Smoothing Properties of Certain Dispersive Nonlinear Partial Differential Equations

By

Erin Compaan

Dissertation

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Doctoral Committee:

Professor M. Burak Erdoğan, Chair
Associate Professor Nikolaos Tzirakis, Director of Research
Professor Jared Bronski
Professor Marius Junge

ABSTRACT

This thesis is primarily concerned with the smoothing properties of dispersive equations and systems. Smoothing in this context means that the nonlinear part of the solution flow is of higher regularity than the initial data. We establish this property, and some of its consequences, for several equations.

The first part of the thesis studies a periodic coupled Korteweg-de Vries (KdV) system. This system, known as the Majda-Biello system, displays an interesting dependancy on the coupling coefficient α linking the two KdV equations. Our main result is that the nonlinear part of the evolution resides in a smoother space for almost every choice of α. The smoothing index depends on number-theoretic properties of α, which control the behavior of the resonant sets. We then consider the forced and damped version of the system and obtain similar smoothing estimates. These estimates are used to show the existence of a global attractor in the energy space. We also use a modified energy functional to show that when the damping is large, the attractor is trivial.

The next chapter studies the Zakharov and related Klein-Gordon-Schrödinger (KGS) systems on Euclidean spaces. Again, the main result is that the nonlinear part of the solution is smoother than the initial data. The proof relies on a new bilinear Bourgain-space estimate, which is proved using delicate dyadic and angular decompositions of the frequency domain. As an application, we give a simplified proof of the existence of global attractors for the KGS flow in the energy space for dimensions two and three. We also use smoothing in conjunction with a high-low decomposition to show global well-posedness of the KGS evolution on $\mathbb{R}^4$ below the energy space for sufficiently small initial data.

In the final portion of the thesis we consider well-posedness and regularity properties of the "good" Boussinesq equation on the half line. We obtain local existence, uniqueness and continuous dependence on initial data in low-regularity spaces. We also establish a smoothing result, obtaining up to half derivative smoothing of the nonlinear term. The results are sharp within the framework of the restricted norm method that we use and match known results on the full line.

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Introduction

In the most broad terms, this work studies the dynamics of nonlinear dispersive partial differential equations (PDE). Linear dispersive equations are characterized by the property that solution components of different wavelengths propagate at different speeds. This means that, on unbounded domains solutions, tend to break apart and decay over time. In the study of nonlinear dispersive PDE, we seek to exploit this behavior and understand how it interacts with the nonlinear effects. Specifically, this thesis is concerned largely with smoothing effects of certain dispersive equations – for some PDE, the nonlinear part of the flow is of higher regularity than the initial data – a smoothing effect of the nonlinearity.

Consider a PDE of the form

$$u_t = \mathcal{L}u + F(u), \tag{1.1}$$

where $\mathcal{L}$ is a dispersive spatial differential operator and $F(u)$ is a nonlinear term. The operator $\mathcal{L}$ is of the form $ih(D)$, where D is the operator $D = -i\nabla$ and h is a real-valued order-m polynomial

$$h(y_1, y_2, \ldots, y_d) = \sum_{|\alpha| \leqslant m} c_\alpha \, y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_d^{\alpha_d}.$$

We will be concerned with the solutions in low-regularity Sobolev spaces. In such spaces, one cannot expect to find classical solutions, but it is possible to study distributional solutions. We say that u is a strong H^s solution if it is in $C_t^0 H_x^s \cap C_t^1 H_x^{s-m}$, where m is the order of the differential operator $\mathcal{L}$, and u satisfies (1.1) in the H^{s-m} sense. We frequently use Duhamel's solution formula

$$u(x,t) = e^{t\mathcal{L}} u_0 + \int_0^t e^{(t-s)\mathcal{L}} F(u(x,s)) \, \mathrm{d}s,$$

where $e^{t\mathcal{L}} u_0$ is the solution to the linear problem with initial data u_0. This formula, together with the Contraction Mapping Theorem, is central to many proofs of existence of solutions for PDE. It will be vital for the smoothing estimates presented in this work since it provides a formula, albeit an implicit integral one, for the nonlinear part of the flow.

For many such dispersive PDE in high-regularity spaces, there are classical well-posedness results. As regularity decreases, however, more refined approaches become necessary. In 1993, Bourgain published his influential restricted-norm papers [19, 20]. The spaces he used, called $X^{s,b}$ spaces, are L^2-based, with weights tailored to the dispersion of the equation. Similar weighted spaces had been used by Beals [9], Rauch and Reed [86], and Klainerman and Machedon [70] in works on the wave equation. Use of these spaces enabled Bourgain to prove local theory results at regularities hitherto unattainable. These spaces continue to be important to dispersive PDE research. In this work, we use $X^{s,b}$ spaces extensively – for problems posed on the torus, $\mathbb{R}^d$, and, in the last chapter, for an equation on the half-line.

In addition to local well-posedness, it may be possible to obtain global existence of solutions. The simplest way to achieve this is via a conservation law which yields an *a priori* bound on the norm of a solution. Iterating the local theory then gives global existence. In the absence of helpful conservation laws, other approaches, such as Bourgain's separation of high and low frequencies [22] or the I-method of Colliander, Keel, Staffilani, Takaoka, and Tao [28] may yield global results. In this work, we give a result which shows how smoothing estimates can be used with a separation of high and low frequencies to obtain global existence.

The first result in this thesis is a smoothing result for the periodic Majda-Biello system (1.2) of coupled KdV-type equations. This is a physical equation used to model global atmospheric and oceanic currents [75]. Note that when one models such phenomena, the periodic case is of particular interest. The system is as follows:

$$
\begin{cases}
u_t + u_{xxx} + \frac{1}{2}(v^2)_x = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R} \\
v_t + \alpha v_{xxx} + (uv)_x = 0, \\
(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0) \in \dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T}).
\end{cases}
\tag{1.2}
$$

The well-posedness theory [79] depends heavily on the arithmetic properties of α, particularly a measure of how well certain paramters $c = c(\alpha)$ can be approximated by rationals. This occurs because for $\alpha \in (0, 1)$ – a physically important case – the system exhibits complex resonance relations, far different than those of a single KdV. The proof of the smoothing estimate uses a refinement of the method of [40], which combines a normal form transformation (see [8]) with $X^{s,b}$

space estimates. This process allows us to use the frequency-space oscillation to overcome the effects of the derivative nonlinearity – the normal forms transformation allows us to quantify the oscillatory effects. However, the transformation also leads to third-order nonlinearities instead of quadratic, and appreciably more difficult resonance relations. These challenges are overcome by a careful analysis of the frequency interactions.

The same smoothing result also holds for the dissipative Majda-Biello system – i.e. the same system (1.2) with added forcing and damping terms. Using smoothing, we show that for almost every α, all solutions will eventually enter a compact set which is an invariant under the evolution, called the global attractor. The existence of such sets has been studied extensively, but proofs of their existence are often long and technical. The smoothing allows us to give an simple and elegant proof in this case. Using a modified energy functional, we also show that when the damping is large in relation to the forcing terms, the attractor is trivial, consisting of only a single function.

In the next chapter, we establish smoothing estimates in higher-dimensional Euclidean spaces for the Zakharov and Klein-Gordon-Schrödinger (KGS) systems. The Zakharov equation (1.3) is a model for Langmuir turbulence in plasma [101], and has been extensively studied. The related KGS system exhibits similar dispersive dynamics, and has applications in particle physics [46].

$$
\begin{cases}
iu_t + \Delta u = un, \quad x \in \mathbb{R}^d,\ t \in \mathbb{R} \\
n_{tt} - \Delta n = \Delta |u|^2 \\
\big(u(\cdot, 0), n(\cdot, 0), n_t(\cdot, 0)\big) = (u_0, n_0, n_1) \in H^s \times H^r \times H^{r-1}.
\end{cases} \tag{1.3}
$$

Deriving smoothing estimates for these models presents challenges because in $\mathbb{R}^d$, one must control resonant hypersurfaces rather than merely resonant points or lines. The approach extends the method used in [26] to prove bilinear estimates for a Schrödinger equation on $\mathbb{R}^2$. A major obstacle was the effect of the wave part of the system, which makes the resonances more difficult to understand and bound. This is overcome by extensive use of Littlewood-Paley and parabolic decompositions in the frequency space. As applications of the smoothing result, we show a global well-posedness for the KGS on $\mathbb{R}^4$ in regularities below the energy space via Bourgain's high-low method. We also provide a simplified proof of the existence of global attractors for the KGS in dimensions two and three using the smoothing property.

Finally, in the last chapter we show well-posedness and smoothing results for the "good" Boussinesq equation (1.4) on the half-line. The work on this equation is joint with N. Tzirakis.

$$
\begin{cases}
u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, & x \in \mathbb{R}^+,\ t \in \mathbb{R}^+ \\
u(0,t) = h_1(t), \qquad u_x(0,t) = h_2(t), \\
u(x,0) = f(x), \qquad u_t(x,0) = g_x(x).
\end{cases}
\tag{1.4}
$$

On the full line, local existence has been shown using standard $X^{s,b}$ space methods for $s > -\frac{1}{4}$ [42], and sharp local existence, using modified $X^{s,b}$ spaces, holds for $s \geqslant -\frac{1}{2}$ [68]. On the half-line, however, the standard Fourier analytic techniques cannot be directly applied, and much less was known. We show local existence, uniqueness, and smoothing results for negative Sobolev indices using the methods of [38, 39]. To do so, we extend the initial data to the full line, and write the Duhamel formula for the solution on $\mathbb{R}$. We then add a forcing term to this formula to ensure that the boundary conditions are met. The Contraction Mapping Theorem in $X^{s,b}$ space is used to obtain a solution. The boundary forcing term requires an explicit solution formula for the linear initial-boundary-value problem, which we derive using Laplace transforms. We then obtain a number of $X^{s,b}$ and Sobolev space estimates to close the contraction argument.

Initial-boundary-value problems present challenges because much of the powerful machinery which has been developed to study dispersive initial-value problems depends heavily on Fourier analytic techniques, and it is not clear how these approaches can be generalized to bounded domains.

For relatively smooth data, more classical approaches may prove applicable. For instance, in 1983, Bona and Winther showed well-posedness of a KdV equation on the half-line via a parabolic regularization argument for initial data in $H^4(\mathbb{R}^+)$ [18]. More recently, Colliander and Kenig [29] and Holmer [57] adapted the real-line theory of Bourgain [20] and Kenig, Ponce, and Vega [63] to show well-posedness of the KdV for $H^{-\frac{3}{4}}(\mathbb{R}^+)$ initial data. Their approach involves recasting the problem as a real line problem with forcing. The forcing is a multiple of the Dirac mass at $x = 0$, and is chosen to enforce the boundary condition. This method has also been applied to the Schrödinger equation [56].

Other recent approaches include Fokas' unified transform method [44, 54, 45]. Bona, Sun, and Zhang have also recently published works on the initial-boundary-value problem for dispersive

models, see e.g. [16, 17] for results concerning the KdV and Schrödinger equations. An advantage of our approach is that we apply $X^{s,b}$ space tools – which *a priori* appear only applicable to $\mathbb{R}^d$ or $\mathbb{T}^d$ – to the half-line problem, without the use of a Dirac mass forcing and ensuing technicalities.

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NOTATION AND BACKGROUND

2.1   NOTATION

The Fourier sequence of a function $u \in L^2(\mathbb{T})$ is defined by

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} \ \mathrm{d}x \quad \text{for } k \in \mathbb{Z}.$$

We use the corresponding periodic Sobolev spaces $H^s(\mathbb{T})$, with their norms given by

$$\|u\|_{H^s} = \|\langle k \rangle^s u_k\|_{\ell_k^2},$$

where $\langle k \rangle = (1 + |k|^2)^{1/2}$. The notation $\dot{H}^s(\mathbb{T})$ indicates the mean-zero counterpart of this space, i.e. $\dot{H}^s(\mathbb{T}) = \{u \in H^s \mid \int u \ \mathrm{d}x = 0\}$.

The Fourier transform of a Euclidean space function $u \in L^2(\mathbb{R}^d)$ is defined similarly:

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} u(x) e^{-i\xi \cdot x} \ \mathrm{d}x.$$

The Euclidean Sobolev spaces $H^s(\mathbb{R}^d)$ have norms given by

$$\|u\|_{H^s} = \|\langle \xi \rangle^s \hat{u}(\xi)\|_{L^2}.$$

We also use the homogeneous space $\dot{H}^s(\mathbb{R}^d)$, with $\|u\|_{\dot{H}^s} = \||\xi|^s \hat{u}(\xi)\|_{L^2}$.

The expression $e^{-tL} u_0$ will denote the solution to the linear problem $u_t + Lu = 0$ with $u(\cdot, 0) = u_0$. Thus, for example, $e^{it\Delta} u_0$ is the linear Schrödinger flow with initial data u_0.

We write $a \lesssim b$ to indicate that there is an absolute constant C such that $a \leqslant Cb$. The symbol $\gtrsim$ is used similarly. The expression $a \approx b$ means that $a \lesssim b$ and $a \gtrsim b$. The notation $a \simeq b$ is used to indicate that $|a - b| \leqslant \delta$ for some small δ determined by the context. We write $a-$ for $a - \epsilon$ when $\epsilon > 0$ is arbitrary; similarly we write $a+$ for $a + \epsilon$.

2.2 $X^{s,b}$ Theory

We now introduce some general $X^{s,b}$ space theory. Consider again a PDE of the form

$$u_t = \mathcal{L}u + F(u).$$

Suppose for the moment that $F = 0$, so that the equation is linear. If the initial condition is a simple wave e^{ikx} for some k, we can find a function $\omega(k)$ such that the wave

$$e^{ik\left(x - \frac{\omega(k)}{k}t\right)}$$

is a solution of the linear PDE. This function ω is called the dispersion relation for the PDE, and the equation is called dispersive if $\omega(k)/k$ is not a constant. For instance, for the Airy equation $u_t + u_{xxx} = 0$, we find $\omega(k) = k^3$, so this equation is dispersive, but for the transport equation $u_t + u_x = 0$, the dispersion relation is $\omega(k) = k$, so it is not dispersive.

We can also think of the dispersion relation in another way. Taking the Fourier transform in space and time, we obtain the equation

$$\tau \hat{u}(\xi, \tau) = \omega(\xi)\hat{u}.$$

This means that the Fourier transform of the solution to the linear PDE is supported on the curve (or hypersurface) $\tau = \omega(\xi)$ in frequency space. If the PDE is nonlinear, of course, this is no longer true. However, one can hope that, for short times at least, the support of the solution in frequency space remains near this curve. The definition of $X^{s,b}$ space is based on this observation. The $X^{s,b}$ norm for functions on $\mathbb{R}^d \times \mathbb{R}$ is defined by

$$\|u\|_{X^{s,b}_{\tau=\omega(\xi)}} = \|\langle\xi\rangle^s\langle\tau - \omega(\xi)\rangle^b \hat{u}(\xi, \tau)\|_{L^2_{\xi,\tau}}.$$

Similarly, for functions on $\mathbb{T}^d \times \mathbb{R}$, the norm is defined by

$$\|u\|_{X^{s,b}_{\tau=\omega(k)}} = \|\langle k\rangle^s\langle\tau - \omega(k)\rangle^b \hat{u}(k, \tau)\|_{\ell^2_k L^2_\tau}.$$

Thus the norm is similar to an H^s norm, but with an additional weight that inflates the norms of functions supported far from the $\tau = \omega(\xi)$ surface. We will drop the $\tau = \omega(\xi)$ subscript and write only $X^{s,b}$ when the dispersion relation is clear from the context. The following alternate definition

of the norm can be illuminating. If $e^{t\mathcal{L}}$ is the linear flow operator corresponding to $u_t = \mathcal{L}u$, the $X^{s,b}$ norm above can be written as

$$\|u\|_{X^{s,b}} = \|e^{-t\mathcal{L}}u\|_{H_x^s H_t^b}.$$

We now address some useful general properties of $X^{s,b}$ spaces. This discussion is based on [94, 50]; proofs and further discussion can be found in these works. To begin, we note that $X^{s,b}$ is a Banach space. Using Parseval's theorem, one can see that that dual of $X^{s,b}_{\tau=\omega(\xi)}$ is $X^{-s,-b}_{\tau=-\omega(-\xi)}$. The change in the dispersion relation is due to complex conjugation, which flips signs on the frequency side.

It's reasonable to ask how $X^{s,b}$ spaces relate to other, more classical, spaces. For instance, if we prove well-posedness in $X^{s,b}$ space, does that say anything about existence in Sobolev spaces? The answer is often yes, due to the following lemma.

Lemma 2.2.1 ([94]). *For $b > \frac{1}{2}$, the $X^{s,b}$ space corresponding to a continuous dispersion relation $\omega(\xi)$ embeds continuously in $C_t^0 H_x^s$.*

Thus if we prove existence in $X^{s,b}$ with $b > \frac{1}{2}$, we have existence in $C_t^0 H_x^s$ automatically. Sometimes, though, to obtain the estimates necessary for well-posedness, one must work with $b \leq \frac{1}{2}$. An example of this appears in the last chapter of this work, where we must take $b < \frac{1}{2}$ in order to complete the well-posedness argument. In such cases some extra effort is required to show that the solutions are also H^s solutions.

We now outline the estimates necessary to prove well-posedness in $X^{s,b}$ spaces. The idea of $X^{s,b}$ space is that the support of nonlinear solutions remains close to that of the linear solutions. This is only expected to hold for short times, so in order to use $X^{s,b}$ spaces, we multiply by a time-localization cut-off function η. The well-posedness argument uses the Contraction Mapping Theorem on the Duhamel operator Φ given by

$$\Phi u(x,t) = \eta(t)e^{t\mathcal{L}}u_0 + \eta(t)\int_0^t e^{(t-t')\mathcal{L}}[\eta(t'/T)F(u(x,t'))]\,\mathrm{d}t. \tag{2.1}$$

The first term of Φ, the localized linear solution, can be bounded in $X^{s,b}$ spaces using the following lemma.

Lemma 2.2.2. *Suppose $u_0 \in H_x^s$, and let η be a smooth compactly supported function. Then we have*

$$\|\eta(t)e^{t\mathcal{L}}u_0\|_{X^{s,b}} \lesssim \|u_0\|_{H^s}.$$

Proof. Notice that

$$\|\eta(t)e^{t\mathcal{L}}u_0\|_{X^{s,b}} = \|\langle\tau - \omega(\xi)\rangle^b \hat{\eta}(\tau - \omega(\xi))\langle\xi\rangle^s \hat{u}_0(\xi)\|_{L^2_\tau L^2_\xi} = \|\eta\|_{H^b}\|u_0\|_{H^s}.$$

Since η is smooth and compactly supported, its H^b norm is certainly finite and we're done. $\qquad\square$

To control the Duhamel integral term, we require the following estimate.

Lemma 2.2.3 ([94])**.** *Let $\mathcal{L}$ be a linear differential operator with a real polynomial dispersion relation. For any $b > \frac{1}{2}$ and smooth compactly supported function η, we have*

$$\left\|\eta(t)\int_0^t e^{(t-t')\mathcal{L}}F(u(x,t'))\,\mathrm{d}t'\right\|_{X^{s,b}} \lesssim \|F(u(x,t))\|_{X^{s,b-1}}.$$

We also have the following result, which allows us to extract a power of the length of the time interval. This is important in closing the contraction mapping argument – it allows us to control the norm by choosing a sufficiently small time interval.

Lemma 2.2.4 ([94])**.** *Let η be a Schwarz function. Then for $-\frac{1}{2} < b - 1 \leqslant b' < \frac{1}{2}$ and any $0 < T \leqslant 1$, we have*

$$\|\eta(t/T)u\|_{X^{s,b-1}} \lesssim T^{1+b'-b}\|u\|_{X^{s,b'}}.$$

Applying these results to (2.1), we have the bound

$$\|\Phi u\|_{X^{s,b}} \lesssim \|u_0\|_{H^s} + T^{1+b'-b}\|F(u)\|_{X^{s,b'}}$$

for $b > \frac{1}{2}$ and $b - 1 \leqslant b' < \frac{1}{2}$. To finish the argument, we need an estimate of the form $\|F(u)\|_{X^{s,b'}} \lesssim \|u\|_{X^{s,b}}^\beta$. The proofs of such estimates can be quite challenging, depending on the specific dispersion relation and nonlinearity under consideration. However, with such an estimate in hand, we can close the contraction on a ball $\{u \ : \ \|u\|_{X_{s,b}} \leqslant C\|u_0\|_{H^s}\}$ by choosing a sufficiently small time T.

2.3 Useful Estimates

The following special case of the Sobolev embedding theorem will be used at several stages in the proofs.

Theorem 2.3.1 (Sobolev Embedding [1])**.** *The space $H^s(\mathbb{R}^n)$ is continuously embedded in $L^\infty(\mathbb{R}^n)$ for $s > \frac{n}{2}$. Similarly, $H^s(\mathbb{T}^n)$ is continuously embedded in $L^\infty(\mathbb{T}^n)$ for $s > \frac{n}{2}$.*

We also require the Gagliardo-Nirenberg-Sobolev inequality.

Theorem 2.3.2 (Gagliardo-Nirenberg-Sobolev Inequality [77])**.** *Suppose that $u \in L^q(\mathbb{R}^n)$ and $D^m u \in L^r(\mathbb{R}^n)$. For $1 \leqslant q, r \leqslant \infty$, we have*

$$\|D^j u\|_{L^p} \lesssim \|D^m u\|_{L^r}^\alpha \|u\|_{L^q}^{1-\alpha},$$

where $\frac{1}{p} = \frac{j}{n} + (\frac{1}{r} - \frac{m}{n})\alpha + \frac{1-\alpha}{q}$ and $\alpha \in [\frac{j}{m}, 1]$.
If $j = 0$, $rm < n$, and $q = \infty$, we require the additional assumption that either u tends to zero at infinity or $u \in L^s$ for some $s > 0$. If $1 < r < \infty$ and $m - j - \frac{n}{r}$ is a non-negative integer, then we require that $\alpha < 1$.

The following calculus lemmata will be used frequently in the proofs. See, e.g., the appendix of [40] for proofs of similar results. For periodic problems, we use the summation estimates; the integral estimates are used for problems posed on Euclidean spaces.

Lemma 2.3.3.   *1. If $\beta \geqslant \gamma > 1$, then*

$$\sum_n \frac{1}{\langle n - k_1 \rangle^\beta \langle n - k_2 \rangle^\gamma} \lesssim \langle k_1 - k_2 \rangle^{-\gamma}.$$

2. If $\beta > \frac{1}{3}$, then

$$\sum_n \frac{1}{\langle n^3 + an^2 + bn + c \rangle^\beta} \lesssim 1,$$

with the implicit constant independent of a, b, and c.

Lemma 2.3.4. *If $\beta \geqslant \gamma \geqslant 0$ and $\beta + \gamma > 1$, then we have*

$$\int \frac{1}{\langle x - a \rangle^\beta \langle x - b \rangle^\gamma} \, \mathrm{d}x \lesssim \langle a - b \rangle^{-\gamma} \varphi_\beta(a - b),$$

where

$$\varphi_\beta(c) = \begin{cases} 1 & \text{if } \beta > 1 \\ \log(1 + \langle c \rangle) & \text{if } \beta = 1 \\ \langle c \rangle^{1-\beta} & \text{if } \beta < 1. \end{cases}$$

2.4 Normal Forms

In our study of the coupled KdV system, we employ a normal form transformation. The idea has its roots in the theory of Poincaré normal forms for ordinary differential equations. This method uses a sequence of polynomial transformations to eliminate the nonlinearity in the ODE system; see, e.g., [6]. The version of the normal forms transformation which we use for our PDE system is similar to the one introduced by Shatah in [91]. In this work, he uses a transformation to raise the degree of the nonlinearity of a Klein-Gordon equation on $\mathbb{R}^3$, thus rendering the equation susceptible to analysis by perturbative methods.

For the periodic KdV equation, the normal forms transform method was introduced as an alternative method to arrive at well-posedness by Babin, Ilyin, and Titi in [8]. Our work using normal forms to prove a smoothing effect for the Majda-Biello system builds on a similar result of Erdoğan and Tzirakis for the KdV and Zakharov equations [36, 40].

There are several steps to the method. The first is to take a Fourier transform in space. This results in a system of ordinary differential equations for the Fourier coefficients. Multiplying by a modulation factor, the system can be reduced to a collection of equation of roughly the form

$$\partial_t v = e^{i\omega t} F(v).$$

In practice, there will be infinitely many $v = v_k$ and corresponding ODE – one for each frequency – and the nonlinearity on the right-hand side will depend on *all* the frequencies. However, this simple equation is helpful for illustration. Notice that an equation of this form can be written as

$$\partial_t v = \partial_t \left[\frac{e^{i\omega t}}{i\omega} F(v) \right] - \frac{e^{i\omega t}}{i\omega} F'(v) v_t$$

$$= \partial_t \left[\frac{e^{i\omega t}}{i\omega} F(v) \right] - \frac{e^{2i\omega t}}{i\omega} F(v) F'(v) \qquad \Rightarrow$$

$$\partial_t \left[v - \frac{e^{i\omega t}}{i\omega} F(v) \right] = -\frac{e^{2i\omega t}}{i\omega} F(v) F'(v).$$

In practice, the form of F is quite complex, and there may be resonances (where $\omega = 0$) which must be treated separately. However, the basic result of the transformation should be apparent – we've increased the power of the nonlinearity from F to FF', but gained an ω in the denominator. For the Majda-Biello system, the nonlinearity becomes cubic rather than quadratic, but ω is a cubic polynomial in the frequency variable. This polynomial in the denominator will allow us to overcome the effect of the derivative on the nonlinearity and obtain a smoothing result.

2.5   Global Attractors

A global attractor for a dynamical system has three features – it is compact, it is invariant under the flow, and it eventually attracts all elements in the phase space. Such sets are of interest in understanding the long term dynamics of dissipative systems. The concept of a global attractor dates back to the work of Auslander, Bhatia, and Seibert [7] in the 1960s. Previous researchers in dynamical systems had considered attracting *points*, but not larger sets which attracted the flow [76]. Since they're compact, and sometimes even finite dimensional, global attractors can be easier to study than the full infinite-dimensional phase space, while still providing valuable insight into solution behavior. For instance, in the 1980s Ghidaglia showed the the dissipative periodic Schrödinger and Korteweg-de Vries equations have global attractors of finite Hausdorff (and fractal) dimension [48, 49].

The following definitions and results from the study of dissipative PDE in classical dynamical systems will be vital to our study of global attractors for the damped Majda-Biello system and Klein-Gordon-Schrödinger systems. In our context, the phase space is the energy space of the system, i.e. $H^1(\mathbb{T}) \times H^1(\mathbb{T})$ for the Majda-Biello system and $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ for the KGS system. The semigroup operator is the solution flow of the PDE system, which is defined globally in time. A global attractor is then defined as follows.

Definition 2.5.1 ([95]). *A compact subset $\mathcal{A}$ of the phase space H is called a global attractor for the semigroup $\{U(t)\}_{t \geqslant 0}$ if $\mathcal{A}$ is invariant under the flow of U and*

$$\lim_{t \to \infty} d(U(t)u_0, \mathcal{A}) = 0 \text{ for every } u_0 \in H.$$

We also have the weaker notion of absorbing sets:

Definition 2.5.2 ([95]). *A bounded subset $\mathcal{B}_0$ of the phase space H is called absorbing if for any bounded $\mathcal{B} \subset H$, there exists a time $T = T(\mathcal{B})$ such that $U(t)\mathcal{B} \subset \mathcal{B}_0$ for all $t \geqslant T$.*

Notice that though a global attractor is necessarily an absorbing set, an absorbing set need not be a global attractor.

Our global attractors will be shown to be the ω-limit set of the absorbing set $\mathcal{B}_0$, which is defined by

$$\omega(\mathcal{B}_0) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} U(t)\mathcal{B}_0}.$$

An alternative, and perhaps more intuitively understandable definition of the ω-limit set is that it is the set of all points u such that there is a function $u_0 \in \mathcal{B}_0$ and an increasing sequence of times $t_n \to \infty$ such that $U(t_n)u_0$ converges to u in the phase space. Taking the ω-limit set will reduce the absorbing set, which may be very large, down to its "essential elements", leaving a compact invariant set.

We will use the following theorem to show that the ω-limit set is indeed a global attractor.

Theorem 2.5.3 ([95]). *Let H be a metric space and $U(t)$ be a continuous semigroup from H to itself for all $t \geqslant 0$. Assume that there is an absorbing set $\mathcal{B}_0$. If the semigroup $\{U(t)\}_{t \geqslant 0}$ is asymptotically compact, i.e. for every bounded sequence $\{x_k\} \subset H$ and every sequence $t_k \to \infty$, the set $\{U(t_k)x_k\}_k$ is relatively compact in H, then $\omega(\mathcal{B}_0)$ is a global attractor.*

Given an absorbing set, this theorem reduces the problem of existence of attractors to showing asymptotic compactness of the flow. However, showing asymptotic compactness is not trivial. Most arguments in previous works required several steps. First, they used functional analysis arguments to show the existence of a weakly-convergent subsequence. Then properties of the PDE, such as decay of an energy functional, are used to upgrade this weak convergence to strong convergence. The arguments can be complex and technical. They often involving proving estimates on truncated domains, and then showing that the estimates are independent of the domain size. In this work, we use smoothing to simplify matters. For the periodic Majda-Biello system, we use the method introduced in [37], which uses smoothing together with the compact embedding of Sobolev spaces to provide a brief and elegant proof of the existence of an attractor. In the case of the Klein-

Gordon-Schrödinger system, we use smoothing to greatly simplify the energy arguments used to upgrade weak convergence to strong convergence.

The Periodic Majda-Biello System

3.1 Introduction

This chapter is concerned with the system (3.1) of coupled KdV-type equations on the torus. The results here have appeared in [30].

$$
\begin{cases}
u_t + u_{xxx} + \frac{1}{2}(v^2)_x = 0, & x \in \mathbb{T} \\[2mm]
v_t + \alpha v_{xxx} + (uv)_x = 0, \\[2mm]
(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0) \in \dot{H}^s(\mathbb{T}) \times H^s(\mathbb{T}).
\end{cases}
\tag{3.1}
$$

This system was introduced by Majda and Biello, [75, 12], as a simplified asymptotic model for the behavior of certain atmospheric Rossby waves. Rossby waves are long atmospheric or oceanic waves which have significant effects on weather patterns and ocean currents. The system (3.1) models such waves in the upper atmosphere. In the model, u corresponds to a Rossby wave with significant energy in the midlatitudes and v corresponds to a Rossby wave confined to the equatorial region. The system is designed to capture the nonlinear interactions between the waves under specific physical conditions – such interaction is relevant in both theoretical atmospheric science and weather prediction. Majda and Biello obtained numerical estimates of 0.899, 0.960, and 0.980 for the coupling parameter α in the physical cases they considered. We note that in the case of atmospheric waves, the periodic problem is physically relevant.

Solutions of the Majda-Biello system have momentum conservation. They also satisfy conservation laws at the L^2 and H^1 levels. Specifically, the following quantities are constant:

$$
E_1 = \int u \ \mathrm{d}x \qquad E_2 = \int v \ \mathrm{d}x \qquad E_3 = \int u^2 + v^2 \ \mathrm{d}x \qquad E_4 = \int u_x^2 + \alpha v_x^2 - uv^2 \ \mathrm{d}x.
\tag{3.2}
$$

The last integral above is the Hamiltonian conservation law. However, unlike the KdV, the system is not completely integrable, even in the relatively simple case $\alpha = 1$ – it has been shown

that there are no higher conservation laws [97]. The system scales like the KdV, leading to a critical Sobolev index of $-\frac{3}{2}$.

The original KdV equation $u_t + u_{xxx} + uu_x = 0$ is a water wave model introduced in the 19th century [23, 71]. It has long been studied as a canonical example of a dispersive equation with derivative nonlinearity. Low-regularity results may be traced back to the work of Bona and Smith, who used parabolic regularization along with energy inequalities to obtain local well-posedness on $\mathbb{R}$ or $\mathbb{T}$ for $s \geqslant 2$ [15]. Kato used similar methods to push this down to $s > \frac{3}{2}$ on the real line [62, 61]. This was improved by Kenig, Ponce, and Vega to $s > \frac{3}{4}$ for the real line problem [65]. Their methods are different, and rely heavily on dispersive decay estimates which fail in the periodic case. Bourgain's restricted norm method gave local well-posedness for $s > -\frac{3}{4}$ on $\mathbb{R}$ and $s \geqslant -\frac{1}{2}$ on $\mathbb{T}$ [20, 63]. Using complete integrability methods, Kappeler and Topalov obtained global existence in H^{-1} on both $\mathbb{R}$ and $\mathbb{T}$ [60]. We also have normal forms methods to prove L^2 local (and global) well-posedness in the periodic setting, due to the work of Babin, Ilyin, and Titi [8], as mentioned previously.

Coupled KdV-type systems have been extensively studied, see e.g. [53, 5, 73, 88, 3], but little of the work addresses periodic problems with coupling parameter $\alpha \neq 1$ such as appears in (3.1). For the Majda-Biello system on $\mathbb{R}$, and systems with similar coupling, more is known. For the related Gear-Grimshaw system [47], a model of gravity waves in stratified fluids, Bona, Ponce, Saut, and Tom proved local well-posedness results in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geqslant \frac{3}{4}$ [13]. In [43], the same result for the Hirota-Satsuma system, another similar coupled KdV system, is proven. In [79], Oh proved global well-posedness for the Majda-Biello system on $\mathbb{R}$ with $s \geqslant 0$.

The well-posedness of (3.1) on $\mathbb{T}$ was also studied in [79], and local well-posedness in H^s for s above a threshold s^* established. The value of s^* is dependent on the arithmetic properties of α, leading to well-posedness results of markedly different types depending on the nature of α. When $\alpha = 1$, the resonant interactions in the system simplify significantly. In this case, the methods used by Kenig, Ponce, and Vega in [63] to prove the local well-posedness of the KdV equation can be applied; see [79]. This gives local well-posedness in $H^{-\frac{1}{2}} \times H^{-\frac{1}{2}}$ for mean zero initial data. A further argument gives the result for general initial data [79]. Oh also shows that for $\alpha < 0$ and $\alpha > 4$, the resonant interactions are easier to control and the KdV theory can be applied.

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For $\alpha \in (0,1) \cup (1,4]$, the behavior is more complex. Oh used the restricted norm method of Bourgain [20] to prove local well-posedness in $H^s \times H^s$ for $s \geqslant \min\left\{1, \frac{1}{2} + \frac{1}{2}\max\{\nu_c, \nu_d\} + \right\}$ with the assumption that the initial data u_0 is mean zero. The values ν_c and ν_d are number-theoretic parameters which depend on the properties of α; generically $\nu_c = \nu_d = 0$ for almost every α. Introducing these parameters gives control over the resonant sets which arise in Bourgain space estimates. For any α, local well-posedness extends to global for $s \geqslant 1$ due to conservation of the Hamiltonian E_4. This implies that the system is globally well-posed in H^s for $s \geqslant 1$ regardless of the value of α. In [78], global well-posedness for $s > s^*(\alpha) \geqslant \frac{5}{7}$ was established using the I-method. Here again, the threshold value depends on properties of α. In the special case $\alpha = 1$, global well-posedness holds for $s > -\frac{1}{2}$.

Here we are concerned with the dynamics of solutions to the Majda-Biello system. In the first part of this chapter, we demonstrate that the difference between the linear evolution and the nonlinear evolution resides in a higher-regularity space. The result follows from a combination of the method of normal forms of Babin, Ilyin, and Titi [8] and the restricted norm method. This approach was first used by Erdoğan and Tzirakis in [36, 40] on the KdV and the Zakharov system. The difficulty in applying their methods to this particular system arises from the complexity of the resonance relations. The coupling of the equations through α makes the resonances significantly more complex than those of the KdV and the Zakharov system. Unlike the KdV case, the resonance equations do not factor neatly, and the coupling interactions are considerably more difficult to control than those of the Zakharov system.

The normal form transformation eliminates the derivative nonlinearity and replaces it with a third-order power nonlinearity. Controlling this requires trilinear $X^{s,b}$ estimates, in contrast to the bilinear estimates necessary for well-posedness. The local theory used multipliers of the form

$$\frac{k\langle k\rangle^s \langle k_1\rangle^{-s}\langle k_2\rangle^{-s}}{\langle \tau - k^3\rangle^{1-b}\langle \tau_1 - \alpha k_1^3\rangle^{1/2}\langle \tau_2 - \alpha k_2^3\rangle^{1/2}},$$

whereas the smoothing results require control over multipliers such as

$$\frac{k(k_1+k_2)\langle k\rangle^{s_1}\langle k_1\rangle^{-s}\langle k_2\rangle^{-s}\langle k_3\rangle^{-s}}{(k^3 - \alpha(k_1+k_2)^3 - \alpha k_3^3)\langle \tau - k^3\rangle^{1-b}\langle \tau_1 - k_1^3\rangle^{1/2}\langle \tau_2 - \alpha k_2^3\rangle^{1/2}\langle \tau_3 - \alpha k_3^3\rangle^{1/2}}.$$

For the latter, we want $s_1 > s$ to obtain smoothing. This means we have no *a priori* bound on $\langle k\rangle^{s_1}\langle k_1\rangle^{-s}\langle k_2\rangle^{-s}\langle k - k_1 - k_2\rangle^{-s}$. Furthermore, the differentiation by parts introduces the term

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$k^3 - \alpha(k_1 + k_2)^3 - \alpha(k - k_1 - k_2)^3$ in the denominator. Unlike the bracketed terms which appear in the local theory multiplier, this can be arbitrarily small. The estimates require precise control of multiple terms to ensure that the multiplier remains bounded. Depending on the characteristics of α, we obtain different levels of smoothing, with a gain of up to half a derivative for $\alpha \neq 1$. Again, the results improve if $\alpha = 1$; the KdV results in [36] can be applied to get a gain of nearly a full derivative.

In the second part, we consider the behavior of the system when forcing and weak damping terms are included:

$$\begin{cases} u_t + u_{xxx} + \gamma u + \frac{1}{2}(v^2)_x = f \\ v_t + \alpha v_{xxx} + \delta v + (uv)_x = g \end{cases} \tag{3.3}$$

We take initial data $u_0,\ v_0 \in H^1$; the functions f and g are in H^1 with mean zero and the coefficients γ and δ are positive. We investigate the long-time dynamics of this equation, and show that for almost every α, the evolution has a global attractor. For the KdV, global attractors were first studied by Ghidaglia in H^2 [49]. Further work by other authors has established the existence below the L^2 level; see the discussion and references in [37]. To obtain an attractor for the Madja-Biello system, we use the method of [37] and [40] along with our smoothing estimate to decompose the solution into two parts: the linear part which decays over time thanks to the damping terms, and the nonlinear part. We then apply smoothing estimates to the nonlinear part to show that it resides in a smoother space. This gives a global attractor for almost every $\alpha \in (0, 1)$. For $\alpha = 1$, the estimates in [37] can be applied directly and one can obtain an L^2 attractor.

