \Delta(G)$. Otherwise, G is said to be of *class 2*.

Total coloring is a combination of proper vertex and proper edge coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest k such we can find a k-coloring of $V(G) \cup E(G)$ that assigns different colors to adjacent/incident elements of $V(G) \cup E(G)$. The almost 50 year old (and still open) Total Coloring Conjecture states that the total chromatic number is either $\Delta(G) + 1$ or $\Delta(G)+2$, classifying graphs in a similar way as Vizing's theorem. We give a brief exposition of this conjecture at the beginning of Chapter 5.

For various types of graph colorings we are often interested in their corresponding list coloring versions. Recall that coloring is a function $c : O \to S$ that assigns colors of S to the elements in O according to some prescribed rule(s) R. If, in addition, we are given a list $L_o \subseteq S$ of colors for each object $o \in O$, then a function $c : O \to S$ that maps every object o to a color from L_o is called a *list coloring*. We then seek the smallest k such that for any assignment of lists of size k to the objects in O, there is a list coloring of O. For example, the *list-chromatic number* $\chi_\ell(G)$ is the least k such that for any assignment of lists of size k to the vertices of G, there is a proper coloring of $V(G)$ such that every vertex $v \in V(G)$ uses a color from its list L_v. The *list-chromatic index* $\chi'_\ell(G)$ and the *total-list-chromatic number* $\chi''_\ell(G)$ are defined analogously, with $O = E(G)$ and $O = V(G) \cup E(G)$, respectively, and the rule that the coloring has to be proper.

Observe that $\chi(G) \leq \chi_\ell(G)$ for any graph G – consider assigning the list $\{1, 2 \ldots, k\}$ to every vertex in G. The list-chromatic number however cannot be bounded in terms of chromatic number. For example, the chromatic number of a complete bipartite graph is always at most 2, but the complete bipartite graph with $\binom{2k-1}{k}$ vertices in each part is not k-list colorable. This follows by assigning all possible k-subsets of $[2k-1]$ as list to vertices in each part. Indeed, if there were a coloring from these lists, one of the two parts would use at most $k-1$ colors (since no color can be used in both parts). But then, the vertex in this part whose list consist of the remaining $2k-1-(k-1)=k$ colors could not have been colored from its list. See Figure 1.3 (left) for the case $k=2$.

Similarly, the complete bipartite graph with k vertices in one part and k^k vertices in the other part is not k-list colorable. Indeed, suppose we assign list $\{i1, i2, \ldots, ik\}$ to the i-th vertex of the smaller part A, and $(1j_1, 2j_2, \ldots, kj_k)$. Then, every choice of colors for the k vertices in A will be in the form $\{1j_1, 2j_2, \ldots, kj_k\}$ for some $(j_1, \ldots, j_k) \in [k]^k$. There are k^k such different sets, so if we assign them to the vertices in the large part B, we will not be able to properly color the vertices from their lists. See Figure 1.3 (right) for the case $k=2$.

A graph G is called *chromatic-choosable* if its chromatic number $\chi(G)$ is equal to its list-chromatic number $\chi_\ell(G)$. Disproving the List Square Coloring Conjecture, Kim and Park [48] found an infinite family of graphs whose squares are not chromatic-choosable. Xuding Zhu asked whether there exists a k such that all k-th power graphs are chromatic-choosable. In Chapter 5, we answer this question in the negative:

Theorem 1.3.3 (Kosar-Petříčková-Reiniger-Yeager [52], 2014)**.** *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded such that*

$$\chi_\ell(G^k) \geq c\chi(G^k) \log \chi(G^k).$$

1.4 Notation and terminology

We use the symbol $[n]$ for the set of all natural numbers from 1 to n, that is $[n] = \{1, \ldots, n\}$. For two sets $\mathcal{A}$ and $\mathcal{B}$, we often write $\mathcal{A} - B$ instead of $\mathcal{A} \setminus \mathcal{B}$ to improve readability. We omit floor and

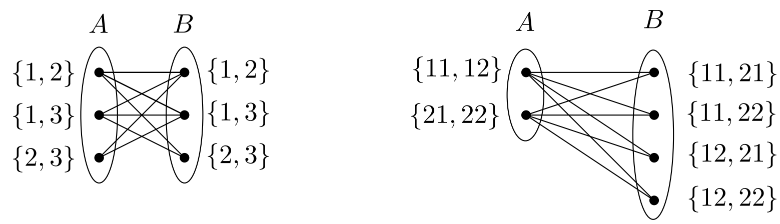


Figure 1.3: The graphs $K_{2,2^2}$ and $K_{3,3}$ are not 2-list-colorable.

ceiling signs when they are not crucial for the sake of clarity of presentation.

Graphs

A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a set of *vertices* and $E(G) \subseteq \binom{V(G)}{2}$ is a set of *edges*. We write uv for an edge $\{u, v\}$ of $E(G)$. Next, we use $v(G)$ and $e(G)$ to denote the number of vertices and edges of G, respectively. The number $v(G)$ is also called the *order* of G and $e(G)$ the *size* of G. Two vertices defining an edge are called the *endpoints* of that edge. We say that a vertex v is *incident with* an edge e if v is an endpoint of e. Two vertices u, v are *adjacent* if $uv \in E(G)$. Two edges are *adjacent* if they share one endpoint. A vertex adjacent to v is called a *neighbor* of v (in G). The set of all neighbors of v is called the *neighborhood* of v (in G), and is denoted by $N_G(v)$, or simply $N(v)$ if no confusion can arise. The *degree of* v, denoted by $d_G(v)$ or $d(v)$, is the number of edges adjacent to v (i.e., $d_G(v) = |N_G(v)|$). The *maximum degree of* G $\max_{v \in V(G)} d(v)$ is denoted by $\Delta(G)$. A graph G is called *r-regular graph*, if all vertices of G have degree r. In particular, a 3-*regular* is called *cubic*.

The set of edges with one endpoint in X and the other endpoint in Y is denoted $E(X, Y)$, and the number of edges in $E(X, Y)$ is denoted $e(X, Y)$.

We say that two graphs G *and* H *are isomorphic*, and write $G \simeq H$, if there exists a bijective function $f : V(G) \to V(H)$ satisfying $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. For a fixed graph G, a *copy* H of G is a graph isomorphic to G with $V(G) \cap V(H) = \emptyset$. A graph is called *$H$-free* if it does not contain a copy of H as a subgraph.

The *union* $G \cup H$ of two graphs G and H is a graph with with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then we call the union a *disjoint union*. If we say that some graph is a disjoint union of G and H, where the vertex sets $V(G)$ and $V(H)$ are not specified, then we automatically assume that they are disjoint.

A graph H is said to be a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If in addition, $E(H)$ contains all edges of $E(G)$ that have endpoints in $V(H)$, then H is called an *induced subgraph* of G or a *graph induced by* $V(H)$, and is denoted by $G[V(H)]$. For a vertex v and an edge e of G we define $G - v = G[V(G) \setminus \{v\}]$ and $G - e = (V(G), E(G) \setminus \{e\})$.

A *path* P is a graph isomorphic (for some $n \geq 1$) to a graph with vertex set $\{x_1, \ldots, x_n\}$ and edge set $\{x_i x_{i+1} : i \in [n-1]\}$. The *length of a path* is the number of its edges. The *distance* $d_G(u, v)$ (or $d(u, v)$) between two vertices u and v in G is the length of a shortest path connecting u and v. A non-empty graph G is *connected* if there is a path connecting any two vertices of G. Maximal connected subgraphs of G are called *components*. A cycle C_n is a graph isomorphic to a graph with vertex set $\{x_1, \ldots, x_n\}$ and edge set $\{x_i x_{i+1} : i \in [n-1]\} \cup \{x_n x_1\}$. A graph without cycles is called a *forest*. A *tree* is a forest with exactly one component.

A graph G is called a *complete graph* if $E(G) = \binom{V(G)}{2}$. A complete graph on n vertices is denoted by K_n. A *clique* of a graph G is a complete subgraph of G. The order of a maximum clique of G is called *the clique number of* G and denoted $\omega(G)$. A set $I \subseteq V(G)$ is an *independent set* of G if no two vertices of I are adjacent, or equivalently, if $G[I]$ has no edges.

A graph is called r*-partite* if we can partition $V(G)$ into r subsets, called *parts*, so that each part induces a graph with no edges. A 2-partite graph is called *bipartite*. A *complete* r*-partite* graph is an r-partite graph with the maximum number of edges. A complete bipartite graph with the two parts of size r and s is denoted by $K_{r,s}$. Denote by K_{r*s} the complete r-partite graph with each part of size s.

A *blow-up* of a graph G is formed by replacing every vertex of G with a finite collection of copies so that the copies of two vertices are adjacent if and only if the originals were.

The *line graph* $L(G)$ of a graph G is a graph with vertex set $E(G)$ and an edge ef if and only if e and f are adjacent (edges) in G. The *total graph* $T(G)$ of G is a graph with vertex set $V(G) \cup E(G)$, where two vertices are adjacent if and only if their corresponding elements are adjacent or incident in G.

The k-th power G^k of a graph G is a graph formed from G by connecting every two vertices of distance at most k (in G) by an edge. For example, the second power of a path of length 3 is the graph $K_4 - e$ (complete graph on 4 vertices with one edge removed).

A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a set of *vertices* and $E(\mathcal{H}) \subseteq \mathcal{P}(V(\mathcal{H})) - \{\emptyset\}$ is a set of *hyperedges*. If $E(\mathcal{H}) \subseteq \binom{V(\mathcal{H})}{r}$, then $\mathcal{H}$ is an *r-uniform hypergraph* or an *r-graph*, and its hyperedges are called *r-edges*.

Posets

A *partially ordered set* $(P, \leq)$ (or *poset*) is a set P together with a binary relation $\leq$ over P that is

- reflexive ($\forall x \in P$: $x \leq x$),
- antisymmetric ($\forall x, y \in P$: $x \leq y$ and $y \leq x$ implies $x = y$), and
- transitive ($\forall x, y, z \in P$: $x \leq y$ and $y \leq z$ implies $x \leq z$).

Note that we will write P to denote the poset $(P, \leq)$ when the order is clear form the context.

Two elements x and y form a *comparable pair* (or are *comparable*) if $x \leq y$ or $y \leq x$. An r-chain is a subset of size r where every two elements are comparable, that is, whose elements can be ordered as $x^1 \leq x^2 \leq \cdots \leq x^r$. An *antichain* is a subset with no comparable pairs.

We say that the poset P is a *graded poset* if it is equipped with a rank function $\text{rk} : P \to \mathbb{N}$ which satisfies that $\text{rk}(x) < \text{rk}(y)$ whenever $x < y$, and $\text{rk}(y) = \text{rk}(x) + 1$ whenever y covers x. Given a graded poset P we write $\ell_i(P)$ for the number of elements in P of rank i. A graded poset of rank n is *rank-symmetric* if $\ell_i(P) = \ell_{n-i}(P)$ for $0 \leq i \leq n$ and it is *rank-unimodal* if $\ell_0(P) \leq \ldots \leq \ell_j(P) \geq \ell_{j+1}(P) \geq \ldots \geq \ell_n(P)$ for some $0 \leq j \leq n$.

In a graded poset P, the *r-th layer* (or *r-th level*) $\mathcal{L}_r(P)$ is the set of elements of rank r, $\ell_r(P)$ is the size of $\mathcal{L}_r(P)$, and $\Sigma_j(P)$ is the size of the j middle layers of P.

We write $\{0, 1, \ldots, k\}^n$ for the poset over $\{0, 1, \ldots, k\}^n$ where for two elements $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ we have the order $A \leq B$ iff $A_i \leq B_i$ for all $1 \leq i \leq n$. We will often write $A \subseteq B$ instead of $A \leq B$.

In the poset $\{0, 1, \ldots, k\}^n$, the r-th layer $\mathcal{L}_r(n, k)$ is the set of vectors in $\{0, 1, \ldots, k\}^n$ whose coordinates sum to r, and the size of $\mathcal{L}_r(n, k)$ is denoted $\ell_r(n, k)$. We write $\Sigma_j(n, k)$ for the total size of the j middle layers of $\{0, 1, \ldots, k\}^n$.

Probability

For each $n \in \mathbb{N}$, let $\mathcal{A}(n)$ be a set of n-vertex graphs that have property P, and let $\mathcal{B}(n)$ be a subset of $\mathcal{A}(n)$. We say that *almost all* graphs of $\mathcal{B} := \bigcup \mathcal{B}(n)$ have property P if $|\mathcal{A}(n) - \mathcal{B}(n)| = o\left(|\mathcal{A}(n)|\right)$.

Lemma 1.4.1 (Chernoff bound)**.** *For independent* $0-1$ *random variables* $X_1, \ldots, X_n$*, let* $X = X_1 + \cdots + X_n$*. For every* $\delta \in (0, 1)$*,*

$$\mathbb{P}[X > (1+\delta)\mathbb{E}[X]] \leq e^{-\delta^2 \mathbb{E}[X]/3} \quad \textit{and} \quad \mathbb{P}[X < (1-\delta)\mathbb{E}[X]] \leq e^{-\delta^2 \mathbb{E}[X]/2}.$$

Chapter 2

Number of maximal triangle-free graphs

The results of this chapter are joint work with József Balogh [17].

The maximum triangle-free graph has $n^2/4$ edges [56]. Hence, the number of triangle-free graphs is at least $2^{n^2/4}$, which was shown to be the correct order of magnitude by Erdős, Kleitman and Rothschild [37]. Moreover, almost every triangle-free graph is bipartite [37], even if there is a restriction on the number of edges (first shown by Osthus-Prömel-Taraz [63], extended by Balogh-Morris-Samotij-Warnke [15]; see [10] and [15] for a more detailed history of the problem). This suggests that most of those graphs are bipartite, and subgraphs of a complete bipartite graph, therefore most of them are not maximal (with respect to inclusion) triangle-free. Paul Erdős suggested the following problem:

Problem 2.0.1 ([71])**.** *Determine or estimate the number $f(n)$ of maximal triangle-free graphs on n vertices.*

The following folklore construction (see [59], but it was known much earlier) shows that $f(n) \geq 2^{n^2/8}$. Let G be a graph on a vertex set $X \cup Y$ with $|X| = |Y| = n/2$ such that X induces a perfect matching, Y is an independent set, and there are no edges between X and Y. For each pair of a matching edge $x_1x_2 \in E(G[X])$ and a vertex $y \in Y$, we add one of the edges x_1y or x_2y to G. Since there are $n/4$ matching edges in $E(G[X])$ and $n/2$ vertices in Y, we obtain $2^{n^2/8}$ triangle-free graphs. These graphs may not be maximal triangle-free, but since no further edges can be added between X and Y, all of there $2^{n^2/8}$ graphs extend to distinct maximal triangle-free graphs.

In this chapter we prove a matching upper bound.

Theorem 2.0.2. *The number of maximal triangle-free graphs on vertex set $[n]$ is at most $2^{n^2/8+o(n^2)}$.*

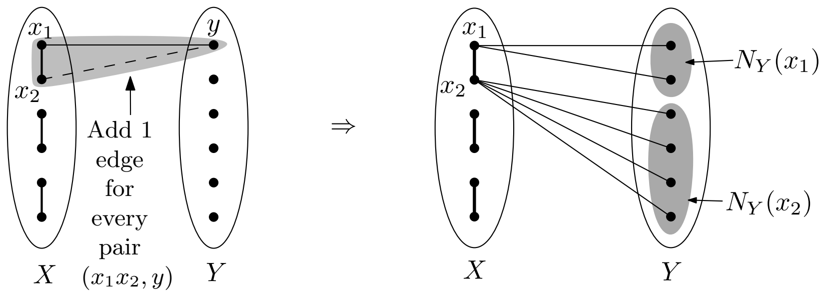


Figure 2.1: A construction of $2^{n^2/8}$ maximal triangle-free graphs.

2.1 Tools

Our first tool is a corollary of recent powerful counting theorems of Balogh-Morris-Samotij [10, Theorem 2.2.], and Saxton-Thomason [69]. We refer to the graphs $F_1, \ldots, F_t$ in the theorem as *containers.*

Theorem 2.1.1. *For each $\delta > 0$ there is $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{F_1, \ldots, F_t\}$ of graphs, each containing at most δn^3 triangles, such that for every triangle-free graph G there is $i \in [t]$ such that $G \subseteq F_i$, where n is sufficiently large.*

We also use the Ruzsa-Szemerédi triangle-removal lemma [68].

Theorem 2.1.2. *For every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that any graph G on n vertices with at most $\delta(\varepsilon)n^3$ triangles can be made triangle-free by removing at most εn^2 edges.*

Our next tool is the following theorem of Hujter and Tuza [45]. Recall that a set $I \in V(G)$ is an *independent set* if no two vertices in I are adjacent. An independent set I is a *maximal independent set* if $I \cup \{v\}$ contains an edge for every $v \in V(G) - I$. Note that we write $v(G)$ for the number of vertices of G.

Theorem 2.1.3. *Every triangle-free graph G has at most $2^{v(G)/2}$ maximal independent sets.*

In the next section we prove our main result, Theorem 2.0.2.

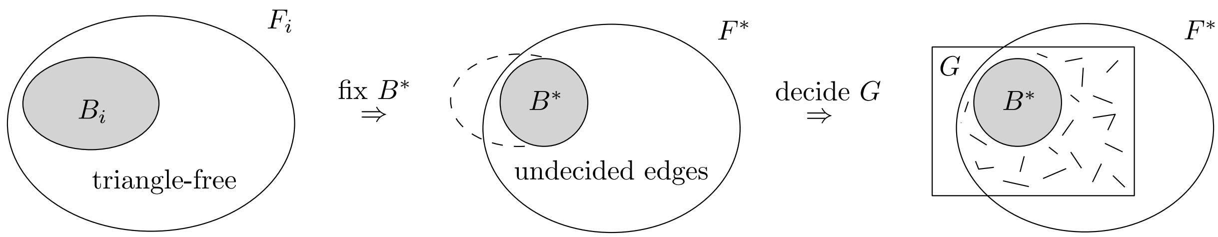


Figure 2.2: The structure of F_i and B_i.

2.2 Proof of Main Theorem

We show that for every $\varepsilon > 0$, the number of maximal triangle-free graphs with vertex set $[n]$ is $2^{n^2/8+2\varepsilon n^2}$ for sufficiently large n. We fix an arbitrarily small constant $\varepsilon > 0$. First we apply Theorem 2.1.2 with this ε, which provides us $\delta := \delta(\varepsilon)$, and then we apply Theorem 2.1.1 with this δ. Each container F_i returned by Theorem 2.1.1 has at most δn^3 triangles, and thus can be made triangle-free by removing at most εn^2 edges. Also, since every triangle-free graph has at most $n^2/4$ edges by Mantel's theorem, each F_i has at most $n^2/4 + \varepsilon n^2$ edges.

For every $i \in [t]$, we count the number of maximal triangle-free graphs G that satisfy $G \subseteq F_i$. Denote by $\mathcal{G}$ the set of maximal triangle-free graphs with vertex set $[n]$, and let $\mathcal{G}_i = \{G \in \mathcal{G} : G \subseteq F_i\}$.

Since $t \leq 2^{\varepsilon n^2}$ for sufficiently large n, we have

$$|\mathcal{G}| \leq \sum_{i=1}^{t} |\mathcal{G}_i| \leq 2^{\varepsilon n^2} \max_{i \in [t]} |\mathcal{G}_i|.$$

Fix an arbitrary $i \in [t]$. By Theorem 2.1.2 applied on F_i, there is $B_i \subseteq E(F_i)$ such that $|B_i| \leq \varepsilon n^2$ and $F_i - B_i$ is triangle-free. For each F_i we fix one such B_i. For every $B^* \subseteq B_i$, define $\mathcal{G}_i(B^*) = \{G \in \mathcal{G}_i : E(G) \cap B_i = B^*\}$.

Now we show that for every choice of B^* we have $|\mathcal{G}_i(B^*)| \leq 2^{e(F_i)/2}$. Fix B^*, and let

$$F^* := F_i - (B_i - B^*) - \{e \in E(F_i) : \exists f, g \in B^* \text{ such that } e, f, g \text{ form a triangle}\}.$$

So, F^* is obtained from F_i by removing edges that are in none of $G \in \mathcal{G}_i(B^*)$. We can assume that B^* is triangle-free since otherwise $\mathcal{G}_i(B^*) = \emptyset$. We now count the number of ways to add edges of $E(F^*) - B^*$ to B^* such that the resulting graph is maximal triangle-free. We construct an auxiliary graph T as follows:

$$V(T) := E(F^*) - B^* \quad \text{and} \quad E(T) := \{ef \mid \exists d \in B^*\colon \{d, e, f\} \text{ spans a triangle in } F^*\}.$$

Claim 1. T is triangle-free.

Proof. Suppose not. Let e, f, g be vertices of a triangle in T. Then $e, f, g \in E(F^*) - B^*$ and there are $d_1, d_2, d_3 \in B^*$ such that the 3-sets $\{d_1, e, f\}$, $\{d_2, e, g\}$, and $\{d_3, f, g\}$ span triangles in F^*. As $F_i - B_i$ is triangle-free and $F^* - B^* \subseteq F_i - B_i$, it follows that the edges e, f, g share a common endpoint in F^*, and that $\{d_1, d_2, d_3\}$ spans a triangle. This is a contradiction since B^* is triangle-free. □

Claim 2. If $G \in \mathcal{G}_i(B^*)$, then $E(G) - B^*$ spans a maximal independent set in T.

Proof. Let $G \in \mathcal{G}_i(B^*)$. We first show that $E(G) - B^*$ spans an independent set in T. If not, then there is an edge ef in $E(T)$ with $e, f \in E(G) - B^*$. By the definition of $E(T)$, there is $d \in B^*$ such that the edges d, e, f form a triangle in F^*, which is clearly in G.

Suppose now that $E(G) - B^*$ is an independent set in T that is not maximal. So, there is $x \in E(F^*) - E(G)$ such that for every $y \in E(G) - B^*$ and for every $z \in B^*$, the edges x, y, z do not span a triangle in F^*. This means that $G \cup \{x\}$ is triangle-free. Hence, G is not maximal. □

By Theorem 2.1.3, the number of maximal independent sets in T is at most $2^{v(T)/2}$. Since $V(T)$ is the edge-set of an n-vertex triangle-free graph, we have $v(T) \leq n^2/4$, and thus

$$|\mathcal{G}_i(B^*)| \leq 2^{v(T)/2} \leq 2^{(n^2/4)/2} = 2^{n^2/8}.$$

The number of ways to choose $B^* \subseteq B_i$ for a given B_i is at most $2^{|B_i|} \leq 2^{\varepsilon n^2}$, so we can conclude that for sufficiently large n,

$$|\mathcal{G}| \le 2^{\varepsilon n^2} \max_{i\in[t]} |\mathcal{G}_i| \le 2^{\varepsilon n^2} \sum_{B^*\subseteq B_i} |\mathcal{G}_i(B^*)| \le 2^{\varepsilon n^2} 2^{\varepsilon n^2} \max_{B^*\subseteq B_i} |\mathcal{G}_i(B^*)| \le 2^{2\varepsilon n^2 + n^2/8}.$$

2.3 Concluding remarks

It would be interesting to have similar results for K_{r+1} as well. Unfortunately, not all steps of our upper bound method work when $r > 2$. We are able to get only the following modest improvement on the trivial $2^{(1-1/r+o(1))n^2/2}$ bound: for every r there is a positive constant c_r such that the number of maximal K_{r+1}-free graphs is at most $2^{(1-1/r-c_r)n^2/2}$ for n sufficiently large. More precisely, if we let $s = 2^{\binom{r+1}{2}-1}$, then the number of maximal K_{r+1}-free graphs is at most $(s-1)^{n^2/(r(r+2))+o(n^2)}$.