One reason for the interest in global attractors is that they can be finite-dimensional even when the phase space of the equation is not, making them useful tools in understanding the dynamics of a system. In the last part of the chapter, we show that the attractor for the Majda-Biello system is trivial, consisting of a single pair of functions $(p, q) \in H^2 \times H^2$, if the damping coefficients δ and γ are sufficiently large in relation to the forcing terms. This is motivated by the corresponding result for the forced and damped KdV [25] and for the Zakharov system [34]. We show that for any α, as long as γ and δ are sufficiently large in relation to $\|f\|_{H^1}$ and $\|g\|_{H^1}$, the time-independent version of (3.3) has a solution in H^1. For values of α at which the system exhibits smoothing, we show that the solutions to (3.3) converge to this stationary solution in H^1. The proof uses a modified version

of H^1 conservation law to obtain control over the difference between a solution and the stationary evolution. We also prove a similar result for the L^2 attractor in the case $\alpha = 1$.

3.2 Notation & Function Spaces

The estimates will require the Bourgain spaces corresponding to the u and v evolutions. These are defined as follows:

$$\|u\|_{X_1^{s,b}} = \|\langle k \rangle^s \langle \tau - k^3 \rangle^b u_k(\tau)\|_{L_\tau^2 \ell_k^2}$$

$$\|v\|_{X_\alpha^{s,b}} = \|\langle k \rangle^s \langle \tau - \alpha k^3 \rangle^b v_k(\tau)\|_{L_\tau^2 \ell_k^2}.$$

We also define restricted versions of the norms:

$$\|u\|_{X_{1,\delta}^{s,b}} = \inf_{u=\tilde{u},\ |t| \leqslant \delta} \|\tilde{u}\|_{X_1^{s,b}} \qquad\qquad \|v\|_{X_{\alpha,\delta}^{s,b}} = \inf_{v=\tilde{v},\ |t| \leqslant \delta} \|\tilde{v}\|_{X_\alpha^{s,b}}.$$

We write $U(t)$ for the semigroup operator corresponding to the Majda-Biello evolution. The phase space of this operator is $\dot{H}^s \times H^s$ for $\alpha \neq 1$; when $\alpha = 1$ we work with the phase space $\dot{L}^2 \times \dot{L}^2$.

The notation $\sum^*$ indicates summation over all terms for which the denominator of the summand is nonzero. To simplify calculations, we use the notation $\mathcal{O}(\epsilon)$ to denote a constant of the form $C\epsilon$, where C may depend on α, but not on any of the variables in the calculation.

3.3 Statement of Results

3.3.1 Background

To study well-posedness, Oh in [79] used the minimal type index ν_ρ, a parameter which quantifies how "close" the number ρ is to being rational. Quantities of this type are heavily studied in the theory of diophantine approximations to irrational numbers. In our case, it is important in controlling the resonances which arise in estimates.

Definition 3.3.1 ([6], [79]). *A number $\rho \in \mathbb{R}$ is said to be of type ν if there exists $K > 0$ such that for all $m, n \in \mathbb{Z}$,*

$$\left| \rho - \frac{m}{n} \right| \geqslant \frac{K}{|n|^{2+\nu}}.$$

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The minimal type index of a number ρ is defined to be

$$\nu_\rho = \begin{cases} \infty & \rho \in \mathbb{Q} \\ \inf\{\nu > 0 \mid \rho \text{ is of type } \nu\} & \rho \notin \mathbb{Q}. \end{cases}$$

Dirichlet's approximation theorem implies that $\nu_\rho \geqslant 0$ for every real number ρ. Furthermore, it is known that $\nu_\rho = 0$ for almost every $\rho \in \mathbb{R}$ [6]. In general, though, determining the minimal type index of a specific number is difficult. In fact, it is not even known whether there is any ρ such that $0 < \nu_\rho < \infty$. However, for irrational algebraic numbers we have $\nu_\rho = 0$ due to the Thue-Siegel-Roth theorem [87].

The local theory depends on the minimal type index of certain parameters c_1, c_2, d_1, and d_2 which arise in the resonance equations. The $X^{s,b}$ estimates yield resonance equations of the form $k^3 - \alpha k_1^3 - \alpha(k - k_1)^3$ and $\alpha k^3 - k_1^3 - \alpha(k - k_1)^3$. The roots of the former equation are $k_1 = c_1 k$, $k_1 = c_2 k$, and $k = 0$, where

$$c_1 = \frac{1}{2} + \frac{\sqrt{-3 + 12/\alpha}}{6} \qquad c_2 = \frac{1}{2} - \frac{\sqrt{-3 + 12/\alpha}}{6}.$$

Note that these are the roots of the quadratic $3\alpha x^2 - 3\alpha x + \alpha - 1$, so they are algebraic for rational α. The solutions to the second resonance equation are $k_1 = d_1 k$, $k_1 = d_2 k$, and $k_1 = 0$, where $d_1 = c_1^{-1}$ and $d_2 = c_2^{-1}$. These are the roots of the quadratic $(1 - \alpha)x^2 + 3\alpha x - 3\alpha$.

For α outside $[0, 4]$, the roots are not real, meaning that the resonances don't cause trouble in the estimates. In this case, the local theory is like that of the KdV. The problem for $\alpha \in (1, 4]$ can be treated in the same way as that for $\alpha \in (0, 1)$. For simplicity, we state results for $\alpha \in (0, 1)$.

To give the local theory precisely, define

$$\nu_c = \nu_{c_1} = \nu_{c_2} \qquad \text{and} \qquad \nu_d = \max\{\nu_{d_1}, \nu_{d_2}\}.$$

Theorem 3.3.2 ([79])**.** *Let $\alpha \in (0, 1)$. For $s \geqslant \min\{1, \frac{1}{2} + \frac{1}{2}\max\{\nu_c, \nu_d\}+\}$, the Majda-Biello initial value problem is locally well-posed in $\dot{H}^s \times H^s$. In particular, for any $(u_0, v_0) \in \dot{H}^s \times H^s$, there exists $T \gtrsim (\|u_0\|_{H^s} + \|v_0\|_{H^s})^{-3}$ such that there is a unique solution (u, v) to (3.1) satisfying*

$$(u, v) \in C([-T, T]; H_x^s(\mathbb{T})) \times C([-T, T]; H_x^s(\mathbb{T}))$$

and

$$\|u\|_{X_{1,T}^{s,1/2}} + \|v\|_{X_{\alpha,T}^{s,1/2}} \lesssim \|u_0\|_{H^s} + \|v_0\|_{H^s}.$$

3.3.2 SMOOTHING ESTIMATE

The smoothing result for the nonlinear part of the Majda-Biello evolution is then as follows.

Theorem 3.3.3. *Fix $\alpha \in (0,1)$ and $s > \frac{1}{2}$. Consider the solution of (3.1) with initial data $(u_0, v_0) \in \dot{H}^s \times H^s$. Let*

$$s_1 - s < \min\left\{\frac{1}{2}, s - \frac{1}{2}, s - \nu_c, s - \nu_d, 2s - 1 - \nu_c, 2s - 1 - \nu_d\right\}.$$

If $\alpha = q^2/(3p(p-q)+q^2)$ for some $p, q \in \mathbb{Z}$ with $p > q$, we must instead take $s^* \leqslant \min\{\frac{1}{2}-, s-1\}$. Then for $s_1 = s + s^*$, we have

$$u(t) - e^{-t\partial_x^3} u_0 \in C_t^0 H_x^{s_1}$$

$$v(t) - e^{-\alpha t\partial_x^3} v_0 \in C_t^0 H_x^{s_1}.$$

In particular, for almost every α, the above statements hold with $s_1 - s < \min\{\frac{1}{2}, s - \frac{1}{2}\}$.

If there is a growth bound

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \lesssim (1 + |t|)^{g(s)},$$

then we also have

$$\|u(T) - e^{-T\partial_x^3} u_0\|_{H^{s_1}} + \|v(T) - e^{-\alpha T\partial_x^3} v_0\|_{H^{s_1}} \leqslant CT^{1+6g(s)},$$

where $C = C(s, s_1, \alpha, \|u_0\|_{H^s}, \|v_0\|_{H^s})$.

Remark 3.3.4. *When α is a rational number which cannot be written in the form $q^2/(3p(p-q)+q^2)$ for some integers $p > q$, the coefficients c_i and d_i are irrational algebraic numbers, implying that $\nu_c = \nu_d = 0$. In this case, the best possible smoothing given by Theorem 3.3.3 is attained. In contrast, for rationals of the form $q^2/(3p(p-q)+q^2)$, the theorem gives no smoothing unless $s > 1$. For examples of such rationals, notice that no rational of the form $\ell/3^k$, where ℓ is not divisible by 3, can be written as $q^2/(3p(p-q)+q^2)$. Thus these rationals form a dense subset of $[0,1]$. The rationals which are of the form $q^2/(3p(p-q)+q^2)$ are also dense.*

Remark 3.3.5. *For $\alpha = 1$, the smoothing results for the KdV contained in [36] can be applied to the system directly as long as we take initial data in $\dot{H}^s \times \dot{H}^s$. This implies that for any $s > -\frac{1}{2}$ the nonlinear part of the evolution is in $C_t^0 H_x^{s_1}$ for $s_1 \leqslant \min\{3s, s+1\}$.*

Remark 3.3.6. *For $s \geqslant 1$, well-posedness holds for* any *choice of α. However, the smoothing in the theorem above is dependent on α, even for large s. It can be shown that the methods used to prove this smoothing cannot be applied to get smoothing for α such that ν_c or ν_d is large and finite, regardless of the size of s. The problem of obtaining smoothing for all α when s is large remains open. The difference between the LWP results and the smoothing arises since well-posedness is proved in $X^{s,b}$ spaces, requiring estimates of multipliers such as*

$$\frac{k\langle k\rangle^s}{\langle k^3 - \alpha k_1^3 - \alpha(k-k_1)^3\rangle^{1-b}\langle k_1\rangle^s\langle k-k_1\rangle^s}.$$

For sufficiently large s, the estimates can be completed without a contribution from the resonant term, i.e., one can estimate $\langle k^3 - \alpha k_1^3 - \alpha(k-k_1)^3\rangle \gtrsim 1$. However, the smoothing estimates are proved using differentiation by parts, which introduces multipliers of the form

$$\frac{k\langle k\rangle^{s_1}}{(k^3 - \alpha k_1^3 - \alpha(k-k_1)^3)^{1-b}\langle k_1\rangle^s\langle k-k_1\rangle^s}.$$

In this case, the denominator can be arbitrarily small, and the estimates cannot be completed without controlling it in some way.

Smoothing estimates can be used to obtain rough bounds on higher-order Sobolev norms by an iterative argument. Such bounds are of particular interest since the system is not completely integrable and no high regularity conservation laws exist.

Corollary 3.3.7. *For almost every $\alpha \in (0,1)$ and for any $s \geqslant 1$, the global solution of* (3.1) *with $\dot{H}^s \times H^s$ initial data satisfies the growth bound*

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leqslant C(1+|t|)^{\tilde{C}},$$

where $C = C(s, \alpha, \|u_0\|_{H^s}, \|v_0\|_{H^s})$ and $\tilde{C} = \tilde{C}(s)$.

Proof. For $s = 1$, the solution is bounded in H^1 for all time by the conservation of the Hamiltonian. Take α such that $\nu_c = \nu_d = 0$. Assume inductively that the statement of the corollary holds for some $s_0 \geqslant 1$. Then for $s \in (s_0, s_0 + \frac{1}{2})$ and initial data in $\dot{H}^s \times H^s$, solutions satisfy

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leqslant \|u_0\|_{H^s} + \|u(t) - e^{-t\partial_x^3}u_0\|_{H^s} + \|v_0\|_{H^s} + \|v(t) - e^{-\alpha t\partial_x^3}v_0\|_{H^s}$$

$$\leqslant C(\|u_0\|_{H^s}, \|v_0\|_{H^s}) + C(s, s_0, \alpha, \|u_0\|_{H^{s_0}}, \|v_0\|_{H^{s_0}})|t|^{1+6C(s_0)}.$$

Repeating this argument, we can obtain the statement of the corollary for any $s \geqslant 1$. $\qquad\square$

Polynomial bounds for higher Sobolev norms of solutions to the KdV equation have been studied by Bourgain [21] and Staffilani [92]. Their methods use careful $X^{s,b}$ space estimates and are much more refined than the simple induction used to prove Corollary 3.3.7. More recently, Kappeler, Schaad, and Topalov obtained uniform bounds for KdV solutions in all Sobolev spaces H^s with $s \geqslant 0$ using perturbative expansions of the Fourier coefficients [59]. Their methods used Birkhoff normal forms, relying on the integrability of the KdV.

3.3.3 Existence of a Global Attractor

We use smoothing estimates for the dissipative version of the Majda-Biello system to derive the existence of a global attractor. In the following, $U(t)$ will denote the evolution operator corresponding to (3.3). Note that the notion of a global attractor is only reasonable when the system is globally well-posed. For the forced and weakly damped system, global well-posedness holds by the restricted norm argument of Bourgain using the estimates established in [79]; see [37, Section 2] for a similar argument.

Recall Definitions 2.5.1–2.5.2, which defined global attractors and absorbing sets. Using energy estimates, we show that the system (3.3) has an absorbing set in $\dot{H}^1 \times H^1$. Our global attractor is the ω-limit set of the absorbing ball. We use Theorem 2.5.3 to establish this. This theorem requires asymptotic compactness of the flow, which we will prove using a smoothing estimate for the dissipative system. This yields the following result.

Theorem 3.3.8. *For almost every $\alpha \in (0,1)$, the dissipative Majda-Biello system (3.3) has a global attractor in $\dot{H}^1 \times H^1$. Moreover, the global attractor is a compact subset of $H^s \times H^s$ for any $s < \frac{3}{2}$.*

Remark 3.3.9. *For $\alpha = 1$ and forcing terms in $\dot{L}^2$, the arguments in [37] immediately yield a global attractor in $\dot{L}^2 \times \dot{L}^2$ which is compact in $H^s \times H^s$ for any $s < 1$.*

When the damping terms are sufficiently large in relation to the forcing, the attractor is trivial:

Theorem 3.3.10. *Assume $\min\{\gamma, \delta\} \geqslant \frac{\sqrt{\alpha C^3}}{8}$, where C the norm of the embedding $H^1 \hookrightarrow L^\infty$. If $\|f\|_{H^1}, \|g\|_{H^1} \ll \left(\min\{\gamma, \delta\}\right)^{4/3}$, the global attractor given by Theorem 4.3.6 consists of a single pair of functions $(p,q) \in H^2(\mathbb{T}) \times H^2(\mathbb{T})$.*

Remark 3.3.11. *When $\alpha = 1$ and the forcing terms f and g are in $\dot{L}^2$, the statement of the theorem holds if $\|f\|_{L^2}, \|g\|_{L^2} \ll \min\{\gamma, \delta\}$.*

This theorem is proved using a modification of the Hamiltonian conservation law to show that a solution to (3.3) converges to the solution of the corresponding time-independent system as $t \to \infty$.

3.4   Proof of Theorem 3.3.3: Smoothing Result

To prove the smoothing estimate, we begin by establishing an equivalent formulation of (3.1) via differentiation by parts. This formulation decomposes the equation into several terms which will be estimated separately.

Proposition 3.4.1. *Assume $u_0 \in \dot{H}^s$. The system (3.1) can be written in the following form:*

$$
\begin{cases}
\partial_t\big[e^{-ik^3 t}(u_k + B_1(v,v)_k)\big] &= e^{-ik^3 t}\big[\rho_1(v,v)_k + R_1(u,v,v)_k\big] \\
\partial_t\big[e^{-i\alpha k^3 t}(v_k + B_2(u,v)_k)\big] &= e^{-i\alpha k^3 t}\big[\rho_2(u,v)_k + R_2(v,v,v)_k + R_3(u,u,v)_k\big],
\end{cases}
$$

where

$$
B_1(u,v)_k = -\frac{k}{2}\sum_{k_1+k_2=k}^{*}\frac{u_{k_1}v_{k_2}}{k^3 - \alpha k_1^3 - \alpha k_2^3}
$$

$$
B_2(u,v)_k = -k\sum_{k_1+k_2=k}^{*}\frac{u_{k_1}v_{k_2}}{\alpha k^3 - k_1^3 - \alpha k_2^3}
$$

$$
R_1(u,v,w)_k = -\frac{i}{3\alpha}\sum_{k_1+k_2+k_3=k}^{*}\frac{(k_1+k_2)u_{k_1}v_{k_2}w_{k_3}}{(k_1+k_2-c_1 k)(k_1+k_2-c_2 k)}
$$

$$
R_2(u,v,w)_k = \frac{ik}{2}\sum_{k_1+k_2+k_3=k}^{*}\frac{(k_1+k_2)u_{k_1}v_{k_2}w_{k_3}}{\alpha k^3 - (k_1+k_2)^3 - \alpha k_3^3}
$$

$$
R_3(u,v,w)_k = ik\sum_{k_1+k_2+k_3=k}^{*}\frac{(k_2+k_3)u_{k_1}v_{k_2}w_{k_3}}{\alpha k^3 - k_1^3 - \alpha(k_2+k_3)^3}
$$

$$
\rho_1(u,v)_k = -ik\big(u_{c_1 k}v_{c_2 k}\big)
$$

$$
\rho_2(u,v)_k = -ik\big(u_{d_1 k}v_{(1-d_1)k} + u_{d_2 k}v_{(1-d_2)k}\big).
$$

Proof. Taking the spatial Fourier transform of (3.1) yields

$$
\begin{cases}
\partial_t u_k - i k^3 u_k + \dfrac{ik}{2} \displaystyle\sum_{k_1+k_2=k} v_{k_1} v_{k_2} = 0 \\[3mm]
\partial_t v_k - i\alpha k^3 v_k + ik \displaystyle\sum_{k_1+k_2=k} u_{k_1} v_{k_2} = 0.
\end{cases}
$$

Change variables by setting $m_k(t) = e^{-ik^3 t} u_k(t)$ and $n_k(t) = e^{-i\alpha k^3 t} v_k(t)$. Then the system becomes

$$
\begin{cases}
\partial_t m_k = \dfrac{-ik}{2} \displaystyle\sum_{k_1+k_2=k} e^{-it(k^3-\alpha k_1^3-\alpha k_2^3)} n_{k_1} n_{k_2} \\[3mm]
\partial_t n_k = -ik \displaystyle\sum_{k_1+k_2=k} e^{-it(\alpha k^3-k_1^3-\alpha k_2^3)} m_{k_1} n_{k_2}.
\end{cases}
$$

Differentiate the equation for $\partial_t m_k$ by parts:

$$
\partial_t m_k = \frac{k}{2} \sum_{k_1+k_2=k}^{*} \frac{\partial_t\left(e^{-it(k^3-\alpha k_1^3-\alpha k_2^3)} n_{k_1} n_{k_2}\right)}{k^3 - \alpha k_1^3 - \alpha k_2^3}
$$
$$
- \frac{k}{2} \sum_{k_1+k_2=k}^{*} \frac{e^{-it(k^3-\alpha k_1^3-\alpha k_2^3)} \partial_t(n_{k_1} n_{k_2})}{k^3 - \alpha k_1^3 - \alpha k_2^3} - ik n_{c_1 k} n_{c_2 k}.
$$

Rewrite second sum using equation for $\partial_t n_k$. Recall that the constants c_1 and c_2 arise from the solving $k^3 - \alpha k_1^3 - \alpha k_2^3 = 0$. The $k = 0$ solution does not appear in the resonant term since we assume that u_0, and hence u, is mean zero. Furthermore, the resonant term only appears when c_1 and c_2 are rational and $c_1 k$ is an integer. In particular, $c_1, c_2 \in \mathbb{Q}$ only if $\alpha = q^2/(3p(p-q) + q^2)$ for some $p, q \in \mathbb{Z}$ with $p > q$.

Using the differential equation, we find that the second sum in $\partial_t m_k$ is

$$
ik \sum_{k_1+k_2+k_3=k}^{*} (k_1 + k_2) \frac{e^{-it(k^3-k_1^3-\alpha k_2^3-\alpha k_3^3)} m_{k_1} n_{k_2} n_{k_3}}{k^3 - \alpha(k_1 + k_2)^3 - \alpha k_3^3}
$$
$$
= -\frac{i}{3\alpha} \sum_{k_1+k_2+k_3=k}^{*} (k_1 + k_2) \frac{e^{-it(k^3-k_1^3-\alpha k_2^3-\alpha k_3^3)} m_{k_1} n_{k_2} n_{k_3}}{(k_1 + k_2 - c_1 k)(k_1 + k_2 - c_2 k)}.
$$

Moving to the equation for $\partial_t n$, differentiate by parts again to find

$$
\partial_t n_k = k \sum_{k_1+k_2=k}^{*} \frac{\partial_t\left(e^{-it(\alpha k^3-k_1^3-\alpha k_2^3)} m_{k_1} n_{k_2}\right)}{\alpha k^3 - k_1^3 - \alpha k_2^3} - k \sum_{k_1+k_2=k}^{*} \frac{\partial_t(m_{k_1} n_{k_2}) e^{-it(\alpha k^3-k_1^3-\alpha k_2^3)}}{\alpha k^3 - k_1^3 - \alpha k_2^3}
$$
$$
- ik m_{d_1 k} n_{(1-d_1)k} - ik m_{d_2 k} n_{(1-d_2)k}.
$$

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Here again, the last terms in the equality only appear when d_1 and d_2 are rational and $d_1 k, d_2 k \in \mathbb{Z}$. Using the differential equation, rewrite the second sum in $\partial_t n_k$ as

$$\frac{ik}{2} \sum_{k_1+k_2+k_3=k}^{*} e^{-it(\alpha k^3 - \alpha k_1^3 - \alpha k_2^3 - \alpha k_3^3)} \frac{(k_1+k_2)n_{k_1} n_{k_2} n_{k_3}}{\alpha k^3 - (k_1+k_2)^3 - \alpha k_3^3}$$

$$+ ik \sum_{k_1+k_2+k_3=k}^{*} e^{-it(\alpha k^3 - k_1^3 - k_2^3 - \alpha k_3^3)} \frac{(k_2+k_3)m_{k_1} m_{k_2} n_{k_3}}{\alpha k^3 - k_1^3 - \alpha(k_2+k_3)^3}.$$

Collecting all these terms and returning to u and v variables gives the statement of the proposition.

$\square$

We use the transformed system to get bounds on the norm of the difference between the linear and nonlinear evolution. First, integrate the new system from 0 to t to obtain

$$
\begin{cases}
u_k(t) - e^{ik^3 t} u_k(0) = & -B_1(v,v)_k(t) + e^{ik^3 t} B_1(v,v)_k(0) \\
& + \int_0^t e^{ik^3(t-r)} \left[R_1(u,v,v)_k(r) + \rho_1(v,v)_k(r) \right] \, dr \\
v_k(t) - e^{i\alpha k^3 t} v_k(0) = & -B_2(u,v)_k(t) + e^{i\alpha k^3 t} B_2(u,v)_k(0) \\
& + \int_0^t e^{i\alpha k^3(t-r)} \left[R_2(v,v,v)_k(r) + R_3(u,u,v)_k(r) + \rho_2(u,v)_k(r) \right] \, dr.
\end{cases}
$$

To control these expressions, we use the following estimates. Propositions 3.4.2 and 3.4.3 are proved in Section 3.7; Proposition 3.4.4 is immediate from the definitions of ρ_1 and ρ_2.

Proposition 3.4.2. *If $s > \frac{1}{2}$ and $s_1 - s < \min\{1, s - \nu_c\}$, then*

$$\|B_1(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

When $\alpha = q^2/(3p(p-q)+q^2)$ for some $p, q \in \mathbb{Z}$ with $p > q$, we only require that $s_1 - s \leqslant 1$.

Proposition 3.4.3. *Let $u \in \dot{H}^s$. If $s > \frac{1}{2}$ and $s_1 - s < \min\{1, s - \nu_d\}$, then*

$$\|B_2(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

When $\alpha = q^2/(3p(p-q)+q^2)$ for some $p, q \in \mathbb{Z}$ with $p > q$, we only need $s_1 - s \leqslant \min\{1, s-\}$.

Proposition 3.4.4. *If $s_1 - s \leqslant s - 1$, then*

$$\|\rho_1(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s} \qquad \text{and} \qquad \|\rho_2(u,v)\|_{H_x^{s_1}} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

Using Propositions 3.4.2-3.4.4 on the equations found above, write, for $s_1 - s$ sufficiently small,

$$\|u(t) - e^{-t\partial_x^3}u_0\|_{H_x^{s_1}} \lesssim \|v(t)\|_{H_x^s}^2 + \|v(0)\|_{H_x^s}^2 + \int_0^t \|v(r)\|_{H_x^s}^2 \, \mathrm{d}r$$

$$+ \left\| \int_0^t e^{-(t-r)\partial_x^3} R_1(u,v,v)(r) \, \mathrm{d}r \right\|_{H_x^{s_1}}$$

$$\|v(t) - e^{-\alpha t\partial_x^3}v_0\|_{H_x^{s_1}} \lesssim \|u(t)\|_{H_x^s}\|v(t)\|_{H_x^s} + \|u(0)\|_{H_x^s}\|v(0)\|_{H_x^s} + \int_0^t \|u(r)\|_{H_x^s}\|v(r)\|_{H_x^s} \, \mathrm{d}r$$

$$+ \left\| \int_0^t e^{-\alpha(t-r)\partial_x^3}[R_2(v,v,v)(r) + R_3(u,u,v)(r)] \, \mathrm{d}r \right\|_{H_x^{s_1}}.$$

To complete the estimates, we need the following bounds for R_1, R_2, and R_3. See Section 3.7 for proofs.

Proposition 3.4.5. *Let $u \in \dot{H}^s$. For $b - \frac{1}{2}$ sufficiently small, $s > \frac{1}{2}$, and $s_1 - s < \min\{1, 2s - 1 - \nu_{c_1}, s - \frac{1}{2}, s + \frac{1}{2} - \nu_{c_1}\}$, we have*

$$\|R_1(u,v,w)\|_{X_1^{s_1,b-1}} \lesssim \|u\|_{X_1^{s,1/2}}\|v\|_{X_\alpha^{s,1/2}}\|w\|_{X_\alpha^{s,1/2}}.$$

When $\alpha = q^2/(3p(p-q) + q^2)$ for $p, q \in \mathbb{Z}$ with $p > q$, we only need $s_1 - s \leqslant \min\{1, s - \frac{1}{2}\}$.

Proposition 3.4.6. *For $b - \frac{1}{2}$ sufficiently small, $s > \frac{1}{2}$, and $s_1 - s < \min\{2s - 1 - \nu_d, s + \frac{1}{2} - \nu_d\}$,*

$$\|R_2(u,v,w)\|_{X_\alpha^{s_1,b-1}} \lesssim \|u\|_{X_\alpha^{s,1/2}}\|v\|_{X_\alpha^{s,1/2}}\|w\|_{X_\alpha^{s,1/2}}.$$

When $\alpha = q^2/(3p(p-q) + q^2)$ for $p, q \in \mathbb{Z}$ with $p > q$, we only need $s_1 - s \leqslant 1$.

Proposition 3.4.7. *Let $u \in \dot{H}^s$. For $b - \frac{1}{2}$ sufficiently small, $s > \frac{1}{2}$, and $s_1 - s < \min\{\frac{1}{2}, s - \frac{1}{2}, 2s - 1 - \nu_d, s + \frac{1}{2} - \nu_d\}$,*

$$\|R_3(u,u,v)\|_{X_\alpha^{s_1,b-1}} \lesssim \|u\|_{X_1^{s,1/2}}^2\|v\|_{X_\alpha^{s,1/2}}.$$

When $\alpha = q^2/(3p(p-q) + q^2)$ for $p, q \in \mathbb{Z}$ with $p > q$, we only need $s_1 - s \leqslant \min\{\frac{1}{2}-, s - \frac{1}{2}\}$.

We will use these estimates, the embeddings $X_1^{s_1,b}$, $X_\alpha^{s_1,b} \hookrightarrow L_t^\infty H_x^{s_1}$ for $b > \frac{1}{2}$, and the following standard lemma to complete the proof. Here η is a smooth function supported on $[-2, 2]$ with $\eta = 1$ on $[-1, 1]$, and η_δ is defined by $\eta_\delta(t) = \eta(t/\delta)$.

Lemma 3.4.8 ([51])**.** *For* $b \in (\frac{1}{2}, 1]$, *we have*

$$\left\| \eta(t) \int_0^t e^{-(t-r)\partial_x^3} F(r) \ dr \right\|_{X_1^{s_1,b}} \lesssim \|F\|_{X_{1,\delta}^{s_1,b-1}}$$

$$\left\| \eta(t) \int_0^t e^{-\alpha(t-r)\partial_x^3} F(r) \ dr \right\|_{X_\alpha^{s_1,b}} \lesssim \|F\|_{X_{\alpha,\delta}^{s_1,b-1}}.$$

Let δ be the existence time for the system from the local theory. Then for $t \in [-\delta/2, \delta/2]$, we have

$$\left\| \int_0^t e^{-(t-r)\partial_x^3} R_1(u,v,v)(r) \ dr \right\|_{H_x^{s_1}} \leqslant \left\| \eta_\delta(t) \int_0^t e^{-(t-r)\partial_x^3} R_1(u,v,v)(r) \ dr \right\|_{L_t^\infty H_x^{s_1}}$$

$$\lesssim \left\| \eta_\delta(t) \int_0^t e^{-(t-r)\partial_x^3} R_1(u,v,v)(r) \ dr \right\|_{X_1^{s_1,b}} \lesssim \|R_1(u,v,v)\|_{X_{1,\delta}^{s_1,b-1}} \lesssim \|u\|_{X_{1,\delta}^{s,1/2}} \|v\|_{X_{\alpha,\delta}^{s,1/2}}^2.$$

Similarly, from the second equation we find

$$\left\| \int_0^t e^{-\alpha(t-r)\partial_x^3} [R_2(v,v,v)(r) + R_3(u,u,v)(r)] \ dr \right\|_{H_x^{s_1}}$$

$$\leqslant \left\| \eta_\delta(t) \int_0^t e^{-\alpha(t-r)\partial_x^3} [R_2(,v,v,v)(r) + R_3(u,u,v)(r)] \ dr \right\|_{L_t^\infty H_x^{s_1}}$$

$$\lesssim \left\| \eta_\delta(t) \int_0^t e^{-\alpha(t-r)\partial_x^3} [R_2(v,v,v)(r) + R_3(u,u,v)(r)] \ dr \right\|_{X_\alpha^{s_1,b}}$$

$$\lesssim \|R_2(v,v,v)\|_{X_{\alpha,\delta}^{s_1,b-1}} + \|R_3(u,u,v)\|_{X_{\alpha,\delta}^{s_1,b-1}} \lesssim \|v\|_{X_{\alpha,\delta}^{s,1/2}}^3 + \|u\|_{X_{1,\delta}^{s,1/2}}^2 \|v\|_{X_{\alpha,\delta}^{s,1/2}}.$$

Thus, collecting these estimates, we have

$$\|u(t) - e^{-t\partial_x^3} u_0\|_{H_x^{s_1}} \lesssim \|v(t)\|_{H_x^s}^2 + \|v(0)\|_{H_x^s}^2 + \int_0^t \|v(r)\|_{H_x^s}^2 \ dr + \|u\|_{X_{1,\delta}^{s,1/2}} \|v\|_{X_{\alpha,\delta}^{s,1/2}}^2$$

$$\|v(t) - e^{-\alpha t\partial_x^3} v_0\|_{H_x^{s_1}} \lesssim \|u(t)\|_{H_x^s} \|v(t)\|_{H_x^s} + \|u(0)\|_{H_x^s} \|v(0)\|_{H_x^s} + \int_0^t \|u(r)\|_{H_x^s} \|v(r)\|_{H_x^s} \ dr$$

$$+ \|v\|_{X_{\alpha,\delta}^{s,1/2}}^3 + \|u\|_{X_{1,\delta}^{s,1/2}}^2 \|v\|_{X_{\alpha,\delta}^{s,1/2}}.$$

Combining the estimates for the two equations, we may write

$$\|u(t) - e^{-t\partial_x^3} u_0\|_{H_x^{s_1}} + \|v(t) - e^{-\alpha t\partial_x^3} v_0\|_{H_x^{s_1}} \lesssim \left(\|u(0)\|_{H_x^s} + \|v(0)\|_{H_x^s} \right)^2$$

$$+ \left(\|u(t)\|_{H_x^s} + \|v(t)\|_{H_x^s} \right)^2 + \int_0^t \left(\|u(r)\|_{H_x^s} + \|v(r)\|_{H_x^s} \right)^2 \ dr + \left(\|u\|_{X_{1,\delta}^{s,1/2}} + \|v\|_{X_{\alpha,\delta}^{s,1/2}} \right)^3.$$

We demonstrate the polynomial growth bound and then the continuity. Fix T large. For $t \leqslant T$, we have the bound

$$\|u(t)\|_{H_x^s} + \|v(t)\|_{H_x^s} \lesssim (1 + |t|)^{g(s)} \lesssim T^{g(s)}.$$

Then for $\delta \sim T^{-3g(s)}$ and any $j \leqslant T/\delta \sim T^{1+3g(s)}$, we have

$$\|u(j\delta) - e^{-t\partial_x^3} u((j-1)\delta)\|_{H_x^{s_1}} + \|v(j\delta) - e^{-\alpha t\partial_x^3} v((j-1)\delta)\|_{H_x^{s_1}} \lesssim T^{3g(s)},$$

using the local theory bound

$$\|u\|_{X_{1,[(j-1)\delta,j\delta]}^{s,1/2}} + \|v\|_{X_{\alpha,[(j-1)\delta,j\delta]}^{s,1/2}} \lesssim \|u((j-1)\delta)\|_{H^s} + \|v((j-1)\delta)\|_{H^s} \lesssim T^{g(s)}.$$

Now let $J = T/\delta \sim T^{1+3g(s)}$ and write

$$\|u(J\delta) - e^{-J\delta\partial_x^3} u_0\|_{H^{s_1}} \leqslant \sum_{j=1}^{J} \|u(j\delta) - e^{-t\partial_x^3} u((j-1)\delta)\|_{H_x^{s_1}} \lesssim T^{1+6g(s)}.$$

The corresponding estimate for v completes the proof of the growth bound.

To prove continuity, write

$$\begin{aligned}
u_k(t) - u_k(t') = (e^{ik^3 t} - e^{ik^3 t'})\big[u_k(0) + B_1(v,v)_k(0)\big] &+ B_1(v,v)_k(t') - B_1(v,v)_k(t) \\
&+ \int_0^t e^{ik^3(t-r)} R_1(u,v,v)_k(r)\ \mathrm{d}r - \int_0^{t'} e^{ik^3(t'-r)} R_1(u,v,v)_k(r)\ \mathrm{d}r \\
&+ \int_0^t e^{ik^3(t-r)} \rho_1(v,v)_k(r)\ \mathrm{d}r - \int_0^{t'} e^{ik^3(t'-r)} \rho_1(v,v)_k(r)\ \mathrm{d}r.
\end{aligned}$$

The continuity follows by applying the estimates stated previously along with the continuity of u in H^s; see [36]. Continuity of v is proved in the same way.

3.5 PROOF OF EXISTENCE OF GLOBAL ATTRACTORS

We will consider the forced and damped version (3.3) of the Majda-Biello system with $\gamma, \delta > 0$. For simplicity, take $\gamma = \delta$; minor modifications to the calculations extend them to the general case. The first step is to obtain bounds on the H^1 norms for the dissipative system. This will imply the existence of an absorbing set (see Definition 2.5.2). Recall the conservation laws (3.2) for the original Majda-Biello system. To get a bound in the dissipative case, we study E_3 and E_4 in the presence of dissipation.

Lemma 3.5.1. *Solutions to* (3.3) *satisfy*

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leqslant C = C(\|u_0\|_{H^1}, \|v_0\|_{H^1}, \|f\|_{H^1} + \|g\|_{H^1}, \gamma, \alpha).$$

Proof. In the following manipulations, C and $\tilde{C}$ are positive constants whose value may change from one side of an inequality to the other. Recall that $E_3 = \int u^2 + v^2 \; \mathrm{d}x$. Then

$$\partial_t E_3 + 2\gamma E_3 = 2 \int fu + gv \; \mathrm{d}x \leqslant 2\|f\|_{L^2}\|u\|_{L^2} + 2\|g\|_{L^2}\|v\|_{L^2} \leqslant 4\big(\|f\|_{L^2} + \|g\|_{L^2}\big)\sqrt{E_3}.$$

Let $F_3 = e^{2\gamma t}E_3$. The above inequality gives $\partial_t F_3 \leqslant 4e^{\gamma t}\big(\|f\|_{L^2} + \|g\|_{L^2}\big)\sqrt{F_3}$, or

$$\partial_t \sqrt{F_3} \leqslant 2e^{\gamma t}\big(\|f\|_{L^2} + \|g\|_{L^2}\big).$$

Integrating this inequality and rewriting F_3 in terms of u and v norms gives

$$\sqrt{\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2} \leqslant e^{-\gamma t}\sqrt{\|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2} + 2\frac{\|f\|_{L^2} + \|g\|_{L^2}}{\gamma}(1 - e^{-\gamma t})$$

$$\leqslant C = C(\|u_0\|_{L^2}, \|v_0\|_{L^2}, (\|f\|_{L^2} + \|g\|_{L^2}), \gamma).$$

Thus the L^2 norms of the u and v are bounded in the dissipative case. Next consider $E_4 = \int u_x^2 + \alpha v_x^2 - uv^2 \; \mathrm{d}x$. First notice that E_4 is bounded below due to the bound on $\|v\|_{L^2}$ and the embedding $H^1 \hookrightarrow L^\infty$. To get an upper bound, use the embedding again to write

$$\partial_t E_4 + 2\gamma E_4 = 2 \int f_x u_x + g_x v_x \; \mathrm{d}x - \int fv^2 + guv \; \mathrm{d}x + \gamma \int uv^2 \; \mathrm{d}x$$

$$\lesssim \|f\|_{H^1}\|u_x\|_{L^2} + \|g\|_{H^1}\|v_x\|_{L^2} + \|f\|_{H^1}\|v\|_{L^2}^2 + \|g\|_{H^1}\|u\|_{L^2}\|v\|_{L^2} + \|u\|_{H^1}\|v\|_{L^2}^2$$

$$\leqslant C + \tilde{C}\big(\|u_x\|_{L^2} + \|v_x\|_{L^2}\big).$$

The constants in second inequality depend on the bounds on $\|u\|_{L^2}$ and $\|v\|_{L^2}$ and on the value of $\|f\|_{H^1}$. Now note that

$$\|u_x\|_{L^2}^2 + \alpha\|v_x\|_{L^2}^2 = E_4 + \int uv^2 \; \mathrm{d}x \leqslant E_4 + C\|v\|_{L^2}^2\|u\|_{H^1}$$

$$\leqslant E_4 + C(\|u_x\|_{L^2}^2 + C)^{1/2} \leqslant (E_4 + C) + \tilde{C}\|u_x\|_2.$$

The second inequality uses the L^2 bounds on u and v. Then we have

$$\|v_x\|_{L^2} \lesssim \sqrt{\big(\|u_x\|_{L^2} - \tilde{C}/2\big)^2 + \alpha\|v_x\|_{L^2}^2} \leqslant \sqrt{E_4 + C + \tilde{C}^2/4} \lesssim \sqrt{|E_4|} + C,$$

and similarly

$$\|u_x\|_{L^2} - \tilde{C}/2 \lesssim \sqrt{\left(\|u_x\|_{L^2} - \tilde{C}/2\right)^2 + \alpha\|v_x\|_{L^2}^2} \lesssim \sqrt{|E_4|} + C.$$

This implies that $\|u_x\|_{L^2} + \|v_x\|_{L^2} \lesssim \sqrt{|E_4|} + C$. Using this bound with the change of variables $F_4 = e^{2\gamma t} E_4$, we have

$$\partial_t F_4 \leqslant e^{\gamma t} \left[C e^{\gamma t} + \tilde{C}\sqrt{|F_4|} \right].$$

Then

$$E_4(t) \leqslant e^{-2\gamma t} E_4(0) + C\frac{1 - e^{-2\gamma t}}{2\gamma} + \tilde{C} \int_0^t e^{-2\gamma(t-s)} \sqrt{|E_4(s)|} \, ds$$

$$\leqslant E_4(0) + C + \tilde{C}\|\sqrt{E_4}\|_{L^\infty([0,t])}.$$

Now take $M \gg 1$, and suppose E_4 attains the value M. Let t be the first time the value is attained. Then $M \leqslant C + \tilde{C}\sqrt{M}$, which is impossible for sufficiently large M. Thus E_4 is bounded above. $\qquad\square$

With this lemma, we conclude that solutions of the dissipative Majda-Biello system remain in a ball, say $\mathcal{B}_0$, in the space $H^1 \times H^1$. We now show that the ω-limit set of the ball,

$$\omega(\mathcal{B}_0) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} U(t)\mathcal{B}_0},$$

is a global attractor in the sense of Definition 2.5.1. Lemma 3.5.1 gives the existence of an absorbing set for (3.3), so by Theorem 2.5.3 we only need prove asymptotic compactness of $U(t)$. To do so, we use the following general smoothing estimate. Notice that it gives a bound on the nonlinear evolution minus a correction involving the resonant terms ρ_i. In Theorem 4.3.6, we consider only the full-measure set of α such that $\rho_1 = \rho_2 = 0$. In this situation, the correction terms vanish.

Theorem 3.5.2. *Consider the solution of* (3.3) *with initial data* $(u_0, v_0) \in \dot{H}^1 \times H^1$. *Then for any* $a < \min\{1 - \nu_c, 1 - \nu_d\}$, *we have*

$$\left\| u(t) - e^{-t\partial_x^3 - \gamma t} u_0 - \int_0^t e^{(-\partial_x^3 - \gamma)(t-r)} \rho_1(v, v)(r) \, dr \right\|_{H^{1+a}}$$

$$+ \left\| v(t) - e^{-\alpha t\partial_x^3 - \gamma t} v_0 - \int_0^t e^{(-\alpha\partial_x^3 - \gamma)(t-r)} \rho_2(u, v)(r) \, dr \right\|_{H^{1+a}}$$

$$\leqslant C(a, \gamma, \|f\|_{H^1}, \|g\|_{H^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1}).$$

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Proof. Taking the Fourier transform of (3.3) yields

$$\begin{cases} \partial_t u_k - ik^3 u_k + \gamma u_k + \dfrac{ik}{2} \displaystyle\sum_{k_1+k_2=k} v_{k_1} v_{k_2} = f_k \\[4mm] \partial_t v_k - i\alpha k^3 v_k + \gamma v_k + ik \displaystyle\sum_{k_1+k_2=k} u_{k_1} v_{k_2} = g_k. \end{cases}$$

Change variables by setting $m_k = e^{-ik^3 t + \gamma t} u_k$ and $n_k = e^{-i\alpha k^3 t + \gamma t} v_k$, with $p_k(t) = e^{-ik^3 t + \gamma t} f_k$ and $q_k(t) = e^{-i\alpha k^3 t + \gamma t} g_k$. After the change of variables, the system is

$$\begin{cases} \partial_t m_k = -\dfrac{ik}{2} \displaystyle\sum_{k_1+k_2=k} e^{-it(k^3-\alpha k_1^3 - \alpha k_2^3)} n_{k_1} n_{k_2} + p_k \\[4mm] \partial_t n_k = -ik \displaystyle\sum_{k_1+k_2=k} e^{-it(\alpha k^3 - k_1^3 - \alpha k_2^3)} m_{k_1} n_{k_2} + q_k. \end{cases}$$

Then differentiating by parts as before gives the equivalent formulation

$$\begin{cases} \partial_t \left[e^{-ik^3 t + \gamma t} \left(u_k + B_1(v,v)_k \right) \right] = e^{-ik^3 t + \gamma t} \left[\rho_1(v,v)_k + R_1(u,v)_k + B_1(g,v)_k + f_k \right] \\[4mm] \partial_t \left[e^{-i\alpha k^3 t + \gamma t} \left(v_k + B_2(u,v)_k \right) \right] = \\[2mm] \qquad e^{-i\alpha k^3 t + \gamma t} \left[\rho_2(u,v)_k + R_2(v,v,v)_k + R_3(u,u,v)_k + B_2(f,v)_k + B_2(g,u)_k + g_k \right], \end{cases}$$

where ρ_j, B_j, and R_j are defined as in Proposition 3.4.1. Integrating from 0 to t yields the equations

$$u_k(t) - e^{ik^3 t - \gamma t} u_k(0) = -B_1(v,v)_k + e^{ik^3 t - \gamma t} B_1(v_0, v_0)$$
$$+ \int_0^t e^{(ik^3 - \gamma)(t-s)} \left[\rho_1(v,v)_k + f_k + R_1(u,v)_k + B_1(v,g)_k \right] ds$$

$$v_k(t) - e^{-\alpha k^3 t - \gamma t} v_k(0) = -B_2(u,v)_k(t) + e^{i\alpha k^3 t - \gamma t} B_2(u_0, v_0)$$
$$+ \int_0^t e^{(i\alpha k^3 - \gamma)(t-s)} \left[\rho_2(u,v)_k + B_2(f,v)_k + B_2(u,g)_k + R_2(v,v,v)_k + R_3(u,u,v)_k + g_k \right] ds.$$

Note that

$$\left\| \int_0^t e^{(ik^3-\gamma)(t-s)} f_k \, ds \right\|_{H^{s+a}} = \left\| \frac{e^{(ik^3-\gamma)t} - 1}{ik^3 - \gamma} f_k \right\|_{H^{s+a}} \lesssim \|f\|_{H^{s-2}}.$$

This, the corresponding estimate for $e^{(i\alpha k^3 - \gamma)(t-s)} g_k$, the estimates used for the previous smoothing result, and Lemma 3.5.1 give the following estimates for $t < \delta$, where δ is the existence time from the local theory:

$$\left\| u(t) - e^{-t\partial_x^3 - \gamma t} u_0 - \int_0^t e^{(-\partial_x^3 - \gamma)(t-r)} \rho_1(v,v)(r) \, dr \right\|_{H^{1+a}}$$

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$$\leqslant C\Big(a, \gamma, \|f\|_{H^1}, \|g\|_{H^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1}\Big)$$

$$\left\| v(t) - e^{-\alpha t \partial_x^3 - \gamma t} v_0 - \int_0^t e^{(-\alpha \partial_x^3 - \gamma)(t-r)} \rho_2(u,v)(r) \, \mathrm{d}r \right\|_{H^{1+a}}$$

$$\leqslant C\Big(a, \gamma, \|f\|_{H^1}, \|g\|_{H^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1}\Big).$$

This bound extends to large times by breaking the time interval down into δ-length pieces. Due to the dissipation, the norm over the short intervals decays over time so that the sum remains uniformly bounded. For details of the argument, see Section 6 in [40]. $\qquad\square$

We now show that U_t is asymptotically compact, i.e., for any bounded sequence $\{(u_{0,k}, v_{0,k})\}$ in $\dot{H}^1 \times H^1$ and sequence of times $t_k \to \infty$, the sequence $\{U(t_k)(u_{0,k}, v_{0,k})\}_k$ has a convergent subsequence in $\dot{H}^1 \times H^1$. It suffices to consider sequences $\{(u_{0,k}, v_{0,k})\}$ which lie within the absorbing set $\mathcal{B}_0$. By Theorem 3.5.2, for any α such that the resonant terms ρ_1 and ρ_2 are zero (i.e. $c_i, d_i \notin \mathbb{Q}$), we have

$$U_{t_k}(u_{0,k}, v_{0,k}) = (e^{-t_k \partial_x^3 - \gamma t_k} u_{0,k}, e^{-\alpha t_k \partial_x^3 - \gamma t_k} v_{0,k}) + N_{t_k}(u_{0,k}, v_{0,k}),$$

where $N_{t_k}(u_{0,k}, v_{0,k})$ is in a ball in $H^{1+a} \times H^{1+a}$. Note we can take $a = \frac{1}{2}-$ for almost every α. In the following, we assume $a = \frac{1}{2}-$.

By Rellich's theorem, there is a subsequence of $\{N_{t_k}(u_{0,k}, v_{0,k})\}$ which converges in $H^1 \times H^1$. Furthermore

$$\|e^{-t_k \partial_x^3 - \gamma t_k} u_{0,n}\|_{H_x^1} + \|e^{-\alpha t_k \partial_x^3 - \gamma t_k} v_{0,n}\|_{H_x^1} \lesssim e^{-\gamma t_k} \Big(\|u_{0,n}\|_{H_x^1} + \|v_{0,n}\|_{H_x^1} \Big) \lesssim e^{-\gamma t_k}$$

converges to zero uniformly as $k \to \infty$. Thus $U_{t_k}(u_{0,k}, v_{0,k})$ has a convergent subsequence and U_t is asymptotically compact.

To show that the attractor is compact in $H^{1+a} \times H^{1+a}$, it suffices, by Rellich's theorem, to show that it is is bounded in $H^{1+a+\epsilon} \times H^{1+a+\epsilon}$ for some $\epsilon > 0$. To do this, choose $\epsilon > 0$ small so that the nonlinear part of the solution lies in $H^{1+a+\epsilon} \times H^{1+a+\epsilon}$, e.g. take $\epsilon = (\frac{1}{2} - a)/2$. We show that the attractor is contained in a closed ball, say $\mathcal{B}_\epsilon$, in this space.

Define $V_\tau = \overline{\cup_{t \geqslant \tau} U_t \mathcal{B}_0}$ so that the attractor is

$$\bigcap_{\tau \geqslant 0} \overline{\bigcup_{t \geqslant \tau} U_t \mathcal{B}_0} = \bigcap_{\tau \geqslant 0} V_\tau.$$

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Using the smoothing result again, elements in V_τ can be broken into two pieces – the linear evolution which is converging uniformly to zero in H^1 by the argument above, and the nonlinear evolution which lives in some ball in $H^{1+a+\epsilon} \times H^{1+a+\epsilon}$.

Thus as a subset of $\dot{H}^1 \times H^1$, the set V_τ is contained in a δ_τ-neighborhood N_τ of a ball $\mathcal{B}_\epsilon$ in $H^{1+a+\epsilon} \times H^{1+a+\epsilon}$. The uniform convergence of the linear parts to zero implies that $\delta_\tau \to 0$ as $\tau \to \infty$. Therefore the attractor is inside $\mathcal{B}_\epsilon$:

$$\bigcap_{\tau \geqslant 0} V_\tau \subset \bigcap_{\tau \geqslant 0} N_\tau = \mathcal{B}_\epsilon.$$

3.6 Trivial Attractor for γ, δ Large

In this section, we show that when the damping is large relative to the forcing terms in the dissipative system (3.3), the global attractor consists of a single function, namely the solution to the time-independent system. We focus on the $\alpha \neq 1$ case with a global attractor in $\dot{H}^1 \times H^1$, noting along the way where the argument differs for $\alpha = 1$ and the $\dot{L}^2 \times \dot{L}^2$ attractor.

Consider the stationary version of the forced and weakly damped Majda-Biello system:

$$\begin{cases} p_{xxx} + \gamma p + qq_x = f \\ \alpha q_{xxx} + \delta q + (pq)_x = g. \end{cases} \tag{3.4}$$

We will take $\gamma = \delta$ to simplify the notation; the arguments can be applied to the general case by replacing γ by $\min\{\gamma, \delta\}$ in the estimates. The first step is to demonstrate the existence of a solution to (3.4) under certain conditions on γ, f, and g.

Proposition 3.6.1. *If $\|f\|_{H^1} \ll \alpha^{1/3}\gamma^{4/3}$ and $\|g\|_{H^1} \ll \alpha^{1/2}\gamma^{4/3}$, then (3.4) has a unique solution on a ball in $H^2(\mathbb{T})$. The same statement holds if $\|f\|_{L^2} \ll \alpha^{1/3}\gamma$ and $\|g\|_{L^2} \ll \alpha^{5/6}\gamma$.*

Proof. The proof uses a fixed point argument. To construct the contraction operator, begin by taking the Fourier transform of the stationary system:

$$\begin{cases} -ik^3 p_k + \gamma p_k + (qq_x)_k = f_k \\ -i\alpha k^3 q_k + \gamma q_k + ((pq)_x)_k = g_k \end{cases}$$

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Define Fourier multiplier operators $\mathcal{M}_1$ and $\mathcal{M}_2$ as follows:

$$\mathcal{M}_1 : w_k \mapsto \frac{w_k}{\gamma - ik^3} \qquad \mathcal{M}_2 : w_k \mapsto \frac{w_k}{\gamma - i\alpha k^3}.$$

We have $\|\mathcal{M}_1 w\|_{H^{s+1}} \lesssim \frac{1}{\gamma^{2/3}} \|w\|_{H^s}$ and $\|\mathcal{M}_2 w\|_{H^{s+1}} \lesssim \frac{1}{\alpha^{1/3}\gamma^{2/3}} \|w\|_{H^s}$. To see this, write

$$\|\mathcal{M}_1 w\|_{H^{s+1}} = \left\| \frac{\langle k \rangle^{s+1} w_k}{\gamma - ik^3} \right\|_{\ell^2} \leqslant \left\| \frac{\langle k \rangle}{\gamma - ik^3} \right\|_{\ell^\infty} \|w\|_{H^s}$$

$$\leqslant \frac{1}{\gamma^{2/3}} \left\| \frac{\langle k \rangle}{(\gamma - ik^3)^{1/3}} \right\|_{\ell^\infty} \|w\|_{H^s} \leqslant \frac{\sqrt{2}}{\gamma^{2/3}} \|w\|_{H^s}.$$

The constant in the last inequality is $\sqrt{2}$ and not $\max\{(1/\gamma)^{1/3}, \sqrt{2}\}$ because we're working with mean zero functions. The arguments go through without this assumption, but the power of γ will change slightly. The other estimate is proved in the same way. Now notice that a solution to (3.4) must satisfy $p = \mathcal{M}_1(f - qq_x)$. Substituting this into the evolution equation for q, we find that q must satisfy

$$q = \mathcal{M}_2(g - (pq)_x) = \mathcal{M}_2\Big(g - \big(\mathcal{M}_1(f - qq_x)q\big)_x\Big).$$

Let $T(q) = \mathcal{M}_2\Big(g - \big(\mathcal{M}_1(f - qq_x)q\big)_x\Big)$. We will find a fixed point of T. Estimate $T(q)$ as follows:

$$\|T(q)\|_{H^2} \lesssim \frac{1}{\alpha^{1/3}\gamma^{2/3}} \|g - (\mathcal{M}_1(f - qq_x)q)_x\|_{H^1} \leqslant \frac{1}{\alpha^{1/3}\gamma^{2/3}} \Big(\|g\|_{H^1} + \|\mathcal{M}_1(f - qq_x)q\|_{H^2} \Big)$$

$$\leqslant \frac{1}{\alpha^{1/3}\gamma^{2/3}} \Big(\|g\|_{H^1} + \|\mathcal{M}_1(f - qq_x)\|_{H^2} \|q\|_{H^2} \Big)$$

$$\leqslant \frac{1}{\alpha^{1/3}\gamma^{2/3}} \Big(\|g\|_{H^1} + \frac{1}{\gamma^{2/3}} \|(f - qq_x)\|_{H^1} \|q\|_{H^2} \Big)$$

$$\lesssim \frac{\|g\|_{H^1}}{\alpha^{1/3}\gamma^{2/3}} + \frac{\|q\|_{H^2}}{\alpha^{1/3}\gamma^{4/3}} \Big(\|f\|_{H^1} + \|q\|_{H^2}^2 \Big).$$

Now we make the contraction estimate:

$$\|T(w) - T(\tilde{w})\|_{H^2} = \left\| \mathcal{M}_2\Big(\big(\mathcal{M}_1(f - \tilde{w}\tilde{w}_x)\tilde{w} - \mathcal{M}_1(f - ww_x)w \big)_x \Big) \right\|_{H^2}$$

$$\lesssim \frac{1}{\alpha^{1/3}\gamma^{2/3}} \left\| \mathcal{M}_1(f - \tilde{w}\tilde{w}_x)\tilde{w} - \mathcal{M}_1(f - ww_x)w \right\|_{H^2}$$

$$= \frac{1}{\alpha^{1/3}\gamma^{2/3}} \left\| \mathcal{M}_1(f - \tilde{w}\tilde{w}_x)(\tilde{w} - w) + \mathcal{M}_1\big((w - \tilde{w})w_x + \tilde{w}(w - \tilde{w})_x\big)w \right\|_{H^2}$$

$$\leqslant \frac{1}{\alpha^{1/3}\gamma^{4/3}} \Big(\|f - \tilde{w}\tilde{w}_x\|_{H^1} \|w - \tilde{w}\|_{H^2} + \|(w - \tilde{w})w_x\|_{H^1} \|w\|_{H^2} + \|\tilde{w}(w - \tilde{w})_x\|_{H^1} \|w\|_{H^2} \Big)$$

$$\lesssim \frac{\|w - \tilde{w}\|_{H^2}}{\alpha^{1/3}\gamma^{4/3}} \Big(\|f\|_{H^1} + \|\tilde{w}\|_{H^2}^2 + \|w\|_{H^2}^2 + \|\tilde{w}\|_{H^2} \|w\|_{H^2} \Big).$$

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Thus to close the contraction on a ball $\{q \in H^2 : \|q\|_{H^2} \leqslant R\}$, two inequalities must hold:

$$\frac{\|g\|_{H^1}}{\alpha^{1/3}\gamma^{2/3}} + \frac{\|f\|_{H^1}R + R^3}{\alpha^{1/3}\gamma^{4/3}} \leqslant \frac{R}{C} \qquad \text{and} \qquad \frac{\|f\|_{H^1} + 3R^2}{\alpha^{1/3}\gamma^{4/3}} < \frac{1}{C}.$$

These can be satisfied by taking $R = \frac{1}{2\sqrt{C}}\alpha^{1/6}\gamma^{2/3}$ as long as $\|f\|_{H^1} < \frac{1}{4C}\alpha^{1/3}\gamma^{4/3}$ and $\|g\|_{H^1} \leqslant \frac{1}{4C^{3/2}}\alpha^{1/2}\gamma^{4/3}$.

The proof for the L^2 statement is similar. The only difference is that one uses the estimates $\|\mathcal{M}_1 f\|_{H^{s+2}} \leqslant \frac{1}{\gamma^{1/3}}\|f\|_{H^s}$ and $\|\mathcal{M}_2 g\|_{H^{s+2}} \leqslant \frac{1}{\alpha^{2/3}\gamma^{1/3}}\|g\|_{H^s}$. $\qquad\square$

Remark 3.6.2. *If $g = 0$, the existence of a stationary solution is trivial; the solution is $(\mathcal{M}_1(f), 0)$. The convergence arguments are also greatly simplified in this case.*

We now show that solutions to (3.3) converge to the stationary solution under certain conditions on f, g, and γ, implying that the attractor is trivial. Let (u, v) be a solution of the dissipative Majda-Biello system (3.3) and define $y = u - p$ and $z = v - q$. We show that if f and g are small relative to γ, then y and z converge to zero in H^1 if $u, v \in H^1$. Notice that y and z satisfy

$$\begin{cases} y_t + y_{xxx} + \gamma y + z z_x + (qz)_x = 0 \\ z_t + \alpha z_{xxx} + \gamma z + (yz)_x + (pz + qy)_x = 0. \end{cases} \tag{3.5}$$

Recall that $\int u^2 + v^2 \ dx$ and $\int u_x^2 + \alpha v_x^2 - uv^2 \ dx$ are conserved for the original Majda-Biello system. Our estimates will be based on these conservation laws. Recall $E_3 = \int y^2 + z^2 \ dx$. Then we have

$$\begin{aligned} \frac{\partial}{\partial t} E_3 &= -2 \int y\left(\gamma y + (qz)_x\right) + z\left(\gamma z + (pz + qy)_x\right) \ dx \\ &= -2\gamma E_3 - 2 \int qy z_x + q_x yz + p_x z^2 + pz z_x + qy_x z + q_x yz \ dx \\ &= -2\gamma E_3 - 2 \int 2q_x yz + q(yz)_x + \frac{1}{2}p_x z^2 \ dx \\ &= -2\gamma E_3 - \int 2q_x yz + p_x z^2 \ dx \\ &\leqslant -2\gamma E_3 + \|p_x\|_{L^\infty}\|z\|_{L^2}^2 + 2\|q_x\|_{L^\infty}\|y\|_{L^2}\|z\|_{L^2} \\ &\leqslant (-2\gamma + C\|p\|_{H^2} + C\|q\|_{H^2})E_3. \end{aligned}$$

So to ensure that $E_3 \to 0$ as $t \to \infty$, i.e. that $(u, v) \to (p, q)$ in L^2, we need $C\|p\|_{H^2} + C\|q\|_{H^2} < 2\gamma$. The contraction argument for the existence of q was carried out in a ball of radius $R = \frac{1}{2\sqrt{C}}\alpha^{1/6}\gamma^{2/3}$, so we have $C\|q\|_{H^2} < \gamma$ as long as $\gamma^{1/3} > \frac{\sqrt{C}\alpha^{1/6}}{2}$. Also notice that

$$C\|p\|_{H^2} \leqslant \frac{C}{\gamma^{2/3}}\left(\|f\|_{H^1} + \|qq_x\|_{H^1}\right) \leqslant \frac{C}{\gamma^{2/3}}\left(\|f\|_{H^1} + \|q\|_{H^2}^2\right).$$

This is bounded by γ when $C\|f\|_{H^1} < \frac{\gamma^{5/3}}{2}$ and $8\gamma > \alpha$. So we have a stationary solution and L^2 convergence to it whenever $\|f\|_{H^1}, \|g\|_{H^1} \ll \gamma^{4/3}$ and $\gamma > \sqrt{C^3\alpha}/8$. The same holds when $\|f\|_{L^2}, \|g\|_{L^2} \ll \gamma$ and $\gamma > \sqrt{C^3\alpha}/8$, which completes the proof for the $\alpha = 1$ case.

For the H^1 convergence, we use a modification of the Hamiltonian integral E_4:

$$H_4 = \int y_x^2 + \alpha z_x^2 - yz^2 - 2qyz - pz^2 \; \mathrm{d}x.$$

The last two terms are added to make the time derivative well-behaved. Calculating this derivative, we find

$$
\begin{aligned}
\frac{\partial}{\partial t}H_4 = {} & -2\int y_x\left(\gamma y_x + (qz)_{xx}\right) \; \mathrm{d}x - 2\alpha \int z_x\left(\gamma z_x + (pz + qy)_{xx}\right) \; \mathrm{d}x \\
& + \int z^2\left(\gamma y + (qz)_x\right) \; \mathrm{d}x + 2\int yz\left(\gamma z + (pz + qy)_x\right) \; \mathrm{d}x \\
& + 2\int qy\left(\alpha z_{xxx} + \gamma z + (yz)_x + (pz + qy)_x\right) \; \mathrm{d}x \\
& + 2\int qz\left(y_{xxx} + \gamma y + zz_x + (qz)_x\right) \; \mathrm{d}x + 2\int pz\left(\alpha z_{xxx} + \gamma z + (yz)_x + (pz + qy)_x\right) \; \mathrm{d}x \\
= {} & -2\gamma H_4 + \gamma \int yz^2 \; \mathrm{d}x.
\end{aligned}
$$

Notice that

$$\int yz^2 \; \mathrm{d}x \lesssim \|y\|_{H^1}\|z\|_{L^2}^2 \lesssim e^{-at}$$

by the embedding $L^\infty \hookrightarrow H^1$, the bound on the H^1 norm of y (which follows from Lemma 3.5.1), and the decay of the L^2 norm of z. Here $a = -2\gamma + C\|p\|_{H^2} + C\|q\|_{H^2} > 0$. Thus we have

$$\partial_t[e^{2\gamma t}H_4] \lesssim e^{-at}.$$

Integrating this inequality gives $H_4(t) \lesssim e^{-2\gamma t}$. Furthermore, since $\|y\|_{L^2}^2 + \|z\|_{L^2}^2 \to 0$ as $t \to \infty$ and the L^2 norms of p, q, y, and z are bounded, we have

$$\left|\int yz^2 + 2qyz + pz^2 \; \mathrm{d}x\right| \lesssim \|y\|_{H^1}\|z\|_{L^2}^2 + 2\|q\|_{H^1}\|y\|_{L^2}\|z\|_{L^2} + \|p\|_{H^1}\|z\|_{L^2}^2 \to 0 \text{ as } t \to \infty.$$

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Thus we have

$$\left| \int y_x^2 + \alpha z_x^2 \ \mathrm{d}x \right| \leqslant |H_4| + \left| \int yz^2 + 2qyz + pz^2 \ \mathrm{d}x \right| \to 0 \text{ as } t \to \infty.$$

This, along with the L^2 convergence show above, implies that $y = u - p$ and $z = v - q$ converge to zero in H^1.

3.7 PROOFS OF SMOOTHING ESTIMATES

Before beginning, we note that the proofs for the cases where $\alpha = \frac{q^2}{3p(p-q)+q^2}$ are much easier than those for the general cases, and are therefore not explicitly included.

3.7.1 PROOF OF PROPOSITION 3.4.2

By symmetry, it suffices to consider $|k_1| \gtrsim |k_2|$. Then we need to bound

$$\left\| \sum_{\substack{k_1+k_2=k \\ |k_1| \gtrsim |k_2|}}^{*} \frac{\langle k \rangle^{1+s_1} u_{k_1} v_{k_2}}{k^3 - \alpha k_1^3 - \alpha k_2^3} \right\|_{\ell_k^2}$$

Case 1. $|k_1 - c_1 k| \geqslant \frac{1}{2}$ and $|k_1 - c_2 k| \geqslant \frac{1}{2}$

Note that

$$|k^3 - \alpha k_1^3 - \alpha k_2^3| = |3\alpha k(k_1 - c_1 k)(k_1 - c_2 k)| \gtrsim |k| \cdot \max\{|k_1 - c_1 k|, |k_1 - c_2 k|\}$$

$$\gtrsim |k| \cdot (c_1 - c_2)|k| \gtrsim |k|^2.$$

Then using $|k_1| \gtrsim k$, the assumption that $s_1 - s - 1 \leqslant 0$, and Young's inequality, we find

$$\|B_1(u,v)\|_{H_x^{s_1}} \lesssim \left\| \sum_{\substack{k_1+k_2=k \\ |k_1| \gtrsim |k_2|}} \langle k \rangle^{s_1-s-1} \frac{|u_{k_1}|\langle k_1 \rangle^s |v_{k_2}|\langle k_2 \rangle^s}{\langle k_2 \rangle^s} \right\|_{\ell_k^2}$$

$$\leqslant \left\| \sum_{k_1+k_2=k} \frac{|u_{k_1}|\langle k_1 \rangle^s |v_{k_2}|\langle k_2 \rangle^s}{\langle k_2 \rangle^s} \right\|_{\ell_k^2} \leqslant \|u\|_{H_x^s} \left\| \frac{v_k \langle k \rangle^s}{\langle k \rangle^s} \right\|_{\ell_k^1}$$

$$\leqslant \|u\|_{H_x^s} \|v\|_{H_x^s} \|\langle k \rangle^{-s}\|_{\ell_k^2} \lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.$$

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Case 2. $|k_1 - c_1 k| < \frac{1}{2}$ or $|k_1 - c_2 k| < \frac{1}{2}$

Assume that $|k_1 - c_1 k| < \frac{1}{2}$. The other case is parallel. Note that $|k_1 - c_1 k| \geqslant M_{\epsilon_0}|k|^{-1-\nu_{c_1}-\epsilon_0}$ for any $\epsilon_0 > 0$, where ν_{c_1} is the minimal type index of c_1. This holds because

$$|k_1 - c_1 k| = |k| \left| \frac{k_1}{k} - c_1 \right| \geqslant |k| \frac{M_{\epsilon_0}}{|k|^{2+\nu_{c_1}+\epsilon_0}}$$

for any positive ϵ_0 by definition of the minimal type. Therefore

$$|k^3 - \alpha k_1^3 - \alpha k_2^3| = 3\alpha|k(k_1 - c_1 k)(k_1 - c_2 k)| \geqslant 3\alpha M_{\epsilon_0}|k|^{-\nu_{c_1}-\epsilon_0}\left((c_1 - c_2)|k| - \frac{1}{2}\right)$$

$$\gtrsim |k|^{1-\nu_{c_1}-\epsilon_0}.$$

In this region there is only one term in the sum – the one with $k_1 \simeq c_1 k$ and $k_2 \simeq (1 - c_1)k$. Using Cauchy-Schwartz with the fact that $|k| \approx |k_1| \approx |k_2|$, we get for this part of the sum

$$\|B_1(u,v)\|_{H_x^{s_1}} \lesssim \left\|\langle k \rangle^{s_1+\nu_{c_1}+\epsilon_0} u_{k_1} v_{k_2}\right\|_{\ell_k^2} \lesssim \left\|\langle k \rangle^{(s_1+\nu_{c_1}+\epsilon_0)/2} u_k\right\|_{\ell_k^4} \left\|\langle k \rangle^{(s_1+\nu_{c_1}+\epsilon_0)/2} v_k\right\|_{\ell_k^4}$$

$$\lesssim \left\|\langle k \rangle^{(s_1+\nu_{c_1}+\epsilon_0)/2} u_k\right\|_{\ell_k^2} \left\|\langle k \rangle^{(s_1+\nu_{c_1}+\epsilon_0)/2} v_k\right\|_{\ell_k^2} \lesssim \|u\|_{H_x^s}\|v\|_{H_x^s},$$

where the third inequality holds when $s_1 - s < s - \nu_{c_1}$.

3.7.2 PROOF OF PROPOSITION 3.4.3

Write

$$\|B_2(u,v)\|_{H_x^{s_1}} \lesssim \left\|\sum_{k_1+k_2=k}^{*} \frac{\langle k \rangle^{1+s_1} u_{k_1} v_{k_2}}{\alpha k^3 - k_1^3 - \alpha k_2^3}\right\|_{\ell_k^2}.$$

Case 1. $|k_1 - d_1 k|, |k_1 - d_2 k| \geqslant \epsilon$ and $|k_1| \geqslant \epsilon|k|$

In this case, $|k_1| \gtrsim |k_2|$ and

$$|\alpha k^3 - k_1^3 - \alpha k_2^3| = |(1 - \alpha)k_1(k_1 - d_1 k)(k_1 - d_2 k)| \gtrsim |kk_1|.$$

The argument in Case 1 of the B_1 estimate gives the bound when $s_1 - s \leqslant 1$ and $s > \frac{1}{2}$.

Case 2. $|k_1| \leqslant \epsilon|k|$

Recall that $k_1 \neq 0$ since u is mean zero and write

$$k_1 = \mu k \text{ for some } |\mu| \in [1/|k|, \epsilon].$$

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Then

$$|\alpha k^3 - k_1^3 - \alpha(k - k_1)^3| = |\mu k^3||3\alpha(1 - \mu) - \mu^2(1 - \alpha)| \geqslant k^2\left(3\alpha(1 - \epsilon) - \epsilon^2(1 - \alpha)\right) \gtrsim k^2.$$

Apply the argument from Case 1 of the B_1 proof again to get the bound when $s_1 - s \leqslant 1$ and $s > \frac{1}{2}$.

Case 3. $|k_1 - d_1 k| \leqslant \epsilon$ or $|k_1 - d_2 k| \leqslant \epsilon$, with $|k_1| \geqslant \epsilon|k|$

Assume $|k_1 - d_1 k| \leqslant \epsilon$. The other case is parallel. Note that in this region $|k| \sim |k_1| \sim |k_2|$ and the values of k_1 and k_2 are determined by k. We need only bound the following sum, where $k_1 \simeq d_1 k$ and $k_1 + k_2 = k$,

$$
\begin{aligned}
\left(\sum_k \langle k \rangle^{2(s_1 + \nu_{d_1} + \epsilon_0)} u_{k_1}^2 v_{k_2}^2\right)^{1/2} &\lesssim \left\|\langle k \rangle^{s_1 + \nu_{d_1} + \epsilon_0} u_k^2\right\|_{\ell_k^2}^{1/2} \left\|\langle k \rangle^{s_1 + \nu_{d_1} + \epsilon_0} v_k^2\right\|_{\ell_k^2}^{1/2} \\
&= \left\|\langle k \rangle^{(s_1 + \nu_{d_1} + \epsilon_0)/2} u_k\right\|_{\ell_k^4} \left\|\langle k \rangle^{(s_1 + \nu_{d_1} + \epsilon_0)/2} v_k\right\|_{\ell_k^4} \\
&\leqslant \left\|\langle k \rangle^{(s_1 + \nu_{d_1} + \epsilon_0)/2} u_k\right\|_{\ell_k^2} \left\|\langle k \rangle^{(s_1 + \nu_{d_1} + \epsilon_0)/2} v_k\right\|_{\ell_k^2} \\
&\lesssim \|u\|_{H_x^s} \|v\|_{H_x^s}.
\end{aligned}
$$

The last inequality holds when $s_1 + \nu_d + \epsilon_0 \leqslant 2s$, i.e. when $s_1 - s < s - \nu_d$.

3.7.3 Proof of Proposition 3.4.5

We need to establish

$$
\left\|\sum_{k_1 + k_2 + k_3 = k}^{*} \frac{(k_1 + k_2)u_{k_1} v_{k_2} w_{k_3}}{(k_1 + k_2 - c_1 k)(k_1 + k_2 - c_2 k)}\right\|_{X_1^{s_1, b-1}} \lesssim \|u\|_{X_1^{s,1/2}} \|v\|_{X_\alpha^{s,1/2}} \|w\|_{X_\alpha^{s,1/2}}. \tag{3.6}
$$

Define the following functions

$$f(k, \tau) = \langle k \rangle^s \langle \tau - k^3 \rangle^{1/2} u_k$$
$$g(k, \tau) = \langle k \rangle^s \langle \tau - \alpha k^3 \rangle^{1/2} v_k$$
$$h(k, \tau) = \langle k \rangle^s \langle \tau - \alpha k^3 \rangle^{1/2} w_k.$$

Then (3.6) amounts to showing that

$$
\left\|\int_{\sum \tau_i = \tau} \sum_{\sum k_i = k}^{*} M \, f(k_1, \tau_1) g(k_2, \tau_2) h(k_3, \tau_3) \, d\tau_1 \, d\tau_2 \, d\tau_3\right\|_{L_\tau^2 \ell_k^2}^2 \lesssim \|f\|_{L_\tau^2 \ell_k^2}^2 \|g\|_{L_\tau^2 \ell_k^2}^2 \|h\|_{L_\tau^2 \ell_k^2}^2 \tag{3.7}
$$

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where the multiplier $M = M(k_1, k_2, k_3, k, \tau_1, \tau_2, \tau_3, \tau)$ is

$$M = \frac{(k_1 + k_2)\langle k \rangle^{s_1}\langle k_1 \rangle^{-s}\langle k_2 \rangle^{-s}\langle k_3 \rangle^{-s}}{(k_1 + k_2 - c_1 k)(k_1 + k_2 - c_2 k)\langle \tau - k^3 \rangle^{1-b}\langle \tau_1 - k_1^3 \rangle^{1/2}\langle \tau_2 - \alpha k_2^3 \rangle^{1/2}\langle \tau_3 - \alpha k_3^3 \rangle^{1/2}}.$$

Apply Cauchy-Schwartz in the τ_1, τ_2, τ_3, k_1, k_2, and k_3 variables to bound the left-hand side of (3.7) by

$$\sup_{k,\tau} \left(\int_{\sum \tau_i = \tau} \sum_{\sum k_i = k}^{*} M^2 \right) \left\| \int_{\sum \tau_i = \tau} \sum_{\sum k_i = k}^{*} f^2(k_1, \tau_1) g^2(k_2, \tau_2) h^2(k_3, \tau_3) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, \mathrm{d}\tau_3 \right\|_{L^1_\tau \ell^1_k}$$

Using Young's inequality twice bounds the $L^1 \ell^1$ norm above by $\|f\|^2_{L^2_\tau \ell^2_k} \|g\|^2_{L^2_\tau \ell^2_k} \|h\|^2_{L^2_\tau \ell^2_k}$. Thus it suffices to show that the supremum on the left is finite. We can further simplify matters by repeatedly using the calculus estimate

$$\int_{\mathbb{R}} \frac{1}{\langle x \rangle^\beta \langle x - b \rangle} \, \mathrm{d}x \lesssim \langle b \rangle^{-\beta},$$

which holds for $\beta \in (0, 1]$ (see Lemma 2.3.4), to eliminate the τ dependence and bound the supremum by

$$\sup_k \sum_{k_1, k_2}^{*} \frac{\langle k \rangle^{2s_1}\langle k_1 \rangle^{-2s}\langle k_2 \rangle^{-2s}\langle k - k_1 - k_2 \rangle^{-2s}|k_1 + k_2|^2}{(k_1 + k_2 - c_1 k)^2 (k_1 + k_2 - c_2 k)^2 \langle k^3 - k_1^3 - \alpha k_2^3 - \alpha(k - k_1 - k_2)^3 \rangle^{2 - 2b-}},$$

or equivalently, using the change of variables $k_2 \mapsto n - k_1$,

$$\sup_k \sum_{k_1, n}^{*} \frac{\langle k \rangle^{2s_1}\langle k_1 \rangle^{-2s}\langle n - k_1 \rangle^{-2s}\langle n - k \rangle^{-2s} n^2}{(n - c_1 k)^2 (n - c_2 k)^2 \langle k^3 - k_1^3 - \alpha(n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b-}}.$$

We will show that this supremum is finite by considering a number of cases. In the following, to simplfiy notation we will write $2 - 2b$ instead of the technically correct $2 - 2b-$. Since we take $b = \frac{1}{2}+$, this ϵ-difference has no effect on the calculations.