A discussion with Alon and Łuczak led to the following construction that gives $2^{(1-1/r)n^2/4+o(n^2)}$ maximal K_{r+1}-free graphs: partition the vertex set $[n]$ into r equal classes, place a perfect matching into $r-1$ of them. Between the classes we have the following connection rule: between two matching edges place exactly 3 edges, and between a vertex (from the class which is an independent set) and a matching edge put exactly 1 edge.

Alon also pointed out that if the number of maximal K_r-free graphs is $2^{c_r n^2+o(n^2)}$, then c_r is monotone (though not clear if strictly monotone) increasing in r.

A similar question was raised by Cameron and Erdős in [27], where they asked how many maximal sum-free sets are contained in $[n]$. They were able to construct $2^{n/4}$ such sets. An upper bound $2^{3n/8+o(n)}$ was proved by Wolfovitz [79]. Our proof method instantly improves this upper bound to $3^{n/6+o(n)}$, as observed in [13]. Balogh-Liu-Sharifzadeh-Treglown [13] pushed the method further to prove a matching upper bound, $2^{n/4+o(n)}$. As [13] contains all the details, we omit further discussion here.

In a later work, Balogh, Liu, Petříčková, and Sharifzadeh proved that almost every maximal triangle-free graph G admits a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set, as in the construction. This will be proved in the next chapter.

Chapter 3

Typical structure of maximal triangle-free graphs

The results in this chapter are joint work with József Balogh, Hong Liu, and Maryam Sharifzadeh and appear in [12].

Recently, settling a question of Erdős, Balogh and Petříčková showed that there are at most $2^{n^2/8+o(n^2)}$ n-vertex maximal triangle-free graphs, matching the previously known lower bound. Here we characterize the typical structure of maximal triangle-free graphs. We show that almost every maximal triangle-free graph G admits a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set.

Our proof uses the Ruzsa-Szemerédi Triangle Removal Lemma, the Erdős-Simonovits Stability Theorem, and recent results of Balogh-Morris-Samotij and Saxton-Thomason on characterization of the structure of independent sets in hypergraphs. The proof also relies on a new bound on the number of maximal independent sets in triangle-free graphs with many vertex-disjoint P_3's, which is of independent interest.

3.1 Background

Given a family of combinatorial objects with certain properties, a fundamental problem in extremal combinatorics is to describe the *typical* structure of these objects. This was initiated in a seminal work of Erdős, Kleitman, and Rothschild [37] in 1976. They proved that almost all triangle-free graphs on n vertices are bipartite, that is, the proportion of n-vertex triangle-free graphs that are not bipartite goes to zero as $n \to \infty$. Since then, various extensions of this theorem have been established. The typical structure of H-free graphs has been studied when H is a large clique [8, 51], H is a fixed color-critical subgraph [65], H is a finite family of subgraphs [7], and H is an induced subgraph [9]. For sparse H-free graphs, analogous problems were examined in [15, 63]. In the context of other combinatorial objects, the typical structure of hypergraphs with a fixed forbidden subgraph is investigated for example in [16, 64]; the typical structure of intersecting families of discrete structures is studied in [11]; see also [3] for a description of the typical sum-free set in finite abelian groups.

In contrast to the family of all n-vertex triangle-free graphs, which has been well-studied, very little was known about the subfamily consisting of all those that are maximal (under graph inclusion) triangle-free. Note that the size of the family of triangle-free graphs on $[n]$ is at least $2^{n^2/4}$ (all subgraphs of a complete balanced bipartite graph), and at most $2^{n^2/4+o(n^2)}$ by the result of Erdős, Kleitman, and Rothschild from 1976. Until recently, it was not even known if the subfamily of maximal triangle-free graphs is significantly smaller. As a first step, Erdős suggested the following problem (as stated in [71]): determine or estimate the number of maximal triangle-free graphs on n vertices. The following folklore construction shows that there are at least $2^{n^2/8}$ maximal triangle-free graphs on the vertex set $[n] := \{1, \ldots, n\}$.

Lower bound construction. Assume that n is a multiple of 4. Start with a graph on a vertex set $X \cup Y$ with $|X| = |Y| = n/2$ such that X induces a perfect matching and Y is an independent set (see Figure 3.1(a)). For each pair of a matching edge x_1x_2 in X and a vertex $y \in Y$, add exactly one of the edges x_1y or x_2y. Since there are $n/4$ matching edges in X and $n/2$ vertices in Y, we obtain $2^{n^2/8}$ triangle-free graphs. These graphs may not be maximal triangle-free, but since no further edges can be added between X and Y, all of these $2^{n^2/8}$ graphs extend to distinct maximal ones.

Balogh and Petříčková [17] recently proved a matching upper bound, that the number of maximal triangle-free graphs on vertex set $[n]$ is at most $2^{n^2/8+o(n^2)}$. Now that the counting problem is resolved, one would naturally ask how do most of the maximal triangle-free graphs look, i.e. what is their typical structure. Our main result provides an answer to this question.

Theorem 3.1.1. *For almost every maximal triangle-free graph G on $[n]$, there is a vertex partition $X \cup Y$ such that $G[X]$ is a perfect matching and Y is an independent set.*

The proof of Theorem 3.1.1 consists of two parts. In the first part we show an asymptotic version of Theorem 3.1.1, which implies that almost all maximal triangle-free graphs have a structure very

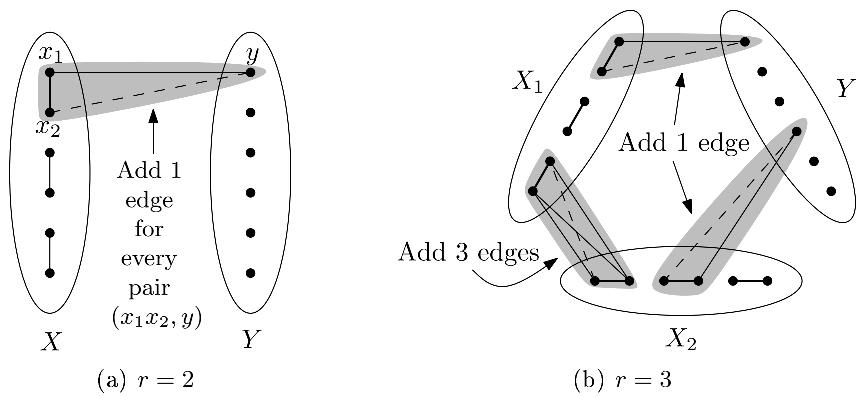


Figure 3.1: Lower bound contruction for maximal K_{r+1}-free graphs.

close to the desired one (see the beginning of Section 3.3 for an outline of the proof). In the second part, we compare directly the size of the family of "bad" maximal triangle-free graphs, i.e. those without the desired structure, with the size of the family of "good" ones (see the beginning of Section 3.4 for the idea of the proof).

It is worth mentioning that once a maximal triangle-free graph has the above partition $X \cup Y$, then there has to be exactly one edge between every matching edge of X and every vertex of Y. Thus Theorem 3.1.1 implies that almost all maximal triangle-free graphs have the same structure as the graphs in the lower bound construction above. Furthermore, our proof yields that the number of maximal triangle-free graphs without the desired structure is exponentially smaller than the number of maximal triangle-free graphs: Let $\mathcal{M}_3(n)$ denote the set of all maximal triangle-free graphs on $[n]$, and $\mathcal{G}(n)$ denote the family of graphs from $\mathcal{M}_3(n)$ that admit a vertex partition such that one part induces a perfect matching and the other is an independent set. Then there exists an absolute constant $c > 0$ such that for n sufficiently large, $|\mathcal{M}_3(n) - \mathcal{G}(n)| \leq 2^{-cn}|\mathcal{M}_3(n)|$.

It would be interesting to have similar results for $\mathcal{M}_r(n)$, the number of maximal K_r-free graphs on $[n]$. Alon pointed out that if the number of maximal K_r-free graphs is $2^{c_r n^2+o(n^2)}$, then c_r is monotone increasing in r, though not necessarily strictly monotone. For the lower bound, a discussion with Alon and Łuczak led to the following construction that gives $2^{(1-1/r+o(1))n^2/4}$ maximal K_{r+1}-free graphs: Assume that n is a multiple of $2r$. Partition the vertex set $[n]$ into r equal classes $X_1, \ldots, X_{r-1}, Y$, and place a perfect matching into each of $X_1, \ldots, X_{r-1}$ (see Figure

3.1(b)). Between the classes we have the following connection rule: between the vertices of two matching edges from different classes X_i and X_j place exactly three edges, and between a vertex in Y and a matching edge in X_i put exactly one edge. For the upper bound, by Erdős, Frankl and Rödl [36], $\mathcal{M}_{r+1}(n) \leq 2^{(1-1/r+o(1))n^2/2}$. A slightly improved bound is given in [17]: For every r there is $\varepsilon(r) > 0$ such that $|\mathcal{M}_{r+1}(n)| \leq 2^{(1-1/r-\varepsilon(r))n^2/2}$ for n sufficiently large. We suspect that the lower bound is the "correct value", i.e. that $|\mathcal{M}_{r+1}(n)| = 2^{(1-1/r+o(1))n^2/4}$.

Related problem. There is a surprising connection between the family of maximal triangle-free graphs and the family of maximal sum-free sets in $[n]$. More recently, Balogh, Liu, Sharifzadeh and Treglown [13] proved that the number of maximal sum-free sets in $[n]$ is $2^{(1+o(1))n/4}$, settling a conjecture of Cameron and Erdős. Although neither of the results imply one another, the methods in both of the papers fall in the same general framework, in which a rough structure of the family is obtained first using appropriate container lemma and removal lemma. These are Theorems 3.2.1 and 3.2.2 in this paper, and a group removal lemma of Green [43] and a granular theorem of Green and Ruzsa [44] in the sum-free case. Both problems can then be translated into bounding the number of maximal independent sets in some auxiliary link graphs. In particular, one of the tools here (Lemma 3.2.4) is also utilized in [14] to give an asymptotic of the number of maximal sum-free sets in $[n]$.

Organization. We first introduce all the tools in Section 3.2, then we prove Lemma 3.3.1, the asymptotic version of Theorem 3.1.1, in Section 3.3. Using this asymptotic result we prove Theorem 3.1.1 in Section 3.4.

Notation. For a graph G, denote by $|G|$ the number of vertices in G and by $e(G)$ the number of its edges. An n-vertex graph G is *t-close to bipartite* if G can be made bipartite by removing at most t edges. Denote by P_k the path on k vertices. Write $\mathrm{MIS}(G)$ for the number of maximal independent sets in G. The *Cartesian product* $G\square H$ of graphs G and H is a graph with vertex set $V(G)\times V(H)$ such that two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and $u'v' \in E(H)$, or $u' = v'$ and $uv \in E(G)$. For a fixed graph G, let $N(v)$ be the set of neighbors of a vertex v in G, and let $d(v) := |N_G(v)|$ and $\Gamma(v) := N(v) \cup \{v\}$. For $v \in V(G)$ and $X \subseteq V(G)$, denote by $N_X(v)$ the set of all neighbors of v in X (i.e. $N_X(v) = N(v) \cap X$), and let $d_X(v) := |N_X(v)|$. Denote by $\Delta(X)$ the maximum degree of the induced subgraph $G[X]$. For two disjoint vertex sets $X, Y \subseteq V$, the edges between X and Y are called $[X,Y]$-*edges* and the number of $[X,Y]$-edges is denoted $e(X,Y)$. A *(vertex) cut* $X \cup Y$ is a partition of the vertex set V into two disjoint subsets X and Y, and $e(X,Y)$ is the *size of the cut* $X \cup Y$. A vertex cut $X \cup Y$ is a *max-cut* if $e(X,Y)$ is not smaller than the size of any other cut. Given a vertex cut $X \cup Y$, the *inner edges* (*of* $X \cup Y$) are the edges in $G[X]$ and $G[Y]$, the *inner neighbors* of a vertex v are its neighbors in the same partite set as v (i.e. $N_X(v)$ if $v \in X$), and the *inner degree* of a vertex is the number of its inner neighbors. We say that a family $\mathcal{F}$ of maximal triangle-free graphs is *negligible* if there exists an absolute constant $C > 0$ such that $|\mathcal{F}| < 2^{-Cn}|\mathcal{M}_3(n)|$.

3.2 Tools

Our first tool is a corollary of recent powerful counting theorems of Balogh-Morris-Samotij [10, Theorem 2.2.], and Saxton-Thomason [69].

Theorem 3.2.1. *For all $\delta > 0$ there is $c = c(\delta) > 0$ such that there is a family $\mathcal{F}$ of at most $2^{c \cdot \log n \cdot n^{3/2}}$ graphs on $[n]$, each containing at most δn^3 triangles, such that for every triangle-free graph G on $[n]$ there is an $F \in \mathcal{F}$ such that $G \subseteq F$, where n is sufficiently large.*

The graphs in $\mathcal{F}$ in the above theorem will be referred to as *containers*. A weaker version of Theorem 3.2.1, which can be concluded from the Szemerédi Regularity Lemma, could be used instead of Theorem 3.2.1 here. The only difference is that the upper bound on the size of $\mathcal{F}$ is $2^{o(n^2)}$.

We need two well-known results. The first is the Ruzsa-Szemerédi Triangle Removal lemma [68] and the second is the Erdős-Simonovits Stability Theorem [38]:

Theorem 3.2.2. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $n_0(\varepsilon) > 0$ such that any graph G on $n > n_0(\varepsilon)$ vertices with at most δn^3 triangles can be made triangle-free by removing at most εn^2 edges.*

Theorem 3.2.3. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $n_0(\varepsilon) > 0$ such that every triangle-free graph G on $n > n_0(\varepsilon)$ vertices with at least $\frac{n^2}{4} - \delta n^2$ edges can be made bipartite by removing* *at most εn^2 edges.*

We also need the following lemma, which is an extension of results of Moon-Moser [58] and Hujter-Tuza [45].

Lemma 3.2.4. *Let G be an n-vertex triangle-free graph. If G contains at least k vertex-disjoint P_3's, then*

$$\mathrm{MIS}(G) \leq 2^{\frac{n}{2} - \frac{k}{25}}. \tag{3.2.1}$$

Proof. The proof is by induction on n. The base case of the induction is $n = 1$ with $k = 0$, for which $\mathrm{MIS}(G) = 1 \leq 2^{\frac{1}{2} - \frac{0}{25}}$.

For the inductive step, let G be a triangle-free graph on $n \geq 2$ vertices with k vertex-disjoint P_3's, and let v be any vertex in G. Observe that $\mathrm{MIS}(G - \Gamma(v))$ is the number of maximal independent sets containing v, and that $\mathrm{MIS}(G - \{v\})$ bounds from above the number of maximal independent sets not containing v. Therefore,

$$\mathrm{MIS}(G) \leq \mathrm{MIS}(G - \{v\}) + \mathrm{MIS}(G - \Gamma(v)).$$

If G has k vertex-disjoint P_3's, then $G - \Gamma(v)$ has at least $k - d(v)$ vertex-disjoint P_3's, and so, by the induction hypothesis,

$$\mathrm{MIS}(G) \leq 2^{\frac{n-1}{2} - \frac{k-1}{25}} + 2^{\frac{n-(d(v)+1)}{2} - \frac{k-d(v)}{25}} \leq 2^{\frac{n}{2} - \frac{k}{25}} \left(2^{-\frac{1}{2} + \frac{1}{25}} + 2^{-\frac{d(v)+1}{2} + \frac{d(v)}{25}} \right).$$

The function $f(x) = 2^{-\frac{1}{2} + \frac{1}{25}} + 2^{-\frac{x+1}{2} + \frac{x}{25}}$ is a decreasing function with $f(3) \approx 0.9987 < 1$. So, if there exists a vertex of degree at least 3 in G, then we have $\mathrm{MIS}(G) \leq 2^{\frac{n}{2} - \frac{k}{25}}$ as desired.

It remains to verify (3.2.1) for graphs with $\Delta(G) \leq 2$. Observe that we can assume that G is connected. Indeed, if $G_1, \ldots, G_l$ are maximal components of G, and each of G_i has n_i vertices and k_i vertex-disjoint P_3's, then

$$\mathrm{MIS}(G) = \prod_i \mathrm{MIS}(G_i) \leq \prod_i 2^{\frac{n_i}{2} - \frac{k_i}{25}} = 2^{\sum_i \frac{n_i}{2} - \sum_i \frac{k_i}{25}} = 2^{\frac{n}{2} - \frac{k}{25}}.$$

Every connected graph with $\Delta(G) \le 2$ and $n \ge 2$ vertices is either a path or a cycle. Suppose first that G is a path P_n. We have $\mathrm{MIS}(P_2) = 2 \le 2^{\frac{2}{2}-\frac{0}{25}}$, $\mathrm{MIS}(P_3) = 2 \le 2^{\frac{3}{2}-\frac{1}{25}}$. By Füredi [42, Example 1.1], $\mathrm{MIS}(P_n) = \mathrm{MIS}(P_{n-2}) + \mathrm{MIS}(P_{n-3})$ for all $n \ge 4$. By the induction hypothesis thus

$$\mathrm{MIS}(P_n) \le 2^{\frac{n-2}{2}-\frac{k-1}{25}} + 2^{\frac{n-3}{2}-\frac{k-1}{25}} \le 2^{\frac{n}{2}-\frac{k}{25}}\left(2^{-1+\frac{1}{25}} + 2^{-\frac{3}{2}+\frac{1}{25}}\right) \le 2^{\frac{n}{2}-\frac{k}{25}}.$$

Let now G be a cycle C_n. We have $\mathrm{MIS}(C_4) = 2 \le 2^{4/2-1/25}$ and $\mathrm{MIS}(C_5) = 5 \le 2^{5/2-1/25}$. By Füredi [42, Example 1.2], $\mathrm{MIS}(C_n) = \mathrm{MIS}(C_{n-2}) + \mathrm{MIS}(C_{n-3})$ for all $n \ge 6$. Therefore, by the induction hypothesis,

$$\mathrm{MIS}(C_n) \le 2^{\frac{n-2}{2}-\frac{k-1}{25}} + 2^{\frac{n-3}{2}-\frac{k-1}{25}} \le 2^{\frac{n}{2}-\frac{k}{25}}.$$

□

Remark 3.2.5. *A disjoint union of C_5's and a matching shows that the constant c for which $\mathrm{MIS}(G) \le 2^{\frac{n}{2}-\frac{k}{c}}$ in Lemma 3.2.4 cannot be smaller than* 5.6.

3.3 Asymptotic result

In this section we prove an asymptotic version of Theorem 3.1.1:

Lemma 3.3.1. *Fix any $\gamma > 0$. Almost every maximal triangle-free graph G on the vertex set $[n]$ satisfies the following: for any max-cut $V(G) = X \cup Y$, there exist $X' \subseteq X$ and $Y' \subseteq Y$ such that*

(i) $|X'| \le \gamma n$ and $G[X - X']$ is an induced perfect matching, and

(ii) $|Y'| \le \gamma n$ and $Y - Y'$ is an independent set.

The outline of the proof is as follows. We observe that every maximal triangle-free graph G on $[n]$ can be built in the following three steps.

(S1) Choose a max-cut $X \cup Y$ for G.

(S2) Choose triangle-free graphs S and T on the vertex sets X and Y, respectively.

(S3) Extend $S \cup T$ to a maximal triangle-free graph by adding edges between X and Y.

We give an upper bound on the number of choices for each step. First, there are at most 2^n ways to fix a max-cut $X \cup Y$ in (S1). For (S2), we show (Lemma 3.3.5) that almost all maximal triangle-free graphs on $[n]$ are $o(n^2)$-close to bipartite, which implies that the number of choices for most of these graphs in (S2) is at most $2^{o(n^2)}$. For fixed X, Y, S, T, we bound, using Claim 3.3.4, the number of choices in (S3) by the number of maximal independent sets in some auxiliary link graph L. This enables us to use Lemma 3.2.4 to force the desired structure on S and T.

Definition 3.3.2 (**Link graph**). Given edge-disjoint graphs S and A on $[n]$, define the *link graph* $L := L_S[A]$ of S on A as follows:

$$V(L) := E(A),$$
$$E(L) := \{a_1a_2 : \exists s \in E(S) \text{ such that } \{a_1, a_2, s\} \text{ forms a triangle}\}.$$

Claim 3.3.3. *Let S and A be two edge-disjoint graphs on $[n]$. If S is triangle-free, then $L_S[A]$ is triangle-free.*

Proof. Indeed, otherwise there exist $a_1, a_2, a_3 \in E(A)$ and $s_1, s_2, s_3 \in E(S)$ such that the 3-sets $\{a_1, a_2, s_1\}$, $\{a_2, a_3, s_2\}$, and $\{a_1, a_3, s_3\}$ span triangles. Since S and A are edge-disjoint, the edges a_1, a_2, a_3 share a common endpoint, and $\{s_1, s_2, s_3\}$ spans a triangle. This is a contradiction since S is triangle-free. □

Claim 3.3.4. *Let S and A be two edge-disjoint triangle-free graphs on $[n]$ such that there is no triangle $\{a, s_1, s_2\}$ in $S \cup A$ with $a \in E(A)$ and $s_1, s_2 \in E(S)$. Then the number of maximal triangle-free subgraphs of $S \cup A$ containing S is exactly* $\mathrm{MIS}(L_S[A])$.

Proof. First observe that by our assumption, every triangle in $S \cup A$ consists of two edges from $E(A)$ and one edge from $E(S)$. It follows that for a subgraph $A' \subseteq A$, the graph $G = S \cup A'$ is triangle-free if and only if $E(A')$ spans an independent set in $L := L_S[A]$.

A triangle-free graph $G = S \cup A'$ is not maximal triangle-free subgraph of $S \cup A$ if and only if there exists an edge $a \in E(A) - (A')$ such that for any two edges $a' \in E(A')$ and $s \in E(S)$, $\{a, a', s\}$ does not form a triangle. By definition of a link graph $L_S[A]$, this is exactly when there exists a vertex $a \in E(A) - (A')$ such that the set $E(A') \cup \{a\}$ is an independent set, i.e. when $E(A')$ is not maximal independent set in $L_S[A]$. $\square$

We fix the following parameters that will be used throughout the rest of the paper. Let $\gamma, \beta, \varepsilon, \varepsilon' > 0$ be sufficiently small constants satisfying the following hierarchy:

$$\varepsilon' \ll \delta_{2.3}(\varepsilon) \ll \varepsilon \ll \beta \ll \delta_{2.3}(\gamma^3) \ll \gamma \ll 1, \tag{3.3.1}$$

where $\delta_{2.3}(x) > 0$ is the constant returned from Theorem 3.2.3 with input x. The notation $x \ll y$ above means that x is a sufficiently small function of y to satisfy some inequalities in the proof. In the following proof, $\delta_{2.2}(x)$ is the constant returned from Theorem 3.2.2 with input x, and in the rest of the paper, we shall always assume that n is sufficiently large, even when this is not explicitly stated.