Case 1. $k_1 = k$

In this case, the supremum becomes

$$\sup_k \sum_n^{*} \frac{\langle k \rangle^{2s_1 - 2s}\langle n - k \rangle^{-4s} n^2}{(n - c_1 k)^2 (n - c_2 k)^2}.$$

Case 1.1. $kn > 0$

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Since $c_2 < 0$, we cancel n^2 with $(n - c_2 k)^2$ to obtain

$$\sup_k \sum_n^* \frac{\langle k \rangle^{2s_1 - 2s} \langle n - k \rangle^{-4s}}{(n - c_1 k)^2}.$$

If $|n - c_1 k| \geqslant \epsilon$, with ϵ small but fixed, then the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 - 2s} \sum_n \frac{\langle n - k \rangle^{-4s}}{\langle n - c_1 k \rangle^2} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s} \langle (c_1 - 1) k \rangle^{-2} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 2},$$

which is finite for $s_1 - s \leqslant 1$. In the first inequality, we used Lemma 2.3.3(a).

If $|n - c_1 k| < \epsilon$, then there's only one term in the sum since $n \simeq c_1$, and we have $|n - k| \gtrsim |k|$. Using the minimal type index, the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 - 6s + 2 + 2\nu_{c_1} + 2\epsilon_0},$$

which is finite when $s_1 - s < 2s - 1 - \nu_{c_1}$.

Case 1.2. $kn < 0$

For this case, cancel n^2 with $(n - c_1 k)^2$ and repeat the argument from Case 1.1.

Case 1.3. $kn = 0$

The supremum is immediately bounded in this case.

Case 2. $kn > 0$ with $k_1 \neq k$

In this region, the supremum is bounded by

$$\sup_k \sum_{k_1, n}^* \frac{\langle k \rangle^{2s_1} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s}}{(n - c_1 k)^2 \langle k^3 - k_1^3 - \alpha(n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}}.$$

Case 2.1 $|n - c_1 k| \geqslant \epsilon |k|$

Here the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 - 2} \sum_{k_1, n} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s} \lesssim \sup_k \langle k \rangle^{2s_1 - 2 - 2s} < \infty$$

for $s_1 - s \leqslant 1$. This estimate comes from applying Lemma 2.3.3(a) repeatedly.

Case 2.2. $\epsilon \leqslant |n - c_1 k| < \epsilon |k|$

Note that $|n| \in \big((c_1 - \epsilon)|k|, (c_1 + \epsilon)|k| \big)$. Choose $\epsilon < c_1 - 1$ so that $|n - k| \gtrsim |k|$. The supremum is then bounded by

$$\sup_k \langle k \rangle^{2s_1 - 2s} \sum_{k_1, |n| \geqslant |k|} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s} \sum_{|n| \geqslant |k|} \langle n \rangle^{-2s} \lesssim \sup_k \langle k \rangle^{2s_1 - 4s + 1}.$$

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This is finite when $s_1 - s \leqslant s - \frac{1}{2}$.

Case 2.3. $|n - c_1 k| < \epsilon$

Case 2.3a. $|k_1|, |k_1 - n| \geqslant \epsilon|k|$

In this case, the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 6s} \sum_{\substack{k_1 \\ (n \simeq c_1 k)}} \langle k^3 - k_1^3 - \alpha(n - k_1)^3 + \alpha(n - k)^3 \rangle^{-(2 - 2b)}$$

$$\lesssim \sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 6s},$$

which is finite for $s_1 - s < 2s - 1 - \nu_{c_1}$. This estimate comes from Lemma 2.3.3(b).

Case 2.3b. $|k_1| < \epsilon|k|$

Note that in this case

$$|k_1 - n| \geqslant c_1|k| - |n - c_1 k| - |k_1| > (c_1 - \epsilon)|k| - \epsilon$$

so that $|k_1 - n| \gtrsim |k|$. Recall $k_1 \neq 0$ by the mean zero assumption on u, and write

$$n = c_1 k + \delta \text{ for some } |\delta| < \epsilon, \qquad k_1 = \mu k \text{ for some } |\mu| \in [1/|k|, \epsilon).$$

Then use the fact that $1 - \alpha = 3\alpha c_1(c_1 - 1)$ to calculate that

$$|k^3 - k_1^3 - \alpha(n - k_1)^3 + \alpha(n - k)^3|$$

$$= |k - k_1| \, |\mu k^2[(1 - \alpha)(1 + \mu) + 3\alpha c_1] + 3\alpha\delta[1 + \mu - 2c_1]k - 3\alpha\delta^2|$$

$$\geqslant |k - k_1| \left[(3\alpha c_1 + (1 - \alpha)(1 - \epsilon))|k| - 3\alpha\epsilon(2c_1 + \epsilon - 1)|k| - 3\alpha\epsilon^2 \right] \gtrsim |k_1 - k||k|.$$

Using Lemma 2.3.3(a) again, the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 4s} \sum_{k_1} \frac{\langle k_1 \rangle^{-2s}}{\langle (k - k_1)k \rangle^{2 - 2b}}$$

$$\lesssim \sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 4s - (2 - 2b)} \sum_{k_1} \frac{\langle k_1 \rangle^{-2s}}{\langle k - k_1 \rangle^{2 - 2b}} \lesssim \sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 4s - 2(2 - 2b)},$$

which is finite when $s_1 - s < s + 1 - 2b - \nu_{c_1}$.

Case 2.3c. $|n - k_1| < \epsilon|k|$

In this case we have $|k_1| \geq |n| - |n - k_1| \geq (c_1 - \epsilon)|k| - \epsilon$ so that $|k_1| \gtrsim |k|$. The supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 4s} \sum_{\substack{k_1 \\ n \simeq c_1 k}} \frac{\langle n - k_1 \rangle^{-2s}}{\langle k^3 - k_1^3 - \alpha(n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}}.$$

We may assume, since $(k, k_1, n) \to (-k, -k_1, -n)$ is a symmetry for the supremum, that $k_1 > 0$. Then in our case of $kn > 0$, we must have $k, n > 0$, since otherwise $|k_1 - n| > |n| \simeq c_1|k|$.

Notice that the following three inequalities hold:

$$k^2 + k_1 k + k_1^2 \geq k^2, \qquad 3\alpha n > 0, \qquad \text{and} \qquad k_1 - (n - k) \geq (1 - \epsilon)k - 2\epsilon.$$

Thus we have

$$(1 - \alpha)(k^2 + k k_1 + k_1^2) - 3\alpha n(n - n - k_1) \gtrsim k^2,$$

which implies that

$$k^3 - k_1^3 - \alpha(n - k_1)^3 - \alpha(k - n)^3 \gtrsim |k - k_1| k^2.$$

The supremum is therefore bounded by

$$\sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 4s} \sum_{\substack{k_1 \\ n \simeq c_1 k}} \frac{\langle n - k_1 \rangle^{-2s}}{\langle k^2(k - k_1) \rangle^{2 - 2b}} \lesssim \sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{c_1} + 2\epsilon_0 - 4s - 6 + 6b},$$

which is finite if $s_1 - s < s + 2 - 3b - \nu_{c_1}$.

Case 3. $kn < 0$ and $k_1 \neq k$

In this case, the supremum can be bounded by

$$\sup_k \langle k \rangle^{2s_1 - 2s} \sum_{\substack{k_1 > 0 \\ n}}^{*} \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s}}{(n - c_2 k)^2 \langle k^3 - k_1^3 \alpha(n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}}.$$

Case 3.1. $|n - c_2 k| \geq \epsilon|k|$

If $s > \frac{1}{2}$ and $s_1 - s \leq 1$, the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 - 2s - 2} \sum_{\substack{k_1 > 0 \\ n}} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 2} < \infty,$$

.

Case 3.2. $\epsilon \leq |n - c_2 k| < \epsilon|k|$

Note that $|n| \geqslant (|c_2| - \epsilon)|k| \approx |k|$. When $s_1 - s \leqslant s - \frac{1}{2}$, the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1 - 2s} \sum_{\substack{k_1 > 0 \\ |n| \gtrsim |k|}} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} < \sup_k \langle k \rangle^{2s_1 - 2s} \sum_{|n| \gtrsim |k|} \langle n \rangle^{-2s} < \infty.$$

Case 3.3. $|n - c_2 k| < \epsilon$

Case 3.3a. $|k_1|, |k_1 - n| \geqslant \epsilon |k|$

As in Case 2.3a, the supremum is finite if $s_1 - s < 2s - 1 - \nu_{c_1}$.

Case 3.3b. $|k_1| < \epsilon |k|$

Here $|k_1 - n| \geqslant (|c_2| - \epsilon)|k| - \epsilon$, so $|k_1 - n| \gtrsim |k|$. Write

$$k_1 = \mu k \text{ for some } |\mu| \in [1/|k|, \epsilon), \qquad n = c_2 k + \delta \text{ for some } |\delta| < \epsilon.$$

Expanding the resonance equation with this notation gives

$$|k^3 - k_1^3 - \alpha(n - k_1)^3 - \alpha(k - n)^3|$$
$$= |k - k_1| \left| \mu k^2 (3\alpha c_2^2 + \mu(1 - \alpha)) + 3\alpha\delta(1 + \mu - 2c_2)k - 3\alpha\delta^2 \right|$$
$$\geqslant |k - k_1| \left[\left(3\alpha c_2^2 - \epsilon(1 - \alpha) - 3\alpha\epsilon[1 + \epsilon - 2c_2] \right) |k| - 3\alpha\epsilon^2 \right] \gtrsim |k - k_1||k|.$$

Notice that, depending on α, we may have $|c_2| \ll 1$, but by choosing ϵ small enough, we can ensure that

$$\left[\left(3\alpha c_2^2 - \epsilon(1 - \alpha) - 3\alpha\epsilon[1 + \epsilon - 2c_2] \right) |k| - 3\alpha\epsilon^2 \right] \gtrsim |k|$$

to get the last inequality above. Then as in Case 2.3b, the supremum is finite if $s_1 - s < s + 1 - 2b - \nu_{c_1}$.

Case 3.3c. $|k_1 - n| \leqslant \epsilon |k|$

Note $|k_1| \geqslant |n| - |k_1 - n| \geqslant (|c_2| - \epsilon)|k| - \epsilon$, so for ϵ small enough, $|k_1| \geqslant \epsilon |k|$. Write

$$n - k_1 = \mu k \text{ for some } |\mu| \in \{0\} \cup [1/|k|, \epsilon], \qquad n = c_2 k + \delta \text{ for some } |\delta| < \epsilon.$$

With this notation,

$$|k^3 - \alpha(n - k_1)^3 - \alpha(k - n)^3| = \left| \alpha(c_2^3 - \mu^3)k^3 + 3\alpha\delta(1 - c_2)^2 k^2 - 3\alpha\delta^2(1 - c_2)k + \alpha\delta^3 \right|$$
$$\geqslant \alpha(|c_2|^3 - \epsilon^3)|k|^3 - 3\alpha\epsilon(1 - c_2)^2 k^2 - 3\alpha\epsilon^2(1 - c_2)|k| - \alpha\epsilon^3 \gtrsim |k|^3,$$

for ϵ small. Then the supremum is finite for $s_1 - s < s + 2 - 3b - \nu_{c_1}$ by the same reasoning as before.

Case 4. $kn = 0$

The bound is immediate in this case.

3.7.4   PROOF OF PROPOSITION 3.4.6

As in the previous proof, it suffices to show that the supremum of the following quantity is finite:

$$\langle k \rangle^{2+2s_1} \sum_{k_1,k_2}^* \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k - k_1 - k_2 \rangle^{-2s} |k_1 + k_2|^2}{(\alpha k^3 - (k_1 + k_2)^2 - \alpha(k - k_1 - k_2)^3)^2 \langle \alpha(k^3 - k_1^3 - k_2^3 - (k - k_1 - k_2)^3) \rangle^{2-2b}}.$$

We will work with the equivalent supremum

$$\sup_k \langle k \rangle^{2+2s_1} \sum_{\substack{k_1 \\ n \neq 0}}^* \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s}}{(n - d_1 k)^2 (n - d_2 k)^2 \langle (k - k_1)(k + k_1 - n)n \rangle^{2-2b}},$$

which results from changing variables $k_2 \mapsto n - k_1$ and canceling a factor of n^2 from the quotient.

Case 1. $k_1 = k$

In this case, the supremum becomes

$$\sup_k \langle k \rangle^{2+2s_1-2s} \sum_{n \neq 0}^* \frac{\langle k - n \rangle^{-4s}}{(n - d_1 k)^2 (n - d_2 k)^2}.$$

Repeat the arguments from Case 1 of the R_1 estimate to show that the supremum is finite if $s_1 - s \leqslant 1$ and $s - s_1 < 2s - \nu_d - 1$.

Case 2. $n - k_1 = k$

The supremum becomes

$$\sup_k \langle k \rangle^{2+2s_1-2s} \sum_{n \neq 0}^* \frac{\langle n - k \rangle^{-4s}}{(n - d_1 k)^2 (n - d_2 k)^2},$$

which is the same as that in Case 1.

Case 3. $(k - k_1)(k + k_1 - n)n \neq 0$

In this case, the supremum is bounded by

$$\sup_k \langle k \rangle^{2+2s_1} \sum_{\substack{k_1 \\ n \neq 0}}^* \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s}}{(n - d_1 k)^2 (n - d_2 k)^2 \langle k - k_1 \rangle^{2-2b} \langle k + k_1 - n \rangle^{2-2b} \langle n \rangle^{2-2b}}.$$

Case 3.1. $kn > 0$

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Here $|n - d_2 k| > k$, so the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1} \sum_{\substack{k_1 \\ n \neq 0}}^{*} \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s}}{(n - d_1 k)^2 \langle k - k_1 \rangle^{2-2b} \langle k + k_1 - n \rangle^{2-2b} \langle n \rangle^{2-2b}}.$$

Case 3.1a. $|n - d_1 k| \geqslant \epsilon |k|$

In this case we have the bound

$$\sup_k \langle k \rangle^{2s_1 - 2} \sum_{\substack{k_1 \\ n \neq 0}} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s} \lesssim \sup_k \langle k \rangle^{2s_1 - s - 2}.$$

This is finite if $s_1 - s \leqslant 1$.

Case 3.1b. $\epsilon \leqslant |n - d_1 k| < \epsilon |k|$

Note that $|n| \in [(d_1 - \epsilon)|k|, (d_1 + \epsilon)|k|]$. Thus for ϵ small, $|n - k| \gtrsim |k|$. The supremum is finite when $s_1 - s \leqslant 3 - 3b$:

$$\sup_k \langle k \rangle^{2s_1 - 2s - 2 + 2b} \sum_{\substack{k_1 \\ n \neq 0}} \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s}}{\langle k - k_1 \rangle^{2-2b} \langle k + k_1 - n \rangle^{2-2b}} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 4 + 4b} \sum_{k_1} \frac{\langle k_1 \rangle^{-2s}}{\langle k - k_1 \rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 6 + 6b} < \infty$$

Case 3.1c. $|n - d_1 k| < \epsilon$

The supremum can be estimated by

$$\sup_k \langle k \rangle^{2s_1 + 2 + 2\nu_{d_1} + 2\epsilon_0 - 2s - (2-2b)} \sum_{\substack{k_1 \\ (n \simeq d_1 k)}} \frac{\langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s}}{\langle k - k_1 \rangle^{2-2b} \langle k + k_1 - n \rangle^{2-2b}}.$$

If all four factors in the summation are of order at least $|k|$, this is easy to estimate. Furthermore, if any one factor in the summation is of order $\ll |k|$, then the other three factors are all $\gtrsim |k|$. This implies that the sum over k_1 can always be controlled by a sum of the form

$$\langle k \rangle^{-2(2-2b) - 2s} \sum_m \langle m \rangle^{-2s} \lesssim \langle k \rangle^{-2(2-2b) - 2s},$$

which means that the supremum is finite whenever

$$2s_1 + 2 + 2\nu_{d_1} + 2\epsilon_0 - 4s - 3(2 - 2b) < 0,$$

which holds when $s_1 - s < s + \frac{1}{2} - \nu_{d_1}$.

Case 3.2. $kn < 0$

Cancel $\langle k \rangle^2$ with $(n - d_1 k)^2$ and repeat the arguments from Case 3.1.

3.7.5 Proof of Proposition 3.4.7

Decompose R_3 into two sums based on whether or not $k_1 + k_2$ is zero:

$$
R_3(u,v)_k = ik \sum_{\substack{k_1+k_2+k_3=k \\ k_1 \neq 0}}^{*} \frac{u_{k_1} u_{k_2} v_{k_3}(k_2+k_3)}{\alpha k^3 - k_1^3 - \alpha(k_2+k_3)^3}
$$

$$
= ik \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0 \\ k_1 \neq 0}}^{*} \frac{u_{k_1} u_{k_2} v_{k_3}(k_2+k_3)}{\alpha k^3 - k_1^3 - \alpha(k_2+k_3)^3} + ikv_k \sum_{k_1 \neq 0}^{*} \frac{u_{k_1} u_{-k_1}(k-k_1)}{\alpha k^3 - k_1^3 - \alpha(k-k_1)^3}
$$

$$
= ik \sum_{\substack{k_1+k_2+k_3=k \\ k_1+k_2 \neq 0 \\ k_1 \neq 0}}^{*} \frac{u_{k_1} u_{k_2} v_{k_3}(k_2+k_3)}{\alpha k^3 - k_1^3 - \alpha(k_2+k_3)^3}
$$

$$
+ ikv_k \sum_{k_1>0}^{*} |u_{k_1}|^2 \left[\frac{k-k_1}{\alpha k^3 - k_1^3 - \alpha(k-k_1)^3} + \frac{k+k_1}{\alpha k^3 + k_1^3 - \alpha(k+k_1)^3} \right]
$$

$$
= \mathrm{I} + \mathrm{II}.
$$

To bound II in $X_\alpha^{s_1,b-1}$, note that the bracketed sum is equal to

$$
\frac{2(1-\alpha)k_1^4}{k_1^2(k_1-d_1k)(k_1-d_2k)(k_1+d_1k)(k_1+d_2k)}
$$

and by an application of Cauchy-Schwartz and Young's inequalities, it suffices to show that

$$
\sup_k \langle k \rangle^{2+2s_1-2s} \sum_{k_1>0}^{*} \frac{\langle k_1 \rangle^{4-4s}}{(k_1-d_1k)^2(k_1-d_2k)^2(k_1+d_1k)^2(k_1+d_2k)^2} < \infty.
$$

Case 1. $k > 0$

In this case, $|k_1 - d_2 k|, |k_1 + d_1 k| > k_1, k$ and the supremum can be estimated by

$$
\sup_k \langle k \rangle^{2s_1-2s} \sum_{k_1>0}^{*} \frac{\langle k_1 \rangle^{2-4s}}{(k_1-d_1k)^2(k_1+d_2k)^2}.
$$

Case 1.1. $|k_1 - d_1 k|, |k_1 + d_2 k| \geqslant \epsilon k$

Here we have the estimates $|k_1| \leqslant \left[(|d_1| + \epsilon)/\epsilon \right] |k_1 - d_1 k|$ and $|k_1| \leqslant \left[(|d_2| + \epsilon)/\epsilon \right] |k_1 + d_2 k|$. The supremum can thus be bounded by

$$
\sup_k \langle k \rangle^{2s_1-2s-2} \sum_{k_1>0} \langle k_1 \rangle^{-4s} < \infty
$$

for $s_1 - s \leqslant 1$.

Case 1.2. $|k_1 - d_1 k| \geqslant \epsilon k$, $\epsilon \leqslant |k_1 + d_2 k| < \epsilon k$ (or vice versa)

In this case, note that $k_1 \geqslant (d_1 - \epsilon) k$ and bound the supremum by

$$\sup_k \langle k \rangle^{2s_1 - 2s - 1} \sum_{k_1 \gtrsim k} \langle k_1 \rangle^{-4s+1} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 1 - 4s + 2} < \infty$$

when $s_1 - s \leqslant 2s - \frac{1}{2}$.

Case 1.3. $|k_1 - d_1 k| \geqslant \epsilon k$, $|k_1 + d_2 k| < \epsilon$ (or vice versa)

There is only one term in the sum in this case since $k_1 \simeq -d_2 k$. The supremum can be bounded by

$$\sup_k \langle k \rangle^{2s_1 - 6s + 2 + 2\nu_{d_2} + 2\epsilon_0},$$

which is finite when $s_1 - s < 2s - 1 - \nu_{d_2}$.

This exhausts the cases with $k > 0$ since by choosing ϵ small, we may ensure that $|k_1 - d_1 k| \leqslant \epsilon k$ and $|k_1 + d_2 k| \leqslant \epsilon k$ cannot occur simultaneously.

Case 2. $k < 0$

Note that $|k_1 - d_1 k|, |k_1 + d_2 k| > k_1, |k|$ and proceed as in Case 1. This completes the proof of the estimate for II.

To complete the proof, we must bound I in $X_\alpha^{s_1, b-1}$. As before, it suffices to show that the following supremum is finite:

$$\sup_k \langle k \rangle^{2 + 2s_1} \sum_{\substack{k_1 \neq 0, k_2 \\ k_1 + k_2 \neq 0}} \frac{\langle k_1 \rangle^{-2s} \langle k_2 \rangle^{-2s} \langle k - k_1 - k_2 \rangle^{-2s} |k - k_1|^2}{(\alpha k^3 - k_1^3 - \alpha(k - k_1)^3)^2 \langle \alpha k^3 - k_1^3 - k_2^3 - \alpha(k - k_1 - k_2)^3 \rangle^{2 - 2b}},$$

or equivalently

$$\sup_k \langle k \rangle^{2 + 2s_1} \sum_{\substack{k_1 \neq 0 \\ n \neq 0}} \frac{\langle k_1 \rangle^{-2s-2} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s} |k - k_1|^2}{(k_1 - d_1 k)^2 (k_1 - d_2 k)^2 \langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}}.$$

Case 1. $|k_1 - d_1 k| < \epsilon$ or $|k_1 - d_2 k| < \epsilon$

Assume that $|k_1 - d_1 k| < \epsilon$. The other case is parallel. Note that

$$|k_1 - d_2 k| \geqslant (d_1 - d_2)|k| - |k_1 - d_1 k|$$

so that $|k_1 - d_2 k| \gtrsim |k|$. Also we have $|k_1 - k| \leqslant |k_1 - d_1 k| + |d_1 - 1||k| \lesssim |k|$. The supremum may thus be bounded by

$$\sup_k \langle k \rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-2s} \sum_{\substack{n \neq 0 \\ (k_1 \simeq d_1 k)}} \frac{\langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s}}{\langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{2-2b}}.$$

Case 1.1. $|n - k_1| \geqslant \epsilon |k|$, $|n - k| \geqslant \epsilon |k|$

In this case, the supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-6s} \sum_{\substack{n \neq 0 \\ (k_1 \simeq d_1 k)}} \langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{-(2-2b)}.$$

This is finite when $2s_1 + 2 + 2\nu_{d_1} - 6s < 0$, i.e. when $s_1 - s < 2s - 1 - \nu_{d_1}$.

Case 1.2. $|n - k| < \epsilon |k|$

Note that in this region $|n - k_1| \gtrsim |k|$ since

$$|n - k_1| \geqslant (1 - d_1)|k| - |k - n| - |k_1 - d_1 k| > (1 - d_1 - \epsilon)|k| - \epsilon.$$

Write

$$n - k = \mu k \text{ for some } |\mu| < \epsilon, \qquad k_1 = d_1 k + \delta \text{ for some } |\delta| < \epsilon.$$

Then

$$\left| \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \right| = \left| \alpha k^3 - (d_1 k + \delta)^3 - [(1 - d_1 + \mu)k - \delta]^3 + \alpha \mu^3 k^3 \right|$$

$$= \left| k^3[\alpha - 1 + 3d_1 - 3d_1^2 + \mathcal{O}(\epsilon)] + \mathcal{O}(\epsilon)k^2 + \mathcal{O}(\epsilon^2)k + \mathcal{O}(\epsilon^3) \right| \gtrsim |k^3|.$$

The supremum is bounded by

$$\sup_k \langle k \rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-4s} \sum_{\substack{n \neq 0 \\ (k_1 \simeq d_1 k)}} \frac{\langle n - k \rangle^{-2s}}{\langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k \rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-4s} \sum_{n \neq 0} \frac{\langle n - k \rangle^{-2s}}{\langle k^3 \rangle^{2-2b}} \lesssim \sup_k \langle k \rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-4s-6+6b} < \infty,$$

for $s_1 - s < s + 2 - 3b - \nu_{d_1}$.

Case 1.3. $|n - k_1| < \epsilon |k|$

In this case, $|n - k| \gtrsim |k|$. Write

$$n - k = \mu k \text{ for some } |\mu| \in [1/|k|, \epsilon] \qquad k_1 - d_1 k = \delta \text{ for some } |\delta| < \epsilon.$$

Then

$$|\alpha k^3 - k_1^3 - (n-k_1)^3 - \alpha(k-n)^3| = |\alpha k^3 - (d_1 k + \delta)^3 - [(1-d_1+\mu)k-\delta]^3 + \alpha\mu^3 k^3|$$

$$= \left|[\alpha - 1 + 3d_1 - 3d_1^2 + \mathcal{O}(\epsilon)]k^3 + \mathcal{O}(\epsilon)k^2 + \mathcal{O}(\epsilon^2)k + \mathcal{O}(\epsilon^3)\right| \gtrsim |k^3|.$$

Thus the supremum is finite when $s_1 - s < s + 2 - 3b - \nu_{d_1}$:

$$\sup_k \langle k\rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-4s} \sum_{\substack{n\neq 0 \\ (k_1\simeq d_1 k)}} \frac{\langle n-k_1\rangle^{-2s}}{\langle k^3\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1+2+2\nu_{d_1}+2\epsilon_0-4s-6+6b} < \infty.$$

Case 2. $\epsilon \leqslant |k_1 - d_1 k| < \epsilon|k|$ or $\epsilon \leqslant |k_1 - d_2 k| < \epsilon|k|$

Assume that $\epsilon \leqslant |k_1 - d_1 k| < \epsilon|k|$; the other case is similar. Note that we have $|k_1 - k| \lesssim |k|$ and $|k_1 - d_2 k| \gtrsim |k|$ so that the supremum is bounded by

$$\sup_k \langle k\rangle^{2s_1} \sum_{\substack{|k_1|\gtrsim|k| \\ n\neq 0}} \frac{\langle k_1\rangle^{-2s}\langle n-k_1\rangle^{-2s}\langle n-k\rangle^{-2s}}{\langle \alpha k^3 - k_1^3 - (n-k_1)^3 - \alpha(k-n)^3\rangle^{2-2b}}.$$

Case 2.1. $|n-k_1| \geqslant \epsilon|k|$, $|n-k| \geqslant \epsilon|k|$

Here the supremum is bounded for $s_1 - s \leqslant 2s - \frac{1}{2}$:

$$\sup_k \langle k\rangle^{2s_1-4s} \sum_{\substack{|k_1|\gtrsim|k| \\ n\neq 0}} \frac{\langle k_1\rangle^{-2s}}{\langle \alpha k^3 - k_1^3 - (n-k_1)^3 - \alpha(k-n)^3\rangle^{2-2b}}$$

$$\lesssim \sup_k \langle k\rangle^{2s_1-4s} \sum_{|k_1|\gtrsim|k|} \langle k_1\rangle^{-2s} \lesssim \sup_k \langle k\rangle^{2s_1-4s-2s+1} < \infty.$$

Case 2.2. $|n-k| < \epsilon|k|$

In this case, notice that $|n-k_1| \geqslant |k_1-k| - |n-k| \geqslant (1-d_1-2\epsilon)|k| \gtrsim |k|$ and write

$$n - k = \mu k \text{ for some } |\mu| < \epsilon \qquad k_1 - d_1 k = \mu' k \text{ for some } |\mu'| < \epsilon.$$

Then

$$|\alpha k^3 - k_1^3 - (n-k_1)^3 - \alpha(k-n)^3| = |k^3[\alpha - (d_1+\mu')^3 - (1-d_1+\mu-\mu')^3 + \alpha\mu^3]|$$

$$= |k^3[(1-\alpha) + 3(d_1^2 - d_1) + \mathcal{O}(\epsilon)]| \gtrsim |k|^3.$$

Thus the supremum is bounded by

$$\sup_k \langle k\rangle^{2s_1-2s} \sum_{\substack{|k_1|\gtrsim|k| \\ |n|\lesssim|k|}} \frac{\langle k_1\rangle^{-2s}}{\langle k^3\rangle^{2-2b}} \lesssim \sup_k \langle k\rangle^{2s_1-2s+1-3(2-2b)-2s+1} < \infty$$

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if $s_1 - s \leqslant s + 2 - 3b$.

Case 2.3. $|n - k_1| < \epsilon|k|$

Here $|n - k| \gtrsim |k|$, so the supremum can be estimated as follows for $s_1 - s \leqslant s - \frac{1}{2}$:

$$\sup_k \langle k \rangle^{2s_1 - 2s} \sum_{\substack{|k_1| \gtrsim |k| \\ n \neq 0}} \frac{\langle k_1 \rangle^{-2s}}{\langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 2s + 1} < \infty.$$

Case 3. $|k_1 - d_1 k| \geqslant \epsilon|k|$, $|k_1 - d_2 k| \geqslant \epsilon|k|$

In this case, we need to bound

$$\sup_k \langle k \rangle^{2s_1 - 2} \sum_{\substack{k_1 \neq 0 \\ n \neq 0}} \frac{\langle k_1 \rangle^{-2s - 2} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s} |k - k_1|^2}{\langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}}.$$

Case 3.1. $|k_1| \geqslant \epsilon|k|$

Here we have $|k - k_1| \lesssim |k_1|$ so that for $s_1 - s \leqslant 1$, the supremum can be estimated by

$$\sup_k \langle k \rangle^{2s_1 - 2} \sum_{\substack{k_1 \neq 0 \\ n \neq 0}} \langle k_1 \rangle^{-2s} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s} \lesssim \sup_k \langle k \rangle^{2s_1 - 2s - 2} < \infty.$$

Case 3.2. $|k_1| \leqslant \epsilon|k|$

Here the supremum may be bounded by

$$\sup_k \langle k \rangle^{2s_1} \sum_{\substack{k_1 \neq 0 \\ n \neq 0}} \frac{\langle k_1 \rangle^{-2s - 2} \langle n - k_1 \rangle^{-2s} \langle n - k \rangle^{-2s}}{\langle \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \rangle^{2 - 2b}}.$$

Case 3.2a. $|n - k_1| \leqslant \epsilon|k|$

Note that we have $|n - k| \geqslant |k_1 - k| - |k_1 - n| \geqslant (1 - 2\epsilon)|k|$ and write

$$k_1 = \mu_1 k \text{ for some } |\mu_1| \in [1/|k|, \epsilon]$$

$$n - k_1 = \mu_2 k \text{ for some } |\mu_2| \in [1/|k|, \epsilon]$$

$$\mu = \mu_1 + \mu_2 \text{ where } |\mu| \in [1/|k|, 2\epsilon].$$

The lower bounds on μ_1 and μ_2 are positive because of the mean zero assumption on u. The lower bound on μ comes from the fact that $n \neq 0$. With this notation,

$$\left| \alpha k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \right| = |\mu k^3| \left| (\alpha - 1)\mu^2 + 3\alpha(1 - \mu) + 3\mu_1 \mu_2 \right| \gtrsim |k^2|.$$

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Then the supremum is bounded by

$$\sup_{k} \langle k \rangle^{2s_1-2s} \sum_{\substack{k_1 \neq 0 \\ |n| \lesssim |k|}} \frac{\langle k_1 \rangle^{-2s-2}}{\langle k^2 \rangle^{2-2b}} \lesssim \sup_{k} \langle k \rangle^{2s_1-2s-4+4b} \sum_{\substack{k_1 \neq 0 \\ |n| \lesssim |k|}} \langle k_1 \rangle^{-2s-2}$$

$$\lesssim \sup_{k} \langle k \rangle^{2s_1-2s-4+4b+1} \sum_{k_1 \neq 0} \langle k_1 \rangle^{-2s-2} \lesssim \sup_{k} \langle k \rangle^{2s_1-2s-4+4b+1},$$

which is finite for $s_1 - s \leqslant \frac{3}{2} - 2b$.

Case 3.2b. $|n - k| \leqslant \epsilon |k|$

In this case, note that $|n - k_1| \geqslant |k - k_1| - |k - n| \geqslant (1 - 2\epsilon)|k| \gtrsim |k|$ and write

$$k_1 = \mu_1 k \text{ for some } |\mu_1| \in [1/|k|, \epsilon], \qquad n - k = \mu_2 k \text{ for some } |\mu_2| \in [0, \epsilon].$$

Then

$$\alpha \left| k^3 - k_1^3 - (n - k_1)^3 - \alpha(k - n)^3 \right| = |k^3| \left| \alpha - \mu_1^3 - (1 - \mu_1 + \mu_2)^3 + \alpha \mu_2^3 \right|$$

$$= |k^3| \left| 1 - \alpha + \mathcal{O}(\epsilon) \right| \gtrsim |k^3|.$$

Thus for $s_1 - s \leqslant \frac{5}{2} - 3b$, the supremum is bounded:

$$\sup_{k} \langle k \rangle^{2s_1-2s} \sum_{\substack{k_1 \neq 0 \\ |n| \lesssim |k|}} \frac{\langle k \rangle^{-2s-2}}{\langle k^3 \rangle^{2-2b}} \lesssim \sup_{k} \langle k \rangle^{2s_1-2s-6+6b+1} \sum_{k_1 \neq 0} \langle k_1 \rangle^{-2s-2}$$

$$\lesssim \sup_{k} \langle k \rangle^{2s_1-2s-6+6b+1} < \infty.$$

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The Klein-Gordon-Schrödinger & Zakharov Systems on $\mathbb{R}^d$

4.1 Introduction

In this work, we derive smoothing estimates for the Klein-Gordon Schrödinger system (KGS) with Yukawa coupling:

$$\begin{cases} iu_t + \Delta u = -uv, \quad x \in \mathbb{R}^d, \ t \in \mathbb{R} \\ v_{tt} + (-\Delta + 1)v = |u|^2 \\ \Big(u(\cdot, 0), v(\cdot, 0), v_t(\cdot, 0)\Big) = (u_0, v_0, v_1) \in H^s \times H^r \times H^{r-1}. \end{cases} \tag{4.1}$$

We also consider the closely-related Zakharov system:

$$\begin{cases} iu_t + \Delta u = un, \quad x \in \mathbb{R}^d, \ t \in \mathbb{R} \\ n_{tt} - \Delta n = \Delta |u|^2 \\ \Big(u(\cdot, 0), n(\cdot, 0), n_t(\cdot, 0)\Big) = (u_0, n_0, n_1) \in H^s \times H^r \times H^{r-1}. \end{cases} \tag{4.2}$$

The results here have appeared in [31]. The KGS system (4.1) is a model from classical particle physics, in which u represents a complex nucleon field and v a real meson field [46]. The Zakharov system (4.2) was introduced in [101] to model Langmuir turbulence in ionized plasma. In it, the function u represents the envelope of a oscillating electric field while n represents the deviation of ion density from its average value.

Solutions of the Klein-Gordon-Schrödinger system conserve the mass and the Hamiltonian energy:

$$M(u) = \|u\|_{L^2}$$

$$E(u, v, v_t) = \|\nabla u\|_{L^2}^2 + \frac{1}{2}\Big(\|v\|_{L^2}^2 + \|v_t\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\Big) - \int |u|^2 v \, dx.$$

Note that the energy space for the KGS system is $H^1 \times H^1 \times L^2$. Similarly, the Zakharov system has the following conservation laws:

$$M(u) = \|u\|_{L^2}$$

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$$\widetilde{E}(u, n, n_t) = \|\nabla u\|_{L^2}^2 + \frac{1}{2}\Big(\|n\|_{L^2}^2 + \|(-\Delta)^{-1/2}n_t\|_{L^2}^2\Big) + \int |u|^2 n \, \mathrm{d}x.$$

This law identifies the energy space as $H^1 \times L^2 \times \dot{H}^{-1}$.

The wellposedness theory for the Zakharov system on Euclidean spaces has been extensively studied. Sulem and Sulem derived existence results for smooth solutions in dimensions $d \leqslant 3$ [93]. The regularity assumptions and dimension restrictions were weakened in [2, 80, 66, 90]. In [51], Ginibre, Tsutsumi, and Velo applied Bourgain's restricted norm method [19] to obtain local existence results in all dimensions, covering the full subcritical regularity range (excluding the endpoints) for $d \geqslant 4$. In dimension $d = 1$, they obtained local existence at the critical regularity $L^2 \times H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$. In [58], local ill-posedness results were obtained for some regularities outside the well-posedness regime established in [51]. In dimensions two and three, the local well-posedness was obtained in the critical space $L^2 \times H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$ in [11] and [10] respectively. These results are sharp in the sense that the data-to-solution map fails to be analytic at lower regularity levels.