Lemma 3.3.5. *Almost all maximal triangle-free graphs on $[n]$ are $2\varepsilon n^2$-close to bipartite.*

Proof. Let $\mathcal{F}$ be the family of graphs obtained from Theorem 3.2.1 using $\delta_{2.2}(\varepsilon')$. Then every triangle-free graph on $[n]$ is a subgraph of some container $F \in \mathcal{F}$.

We first show that the family of maximal triangle-free graphs in small containers is negligible. Consider a container $F \in \mathcal{F}$ with $e(F) \leq n^2/4 - 6\varepsilon' n^2$. Since F contains at most $\delta_{2.2}(\varepsilon')n^3$ triangles, by Theorem 3.2.2, we can find A and B, subgraphs of F, such that $F = A \cup B$, where A is triangle-free, and $e(B) \leq \varepsilon' n^2$. For each $F \in \mathcal{F}$, fix such a pair (A, B). Then every maximal triangle-free graph in F can be built in two steps:

(i) Choose a triangle-free $S \subseteq B$;

(ii) Extend S in A to a maximal triangle-free graph.

The number of choices in (i) is at most $2^{e(B)} \leq 2^{\varepsilon' n^2}$. Observe that any edge $A \in E(A)$ that is in a triangle containing two edges from S cannot be added in step (ii). Therefore we remove all such edges from A and call the resulting graph A'. Let $L := L_S[A']$ be the link graph of S on A'. By Claim 3.3.3, L is triangle-free. Claim 3.3.4 implies that the number of maximal triangle-free graphs in $S \cup A$ containing S (i.e. the number of extensions in (ii)) is at most $\mathrm{MIS}(L)$. Thus, by

Lemma 3.2.4,

$$\mathrm{MIS}(L) \leq 2^{|A'|/2} \leq 2^{n^2/8-3\varepsilon' n^2}.$$

Therefore, the number of maximal triangle-free graphs in small containers is at most

$$|\mathcal{F}| \cdot 2^{\varepsilon' n^2} \cdot 2^{n^2/8-3\varepsilon' n^2} \leq 2^{n^2/8-\varepsilon' n^2}.$$

From now on, we may consider only maximal triangle-free graphs contained in containers of size at least $n^2/4 - 6\varepsilon' n^2$. Let F be any large container. Recall that by Theorem 3.2.2, $F = A \cup B$, where A is triangle-free with $e(A) \geq n^2/4 - 7\varepsilon' n^2$ and $e(B) \leq \varepsilon' n^2$. Since $\varepsilon' \ll \delta_{2.3}(\varepsilon)$, by Theorem 3.2.3, A can be made bipartite by removing at most εn^2 edges. Since $\varepsilon' \ll \varepsilon$, F can be made bipartite by removing at most $(\varepsilon' + \varepsilon)n^2 \leq 2\varepsilon n^2$ edges. Therefore, every maximal triangle-free graphs contained in F is $2\varepsilon n^2$-close to bipartite. □

Fix X, Y, S, T as in steps (S1) and (S2). Let A be the complete bipartite graph with parts X and Y. By Claim 3.3.4, the number of ways to extend $S \cup T$ in (S3) is at most $\mathrm{MIS}(L_{S\cup T}[A])$. The number of ways to fix X and Y is at most 2^n, and by Lemma 3.3.5, the number of ways to fix S and T is at most $\binom{n^2}{2\varepsilon n^2}$. It follows that if $\mathrm{MIS}(L_{S\cup T}[A])$ is smaller than $2^{n^2/8-cn^2}$ for some $c \gg \varepsilon$, then the family of maximal triangle-free graphs with such (X, Y, S, T) is negligible.

Claim 3.3.6. *$L_{S\cup T}[A] = S \square T$.*

Proof. Note that $V(L_{S\cup T}[A]) = E(A) = \{(x, y) : x \in X, y \in Y\} = V(S \square T)$. Using the definition of the Cartesian product, (x, y) and (x', y') are adjacent in $S \square T$ if and only if $x = x'$ and $\{y, y'\} \in E(T)$, or $y = y'$ and $\{x, x'\} \in E(S)$, i.e. if and only if $\{x = x', y, y'\}$ or $\{x, x', y = y'\}$ form a triangle in $S \cup A$. But by the definition of $L_{S\cup T}[A]$, this is exactly when (x, y) and (x', y') are adjacent in $L_{S\cup T}[A]$. □

Claim 3.3.6 allows us to rule out certain structures of S and T since, by Lemma 3.2.4, if $S \square T$ has many vertex disjoint P_3's then the number of maximal-triangle free graphs with $S = G[X]$ and $T = G[Y]$ is much smaller than $2^{n^2/8}$.

Claim 3.3.7. *For almost all maximal triangle-free n-vertex graphs G with a max-cut $X \cup Y$,* *(i)* $|X|, |Y| \geq n/2 - \beta n$, *and*

(ii) $\Delta(X), \Delta(Y) \leq \beta n$.

Proof. Let G be a maximal triangle-free graph with a max-cut $X \cup Y$. By Lemma 3.3.5, almost all maximal triangle-free graphs are $2\varepsilon n^2$-close to bipartite, which implies that the number of choices for $G[X]$ and $G[Y]$ is at most $\binom{n^2}{2\varepsilon n^2}$. Denote by A the complete bipartite graph with partite sets X and Y.

For (i), suppose that $|X| \leq n/2 - \beta n$. Then $|X||Y| \leq n^2/4 - \beta^2 n^2$, and for any fixed S on X and T on Y, Lemma 3.2.4 implies $\mathrm{MIS}(L_{S\cup T}[A]) \leq 2^{n^2/8 - \beta^2 n^2/2}$. Since $\beta \gg \varepsilon$, it follows from the discussion before Claim 3.3.6 that the family of maximal triangle-free graphs with such max-cut $X \cup Y$ is negligible.

For (ii), suppose that G has a vertex $x \in X$ of inner degree at least βn. Since $X \cup Y$ is a max-cut, $|N_Y(x)| \geq |N_X(x)| \geq \beta n$. Since G is triangle-free, there is no edge in between $N_X(x)$ and $N_Y(x)$. Let $A' \subseteq A$ be a graph formed by deleting all edges between $N_X(x)$ and $N_Y(x)$ from A. Define a link graph $L' := L_{S\cup T}[A']$ of $S \cup T$ on A'. In this case, the number of choices for (S3) is at most $\mathrm{MIS}(L')$. Since L' is triangle-free (Claim 3.3.3) and $|L'| = e(A') \leq |X||Y| - |N_X(x)||N_Y(x)| \leq \frac{n^2}{4} - \beta^2 n^2$, it follows from Lemma 3.2.4 that

$$\mathrm{MIS}(L') \leq 2^{|L'|/2} \leq 2^{n^2/8 - \beta^2 n^2/2}.$$

□

Proof of Lemma 3.3.1. First, we show that for almost every maximal triangle-free graph G on $[n]$ with max-cut $X \cup Y$ and with $G[X] = S$ and $G[Y] = T$, there are very few vertex-disjoint P_3's in $S \cup T$. Suppose that there exist βn vertex-disjoint P_3's in S or in T, say in S. Since $L_{S\cup T}[A] = S \Box T$ by Claim 3.3.6, and for each of the βn vertex-disjoint P_3's in S we obtain $|T|$ vertex-disjoint P_3's in $S \Box T$, the number of vertex-disjoint P_3's in $L_{S\cup T}[A]$ is at least $\beta n|T| = \beta n|Y|$. By Claim 3.3.7(i), $\beta n|Y| \geq \beta n(n/2 - \beta n) \geq \beta n^2/3$. Then by Lemma 3.2.4,

$$\mathrm{MIS}(L_{S\cup T}[A]) \leq 2^{|S\Box T|/2 - \beta n^2/75} \leq 2^{n^2/8 - \beta n^2/75}.$$

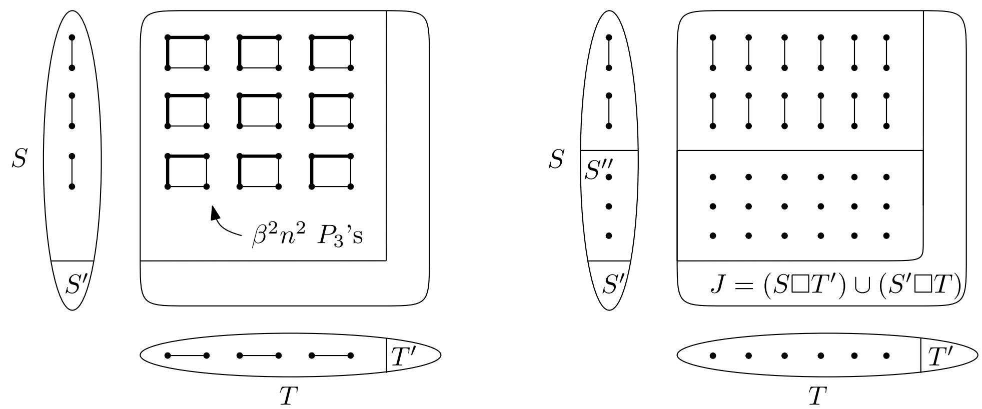


Figure 3.2: Forbidden structures in S and T.

Since $\beta \gg \varepsilon$, the family of maximal triangle-free graphs with such (X, Y, S, T) is negligible. Hence, for almost every maximal triangle-free graph G with some (X, Y, S, T), we can find some induced subgraphs $S' \subseteq S$ and $T' \subseteq T$ with $|S'| \leq 3\beta n$ and $|T'| \leq 3\beta n$ such that both $S - S'$ and $T - T'$ are P_3-free. This implies that each of $S - S'$ and $T - T'$ is a union of a matching and an independent set.

Next, we show that at most one of the graphs S and T can have a large matching. Suppose both S and T have a matching of size at least βn, then there are at least $\beta^2 n^2$ vertex-disjoint C_4's in $S\Box T$, each of which contains a copy of P_3 (see Figure 3.2(a)). It follows that the family of such graphs is negligible since $\mathrm{MIS}(L_{S\cup T}[A]) \leq 2^{n^2/8-\beta^2 n^2/25}$ and $\beta \gg \varepsilon$. Hence, we can assume that all but $2\beta n$ vertices in T form an independent set. Redefine T' so that $|T'| \leq 2\beta n$ and $V(T - T')$ is an independent set.

Lastly, we show that there are very few isolated vertices in the graph $S - S'$. Suppose that there are $\gamma n/2$ isolated vertices in $S - S'$, spanning a subgraph S'' of S. We count $\mathrm{MIS}(S\Box T)$ as follows. Let $J := (S\Box T') \cup (S'\Box T)$ and $L' := S\Box T - J$. Every maximal independent set in $S\Box T$ can be built by

(i) choosing an independent set in J, and

(ii) extending it to a maximal independent set in L'.

Since $|J| \leq |S'||T| + |T'||S| \leq 3\beta n \cdot n + 2\beta n \cdot n = 5\beta n^2$, there are at most $2^{|J|} = 2^{5\beta n^2}$ choices for (i). Note that L' consists of isolated vertices from $S''\Box(T - T')$ and an induced matching from

$(S-S'-S'')\square(T-T')$ (see Figure 3.2(b)). Thus the number of extensions in (ii) is at most $\text{MIS}((S-S'-S'')\square(T-T'))$. The graph $(S-S'-S'')\square(T-T')$ is a perfect matching with at most

$$\frac{1}{2}|S-S''||T| \le \frac{1}{2}\left(|S|-\frac{\gamma n}{2}\right)(n-|S|) \le \frac{1}{2}\left(\frac{n}{2}-\frac{\gamma n}{4}\right)^2 \le \frac{n^2}{8}-\frac{\gamma n^2}{16}$$

edges, and so choosing one vertex for each matching edge gives at most $2^{n^2/8-\gamma n^2/16}$ maximal independent sets. Since $\beta \ll \gamma$, it follows that $\text{MIS}(S\square T) \le 2^{5\beta n^2} \cdot 2^{n^2/8-\gamma n^2/16} \le 2^{n^2/8-\gamma n^2/17}$. Thus, such family of maximal triangle-free graphs is negligible, and we may assume that $|S''| \le \gamma n/2$.

The statement of Lemma 3.3.1 follows by setting $X' := V(S' \cup S'')$ and $Y' := V(T')$. Indeed, $|X'| \le 3\beta n + \gamma n/2 \le \gamma n$, $|Y'| \le 2\beta n \le \gamma n$, $G[X-X'] = S-S'-S''$ is a perfect matching, and $Y - Y' = V(T) - V(T')$ is an independent set. $\square$

3.4 Proof of Main Theorem

In this section, we will prove Theorem 3.1.1. Recall the hierarchy of parameters fixed in Section 3.3:

$$\varepsilon' \ll \delta_{2.3}(\varepsilon) \ll \varepsilon \ll \beta \ll \delta_{2.3}(\gamma^3) \ll \gamma \ll 1, \tag{3.4.1}$$

We will in fact show that there are exponentially fewer "bad" graphs, i.e. maximal triangle-free graphs without the desired structure. We do so by first grouping graphs by some triple (X, Y, M) (see the definitions below). Then we compare the number of "bad" graphs to the number of "good" graphs within each group by showing that there are not many "bad" ones (Lemmas 3.4.2 and 3.4.3), while there are many "good" ones (Lemma 3.4.5). There might be an overcounting issue due to overlaps among groups. This is taken care of by Lemma 3.4.4.

Definition 3.4.1. Fix a vertex partition $V = X \cup Y$, a perfect matching M on the vertex set X (in case $|X|$ is odd, M is an almost perfect matching covering all but one vertex of X), and non-negative integers r, s and t.

1. Denote by $\mathcal{B}(X, Y, M, s, t)$ the class of maximal triangle-free graphs G with max-cut $X \cup Y$ satisfying the following three conditions:

(i) The subgraph $G[X]$ has a maximum matching $M' \subseteq M$ covering all but at most γn vertices in X;

(ii) The size of a largest family of vertex-disjoint P_3's in $S := G[X]$ is s;

(iii) The size of a maximum matching in $T := G[Y]$ is t.

2. Denote by $\mathcal{B}(X, Y, M, r) \subseteq \mathcal{B}(X, Y, M, 0, 0)$ the subclass consisting of all graphs in $\mathcal{B}(X, Y, M, 0, 0)$ with exactly r isolated vertices in $G[X]$.

3. When $|X|$ is even, denote by $\mathcal{G}(X, Y, M)$ the class of all maximal triangle-free graphs G with max-cut $X \cup Y$, $G[X] = M$, and Y an independent set.

4. When $|X|$ is even, denote by $\mathcal{H}(X, Y, M)$ the class of maximal triangle-free graphs G that are constructed as follows:

(P1) Add M to X;

(P2) For every edge $x_1x_2 \in M$ and every vertex $y \in Y$, add either the edge x_1y or x_2y;

(P3) Extend each of the $2^{|X||Y|/2}$ resulting graphs to a maximal triangle-free graph by adding edges in X and/or Y.

By Lemmas 3.3.1, 3.3.5 and Claim 3.3.7, throughout the rest of the proof, we may only consider maximal triangle-free graphs in $\bigcup_{X,Y,M,s,t} \mathcal{B}(X, Y, M, s, t)$ that are βn^2-close to bipartite, $|X|, |Y| \geq n/2 - \beta n$ and $\Delta(X), \Delta(Y) \leq \beta n$. We may further assume from the proof of Lemma 3.3.1 that $s, t \leq \beta n$.

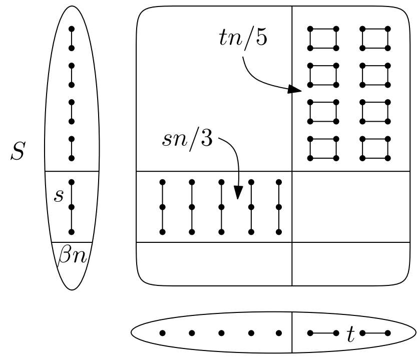
Notice that graphs from $\mathcal{G}(X, Y, M) = \mathcal{B}(X, Y, M, 0)$ are precisely those with the desired structure. We will show that the number of graphs without the desired structure is exponentially smaller. The set of "bad" graphs consists of the following two types:

(i) when $|X|$ is even, $\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t) - \mathcal{B}(X, Y, M, 0)$;

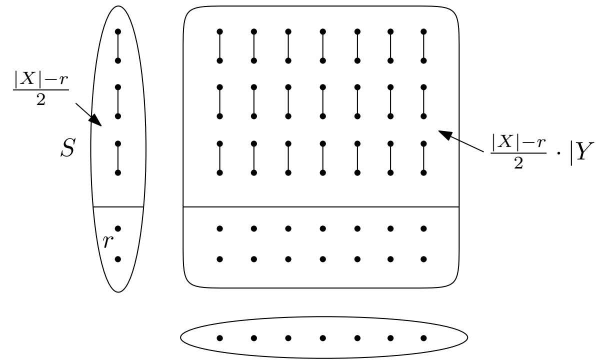
(ii) when $|X|$ is odd, $\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t)$.

Fix an arbitrary choice of (X, Y, M). For simplicity, let $\mathcal{B}(s, t) := \mathcal{B}(X, Y, M, s, t)$ and $\mathcal{B}(r) := \mathcal{B}(X, Y, M, r)$. Let A be the complete bipartite graph with parts X and Y.

Lemma 3.4.2. *If $s + t \geq 1$, then $|\mathcal{B}(s, t)| \leq 2^{|X||Y|/2 - n/200}$.*



(a) The number of vertex-disjoint P_3's in $S\Box T$ is at least $sn/3 + tn/5$ (Lemma 3.4.2)



(b) $\mathrm{MIS}(S\Box T) \leq 2^{(|X|-r)|Y|/2}$ if $s = t = 0$ and X has r isolated vertices (Lemma 3.4.3).

Figure 3.3: Forbidden structures in S and T.

Proof. Let s and t be two non-negative integers, at least one of which is nonzero. We first bound the number of ways to choose S and T, i.e. the number of ways to add inner edges. The number of ways to choose the vertex set of the s vertex-disjoint P_3's in S and the t matching edges in T is at most $\binom{n}{3s}\binom{n}{2t}$. Recall that by definition of $\mathcal{B}(s,t)$, the maximum matching $M' \subseteq M$ covers all but at most γn vertices of X. So the number of ways to choose the independent vertices in X is at most $\binom{n}{\gamma n}$. Since $\Delta(X), \Delta(Y) \leq \beta n$, each of the $3s+2t$ chosen vertices has inner degree at most βn. Therefore, the number of ways to choose their inner neighbors is at most

$$\binom{n}{\beta n}^{3s+2t} \leq \left(\left(\frac{en}{\beta n}\right)^{\beta n}\right)^{3s+2t} \leq 2^{\beta \log(e/\beta)\cdot(3s+2t)n}.$$

The number of ways to add the $[X,Y]$-edges is $\mathrm{MIS}(L_{S\cup T}(A))$. We claim that the link graph $L := L_{S\cup T}(A) = S\Box T$ has at least $(s+t)n/5$ vertex-disjoint P_3's. Indeed, recall that $|S| = |T| \geq n/2 - \beta n$ and $s, t \leq \beta n$, thus in $S\Box T$ (see Figure 3.3(a)), we have at least $s(|T| - 2t) \geq sn/3$ vertex-disjoint P_3's coming from s vertex-disjoint P_3's in S and at least $\frac{1}{2}(|S| - \beta n - 3s)\cdot t \geq tn/5$ vertex-disjoint P_3's coming from the Cartesian product of a matching in S and a matching in T. So by Lemma 3.2.4,

$$\mathrm{MIS}(L) \leq 2^{|X||Y|/2-(s+t)n/125}.$$

Since $s+t \geq 1$ and γ and β are sufficiently small,

$$|\mathcal{B}(s,t)| \leq \binom{n}{3s}\binom{n}{2t}\binom{n}{\gamma n} \cdot 2^{\beta \log(e/\beta)\cdot(3s+2t)n} \cdot 2^{|X||Y|/2-(s+t)n/125} \leq 2^{|X||Y|/2-n/200}.$$

$\square$

Lemma 3.4.3. *If $s=t=0$ and $r \in \mathbb{Z}^+$, then $|\mathcal{B}(r)| \leq 2^{|X||Y|/2-n/6}$.*

Proof. By the definition of $\mathcal{B}(r)$, X consists of r isolated vertices and a matching of size $(|X|-r)/2$, and Y is an independent set. Hence the graph $L_{S\cup T}(A) = S\square T$ consists of a matching of size $(|X|-r)|Y|/2$ and isolated vertices (see Figure 3.3(b)). There are at most $\binom{n}{r}$ ways to pick the isolated vertices in X and at most $\mathrm{MIS}(L_{S\cup T}(A))$ ways to choose the $[X,Y]$-edges. Recall that $|Y| \geq n/2 - \beta n$. Thus we have

$$|\mathcal{B}(r)| \leq \binom{n}{r} \cdot 2^{(|X|-r)|Y|/2} \leq 2^{|X||Y|/2+r\log n - rn/5} \leq 2^{|X||Y|/2-rn/6} \leq 2^{|X||Y|/2-n/6}.$$

$\square$

Case 1: $|X|$ is even.

Lemma 3.4.4. *A maximal triangle-free graph G on $[n]$ is in at most n^2 different classes $\mathcal{G}(X,Y,M)$.*

Proof. Let $G \in \mathcal{G}(X,Y,M)$. Recall that $G[X]=M$ and Y is an independent set. Thus G can be in a different class $\mathcal{G}(X',Y',M')$ if and only if $X' \neq X$, $Y' \neq Y$ and $M' \neq M$. Since $M' \neq M$ and Y is an independent set, there exists an edge xy in M' with $x \in X$ and $y \in Y$. There are at most n^2 ways to choose such an edge. We claim that once we pick the edge $xy \in M'$, the sets X' and Y' (and thus also $M' = G[X']$) are already decided. Recall that since G is a maximal triangle-free graph, every vertex in Y is adjacent to exactly one vertex from each edge in M.