In one dimension, the Hamiltonian conservation law upgrades local existence to global for initial data in $H^1 \times L^2 \times (-\Delta)^{\frac{1}{2}}L^2$. This result was improved in [83, 82] using Bourgain's high-low decomposition [22] method and the I-method [28] respectively. It was lowered further to global existence in $L^2 \times H^{-1/2} \times H^{-3/2}$ in [27] using an iteration method relying on the L^2 conservation of u. In two and three dimensions, global existence in the energy space follows from the Hamiltonian conservation as long as $\|u_0\|_{L^2}$ is sufficiently small. In two dimensions, global well-posedness for some regularities below the energy space was obtained in [67] using the I-method.

Unlike the Zakharov, the nonlinearity in the wave part of the Klein-Gordon-Schrödinger system contains no derivative. Thus we have well-posedness at somewhat lower regularity levels for this system. For the two-dimensional KGS, local well-posedness holds in $H^{-\frac{1}{4}+} \times H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$; see [85]. The same result, up to endpoints, hold in three dimensions [84]. Local existence in higher dimensions follows from the estimates derived for the Zakharov in [51].

For the Klein-Gordon-Schrödinger system in dimensions $d \leqslant 3$, global existence in $H^1 \times H^1 \times L^2$ follows from the Hamiltonian conservation law. In three dimensions, global existence somewhat below the energy was proved using Bourgain's high-low decomposition method in [81]. This was improved in [96], where the I-method was used to obtain global existence below the energy for $d \leqslant 3$. Global existence for the three-dimensional KGS in $L^2 \times L^2 \times H^{-1}$ was obtained in [27],

again relying on the L^2 conservation law for u. This was lowered to $L^2 \times H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$ for $d = 2$ and to $L^2 \times H^{-\frac{1}{2}+} \times H^{-\frac{3}{2}+}$ for $d = 3$ in [85]. We also note that global existence for the closely related wave-Schrödinger system on $H^s \times H^r \times H^{r-1}$ for some $s, r < 0$ was shown in [4].

This chapter is concerned with the dynamics of solutions to (4.1) and (4.2). The main result is that the difference between the linear evolution and the nonlinear evolution resides in a higher-regularity space. This follows from a refinement of the bilinear $X^{s,b}$ local theory estimates, similar to that contained in [26] for two dimensional nonlinear Schrödinger equations with quadratic non-linearities. The difficulty in this case is that the addition of a Klein-Gordon or wave equation to the Schrödinger to obtain (4.1) or (4.2) respectively complicates the resonance relations in the system, making the estimates more challenging. As in [26], the proof depends on delicate decompositions of the frequency space to control the nonlinear interactions, with especial care being required near the resonant sets of the interaction.

In the remainder of the chapter we present some consequences of the smoothing estimate. One of these is a simplified proof of the existence of a global attractor for the forced and damped Klein-Gordon-Schrödinger equation in dimensions $d = 2, 3$. This result is known [74], but the existing proof relies on truncation arguments to obtain the necessary compactness. The truncation step can be eliminated by the employment of the smoothing effect of the nonlinear flow, significantly simplifying the argument. Secondly, we show global existence below the energy space for the four-dimensional Klein-Gordon-Schrödinger system for $\|u_0\|_{L^2}$ sufficiently small using a variant of Bourgain's high-low argument [22]. Similar smoothing estimates have been used with high-low decomposition method to prove global existence for other equations – see e.g. [33] for results on the periodic fractional Schrödinger equation. We remark that method of [27] to obtain global existence for the Klein-Gordon-Schrödinger does not apply; in four dimensions, there is not sufficient slack in the wave equation estimates to iterate that scheme. The refinement used in [85], which uses $X^{s,b}$ estimates instead of Strichartz space controls, also cannot be directly applied. Smoothing estimates provide a straightforward proof of the global existence.

The chapter is organized as follows. In Section 4.2, we introduce the function spaces required for the estimates, and in Section 4.3, we state our results. Sections 4.4, 4.5, and 4.6 contain the proofs of the main smoothing estimate, the existence of the Klein-Gordon-Schrödinger attractor,

and global well-posedness in $\mathbb{R}^4$ respectively. Finally, in Section 4.7, we prove the main bilinear estimate.

4.2   Notation & Function Spaces

To prove the desired estimates, we work with transformed versions of the systems (4.1) and (4.2). Define $A = (1 - \Delta)^{1/2}$. For the Klein-Gordon-Schrödinger system, let $v^\pm = v \pm iA^{-1}v_t$. Under this transformation, (4.1) becomes

$$
\begin{cases}
iu_t + \Delta u = -\frac{1}{2}u(v^+ + v^-), \\
iv_t^\pm \mp Av^\pm = \mp A^{-1}|u|^2 \\
\left(u(\cdot, 0), v^\pm(\cdot, 0)\right) = (u_0, v_0^\pm) \in H^s \times H^r.
\end{cases}
\tag{4.3}
$$

For the Zakharov system (4.2), we similarly define $n^\pm = n \pm iA^{-1}n_t$. After this transformation, (4.2) becomes

$$
\begin{cases}
iu_t + \Delta u = \frac{1}{2}u(n^+ + n^-), \\
in_t^\pm \mp An^\pm = \mp A^{-1}\Delta|u|^2 \mp A^{-1}\operatorname{Re}n^\pm \\
\left(u(\cdot, 0), n^\pm(\cdot, 0), n_t(\cdot, 0)\right) = (u_0, n_0^\pm) \in H^s \times H^r.
\end{cases}
\tag{4.4}
$$

Notice that we can recover the original function v and n by taking the real part of $v^\pm$ and $n^\pm$ respectively. The corresponding Bourgain spaces are defined by the norms

$$
\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s\langle\tau + |\xi|^2\rangle^b\hat{u}(\xi, \tau)\|_{L^2_{\xi,\tau}}
$$

$$
\|v\|_{X^{s,b}_\pm} = \|\langle\xi\rangle^s\langle\tau \pm |\xi|\rangle^b\hat{u}(\xi, \tau)\|_{L^2_{\xi,\tau}}.
$$

The multiplier for the Klein-Gordon evolution is technically $\langle\tau \pm \langle\xi\rangle\rangle$ rather than $\langle\tau \pm |\xi|\rangle$, but $\langle\tau \pm \langle\xi\rangle\rangle \approx \langle\tau \pm |\xi|\rangle$ and using the latter multiplier results in a cleaner exposition. We also define the time-restricted versions of these norms:

$$
\|u\|_{X^{s,b}_\delta} = \inf_{u=\tilde{u}, |t|\leqslant\delta} \|\tilde{u}\|_{X^{s,b}} \qquad\qquad \|v\|_{X^{s,b}_{\pm,\delta}} = \inf_{v=\tilde{v}, |t|\leqslant\delta} \|\tilde{v}\|_{X^{s,b}_\pm}.
$$

4.3 STATEMENT OF RESULTS

In the first part of this section, we give the main theorems which demonstrate the smoothing effect of the nonlinear flow. We then give two results which show some of the implications of smoothing for the global dynamics of the system. First, we state the theorem for the Zakharov system.

Theorem 4.3.1. *Consider the Zakharov evolution* (4.4) *on* $\mathbb{R}^d$. *If* $d = 2, 3$, *assume that* $r \geqslant -\frac{1}{2}$ *with* $2s - r \geqslant \frac{1}{2}$ *and* $r < s < r + 1$. *Then*

$$u(t) - e^{it\Delta}u_0 \in C_t H_x^{s+\alpha}$$
$$n^{\pm}(t) - e^{\mp itJ}n_0^{\pm} \in C_t H_x^{r+\beta}$$

on the interval of existence as long as $\alpha < \min\{\frac{1}{2}, r-s+1, r+2-\frac{d}{2}\}$ *and* $\beta < \min\{2s-r-\frac{1}{2}, s-r\}$. *If* $d \geqslant 4$, *assume* $r > \frac{d-4}{4}$ *and* $2s - r > \frac{d-2}{2}$ *with* $r \leqslant s \leqslant r + 1$. *Then the same statement holds if* $\beta < \min\{2s-r-\frac{d-2}{2}, s-r\}$.

The restrictions on r and s are necessary to ensure well-posedness of the equation. The values α and β represent the smoothing effect. For instance, in dimensions $d = 2, 3$, for initial data in $H^{\frac{1}{2}} \times L^2$, the nonlinear part of the evolution lies in $H^{1-} \times H^{\frac{1}{2}-}$. A similar result holds for the Klein-Gordon-Schrödinger system.

Theorem 4.3.2. *Consider the Klein-Gordon-Schrödinger evolution* (4.3) *on* $\mathbb{R}^d$. *If* $d = 2, 3$, *assume* $s > -\frac{1}{4}$ *and* $r > -\frac{1}{2}$ *with* $2s - r \geqslant -\frac{3}{2}$ *and* $r - 2 < s < r + 1$. *Then we have*

$$u(t) - e^{it\Delta}u_0 \in C_t H_x^{s+\alpha}$$
$$v^{\pm}(t) - e^{\mp itA}v_0^{\pm} \in C_t H_x^{r+\beta}$$

on the interval of existence as long as $\alpha < \min\{\frac{1}{2}, r-s+1, r+2-\frac{d}{2}\}$ *and* $\beta < \min\{2s-r+\frac{3}{2}, s-r+2\}$. *If* $d \geqslant 4$, *assume* $r > \frac{d-4}{4}$ *and* $2s - r > \frac{d-6}{2}$ *with* $r - 2 \leqslant s \leqslant r + 1$. *Then the same statement holds if* $\beta < \min\{2s-r-\frac{d-6}{2}, s-r+2\}$.

For the Klein-Gordon-Schrödinger system, the smoothing effect on the wave part is much stronger than that on the Zakharov because of the lack of derivatives in the nonlinearity. For instance, in dimensions $d = 2, 3$ and initial data in $L^2 \times L^2$, the nonlinear part is in $H^{\frac{1}{2}-} \times H^{\frac{3}{2}-}$.

The proof of these results is in Section 4.4. It depends on the following new bilinear estimate for the Schrödinger nonlinearity, together with the known local theory estimates for the wave equation nonlinearity.

Proposition 4.3.3. *Assume $d \geqslant 2$ and $b = \frac{1}{2}+$ with $s, r > -\frac{1}{2}$. Then the estimate*

$$\|uv\|_{X^{s+\alpha,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{r,b}_\pm} \tag{4.5}$$

holds for $\alpha < \min\{\frac{1}{2}, r - s + 1, r + 2 - \frac{d}{2}\}$. The same result holds with the restricted versions of the norms.

We also state the estimates for the wave and Klein-Gordon nonlinear terms:

Proposition 4.3.4 ([10, 11, 51])**.** *Let $b = \frac{1}{2}+$. If $d = 2, 3$, assume $s > -\frac{1}{4}$ with $2s - \sigma > \frac{1}{2}$ and $s > \sigma$. If $d \geqslant 4$, assume $2s - \sigma > \frac{d-2}{2}$ and $\sigma \leqslant s$, $s \geqslant 0$. Then*

$$\|A(|u|^2)\|_{X^{\sigma,b-1}_\pm} \lesssim \|u\|^2_{X^{s,b}}.$$

The same result holds for the restricted versions of the norms.

Proposition 4.3.5 ([51, 85, 84])**.** *Let $b = \frac{1}{2}+$. If $d = 2, 3$, assume $s > -\frac{1}{4}$ with $2s - \sigma > -\frac{3}{2}$ and $\sigma - 2 < s$. If $d \geqslant 4$, assume that $2s - \sigma > \frac{d-6}{2}$ and $\sigma - 2 \leqslant s$, $s \geqslant 0$. Then*

$$\|A^{-1}(|u|^2)\|_{X^{\sigma,b-1}_\pm} \lesssim \|u\|^2_{X^{s,b}}. \tag{4.6}$$

The same result holds for the restricted versions of the norms.

We remark that a half derivative gain is the best that can be hoped for in the Schrödinger evolution from the use of such bilinear estimates. To see that the bilinear estimate (4.5) fails for $\alpha > \frac{1}{2}$, let $\hat{u} = \chi_{B_1}$ and $\hat{v} = \chi_{B_2}$, where

$$B_1 = \left\{ (\xi_1, \xi_2, \dots, \xi_d, \tau) \in \mathbb{R}^{d+1} : |\xi_1 - N| < N^{-1}, \ |\xi_i| < 1 \text{ for } i \geqslant 2, \ |\tau + N^2| < 1 \right\},$$
$$B_2 = \left\{ (\xi_1, \xi_2, \dots, \xi_d, \tau) \in \mathbb{R}^{d+1} : \quad |\xi_1| < N^{-1}, \ |\xi_i| < 1 \text{ for } i \geqslant 2, \qquad |\tau| < 1 \right\}$$

for some $N \gg 1$. Then $\|u\|_{X^{s,b}} \approx N^{s-\frac{1}{2}}$ and $\|v\|_{X^{s,b}_\pm} \approx N^{-\frac{1}{2}}$, while $\widehat{uv}$ is roughly $N^{-1}\chi_{B_1}$, so that $\|uv\|_{X^{s+\alpha,b}} \approx N^{s+\alpha-\frac{3}{2}}$. This can only be bounded by $\|u\|_{X^{s,b}} \|v\|_{X^{s,b}_\pm} \approx N^{s-1}$ when $\alpha \leqslant \frac{1}{2}$.

As an application of the smoothing estimate, we study the existence of global attractors for the dissipative Klein-Gordon-Schrödinger evolution. Proofs generally use the dissipative property to obtain decay of solutions, followed by a weak-convergence argument to show compactness of the absorbing set. This second step is particularly challenging on noncompact spaces such as $\mathbb{R}^d$, where proving compactness can be difficult. For the dissipative Klein-Gordon-Schrödinger evolution on $\mathbb{R}^d$, $d \leqslant 3$, the existence of a global attractor was proved in [74]. In the following, we simplify the proof using our smoothing estimate.

With the addition of damping and forcing terms, the Klein-Gordon-Schrödinger (4.1) system becomes

$$
\begin{cases}
iu_t + \Delta u + i\gamma u = -uv + f, \quad x \in \mathbb{R}^d \\
v_{tt} + (-\Delta + 1)v + \delta v_t = |u|^2 + g.
\end{cases}
\tag{4.7}
$$

We will be concerned with $d = 2, 3$ and initial data $\big(u(x,0), v(x,0), v_t(x,0)\big)$ in the energy space $H^1 \times H^1 \times L^2$ with damping coefficients $\gamma, \delta > 0$ and forcing terms $f, g \in H^1$. In the following, $U(t)$ will denote the evolution operator corresponding to (4.7). Note that the notion of a global attractor is only reasonable when the system is globally well-posed. For the forced and weakly damped system, global well-posedness holds in the energy space $H^1 \times H^1 \times L^2$ by a minor modification of the nondissipative local theory arguments together with decay of the Hamiltonian energy (see [40] for details).

Recall Definitions 2.5.1-2.5.2, which defined the global attractor and absorbing set. Using energy estimates, it can be shown that (4.7) has an absorbing set in the energy space $H^1 \times H^1 \times L^2$. We will show, using Theorem 2.5.3, that the ω-limit set of this absorbing set is a global attractor. This theorem uses asymptotic compactness of the solution flow, which we will demonstrate using a smoothing estimate for the dissipative system. We obtain the following result.

Theorem 4.3.6. *The Klein-Gordon-Schrödinger evolution in dimensions $d = 2, 3$ has a global attractor in $H^1 \times H^1 \times L^2$ which is compact in $H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}$.*

The existence of a global attractor is known [74]. However, the compactness statement appears to be new. We remark that the existence of a global attractor for the dissipative Zakharov system (without a mass term) on Euclidean spaces appears to be an interesting open problem. The

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methods we use cannot be applied to the Zakharov because of difficulties in controlling the low-frequency components of the wave equation. We also remark that our proof method also applies to (4.7) with forcing $f, g \in H^{-\frac{1}{2}+}$. In this case, we obtain a global attractor which is compact in $H^{\frac{3}{2}-} \times H^{\frac{3}{2}+} \times H^{\frac{1}{2}+}$.

As a second application, we use a variant of the high-low decomposition method together with the smoothing estimate to obtain global existence for the Klein-Gordon-Schrödinger equation in four dimensions.

Theorem 4.3.7. *The Klein-Gordon-Schrödinger evolution* (4.3) *is globally well-posed on* $H^s \times H^r$ *for* $s, r > 9/10$ *as long as* $\|u_0\|_{L^2} < \sqrt{2} C_1 C_2^2$, *where* C_1 *and* C_2 *are the optimal constants in the four-dimensional* L^4 *and* $L^{8/3}$ *Gagliardo-Nirenberg-Sobolev inequalities respectively.*

The constraint on the norm of the u_0 is necessary to ensure that the energy functional is positive definite. The optimal constants in the Gagliardo-Nirenburg-Sobolev inequalities have been established by Weinstein [98]. The proof of this result is in Section 4.6.

4.4 Proof of Theorems 4.3.1 & 4.3.2: Smoothing Results

In this section, we give the proof of the smoothing theorem for the Klein-Gordon-Schrödinger flow. The proof for the Zakharov equation has the same structure; it is obtained by adding two derivatives to the wave nonlinearity which appears in the Klein-Gordon-Schrödinger system. Since the calculations for the Zakharov equation are similar, they are omitted.

Writing the solution to the transformed Klein-Gordon-Schrödinger equation (4.3) in its Duhamel form yields

$$u(t) - e^{it\Delta} u_0 = -\frac{1}{2} \int_0^t e^{i(t-t')\Delta} \Big(u(v^+ + v^-) \Big) \, \mathrm{d}t'$$
$$v^\pm(t) - e^{\mp itA} v_0^\pm = \mp \int_0^t e^{\mp i(t-t')A} \Big(A^{-1} |u|^2 \Big) \, \mathrm{d}t'.$$

Let δ be the local existence time of the solution. Then on $[0, \delta]$ we have

$$\|u\|_{X_\delta^{s,b}} + \|v^\pm\|_{X_{\pm,\delta}^{r,b}} \lesssim \|u_0\|_{H^s} + \|v_0^\pm\|_{H^r}. \tag{4.8}$$

To control the Duhamel integral terms, we use the embeddings $X^{s,b} \hookrightarrow C^0 H^s$ and $X_{\pm}^{r,b} \hookrightarrow C^0 H^r$, which hold for $b > \frac{1}{2}$, along with Lemma 2.2.3 for the Schrödinger and Klein-Gordon $X^{s,b}$ space norms, which can be found explicitly in [51]. Using these estimates yields

$$\|u(t) - e^{it\Delta}u_0\|_{L_{[0,\delta]}^\infty H^{s+\alpha}} \lesssim \|uv^+\|_{X_\delta^{s+\alpha,b-1}} + \|uv^-\|_{X_\delta^{s+\alpha,b-1}}$$

$$\|v^\pm(t) - e^{\mp itA}v_0^\pm\|_{L_{[0,\delta]}^\infty H^{r+\beta}} \lesssim \||A^{-1}|u|^2\|_{X_{\pm,\delta}^{r+\beta,b-1}}.$$

Using the estimates from Propositions 4.3.3 and 4.3.5, we have

$$\|u(t) - e^{it\Delta}u_0\|_{L_{[0,\delta]}^\infty H^{s+\alpha}} \lesssim \|u\|_{X_\delta^{s,b}}\left(\|v^+\|_{X_{+,\delta}^{r,b}} + \|v^-\|_{X_{-,\delta}^{r,b}}\right)$$

$$\|v^\pm(t) - e^{\mp itA}v_0^\pm\|_{L_{[0,\delta]}^\infty H^{r+\beta}} \lesssim \|u\|_{X_\delta^s}^2.$$

Using the local theory bound (4.8), we conclude

$$\|u(t) - e^{it\Delta}u_0\|_{L_{[0,\delta]}^\infty H^{s+\alpha}} \lesssim \left(\|u_0\|_{H^s} + \|v_0^\pm\|_{H^r}\right)^2$$

$$\|v^\pm(t) - e^{\mp itA}v_0^\pm\|_{L_{[0,\delta]}^\infty H^{r+\beta}} \lesssim \left(\|u_0\|_{H^s} + \|v_0^\pm\|_{H^r}\right)^2.$$

Repeating this process shows that the nonlinear part of the solution remains in $H^{s+\alpha} \times H^{r+\beta}$ for the full interval of existence.

To prove continuity, write

$$\left(u(t) - e^{-t\Delta}u_0\right) - \left(u(t+\epsilon) - e^{i(t+\epsilon)\Delta}u_0\right)$$

$$= \frac{1}{2}\int_0^{t+\epsilon} e^{i(t+\epsilon-t')\Delta}\left(u(v^+ + v^-)\right)dt' - \frac{1}{2}\int_0^t e^{i(t-t')\Delta}\left(u(v^+ + v^-)\right)dt'$$

$$= \frac{1}{2}\left(e^{i\epsilon\Delta} - \mathrm{Id}\right)\int_0^t e^{i(t-t')\Delta}\left(u(v^+ + v^-)\right)dt' + \frac{1}{2}\int_t^{t+\epsilon} e^{i(t+\epsilon-t')\Delta}\left(u(v^+ + v^-)\right)dt'$$

The continuity follows by applying the estimates stated previously along with the continuity of $(u, v^\pm)$ in $H^s \times H^r$; see [36]. Continuity of the nonlinear part of v is proved in the same way.

4.5 PROOF OF THE EXISTENCE OF A GLOBAL ATTRACTOR

In this section, we use smoothing estimates to simplify the proof of the existence of a global attractor for the dissipative Klein-Gordon-Schrödinger flow in two and three dimensions. To prove this result,

we need to establish boundedness and asymptotic compactness of the flow. The boundedness follows from the energy equation; compactness is the challenging part. To prove this, we use boundedness to obtain a weakly convergent sequence of solutions. The energy equation is used to upgrade the weak convergence to strong convergence, yielding the desired compactness. The energy functional contains cubic terms which can easily be bounded using our smoothing result and the embedding $H^{\frac{3}{2}+} \hookrightarrow L^\infty$. In the existing proof, an extensive argument, involving uniform estimates of the solution restricted to compact sets, is required to control these terms.

First we establish a weak continuity result for the evolution operator which will be needed to work with the energy equations. A slightly weaker form of the following lemma is in [74, Lemma 3.1].

Lemma 4.5.1. *Let $d = 2, 3$. Let $\S(t)$ denote the semigroup operator for (4.7), and let $L(t)$ denote the linear part of the semigroup operator. If $(u_0^n, v_0^n, w_0^n) \rightharpoonup (u_0, v_0, w_0)$ weakly in $H^1 \times H^1 \times L^2$, then for any $T > 0$,*

$$L(t)(u_0^n, v_0^n, w_0^n) \rightharpoonup L(t)(u_0, v_0, w_0) \quad \text{weakly in} \quad L^2([0, T], H^1 \times H^1 \times L^2)$$

$$\big[\S(t) - L(t)\big](u_0^n, v_0^n, w_0^n) \rightharpoonup \big[\S(t) - L(t)\big](u_0, v_0, w_0) \quad \text{weakly in} \quad L^2([0, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}).$$

Furthermore, we have pointwise weak convergence: for any $t \in [0, T]$,

$$L(t)(u_0^n, v_0^n, w_0^n) \rightharpoonup L(t)(u_0, v_0, w_0) \quad \text{weakly in} \quad H^1 \times H^1 \times L^2$$

$$\big[\S(t) - L(t)\big](u_0^n, v_0^n, w_0^n) \rightharpoonup \big[\S(t) - L(t)\big](u_0, v_0, w_0) \quad \text{weakly in} \quad H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}.$$

Proof. The statements for the linear part of the flow can be verified using the Fourier multiplier representation of the linear solutions. To work on the nonlinear part, we transform the equation (4.7). Set $\tilde{f} = (1 - \Delta)^{-1} f$ and $\tilde{g} = (1 - \Delta)^{-1} g$. Let $\tilde{u} = u + \tilde{f}$ and $\tilde{v} = v - \tilde{g}$ with $w = av + v_t$. We will choose $0 < a \ll 1$ later. This transformation yields

$$
\begin{cases}
i\tilde{u}_t + \Delta\tilde{u} + i\gamma\tilde{u} = -(\tilde{u} - \tilde{f})(\tilde{v} + \tilde{g}) + (1 + i\gamma)\tilde{f} \\
\tilde{v}_t + a\tilde{v} + a\tilde{g} = w \\
w_t + (\delta - a)w + \Big(1 + a(a - \delta) - \Delta\Big)\tilde{v} = |\tilde{u} - \tilde{f}|^2 - a(a - \delta)\tilde{g}.
\end{cases}
\tag{4.9}
$$

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The transformation allows us to replace the H^1 forcing terms by H^3 forcing terms, in exchange for more complex nonlinearities. The introduction of w is convenient for energy calculations.

Consider the homogeneous linear system

$$\begin{cases} ip_t + \Delta p + i\gamma p = 0 \\ q_t + aq = r \\ r_t + (\delta - a)r + \left(1 + a(a - \delta) - \Delta\right)q = 0. \end{cases} \tag{4.10}$$

In the following, we abuse notation and let L also denote the semigroup associated with this equation. The nonlinear parts $(U, V, W) = (\tilde{u} - p, \tilde{v} - q, w - r)$ satisfy

$$\begin{cases} iU_t + \Delta U + i\gamma U = -(U + p - \tilde{f})(V + q + \tilde{g}) + (1 + i\gamma)\tilde{f} \\ V_t + aV + a\tilde{g} = W \\ W_t + (\delta - a)W + \left(1 + a(a - \delta) - \Delta\right)V = |U + p - \tilde{f}|^2 - a(a - \delta)\tilde{g}, \end{cases} \tag{4.11}$$

with zero initial data. Just as in Section 4.4, we can use bilinear estimates, together with the smoothness of $\tilde{f}$ and $\tilde{g}$, to conclude that $(U, V, W) \in H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}$ for initial data $H^1 \times H^1 \times L^2$ in dimensions $d = 2, 3$.

We will show that every subsequence of $\left[\S(t) - L(t)\right](u_0^n, v_0^n, w_0^n)$ has a further subsequence which converges weakly to the solution of the KGS, which implies that the full sequence converges weakly to that solution.

Note that the sequence (u_0^n, v_0^n, w_0^n) is uniformly bounded in the energy space. Denote by (p^n, q^n, r^n) the solution to linear system (4.10) with initial data (u_0^n, v_0^n, w_0^n). Let (U^n, V^n, W^n) be the nonlinear part of the flow. For the nonlinear part, smoothing estimates along with the uniform bound on the initial data imply that, for any $T > 0$,

$$\left\{(U^n, V^n, W^n)\right\} \text{ is bounded in}$$

$$C([-T, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}) \cap C^1([-T, T], H^{-\frac{1}{2}-} \times H^{2-} \times H^{1-}). \tag{4.12}$$

This has several implications:

(i) The Banach-Alaoglu theorem implies weak* convergence of a subsequence of

$$\{(U^n, V^n, W^n)\} \quad \text{in} \quad L^\infty([-T, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}).$$

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(ii) The Arzela-Ascoli theorem implies that $\{(U^n, V^n, W^n)\}$ is precompact in $C([-T,T], H_{\text{loc}}^{-\frac{1}{2}-} \times H_{\text{loc}}^{2-} \times H_{\text{loc}}^{1-})$. By interpolation between this and (4.12), we find strongly convergent subsequence of

$$\{(U^n, V^n, W^n)\} \quad \text{in} \quad C([-T,T], H_{\text{loc}}^{\frac{3}{2}-} \times H_{\text{loc}}^{3-} \times H_{\text{loc}}^{2-}).$$

Similar statements hold for the linear parts (p^n, q^n, r^n) in $H^1 \times H^1 \times L^2$. By passing to a further subsequence, we obtain a sequence, which we still call $\{(U^n, V^n, W^n)\}$, which is weak* convergent in $L^\infty([-T,T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-})$ and strongly convergent in $C([-T,T], H_{\text{loc}}^{\frac{3}{2}-} \times H_{\text{loc}}^{3-} \times H_{\text{loc}}^{2-})$. Denote the limit by (U, V, W).

To see that the limit is a distributional solution, multiply the equations for (U^n, V^n, W^n) by an arbitrary test function $\phi \in C_c^\infty([-T,T] \times \mathbb{R}^d)$, integrate in space and time, and take the limit in n. For U^n, we have

$$\iint \left[-iU\phi_t + U\Delta\phi + i\gamma U\phi + \phi\Big((U + p - \tilde{f})(V + q + \tilde{g}) - (1 + i\gamma)\tilde{f}\Big) \right] \mathrm{d}x\,\mathrm{d}t$$
$$= \lim_{n\to\infty} \iint \left[-iU^n\phi_t + U^n\Delta\phi + i\gamma U^n\phi + \phi\Big((U^n + p^n - \tilde{f})(V^n + q^n + \tilde{g}) - (1 + i\gamma)\tilde{f}\Big) \right] \mathrm{d}x\,\mathrm{d}t = 0.$$

The equality is a consequence of the local strong convergence ((ii)) of U^n and V^n and strong local convergence of p^n and q^n in $C([-T,T], H^{1-})$. To verify the limit for the nonlinear term, note that

$$\left| \iint \phi \Big[(U^n + p^n - \tilde{f})(V^n + q^n + \tilde{g}) - (U + p - \tilde{f})(V + q + \tilde{g}) \Big] \mathrm{d}x\,\mathrm{d}t \right|$$
$$= \left| \iint \phi \Big[(U^n + p^n - \tilde{f})[V^n - V + q^n - q] + [U^n - U + p^n - p](V + q + \tilde{g}) \Big] \mathrm{d}x\,\mathrm{d}t \right|$$
$$\leq \|\phi\|_{L_{x,t}^\infty} \left| \iint_{\text{supp}\,\phi} \Big[(U^n + p^n - \tilde{f})[V^n - V + q^n - q] + [U^n - U + p^n - p](V + q + \tilde{g}) \Big] \mathrm{d}x\,\mathrm{d}t \right|,$$

which decays by local strong convergence. For V^n and W^n, we have

$$\iint \left[-V\phi_t + \phi\Big(aV + a\tilde{g} - W\Big) \right] \mathrm{d}x\,\mathrm{d}t = \lim_{n\to\infty} \iint \left[-V^n\phi_t + \phi\Big(aV^n + a\tilde{g} - W^n\Big) \right] \mathrm{d}x\,\mathrm{d}t = 0$$

and

$$\iint \left[-W\phi_t - V\Delta\phi + \phi\Big((\delta - a)W + \big(1 + a(a - \delta)\big)V\Big) - \phi\Big(|U + p + \tilde{f}|^2 - a(a - \delta)\tilde{g}\Big) \right] \mathrm{d}x\,\mathrm{d}t$$
$$= \lim_{n\to\infty} \iint \left[W^n\phi_t - V^n\Delta\phi + \phi\Big((\delta - a)W^n + \big(1 + a(a - \delta)\big)V^n\Big) \right.$$

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$$- \phi\Big(|U^n + p^n + \tilde{f}|^2 - a(a - \delta)\tilde{g}\Big)\bigg]\mathrm{d}x\,\mathrm{d}t = 0.$$

Again, convergence of the nonlinear terms follows from strong local convergence. Thus (U, V, W) is a distributional solution of (4.11) with $(U(0), V(0), W(0)) = (0, 0, 0)$. Furthermore, by the weak* convergence ((i)), we see that (U, V, W) is in the uniqueness class $C([-T, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-})$. Thus $(U, V, W) = \big[\S(t) - L(t)\big](u_0, v_0, w_0)$. The weak* convergence ((i)) implies weak convergence in $L^2([-T, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-})$ as desired since $L^2([-T, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-})$ is contained in the dual of $C([-T, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-})$.

To show pointwise weak convergence, fix a $t_0 \in [0, T]$. By again applying the Banach-Alaoglu theorem and passing to a further subsequence if necessary, we can ensure that the convergence described above still holds, along with weak $H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}$ convergence of $\big[\S(t_0) - L(t_0)\big](u_0^n, v_0^n, w_0^n)$, say to (u^*, v^*, w^*). Recall that we have shown weak* convergence of $\big[\S(t) - L(t)\big](u_0^n, v_0^n, w_0^n)$ to $(U, V, W) = \big[\S(t) - L(t)\big](u_0, v_0, w_0)$ in $C([0, T], H^{\frac{3}{2}-} \times H^{3-} \times H^{2-})$. Thus we have $(u^*, v^*, w^*) = \big[\S(t) - L(t)\big](u_0, v_0, w_0)$. $\qquad\square$

In the remainder of this section, we work with the following transformation of (4.7):

$$\begin{cases} iu_t + \Delta u + i\gamma u = -uv + f \\ v_t + av = w \\ w_t + (\delta - a)w + \Big(1 + a(a - \delta) - \Delta\Big)v = |u|^2 + g. \end{cases} \tag{4.13}$$

Again, let $\S(t)$ denote the semigroup operator for (4.13), and let $L(t)$ denote the linear flow operator. The evolution (4.13) has the absorbing ball property in dimensions $d = 2, 3$. For a proof of this, see [74, Section 2]. Thus to obtain a global attractor, it suffices to prove that the evolution is asymptotically compact – that is, that for every sequence of initial data $\{(u_0^n, v_0^n, w_0^n)\}_n$ in the energy space with corresponding solutions $\{(u^n, v^n, w^n)\}_n$ and every sequence of times $t_n \to \infty$, the sequence $\{(u^n(t_n), v^n(t_n), w^n(t_n))\}_n$ has a convergent subsequence in the energy space.

Let $(u_n, v_n, w_n) \in H^1 \times H^1 \times L^2$ be a sequence of initial data. We may assume that the data lies within the absorbing ball. Also let $t_n \to \infty$ be a sequence of times. The Banach-Alaoglu theorem implies that the sequence $\S(t_n)(u_n, v_n, w_n)$ has weakly convergent subsequence in $H^1 \times H^1 \times L^2$. Smoothing estimates together with bounds on the initial data imply that the nonlinear parts are

bounded in $H^{\frac{3}{2}-} \times H^{3-} \times H^{2-}$. Thus we may choose a subsequence such that these nonlinear parts $\S(t_n)(u_n, v_n, w_n) - L(t_n)(u_n, v_n, w_n)$ also converge weakly in $H^{3/2-} \times H^{3-} \times H^{2-}$. Since the linear part decays to zero, these two limits must be equal. Call the limit (u, v, w).

For any T, we can, by passing to a further subsequence, conclude that $\S(t_n - T)(u_n, v_n, w_n)$ converges weakly in $H^1 \times H^1 \times L^2$ and that $\S(t_n - T)(u_n, v_n, w_n) - L(t_n - T)(u_n, v_n, w_n)$ converges weakly in $H^{3/2-} \times H^{3-} \times H^{2-}$. Again, dissipative decay of the linear part implies that the two limits are equal; we denote the limit by (u_T, v_T, w_T). Weak continuity of the semigroup (Lemma 4.5.1) implies that $\S(T)(u_T, v_T, w_T) = (u, v, w)$. Note that by a diagonalization argument, we can obtain such weak convergence of $\S(t_n - T)(u_n, v_n, w_n)$ and $\S(t_n - T)(u_n, v_n, w_n) - L(t_n - T)(u_n, v_n, w_n)$ as above for a countable set of T simultaneously, e.g. $\{T \in \mathbb{N}\}$. This will be important later when we take $T \to \infty$.

The L^2 law for the evolution of $\S(t)$ gives

$$\|\S(t_n)u_n\|_{L^2}^2 = e^{-2\gamma T}\|\S(t_n - T)u_n\|_{L^2}^2 - 2\operatorname{Re} i \int_0^T e^{2\gamma(s-T)}\Big\langle \S(t_n - T + s)u_n, \, f\Big\rangle_{L_x^2} ds,$$

$$\|\S(T)u_T\|_{L^2}^2 = e^{-2\gamma T}\|u_T\|_{L^2}^2 - 2\operatorname{Re} i \int_0^T e^{2\gamma(s-T)}\Big\langle \S(s)u_T, \, f\Big\rangle_{L_x^2} ds.$$

Combining the two equations yields

$$\|\S(t_n)u_n\|_{L^2}^2 - \|\S(T)u_T\|_{L^2}^2 = e^{-2\gamma T}\Big(\|\S(t_n - T)u_n\|_{L^2}^2 - \|u_T\|_{L^2}^2\Big)$$
$$+ 2\operatorname{Re} i \int_0^T e^{2\gamma(s-T)}\Big\langle \S(s)u_T - \S(t_n - T + s)u_n, \, f\Big\rangle_{L_x^2} ds.$$

The first term on the right-hand side can be made arbitrarily small by increasing T since the u_n are uniformly bounded in L^2. The second term decays to zero as $n \to \infty$ by the weak continuity of $\S$ in $L_t^2 H_x^1$. Thus we conclude that

$$\limsup_{n\to\infty}\Big[\|\S(t_n)u_n\|_{L^2}^2 - \|\S(T)u_T\|_{L^2}^2\Big] = \limsup_{n\to\infty}\Big[\|\S(t_n)u_n\|_{L^2}^2 - \|u\|_{L^2}^2\Big] \leq 0.$$

With the weak convergence of $S(t_n)u_n$ to u, this implies that $\S(t_n)u_n \to u$ strongly in L^2.