Observe that the neighbor x^* of x in X has to be in Y' since otherwise there would be a path x^*xy in $G[X']$ (see Figure 3.4). Let $vv^* \in M - \{xx^*\}$ such that $vy \in E(G)$. Then $v \in Y'$ since otherwise there would be a path vyx in X'. The set Y' is independent, and so $v^* \in X'$. It remains to decide whether $w \in X'$ or $w \in Y'$ for every vertex $w \in Y - \{y\}$. If $xw \in E(G)$, then $w \in Y'$

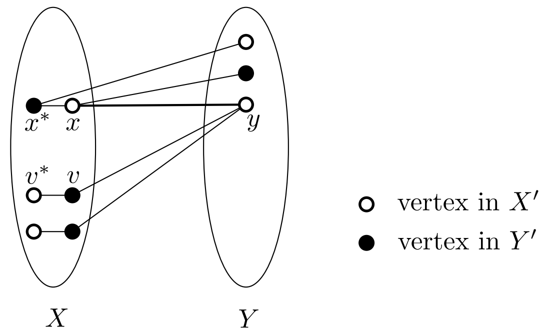


Figure 3.4: (X', Y', M') is uniquely determined after choosing $xy \in M'$ (Lemma 3.4.4).

since otherwise we would have a path wxy in $G[X']$. Otherwise $x^*w \in E(G)$ and so $w \in X'$ since otherwise there would be an edge wx^* in $G[Y']$. $\square$

By Lemma 3.4.4, it is sufficient to show that for any choice of (X, Y, M) with $|X|$ even,

$$\frac{|\bigcup_{s,t} \mathcal{B}(X,Y,M,s,t) - \mathcal{B}(X,Y,M,0)|}{|\mathcal{G}(X,Y,M)|} \leq 2^{-n/300}. \tag{3.4.2}$$

For simplicity, let $\mathcal{G} := \mathcal{G}(X, Y, M)$ and $\mathcal{H} := \mathcal{H}(X, Y, M)$.

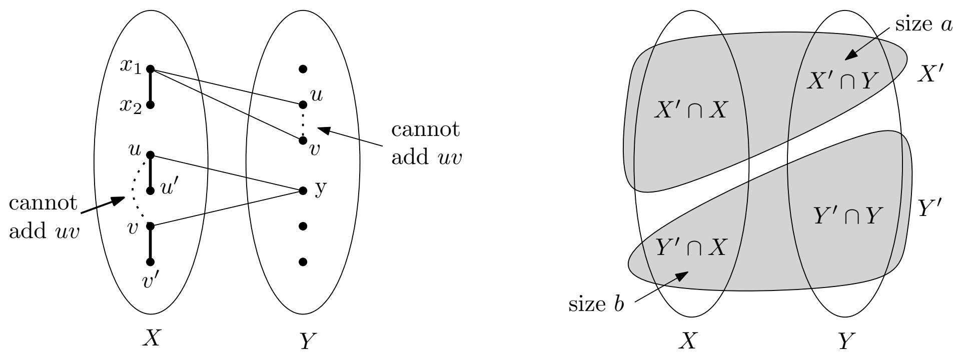
Lemma 3.4.5. *We have* $|\mathcal{G}| \geq (1 + o(1))2^{|X||Y|/2}$.

Proof. Recall that $|X|, |Y| \geq n/2 - \beta n$, and therefore $|\mathcal{H}| = 2^{|X||Y|/2} \gg 2^{n^2/8 - \beta n^2}$. Running the same proof as Lemma 3.3.5 (start the proof by invoking Theorem 3.2.1 with $\delta_{2.2}(\beta)$, replace ε' by β and ε by γ^3) implies that almost all graphs in $\mathcal{H}$ are $2\gamma^3 n^2$-close to bipartite. Let $\mathcal{H}' \subseteq \mathcal{H}$ be the subfamily consisting of all those that are $2\gamma^3 n^2$-close to bipartite. Then it is sufficient to show $|\mathcal{H}' - \mathcal{G}| = o(2^{|X||Y|/2})$. There are two types of graphs in $\mathcal{H}' - \mathcal{G}$:

(i) $\mathcal{H}_1$: those that are not maximal after (P2),

(ii) $\mathcal{H}_2$: those that are maximal after (P2), but $X \cup Y$ is not one of its max-cut.

We first bound the number of graphs in $\mathcal{H}_1$. For any graph $G \in \mathcal{H}_1$, some inner edges were added in (P3). Suppose that $[X, Y]$-edges added in (P2) were chosen randomly (each of x_1y and x_2y with probability 1/2). Clearly, uv can be added in (P3) if and only if u and v have no common neighbor. Consider the case when $u, v \in X$ and let uu', vv' be the corresponding edges in M (see Figure 3.5(a)). Every $y \in Y$ is adjacent to exactly one of u, u' and exactly one of v, v'. Thus the



(a) Two examples of uv that cannot be added when forming a graph from $\mathcal{H}_1$.

(b) Bounding $|\mathcal{H}_2|$: the probability that $e(X', Y') > e(X, Y)$ is $o(1)$.

Figure 3.5: Bounding the number of graphs in $\mathcal{H}_1$ and $\mathcal{H}_2$ in Lemma 3.4.5.

probability that y is a common neighbor of u and v is $1/4$, which implies that uv can be added with probability $(3/4)^{|Y|}$. Let now $u, v \in Y$. Then u and v have no common neighbor if and only if for every $x_1x_2 \in M$, u and v chose different neighbors among x_1 and x_2. So in this case we can add u, v with probability $(1/2)^{|X|/2}$. Summing over all possible outcomes of (P2) and all possible choices for uv implies

$$|\mathcal{H}_1| \leq 2^{|X||Y|/2} \cdot \binom{n}{2} \cdot \left(\left(\frac{1}{2}\right)^{|X|/2} + \left(\frac{3}{4}\right)^{|Y|} \right) \ll 2^{|X||Y|/2 - n/5}.$$

We can bound $|\mathcal{H}_2|$ in a similar way. It suffices to show that if the $[X, Y]$-edges added in (P2) were chosen uniformly at random, then the probability that $X \cup Y$ is not a max-cut is $o(1)$. Let $X' \cup Y'$ be a different vertex cut, where we may assume that $|X' \cap X| \geq |Y' \cap X|$ (see Figure 3.5(b)). Then $|X' \cap X| \geq |X|/2 > n/5$. Let $M_{X',Y'}$ be the event that $X' \cup Y'$ is a cut greater than $X \cup Y$ and let $a := |X' \cap Y|$ and $b := |Y' \cap X|$. Recall that the number of inner edges of $X \cup Y$ is $e(G[X]) = |X|/2 < n/3$. If $a \geq 200$, then the expected number of edges in $G[X']$ is at least $\mathbb{E}[e([X' \cap X, X' \cap Y])] \geq \frac{1}{2} \cdot \frac{|X' \cap X|}{2} \cdot a \geq 10n$. Therefore, by Chernoff bound (Lemma 1.4.1), $\mathbb{P}[M_{X',Y'}] \leq \mathbb{P}[e(G[X']) < n/3] = o(2^{-n})$. We may thus assume that $a \leq 200$, which implies $|Y' \cap Y| = |Y| - a > n/5$. If $b \geq 200$, then $\mathbb{E}[e(G[Y'])] \geq \frac{1}{2} \cdot \frac{b}{2} \cdot |Y' \cap Y| \geq 10n$, and so $\mathbb{P}[M_{X',Y'}] \leq \mathbb{P}[e(G[Y']) < n/3] = o(2^{-n})$. Hence, we may further assume that $b \leq 200$. Note now that both $X' \cap X$ and $Y' \cap Y$ have size at least $n/2 - \beta n - 200 \geq n/2 - 2\beta n$. Since $X' \cup Y' \neq X \cup Y$, we have

$a+b \geq 1$. Hence, the expected number of inner edges of $X' \cup Y'$ is at least

$$\begin{aligned} & \mathbb{E}[e([X' \cap X, X' \cap Y])] + \mathbb{E}[e(G[Y'])] + e(G[X' \cap X]) \\ \geq \; & \frac{1}{2} \cdot \frac{|X' \cap X|}{2} \cdot a + \frac{1}{2} \cdot \frac{b}{2} \cdot |Y' \cap Y| + (e(G[X]) - b) \\ \geq \; & \frac{1}{4} \cdot (a+b) \cdot \left(\frac{n}{2} - 2\beta n\right) + \frac{|X|}{2} - b \geq \frac{3n}{8} - 300\beta n. \end{aligned}$$

Thus, by Chernoff bound (Lemma 1.4.1), $\mathbb{P}[M_{X',Y'}] \leq \mathbb{P}[e(G[X']) + e(G[Y']) < n/3] \leq 2^{-n/1000}$. The number of $X' \cup Y'$ with $a, b \leq 200$ is only at most $\binom{n}{200}^2 \leq n^{400}$. $\square$

Since $s, t, r \leq n$, Lemmas 3.4.2, 3.4.3 and 3.4.5 imply (3.4.2):

$$\frac{\left|\bigcup_{s,t} \mathcal{B}(s,t) - \mathcal{B}(0)\right|}{|\mathcal{G}|} = \frac{\sum_{s+t \geq 1} |\mathcal{B}(s,t)| + \sum_{r \geq 1} |\mathcal{B}(r)|}{|\mathcal{G}|} \leq \frac{(n^2+n) \cdot 2^{|X||Y|/2 - n/200}}{(1+o(1))2^{|X||Y|/2}} \leq 2^{-n/300}.$$

Case 2: $|X|$ is odd.

Fix an arbitrary choice of X, Y, M with $|X|$ odd and let $x \in X$ be the vertex not covered by M. By Lemmas 3.4.2 and 3.4.3,

$$\left|\bigcup_{s,t} \mathcal{B}(X,Y,M,s,t)\right| \leq \sum_{s,t:\ s+t \geq 1} |\mathcal{B}(X,Y,M,s,t)| + \sum_{r \geq 1} |\mathcal{B}(X,Y,M,r)| \leq 2^{|X||Y|/2 - n/300}.$$

Pick an arbitrary vertex $y \in Y$, define $X_0 = X \cup \{y\}, Y_0 = Y - \{y\}$ and $M_0 = M \cup \{xy\}$. Then by Lemma 3.4.5, we have

$$|\mathcal{G}(X_0, Y_0, M_0)| \geq (1+o(1))2^{|X_0||Y_0|/2} \geq 2^{|X||Y|/2 - (|X|-|Y|)/2 - 1} \geq 2^{|X||Y|/2 - 2\beta n},$$

since $|X| - |Y| \leq 2\beta n$. Notice that any (X_0, Y_0, M_0) with $|X_0|$ even can be obtained from at most n different triples (X, Y, M) with $|X|$ odd in this way. Together with Lemma 3.4.4, it is sufficient to show that $\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t)$ is negligible compared to $\mathcal{G}(X_0, Y_0, M_0)$:

$$\frac{\left|\bigcup_{s,t} \mathcal{B}(X, Y, M, s, t)\right|}{|\mathcal{G}(X_0, Y_0, M_0)|} \leq \frac{2^{|X||Y|/2 - n/300}}{2^{|X||Y|/2 - 2\beta n}} \leq 2^{-n/400}.$$

This completes the proof of Theorem 3.1.1.

Chapter 4

Families in posets minimizing the number of comparable pairs

The results of this chapter are joint work with József Balogh and Adam Zsolt Wagner [18] (in preparation).

Given a poset P we say a family $\mathcal{F} \subseteq P$ is *centered* if it is obtained by 'taking sets as close to the middle layer as possible'. A poset P is said to have the *centeredness property* if for any M, among all families of size M in P, centered families contain the minimum number of comparable pairs. Kleitman showed that the Boolean lattice $\{0,1\}^n$ has the centeredness property. It was conjectured by Noel, Scott, and Sudakov, and by Balogh and Wagner, that the poset $\{0,1,\dots,k\}^n$ also has the centeredness property, provided n is sufficiently large compared to k. We show that this conjecture is false for all $k \geq 2$ and investigate the range of M for which it holds. Further, we improve a result of Noel, Scott, and Sudakov by showing that the poset of subspaces of $\mathbb{F}_q^n$ has the centeredness property. Several open questions are also given.

4.1 Introduction

Given a poset P, we say that two elements $A, B \in P$ form a *comparable pair* if $A \leq B$ or $B \leq A$. The study of families of sets containing few comparable pairs started with Sperner's Theorem, a cornerstone result of combinatorics. It states that the largest antichain (i.e. family containing no comparable pairs) in the Boolean lattice $\mathcal{P}(n) = \{0,1\}^n$ has size $\binom{n}{\lfloor n/2 \rfloor}$. The following natural question was first posed by Erdős and Katona for $r = 2$ and then extended by Kleitman [49] some fifty years ago: Given a poset $\mathcal{P}(n)$ and an integer M, what is the minimum number of r-chains that a family of M elements in $\mathcal{P}(n)$ must contain? For $r = 2$, the case of comparable pairs, the question was completely resolved by Kleitman [49]. For $r \geq 3$, we refer the reader to [20, 31, 32].

Here we are interested in the case $r = 2$, but for a general poset P.

Centered families in $\{0, 1, \dots, k\}^n$

We say that a family $\mathcal{F} \subseteq \{0,1\}^n$ is *centered* if for any two sets $A, B \in \{0,1\}^n$ with $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ we have that

$$\left| |A| - \frac{n}{2} \right| \leq \left| |B| - \frac{n}{2} \right|,$$

where $|A|$ denotes the number of 1-coordinates in A. That is, $\mathcal{F}$ is centered if it is constructed by "taking sets that are as close to the middle layer as possible". This same notion can be extended to the poset $\{0, 1, \dots, k\}^n$ where $A \leq B$ if $A_i \leq B_i$ for all $i \in [n]$, where A_i and B_i are the ith coordinates of A and B. We say that a family $\mathcal{F} \subseteq \{0, 1, \dots, k\}^n$ is *centered* if for any two sets $A, B \in \{0, 1, \dots, k\}^n$ with $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ we have that

$$\left| \sum_{i=1}^{n} A_i - \frac{nk}{2} \right| \leq \left| \sum_{i=1}^{n} B_i - \frac{nk}{2} \right|.$$

Denote by $\mathrm{comp}(\mathcal{F})$ the number of comparable pairs in $\mathcal{F} \subseteq P$. A family $\mathcal{F} \subseteq P$ of size M is *M-optimal* if for all families $\mathcal{F}' \subseteq P$ of size M we have $\mathrm{comp}(\mathcal{F}) \leq \mathrm{comp}(\mathcal{F}')$. A poset $P \in \mathbf{P}$ has the *centeredness property* if for all $M \leq |P|$ there exists an M-optimal centered family. Using this terminology, Kleitman's celebrated theorem [49] can be stated as follows:

Theorem 4.1.1 (Kleitman [49], 1966)**.** *The poset $\{0,1\}^n$ has the centeredness property for all $n \in \mathbb{N}$.*

In [31] the authors characterised precisely which families achieve the minimum number of contained comparable pairs. It is natural to ask whether Theorem 4.1.1 holds for the poset $\{0, 1, \dots, k\}^n$ with $k \geq 2$ as well. It was showed in [20] that there exists a counterexample with $n = 2$ and $k = 16$. The following conjecture was raised independently in [62] and [20]:

Conjecture 4.1.2 (Noel–Scott–Sudakov [62], Balogh–Wagner [20])**.** *For every k there exists an n_0 such that if $n \geq n_0$ then the poset $\{0, 1, \dots, k\}^n$ has the centeredness property.*

Our main result is the construction of two different classes of explicit counterexamples to this natural generalisation of Theorem 4.1.1. We show that for every k, if n is sufficiently large, then there exists a suitable choice of M and a family $\mathcal{F}$ of size M that contains strictly fewer comparable pairs than the centered families of the same size.

Denote by $\mathcal{L}_r(n,k)$ the *r-th layer* of $\{0,1,\ldots,k\}^n$, i.e. the set of vectors in $\{0,1,\ldots,k\}^n$ whose coordinates sum to r, and let $\ell_r(n,k) := |\mathcal{L}_r(n,k)|$. Write $\Sigma_r(n,k)$ for the total size of the r middle layers of $\{0,1,\ldots,k\}^n$. For $M \leq \Sigma_1(n,k)$ there exists an antichain of size M in the middle layer $\mathcal{L}_{\lfloor nk/2 \rfloor}(n,k)$ and hence Conjecture 4.1.2 trivially holds.

Our main result for the poset $\{0,1,2\}^n$ is the following.

Theorem 4.1.3. *(a) Let $\varepsilon > 0$, n be sufficiently large, and $M \leq (1-\varepsilon)\Sigma_3(n,2)$. Then there exists an M-optimal centered family in $\{0,1,2\}^n$.*

(b) Let n be sufficiently large and $M = \Sigma_6(n,2) - \binom{n}{3} - 1$. Then none of the centered families in $\{0,1,2\}^n$ are M-optimal.

Theorem 4.1.3 says that the smallest $M = M_0$ for which Conjecture 4.1.2 breaks down (for $k=2$) satisfies $(1-\varepsilon)\Sigma_3(n,2) < M_0 < \Sigma_6(n,2) - \binom{n}{3}$. For $k=2$ and M slightly larger than $\Sigma_1(n,2)$ it was previously shown by Noel–Scott–Sudakov [62] that centered families contain asymptotically the optimal number of comparable pairs. They also obtained good lower bounds for the number of comparable pairs in larger families.

Theorem 4.1.4 (Noel–Scott–Sudakov [62])**.** *Let r be a fixed positive integer. Then there exists a constant $n_0(r)$ such that if $n \geq n_0(r)$ and $\mathcal{F} \subseteq \{0,1,2\}^n$ has cardinality at least $\Sigma_r(n,2) + t$ then*

$$\mathrm{comp}(\mathcal{F}) \geq \left(\frac{\ell_{3r-1}(n,2)}{\ell_{2r-1}(n,2)} - 1 \right) t.$$

While at first sight it may seem feasible that Conjecture 4.1.2 holds for much larger M, Theorem 4.1.5 shows that this is not the case.

Theorem 4.1.5. *Let $k \geq 2$ and $\varepsilon > 0$. There exists a constant $n_0 = n_0(k,\varepsilon)$ such that for every $n \geq n_0$, if $M = \Sigma_j(n,k)$, where $(1+\varepsilon)\log_2 n \leq j \leq \sqrt{n}/\log_2 n$, then none of the centered families in $\{0,1,\ldots,k\}^n$ are M-optimal.*

Centered families in other posets

The notion of centeredness can be readily extended to several other common posets that satisfy some nice properties. In a poset P, y *covers* x if $x < y$ and there is no element z such that $x < z < y$. We say that the poset P is a *graded poset* if it is equipped with a rank function rk: $P \to \mathbb{N}$ which satisfies that $\mathrm{rk}(x) < \mathrm{rk}(y)$ whenever $x < y$, and $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ whenever y covers x. The *rank of a poset* P is the maximum rank of an element of P. Given a graded poset P, the r-th layer $\mathcal{L}_r(P)$ is the collection of elements in P of rank r, $\ell_r(P)$ is the size of $\mathcal{L}_r(P)$, and $\Sigma_r(P)$ is the total number of elements of P in the middle r layers. A graded poset of rank n is *rank-symmetric* if $\ell_i(P) = \ell_{n-i}(P)$ for $0 \le i \le n$ and it is *rank-unimodal* if $\ell_0(P) \le \ldots \le \ell_j(P) \ge \ell_{j+1}(P) \ge \ldots \ge \ell_n(P)$ for some $0 \le j \le n$. Denote by $\mathbf{P}$ the family of all graded posets that are rank-symmetric and rank-unimodal, and by $\mathbf{P}(n)$ the posets in $\mathbf{P}$ of rank n.

We will extend the notion of centeredness only to the posets in $\mathbf{P}$. Note that every $P \in \mathbf{P}(n)$ satisfies that its largest layer is $\mathcal{L}_{\lfloor n/2 \rfloor}(P)$ and its k largest layers are the k layers closest to the middle layer. Examples of such posets include $\{0, 1, \ldots, k\}^n$ where $(A_1, \ldots, A_n) \le (B_1, \ldots, B_n)$ if $A_i \le B_i$ for all $1 \le i \le n$, and the poset $\mathcal{V}(q, n)$ of subspaces of $\mathbb{F}_q^n$ ordered by inclusion where q is a prime power.

Similarly as before, given a poset $P \in \mathbf{P}(n)$, we say that a family $\mathcal{F} \subseteq P$ is *centered* if for any two sets $A, B \in P$ with $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ we have that their ranks $\mathrm{rk}(A), \mathrm{rk}(B)$ satisfy

$$\left|\mathrm{rk}(A) - \frac{n}{2}\right| \le \left|\mathrm{rk}(B) - \frac{n}{2}\right|.$$

In other words, $\mathcal{F}$ is centered if it is constructed by "taking sets that are as close to the middle layer as possible". Note that if $P = \{0, 1, \ldots, k\}^n$, then this definition is the same as the definition of 'centered' introduced in the previous section (where the rank of P was nk).

Consider now for a prime power q the poset $\mathcal{V}(q, n)$ of subspaces of $\mathbb{F}_q^n$ ordered by inclusion. Denote by $\begin{bmatrix} n \\ i \end{bmatrix}_q$ the number of subspaces of $\mathbb{F}_q^n$ of dimension i. Note that $\begin{bmatrix} n \\ i \end{bmatrix}_q = \prod_{j=0}^{i-1} \frac{1-q^{n-j}}{1-q^{j+1}}$. The following result of Noel, Scott, and Sudakov [62] provides a lower bound on $\mathrm{comp}(\mathcal{F})$ for $\mathcal{F} \subseteq \mathcal{V}(q, n)$.

Theorem 4.1.6 (Noel–Scott–Sudakov [62]). *Let q be a prime power and k be a fixed positive integer.*

There exists a constant $n_0(k)$ such that for $n \geq n_0(k)$ and $\mathcal{F} \subseteq \mathcal{V}(q,n)$,

$$\text{If} \quad |\mathcal{F}| \geq \sum_{r=0}^{k-1} \begin{bmatrix} n \\ \lceil \frac{n-k+1+2r}{2} \rceil \end{bmatrix}_q + t, \quad \text{then} \quad \mathrm{comp}(\mathcal{F}) \geq t \begin{bmatrix} \lceil (n+k)/2 \rceil \\ k \end{bmatrix}_q .$$

They pointed out that this bound is attained by a centered family and hence best possible when $k = 1$ and $0 \leq t \leq \left[\begin{smallmatrix} n \\ \lfloor (n-1)/2 \rfloor \end{smallmatrix}\right]_q$. We show that centered families are best for all sizes.

Theorem 4.1.7. *Let q be a prime power and $n \geq 1$. Then the poset $\mathcal{V}(q,n)$ has the centeredness property.*

Our proofs of Theorem 4.1.3 (a) and Theorem 4.1.7 are heavily based on the compression techniques of Kleitman [49]. The proof of Theorem 4.1.3 (b) arose when we attempted to prove that Conjecture 4.1.2 holds in the range $M \leq \Sigma_{(1-\varepsilon)\log_2 n}(n,2)$ – all our proof attempts kept breaking down and they eventually led us to this counterexample. The construction in Theorem 4.1.5 came from the observation that for large enough M, centered families are not even locally optimal, and in fact by replacing one of its elements in an appropriate way we can decrease the number of comparable pairs in the family.

For the corresponding maximization question, i.e. determining the maximum possible number of comparable pairs among families of size M in $\mathcal{P}(n)$ we refer the reader to [2].

4.2 Proof of Theorem 4.1.3 (a)

Whenever $A = (A_1, \ldots, A_n)$ is an element of $\{0,1,2\}^n$, we will define the *size* (or *rank*) of A by $|A| := \sum_{i=1}^n A_i$. We will use a_0, a_1 and a_2 to denote the number of 0-, 1-, and 2-coordinates of A (that is, $a_i := |\{j : A_j = i\}|$). Similarly for $B \in \{0,1,2\}^n$ we will use the variables b_0, b_1, b_2 in the same fashion. The *complement* of a set $A \in \{0,1,2\}^n$ is defined as $A^c := (2 - A_1, \ldots, 2 - A_n)$. For a permutation $\pi \in S_n$ and a set $A \in \{0,1,2\}^n$ we denote by $\pi(A)$ the set $(A_{\pi(1)}, \ldots, A_{\pi(n)})$. For a family $\mathcal{F} \subseteq \{0,1,2\}^n$ and integer $0 \leq r \leq 2n$, we write $\mathcal{F}_r = \{A \in \mathcal{F} : |A| = r\}$ and $N_r(A) := \{B : |B| = r, B \subseteq A \text{ or } A \subseteq B\}$. Recall that in the poset $\{0,1,2\}^n$, $\mathcal{L}_r(n,2)$ denotes the r-th layer and $\Sigma_j(n,2)$ the total size of the j middle layers. In this section, we will often shorten

$\mathcal{L}_r(n,2)$ to $\mathcal{L}_r$ and $\Sigma_j(n,2)$ to Σ_j. Recall also that a family $\mathcal{F} \subseteq \{0,1,2\}^n$ of size M is called M-optimal if there is no other family $\mathcal{F}' \subseteq \{0,1,2\}^n$ of size M that contains strictly fewer comparable pairs than $\mathcal{F}$. Our goal is to show that there exists an M-optimal family that is centered.

Let $\varepsilon > 0$, let n be sufficiently large so that all the following estimates hold, and fix an $M \leq (1-\varepsilon)\Sigma_3(n,2)$. The proof is by induction on M, with the base case $M \leq \Sigma_1(n,2)$ in which case there is an antichain in $\mathcal{L}_n$ of size M and the claim follows. Hence we will assume that there exists an $(M-1)$-optimal centered family, and show that there exists an M-optimal centered family. Our first goal is to show that there exist M-optimal families that are contained in the middle three layers of $\{0,1,2\}^n$.

The following claim will be useful for us:

Claim 4.2.1. *Let $A, B \in \{0,1,2\}^n$ such that $B \subsetneq A$. If $|A|, |B| \geq n$, then for every $i \in \{1, \ldots, |A| - |B|\}$:*

$$|N_{|B|+i}(B)| \leq |N_{|A|-i}(A)|.$$

Proof. Suppose that $|A|, |B| \geq n$. We show that B^c has at most as many 2's and at least as many 0's as A. This implies that there exists a permutation $\pi(B^c)$ of the coordinates of B^c such that $\pi(B^c) \subsetneq A$. Thus, $\pi(B^c)$ has at most as many neighbors in level $\mathcal{L}_{\pi(B^c)-i}$ as A does in level $\mathcal{L}_{A-i}$, for every $i \in \mathbb{N} \cup \{0\}$, so

$$|N_{|B|+i}(B)| = |N_{|B^c|-i}(B^c)| = |N_{|B^c|-i}(\pi(B^c))| \leq |N_{|A|-i}(A)|.$$

The number of 0's in B^c is equal to b_2 and the number of 2's in B^c is equal to b_0. Hence we want to show that $b_0 \leq a_2$ and $b_2 \geq a_0$. Note first that since $B \subseteq A$, we have $b_2 \leq a_2$ and $b_0 \geq a_0$.

Let k, l be such that $|A| = n + k$ and $|B| = n + l$. From $|A| > |B| \geq n$ we have that $k > l \geq 0$. Since $a_0 + a_1 + a_2 = n$ and $a_1 + 2a_2 = n + k$, we have $a_2 - a_0 = k$, and similarly $b_2 - b_0 = l$.

$$b_0 = b_2 - l \leq a_2 - l \leq a_2 \qquad \text{and} \qquad b_2 = b_0 + l \geq b_0 \geq a_0.$$

$\square$

A family $\mathcal{F}$ in a poset $P \in \mathbf{P}$ is *compressed* if for every element $A \in \mathcal{F}$, every element comparable with A that is closer to the middle than A is in $\mathcal{F}$. Kleitman proved that every family in the Boolean lattice "can be compressed" without increasing the number of comparable pairs. It is not clear why this would be the case for $\{0,1,\ldots,k\}^n$ with $k > 2$. In the poset $\{0,1,2\}^n$ we can however at least obtain an analogous result for a weaker notion of top- and bottom-compressed, given in the following definition.

Definition 4.2.2. A family $\mathcal{F} \subseteq \{0,1,2\}^n$ is *top-compressed* if the following condition holds:

(T) If $A \in \mathcal{F}$ with $|A| > n$ and $B \subseteq A$ with $|B| \geq n$, then $B \in \mathcal{F}$.

A family $\mathcal{F} \subseteq \{0,1,2\}^n$ is *bottom-compressed* if the following condition holds:

(B) If $A \in \mathcal{F}$ with $|A| < n$ and $B \supseteq A$ with $|B| \leq n$, then $B \in \mathcal{F}$.

Lemma 4.2.3. *For every natural number $M \leq 3^n$, there exists an M-optimal family that is top- and bottom-compressed.*

Proof. Let $\mathcal{F}$ be an M-optimal family. Suppose that there exist elements $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ that violate condition (T). Pick such A for which $|A|$ is maximum, and then pick such B for which $|A| - |B|$ is minimal, and let $a = |A|$ and $b = |B|$. Then all elements in levels $\mathcal{L}_{b+1}, \ldots, \mathcal{L}_{a-1}$ that are comparable with A are in $\mathcal{F}$.

We form a bipartite graph with parts $\mathcal{F}_a = \mathcal{F} \cap \mathcal{L}_a$ and $\overline{\mathcal{F}_b} = \mathcal{L}_b \setminus \mathcal{F}$ and with edges between comparable pairs. We write $N_{\mathcal{X}}(A)$ for the set of elements in $\mathcal{X}$ comparable with A. Additionally, let $N_r(A) := N_{\mathcal{L}_r}(A)$, $N(A) := N_{\{0,1,2\}^n}(A)$, and $N_{\mathcal{X}}(\mathcal{A}) := \cup_{A \in \mathcal{A}} N_{\mathcal{X}}(A)$.

We will show that we can iteratively replace some elements of $\mathcal{F}_a$ by elements of $\overline{\mathcal{F}_b}$ without increasing the number of comparable pairs. We will consider several cases based on sizes of $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ and the existence of "good" matchings that allow us to top-compress $\mathcal{F}$. Since $b < a$, the total value $\sum_{C \in \mathcal{F}} ||C| - n|$ of the family strictly decreases, ensuring that this process will terminate.

Suppose that we have families $\mathcal{A} \subseteq \mathcal{F}_a$ and $\mathcal{B} \subseteq \overline{\mathcal{F}_b}$ such that there is a perfect matching f between $\mathcal{A}$ and $\mathcal{B}$. We define a new family $\mathcal{G} = (\mathcal{F} \setminus \mathcal{A}) \cup \mathcal{B}$ and show that the new family $\mathcal{G}$ has no more comparable pairs than $\mathcal{F}$ does. We compare the sizes of neighborhoods of $\mathcal{A}$ and $\mathcal{B}$ in the following four parts of the poset $\{0,1,2\}^n$:

1. In levels $\mathcal{L}_{a+1}, \ldots, \mathcal{L}_{2n}$: Since A is a greatest element of $\mathcal{F}$, no elements of $\mathcal{F}$ are in these levels.

2. In levels $\mathcal{L}_0, \ldots, \mathcal{L}_{b-1}$: Let $A \in \mathcal{A}$ and $B \coloneqq f(A)$. Since $B \subseteq A$, if $C \subseteq B$ then $C \subseteq A$. So the number of comparable pairs cannot increase here.

3. In levels $\mathcal{L}_{b+1}, \ldots, \mathcal{L}_{a-1}$: Since all elements in these levels are in $\mathcal{F}$, by Claim 4.2.1, for every $i \in [a-b-1]$,

$$|\mathcal{F} \cap N_{b+i}(B)| \leq |N_{b+i}(B)| \leq |N_{a-i}(A)| = |\mathcal{F} \cap N_{a-i}(A)|.$$

   Thus, every element $B \in \mathcal{B}$ has at most as many neighbors in $\mathcal{L}_{b+1} \cup \cdots \cup \mathcal{L}_{a-1}$ as every $A \in \mathcal{A}$ does.

4. In levels $\mathcal{L}_a$ and $\mathcal{L}_b$: This will be checked in each case separately.

In each case below, we present suitable sets $\mathcal{A} \in \mathcal{F}_a$ and $\mathcal{B} \in \overline{\mathcal{F}_b}$ with a perfect matching f between $\mathcal{A}$ and $\mathcal{B}$ for which

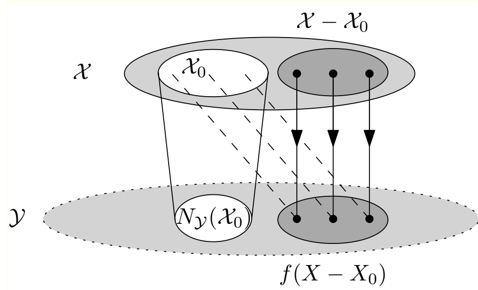
$$e(\mathcal{B}, \mathcal{G}_a) \leq e(\mathcal{A}, \mathcal{F}_b), \tag{4.2.1}$$

where $e(\mathcal{C}, \mathcal{D})$ denotes the number of edges between the families $\mathcal{C}$ and $\mathcal{D}$.

Suppose first that there exists a matching f between $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ covering $\mathcal{F}_a$. Let $\mathcal{A} = \mathcal{F}_a$ and $\mathcal{B} = f(\mathcal{F}_a)$. Then there are no elements of $\mathcal{G}$ in $\mathcal{F}_a$, so $e(\mathcal{B}, \mathcal{G}_a) = 0$. Henceforth we assume that there is no matching f between $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ covering $\mathcal{F}_a$, and we restrict our attention to the bipartite graph $(\mathcal{X}, \mathcal{Y})$, where

$$\mathcal{X} \coloneqq \mathcal{F}_a \quad \text{and} \quad \mathcal{Y} \coloneqq N(\mathcal{F}_a) \cap \overline{\mathcal{F}_b}.$$

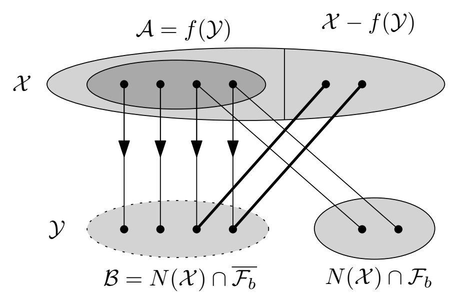
Case 1: $|\mathcal{X}| \leq |\mathcal{Y}|$. By Hall's theorem, since there is no matching between $\mathcal{X}$ and $\mathcal{Y}$ covering $\mathcal{X}$, there must be a vertex set $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $|N_{\mathcal{Y}}(\mathcal{X}_0)| < |\mathcal{X}_0|$. Choose $\mathcal{X}_0$ to be a maximal such vertex set. Then there must exist a matching f between $\mathcal{X} \setminus \mathcal{X}_0$ and $\mathcal{Y} \setminus N_{\mathcal{Y}}(\mathcal{X}_0)$ covering $\mathcal{X} \setminus \mathcal{X}_0$. Define $\mathcal{A} = \mathcal{X} \setminus \mathcal{X}_0$ and $\mathcal{B} = f(\mathcal{X} \setminus \mathcal{X}_0)$. Since there is no edge between $\mathcal{B} = f(\mathcal{X} \setminus \mathcal{X}_0)$ and $\mathcal{G}_a = \mathcal{X}_0$, the relation (4.2.1) holds.



Case 2: $|\mathcal{X}|>|\mathcal{Y}|$. Suppose first that there exists a matching f between $\mathcal{X}$ and $\mathcal{Y}$ covering $\mathcal{Y}$. Let $\mathcal{A}=f(\mathcal{Y})$ and $\mathcal{B}=\mathcal{Y}$. By Claim 4.2.1 applied with $i=a-b$ on every pair $(A,f(A))\in(\mathcal{A},\mathcal{B})$, we have $e(\mathcal{B},\mathcal{L}_a)\le e(\mathcal{A},\mathcal{L}_b)$, so

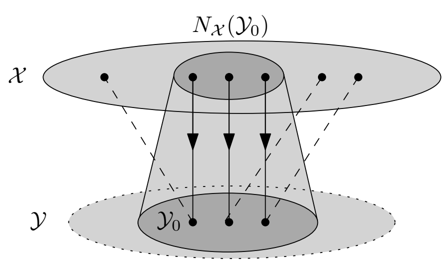
$$e(\mathcal{B},\mathcal{G}_a)+e(\mathcal{B},\mathcal{A})=e(\mathcal{B},\mathcal{F}_a)\le e(\mathcal{B},\mathcal{L}_a)\le e(\mathcal{A},\mathcal{L}_b)=e(\mathcal{A},\mathcal{F}_b)+e(\mathcal{A},\mathcal{B}).$$

The inequality (4.2.1) follows by subtracting $e(\mathcal{A},\mathcal{B})$ on both sides.



Suppose now that there is no matching covering $\mathcal{Y}$. By Hall's theorem, there must exist a minimal vertex set $\mathcal{Y}_0\subseteq\mathcal{Y}$ such that $|N_{\mathcal{X}}(\mathcal{Y}_0)|<|\mathcal{Y}_0|$. Consider the following two subcases:

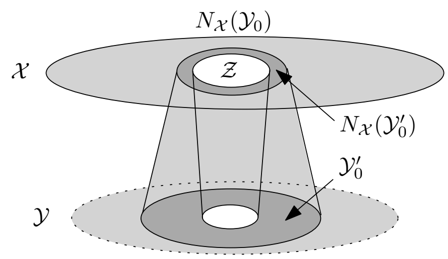
a) There is a matching f between $\mathcal{Y}_0$ and $N_{\mathcal{X}}(\mathcal{Y}_0)$ covering $N(\mathcal{Y}_0)$. Let $\mathcal{A}=N_{\mathcal{X}}(\mathcal{Y}_0)$ and $\mathcal{B}=f(N_{\mathcal{X}}(\mathcal{Y}_0))$. There is no edge between $\mathcal{B}$ and $\mathcal{G}_a=\mathcal{F}_a\setminus\mathcal{A}$, hence $e(\mathcal{B},\mathcal{G}_a)=0$ and the inequality (4.2.1) trivially holds.



b) There is no matching between $\mathcal{Y}_0$ and $N_{\mathcal{X}}(\mathcal{Y}_0)$ covering $N_{\mathcal{X}}(\mathcal{Y}_0)$. By Hall's theorem, there exists a vertex set $\mathcal{Z} \subseteq N_{\mathcal{X}}(\mathcal{Y}_0)$ with $|N_{\mathcal{Y}_0}(\mathcal{Z})| < |\mathcal{Z}|$. Then $\mathcal{Y}_0' := \mathcal{Y}_0 \setminus N_{\mathcal{Y}_0}(\mathcal{Z})$ is smaller than $\mathcal{Y}_0$. Since $|N_{\mathcal{X}}(\mathcal{Y}_0)| < |\mathcal{Y}_0|$ and $|\mathcal{Z}| > |N_{\mathcal{Y}_0}(\mathcal{Z})|$, we also have

$$|N_{\mathcal{X}}(\mathcal{Y}_0')| \leq |N_{\mathcal{X}}(\mathcal{Y}_0)| - |\mathcal{Z}| < |\mathcal{Y}_0| - |N_{\mathcal{Y}_0}(\mathcal{Z})| = |\mathcal{Y}_0'|,$$

and we can conclude that $\mathcal{Y}_0$ was not a minimal set with $|N_{\mathcal{X}}(\mathcal{Y}_0)| < |\mathcal{Y}_0|$.



We showed that there exists an M-optimal family $\mathcal{F}$ that is top-compressed. The proof that $\mathcal{F}$ can "be made" bottom-compressed without increasing the number of comparable pairs follows by the above proof applied on $\mathcal{F}^c = \{A^c : A \in \mathcal{F}\}$.

□

Lemma 4.2.3 ensures the existence of an M-optimal top- and bottom-compressed family. Although we will use the lemma only for $M \leq (1-\varepsilon)\Sigma_3$, we emphasize that the result holds for any M, which might be of independent interest. Our next goal is to find an M-optimal family which additionally satisfies conditions (C1) and (C2) in the following definition.

Definition 4.2.4. We say that a family $\mathcal{F} \subseteq \{0,1,2\}^n$ of size M is 3-*compressed* if $\mathcal{F}$ is top-compressed, bottom-compressed, and additionally the following two conditions hold:

(C1) If A is a maximal element of $\mathcal{F}$ with $|A| = n+2$ and $B \subseteq A$ is such that $|B| = n-1$ and $b_0 > a_0$ then $B \in \mathcal{F}$.

(C2) If A is a minimal element of $\mathcal{F}$ with $|A| = n-2$ and $B \supseteq A$ is such that $|B| = n+1$ and $b_2 > a_2$ then $B \in \mathcal{F}$.

The following claim is an analogue statement to Claim 4.2.1.

Claim 4.2.5. *Let $A, B \in \{0,1,2\}^n$ such that $B \subseteq A$. If $|A| = n+2, |B| = n-1$, and $b_0 \neq a_0$, then for every $i \in \{1,2,3\}$,*

$$|N_{|B|+i}(B)| \leq |N_{|A|-i}(A)|.$$

Proof. Suppose that $|A| = n+2$ and $|B| = n-1$. Since $b_0 \neq a_0$, we only need to consider the following two cases:

Case 1: $b_2 = a_2$. The number of elements in levels $n+1$, n, and $n-1$, comparable with A, are

$$\alpha_1 \coloneqq a_2 + a_1, \quad \alpha_2 \coloneqq \binom{a_2 + a_1}{2} + a_2, \text{ and } \alpha_3 \coloneqq \binom{a_2 + a_1}{3} + a_2 \cdot (a_1 + a_2 - 1),$$

respectively. Similarly, the number of elements in levels n, $n+1$, and $n+2$, comparable with B, is

$$\beta_1 \coloneqq b_0 + b_1, \text{ and } \beta_2 \coloneqq \binom{b_0 + b_1}{2} + b_0, \text{ and } \beta_3 \coloneqq \binom{b_0 + b_1}{3} + b_0 \cdot (b_1 + b_0 - 1),$$

respectively. Note that $a_1 = b_1 + 3$ and $a_2 = b_2 = b_0 - 1$, and so $a_2 + a_1 = b_0 + b_1 + 2$. We show that $\alpha_1 \geq \beta_1$, $\alpha_2 \geq \beta_2$, and $\alpha_3 \geq \beta_3$.

$$\begin{aligned}
\alpha_1 - \beta_1 &= a_2 + a_1 - (b_0 + b_1) = b_0 + b_1 + 2 - (b_0 + b_1) > 0, \\
\alpha_2 - \beta_2 &= \tbinom{b_0+b_1+2}{2} + (b_0 - 1) - \left(\tbinom{b_0+b_1}{2} + b_0\right) = 2(b_0 + b_1) \geq 0, \\
\alpha_3 - \beta_3 &= \tbinom{b_0+b_1+2}{3} + (b_0 - 1)(b_1 + b_0 + 1) - \left(\tbinom{b_0+b_1}{3} + b_0(b_1 + b_0 - 1)\right) \\
&= b_0^2 + 2b_0 b_1 + b_1^2 + b_0 - b_1 - 1.
\end{aligned}$$

The last expression is negative only if $b_0 = 0$ and $b_1 = 1$, which is not possible since every element $B \in \mathcal{L}_{n-1}$ must contain at least one 0-coordinate.

Case 2: $b_2 \leq a_2 - 1$ and $b_0 \geq a_0 + 1$. Then

$$b_0 = b_2 + 1 \leq a_2 \qquad \text{and} \qquad b_2 = b_0 - 1 \geq a_0.$$

So B^c has at most as many 2's and at least as many 0's as A, which implies that there exists a permutation $\pi(B^c)$ of the coordinates of B^c such that $\pi(B^c) \subseteq A$. This implies that for every

$i \in \{1, 2, 3\}$,

$$|N_{|B|+i}(B)| = |N_{|B^c|-i}(B^c)| = |N_{|B^c|-i}(\pi(B^c))| \leq |N_{|A|-i}(A)|.$$

□

Lemma 4.2.6. *For every natural number $M \leq 3^n$, there exists an M-optimal family that is 3-compressed.*

Proof. Let $\mathcal{F}$ be an M-optimal family in $\{0, 1, 2\}^n$ that is top- and bottom-compressed, whose existence is guaranteed by Lemma 4.2.3. If $\mathcal{F}$ is not 3-compressed, then at least one of the conditions (C1) and (C2) fails. We assume that (C1) does not hold, keeping in mind that in the other case we can apply the same proof on $\mathcal{F}^c$. Suppose that there exists a comparable pair (A, B) in $\mathcal{F}$ such that A is a maximal element with $|A| = n + 2$, $|B| = n - 1$, and $b_0 > a_0$. Let $a = |A|$ and $b = |B|$.

Let G be a bipartite graph with parts $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ and with edges between comparable pairs (A, B) for which $b_0 \neq a_0$. As in the proof of Lemma 4.2.3, we can iteratively replace some elements of $\mathcal{F}_a$ by elements of $\overline{\mathcal{F}_b}$ without increasing the number of comparable pairs. We need to consider several cases based on sizes of $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ and existence of "good" matchings in G that allow us to compress $\mathcal{F}$. Since $b < a$, the total value $\sum_{C \in \mathcal{F}} ||C| - n|$ of the family strictly decreases, ensuring that this process will terminate. These cases are the same as in the proof of Lemma 4.2.3, except now we only consider matchings in the graph G (in which all pairs with $b_0 = a_0$ are removed), and we apply Claim 4.2.5 at every place we applied Claim 4.2.1 before.

□

We are almost ready to tackle Theorem 4.1.3 (a). We will need to make use of the fact that a typical set in $\{0, 1, 2\}^n$ of size n has about $n/3$ zeros $n/3$ ones, and $n/3$ twos.

Claim 4.2.7. *For every $\varepsilon > 10\left(\frac{1}{1.1}\right)^{0.005n}$,*

$$\left|\left\{A \in \mathcal{L}_{n+1} : \frac{0.9}{3}n \leq a_0 \leq \frac{1.1}{3}n\right\}\right| \geq (1 - \varepsilon^2)\ell_{n+1}.$$

Proof. For an integer $c \geq 0$, let $f(c) := |\{A \in \mathcal{L}_{n+1} : a_0 = c\}|$. Note that $f(c) = \binom{n}{c}\binom{n-c}{c+1}$, and hence

$$\frac{f(c)}{f(c+1)} = \frac{(c+1)(c+2)}{(n-2c-2)(n-2c-1)}.$$

If $c > \frac{1.07}{3}n$ we get $f(c)/f(c+1) > 1.1$ and if $c < \frac{0.93}{3}n = 0.31n$ we have $f(c)/f(c-1) > 1.1$. This means that

$$\sum_{i \leq 0.3n} f(i) \leq \frac{1}{1-\frac{1}{1.1}} \cdot f(0.3n) \leq 11 \cdot \left(\frac{1}{1.1}\right)^{0.01n} \cdot f(0.31n) \leq \frac{\varepsilon^2}{2} \cdot \ell_{n+1}.$$

A similar computation gives $\sum_{i \geq 1.1n/3} f(i) \leq \frac{\varepsilon^2}{2} \cdot \ell_{n+1}$, and the claim follows. $\square$

The next claim shows that for slightly varying values of M, the M-optimal families contain about the same number of comparable pairs. For an integer N, write $\mathrm{comp}(N)$ for the number of comparable pairs in an N-optimal family:

$$\mathrm{comp}(N) \coloneqq \min\{\mathrm{comp}(\mathcal{F}) : \mathcal{F} \subseteq \{0,1,2\}^n, |\mathcal{F}| = N\}.$$

Claim 4.2.8. *If* $M \leq (1-\varepsilon)\sum_3(n,2)$, *then* $\mathrm{comp}(M) \leq \mathrm{comp}(M-1) + \frac{n^2}{4}$.