Now consider the full $\dot{H}^1 \times H^1 \times L^2$ energy equation. Define the energy functional $H = H(u_0, v_0, w_0)(t)$ as follows:

$$H = 2\|\nabla\S(t)u_0\|_{L^2}^2 + \Big(1 + a(a - \delta)\Big)\|\S(t)v_0\|_{L^2}^2 + \|\nabla\S(t)v_0\|_{L^2}^2 + \|\S(t)w_0\|_{L^2}^2$$

$$- 2 \int |\S(t)u_0|^2 \S(t)v_0 \, \mathrm{d}x + 4 \int f\overline{S(t)u_0} \, \mathrm{d}x.$$

Then the time derivative $\mathrm{d}H/\mathrm{d}t$ is given by

$$-4\gamma\|\nabla\S(t)u_0\|_{L^2}^2 - 2a\Big(1 + a(a-\delta)\Big)\|\S(t)v_0\|_{L^2}^2 - 2a\|\nabla\S(t)v_0\|_{L^2}^2 - 2(\delta - a)\|\S(t)w_0\|_{L^2}^2$$
$$+(4\gamma + 2a)\int |\S(t)u_0|^2\S(t)v_0 \, \mathrm{d}x - 4\gamma\,\mathrm{Re}\int f\overline{S(t)u_0}\,\mathrm{d}x + 2\int gS(t)w_0\,\mathrm{d}x.$$

This implies that

$$H(u_n, v_n, w_n)(t_n) - H(u_T, v_T, w_T)(T) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V},$$

where

$$\mathrm{I} = e^{-2aT}\Big(H(u_n, v_n, w_n)(t_n - T) - H(u_T, v_T, w_T)(0)\Big)$$

$$\mathrm{II} = -4(\gamma - a)\int_0^T e^{2a(s-T)}\Big[\|\nabla\S(t_n - T + s)u_n\|_{L^2}^2 - \|\nabla\S(s)u_T\|_{L^2}^2\Big]\,\mathrm{d}s$$
$$-2(\delta - 2a)\int_0^T e^{2a(s-T)}\Big[\|\S(t_n - T + s)w_n\|_{L^2}^2 - \|S(s)w_T\|_{L^2}^2\Big]\,\mathrm{d}s$$

$$\mathrm{III} = 2(2\gamma - a)\int_0^T\!\!\int e^{2a(s-T)}\Big[\big|\S(t_n - T + s)u_n\big|^2\S(t_n - T + s)v_n - \big|\S(s)u_T\big|^2\S(s)v_T\Big]\,\mathrm{d}x\,\mathrm{d}s$$

$$\mathrm{IV} = -4(\gamma - 2a)\,\mathrm{Re}\int_0^T e^{2a(s-T)}\Big\langle \S(t_n - T + s)u_n - S(s)u_T,\ f\Big\rangle_{L_x^2}\,\mathrm{d}s$$

$$\mathrm{V} = 2\int_0^T e^{2a(s-T)}\Big\langle \S(t_n - T + s)w_n - S(s)w_T,\ g\Big\rangle_{L_x^2}\,\mathrm{d}s.$$

The term I is negligible for large T. For II, weak convergence implies that

$$\liminf_{n\to\infty}\|\nabla\S(t_n - T + s)u_n\|_{L^2}^2 - \|\nabla\S(s)u_T\|_{L^2}^2 \geqslant 0,$$

$$\liminf_{n\to\infty}\|\S(t_n - T + s)w_n\|_{L^2}^2 - \|\S(s)w_T\|_{L^2}^2 \geqslant 0$$

for each s, so the $\limsup$ over n of II is nonpositive. Write the integral in III as

$$\int_0^T\!\!\int e^{2a(s-T)}\Big[\big|\S(t_n - T + s)u_n\big|^2 - \big|\S(s)u_T\big|^2\Big]L(t_n - T + s)v_n\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^T\!\!\int e^{2a(s-T)}\Big[\big|\S(t_n - T + s)u_n\big|^2 - \big|\S(s)u_T\big|^2\Big]\big[\S - L\big](t_n - T + s)v_n\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^T\!\!\int e^{2a(s-T)}\big|\S(s)u_T\big|^2\Big[\S(t_n - T + s)v_n - \S(s)v_T\Big]\,\mathrm{d}x\,\mathrm{d}s.$$

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To see that the first line vanishes in the limit, apply the L^3 Gagliardo-Nirenberg inequality $\|h\|_{L^3} \lesssim \|\nabla h\|_{L^2}^{d/6} \|h\|_{L^2}^{(6-d)/6}$ with the fact that $L(t_n - T + s)v_n \to 0$ uniformly in H^1. For the second line, extract $\big[\S - L\big](t_n - T + s)v_n$ in the $H^{3-} \hookrightarrow L^\infty$ norm and use the strong L^2 convergence of $\S(t_n)u_n$ to $\S(T)u_T$ and strong continuity of $\S(s - T)$. The last line decays by weak continuity of $\S(s)$ since $|\S(s)u_T|^2$ is an L^2 function by the Gagliardo-Nirenberg inequality $\|h\|_{L^4} \lesssim \|\nabla h\|_{L^2}^{d/4} \|h\|_{L^2}^{(4-d)/4}$. The remaining terms IV and V vanish in the limit by weak continuity of the semigroup.

Thus we conclude that

$$\limsup_{n\to\infty}\Big[H(u_n, v_n, w_n)(t_n) - H(u_T, v_T, w_T)(T)\Big] = \limsup_{n\to\infty}\Big[H(u_n, v_n, w_n)(t_n) - H(u, v, w)(0)\Big] \leqslant 0.$$

This, together with the weak convergence of $S(t_n)(u_0^n, v_0^n, w_0^n)$ to (u, v, w) in $H^1 \times H^1 \times L^2$, implies that $S(t_n)(u_0^n, v_0^n, w_0^n)$ converges strongly to (u, v, w) in $H^1 \times H^1 \times L^2$. This completes the proof of asymptotic compactness, and thus of the existence of a global attractor.

4.6 Proof of Global Existence in $\mathbb{R}^4$

In this section, we prove global existence for the Klein-Gordon-Schrödinger system in four dimensions. We work with the form of the equation given in (4.3) in dimension $d = 4$. In the following, we drop the $\pm$ superscripts on n to simplify the notation. Suppose we have $(u_0, n_0) \in H^s \times H^r$ for some $s, r > 9/10$ with $\|u_0\|_{L^2}$ small. Fix T large. We wish to show that the solution (u, n) exists on $[0, T]$. To do so, we decompose the solution into two parts: one with low-frequency initial data and one with the complementary high-frequency data. Specifically, recall that $A = (1 - \Delta)^{1/2}$ and write $u = \phi + \mu$ and $n = \psi + \lambda$, where

$$\begin{cases} i\phi_t + \Delta\phi = -\frac{1}{2}\operatorname{Re}(\psi)\phi \\ i\psi_t \mp A\psi = \mp A^{-1}|\phi|^2, \end{cases} \tag{4.14}$$

$$\begin{cases} i\mu_t + \Delta\mu = -\frac{1}{2}\operatorname{Re}(\lambda + \psi)\mu - \frac{1}{2}\operatorname{Re}(\lambda)\phi \\ i\lambda_t \mp A\lambda = \mp A^{-1}|\mu|^2 \mp 2\operatorname{Re}A^{-1}\mu\overline{\phi}. \end{cases} \tag{4.15}$$

The initial data for these two systems is $(\phi_0, \psi_0) = (P_{\leqslant N}u_0, P_{\leqslant N}n_0)$ and $(\mu_0, \lambda_0) = (u_0 - \phi_0, n_0 - \psi_0)$, where $P_{\leqslant N}$ is the projection onto Fourier modes less than N. We will allow these equations to

evolve a local-theory time step δ. Then we add the nonlinear part of (μ, λ) to (ϕ, ψ) and start again, i.e. evolve (4.14) and (4.15) another local time step with initial data

$$(\phi_1, \psi_1) = \Big(\phi(\delta) + \big[\mu(\delta) - e^{i\delta\Delta}\mu_0\big], \ \psi(\delta) + \big[\lambda(\delta) - e^{\mp i\delta A}\lambda_0\big]\Big)$$

$$(\mu_1, \lambda_1) = (e^{i\delta\Delta}\mu_0, \ e^{\mp i\delta A}\lambda_0).$$

To iterate this process, we use smoothing estimates to show that the nonlinear part of (μ, λ) is in $H^1 \times H^1$ and that relevant norms do not grow too rapidly, so that a uniform time step δ can be used to cover $[0, T]$.

We will need the following observations, which hold for $s_0 \leqslant s \leqslant 1$ and $r_0 \leqslant r \leqslant 1$ to carry out the local theory estimates:

$$
\begin{aligned}
\|\phi_0\|_{H^1} &\leqslant N^{1-s}\|u_0\|_{H^s} \lesssim N^{1-s} & \|\psi_0\|_{H^1} &\leqslant N^{1-r}\|n_0\|_{H^r} \lesssim N^{1-r} \\
\|\mu_0\|_{H^{s_0}} &\leqslant N^{s_0-s}\|u_0\|_{H^s} \lesssim N^{s_0-s} & \|\lambda_0\|_{H^{r_0}} &\leqslant N^{r_0-r}\|n_0\|_{H^r} \lesssim N^{r_0-r}.
\end{aligned}
\tag{4.16}
$$

In the following, we let $m = \min\{s, r\}$ and $s_0 = r_0 = \frac{1}{2}+$.

In four dimensions, the relevant nonlinear estimates for the local theory are in [51]. In particular, from [51, Lemma 3.4], we obtain

$$\|un\|_{X^{k, -\frac{1}{2}+\epsilon}} \lesssim T^{\theta}\|u\|_{X^{k, \frac{1}{2}+\epsilon}}\|n\|_{X_{\pm}^{\ell, \frac{1}{2}+\epsilon}}$$

where $\theta = \min\{\frac{1}{2}, \frac{\ell}{2} - (1 - \frac{1}{1+2\epsilon})\}$ and $k \leqslant \ell + 1 - 2\epsilon$. From [51, Lemma 3.5], we have also

$$\|A^{-1}|u|^2\|_{X_{\pm}^{\ell, -\frac{1}{2}+\epsilon}} = \||u|^2\|_{X_{\pm}^{\ell-1, -\frac{1}{2}+\epsilon}} \lesssim T^{\theta}\|u\|_{X^{k, \frac{1}{2}+\epsilon}}^2$$

as long as $2k - \ell + 1 > 0$ with $\theta = (2k - \ell + 1)/2 - 2\epsilon/(1 + 2\epsilon)$. Thus a time step

$$\delta \lesssim N^{-2(1-m)/r_0-} \lesssim \big\langle \|\mu_0\|_{H^{s_0}} + \|\lambda_0^{\pm}\|_{H^{r_0}} + \|\phi_0\|_{H^1} + \|\psi_0^{\pm}\|_{H^1} \big\rangle^{-2/r_0-}.$$

yields local existence for (4.14) in $H^1 \times H^1$ on $[0, \delta]$ with

$$\|\phi\|_{X_{\delta}^{1, \frac{1}{2}+}} + \|\psi\|_{X_{\pm, \delta}^{1, \frac{1}{2}+}} \lesssim \|\phi_0\|_{H^1} + \|\psi_0\|_{H^1} \lesssim N^{1-m},$$

and local existence for (4.15) in $H^{s_0} \times H^{r_0}$ with

$$\|\mu\|_{X_{\delta}^{s_0, \frac{1}{2}+}} + \|\lambda\|_{X_{\pm, \delta}^{r_0, \frac{1}{2}+}} \lesssim \|\mu_0\|_{H^{s_0}} + \|\lambda_0\|_{H^{r_0}} \lesssim N^{\max\{s_0-s, r_0-r\}} = N^{1/2-m+}.$$

Next write $\mu = e^{it\delta\Delta}\mu_0 + w(t)$ and $\lambda = e^{\mp itA}\lambda_0 + z(t)$, where w and z are the Duhamel terms

$$w(t) = -\frac{1}{2}\int_0^t e^{i(t-s)\Delta}\Big[\mathrm{Re}(\lambda+\psi)\mu + \mathrm{Re}(\lambda)\phi\Big]\,ds,$$

$$z(t) = \int_0^t e^{\mp i(t-s)A}\Big[\mp A^{-1}|\mu|^2 \mp 2\,\mathrm{Re}\,A^{-1}\mu\overline{\phi}\Big]\,ds.$$

Then for $r_0 = s_0 = \frac{1}{2}+$ and $m = \min\{s, r\}$, we have, using the estimates (4.5) and (4.6) along with the local theory bounds,

$$
\begin{aligned}
\|w\|_{L^\infty_{[0,\delta]}H^1} &\lesssim \|w\|_{X^{1,\frac{1}{2}+}_\delta} \lesssim \|\mu\|_{X^{s_0,\frac{1}{2}+}_\delta}\left(\|\lambda\|_{X^{r_0,\frac{1}{2}+}_{\pm,\delta}} + \|\psi\|_{X^{1,\frac{1}{2}+}_{\pm,\delta}}\right) + \|\phi\|_{X^{1,\frac{1}{2}+}_\delta}\|\lambda\|_{X^{r_0,\frac{1}{2}+}_{\pm,\delta}} \\
&\lesssim N^{\max\{s_0-s,r_0-r\}}N^{1-m} = N^{3/2-2m+} \\
\|z\|_{L^\infty_{[0,\delta]}H^1} &\lesssim \|w\|_{X^{1,\frac{1}{2}+}_{\pm,\delta}} \lesssim \|\mu\|_{X^{s_0,\frac{1}{2}+}_\delta}\left(\|\mu\|_{X^{s_0,\frac{1}{2}+}_\delta} + \|\phi\|_{X^{1,\frac{1}{2}+}_\delta}\right) \\
&\lesssim N^{\max\{s_0-s,r_0-r\}}N^{1-m} = N^{3/2-2m+}.
\end{aligned}
\tag{4.17}
$$

To iterate, we must ensure that estimates (4.16) and (4.17) remain valid for each time step. This is immediate for the (μ, λ) initial data, since it is always simply a linear flow. Thus proving the requisite bounds amounts to showing that the $H^1 \times H^1$ norm of (ϕ, ψ) is bounded by N^{1-m} over each time step. To do so, we use the L^2 conservation and the Hamiltonian energy. Notice that the initial data for ϕ at time step k is $u(k\delta) - e^{ik\delta\Delta}\mu_0$, and the L^2 conservation gives $\|u - e^{ik\delta\Delta}\mu_0\|_{L_2} \leqslant \|u_0\|_{L_2}(1 + N^{-s})$. Thus we have uniform control over $\|\phi\|_{L^2}$. To control the remaining components of the $H^1 \times H^1$ norm, use the Hamiltonian

$$E(u, n) = \|An\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 - 2\int |u|^2\,\mathrm{Re}(n)\,dx.$$

This is conserved for the flow of (4.14), so we need only check that it does not grow too much due to the addition of the nonlinear terms. Using the Gagliardo-Nirenberg inequality and Cauchy-Schwarz, the increment of the energy is bounded as follows, where the norms are all evaluated at time δ:

$$
\begin{aligned}
&\Big|E(\phi(\delta) + w(\delta), \psi^\pm(\delta) + z^\pm(\delta)) - E(\phi(\delta), \psi^\pm(\delta))\Big| \\
&\lesssim \|Az^\pm\|_{L^2}\left(\|Az^\pm\|_{L^2} + 2\|An^\pm\|_{L^2}\right) + 2\|\nabla w\|_{L^2}\left(\|\nabla w\|_{L^2} + 2\|\nabla\phi\|_{L^2}\right) \\
&\quad + \|\nabla z^\pm\|_{L^2}\|\phi + w\|_{L^2}\|\nabla(\phi + w)\|_{L^2}
\end{aligned}
$$

$$+ \|\nabla \psi^\pm\|_{L^2} \Big(\|w\|_{L^2} \|\nabla w\|_{L^2} + \|\phi + w\|_{L^2} \|\nabla w\|_{L^2} \Big)$$

Noting that $\|\phi + w\|_{L^2} = \|u - e^{it\Delta}\mu_0\|_{L^2} \lesssim 1$ and $\|w\|_{L^2} \lesssim 1$, this quantity can be controlled by $N^{3/2-2m+} N^{1-m}$, so for E to remain bounded by $N^{2(1-m)}$, we require

$$N^{3/2-2m+} N^{1-m} N^{2(1-m)/r_0+} \lesssim N^{2(1-m)},$$

which holds if $m > 9/10$. To complete the proof, we need to show that the energy controls the $\dot{H}^1 \times H^1$ norm at each time step. This depends on the smallness assumption on u in L^2. By the Gagliardo-Nirenberg inequality, we have

$$\left| \int |\phi|^2 \psi \, dx \right| \leq \|\phi\|_{L^{8/3}}^2 \|\psi\|_{L^4} \leq C_1 C_2^2 \|\phi\|_{L^2} \|\nabla \phi\|_{L^2} \|\nabla \psi\|_{L^2} \leq C_1 C_2^2 \|\phi\|_{L^2} \|\nabla \phi\|_{L^2} \|A\psi\|_{L^2},$$

where C_1 and C_2 are the sharp constants of the following inequalities:

$$\|f\|_{L^4(\mathbb{R}^4)} \leq C_1 \|\nabla f\|_{L^2(\mathbb{R}^4)} \qquad \|f\|_{L^{8/3}(\mathbb{R}^4)} \leq C_2 \|f\|_{L^2(\mathbb{R}^4)}^{1/2} \|\nabla f\|_{L^2(\mathbb{R}^4)}^{1/2}.$$

By our construction of ϕ and the assumption that $\|u_0\|_{L^2} < \sqrt{2}/(C_1 C_2^2)$, we have at each time step $\|\phi\| < (1 + N^{-s})\sqrt{2}/(C_1 C_2^2)$. By choosing N large, we also have $\|\phi\|_{L^2} < \sqrt{2}/(C_1 C_2^2)$ at each step. Choose $C_0 < 1$ so that $\|\phi\|_{L^2} C_1 C_2^2 < \sqrt{2} C_0$. Then we have

$$E(\phi, \psi) = \|A\psi\|_{L^2}^2 + 2\|\nabla \phi\|_{L^2}^2 - 2\sqrt{2} C_0 \|\nabla \phi\|_{L^2} \|A\psi\|_{L^2} \gtrsim \|A\psi\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2.$$

Thus $E(\phi, \psi) \approx \|\phi\|_{\dot{H}^1}^2 + \|\psi\|_{H^1}^2$ at each time step.

4.7 Proof of Proposition 4.3.3: Bilinear Estimate

By duality, to obtain the smoothing estimate (4.5) it suffices to show that

$$\iint uvw \, dx \, dt = \iint \widehat{uv}(\xi_0, \tau_0) \breve{w}(\xi_0, \tau_0) \, d\xi_0 \, d\tau_0 \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{r,b}_\pm} \|w\|_{X^{-(s+\alpha),1-b}}.$$

We introduce the functions f_i, which allow us to state the estimate in terms of L^2 norms:

$$f_1 = \langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^b \hat{u}, \quad f_2 = \langle \xi \rangle^r \langle \tau \pm |\xi| \rangle^b \hat{v}, \quad \text{and} \quad f_3 = \langle \xi \rangle^{-(s+\alpha)} \langle \tau + |\xi|^2 \rangle^{1-b} \hat{w}.$$

Using these functions and the convolution structure of $\widehat{uv}$, the required estimate takes the form

$$\iiiint_{\substack{\sum \xi_i = 0 \\ \sum \tau_i = 0}} \frac{\langle \xi_0 \rangle^{s+\alpha} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-r} f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_0 - |\xi_0|^2 \rangle^{1-b} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 \pm |\xi_2| \rangle^b} \, d\xi_1 \, d\xi_2 \, d\tau_1 \, d\tau_2 \lesssim \prod_{i=0}^{2} \|f_i\|_{L^2_{\xi,\tau}}. \qquad (4.18)$$

We proceed with the proof by breaking the integration region into many components and considering each separately.

CASE 0. $|\xi_1|, |\xi_2| \lesssim 1$. We ignore the order one multipliers $\langle \xi_0 \rangle^{s+\alpha} \langle \xi_1 \rangle^{-s} \langle \xi_2 \rangle^{-r}$ on the left-hand side of (4.18) and work with

$$\iiiint_{\substack{\sum \xi_i = 0 \\ \sum \tau_i = 0}} \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_0 - |\xi_0|^2 \rangle^{1-b} \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 \pm |\xi_2| \rangle^b} \, d\xi_1 \, d\xi_2 \, d\tau_1 \, d\tau_2$$

$$\lesssim \|f_0\|_{L^2} \left\| \langle \tau_0 - |\xi_0|^2 \rangle^{b-1} \iint \frac{f_1(\xi_1, \tau_1) f_2(-\xi_0 - \xi_1, -\tau_0 - \tau_1)}{\langle \tau_1 + |\xi_1|^2 \rangle^b \langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^b} \, d\xi_1 \, d\tau_1 \right\|_{L^2_{\xi_0, \tau_0}}.$$

Using Cauchy-Schwartz in $d\xi_1 \, d\tau_1$ and then in $d\xi_0 \, d\tau_0$, the ξ_0, τ_0 norm in the previous line is bounded by

$$\left\| \iint f_1^2(\xi_1, \tau_1) f_2^2(-\xi_0 - \xi_1, -\tau_0 - \tau_1) \, d\xi_1 \, d\tau_1 \iint \frac{\langle \tau_0 - |\xi_0|^2 \rangle^{2b-2} \, d\xi_1 \, d\tau_1}{\langle \tau_1 + |\xi_1|^2 \rangle^{2b} \langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{2b}} \right\|_{L^1_{\xi_0, \tau_0}}^{1/2}$$

$$\lesssim \left(\sup_{\xi_0, \tau_0} \iint \frac{\langle \tau_0 - |\xi_0|^2 \rangle^{2b-2}}{\langle \tau_1 + |\xi_1|^2 \rangle^{2b} \langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{2b}} \, d\tau_1 \, d\xi_1 \right)^{1/2}$$

$$\times \left\| \iint f_1^2(\xi_1, \tau_1) f_2^2(-\xi_0 - \xi_1, -\tau_0 - \tau_1) \, d\xi_1 \, d\tau_1 \right\|_{L^1_{\xi_0, \tau_0}}^{1/2}.$$

Notice that the L^1 norm on the last line is $\left(\|f_1^2\|_{L^1_{\xi,\tau}} \|f_2^2\|_{L^1_{\xi,\tau}} \right)^{1/2} = \|f_1\|_{L^2_{\xi,\tau}} \|f_2\|_{L^2_{\xi,\tau}}$, so we need only show that the supremum is finite. This is simple when $|\xi_1| \lesssim 1$. First use the fact that $\langle a + b \rangle \lesssim \langle a \rangle \langle b \rangle$ to obtain

$$\iint \frac{\langle \tau_0 - |\xi_0|^2 \rangle^{2b-2} \langle \tau_1 + |\xi_1|^2 \rangle^{-2b}}{\langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{2b}} \, d\tau_1 \, d\xi_1 \lesssim \iint \frac{\langle \tau_0 + \tau_1 - |\xi_0|^2 + |\xi_1|^2 \rangle^{2b-2}}{\langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{2b}} \, d\tau_1 \, d\xi_1.$$

Apply Lemma 2.3.4 and the fact that the integral is constrained to the region $|\xi_1| \lesssim 1$ to bound the supremum by

$$\sup_{\xi_0} \int \langle |\xi_1|^2 - |\xi_0|^2 \pm |\xi_0 + \xi_1| \rangle^{2b-2} \, d\xi_1 \lesssim 1.$$

This finishes with the region where all the ξ_i are small. The argument holds in any dimension and puts no constraints on s, r, or α.

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For the remaining cases, the resonances of the equation play a significant role. To deal with them, we need a few definitions. Let α denote the angle between ξ_1 and ξ_2. We define the maximum modulation M as follows, and use the fact that $\tau_0 + \tau_1 + \tau_2 = 0$ and $\xi_0 + \xi_1 + \xi_2 = 0$ to bound it:

$$M = \max\left\{|\tau_0 - |\xi_0|^2|, |\tau_1 + |\xi_1|^2|, |\tau_2 \pm |\xi_2||\right\} \gtrsim \left||\xi_0|^2 - |\xi_1|^2 \mp |\xi_2|\right|$$

$$= 2|\xi_1||\xi_2|\left|\cos\alpha + \frac{|\xi_2| \mp 1}{2|\xi_1|}\right| = 2|\xi_1||\xi_2|A.$$

We also need a dyadic decomposition. Let $f_i^{M^j} = f_i\big|_{\{|\xi|\approx M^j\}}$ for M^j dyadic so that $f_i = \sum_{M^j} f_i^{M^j}$. Then from (4.18), it suffices to show that

$$\sum_{\substack{M_i^j \\ \text{dyadic}}} \iiiint_{\substack{\sum \xi_i=0 \\ \sum \tau_i=0}} \frac{f_0^{M_0^j}(\xi_0,\tau_0) f_1^{M_1^j}(\xi_1,\tau_1) f_2^{M_2^j}(\xi_2,\tau_2)\, d\xi_1\, d\xi_2\, d\tau_1\, d\tau_2}{\langle\xi_0\rangle^{-(s+\alpha)}\langle\xi_1\rangle^s\langle\xi_2\rangle^r\langle\tau_0 - |\xi_0|^2\rangle^{1-b}\langle\tau_1 + |\xi_1|^2\rangle^b\langle\tau_2 \pm |\xi_2|\rangle^b} \lesssim \prod_{i=0}^{2} \|f_i\|_{L^2_{\xi,\tau}}. \qquad (4.19)$$

In the following cases, we drop the M_i^j superscript to lighten the notation, and implicitly assume that f_i is supported on the dyadic shell $|\xi| \approx M_i$. This results in estimates which depend on the M_i. To finish the proof, we show in each case that these estimates can be dyadically summed to yield (4.19).

<u>**CASE 1.**</u> $M = |\tau_0 - |\xi_0|^2|$. In this case, we must control

$$\langle M_0\rangle^{s+\alpha}\langle M_1\rangle^{-s}\langle M_2\rangle^{-r} \iiiint_{\substack{\sum \xi_i=0 \\ \sum \tau_i=0}} \frac{f_0(\xi_0,\tau_0) f_1(\xi_1,\tau_1) f_2(\xi_2,\tau_2)}{\langle M\rangle^{1-b}\langle\tau_1 + |\xi_1|^2\rangle^b\langle\tau_2 \pm |\xi_2|\rangle^b}\, d\xi_1\, d\xi_2\, d\tau_1\, d\tau_2. \qquad (4.20)$$

Recall that $M \gtrsim M_1 M_2 |A|$. We first consider the nonresonant regions, i.e. where $|A| \gtrsim 1$.

<u>**Case 1.1.**</u> $M_0 \lesssim M_1 \approx M_2$ and $|A| \gtrsim 1$. Estimate the maximum M by $M_1 M_2$. Decompose the functions f_1 and f_2 parabolically:

$$f_1 = \sum_{n\in\mathbb{Z}} f_1^n, \quad \text{where} \quad f_1^n = f_1\chi_{\{\tau_1 + |\xi_1|^2 = n + \mathcal{O}(1)\}}$$

$$f_2 = \sum_{m\in\mathbb{Z}} f_2^m, \quad \text{where} \quad f_2^m = f_2\chi_{\{\tau_2 \pm |\xi_2| = m + \mathcal{O}(1)\}}.$$

Using this decomposition, controlling the integral in (4.20) amounts to bounding

$$\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\langle n\rangle^{-b}\langle m\rangle^{-b}\iiiint_{\theta_i=\mathcal{O}(1)} f_0(-\xi_1-\xi_2,|\xi_1|^2\pm|\xi_2|-n-m-\theta_1-\theta_2) \tag{4.21}$$

$$\times f_1^n(\xi_1,-|\xi_1|^2+n+\theta_1)f_2^m(\xi_2,\mp|\xi_2|+m+\theta_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

To do so, recall we're assuming $M_0\lesssim M_1\approx M_2$, and decompose the supports of f_1 and f_2 into squares (or in higher dimensions, hypercubes) of side length $L\approx M_0$. Denote these squares by $\{|\xi_1|\approx M_1\}=\bigcup_i Q_i$ and $\{|\xi_2|\approx M_2\}=\bigcup_j R_j$. Note that since $\sum\xi_i=0$ and $M_0\lesssim M_1,M_2$, the square $R_j=R_{j(i)}$ is essentially determined by the square Q_i. Technically, each region Q_i could correspond to up to 3^d of the R_j regions, but this factor does not harm the estimates. Let $f_{1,Q_i}^n=f_1^n\chi_{\{\xi_1\in Q_i\}}$ and $f_{2,R_j}^m=f_2^m\chi_{\{\xi_2\in R_j\}}$.

For the moment, consider only the inner $\mathrm{d}\xi_1\,\mathrm{d}\xi_2$ integral in (4.21). For fixed θ_i, n, and m, change variables by letting $u=-\xi_1-\xi_2$ and $v=|\xi_1|^2\pm|\xi_2|-n-m-\theta_1-\theta_2$. We will use u and v to replace ξ_1 and one component of ξ_2. Let $\xi_i=(\xi_{i,1},\xi_{i,2},\ldots,\xi_{i,d})$. Computing the Jacobian matrix for the change of variables in the two-dimensional case gives

$$\begin{bmatrix} \frac{\mathrm{d}u_1}{\mathrm{d}\xi_{1,1}} & \frac{\mathrm{d}u_1}{\mathrm{d}\xi_{1,2}} & \frac{\mathrm{d}u_1}{\mathrm{d}\xi_{2,1}} & \frac{\mathrm{d}u_1}{\mathrm{d}\xi_{2,2}} \\ \frac{\mathrm{d}u_2}{\mathrm{d}\xi_{1,1}} & \frac{\mathrm{d}u_2}{\mathrm{d}\xi_{1,2}} & \frac{\mathrm{d}u_2}{\mathrm{d}\xi_{2,1}} & \frac{\mathrm{d}u_2}{\mathrm{d}\xi_{2,2}} \\ \frac{\mathrm{d}v}{\mathrm{d}\xi_{1,1}} & \frac{\mathrm{d}v}{\mathrm{d}\xi_{1,2}} & \frac{\mathrm{d}v}{\mathrm{d}\xi_{2,1}} & \frac{\mathrm{d}v}{\mathrm{d}\xi_{2,2}} \\ \frac{\mathrm{d}\xi_{2,2}}{\mathrm{d}\xi_{1,1}} & \frac{\mathrm{d}\xi_{2,2}}{\mathrm{d}\xi_{1,2}} & \frac{\mathrm{d}\xi_{2,2}}{\mathrm{d}\xi_{2,1}} & \frac{\mathrm{d}\xi_{2,2}}{\mathrm{d}\xi_{2,2}} \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 2\xi_{1,1} & 2\xi_{1,2} & \pm\frac{\xi_{2,1}}{|\xi_2|} & \pm\frac{\xi_{2,2}}{|\xi_2|} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

when we replace ξ_1 and $\xi_{2,1}$ by u and v. The result in the case when we replace $\xi_{2,2}$ instead of $\xi_{2,1}$ is similar – the one in the final row just moves a column to the left. Computing the determinant of the Jacobian matrix, we see that

$$\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\xi_{2,2}=\left|2\xi_{1,1}\mp\frac{\xi_{2,1}}{|\xi_2|}\right|\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \quad\text{and}\quad \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\xi_{2,1}=\left|2\xi_{1,2}\mp\frac{\xi_{2,2}}{|\xi_2|}\right|\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

depending on which component of ξ_2 we retain. In higher dimensions the result is parallel:

$$\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\xi_{2,1}\cdots\mathrm{d}\xi_{2,j-1}\,\mathrm{d}\xi_{2,j+1}\cdots\mathrm{d}\xi_{2,d}=\left|2\xi_{1,j}\pm\frac{\xi_{2,j}}{|\xi_2|}\right|\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

We may assume that $M_1\gg 1$; Case 0 dealt with the region where all M_i are small. Then we have $|\xi_{1,j}|\approx M_1\gg 1$ for some j. Without loss of generality, assume that $|\xi_{1,1}|\approx M_1$. Let

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$\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the projection onto the last $d-1$ components. Define

$$H(u, v, \xi_{2,2}, \xi_{2,3}, \ldots, \xi_{2,d}) = f_{1,Q_i}^n(\xi_1, -|\xi_1|^2 + n + \theta_1) f_{2,R_{j(i)}}^m(\xi_2, \mp|\xi_2| + m + \theta_2).$$

Then the $\mathrm{d}\xi_1 \, \mathrm{d}\xi_2$ integral in (4.21) is bounded by

$$\sum_{Q_i} \iiint_{(\xi_{2,2},\ldots,\xi_{2,d})\in\pi(R_{j(i)})} f_0(u,v)H(u,v,\xi_{2,2},\ldots,\xi_{2,d}) \left|2\xi_{1,1} \mp \frac{\xi_{2,1}}{|\xi_2|}\right|^{-1} \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\xi_{2,2}\ldots\mathrm{d}\xi_{2,d}$$

$$\lesssim M_1^{-1}\|f_0\|_{L^2} \sum_{Q_i} \left\| \int_{(\xi_{2,2},\ldots,\xi_{2,d})\in\pi(R_{j(i)})} H(u,v,\xi_{2,2},\ldots,\xi_{2,d})\,\mathrm{d}\xi_{2,2}\ldots\mathrm{d}\xi_{2,d} \right\|_{L_{u,v}^2}$$

$$\leq L^{(d-1)/2}M_1^{-1}\|f_0\|_{L^2} \sum_{Q_i} \|H(u,v,\xi_{2,2},\ldots,\xi_{2,d})\|_{L_{u,v,\xi_{2,2},\ldots,\xi_{2,d}}^2}$$

$$\lesssim L^{(d-1)/2}M_1^{-1/2}\|f_0\|_{L_{\xi,\tau}^2} \sum_{Q_i} \|H(\xi_1,\xi_2)\|_{L_{\xi_1,\xi_2}^2}$$

$$= L^{(d-1)/2}M_1^{-1/2}\|f_0\|_{L_{\xi,\tau}^2} \sum_{Q_i} \|f_{1,Q_i}^n(\xi_1, -|\xi_1|^2 + n + \theta_1)\|_{L_{\xi_1}^2} \|f_{2,R_{j(i)}}^m(\xi_2, \mp|\xi_2| + m + \theta_2)\|_{L_{\xi_2}^2}$$

$$\leq L^{(d-1)/2}M_1^{-1/2}\|f_0\|_{L_{\xi,\tau}^2} \|f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1)\|_{L_{\xi_1}^2} \|f_2^m(\xi_2, \mp|\xi_2| + m + \theta_2)\|_{L_{\xi_2}^2}.$$

The last inequality follows from applying Cauchy-Schwarz to the Q_i sum. Thus (4.21) is bounded by

$$L^{(d-1)/2}M_1^{-1/2}\|f_0\|_{L_{\xi,\tau}^2} \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \langle n\rangle^{-b}\langle m\rangle^{-b} \iint_{\theta_i=\mathcal{O}(1)} \|f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1)\|_{L_{\xi_1}^2}$$

$$\times \|f_2^m(\xi_2, \mp|\xi_2| + m + \theta_2)\|_{L_{\xi_2}^2} \, \mathrm{d}\theta_i$$

By Cauchy-Schwarz in θ_1 and θ_2, using the fact that $\theta_i = \mathcal{O}(1)$, and then in n and m using the fact that $b > \frac{1}{2}$, bound this by

$$L^{(d-1)/2}M_1^{-1/2}\|f_0\|_{L_{\xi,\tau}^2} \sum_{n\in\mathbb{Z}}\langle n\rangle^{-b}\|f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1)\|_{L_{\xi_1,\theta_1}^2 (\theta_1=\mathcal{O}(1))}$$

$$\times \sum_{m\in\mathbb{Z}}\langle m\rangle^{-b}\|f_2^m(\xi_2, \mp|\xi_2| + m + \theta_2)\|_{L_{\xi_2,\theta_2}^2 (\theta_2=\mathcal{O}(1))}$$

$$\lesssim L^{(d-1)/2}M_1^{-1/2}\prod_{i=0}^2 \|f_i\|_{L_{\xi,\tau}^2}$$

So in this case, the left-hand side of (4.20) is bounded by

$$\langle M_0\rangle^{s+\alpha}\langle M_1\rangle^{-s-r-2(1-b)-\frac{1}{2}}L^{\frac{d-1}{2}} \approx \langle M_0\rangle^{s+\alpha}M_0^{\frac{d-1}{2}}\langle M_1\rangle^{-s-r-2(1-b)-\frac{1}{2}}.$$

This is dyadically summable if $\alpha < r + 2 - \frac{d}{2}$ for $b - \frac{1}{2} > 0$ sufficiently small.

Case 1.2. $M_1 \ll M_0 \approx M_2$ and $|A| \gtrsim 1$. In this case, we have $M_0 \approx M_2 \gg 1$. Thus $M \gtrsim ||\xi_0|^2 - |\xi_1|^2 \mp |\xi_2|| \approx M_0^2$.

If $M_1 \gg 1$, we proceed just as in the previous case. (The restriction $M_1 \gg 1$ is necessary to ensure that the Jacobian is nonzero.) Break the shells $\{\xi_0 : |\xi_0| \approx M_0\}$ and $\{\xi_2 : |\xi_2| \approx M_2\}$ into squares (or hypercubes) of side length $L \approx M_1$ and change variables as in Case 1.1. This results in the bound

$$\langle M_0 \rangle^{s+\alpha-r-2(1-b)} \langle M_1 \rangle^{-s} M_1^{-\frac{1}{2}} L^{\frac{d-1}{2}} \approx \langle M_0 \rangle^{s+\alpha-r-2(1-b)} \langle M_1 \rangle^{-s+\frac{d-2}{2}}$$

for the left-hand side of (4.20), which is dyadically summable $s + \alpha < r + 1 + \min\{0, s - \frac{d-2}{2}\}$, i.e. when $\alpha < \min\{r - s + 1, r - \frac{d-4}{2}\}$, for $b - \frac{1}{2}$ sufficiently small.

When $M_1 \lesssim 1$, we can use an argument similar to that in Case 0. Notice that

$$\iiiint_{\substack{\sum \xi_i = 0 \\ \sum \tau_i = 0}} \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 \pm |\xi_2| \rangle^b} \, d\xi_1 \, d\xi_2 d\tau_1 \, d\tau_2$$

$$\lesssim \|f_0\|_{L^2_{\xi,\tau}} \left\| \iint \frac{f_1(\xi_1, \tau_1) f_2(-\xi_0 - \xi_1, -\tau_0 - \tau_1)}{\langle \tau_1 + |\xi_1|^2 \rangle^b \langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^b} \, d\xi_1 \, d\tau_1 \right\|_{L^2_{\xi_0, \tau_0}(|\xi_0| \approx M_0)}.$$

Using Cauchy-Schwartz in $d\xi_1 \, d\tau_1$ and then in $d\xi_0 \, d\tau_0$, the last norm above is bounded by

$$\left\| \iint f_1^2(\xi_1, \tau_1) f_2^2(-\xi_0 - \xi_1, -\tau_0 - \tau_1) \, d\xi_1 \, d\tau_1 \iint_{|\xi_1| \approx M_1} \frac{\langle \tau_1 + |\xi_1|^2 \rangle^{-2b}}{\langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{2b}} \, d\xi_1 \, d\tau_1 \right\|_{L^1_{\xi_0, \tau_0}(|\xi_0| \approx M_0)}^{1/2}$$

$$\lesssim \left(\sup_{\substack{|\xi_0| \approx M_0 \\ \tau_0}} \iint_{|\xi_1| \approx M_1} \frac{\langle \tau_1 + |\xi_1|^2 \rangle^{-2b}}{\langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{2b}} \, d\tau_1 \, d\xi_1 \right)^{1/2} \|f_1^2(-\cdot, -\cdot) * f_2^2(\cdot, \cdot)\|_{L^1_{\xi_0, \tau_0}}^{1/2}$$

$$\lesssim M_1^{d/2} \|f_1^2\|_{L^1}^{1/2} \|f_2^2\|_{L^1}^{1/2} = M_1^{d/2} \|f_1\|_{L^2} \|f_2\|_{L^2}.$$

The rough bound on the supremum is obtained as follows. Using Lemma 2.3.4 a) gives

$$\sup_{\substack{|\xi_0| \approx M_0 \\ \tau_0}} \iint_{|\xi_1| \approx M_1} \langle \tau_1 + |\xi_1|^2 \rangle^{-2b} \langle -\tau_0 - \tau_1 \pm |\xi_0 + \xi_1| \rangle^{-2b} \, d\tau_1 \, d\xi_1$$

$$\lesssim \sup_{\xi_0, \tau_0} \iint_{|\xi_1| \approx M_1} \langle |\xi_1|^2 \pm |\xi_0 + \xi_1| - \tau_0 \rangle^{-2b} \, d\xi_1 \lesssim M_1^d.$$

In this case, we thus bound the left-hand side of (4.20) by

$$\langle M_0 \rangle^{s+\alpha-r-2(1-b)} \langle M_1 \rangle^{-s} M_1^{\frac{d}{2}},$$

which is summable when $s + \alpha < r + 1$ for $M_1 \lesssim 1$ as long as $b - \frac{1}{2}$ is sufficiently small.

Case 1.3. $\underline{M_2 \ll M_0 \approx M_1}$ and $|A| \gtrsim 1$. The same procedure as in Case 1.1 works here, with the simplification that no decomposition of the integration regions into squares is required. (The decomposition served to ensure that the projection on the integration region in ξ_2 onto any axis had measure at most $\min\{M_0, M_1, M_2\}$, which is automatically true when $M_2 = \min\{M_0, M_1, M_2\}$). Repeating the change of variables and ensuing argument gives

$$\iiiint_{\substack{\sum \xi_i = 0 \\ \sum \tau_i = 0}} \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 \pm \langle \xi_2 \rangle \rangle^b} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \lesssim M_2^{\frac{d-1}{2}} M_0^{-\frac{1}{2}} \prod_{i=0}^{2} \|f_i\|_{L^2_{x,t}}.$$

Thus the left-hand size of (4.20) can be estimated by

$$\langle M_0 \rangle^{\alpha - (1-b) - \frac{1}{2}} \langle M_2 \rangle^{-r} M_2^{\frac{d-1}{2} - (1-b)}.$$

For $M_2 \lesssim 1$, this sums as long as $\alpha < 1$. When $M_2 \gg 1$, the product is summable when $\alpha < 1 + \min\{0, r - \frac{d-2}{2}\}$, i.e. when $\alpha < \min\{1, r - \frac{d-4}{4}\}$.

Case 1.4. Resonance. $\underline{|A| \ll 1}$. Recall that $A = \cos \alpha + \frac{|\xi_2| \mp 1}{2|\xi_1|}$, so when A is small, $|\xi_2| \lesssim |\xi_1|$. Thus we assume that $|\xi_1| \gg 1$, since otherwise all $|\xi_i| \lesssim 1$. That region was addressed in Case 0.

Decompose parabolically as in Case 1.1 and take another dyadic decomposition about the resonant surface: assume $|A| \approx \nu \ll 1$ dyadic. Then we need to control

$$\sum_{\nu \ll 1} \frac{1}{\nu^{1-b}} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle n \rangle^{-b} \langle m \rangle^{-b} \iiiint_{\substack{\theta_i = \mathcal{O}(1) \\ |A| \approx \nu}} f_0(-\xi_1 - \xi_2, |\xi_1|^2 \pm |\xi_2| - n - m - \theta_1 - \theta_2)$$

$$\times f_1^n(\xi_1, -|\xi_1|^2 + n + \theta_1) f_2^m(\xi_2, \mp |\xi_2| + m + \theta_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2.$$

(4.22)

To visualize the region of integration, consider a fixed ξ_1. The resonant surface $A = 0$ in ξ_2-space is then a slightly distorted version of a hypersphere of radius $|\xi_1|$ centered at $-\xi_1$. This sphere has equation $|\xi_2|^2 + 2|\xi_1||\xi_2| \cos \alpha = 0$, while the actual resonant surface satisfies $|\xi_2|^2 + 2|\xi_1||\xi_2| \cos \alpha \mp$

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Figure 4.1: Integration region in ξ_2

|$\square$ Resonant curve $\blacksquare$ Integration region|
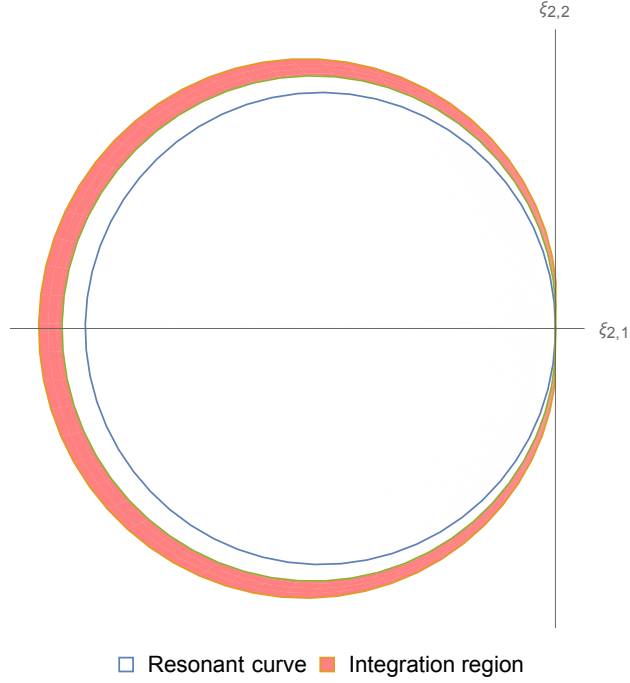
$|\xi_2| = 0$. The region of integration in ξ_2 is a shell centered on this curve, with thickness $\lesssim \nu M_1$. This holds since for a fixed ξ_1 and a fixed angle α,

$$A \in [\nu, 2\nu] \quad \Rightarrow \quad |\xi_2| \in \left[2|\xi_1|(\nu - \cos\alpha) \pm 1, 2|\xi_1|(2\nu - \cos\alpha) \pm 1\right],$$

an interval of length $2\nu|\xi_1|$. See Figure 4.1 for a plot of this region in $\mathbb{R}^2$ for $\xi_1 \in \mathbb{R}^+$.

Decompose the annulus $\{|\xi_1| \approx M_1\}$ into two parts – a set B where $|\xi_{1,i}| \approx M_1$ for each i, and its complement. In two dimensions, this decomposition can be described explicitly by taking

$$B = \left\{\xi_1 \ : \ |\xi_1| \approx M_1, \ \arg(\xi_1) \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right) \cup \left[\frac{5\pi}{8}, \frac{7\pi}{8}\right) \cup \left[\frac{9\pi}{8}, \frac{11\pi}{8}\right) \cup \left[\frac{13\pi}{8}, \frac{15\pi}{8}\right)\right\}.$$

Notice that the complement of B is simply a rotation of B about the origin. In higher dimensions, the set B is similar – if we describe the space in hyperspherical coordinates, we require all $d - 1$ angular variables to be bounded away from multiples of $\pi/2$ – specifically to fall in the intervals $\left[\frac{n\pi}{8}, \frac{(n+2)\pi}{8}\right)$ given above. The complement of B then consists of $2^{d-1} - 1$ copies of B, each of which can be obtained from B by a sequence of $\pi/4$ radian rotations.

The remainder of this calculation will consider the two-dimensional case. There is no fundamental difference in higher dimensions; only much more onerous notation. We perform a rotation so that (4.22) can be written as a sum of two integrals over B. In the following R_y denotes a rotation by y radians.

$$\sum_{k=0}^{1}\sum_{\nu\ll 1}\frac{1}{\nu^{1-b}}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\langle n\rangle^{-b}\langle m\rangle^{-b}\iiiint_{\substack{\theta_i=\mathcal{O}(1)\\|A|\approx\nu\\\xi_1\in B}}f_0(R_{\frac{k\pi}{4}}(-\xi_1-\xi_2),|\xi_1|^2\pm|\xi_2|-n-m-\theta_1-\theta_2)$$

$$\times f_1^n(R_{\frac{k\pi}{4}}(\xi_1),-|\xi_1|^2+n+\theta_1)f_2^m(R_{\frac{n\pi}{4}}(\xi_2),\mp|\xi_2|+m+\theta_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2.$$

Now break the $\mathrm{d}\xi_1\,\mathrm{d}\xi_2$ integration into two additional cases: one where for fixed ξ_1 and $\xi_{2,1}$, the projection of the integration region onto the $\xi_{2,2}$ axis is length $\lesssim \nu M_1$, and one where for fixed ξ_1 and $\xi_{2,2}$, the projection onto the $\xi_{2,1}$ axis is length $\lesssim \nu M_1$. Once again use the change of variables from Case 1.1: set $u=-\xi_1-\xi_2$ and $v=|\xi_1|^2\pm|\xi_2|-n-m-\theta_1-\theta_2$. In the first region, when the projection onto the $\xi_{2,2}$ axis is small, change variables to replace $\mathrm{d}\xi_1\,\mathrm{d}\xi_2$ with $\mathrm{d}\xi_{2,2}\,\mathrm{d}u\,\mathrm{d}v$. When the projection onto the $\xi_{2,1}$ axis is small, use $\mathrm{d}\xi_{2,1}\,\mathrm{d}u\,\mathrm{d}v$.

Following exactly the same steps as in Case 1.1, we bound the (4.22) by

$$\sum_{\nu\ll 1}\frac{1}{\nu^{1-b}}(\nu M_1)^{\frac{1}{2}}/M_1^{\frac{1}{2}}\lesssim 1,$$

using the fact that $b>\frac{1}{2}$. In general dimensions, the bound is $M_2^{\frac{d-2}{2}}$. Thus the quantity to be dyadically summed is

$$\langle M_0\rangle^{s+\alpha}\langle M_1\rangle^{-s}\langle M_2\rangle^{-r}M_1^{b-1}M_2^{b-1}M_2^{\frac{d-2}{2}}.$$

When A is small, there are only two possibilities: either $M_0\approx M_1\approx M_2\gg 1$ or $M_2\ll M_0\approx M_1$. In the first case, we must sum $\langle M_0\rangle^{\alpha-r-2(1-b)+\frac{d-2}{2}}$, which is possible when $\alpha<r+2-\frac{d}{2}=r-\frac{d-4}{2}$.

In the second case, when $M_2\gtrsim 1$ we get $\langle M_0\rangle^{\alpha-(1-b)}\langle M_2\rangle^{-r+\frac{d-2}{2}-(1-b)}$, which sums as long as $\alpha<\frac{1}{2}+\min\{0,r-\frac{d-3}{2}\}=\min\{\frac{1}{2},r-\frac{d-4}{2}\}$ for $b-\frac{1}{2}$ sufficiently small. When $M_2\ll 1$, we estimate the maximum modulation multiplier by one instead of AM_1M_2, and use the argument in Case 1.3 (merely drop the $M_i^{-(1-b)}$ factors) to get convergence when $\alpha<\frac{1}{2}$.

CASE 2. $M=|\tau_1+|\xi_1|^2|.$

Case 2.1. $M_0 \ll M_1 \approx M_2$ and $|A| \gtrsim 1$. When $M_0 \lesssim 1$, the supremum argument in Case 1.2 applies. It gives the multiplier $\langle M_0 \rangle^{s+\alpha} \langle M_1 \rangle^{-s-r-2(1-b)} M_0^{\frac{d}{2}}$, which sums when $s + r > -1$, a condition which is always met when $s, r > -\frac{1}{2}$. When $M_0 \gg 1$, decompose the M_1 and M_2 annuli into squares of scale M_0 and use a change of variables just as in Case 1.1 to get the multiplier $\langle M_0 \rangle^{s+\alpha+\frac{d-2}{2}} \langle M_1 \rangle^{-s-r-2(1-b)}$. This sums when $\max\{s + \alpha + \frac{d-2}{2}, 0\} < s + r + 1$, which holds when $s, r > -\frac{1}{2}$ and $\alpha < r - \frac{d-4}{2}$.

Case 2.2. $M_1 \lesssim M_0 \approx M_2$ and $|A| \gtrsim 1$. Apply the argument in Case 1.1 to obtain the multiplier $\langle M_0 \rangle^{s+\alpha-r-2(1-b)-\frac{1}{2}} \langle M_1 \rangle^{-s} M_1^{\frac{d-1}{2}}$. When $M_1 \lesssim 1$, this sums if $\alpha < r - s + \frac{3}{2}$. If $M_1 \gg 1$, we need $s + \alpha < \min\{s - \frac{d-1}{2}, 0\} + r + \frac{3}{2}$, i.e. $\alpha < \min\{r - s + \frac{3}{2}, r - \frac{d-4}{2}\}$.

Case 2.3. $M_2 \ll M_0 \approx M_1$ and $|A| \gtrsim 1$. Proceed as in Case 1.3 to obtain the multiplier

$$\langle M_0 \rangle^{\alpha-(1-b)-\frac{1}{2}} \langle M_2 \rangle^{-r} M_2^{\frac{d-1}{2}-(1-b)}.$$

When $M_2 \lesssim 1$, this sums as long as $\alpha < 1$. When $M_2 \gg 1$, we require $\alpha < 1 + \min\{r - \frac{d-2}{2}, 0\} = \min\{1, r - \frac{d-4}{2}\}$.

Case 2.4. Resonance. $|A| \ll 1$. The procedure is the same as that in Case 1.4, merely exchanging the roles of (ξ_1, τ_1) and (ξ_2, τ_2), and yields the same constraints.

CASE 3. $M = |\tau_2 \pm \langle \xi_2 \rangle|$. Here we must control

$$\langle M_0 \rangle^{s+\alpha} \langle M_1 \rangle^{-s} \langle M_2 \rangle^{-r} M_1^{-(1-b)} M_2^{-(1-b)} \iiiint_{\substack{\sum \xi_i = 0 \\ \sum \tau_i = 0}} \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{|A|^{1-b} \langle \tau_0 - |\xi_0|^2 \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^b} \, d\xi_1 \, d\xi_2 \, d\tau_1 \, d\tau_2.$$

For the 2d case, the estimates in [26] give

$$\iiiint_{\substack{\sum \xi_i = 0 \\ \sum \tau_i = 0}} \frac{f_0(\xi_0, \tau_0) f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2)}{\langle \tau_0 - |\xi_0|^2 \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^b} \, d\xi_1 \, d\xi_2 \, d\tau_1 \, d\tau_2 \lesssim \left(\frac{\min\{M_0, M_1\}}{\max\{M_0, M_1\}} \right)^{1/2} \prod_{i=0}^{2} \|f_i\|_{L^2_{x,t}}.$$

However, for some cases the argument relies on the $L^4 L^4$ Strichartz estimate, which does not hold for $d \neq 2$.

Case 3.1. $M_0 \lesssim M_1 \approx M_2$ and $|A| \gtrsim 1$. In this case, the arguments from [26] can be applied directly to give a multiplier of $M_0^{\frac{d-1}{2}} M_1^{-\frac{1}{2}}$. This means that we must dyadically sum

$$\langle M_0 \rangle^{s+\alpha} M_0^{\frac{d-1}{2}} \langle M_1 \rangle^{-s-r-2(1-b)-\frac{1}{2}}.$$

When $M_0 \lesssim 1$, this sums as long as $s + r > -\frac{3}{2}$, a condition which is certainly met for $s, r > -\frac{1}{2}$. When $M_0 \gg 1$, we need $\alpha < r - \frac{d-4}{2}$.

Case 3.2. $M_1 \ll M_0 \approx M_2$ and $|A| \gtrsim 1$. Here again the results from [26] can be applied. Doing so yields $\langle M_0 \rangle^{s+\alpha-r-2(1-b)-\frac{1}{2}} \langle M_1 \rangle^{-s} M_1^{\frac{d-1}{2}}$. When $M_1 \lesssim 1$, we require $\alpha < r - s + \frac{3}{2}$. When $M_1 \gg 1$, we need $s+\alpha-r-2(1-b)-\frac{1}{2}+\max\{\frac{d-1}{2}-s, 0\} < 0$, which holds when $\alpha < r-s+\frac{3}{2}$ and $\alpha < r-\frac{d-4}{2}$.

Case 3.3. $M_2 \ll M_0 \approx M_1$ and $|A| \gtrsim 1$. When $M_2 \lesssim 1$, use the supremum argument which appears in Case 1.2. This yields $\langle M_0 \rangle^{s+\alpha-s-(1-b)} \langle M_2 \rangle^{-r} M_2^{-(1-b)} M_2^{\frac{d-1}{2}}$. This sums when $\alpha < \frac{1}{2}$ for $b - \frac{1}{2} > 0$ sufficiently small. When $M_2 \gg 1$, decompose the M_0 and M_1 annuli into squares of scale M_2 and change variables. Unlike the previous cases though, the change of variables here gives a Jacobian of order M_2 (see [26] for details). Thus we arrive at $\langle M_0 \rangle^{s+\alpha-s-(1-b)} \langle M_2 \rangle^{-r+\frac{d-2}{2}-(1-b)}$. To sum this dyadically, we need $\alpha < \frac{1}{2} + \min\{0, r - \frac{d-3}{2}\} = \min\{\frac{1}{2}, r - \frac{d-4}{2}\}$.

Case 3.4. Resonance. $|A| \ll 1$. When $1 \ll M_2 \lesssim M_0 \approx M_1$, proceed as in Case 1.4. Dyadically decompose f_0 and f_1 and then change variables by letting $u = -\xi_0 - \xi_1$ and $v = |\xi_1|^2 - |\xi_0|^2 - n - m - \theta_1 - \theta_2$. This leads to a Jacobian of

$$\mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\xi_{0,1} = |\xi_{0,2} + \xi_{1,2}| \, \mathrm{d}\xi_0 \, \mathrm{d}\xi_1 \quad \text{or} \quad \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\xi_{0,2} = |\xi_{0,1} + \xi_{1,1}| \, \mathrm{d}\xi_0 \, \mathrm{d}\xi_1.$$

The result in higher dimensions is similar:

$$\mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\xi_{0,1} \cdots \mathrm{d}\xi_{0,j-1} \, \mathrm{d}\xi_{0,j+1} \cdots \mathrm{d}\xi_{0,d} = |\xi_{2,j}| \, \mathrm{d}\xi_0 \, \mathrm{d}\xi_1.$$

Proceed just as in Case 1.4 to arrive at $\langle M_0 \rangle^{\alpha-(1-b)} \langle M_2 \rangle^{-r-(1-b)+\frac{d-2}{2}}$, which sums when $\alpha < \frac{1}{2} + \min\{0, r - \frac{d-3}{2}\} = \min\{\frac{1}{2}, r - \frac{d-4}{2}\}$.

When $M_2 \lesssim 1$, estimate the maximum modulation by 1 instead of by $M_1^{-(1-b)} M_2^{-(1-b)} A$ and apply the argument from Case 1.3 (again merely removing the $M_i^{-(1-b)}$ factors) to conclude summability when $\alpha < \frac{1}{2}$.

CHAPTER 5

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THE "GOOD" BOUSSINESQ EQUATION ON $\mathbb{R}^+$

5.1 INTRODUCTION

In this chapter, we are concerned with the following initial-boundary value problem on the half line, known as the "good" Boussinesq equation:

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, & x \in \mathbb{R}^+,\ t \in \mathbb{R}^+ \\ u(0,t) = h_1(t), \qquad u_x(0,t) = h_2(t), \\ u(x,0) = f(x), \qquad u_t(x,0) = g_x(x). \end{cases} \tag{5.1}$$

The work is joint with N. Tzirakis [32]. The data (f, g, h_1, h_2) will be taken in the space $H_x^s(\mathbb{R}^+) \times H_x^{s-1}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{4}}(\mathbb{R}^+) \times H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)$ with the additional compatibility conditions $h_1(0) = f(0)$ when $\frac{1}{2} < s_0 \leqslant \frac{3}{2}$ and $h_1(0) = f(0), \quad h_2(0) = f'(0)$ when $\frac{3}{2} < s_0 \leqslant \frac{5}{2}$. These compatibility conditions are necessary since the solutions we are interested in are continuous space-time functions when $s > \frac{1}{2}$.

This equation is known as the "good" Boussinesq, in contrast to that with the opposite sign in front of the fourth derivative, which was derived by Boussinesq [24] as a water wave model. It also appears as a model of a nonlinear string [102]. This original Boussinesq equation is linearly unstable because of exponential growth in Fourier modes. The "good" Boussinesq (5.1) has appeared in studies of shape-memory alloys [41], and has been extensively studied on $\mathbb{R}$ and $\mathbb{T}$. Bona and Sachs showed well-posedness for data $(f, g) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for $s > \frac{5}{2}$ [14]. Linares established well-posedness for data in $L^2(\mathbb{R}) \times H^{-1}(\mathbb{R})$ [72] using Strichartz estimates and the theory which Kenig, Ponce and Vega developed for the KdV equation in [65]. Well-posedness in $H^{-\frac{1}{4}}(\mathbb{R}) \times H^{-\frac{5}{4}}(\mathbb{R})$ was shown in [42], where the restricted norm method of Bourgain ($X^{s,b}$ method) was used. The result in [42] is sharp in the sense that the key bilinear estimate used in the $X^{s,b}$ theory fails for any $s < -\frac{1}{4}$. A simple gauge transformation, [69], reduces the "good" Boussinesq equation into a quadratic nonlinear Schrödinger equation, but it is not clear how one can take advantage

of this transformation on the half-line. Later in [69] and [68], a modification of the restricted norm method of Bourgain was introduced. The well-posedness theory was then improved for both the real line and the torus. In particular, for the real line local well-posedness was established in $H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$. The well-posedness theory at the $H^{-\frac{1}{2}} \times H^{-\frac{3}{2}}$ level is known to be sharp, [68]. Our result is sharp, up to an endpoint, in the sense that we also obtain local well-posedness in $H^{-\frac{1}{4}+}(\mathbb{R}+) \times H^{-\frac{5}{4}+}(\mathbb{R}+)$, noting that it is not obvious how one can modify the $X^{s,b}$ norm and use an appropriate transformation to simplify the equation in the case of the initial-boundary value problem.

In this work we continue the program initiated in [38] of establishing the regularity properties of nonlinear dispersive partial differential equations (PDE) on a half line using the tools that are available in the case of the real line, where the PDE are fully dispersive. To this end, we extend the data into the whole line and use Laplace transform methods to set up an equivalent integral equation (on $\mathbb{R} \times \mathbb{R}$) for the solution; see (5.5) below. We analyze the integral equation using the restricted norm method and multilinear L^2 convolution estimates. To state the main theorem, we start with a definition.

Definition 5.1.1. *We say that the Boussinesq equation* (5.1) *is locally well-posed in* $H^s(\mathbb{R}^+)$ *if for any* $(f, g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_x^{s-1}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{4}}(\mathbb{R}^+) \times H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)$, *with the additional compatibility conditions mentioned above, the equation* $\Phi(u) = u$, *where* Φ *is defined by* (5.5), *has a unique solution in*

$$X_T^{s,b} \cap C_t^0 H_x^s \cap C_x^0 H_t^{\frac{2s+1}{4}},$$

for some $b < \frac{1}{2}$ *and some sufficiently small* T, *dependent only on the norms of the initial and boundary data. Furthermore, the solution depends continuously on the initial and boundary data.*

Our main theorem is below. Note that it extends the result in [54], which established well-posedness for $s > \frac{1}{2}$. In addition we prove that the nonlinear part of the solution is smoother that the initial data. As expected the smoothing disappears at the upper endpoint $s = \frac{5}{2}$ but not on the lower endpoint $s = -\frac{1}{4}$, where one can still gain a quarter of a derivative. We consider this as an indication (along with the smoothing of order $s + \frac{1}{2}$) that the "good" Boussinesq equation should be well-posed in $H^{-\frac{1}{2}}(\mathbb{R}+) \times H^{-\frac{3}{2}}(\mathbb{R}+)$, although a modification of our method will be definitely

needed to overcome the failure of the bilinear estimates below $H^{-\frac{1}{4}}$. The reader can consult [35] for many examples of dispersive PDE that enjoy nonlinear smoothing properties at regularities equal to the regularities of the sharp local well-posedness theory. We finally note that the operator W_0^t is the linear part of the solution of the equation (5.1), see Section 3 below.

Theorem 5.1.2. *For any $s \in \left(-\frac{1}{4}, \frac{5}{2}\right)$, $s \neq \frac{1}{2}, \frac{3}{2}$, the equation (5.1) is locally well-posed in $H^s(\mathbb{R}^+)$. Moreover, we have the following smoothing estimate. For $a < \min\{\frac{1}{2}, s + \frac{1}{2}, \frac{5}{2} - s\}$,*

$$u - W_0^t(f, g, h_1, h_2) \in C_t^0 H_x^{s+a}.$$

In addition, the solutions are independent of the extensions of the initial data.

To prove the above theorems we rely on a Duhamel formulation of the nonlinear system adapted to the boundary conditions. This expresses the nonlinear solution as the superposition of the linear evolutions which incorporate the boundary and the initial data with the nonlinearity. Thus, we first solve a linear problem by a combination of Fourier and Laplace transforms, [38, 17], after extending the initial data to the whole line. The idea is then to use the restricted norm method in the Duhamel formula. The uniqueness of the solutions thus constructed is not immediate since we do not know that the fixed points of the Duhamel operators have restrictions on the half line which are independent of the extension of the data. For the case of more regular data the uniqueness property of the solution is proved in [54]. For less regular data we take advantage of the smoothing estimate we establish in Theorem 5.1.2 to obtain uniqueness all the way down to the local theory threshold $H^{-\frac{1}{4}+}(\mathbb{R}+) \times H^{-\frac{5}{4}+}(\mathbb{R}+)$. We remark that this iteration is successful because the full nonlinear estimate we provide remains valid for any $s > -\frac{1}{4}$, matching thus the regularity of the local theory.

As we have already mentioned our result improves the result in [54]. The initial and boundary value problem (IVBP) for the "good" Boussinesq equation on the half line has also been considered in [99] and [100]. In the first paper the author obtained local well-posedness for any $s > \frac{1}{2}$ (having a different set of boundary data than [54]), while in the second paper the same author obtained local well-posedness for L^2 solutions. As far as we know our work is the first result where well-posed solutions are constructed below the L^2 space for the "good" Boussinesq equation. At this level of

regularity, Strichartz type estimates available on the full line are not useful in the construction of solutions obtained through fixed point theorems.

We now discuss briefly the organization of the chapter. In Section 5.2, we introduce some notation and the function spaces that we use to obtain the well-posedness of the IBVP. In Section 5.3 we define the notion of the solution. More precisely we set up the integral representation (Duhamel's formula) of the nonlinear solution map that we later prove is a contraction in an appropriate metric space. We obtain the solution as a superposition of a linear and a nonlinear evolution. The solution of the linear IBVP can be found by a direct application of the Fourier and the Laplace transform methods. Section 5.4 states the linear and nonlinear a priori estimates that we use to iterate the solution using the restricted norm method appropriately modified for our needs. In Section 5.5 we put all the estimates together and show why the solution map is a contraction thus proving the first part of Theorem 5.1.2. Uniqueness is proved on Section 5.6. Section 5.7 is the main body of the work, where all the estimates, linear and nonlinear, are established. Finally in Section ?? we justify the application of the Laplace transform on the half line and the representation formula for the solution of the linear problem with zero initial data.

5.2 Notation & Function Spaces

We define the one-dimensional Fourier transform by

$$\widehat{f}(\xi) = \mathcal{F}_x f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, \mathrm{d}x.$$

We set $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. The characteristic function on $[0, \infty)$ is denoted by χ. Sobolev spaces $H^s(\mathbb{R}^+)$ on the half-line for $s > -\frac{1}{2}$ are defined by

$$H^s(\mathbb{R}^+) = \Big\{ g \in \mathcal{D}(\mathbb{R}^+) \ : \ \text{there exists } \tilde{g} \in H^s(\mathbb{R}) \text{ with } \tilde{g}\chi = g \Big\},$$

$$\|g\|_{H^s(\mathbb{R}^+)} = \inf\Big\{ \|\tilde{g}\|_{H^s(\mathbb{R})} \ : \ \tilde{g}\chi = g \Big\}.$$

The restriction $s > -\frac{1}{2}$ is needed because multiplication with characteristic functions is not defined for H^s distributions when $s \leqslant -\frac{1}{2}$. We will also use the $X^{s,b}$ spaces corresponding to the Boussinesq flow. These are defined for functions on the full space $\mathbb{R}_x \times \mathbb{R}_t$ by the norm

$$\|u\|_{X^{s,b}} = \left\| \langle \xi \rangle^s \big\langle |\tau| - \sqrt{\xi^2 + \xi^4} \big\rangle^b \widehat{u}(\xi, \tau) \right\|_{L^2_\xi L^2_\tau}.$$

It is helpful to note ([42]) that there exists c such that

$$\frac{1}{c} \leqslant \frac{\langle a - \sqrt{b+b^2} \rangle}{\langle a-b \rangle} \leqslant c \quad \text{for all} \quad a, b \geqslant 0,$$

so the above $X^{s,b}$ norm is equivalent to $\left\| \langle \xi \rangle^s \langle |\tau| - \xi^2 \rangle^b \hat{u}(\xi, \tau) \right\|_{L_\xi^2 L_\tau^2}$.

The solution to the linear problem $w_{tt} - w_{xx} + w_{xxxx} = 0$ on $\mathbb{R}$ with initial data $w(x,0) = f(x)$ and $w_t(x,0) = g_x(x)$ will be denoted by

$$W_R^t \big(f(x), g(x) \big) = W_{R,1}^t f(x) + W_{R,2}^t g_x(x),$$

where $W_{R,1}^t$ and $W_{R,2}^t$ are the Fourier multiplier operators with multipliers $\operatorname{Re} e^{it\sqrt{\xi^2+\xi^4}}$ and $\operatorname{Im} e^{it\sqrt{\xi^2+\xi^4}}(\xi^2 + \xi^4)^{-1/2}$ respectively.

Let $\rho \in C^\infty$ be a cut-off function such that $\rho = 1$ on $[0, \infty)$ and $\operatorname{supp} \rho \subset [-1, \infty)$. Let $\eta \in C^\infty$ be a bump function such that $\eta = 1$ on $[-1, 1]$ and $\operatorname{supp} \eta \subset [-2, 2]$. The notation D_0 represents evaluation at $x = 0$, i.e.

$$D_0 \big[u(x, t) \big] = u(0, t).$$

5.3   STATEMENT OF RESULTS

To obtain solutions of (5.1), we begin by constructing the solution of the linear initial-boundary-value problem:

$$\begin{cases} v_{tt} - v_{xx} + v_{xxxx} = 0 \\ v(0,t) = h_1(t), \qquad v_x(0,t) = h_2(t), \\ v(x,0) = f(x), \qquad v_t(x,0) = g(x), \end{cases} \tag{5.2}$$

with the compatibility condition $h_1(0) = f(0)$ for $\frac{1}{2} < s \leqslant \frac{3}{2}$, and the additional condition $f'(0) = h_2(0)$ for $\frac{3}{2} < s \leqslant \frac{5}{2}$. Denote this solution by $W_0^t(f, g, h_1, h_2)$. For extensions f^e and g^e to the full line $\mathbb{R}$ of the functions f and g, we may write

$$W_0^t(f, g, h_1, h_2) = W_0^t(0, 0, h_1 - p_1, h_2 - p_2) + W_R^t(f^e, g^e),$$

where $p_1(t) = D_0 \big[W_R^t(f^e, g^e) \big]$ and $p_2(t) = D_0 \big[W_R^t(f_e, g_e) \big]_x$. We thus decompose the solution operator as a sum of a modified boundary operator, which incorporates zero initial data, and the

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free propagator defined on the whole real line. For $x > 0$, this solution formula expresses the unique solution of (5.2). Note that $W_0^t(0, 0, h_1, h_2)$ is the solution to the following problem:

$$
\begin{cases}
v_{tt} - v_{xx} + v_{xxxx} = 0 \\
v(0, t) = h_1(t), \qquad v_x(0, t) = h_2(t), \\
v(x, 0) = 0, \qquad\quad v_t(x, 0) = 0.
\end{cases}
\tag{5.3}
$$

We will use the following explicit representation of $W_0^t(0, 0, h_1, h_2)$ extensively. It is proved in Section 5.8 using a Laplace transform argument. Similar expressions have been derived in [99, 100] using the Laplace transform and in [54] using Fokas' unified transform method.

Lemma 5.3.1. *Suppose h_1 and h_2 are Schwarz functions. The solution to (5.3) on $\mathbb{R}^+ \times \mathbb{R}^+$ can be written in the form $v(x, t) = \frac{1}{2\pi}(-A - B + C + D)$, where*

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} \frac{e^{it\omega\sqrt{\omega^2+1}-x\sqrt{\omega^2+1}}}{\sqrt{1+\omega^2}} i\omega\left(i\omega + \sqrt{1+\omega^2}\right) \widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right) \rho\left(x\sqrt{\omega^2+1}\right) d\omega \\
B &= \int_{-\infty}^{\infty} \frac{e^{it\omega\sqrt{\omega^2+1}-x\sqrt{\omega^2+1}}}{\sqrt{1+\omega^2}} \left(i\omega + \sqrt{1+\omega^2}\right) \widehat{h_2}\left(\omega\sqrt{\omega^2+1}\right) \rho\left(x\sqrt{\omega^2+1}\right) d\omega \\
C &= \int_{-\infty}^{\infty} e^{it\omega\sqrt{\omega^2+1}-ix\omega}\left(i\omega + \sqrt{1+\omega^2}\right) \widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right) d\omega \\
D &= \int_{-\infty}^{\infty} \frac{e^{it\omega\sqrt{\omega^2+1}-ix\omega}}{\sqrt{1+\omega^2}} \left(i\omega + \sqrt{1+\omega^2}\right) \widehat{h_2}\left(\omega\sqrt{\omega^2+1}\right) d\omega.
\end{aligned}
\tag{5.4}
$$

Here by an abuse of notation, $\widehat{h_i}$ denotes the Fourier transform of χh_i.

This explicit form will be used to establish bounds on $W_0^t(0, 0, h_1, h_2)$ in the subsequent sections. Notice that the integrals A, B, C, and D are defined on the entire space $\mathbb{R}_x \times \mathbb{R}_t$ thanks to the inclusion of the cut-off function ρ.

It is now clear that the solution to the full initial-boundary-value problem (5.1) satisfies, for $t \leqslant T$, the equation $\Phi(u) = u$, where the operator Φ is given by

$$
\Phi\big(u(x, t)\big) = \eta(t/T) W_R^t\big(f^e(x), g^e(x)\big) + \eta(t/T) \int_0^t W_{R,2}^{t-t'} G(u) \, dt'
$$
$$
+ \eta(t/T) W_0^t(0, 0, h_1 - p_1 - q_1, h_2 - p_2 - q_2),
\tag{5.5}
$$

with

$$G(u) = \eta(t/T)(u^2)_{xx}, \tag{5.6}$$

$$p_1(t) = \eta(t/T)D_0\Big[W_R^t\big(f^e(x), g_x^e(x)\big)\Big], \quad q_1(t) = \eta(t/T)D_0\Big[\int_0^t W_{R,2}^{t-t'}G(u)\,\mathrm{d}t'\Big],$$

$$p_2(t) = \eta(t/T)D_0\Big[W_R^t\big(f^e(x), g_x^e(x)\big)\Big]_x, \quad q_2(t) = \eta(t/T)D_0\Big[\int_0^t W_{R,2}^{t-t'}G(u)\,\mathrm{d}t'\Big]_x. \tag{5.7}$$

In the following, we will use a fixed point argument to obtain a unique solution to $\Phi(u) = u$ in a suitable function space on $\mathbb{R} \times \mathbb{R}$ for sufficiently small T. The restriction of u to $\mathbb{R}^+ \times \mathbb{R}$ is a distributional solution of (5.1). Furthermore, smooth solutions of $\Phi(u) = u$ are classical solutions of (5.1).

The contraction argument is carried out in $X^{s,b}$ spaces. To bound the solution to the linear Boussinesq on $\mathbb{R}$ and the Duhamel term, we will use the following estimates from [42]. For any s and b, we have

$$\|\eta(t)W_R^t\big(f, g\big)\|_{X^{s,b}} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}. \tag{5.8}$$

Furthermore, for any $-\frac{1}{2} < b' \leqslant 0 \leqslant b \leqslant b' + 1$ and $0 < T < 1$, the estimate

$$\Big\|\eta(t/T)\int_0^t W_{R,2}^{t-t'}G(u)\,\mathrm{d}t'\Big\|_{X^{s,b}} \lesssim T^{1-(b-b')}\Big\|\mathcal{M}(G(u))\Big\|_{X^{s,b'}} \tag{5.9}$$

holds, where $\mathcal{M}$ is the Fourier multiplier operator defined by $\widehat{\mathcal{M}(f)} = (\xi^2 + \xi^4)^{-1/2}\widehat{f}$. for any $-\frac{1}{2} < b_1 < b_2 < \frac{1}{2}$ [35]. Finally, we require the following lemma regarding extensions of $H^s(\mathbb{R}^+)$ functions. It will be used to bound the explicit linear solution given in Lemma 5.3.1, which is given in terms of the Fourier transforms of χh_i.

Lemma 5.3.2. *[38] Assume $h \in H^s(\mathbb{R}^+)$.*

1. *If $-\frac{1}{2} < s < \frac{1}{2}$, then $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.*

2. *If $\frac{1}{2} < s < \frac{3}{2}$ and $h(0) = 0$, then $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.*

5.4   A Priori Estimates

To close the contraction argument, we need a number of estimates on the terms in (5.5).

5.4.1 Linear Estimates

First, we give a Kato smoothing inequality, which is proved in Section 5.7.1. Similar results are stated in [99]. This estimate is necessary to ensure that $\Phi(u)$ lies in $L_x^\infty H_t^{\frac{2s+1}{4}}$ and to control the terms p_i defined in (5.7).

Lemma 5.4.1. *For any s,*

$$\|\eta(t)W_R^t(f,g)\|_{L_x^\infty H_t^{\frac{2s+1}{4}}} \lesssim \|f\|_{H_x^s} + \|g\|_{H_x^{s-1}}$$

$$\|\eta(t)[W_R^t(f,g)]_x\|_{L_x^\infty H_t^{\frac{2s-1}{4}}} \lesssim \|f\|_{H_x^s} + \|g\|_{H_x^{s-1}}.$$

For the solution to the linear initial-boundary-value problem we have the following estimates, which are proved in Sections 5.7.2 and 5.7.3. These are used to bound the W_0^t term in $\Phi(u)$, and to ensure that this term lies in the desired space $C_t^0 H_x^s \cap C_x^0 H_t^{\frac{2s+1}{4}}$.