Proof. By the induction hypothesis, there exists an $(M-1)$-optimal centered family $\mathcal{G}$. Since $M \leq (1-\varepsilon)\sum_3$, the family $\mathcal{G}$ consists of all elements in layer $\mathcal{L}_n$ and some elements in layers $\mathcal{L}_{n-1}$ and $\mathcal{L}_{n+1}$. Define

$$\mathcal{G}_1 \coloneqq \{B \in \mathcal{L}_{n+1} : b_0 \geq \frac{0.9}{3}n\} \quad \text{and} \quad \mathcal{G}_2 \coloneqq \{B \in \mathcal{L}_{n-1} : b_2 \geq \frac{0.9}{3}n\}.$$

Claim 4.2.7 implies $|\mathcal{G}_1|, |\mathcal{G}_2| \geq (1-\varepsilon^2)|\mathcal{L}_{n+1}|$. For $M \leq (1-\varepsilon)\Sigma_3$ we thus have $M < |\mathcal{L}_n| + |\mathcal{G}_1| + |\mathcal{G}_2|$. Add an element $B \in (\mathcal{G}_1 \cup \mathcal{G}_2) \setminus \mathcal{G}$ to $\mathcal{G}$. The element B is in at most $\binom{2.1n/3}{2} + n \leq \frac{n^2}{4}$ comparable pairs of $\mathcal{G} \cup \{B\}$, hence

$$\mathrm{comp}(M) \leq \mathrm{comp}(\mathcal{G} \cup \{B\}) \leq \mathrm{comp}(\mathcal{G}) + \frac{n^2}{4} = \mathrm{comp}(M-1) + \frac{n^2}{4}.$$

$\square$

We are ready to finish the proof of Theorem 4.1.3 (a). Let $\mathcal{F}$ be an M-optimal family that is 3-compressed, whose existence is guaranteed by Lemma 4.2.6, and assume that $\mathcal{F}$ is not centered. This can mean one of two things:

1. The first possibility is that there exists an $A \notin \mathcal{F}$ of size $|A| = n$. Since $\mathcal{F}$ is both top- and bottom-compressed, this means that there is no $B \in \mathcal{F}$ with $A \subseteq B$ or $B \subseteq A$, hence unless $\mathcal{F}$ itself is an antichain we may decrease the number of comparable pairs in $\mathcal{F}$ by replacing one of its elements by A.

2. The second possibility is that $\mathcal{L}_n \subseteq \mathcal{F}$ but $\mathcal{F} \not\subseteq \mathcal{L}_{n-1} \cup \mathcal{L}_n \cup \mathcal{L}_{n+1}$. Then there exists an element $A \in \mathcal{F}$ of size at least $n+2$ or at most $n-2$. By symmetry we may assume that there is an $A \in \mathcal{F}$ with $|A| \geq n+2$. Since $\mathcal{F}$ is 3-compressed, the number of elements in $\mathcal{F}_{a-1} \cup \mathcal{F}_{a-2} \cup \mathcal{F}_{a-3}$ comparable with A is at least

$$(a_1 + a_2) \quad + \quad \left(\binom{a_1+a_2}{2} + a_2\right) \quad + \quad \left(\binom{a_1+a_2}{3} + a_2(a_1+a_2-1) - \binom{a_2}{3}\right). \tag{4.2.2}$$

The term $\binom{a_2}{3}$ accounts for the elements of $\mathcal{L}_{n-1}$ comparable with A that have a_0 zeros, which are not necessarily in $\mathcal{F}$ by the definition of 3-compressed. Observe that every such element is formed by decreasing three 2-coordinates of A to 1-coordinates, giving $\binom{a_2}{3}$ choices. Since $a_1+a_2 \geq n+2-a_2$, the quantity (4.2.2) is minimized when $a_1 = 0$ and $a_2 = \frac{n+2}{2}$. It follows that this quantity is at least $a_2 + \binom{a_2}{2} + a_2^2 \geq \frac{3}{2}a_2^2 \geq \frac{3}{8}n^2$. But then $\mathrm{comp}(\mathcal{F}) > \mathrm{comp}(\mathcal{F} \setminus \{A\}) + \frac{3n^2}{8}$, and $\mathcal{F}$ was not M-optimal (by Claim 4.2.8), a contradiction.

$\square$

4.3 Proof of Theorem 4.1.3 (b)

Recall that for an integer $a \geq 0$ and a family $\mathcal{G} \subseteq \{0,1,2\}^n$ we have the notation $\mathcal{G}_a = \{A \in \mathcal{G} : |A| = a\}$ and $\mathcal{L}_a = \mathcal{L}_a(n,2)$. We say that a centered family $\mathcal{G} \subseteq P$ is *canonical centered* if there exists at most one $\ell \geq 0$ with $0 < |\mathcal{G}_\ell| < |\mathcal{L}_\ell(P)|$, i.e. if it has at most one partial layer (while centered families could have two). As in Section 4.2, whenever A and B are elements of $\{0,1,2\}^n$, we write a_0, a_1, a_2 and b_0, b_1, b_2 for the number of 0-, 1-, 2-coordinates in A and

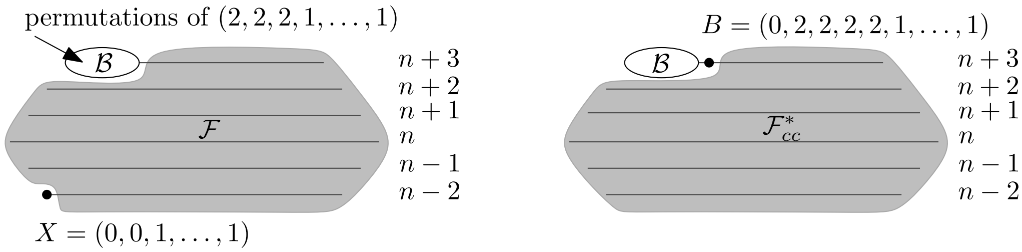


Figure 4.1: A non-centered family $\mathcal{F}$ and a canonical centered family $\mathcal{F}^*_{cc}$ such that $\text{comp}(\mathcal{F}) < \text{comp}(\mathcal{F}^*_{cc}) \leq \text{comp}(F_{cc})$ for every canonical centered family $\mathcal{F}_{cc}$ of size $M = \Sigma_6 - \binom{n}{3} - 1$.

B respectively. For an element $A \in \{0,1,2\}^n$ and family $\mathcal{G} \subseteq \{0,1,2\}^n$, we use the notation $\text{comp}(A,\mathcal{G}) := |\{B \in \mathcal{G} : B \subsetneq A \text{ or } A \subsetneq B\}|$ and $\text{Comp}(\mathcal{G}) := \{(A,B) \in \mathcal{G} \times \mathcal{G} : A \subset B\}$, so that $|\text{Comp}(\mathcal{G})| = \text{comp}(\mathcal{G})$.

Let $X = (0,0,1,1,\dots,1) \in \mathcal{L}_{n-2}$, $\mathcal{B} = \{B \in \mathcal{L}_{n+3} : b_0 = 0\}$, and $\mathcal{C} := \{C \in \{0,1,2\}^n : n-2 \leq |C| \leq n+3\}$. Finally, let $\mathcal{F} := \mathcal{C} \setminus (\mathcal{B} \cup \{X\})$ (see Figure 4.1). Then $\mathcal{F}$ is not centered, but we claim that $\mathcal{F}$ contains fewer comparable pairs than every centered family of size $M = |\mathcal{F}| = \Sigma_6(n,2) - \binom{n}{3} - 1$. The proof of this claim goes in two stages. First we show that $\mathcal{F}$ contains fewer comparable pairs than the best canonical centered family of this size (Claim 4.3.1), and next we show that among centered families of this size the canonical families are the best (Lemma 4.3.2).

Claim 4.3.1. *Whenever $\mathcal{F}_{cc}$ is a canonical centered family of size $M = \Sigma_6(n,2) - \binom{n}{3} - 1$ we have $\text{comp}(\mathcal{F}) < \text{comp}(\mathcal{F}_{cc})$.*

Proof. Every canonical centered family $\mathcal{F}_{cc}$ of size $M = \Sigma_6(n,2) - \binom{n}{3} - 1$ consists of all elements in levels $\mathcal{L}_{n-2}, \dots, \mathcal{L}_{n+2}$ and $\ell_{n+3} - \binom{n}{3} - 1$ elements in $\mathcal{L}_{n+3}$ (or $\ell_{n-3} - \binom{n}{3} - 1$ elements in $\mathcal{L}_{n-3}$, in which case the proof is symmetrical). Let $B = (0,2,2,2,2,1,1,\dots,1) \in \mathcal{L}_{n+3}$ and note that $\mathcal{F}^*_{cc} := \mathcal{F} \cup \{X\} \setminus \{B\}$ is one of the canonical centered families of size M with the least number of contained comparable pairs. Indeed, removing all elements with no 0-coordinates plus one element with one 0-coordinate from $\mathcal{L}_{n+3}$ ensures the smallest possible number of comparable pairs. This can be seen because it is always better to replace a 2-coordinate and a 0-coordinate by two 1-coordinates, or directly from the formula (4.2.2).

It suffices to show that $\text{comp}(B,\mathcal{F}) < \text{comp}(X,\mathcal{F})$ since then we can improve $\mathcal{F} \cup \{X\} \setminus \{B\}$ by deleting X and adding B. Now, $\mathrm{comp}(X,\mathcal{F}) \geq \binom{n}{5} + \binom{n}{4}$ whereas $\mathrm{comp}(B,\mathcal{F}) = \binom{n-1}{5} + \binom{n-1}{4} + O(n^3)$, which is $\Theta(n^4)$ smaller than $\mathrm{comp}(X,\mathcal{F})$ and the claim follows. $\square$

Lemma 4.3.2. *Among centered families of size* $M = \Sigma_6(n,2) - \binom{n}{3} - 1$ *the function* $\mathrm{comp}(\cdot)$ *attains its minimum on a canonical centered family.*

Proof. Define a partial order on the collection of centered families of size M by letting $\mathcal{H} < \mathcal{H}'$ if $\mathrm{comp}(\mathcal{H}) < \mathrm{comp}(\mathcal{H}')$, or if $\mathrm{comp}(\mathcal{H}) = \mathrm{comp}(\mathcal{H}')$ and $|\mathcal{H}_{n+3}| > |\mathcal{H}'_{n+3}|$. We will show that one of the minimal elements of this partial order is canonical centered, which immediately implies Lemma 4.3.2. Let $\mathcal{G}$ be a centered family of size $M = \Sigma_6(n,2) - \binom{n}{3} - 1$ that is minimal according to this ordering. Note that $\mathcal{L}_{n-2} \cup \cdots \cup \mathcal{L}_{n+2} \subseteq \mathcal{G} \subseteq \mathcal{L}_{n-3} \cup \cdots \cup \mathcal{L}_{n+3}$.

Given a permutation $\pi \in S_n$ of order 2 (i.e., $\pi^2 = 1$) define the π-*compression* of $\mathcal{G}$ by "replace $A \in \mathcal{G}_{n-3}$ by $\pi(A^c)$ unless it is already in $\mathcal{G}_{n+3}$". That is,

$$\mathrm{cpr}_\pi(\mathcal{G}) = \mathcal{G} \cup \{\pi(A^c) \in \mathcal{L}_{n+3} : A \in \mathcal{G}_{n-3}\} \setminus \{A \in \mathcal{G}_{n-3} : \pi(A^c) \notin \mathcal{G}_{n+3}\}.$$

Claim 4.3.3. *For every* $\pi \in S_n$ *of order* 2 *we have* $cpr_\pi(\mathcal{G}) < \mathcal{G}$, *unless* $\mathcal{G} = cpr_\pi(\mathcal{G})$. *That is,* π*-compression improves the family unless it is already* π*-compressed.*

Proof. Note first that unless $\mathcal{G} = \mathrm{cpr}_\pi(\mathcal{G})$ we have that $|\mathrm{cpr}_\pi(\mathcal{G})_{n+3}| > |\mathcal{G}_{n+3}|$. It thus remains to show $\mathrm{comp}(\mathrm{cpr}_\pi(\mathcal{G})) \leq \mathrm{comp}(\mathcal{G})$. Suppose that $B \subset \pi(A^c)$ is a new comparable pair. Then A was replaced by $\pi(A^c)$, so $A \in \mathcal{G} \setminus \mathrm{cpr}_\pi(\mathcal{G})$. The element B was not replaced by $\pi(B^c)$, so $\pi(B^c) \in \mathcal{G}$. Observe that for every $\pi \in \mathcal{S}_n$, $B \subset \pi(A^c)$ implies $A \subset \pi^{-1}(B^c)$. Since our permutation π is of order 2, we have $\pi^{-1}(B^c) = \pi(B^c)$, and thus $A \subset \pi(B^c)$. Together, for every new comparable pair $B \subset \pi(A^c)$ there is an old comparable pair $A \subset \pi(B^c)$ which got deleted during the compression. This defines an injection from $\mathrm{Comp}(\mathrm{cpr}_\pi(\mathcal{G})) \setminus \mathrm{Comp}(\mathcal{G})$ into $\mathrm{Comp}(\mathcal{G}) \setminus \mathrm{Comp}(\mathrm{cpr}_\pi(\mathcal{G}))$ and the claim follows. $\square$

We sketch the idea of the remaining part of the proof. By Claim 4.3.3 and the minimality of $\mathcal{G}$, the family $\mathcal{G}$ is π-compressed for all permutations π of order 2. For $A \in \mathcal{G}_{n-3}$, define

$$\Pi(A^c) \coloneqq \{\pi(A^c) \in \mathcal{L}_{n+3} : \pi \in S_n \wedge \pi^2 = 1\},$$

and count the elements of $\Pi(A^c)$ comparable with A. Every such element has to be in $\mathcal{G}_{n+3}$ by definition of π-compression. To obtain a superset of A in $\Pi(A^c)$, we first need to switch all 0-coordinates of A^c with some of its 2-coordinates. After that we can freely switch any of the remaining three 2-coordinates with any three 1-coordinates. Any permutation that is formed in this fashion is obviously of order 2. The number of such permutations is $\binom{a_0}{3}\binom{a_1}{3}$. It follows that if the number of 0's and 1's in A is (close to) linear in n, then the number of elements in $\mathcal{G}_{n+3}$ comparable with A is of order (close to) n^6. Therefore, $\mathcal{G}_{n-3}$ cannot have many such elements since otherwise we could replace $\mathcal{G}_{n-3}$ by elements of $\mathcal{L}_{n+3} \setminus \mathcal{G}$ and the number of comparable pairs would decrease. We partition $\mathcal{G}$ into $\mathcal{G}'$, $\mathcal{G}''$, and $\mathcal{G}^*$ as follows:

$$\mathcal{G}' = \left\{A \in \mathcal{G}_{n-3} : a_2 \leq n^{2/3} \log n\right\},\ \mathcal{G}'' = \left\{A \in \mathcal{G}_{n-3} : a_2 \geq \frac{n}{2} - n^{2/3} \log n\right\},\ G^* = \mathcal{G}_{n-3} \setminus \left(\mathcal{G}' \cup \mathcal{G}''\right).$$

Observe that $\mathcal{G}'$ contains elements with a small number of 0- and 2-coordinates while $\mathcal{G}''$ contains elements with small number of 1-coordinates. Claim 4.3.4 states that there cannot be more elements in $\mathcal{G}^*$ than in $\mathcal{G}' \cup \mathcal{G}''$. Claim 4.3.5 uses a similar averaging argument to bound $|\mathcal{G}' \cup \mathcal{G}''|$ by $2|\mathcal{H}' \cup \mathcal{H}''|$, where $\mathcal{H}' \cup \mathcal{H}''$ is the family of sets in $\mathcal{G}' \cup \mathcal{G}''$ that are in a small number of comparable pairs in $\mathcal{G}$. Claim 4.3.6 then implies that $\mathcal{H}' \cup \mathcal{H}''$ must be empty, and we conclude that $\mathcal{G}$ is canonical centered.

Claim 4.3.4. $|\mathcal{G}_{n-3}| \leq 2|\mathcal{G}' \cup \mathcal{G}''|$.

Proof. Let A be an element of $\mathcal{G}^*$ and consider all its supersets of the form $\pi(A^c)$ with $\pi^2 = 1$. Since $\mathcal{G}$ is π-compressed for every involution π, we know that all these supersets are in $\mathcal{G}$. Let Π_A be the set of a permutations π of order 2 such that each π switches all 0-coordinates of A^c with all but three of its 2-coordinates, and the remaining three 2-coordinates with three arbitrary 1-coordinates. Equivalently, for every $\pi \in \Pi_A$, the element $\pi(A^c)$ is formed from A by increasing three 0-coordinates and three 1-coordinates by one. We thus always have $A \subset \pi(A^c)$, and hence the number of supersets of A in $\Pi(A^c)$ is at least $|\Pi_A| = \binom{a_0}{3}\binom{a_1}{3}$. Since $A \notin G' \cup \mathcal{G}''$ and $a_0 = a_2 + 3$, we have

$$n^{2/3} \log n + 3 \leq a_0 \leq \frac{n}{2} - n^{2/3} \log n + 3.$$

From $a_0 + a_1 + a_2 = n$ we have $a_1 = n - 2a_0 + 3$, and thus

$$a_1 \geq n - 2\left(\frac{n}{2} - n^{2/3}\log n + 3\right) + 3 = 2n^{2/3}\log n - 3.$$

As either a_0 or a_1 is larger than $n/10$, we have

$$|\Pi_A| = \binom{a_0}{3}\binom{a_1}{3} \geq n^5 \log^2 n.$$

We claim that the elements of $\mathcal{G}_{n-3}$ are in at most n^5 comparable pairs each on average. Indeed, otherwise we could replace $\mathcal{G}_{n-3}$ by an arbitrary subset of $\overline{\mathcal{G}_{n+3}} = \mathcal{L}_{n+3} \setminus \mathcal{G}_{n+3}$ of size $|\mathcal{G}_{n-3}|$ and obtain a canonical centered family with a smaller number of comparable pairs. Because each element of $\mathcal{G}^*$ is in at least $n^5\log^2 n$ comparable pairs, we have $|\mathcal{G}^*| \leq |\mathcal{G}' \cup \mathcal{G}''|$, and the claim follows. $\square$

Let

$$\mathcal{H}' = \left\{A \in \mathcal{G}' : \operatorname{comp}(A, \mathcal{G}) \leq 2n^5\right\} \quad \text{and} \quad \mathcal{H}'' = \left\{A \in \mathcal{G}'' : \operatorname{comp}(A, \mathcal{G}) \leq 2n^5\right\}.$$

Claim 4.3.5. $|\mathcal{G}' \cup \mathcal{G}''| \leq 2|\mathcal{H}' \cup \mathcal{H}''|$.

Proof. As before, the elements of $\mathcal{G}_{n-3}$ must be in at most n^5 comparable pairs each on average since otherwise we could replace $\mathcal{G}_{n-3}$ by an arbitrary subset of $\overline{\mathcal{G}_{n+3}}$. Recall that the family $\mathcal{G}_{n-3}$ is partitioned into $\mathcal{G}'$, $\mathcal{G}''$ and $\mathcal{G}^*$, and that every element of $\mathcal{G}^*$ is in at least $n^5\log^2 n$ comparable pairs of $\mathcal{G}$ (see proof of Claim 4.3.4). We thus necessarily have $|\mathcal{G}' \cup \mathcal{G}''| \leq 2|\mathcal{H}' \cup \mathcal{H}''|$. $\square$

Claim 4.3.6.

$$|\mathcal{H}'|, |\mathcal{H}''| \leq \frac{\log^8 n}{n^2} \cdot |\overline{\mathcal{G}_{n+3}}|.$$

Proof. We first count the number E'' of comparable pairs $(A, B) \in \mathcal{H}'' \times \overline{\mathcal{G}_{n+3}}$ such that $a_2 = b_2$. We count E'' two ways:

1. Let $A \in \mathcal{H}'' \subseteq \mathcal{G}''$. Then $a_0 = a_2 + 3 \geq \frac{n}{2} - n^{2/3}\log n + 3$ by the definition of $\mathcal{G}''$. We need to count the number of sets $B \in \overline{\mathcal{G}_{n+3}}$ formed from A by increasing six of its 0-coordinates

to 1-coordinates. Since $\text{comp}(A, \mathcal{G}_{n+3}) \leq 2n^5$ by the definition of $\mathcal{H}''$, this number is at least $\binom{a_0}{6} - 2n^5 \geq \binom{n/3}{6} \geq n^6/10^9$.

2. Let now $B \in \overline{\mathcal{G}_{n+3}}$ for which there exists an $A \in \mathcal{H}''$ with $a_2 = b_2$. Then

$$b_1 = n + 3 - 2b_2 = n + 3 - 2a_2 \leq n + 3 - 2\left(\frac{n}{2} - n^{2/3}\log n\right) \leq 3n^{2/3}\log n.$$

Therefore, the number of sets A formed from B by decreasing six of its 1-coordinates to 0-coordinates is at most $\binom{3n^{2/3}\log n}{6} \leq n^4 \log^7 n$.

Together we obtain

$$|\mathcal{H}''| \cdot \frac{n^6}{10^9} \leq E'' \leq |\overline{\mathcal{G}_{n+3}}| \cdot n^4 \log^7 n, \tag{4.3.1}$$

and the second inequality in Claim 4.3.6 follows.

Similarly, we count the number E' of comparable pairs $(A, B) \in \mathcal{H}' \times \overline{\mathcal{G}_{n+3}}$ such that $a_0 = b_0$.

1. Let $A \in \mathcal{H}' \subseteq \mathcal{G}'$. Then $a_1 = n - 3 - 2a_2 \geq n - 3 - 2n^{2/3}\log n$ by the definition of $\mathcal{G}'$. The number of sets $B \in \overline{\mathcal{G}_{n+3}}$ formed from A by increasing six of its 1-coordinates to 2-coordinates is at least $\binom{a_1}{6} - 2n^5 \geq \binom{n/3}{6} \geq n^6/10^9$.

2. Let now $B \in \overline{\mathcal{G}_{n+3}}$ for which there exists an $A \in \mathcal{H}'$ with $a_0 = b_0$. Then $b_2 = a_2 + 6 \leq n^{2/3}\log n + 6$. Therefore, the number of sets A formed from B by decreasing six of its 1-coordinates to 0-coordinates is at most $\binom{n^{2/3}\log n + 6}{6} \leq n^4 \log^7 n$.

Similarly to 4.3.1 we have

$$|\mathcal{H}'| \cdot \frac{n^6}{10^9} \leq E' \leq |\overline{\mathcal{G}_{n+3}}| \cdot n^4 \log^7 n,$$

and the first inequality in Claim 4.3.6 follows. $\square$

We are ready to finish the proof of Lemma 4.3.2. Applying the previous three claims, we obtain

$$|\mathcal{G}_{n-3}| \overset{C4.3.4}{\leq} 2|\mathcal{G}' \cup \mathcal{G}''| \overset{C4.3.5}{\leq} 4|\mathcal{H}' \cup \mathcal{H}''| \overset{C4.3.6}{\leq} \frac{\log^9 n}{n^2} \cdot |\overline{\mathcal{G}_{n+3}}| = \frac{\log^9 n}{n^2}\left(|\mathcal{G}_{n-3}| + \binom{n}{3} + 1\right),$$

and therefore

$$|\mathcal{G}_{n-3}| \leq n \log^{10} n. \tag{4.3.2}$$

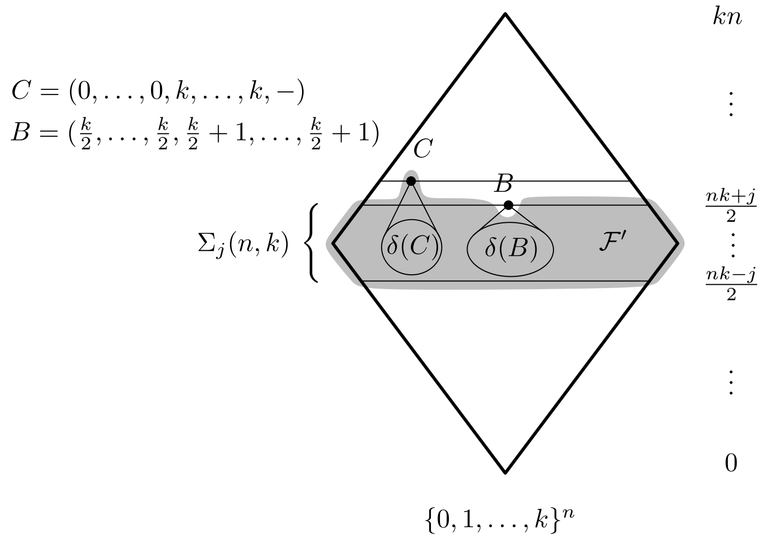


Figure 4.2: A non-centered family $\mathcal{F}' \subseteq \{0, \ldots, k\}^n$ which has smaller number of comparable pairs than the centered family $\mathcal{F}$.

Assume that $\mathcal{H}'' \neq 0$ and let $A \in \mathcal{H}''$. As in the proof of Claim 4.3.6, $\mathrm{comp}(A, \overline{\mathcal{G}_{n+3}}) \geq \binom{a_0}{6} - 2n^5 \geq n^6/10^9$, and so $|\overline{\mathcal{G}_{n+3}}| \geq n^6/10^9$. This implies $|\mathcal{G}_{n-3}| \geq n^6/10^{10}$, which contradicts equation (4.3.2). By the same argument we have $\mathcal{H}' = \emptyset$. Hence $\mathcal{G}_{n-3} = \emptyset$ by Claims 4.3.4 and 4.3.5, and we conclude that $\mathcal{G}$ is canonical centered, proving the lemma. $\square$

4.4 Proof of Theorem 4.1.5

Let $P = \{0, 1, \ldots, k\}^n$ where k is a fixed constant, $0 < \varepsilon < 0.01$, and n be sufficiently large so that all following estimates hold. We are given an integer j with $(1+\varepsilon)\log_2 n \leq j \leq \sqrt{n}/\log_2 n$ and we have $M = \Sigma_j(n,k)$. For simplicity we will assume $nk + j$ is even, the odd case is very similar, and we omit the details. Let

$$\mathcal{F} := \left\{ A \in P : \frac{nk-j}{2} < |A| \leq \frac{nk+j}{2} \right\}.$$

Let B be such that $|B| = \frac{nk+j}{2}$ and every coordinate of B is either $\lfloor \frac{k}{2} \rfloor$ or $\lfloor \frac{k}{2} + 1 \rfloor$. Let C be such that $|C| = \frac{nk+j}{2} + 1$ and every coordinate of C is k or 0, except possibly one. Note that C has at least $\frac{n-j}{2}$ zeros and at most $\frac{n+j}{2}$ non-zeros. Now define

$$\mathcal{F}' := \mathcal{F} \cup \{C\} \setminus \{B\},$$

so that $\mathcal{F}'$ is not a centered family (see Figure 4.2). We claim that $\mathrm{comp}(\mathcal{F}') < \mathrm{comp}(\mathcal{F})$. We only need to compare the number of subsets of B and C that are contained in $\mathcal{F}$ (or $\mathcal{F}'$). For a set D and an integer ℓ, write

$$\delta_\ell(D) := \{A \in \mathcal{F} : A \subseteq D, |A| = |D| - \ell\},$$

that is, the collection of subsets of D that are in $\mathcal{F}$, and are ℓ levels below D. Let

$$\delta(D) = \bigcup_{\ell=0}^{n} \delta_\ell(D).$$

We have the estimate

$$|\delta(B)| = \sum_{\ell=0}^{j-1} |\delta_\ell(B)| > |\delta_{j-1}(B)| > \binom{n}{j-1}.$$

Note that for $0 \leq \ell \leq j$ we have

$$|\delta_\ell(C)| \leq \binom{\frac{n+j}{2} + \ell - 1}{\ell} = (1 + o(1))\binom{\frac{n+j}{2}}{\ell},$$

since the right hand side of the first inequality counts the number of non-negative solutions to the equation $a_1 + \ldots + a_{(n+j)/2} = \ell$. Hence we get

$$\begin{aligned}
|\delta(C)| = \sum_{\ell=0}^{j} |\delta_\ell(C)| &\leq (1 + o(1)) \sum_{\ell=0}^{j} \binom{\frac{n+j}{2}}{\ell} \leq 2\binom{(0.5 + \varepsilon^{3/2})n}{j} \\
&\leq n \cdot (0.5 + \varepsilon^{4/3})^{(1+\varepsilon)\log_2 n} \binom{n}{j-1} < |\delta(B)|,
\end{aligned}$$

where the last inequality holds because $(0.5 + \varepsilon^{4/3})^{(1+\varepsilon)} < \frac{1}{2}$ for $\varepsilon < 0.01$. Hence $\mathrm{comp}(\mathcal{F}') < \mathrm{comp}(\mathcal{F})$ and this completes the proof. $\square$

4.5 Proof of Theorem 4.1.7

Recall that $\mathbf{P}(n)$ denotes the collection of posets of order n that are rank-symmetric and rank-unimodal, and let $P \in \mathbf{P}(n)$. Furthermore, recall that $|A|$ denotes the rank of an element $A \in P$, $\mathrm{comp}(A, \mathcal{G}) \coloneqq |\{B \in \mathcal{G} : B \subset A \text{ or } A \subset B\}|$, and $N_r(A) \coloneqq \{B : |B| = r, B \subseteq A \text{ or } A \subseteq B\}$.

A poset P of rank n has *property* (Q) if all of the following hold:

(Q1) If $|B| < |A|$ and $||B| - n/2| < ||A| - n/2|$, then $|N_{|B|+i}(B)| \leq |N_{|A|-i}(A)|$ for every $i \in \{1, \ldots, |A| - |B|\}$.

(Q2) If $|B| > |A|$ and $||B| - n/2| < ||A| - n/2|$, then $|N_{|B|-i}(B)| \leq |N_{|A|+i}(A)|$ for every $i \in \{1, \ldots, |B| - |A|\}$.

(Q3) If $n/2 \leq |B| < |A|$, then $|N_{|B|-i}(B)| \leq |N_{|A|-i}(A)|$ for every $i \geq 1$.

(Q4) If $n/2 \geq |B| > |A|$, then $|N_{|B|+i}(B)| \leq |N_{|A|+i}(A)|$ for every $i \geq 1$.

The key result of this section is the lemma below, which will easily imply Theorem 4.1.7.

Lemma 4.5.1. *If a rank-symmetric and rank-unimodal poset P of rank n has Property (Q), then P has the centeredness property.*

Proof. Let $P \in \mathbf{P}(n)$ that has Property (Q). We say that a family $\mathcal{F} \subseteq P$ is *mid-compressed* if for every comparable pair $(A, B) \in \mathrm{Comp}(\mathcal{F})$ such that $||B| - n/2| < ||A| - n/2|$, $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

Claim 4.5.2. *For every $M \in \{1, \ldots, |P|\}$, there exists an M-optimal family in P that is mid-compressed.*

Proof. The proof of this claim is essentially the same as Kleitman's proof [49] of Theorem 4.1.1 and hence similar to our proof of Lemma 4.2.3, so we only give a sketch here. We show by induction on M that there exists an M-optimal family that is centered. The base case is $M \leq \Sigma_1(n, k)$, in which case there exists an antichain in $\mathcal{L}_{n/2}$ of size M.

Let now $M > \Sigma_1(n, k)$, and define an order relation on the collection of subsets of P of order M by setting $\mathcal{G} < \mathcal{F}$ if

- $\mathrm{comp}(\mathcal{G}) < \mathrm{comp}(\mathcal{F})$, or

- $\mathrm{comp}(\mathcal{G}) = \mathrm{comp}(\mathcal{F})$ and $\sum_{G\in\mathcal{G}} ||G| - n/2| < \sum_{F\in\mathcal{F}} ||F| - n/2|$.

Given a family $\mathcal{F} \subset P$ of size M that is not mid-compressed we will find a family $\mathcal{G}$ of size M that improves $\mathcal{F}$ (that is, $\mathcal{G} < \mathcal{F}$). Since only mid-compressed families cannot be improved this way this will show that there exists an M-optimal mid-compressed family.

Let $\mathcal{F} \subset P$ be a family of size M that is not mid-compressed. Then there exist elements A and B such that $A \in \mathcal{F}$, $B \notin \mathcal{F}$, and $||B| - n/2| < ||A| - n/2|$. W.l.o.g. there exists such a pair with $|A| > n/2$. Among all such pairs (A, B) consider the pairs with $|A|$ is maximal, and then among these pick one with $|B|$ maximal. Note that this implies that whenever $C \in P$ is such that $C \subset A$ and $|C| > |B|$ then $C \in \mathcal{F}$. Moreover whenever $C \in P$ is such that $B \subset C$ and $|C| > |A|$ then $C \notin \mathcal{F}$. Let $a := |A|$ a and $b := |B|$.

Form a bipartite graph with vertex sets $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ with edges between comparable pairs. If there exists a matching f between $\mathcal{F}_a$ and $\overline{\mathcal{F}_b}$ covering $\mathcal{F}_a$, then replacing $\mathcal{F}_a$ with the matching elements $f(\mathcal{F}_a)$ does not increase the number comparable pairs in $\mathcal{F}$ (since P has Property (Q1)), but decreases $\sum_{F\in\mathcal{F}} ||F| - n/2|$ and hence improves the family. From now on suppose that there is no such matching. Let $\mathcal{X} = \mathcal{F}_a$ and let $\mathcal{Y}$ be the family of neighbors of $\mathcal{F}_a$ in $\overline{\mathcal{F}_b}$.

Case 1: $|\mathcal{X}| \le |\mathcal{Y}|$. Since there is no matching between $\mathcal{X}$ and $\mathcal{Y}$ covering $\mathcal{X}$, we can find a maximal vertex set $\mathcal{X}_0 \subset \mathcal{X}$ such that $|N(\mathcal{X}_0)| < |\mathcal{X}_0|$. Let f be a matching between $\mathcal{F}_a - \mathcal{X}_0$ and $\mathcal{Y} - N(\mathcal{X}_0)$ covering $\mathcal{X} - \mathcal{X}_0$, which exists by the maximality of $\mathcal{X}_0$. Then $\mathcal{G} := \mathcal{F} \cup f(\mathcal{X} - \mathcal{X}_0) - (\mathcal{X} - \mathcal{X}_0)$ satisfies $\mathcal{G} < \mathcal{F}$ (again using that P has Property (Q1)).

Case 2: $|\mathcal{X}| > |\mathcal{Y}|$. If there exists a matching f covering $\mathcal{Y}$ then replacing $f(\mathcal{Y})$ by $\mathcal{Y}$ improves $\mathcal{F}$. Otherwise, let $\mathcal{Y}_0 \subset \mathcal{Y}$ be minimal such that $|N(\mathcal{Y}_0)| < |\mathcal{Y}_0|$. Consider the following two cases:

a) If there is a matching f between $\mathcal{Y}_0$ and $N_{\mathcal{X}}(\mathcal{Y}_0)$ covering $N(\mathcal{Y}_0)$, then let $\mathcal{G} := (\mathcal{F} \setminus N_{\mathcal{X}}(\mathcal{Y}_0)) \cup f(N_{\mathcal{X}}(\mathcal{Y}_0))$. Since there is no edge between $f(N_{\mathcal{X}}(\mathcal{Y}_0))$ and $\mathcal{F}_a$, we have $\mathrm{comp}(\mathcal{G}) < \mathrm{comp}(\mathcal{F})$.

b) Otherwise, there exists a vertex set $\mathcal{Z} \subseteq N_{\mathcal{X}}(\mathcal{Y}_0)$ with $|N_{\mathcal{Y}_0}(\mathcal{Z})| < |\mathcal{Z}|$. Then $\mathcal{Y}_0' := \mathcal{Y}_0 \setminus N_{\mathcal{Y}_0}(\mathcal{Z})$ is smaller than $\mathcal{Y}_0$ and it is easy to check that $|N_{\mathcal{X}}(\mathcal{Y}_0)| < |\mathcal{Y}_0|$, a contradiction with minimality of $\mathcal{Y}_0$.

This finishes the proof of the claim that there exists an M-optimal mid-compressed family. $\square$

From now on we assume that there exists an M-optimal mid-compressed family $\mathcal{F}^*$ that is not centered. Recall that $\Sigma_r(P)$ denotes the total size of the middle r layers of P. Define the integer $j \geq 0$ such that $\Sigma_{j-1}(P) < M \leq \Sigma_j(P)$. Let $\mathcal{G} \subset P$ be the centered family of size $\Sigma_j(P)$ and write $\Delta(\mathcal{G}) := \max\{\text{comp}(A, \mathcal{G}) : A \in \mathcal{G}\}$ for the maximum degree of the graph with vertex set $\mathcal{G}$ and edges corresponding to comparable pairs in P. Let $\text{comp}(M-1) := \min\{\text{comp}(\mathcal{F}) : \mathcal{F} \subseteq P, |\mathcal{F}| = M-1\}$. The following statement is very similar to Claim 4.2.8:

Claim 4.5.3. *We have* $\text{comp}(\mathcal{F}^*) \leq \text{comp}(M-1) + \Delta(\mathcal{G})$.

Proof. It suffices to construct a family $\mathcal{F}$ of size M with at most $\text{comp}(M-1) + \Delta(\mathcal{G})$ comparable pairs. As $\mathcal{F}^*$ is M-optimal it contains at most this many comparable pairs. By induction we know there exists a centered $(M-1)$-optimal family $\mathcal{H}$. Since $\mathcal{H} \subset \mathcal{G}$, adding to it any element of $\mathcal{G} \setminus \mathcal{H}$ increases the number of comparable pairs by at most $\Delta(\mathcal{G})$. □

Since $\mathcal{F}^*$ is not centered, it contains an element A such that for all elements $B \in \mathcal{G}$ we have $||A| - n/2| > ||B| - n/2|$. Since $\mathcal{F}^*$ is mid-compressed and P has properties (Q3) and (Q4), this implies that $\text{comp}(A, \mathcal{F}^*) \geq \Delta(\mathcal{G})$. Hence $\text{comp}(\mathcal{F}^*) \geq \text{comp}(M-1) + \Delta(\mathcal{G})$. By Claim 4.5.3 this implies that every family of size M contains at least $\text{comp}(M-1) + \Delta(\mathcal{G})$ comparable pairs. As shown in the proof of Claim 4.5.3 this value can be achieved by a centered family, completing the proof of Lemma 4.5.1. □

One well-known poset that satisfies the assumptions of Lemma 4.5.1 is the Boolean lattice $\mathcal{P}(n)$. Therefore, Lemma 4.5.1 implies Theorem 4.1.1 – rather unsurprisingly since the proof of Lemma 4.5.1 was motivated by Kleitman's proof of Theorem 4.1.1.

Let q be a prime power and let $n \geq 1$. To finish the proof of Theorem 4.1.7, we only need to check that the assumptions of Lemma 4.5.1 hold for $\mathcal{V}(q, n)$.

Claim 4.5.4. $\mathcal{V}(q, n)$ *is rank-symmetric.*

Proof. The map $V \to \mathbb{F}_q^n \setminus V$ takes the set of subspaces of dimension k into the set of subspaces of dimension $n - k$ bijectively. □

Claim 4.5.5. $\mathcal{V}(q, n)$ *is rank-unimodal.*

Proof. Note that the number of subspaces of $\mathcal{V}(q,n)$ of dimension k, written as $\begin{bmatrix} n \\ k \end{bmatrix}_q$, can be expressed as (see e.g. [73]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!},$$

where

$$[n]! = [1] \cdot [2] \cdot \ldots \cdot [n], \quad \text{and} \quad [i] = q^i - 1.$$

Rank-unimodality of $\mathcal{V}(q,n)$ is easily seen to follow from this formula. □

Claim 4.5.6. *$\mathcal{V}(q,n)$ has Property (Q).*

Proof. Properties (Q1)–(Q4) follow from the observation that if S is a subspace of $\mathbb{F}_q^n$ of dimension m then the number of spaces $S' \subset S$ of dimension $m-k$ is $\begin{bmatrix} m \\ k \end{bmatrix}_q$ and the number of spaces S' with $S \subset S'$ and $\dim(S') = m+k$ is $\begin{bmatrix} n-m \\ k \end{bmatrix}_q$. □

4.6 Open problems

Recall that $\mathbf{P}$ is the collection of posets that are rank-symmetric and rank-unimodal and let $\mathbf{C} \subset \mathbf{P}$ be the collection of posets which have the centeredness property. The main open problem that this paper has only barely begun to explore asks for an easy way to decide whether a poset $P \in \mathbf{P}$ is in $\mathbf{C}$. We know that $\{0,1\}^n \in \mathbf{C}$ and $\mathcal{V}(q,n) \in \mathbf{C}$ but for $k \geq 2$ and n large we have $\{0,1,\ldots,k\}^n \in \mathbf{P} \setminus \mathbf{C}$.

Now let P_G be the lattice of subgroups of a finite Abelian group G. It was shown in [26] that P_G is rank-unimodal. The following general question is likely to be difficult to solve in full generality but any progress would be interesting.

Question 4.6.1. *For what Abelian groups G is it true that $P_G \in \mathbf{C}$?*

Observe that most results of this paper are special cases of Question 4.6.1:

- if $G = C_{p_1} \times C_{p_2} \times \ldots \times C_{p_n}$ for distinct primes $p_1, p_2, \ldots, p_n$ then P_G is (isomorphic to) the Boolean lattice and hence $P_G \in \mathbf{C}$,
- if $G = C_{p_1^k} \times C_{p_2^k} \times \ldots \times C_{p_n^k}$ for distinct primes $p_1, p_2, \ldots, p_n$ then P_G is isomorphic to the lattice $\{0,1,\ldots,k\}^n$ under inclusion and hence if $n \geq n_0(k)$ then $P_G \in \mathbf{P} \setminus \mathbf{C}$.

- if $G = (C_p)^n$ for p prime then P_G is isomorphic to $\mathcal{V}(p,n)$ and hence $P_G \in \mathbf{C}$.

Question 4.6.1 can be asked for other members of $\mathbf{P}$, see e.g. [73]. A natural generalization of the centeredness property is as follows. For an integer $r \geq 2$ say that a poset $P \in \mathbf{P}$ has the *r-centeredness property* if for all M with $0 \leq M \leq |P|$, among all families $\mathcal{F} \subset P$ of size M, the number of r-chains contained in P is minimized by a centered family. Denote the collection of posets with the r-centeredness property by $\mathbf{C}_r$ and note that $\mathbf{C} = \mathbf{C}_2$. A long-standing conjecture in this area due to Kleitman [49] is that $\{0,1\}^n \in \mathbf{C}_r$ for all n, r. For recent progress on this conjecture we refer the reader to [20, 31, 32]. Asking for a characterisation of $\mathbf{C}_r$ is currently out of reach, but finding interesting necessary and/or sufficient conditions for a poset $P \in \mathbf{P}$ to be in $\mathbf{C}_r$ could be a fine result.

In a different direction one could improve Theorem 4.1.3 and investigate further for which M Conjecture 4.1.2 holds.

Question 4.6.2. *For which k and M is there an M-optimal centered family in $\{0,1,\ldots,k\}^n$?*

The same question can be asked for 'centered' replaced by 'canonical centered' (i.e. centered families with at most one partially filled layer). We expect that for $k = 2$ the answer to Question 4.6.2 contains the interval $[0, \Sigma_5(n,2)]$. It seems plausible that for $M \leq \Sigma_{\log_2 n}(n,k)$ the centered families are not too far from being best possible, but for much larger M we do not even have a guess what the best families could be. The following question is open whenever $\sqrt{n}$ is replaced by any value between $\log_2 n$ and n.

Question 4.6.3. *Let $M = \Sigma_{\sqrt{n}}(n,2)$. What do the M-optimal families in $\{0,1,2\}^n$ look like?*

Chapter 5

Chromatic-choosability of graph powers

The results in this chapter are joint work with Nicholas Kosar, Benjamin Reiniger, and Elyse Yeager and appear in [52].

Kim and Park found an infinite family of graphs whose squares are not chromatic-choosable, that is $\chi_\ell(G^2) > \chi(G^2)$. Xuding Zhu asked whether there is some k such that all k-th power graphs are chromatic-choosable. We answer this question in the negative: we show that there is a positive constant c such that for any natural number k, there is a family of graphs G with $\chi(G^k)$ unbounded and $\chi_\ell(G^k) \geq c\chi(G^k)\log\chi(G^k)$. Furthermore, we provide an upper bound, $\chi_\ell(G^k) < \chi(G^k)^3$ for $k > 1$.

5.1 Introduction

The *list-chromatic number* (or *choosability*) of a graph G, denoted $\chi_\ell(G)$, is the least l such that for any assignment of lists of size l to the vertices of G, there is a proper coloring of $V(G)$ where the color at each vertex is in that vertex's list. A graph is said to be *chromatic-choosable* if $\chi_\ell(G) = \chi(G)$. The *k-th power* of a graph G, denoted by G^k, is the graph on the same vertex set as G such that uv is an edge if and only if the distance from u to v in G is at most k.

Recall that the *line graph* $L(G)$ of G is a graph on the vertex $E(G)$ where two vertices are adjacent if and only if their corresponding edges are incident in G. The *total graph* $T(G)$ of G is a graph on the vertex $V(G) \cup E(G)$ where two vertices are adjacent if and only if their corresponding elements are adjacent or incident in G.

Several conjectures on the chromatic-choosability of various classes of graphs have been made. The List-Edge-Coloring Conjecture (LECC) asserts that $\chi'_\ell(G) = \chi'(G)$ for every graph G. According to Jensen and Toft [46], the LECC first appeared in a paper by Bollobás and Harris [23], but it was thought of earlier by several other authors. Since $\chi'_\ell(G) = \chi_\ell(L(G))$ and $\chi'(G) = \chi(L(G))$, the LECC can be stated as follows:

LECC Conjecture (Bollobás–Harris [23]; Vizing; Gupta; Albertson–Collins). *$L(G)$ is chromatic-choosable for every graph G.*

A generalization of the LECC is the The List-Total-Coloring Conjecture (LTCC), which states that $\chi''_\ell(G) = \chi''(G)$ for every graph G. Since $\chi''_\ell(G) = \chi_\ell(T(G))$ and $\chi''(G) = \chi(T(G))$, we can rewrite the LTCC in the following form:

LTCC Conjecture (Borodin–Kostochka–Woodall [24], 1997). *$T(G)$ is chromatic-choosable for every graph G.*

The List-Square-Coloring Conjecture(LSCC) suggests that even more is true.

LSCC Conjecture (Kostochka–Woodall [53], 2001). *G^2 is chromatic-choosable for every graph G.*

The LSCC is stronger than the LTCC since, given a graph G, its total graph $T(G)$ can be obtained by subdividing each edge of G and taking the square. The LSCC was recently disproved by Kim and Park [48], who constructed an infinite family of counter examples to the conjecture, and showed that the value $\chi_\ell(G^2) - \chi(G^2)$ can be arbitrarily large. Let K_{r*s} denote the complete r-partite graph with each part of size s.

Theorem 5.1.1 (Kim-Park [48], 2015). *For each prime $n \geq 3$, there exists a graph G such that G^2 is the complete multipartite graph $K_{(2n-1)*n}$.*

Since $\chi(K_{(2n-1)*n}) = 2n - 1$ and $\chi_\ell(K_{(2n-1)*n}) \geq (n-1)\lfloor\frac{4n-3}{n}\rfloor$ by [77], the authors conclude that there exists a graph G such that $\chi_\ell(G^2) - \chi(G^2) \geq n - 1$ for any prime $n \geq 3$. Xuding Zhu asked whether there is any k such that all k-th powers are chromatic-choosable [81]. We give a negative answer to Zhu's question, with a lower bound on $\chi_\ell(G^k)$ that matches that of Kim and Park for $k = 2$.

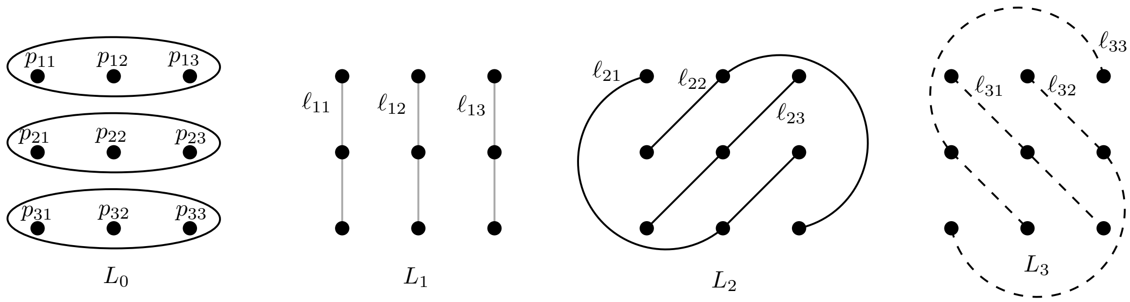


Figure 5.1: An affine plane for $n = 3$ and the decomposition of $\mathcal{L}$ into $\{L_0, L_1, L_2, L_3\}$.

Theorem 5.3.5. *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded such that*

$$\chi_\ell(G^k) \geq c\chi(G^k) \log \chi(G^k).$$

While preparing this note, it has come to our attention that Kim, Kwon, and Park have arrived at a similar result [47]. They have found, for each k, an infinite family of graphs G whose k-th powers satisfy $\chi_\ell(G^k) \geq \frac{10}{9}\chi(G^k) - 1$.

Let $f_k(m) = \max\{\chi_\ell(G^k) : \chi(G^k) = m\}$. Then Theorem 5.3.5 says that $f_k(m) \geq cm \log m$. Kwon (see [61]) observed that $f_2(m) < m^2$. We extend this observation to larger k in section 5.4.

Theorem 5.4.1. *Let $k > 1$. If k is even, then $f_k(m) < m^2$. If k is odd, then $f_k(m) < m^3$.*

5.2 Construction

The example of Kim and Park [48] for $k = 2$ is based on complete sets of mutually orthogonal latin squares. We will use this structure to find examples for all k, but we find the language of affine planes to be more convenient.

Take an affine plane $(\mathcal{P}, \mathcal{L})$ on n^2 points. Let $\{L_0, L_1, \ldots, L_n\}$ be the decomposition of $\mathcal{L}$ into parallel classes. Recall that we call the elements of $\mathcal{P}$ the *points* and the elements of $\mathcal{L}$ the *lines* of the plane, and that we have the following properties (see for instance [28]):

- Each line is a set of n points.

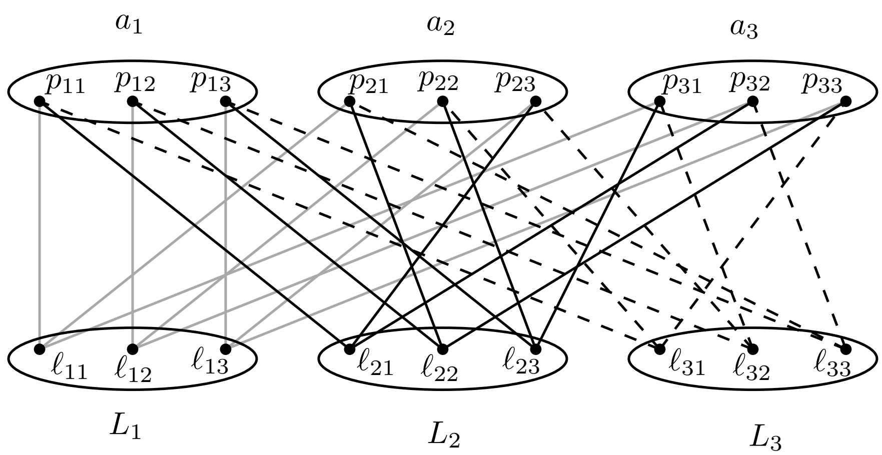


Figure 5.2: The graph H, here with $n = 3$.

- For each pair of points, there is a unique line containing them.
- The set of lines admits a partition into $n+1$ *parallel classes* $L_0, \ldots, L_n$ of equal size such that two lines in the same parallel class do not intersect and two lines in different parallel classes intersect in exactly one point.
- Such a plane exists whenever n is a (positive) power of a prime.

Form the bipartite graph H with parts $\mathcal{P}$ and $B = \mathcal{L} - L_0$, with $p\ell \in E(H)$ if and only if $p \in \ell$. Let $a_1, \ldots, a_n$ denote the lines of L_0. Consider the refinement $\mathcal{V}'$ of the bipartition of H obtained by partitioning $\mathcal{P}$ into $a_1, \ldots, a_n$ and B into $L_1, \ldots, L_n$. Note that the set of edges between a_i and L_j is a matching for each i and j. In Figure 5.2, the graph H is shown with $n = 3$. Edges are drawn differently according to which parallel class their line-endpoint belongs to, and the parts of $\mathcal{V}'$ are indicated.

Let $k \geq 2$. Subdivide the edges of H into paths of different lengths: edges incident to L_1 are subdivided into paths of length k, while edges not incident to L_1 are subdivided into paths of length $k+1$. For an edge $p\ell \in E(H)$, denote the vertices along the subdivision path as $p = (p\ell)_0, (p\ell)_1, (p\ell)_2, \ldots$. If $\ell \in L_1$, then $(p\ell)_k = \ell$, and if $\ell \notin L_1$, then $(p\ell)_{k+1} = \ell$. For a vertex $(p\ell)_i$, say its *level* is i, its *point* is p, and its *line* is ℓ (levels are well-defined, and points and lines of vertices of degree 2 are well-defined). Form the graph G by, for each $\ell \in \bigcup_{2 \leq i \leq n} L_i$, adding edges to make the neighborhood of ℓ a clique and then deleting ℓ. For each $i, j \in [n]$ and $m \in \{0, \ldots, k\}$, let

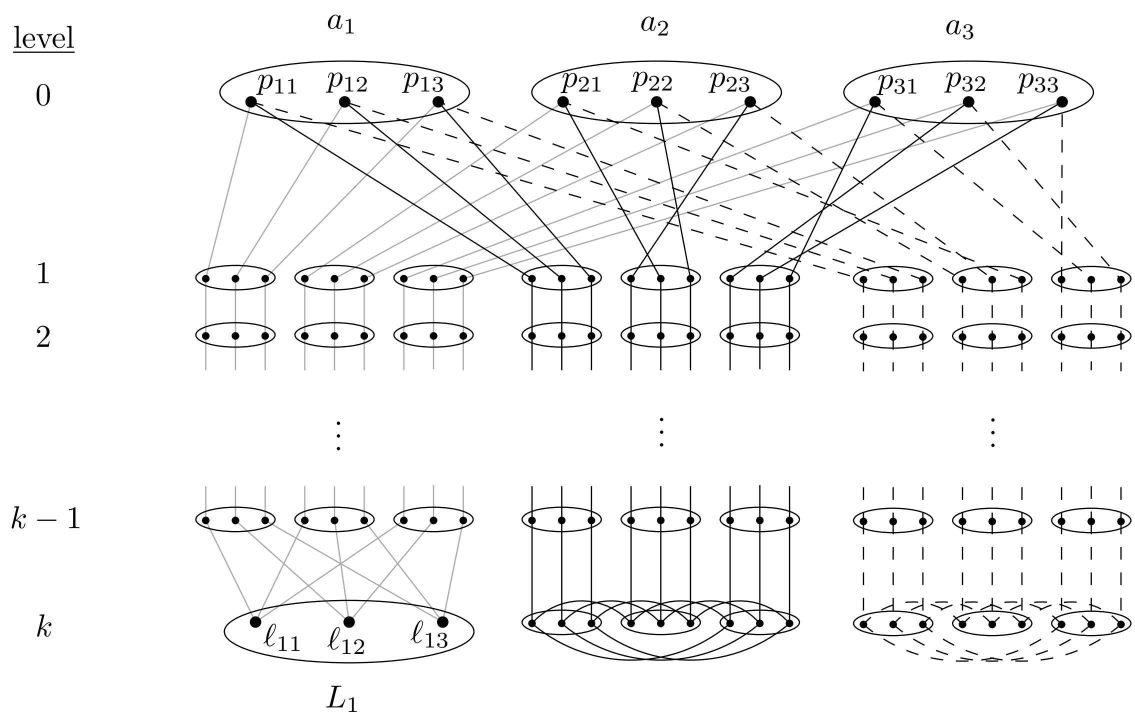


Figure 5.3: The graph G when $n = 3$.

$V_{i,j,m} = \{(p\ell)_m : p\ell \in E(H), p \in a_i, \ell \in L_j\}$; then $\{V_{i,j,m} : i, j \in [n], m \in \{0, \ldots, k\}\}$ is a partition of $V(G)$ into sets of size n, which we call $\mathcal{V}$. In Figure 5.3, the graph G is shown. Again we use $n = 3$, and here the parts of $\mathcal{V}$ are indicated.

5.3 Proof of Main Theorem

Lemma 5.3.1. *G^{4k} is multipartite with partition $\mathcal{V}$.*

Proof. Let p and q be two points in some a_i. Any path from p to q must start by increasing levels, arriving at $(p\ell)_k$. If $\ell \notin L_1$, then the path must move from $(p\ell)_k$ to $(p'\ell)_k$ for some p' not on a_i. Continuing along the path to level 0, we arrive at p'. Since p' is not on a_i, p' and q are on a common line $\ell' \in \bigcup_{i=1}^n L_i$. If $\ell' \in L_1$, the shortest path from p' to q is to increase levels to ℓ' and decrease levels to q. If $\ell' \in \bigcup_{i=2}^n L_i$, the shortest path from p' to q is to increase levels to $(p'\ell')_k$, move over to $(q\ell')_k$, and then decrease levels to q. Notice, if p and p' are on a common line in L_1, p' and q cannot be on a common line in L_1 because then p and q would be on a common line in L_1. Thus, the path uses at least 3 vertices in level k, and so has length at least $4k + 1$.

Let $\ell_1, \ell_2 \in L_1$. Any path would have to have both ends decrease to level 0. If both ℓ_1 and ℓ_2 connect to points in some a_i, then since these vertices are a distance at least $4k+1$ apart, the path between ℓ_1 and ℓ_2 would have length at least $4k+1$. Otherwise, the paths from ℓ_1 and ℓ_2 arrive at points on different lines in L_0, say p and q, respectively. These two points are on a common line not in L_0 or L_1, say ℓ. The shortest path between p and q is to go from p to $(p\ell)_k$, over to $(q\ell)_k$, and finally to q. However, this results in a path between ℓ_1 and ℓ_2 of length at least $4k+1$.

Let $(p\ell_1)_k, (q\ell_2)_k$ be two vertices in the same part other than L_1; that is, p, q are both on some a_i and ℓ_1, ℓ_2 are two lines in the same parallel line class. If a path joining them starts by decreasing levels from both ends to level 0, that is connects $(p\ell_1)_k$ to p and $(q\ell_2)_k$ to q, then since p and q are a distance at least $4k+1$ apart, the path between $(p\ell_1)_k$ and $(q\ell_2)_k$ would have length at least $4k+1$. Otherwise, at least one of $(p\ell_1)_k$ or $(q\ell_2)_k$ must first go to $(p'\ell_1)_k$ or $(q'\ell_2)_k$. Without loss of generality connect $(p\ell_1)_k$ to $(p'\ell_1)_k$. Now, any path must connect $(p'\ell_1)_k$ to p' and $(q\ell_2)_k$ to q. These are on a common line not in L_0, however, increasing levels from each of p' and q to level k results in a total of at least $4k+1$ steps.

Now consider two degree-two vertices in the same part. Any path joining them has ends that either increase or decrease levels from the endpoint. If the path increases levels from both ends or decreases levels from both ends, then we arrive at different vertices in the same level 0 or level k part. Since the rest of the path must have length at least $4k+1$, the total path must have length at least $4k+1$. Otherwise, one end increases levels and the other decreases levels. The resulting point, p, is not on the resulting line, ℓ. The path must next increase levels from p to a line. If this line is in the same parallel line class as ℓ, then the resultant path has length over $4k+1$. Otherwise, since this line is not in the same class as ℓ, these two lines share a common point. The shortest completion of the path is through this point. However, since at least one of these lines is not in L_1, the path must contain at least 3 vertices in level k. Thus, the path has length at least $4k+1$. $\square$

Lemma 5.3.2. *The subgraph of G^{4k} induced by the vertices in levels 0 through $k-1$ is complete multipartite with partition $\mathcal{V}$ restricted to those levels.*

Proof. Consider two points p, q on different lines in L_0. They are on a common line $\ell \in \bigcup_{i=1}^{n} L_i$. If $\ell \in L_1$, connect p to ℓ then ℓ to q. If $\ell \notin L_1$, connect p to $(p\ell)_k$ to $(q\ell)_k$ to q. In each case the path has length at most $2k+1<4k$.

Consider two vertices in different parts at level i, $1 \le i \le k-1$. Either their points are on different lines in L_0 or their lines are from different parallel classes. If their points are from different lines in L_0, go to these points. These points share a common line not in L_0. Connect via the path between this line. This takes at most $2i+2k+1 \le 4k-1$ steps. If their lines are from different parallel classes, increase levels to level k. These two lines share a common point. By, if necessary, first changing vertices at level k, connecting through this point, we get a path of length at most $2(k-i)+2+2k = 4k-2i+2 \le 4k$.

Finally, consider two vertices in levels i and j, $0 \le i < j < k$. Start a path joining them by decreasing levels from the lower-level vertex, and increasing levels from the larger-level vertex. Let the point we arrive at from decreasing the lower-level vertex be p. If the increasing from the larger-level vertex takes us to a line in L_1, we can connect from this line to a point on a different line of L_0 than p, say q. Now p and q are on a common line not in L_0. Connecting through this gives us a path of length at most $k-1+k+2k+1 = 4k$. If instead the increasing from the larger-level vertex takes us to a vertex of the form $(q\ell)_k$, $\ell \notin L_1$, then let ℓ' be the line through p in L_1. Now ℓ and ℓ' intersect at a point, say q'. We can complete the path by going from $(q\ell)_k$ to $(q'\ell)_k$ to q' to ℓ' to p. This takes a total of at most $k-1+1+3k = 4k$ steps. $\square$

We will use the following result of Alon.

Theorem 5.3.3 (Alon [1]). *Let K_{r*s} denote the complete r-partite graph with each part of size s. There are two constants, d_1 and d_2, such that*

$$d_1 r \log s \le \chi_\ell(K_{r*s}) \le d_2 r \log s.$$

The proof of the lower bound in Theorem 5.3.3 is based on a probabilistic argument that gives the following lemma:

Lemma 5.3.4 (Alon [1]). *There is a constant c such that for every $r, s \ge 2$ there is a set S of cardinality $cr \log s$ and a family $\mathcal{F}$ of s subsets of S, each of size at least $|S|/20$, so that there is no $X \subset S$ of size $|X| \le c \log s$ that intersects each member of $\mathcal{F}$.*

We sketch how the above lemma implies that $\chi_\ell(K_{r*s}) \geq d_1 r \log s$ with $d_1 = c/20$. Let $V_1, \ldots, V_r$ be the vertex classes in K_{r*s}. Consider using the set S as the set of $cr \log s$ colors, and the family $\mathcal{F} = \{F_1, \ldots, F_s\}$ as the family of lists. For each $j \in [s]$, assign the list F_j to the j-th vertex in each $V_1, \ldots, V_r$. Suppose that there exists a coloring of K_{r*s} from these lists, and for each $i \in [r]$, let X_i be the set of colors used on the vertex set V_i. Since X_i's are pairwise disjoint, $|X_1| + \cdots + |X_r| \leq |S|$, and so there exists i such that $|X_i| \leq |S|/r = c \log s$. By the lemma, there exists $j \in [s]$ such that $X_i \cap F_j = \emptyset$. But then the i-th vertex in part V_i could not have been colored, a contradiction.

Everything is now in place to complete the proof of our main theorem.

Theorem 5.3.5. *There is a positive constant c such that for every $k \in \mathbb{N}$, there is an infinite family of graphs G with $\chi(G^k)$ unbounded such that*

$$\chi_\ell(G^k) \geq c\chi(G^k) \log \chi(G^k).$$

Proof. Since G^{4k} is multipartite on $kn^2 + 1$ parts, $\chi(G^{4k}) \leq kn^2 + 1$, and so $n \geq \sqrt{(\chi(G^{4k}) - 1)/k}$. Since G^{4k} contains a complete multipartite subgraph with $(k-1)n^2$ parts of size n, we have from Theorem 5.3.3 that

$$\begin{aligned}
\chi_\ell(G^{4k}) &\geq d_1(k-1)n^2 \log n \\
&\geq d_1 \frac{k-1}{k} \left(\chi(G^{4k}) - 1\right) \log \sqrt{\frac{\chi(G^{4k}) - 1}{k}} \\
&= \frac{d_1}{2} \frac{k-1}{k} \left(\chi(G^{4k}) - 1\right) \left(\log(\chi(G^{4k}) - 1) - \log k\right) \\
&\geq \frac{d_1}{4} \left(\chi(G^{4k}) - 1\right) \left(\log(\chi(G^{4k}) - 1) - \log k\right).
\end{aligned}$$

Taking n large enough makes $\chi(G^{4k})$ as large as we like, and so by taking a constant c just smaller than $d_1/4$ and taking n sufficiently large, we obtain

$$\chi_\ell(G^{4k}) \geq c\chi(G^{4k}) \log \chi(G^{4k}).$$

The family $\{G^4\}$ is an infinite family of graphs whose k-th powers have the desired properties. $\square$

5.4 Upper bound

We now provide an upper bound on $\chi_\ell(G^k)$ in terms of $\chi(G^k)$.

Theorem 5.4.1. *Let $k > 1$. If k is even, then $f_k(m) < m^2$. If k is odd, then $f_k(m) < m^3$.*

When k is even, this follows from Kwon's observation (see [61]) that $\chi_l(G^2) < \chi(G^2)^2$ for every graph G. Since $G^{2k} = (G^k)^2$, we have $\chi_l(G^{2k}) < \chi(G^{2k})^2$ and so $f_{2k}(m) < m^2$.

When k is odd, we generalize Kwon's argument and prove the following.

Lemma 5.4.2. *Let $k \geq 3$, k odd. Then for any G, $\chi_\ell(G^k) \leq \Delta(G) \cdot \chi(G^k)^2$.*

Theorem 5.4.1 follows by noting that $\Delta(G) < \omega(G^k) \leq \chi(G^k)$ when $k > 1$.

Proof of Lemma 5.4.2. Let x be a vertex with maximum degree in G^k. Let A be the set of vertices at distance $\lceil k/2 \rceil$ from x in G. Let $B(v, r)$ denote the ball of radius r centered at v in G. Note that $\Delta(G^k) = \max\{|B(v,k)| : v \in V(G)\}$ and $\omega(G^k) \geq \max\{|B(v, \lfloor k/2 \rfloor)| : v \in V(G)\}$.

Since k is odd and bigger than 1, we have

$$B(x,k) - B(x, \lfloor k/2 \rfloor) \subseteq \bigcup_{y \in A} B(y, \lfloor k/2 \rfloor). \tag{5.4.1}$$

Hence

$$\begin{aligned}
\chi_\ell(G^k) &\leq 1 + \Delta(G^k) && \text{(degeneracy)} \\
&= |B(x,k)| \\
&\leq |B(x, \lfloor k/2 \rfloor)| + \sum_{y \in A} |B(y, \lfloor k/2 \rfloor)| && \text{(equation (5.4.1))} \\
&\leq (1 + |A|) \max_{v \in V(G)} |B(v, \lfloor k/2 \rfloor)| && \text{(bounding terms in sum)} \\
&\leq \Big(1 + (\Delta(G) - 1)\omega(G^k)\Big)\, \omega(G^k) \\
&\leq \Delta(G)\, \omega(G^k)^2 \\
&\leq \Delta(G)\, \chi(G^k)^2. && \square
\end{aligned}$$

5.5 Concluding Remarks

Using constructions similar to that of section 5.2, we have found infinite families of graphs G whose k-th powers are complete multipartite on roughly $kn^2/4$ parts each of size n, but only when $k \not\equiv 0 \mod 4$. The construction presented here is messier and does not yield complete multipartite powers, but it proves the theorem for all values of k simultaneously.

Question 5.5.1. *What is the correct order of magnitude of $f_k(m)$? Does it depend on k?*

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