Lemma 5.4.2. *For any compactly supported smooth function η and any $s \geqslant -\frac{1}{2}$ with $b < \frac{1}{2}$,*

$$\|\eta(t)W_0^t(0,0,h_1,h_2)\|_{X^{s,b}} \lesssim \|\chi h_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} + \|\chi h_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}.$$

Lemma 5.4.3. *For any $s \geqslant -1$ and intial data (h_1, h_2) such that $(\chi h_1, \chi h_2) \in H^{\frac{2s+1}{4}}(\mathbb{R}) \times H^{\frac{2s-1}{4}}(\mathbb{R})$, we have*

$$W_0^t(0,0,h_1,h_2) \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$$

$$\eta(t)W_0^t(0,0,h_1,h_2) \in C_x^0 H_t^{\frac{2s+1}{4}}(\mathbb{R} \times \mathbb{R}).$$

5.4.2 Nonlinear Estimates

Lemma 5.4.4. *Let $\mathcal{M}$ be the Fourier multiplier operator with multiplier $(\xi^2 + \xi^4)^{-1/2}$. For $s > -\frac{1}{4}$ with $a < \min\{\frac{1}{2}, s + \frac{1}{2}\}$ and $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\|\mathcal{M}(uv)_{xx}\|_{X^{s+a,-b}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

This lemma is proved in Section 5.7.4. The next requirement is to control the Duhamel part of the correction term. This is accomplished with the following estimate, which is proved in Section 5.7.5.

Lemma 5.4.5. *For $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\left\| \eta(t) \int_0^t W_{R,2}^{t-t'} G \, \mathrm{d}t' \right\|_{L_x^\infty H_t^{\frac{2s+1}{4}}} + \left\| \eta(t) \left[\int_0^t W_{R,2}^{t-t'} G \, \mathrm{d}t' \right]_x \right\|_{L_x^\infty H_t^{\frac{2s-1}{4}}}$$

$$\lesssim \begin{cases} \|\mathcal{M}(G)\|_{X^{s,-b}} + \left\| \langle \tau \rangle^{\frac{2s-1}{4}} \int \chi_Q(\xi,\tau) \langle \xi \rangle^{-1} |\widehat{\mathcal{M}(G)}(\xi,\tau)| \, \mathrm{d}\xi \right\|_{L_\tau^2} & \text{if } -\frac{1}{2} \leqslant s \leqslant \frac{1}{2} \\[3mm] \|\mathcal{M}(G)\|_{X^{s,-b}} + \left\| \int \chi_R(\xi,\tau) \langle |\tau| - \xi^2 \rangle^{\frac{2s-3}{4}} |\widehat{\mathcal{M}(G)}(\xi,\tau)| \, \mathrm{d}\xi \right\|_{L_\tau^2} & \text{if } s > \frac{1}{2}, \end{cases}$$

where $Q = \{|\tau| \ll \xi^2\} \cap \{|\xi| \gtrsim 1\}$ and $R = \{|\tau| \gg \xi^2\} \cup \{|\xi| \lesssim 1\}$.

It remains to bound the left hand side of the inequality in Lemma 5.4.5. We use Lemma 5.4.4 to control the $X^{s,b}$ norms; the other terms are bounded using the following lemmata, which are proved in Sections 5.7.6 and 5.7.7 respectively.

Lemma 5.4.6. *Let Q be the set $\{|\tau| \ll \xi^2\} \cap \{|\xi| \gtrsim 1\}$. For $-\frac{1}{4} < s + a \leqslant \frac{1}{2}$ and $0 \leqslant a < s + \frac{1}{2}$, we have*

$$\left\| \langle \tau \rangle^{\frac{2(s+a)-1}{4}} \int \chi_Q(\xi,\tau) \langle \xi \rangle^{-1} \frac{\xi^2}{\sqrt{\xi^2 + \xi^4}} |\widehat{uv}(\xi,\tau)| \, \mathrm{d}\xi \right\|_{L_\tau^2} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}.$$

Lemma 5.4.7. *Let R be the set $\{|\tau| \gg \xi^2\} \cup \{|\xi| \lesssim 1\}$. For $\frac{1}{2} < s + a \leqslant \frac{5}{2}$ and $a < \min\{1, s + \frac{1}{2}\}$, we have*

$$\left\| \int \chi_R(\xi,\tau) \langle |\tau| - \xi^2 \rangle^{\frac{2(s+a)-3}{4}} \frac{\xi^2}{\sqrt{\xi^2 + \xi^4}} |\widehat{uv}(\xi,\tau)| \, \mathrm{d}\xi \right\|_{L_\tau^2} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}.$$

5.5 LOCAL THEORY: PROOF OF THEOREM 5.1.2

We will first show that the map Φ defined in (5.5) has a unique fixed point in $X^{s,b}$. Let $f^e \in H^s(\mathbb{R})$ and $g^e \in H^{s-1}(\mathbb{R})$ be extensions of f and g such that $\|f^e\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R}^+)}$ and $\|g^e\|_{H^{s-1}(\mathbb{R})} \lesssim \|g\|_{H^{s-1}(\mathbb{R}^+)}$. Recall that

$$\Phi\big(u(x,t)\big) = \eta(t/T) W_R^t \big(f^e(x), g^e(x)\big) + \eta(t/T) \int_0^t W_{R,2}^{t-t'} G(u) \, \mathrm{d}t' \tag{5.10}$$
$$+ \eta(t/T) W_0^t \big(0, 0, h_1 - p_1 - q_1, h_2 - p_2 - q_2\big),$$

where $G(u)$, p_i, and q_i are defined in (5.6)-(5.7). To bound the first summand in Φ, apply (5.8) to obtain

$$\big\| \eta(t/T) W_R^t \big(f^e, g^e\big) \big\|_{X^{s,b}} \lesssim \|f^e\|_{H^s(\mathbb{R})} + \|g^e\|_{H^{s-1}(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R}^+)} + \|g\|_{H^{s-1}(\mathbb{R}^+)}.$$

For the Duhamel term, we apply (5.9) and Lemma 5.4.4 to obtain

$$\left\| \eta(t/T) \int_0^t W_{R,2}^{t-t'} G(u) \, \mathrm{d}t' \right\|_{X^{s,b}} \lesssim T^{1-2b} \|\mathcal{M}(u^2)_{xx}\|_{X^{s,-b}} \lesssim T^{1-2b} \|u\|_{X^{s,b}}^2.$$

Finally, for the W_0^t term, we apply Lemma 5.4.2 and Lemma 5.3.2 to obtain

$$\left\| \eta(t/T) W_0^t\big(0, 0, h_1 - p_1 - q_1, h_2 - p_2 - q_2\big) \right\|_{X^{s,b}}$$

$$\lesssim \|\chi(h_1 - p_1 - q_1)\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} + \|\chi(h_2 - p_2 - q_2)\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})}$$

$$\lesssim \|h_1 - p_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|q_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|h_2 - p_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)} + \|q_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)}.$$

By Kato smoothing, Lemma 5.4.1, we have

$$\|p_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} + \|p_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})} \lesssim \|f^e\|_{H^s(\mathbb{R})} + \|g^e\|_{H^{s-1}(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R}^+)} + \|g\|_{H^{s-1}(\mathbb{R}^+)}.$$

To bound the q_i norms, we apply Lemma 5.4.5, Lemma 2.2.4, and Lemma 5.4.4, Lemma 5.4.6, and Lemma 5.4.7 to obtain the bounds

$$\|q_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} + \|q_2\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} \lesssim T^{\frac{1}{2}-b-} \|u\|_{X^{s,b}}^2.$$

Combining these estimates, we find that

$$\|\Phi(u)\|_{X^{s,b}} \lesssim \|f\|_{H^s(\mathbb{R}^+)} + \|g\|_{H^{s-1}(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|u\|_{X^{s,b}}^2.$$

This, together with similar estimates for the difference $\Phi(u) - \Phi(v)$, yields the existence of a fixed point of Φ for T sufficiently small:

$$T = T\big(\|f\|_{H^s(\mathbb{R}^+)}, \|g\|_{H^{s-1}(\mathbb{R}^+)}, \|h_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)}, \|h_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)}\big).$$

Next, we establish continuity in H^s. For the W_0^t term, this follows from Lemma 5.4.3. The first term of Φ, the linear flow on $\mathbb{R}$, can be seen to be continuous from its Fourier multiplier formula. Continuity of the Duhamel term follows from the embedding $X^{s,b} \subset C_t^0 H_x^s$ for $b > \frac{1}{2}$ along with (5.9) and Lemma 5.4.4. The fact that the solution lies in $C_x^0 H_t^{\frac{2s+1}{4}}$ follows from Lemma 5.4.3 for the W_0^t term, from Kato smoothing (Lemma 5.4.1) for the linear flow on $\mathbb{R}$, and from Lemmata 5.4.5-5.4.7 for the Duhamel term.

5.6   Uniqueness of Solutions

In this section, we show that solutions to (5.1) derived in the previous section are unique. For $s > \frac{1}{2}$, uniqueness of $C^0([0,T], H^s(\mathbb{R}^+))$ solutions to (5.1) holds by [54]. The solutions obtained in the previous section also lie in this space after restriction to $x \in \mathbb{R}^+$. Thus we have uniqueness for $s > \frac{1}{2}$.

Using the smoothing estimates in Theorem 5.1.2, we can now obtain uniqueness of local solutions for the full range of Sobolev exponents in the local theory. First consider initial data $(f, g, h_1, h_2) \in H_x^s(\mathbb{R}^+) \times H_x^{s-1}(\mathbb{R}^+) \times H_t^{\frac{2s+1}{4}}(\mathbb{R}^+) \times H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)$ for some $s \in \left(0, \frac{1}{2}\right)$. Suppose f^e and $\widetilde{f}^e$ are two $H^s(\mathbb{R})$ extensions of f, and g^e and $\widetilde{g}^e$ are two $H^{s-1}(\mathbb{R})$ extensions of g. Let u and $\widetilde{u}$ be the corresponding solutions of the fixed-point equation for Φ. Take a sequence $f_k \in H^{\frac{1}{2}+}(\mathbb{R}^+)$ converging to f in $H^s(\mathbb{R}^+)$. Let f_k^e and $\widetilde{f}_k^e$ be $H^{\frac{1}{2}+}(\mathbb{R})$ extensions of f_k which converge to f^e and $\widetilde{f}^e$ respectively in $H^r(\mathbb{R})$ for $r < \frac{1}{2}-$. This is possible by Lemma 5.6.1 below. Similarly, obtain a sequence g_k in $H^{-\frac{1}{2}+}(\mathbb{R}^+)$ converging to g in $H^{s-1}(\mathbb{R}^+)$, and extensions g_k^e and $\widetilde{g}_k^e$ of g_k converging to g^e and $\widetilde{g}^e$ respectively.

Using a contraction argument on the set

$$\left\{ u \ : \ \|u\|_{X^{\frac{1}{2}+,b}} \leqslant C\left(\|f_k\|_{H_x^{1/2+}(\mathbb{R}^+)} + \|g_k\|_{H_x^{-1/2+}(\mathbb{R}^+)} + \|h_1\|_{H_t^{1/2+}(\mathbb{R}^+)} + \|h_2\|_{H_t^{0+}(\mathbb{R}^+)}\right) \right\}$$
$$\cap \left\{ u \ : \ \|u\|_{X^{s,b}} \leqslant C\left(\|f\|_{H_x^s(\mathbb{R}^+)} + \|g\|_{H_x^{s-1}(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|h_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)}\right) \right\},$$

we construct $H^{\frac{1}{2}+}$ solutions of u_k and $\widetilde{u}_k$ of the Boussinesq (5.1) using the extensions (f_k^e, g_k^e) and $(\widetilde{f}_k^e, \widetilde{g}_k^e)$ respectively. The smoothing estimates give us a time of existence proportional to the data in the lower norm: $T = T\left(\|f\|_{H^s(\mathbb{R}^+)}, \|g\|_{H^{s-1}(\mathbb{R}^+)}, \|h_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R}^+)}, \|h_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R}^+)}\right)$. By uniqueness of $H^{\frac{1}{2}+}$ solutions, u_k and $\widetilde{u}_k$ are equal on $\mathbb{R}^+ \times \mathbb{R}^+$. By the fixed-point argument, u_k and $\widetilde{u}_k$ converge in H^{s-} to u and $\widetilde{u}$ respectively. Thus $u = \widetilde{u}$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Iterating this argument, we obtain uniqueness for $s > -\frac{1}{4}$.

Lemma 5.6.1. *[39] Fix* $-\frac{1}{2} < s < \frac{1}{2}$ *and* $k > s$. *Let* $p \in H^s(\mathbb{R}^+)$ *and* $q \in H^k(\mathbb{R}^+)$. *Let* p^e *be an* H^s *extension of* p *to* $\mathbb{R}$. *Then there is an* H^k *extension* q^e *of* q *to* $\mathbb{R}$ *such that*

$$\|p^e - q^e\|_{H^r(\mathbb{R})} \lesssim \|p - q\|_{H^s(\mathbb{R}^+)} \quad \text{for } r < s.$$

5.7 Proofs of Estimates

5.7.1 Proof of Lemma 5.4.1: Kato Smoothing

We wish to show that

$$\|\eta(t) W_R^t(f,g)\|_{L_x^\infty H_t^{\frac{2s+1}{4}}} \lesssim \|f\|_{H_x^s} + \|g\|_{H_x^{s-1}},$$

$$\|\eta(t) [W_R^t(f,g)]_x\|_{L_x^\infty H_t^{\frac{2s-1}{4}}} \lesssim \|f\|_{H_x^s} + \|g\|_{H_x^{s-1}}.$$

It suffices to consider evaluation at $x = 0$ since Sobolev norms are invariant under translations. Using the Fourier multiplier form of the linear flow, write

$$2\mathcal{F}_t\big(\eta W_R^t(f,g)\big)(0,\tau) = \int \widehat{\eta}(\tau - \sqrt{\omega^2 + \omega^4})\, \widehat{f}(\omega)\, \mathrm{d}\omega + \int \widehat{\eta}(\tau + \sqrt{\omega^2 + \omega^4})\, \widehat{f}(\omega)\, \mathrm{d}\omega$$

$$+ \int \widehat{\eta}(\tau - \sqrt{\omega^2 + \omega^4}) \frac{\omega}{\sqrt{\omega^2 + \omega^4}}\, \widehat{g}(\omega)\, \mathrm{d}\omega - \int \widehat{\eta}(\tau + \sqrt{\omega^2 + \omega^4}) \frac{\omega}{\sqrt{\omega^2 + \omega^4}}\, \widehat{g}(\omega)\, \mathrm{d}\omega.$$

On the region where $|\omega| \leqslant 1$, these terms can easily be bounded in $H_t^{\frac{2s+1}{4}}$ since η is a Schwarz function. When $|\omega| > 1$, change variables by setting $\lambda = \omega\sqrt{\omega^2 + 1}$. The first two integrals in the above sum are then of the form

$$\int_{|\lambda| > 2} \widehat{\eta}(\tau \pm |\lambda|)\, \widehat{f}(\omega(\lambda)) \frac{\sqrt{1 + \omega(\lambda)^2}}{2\omega(\lambda)^2 + 1}\, \mathrm{d}\lambda,$$

and we wish to bound

$$\left\| \int_{|\lambda| > 2} \langle\tau\rangle^{\frac{2s+1}{4}} \widehat{\eta}(\tau \pm |\lambda|)\, \widehat{f}(\omega(\lambda)) \frac{\sqrt{1 + \omega(\lambda)^2}}{2\omega(\lambda)^2 + 1}\, \mathrm{d}\lambda \right\|_{L_\tau^2}.$$

Note that the inequality $\langle a + b \rangle \lesssim \langle a \rangle \langle b \rangle$ implies that for any α, we have $\langle a + b \rangle^\alpha \lesssim \langle a \rangle^{|\alpha|} \langle b \rangle^\alpha$. Using this, the quantity above is bounded by

$$\left\| \int_{|\lambda| > 2} \langle\tau \pm |\lambda|\rangle^{\frac{|2s+1|}{4}} \widehat{\eta}(\tau \pm |\lambda|) \langle\lambda\rangle^{\frac{2s+1}{4}} \widehat{f}(\omega(\lambda)) \frac{\sqrt{1 + \omega(\lambda)^2}}{2\omega(\lambda)^2 + 1}\, \mathrm{d}\lambda \right\|_{L_\tau^2}.$$

Since $\widehat{\eta}$ is a Schwarz function, we may use Young's inequality and then change variables back to ω to bound this quantity by

$$\left\| \langle\lambda\rangle^{\frac{2s+1}{4}} \widehat{f}(\omega(\lambda)) \frac{\sqrt{1 + \omega(\lambda)^2}}{2\omega(\lambda)^2 + 1} \right\|_{L_{|\lambda| > 2}^2} \lesssim \|f\|_{H^s(\mathbb{R})},$$

as desired. The remaining integrals, those involving g, can be treated in exactly the same way and bounded by $\|g\|_{H^{s-1}(\mathbb{R})}$. We obtain the bound on $\|\eta(t) [W_R^t(f,g)]_x\|_{L_x^\infty H_t^{\frac{2s-1}{4}}}$ by the same argument.

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5.7.2 Proof of Lemma 5.4.2: Bounds on Linear Solution

Recall that we wish to establish

$$\|\eta(t)W_0^t(0,0,h_1,h_2)\|_{X^{s,b}} \lesssim \|\chi h_1\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})} + \|\chi h_2\|_{H_t^{\frac{2s-1}{4}}(\mathbb{R})},$$

where $2\pi W_0^t(0,0,h_1,h_2) = -A - B + C + D$, and the terms A, B, C, and D are given in (5.4). Notice that

$$C = L^t\phi_C, \qquad\qquad \text{where} \quad \widehat{\phi_C}(\omega) = \left(i\omega + \sqrt{\omega^2 + 1}\right)\widehat{h_1}\left(\omega\sqrt{\omega^2 + 1}\right),$$

$$D = L^t\phi_D, \qquad\qquad \text{where} \quad \widehat{\phi_D}(\omega) = \frac{i\omega + \sqrt{\omega^2 + 1}}{\sqrt{\omega^2 + 1}}\widehat{h_2}\left(\omega\sqrt{\omega^2 + 1}\right),$$

and L^t is the spatial Fourier multiplier operator with multiplier $e^{it\omega\sqrt{1+\omega^2}}$. The proof of (5.8) implies that $\|\eta(t)C\|_{X^{s,b}} \lesssim \|\phi_C\|_{H_x^s}$ and $\|\eta(t)D\|_{X^{s,b}} \lesssim \|\phi_D\|_{H_x^s}$. Now

$$\begin{aligned}
\|\phi_C\|_{H_x^s}^2 &= \int_{-\infty}^{\infty} (2\omega^2 + 1)\langle\omega\rangle^{2s}\left|\widehat{h_1}\left(\omega\sqrt{\omega^2 + 1}\right)\right|^2 d\omega \\
&= \int_{-\infty}^{\infty} \frac{(\omega^2 + 1)^{1/2}\langle\omega\rangle^{2s}}{\langle\omega\sqrt{\omega^2 + 1}\rangle^{\frac{2s+1}{2}}}\langle\omega\sqrt{\omega^2 + 1}\rangle^{\frac{2s+1}{2}}\left|\widehat{h_1}\left(\omega\sqrt{\omega^2 + 1}\right)\right|^2 \frac{2\omega^2 + 1}{\sqrt{\omega^2 + 1}} d\omega \\
&\lesssim \int_{-\infty}^{\infty}\langle\omega\sqrt{\omega^2 + 1}\rangle^{\frac{2s+1}{2}}\left|\widehat{h_1}\left(\omega\sqrt{\omega^2 + 1}\right)\right|^2 \frac{2\omega^2 + 1}{\sqrt{\omega^2 + 1}} d\omega \\
&= \int_{-\infty}^{\infty}\langle z\rangle^{\frac{2s+1}{2}}\left|\widehat{h_1}(z)\right|^2 dz = \|\chi h_1\|_{H^{\frac{2s+1}{4}}(\mathbb{R})},
\end{aligned}$$

where we used the change of variable $z = \omega\sqrt{\omega^2 + 1}$. Similarly,

$$\begin{aligned}
\|\phi_D\|_{H_x^s}^2 &= \int_{-\infty}^{\infty}\frac{(2\omega^2 + 1)\langle\omega\rangle^{2s}}{\omega^2 + 1}\left|\widehat{h_2}\left(\omega\sqrt{\omega^2 + 1}\right)\right|^2 d\omega \\
&= \int_{-\infty}^{\infty}\frac{\langle\omega\rangle^{2s}}{(\omega^2 + 1)^{1/2}\langle\omega\sqrt{\omega^2 + 1}\rangle^{\frac{2s-1}{2}}}\langle\omega\sqrt{\omega^2 + 1}\rangle^{\frac{2s-1}{2}}\left|\widehat{h_2}\left(\omega\sqrt{\omega^2 + 1}\right)\right|^2 \frac{2\omega^2 + 1}{\sqrt{\omega^2 + 1}} d\omega \\
&\lesssim \int_{-\infty}^{\infty}\langle z\rangle^{\frac{2s-1}{2}}\left|\widehat{h_2}(z)\right|^2 dz = \|\chi h_2\|_{H^{\frac{2s-1}{4}}(\mathbb{R})}.
\end{aligned}$$

Thus we have the desired bounds on C and D. Now we move on to A and B. Assume first that $s = 0$ and $b = \frac{1}{2}-$. Let $f(y) = e^{-y}\rho(y)$. Then

$$A(x,t) = \int_{-\infty}^{\infty}\frac{i\omega\left(i\omega + \sqrt{1 + \omega^2}\right)}{\sqrt{\omega^2 + 1}}e^{it\omega\sqrt{\omega^2 + 1}}\widehat{h_1}\left(\omega\sqrt{\omega^2 + 1}\right)f\left(x\sqrt{\omega^2 + 1}\right)d\omega,$$

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$$\widehat{\eta A}(\xi,\tau) = \int_{-\infty}^{\infty} \widehat{\eta}(\tau - \omega\sqrt{\omega^2+1})\frac{i\omega(i\omega+\sqrt{1+\omega^2})}{\sqrt{\omega^2+1}}\,\widehat{h_1}(\omega\sqrt{\omega^2+1})\mathcal{F}_x\Big(f(x\sqrt{\omega^2+1})\Big)(\xi)\,\mathrm{d}\omega$$

$$= \int_{-\infty}^{\infty} \widehat{\eta}(\tau - \omega\sqrt{\omega^2+1})\frac{i\omega(i\omega+\sqrt{1+\omega^2})}{\omega^2+1}\,\widehat{h_1}(\omega\sqrt{\omega^2+1})\widehat{f}\Big(\xi/\sqrt{\omega^2+1}\Big)\,\mathrm{d}\omega.$$

Since f is a Schwarz function, we have

$$|\widehat{f}(\xi/\sqrt{\omega^2+1})| \lesssim \frac{1}{1+\xi^2/(\omega^2+1)} = \frac{\omega^2+1}{1+\omega^2+\xi^2}.$$

Note also that since η is a Schwarz function,

$$|\widehat{\eta}(\tau-\omega\sqrt{\omega^2+1})| \lesssim \langle\tau-\omega\sqrt{\omega^2+1}\rangle^{-5/2+}$$

$$\lesssim \langle\tau-\omega\sqrt{\omega^2+1}\rangle^{-2}\langle\tau-\xi^2\rangle^{-1/2+}\langle\omega\sqrt{\omega^2+1}-\xi^2\rangle^{1/2-}.$$

Therefore, using the bounds for f, those for η, and then moving the ξ norm inside the integral,

$$\|\eta A\|_{X^{0,\frac{1}{2}-}} \lesssim \left\|\langle\tau-\xi^2\rangle^{1/2-}\int_{-\infty}^{\infty}\left|\widehat{\eta}(\tau-\omega\sqrt{\omega^2+1})\right|\frac{|\omega|\sqrt{1+2\omega^2}}{1+\omega^2+\xi^2}\left|\widehat{h_1}(\omega\sqrt{\omega^2+1})\right|\mathrm{d}\omega\right\|_{L^2_{\xi,\tau}}$$

$$\lesssim \left\|\int_{-\infty}^{\infty}\langle\tau-\omega\sqrt{\omega^2+1}\rangle^{-2}\frac{|\omega|\sqrt{1+2\omega^2}}{(1+\omega^2+\xi^2)^{1/2+}}\left|\widehat{h_1}(\omega\sqrt{\omega^2+1})\right|\mathrm{d}\omega\right\|_{L^2_{\xi,\tau}}$$

$$\lesssim \left\|\int_{-\infty}^{\infty}\langle\tau-\omega\sqrt{\omega^2+1}\rangle^{-2}\frac{|\omega|\sqrt{1+2\omega^2}}{(1+\omega^2)^{1/4}}\left|\widehat{h_1}(\omega\sqrt{\omega^2+1})\right|\mathrm{d}\omega\right\|_{L^2_{\tau}}$$

$$\lesssim \left\|\int_{-\infty}^{\infty}\langle\tau-\omega\sqrt{\omega^2+1}\rangle^{-2}\langle\omega\sqrt{\omega^2+1}\rangle^{1/4}\left|\widehat{h_1}(\omega\sqrt{\omega^2+1})\right|\frac{2\omega^2+1}{\sqrt{\omega^2+1}}\,\mathrm{d}\omega\right\|_{L^2_{\tau}}$$

$$= \left\|\int_{-\infty}^{\infty}\langle\tau-z\rangle^{-2}\langle z\rangle^{1/4}\left|\widehat{h_1}(z)\right|\mathrm{d}z\right\|_{L^2_{\tau}}$$

$$\lesssim \int_{-\infty}^{\infty}\langle z\rangle^{1/2}\left|\widehat{h_1}(z)\right|^2\mathrm{d}z = \|\chi h_1\|_{H^{\frac{1}{4}}(\mathbb{R})}.$$

The last line follows from an application of Young's inequality.

The procedure for B is exactly the same – we drop the factor of ω and replace h_1 with h_2 in the integrals above to arrive at a bound of

$$\int_{-\infty}^{\infty}\langle z\rangle^{-3/2}\left|\widehat{h_2}(z)\right|^2\mathrm{d}z = \|\chi h_2\|_{H^{-\frac{1}{4}}(\mathbb{R})}.$$

It remains to obtain bounds on A and B in $X^{s,\frac{1}{2}-}$ for general s. Notice that for any $s \in \mathbb{N}$, the derivative $\partial_x^s(\eta A)$ is

$$\eta(t)\int_{-\infty}^{\infty}\frac{i\omega(i\omega+\sqrt{\omega^2+1})}{\sqrt{\omega^2+1}}e^{it\omega\sqrt{\omega^2+1}}\,\widehat{h_1}\Big(\omega\sqrt{\omega^2+1}\Big)f^{(s)}\Big(x\sqrt{\omega^2+1}\Big)(\omega^2+1)^{s/2}\,\mathrm{d}\omega,$$

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with a similar formula for $\partial_x^s(\eta B)$. Since $(\omega^2 + 1)^{s/2} \lesssim \langle \omega\sqrt{\omega^2 + 1}\rangle^{s/2}$, the desired result follows for $s \in \mathbb{N}$. By interpolation, we obtain the bound for any $s > 0$.

For $s < 0$, let $\langle \partial \rangle_x^{-1/2}$ be the Fourier multiplier operator $\langle \xi \rangle^{-1/2}$. Then $\langle \partial \rangle_x^{-1/2}(\eta A)$ is equal to

$$\eta(t) \int_{-\infty}^{\infty} \frac{i\omega(i\omega + \sqrt{\omega^2 + 1})}{\sqrt{\omega^2 + 1}} e^{it\omega\sqrt{\omega^2+1}} \, \widehat{h_1}\left(\omega\sqrt{\omega^2 + 1}\right) \langle \partial \rangle_x^{-1/2}\left[f\left(x\sqrt{\omega^2 + 1}\right)\right] d\omega,$$

again with a similar statement for B. Now notice that

$$\mathcal{F}_x\left(\langle \partial \rangle_x^{-1/2}\left[f\left(x\sqrt{\omega^2 + 1}\right)\right]\right) = \mathcal{F}_x\left(\left[\langle \partial \rangle_x^{-1/2}f\right]\left(x\sqrt{\omega^2 + 1}\right)\right)(\xi) \frac{\langle \xi/\sqrt{\omega^2 + 1}\rangle^{1/2}}{\langle \xi \rangle^{1/2}}.$$

Noting the $\langle \partial \rangle_x^{-1/2}f$ is also a Schwarz function, we proceed just as in the case $s = 0$. In that situation, we moved the L_ξ^2 norm inside the integral and used the fact that $\|\frac{1}{(1+\omega^2+\xi^2)^{1/2+}}\|_{L_\xi^2} \approx (1 + \omega^2)^{-1/4-} \leqslant (1 + \omega^2)^{-1/4}$. In this case, we use

$$\left\|\frac{\langle \xi/\sqrt{\omega^2 + 1}\rangle^{1/2}}{\langle \xi \rangle^{1/2}(1 + \omega^2 + \xi^2)^{1/2+}}\right\|_{L_\xi^2} = \left\|\frac{1}{(1 + \omega^2)^{1/4}(1 + \xi^2)^{1/4}(1 + \omega^2 + \xi^2)^{1/4+}}\right\|_{L_\xi^2} \lesssim \frac{1}{(1 + \omega^2)^{1/2}}.$$

This bound holds since

$$\int \frac{1}{(1 + \xi^2)^{1/2}(1 + \omega^2 + \xi^2)^{1/2+}} \, d\xi \approx \int \frac{1}{\langle \xi \rangle \langle |\omega| + |\xi| \rangle^{1+}} \, d\xi \lesssim \langle \omega \rangle^{-1}$$

by Lemma 2.3.4. Then the same argument we used previously yields the bound

$$\int_{-\infty}^{\infty} \left|\widehat{h_1}(z)\right|^2 dz.$$

We obtain a similar bound for B. Interpolating between the $s = -\frac{1}{2}$ and $s = 0$ estimates completes the proof.

5.7.3 Proof of Lemma 5.4.3: Continuity of Linear Flow

Recall that $2\pi \, W_0^t(0, 0, h_1, h_2) = -A - B + C + D$. We start with the claim $A, B \in C_t^0 H_x^s(\mathbb{R} \times \mathbb{R})$. Note that

$$A = \int_{-\infty}^{\infty} f\left(x\sqrt{\omega^2 + 1}\right) \mathcal{F}(L^t \phi_A)(\omega) \, d\omega,$$

$$B = \int_{-\infty}^{\infty} f\left(x\sqrt{\omega^2 + 1}\right) \mathcal{F}(L^t \phi_B)(\omega) \, d\omega$$

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where

$$\widehat{\phi_A} = \frac{i\omega\left(i\omega + \sqrt{1+\omega^2}\right)}{\sqrt{\omega^2+1}}\widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right), \qquad \widehat{\phi_B} = \frac{i\omega + \sqrt{1+\omega^2}}{\sqrt{\omega^2+1}}\widehat{h_2}\left(\omega\sqrt{\omega^2+1}\right),$$

the function f is given by $f(x) = e^{-x}\rho(x)$, and L^t is the Fourier multiplier operator with multiplier $e^{i\omega\sqrt{\omega^2+1}t}$. Now

$$\|\phi_A\|_{H_x^s}^2 = \int_{-\infty}^{\infty} \frac{\omega^2(2\omega^2+1)}{\omega^2+1}\langle\omega\rangle^{2s}\left|\widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right)\right|^2 d\omega$$

$$= \int_{-\infty}^{\infty} \frac{\omega^2\langle\omega\rangle^{2s}\langle\omega\sqrt{\omega^2+1}\rangle^{\frac{2s+1}{2}}}{\sqrt{\omega^2+1}\langle\omega\sqrt{\omega^2+1}\rangle^{\frac{2s+1}{2}}}\left|\widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right)\right|^2 \frac{2\omega^2+1}{\sqrt{\omega^2+1}} d\omega$$

$$\lesssim \int_{-\infty}^{\infty} \langle\omega\sqrt{\omega^2+1}\rangle^{\frac{2s+1}{2}}\left|\widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right)\right|^2 \frac{2\omega^2+1}{\sqrt{\omega^2+1}} d\omega$$

$$= \int_{-\infty}^{\infty} \langle z\rangle^{\frac{2s+1}{2}}\left|\widehat{h_1}(z)\right|^2 dz = \|\chi h_1\|_{H^{\frac{2s+1}{4}}(\mathbb{R})}^2,$$

and similarly

$$\|\phi_B\|_{H_x^s}^2 \lesssim \|\chi h_2\|_{H^{\frac{2s-1}{4}}(\mathbb{R})}^2.$$

Thus, using time continuity of the linear operator L^t, it suffices to show that the map

$$g \mapsto T(g) = \int_{-\infty}^{\infty} f\left(x\sqrt{\omega^2+1}\right)\widehat{g}(\omega) d\omega$$

is bounded from H^s to H^s. Consider first $s = 0$.

Rewrite $Tg(x)$ as follows using the change of variables $z = x\sqrt{\omega^2+1}$:

$$Tg(x) = \int_{-\infty}^{\infty} f\left(x\sqrt{\omega^2+1}\right)\widehat{g}(\omega) d\omega$$

$$= \int_x^{\text{sgn}(x)\infty} f(z)\left[\widehat{g}\left(\sqrt{(z/x)^2-1}\right) + \widehat{g}\left(-\sqrt{(z/x)^2-1}\right)\right]\frac{z/x^2}{\sqrt{(z/x)^2-1}} dz.$$

Then

$$\|Tg\|_{L_x^2} \lesssim \int |f(z)| \left\|\chi_{[0,z]}(x)\,\widehat{g}\left(\pm\sqrt{(z/x)^2-1}\right)\frac{z/x^2}{\sqrt{(z/x)^2-1}}\right\|_{L_x^2} dz,$$

and, expanding the L_x^2 norm,

$$\int_0^z \left|\widehat{g}\left(\pm\sqrt{(z/x)^2-1}\right)\right|^2 \frac{z^2/x^4}{(z/x)^2-1} dx = \frac{1}{|z|}\int_0^{\pm\infty} |\widehat{g}(y)|^2 \frac{\sqrt{1+y^2}}{y} dy.$$

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On the region where $|y| \geqslant 1$, the right hand side above is bounded by $\frac{1}{|z|}\|g\|_{L^2}^2$. Since $|f(z)|/\sqrt{|z|}$ is in L^1, this yields the desired bound. For the case when $|y| \leqslant 1$, go back to the form $Tg(x) = \int f\left(x\sqrt{\omega^2+1}\right)\widehat{g}(\omega)\,d\omega$. The region $|y| \leqslant 1$ corresponds to $|\omega| \leqslant 1$ in this integral. So we consider the following norm, which we bound by applying Cauchy-Schwarz in ω and then using the change of variables $y = x\sqrt{1+\omega^2}$ to replace the integration in x:

$$\left\|\int_{-1}^{1} f\left(x\sqrt{\omega^2+1}\right)\widehat{g}(\omega)\,d\omega\right\|_{L_x^2}^2 \lesssim \|g\|_{L^2}^2 \left\|\chi_{[0,1]}(\omega)f\left(x\sqrt{\omega^2+1}\right)\right\|_{L_{x,\omega}^2}^2$$

$$= \|g\|_{L^2}^2 \int_{-1}^{1} \frac{1}{\sqrt{1+\omega^2}}\int f^2(y)\,dy\,d\omega$$

$$\lesssim \|g\|_{L^2}^2.$$

This completes the proof that $A, B \in C_t^0 H_x^s$ for $s = 0$.

For $s > 0$, notice that for any $s \in \mathbb{N}$, we have

$$\partial_x^s Tg(x) = \int_0^\infty f^{(s)}\left(x\sqrt{\omega^2+1}\right)(\omega^2+1)^{s/2}\,\widehat{g}(\omega)\,d\omega.$$

This and interpolation imply the desired bounds for A and B in H_x^s for positive s.

Also, if we choose ρ such that $\int f\,dx = 0$ so that $\partial_x^{-1} f$ is a Schwarz function, then we have

$$\partial_x^{-1} Tg(x) = \int_0^\infty \partial_x^{-1} f\left(x\sqrt{\omega^2+1}\right)(\omega^2+1)^{-1/2}\,\widehat{g}(\omega)\,d\omega.$$

Combining this with the $s = 0$ result and interpolation, we obtain the bound for $s \geqslant -1$.

Next, recall that

$$C = L^t\phi_C(x) \quad \text{where} \quad \widehat{\phi_C}(\omega) = \left(i\omega + \sqrt{\omega^2+1}\right)\widehat{h_1}\left(\omega\sqrt{\omega^2+1}\right),$$

$$D = L^t\phi_D(x) \quad \text{where} \quad \widehat{\phi_D}(\omega) = \frac{i\omega + \sqrt{\omega^2+1}}{\sqrt{\omega^2+1}}\widehat{h_2}\left(\omega\sqrt{\omega^2+1}\right).$$

The $C_t^0 H_x^s$ bounds for these terms follow from the continuity of the linear operator F^t and the bounds for ϕ_C and ϕ_D which were proved in Lemma 5.4.2.

It remains to prove that $\eta W_0^t(0, 0, h_1, h_2)$ is in $C_x^0 H_t^{\frac{2s+1}{4}}$. Recall the form of C and D as linear flows and apply Lemma 5.4.1 to obtain the desired bound for these terms. For A and B, write

$$A(x,t) = \int \mathcal{F}_\omega\left(f\big(x\sqrt{\omega^2+1}\big)\right)(y)\, L^t\phi_A(y)\,dy,$$

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$$B(x,t) = \int \mathcal{F}_\omega \left(f\left(x\sqrt{\omega^2+1}\right)\right)(y)\, L^t \phi_B(y)\, \mathrm{d}y$$

where ϕ_A and ϕ_B are defined as before. Then A is equal to

$$\int \frac{1}{x} \mathcal{F}_z \left(f(\mathrm{sgn}(x)\sqrt{x^2+z^2})\right) \left(\frac{y}{x}\right) L^t \phi_A(y)\, \mathrm{d}y = \int \mathcal{F}_z \left(f(\mathrm{sgn}(x)\sqrt{x^2+z^2})\right)(y) L^t \phi_A(xy)\, \mathrm{d}y,$$

with a parallel statement for B. By Kato smoothing, Lemma 5.4.1, it suffices to show that the function $\mathcal{F}_z \left(f\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)\right)(y)$ is in $L_x^\infty L_y^1$. It is enough to show that $f\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)$ is in $L_x^\infty H_z^1$ since

$$\int |\widehat{k}(y)|\, \mathrm{d}y = \int \langle y\rangle |\widehat{k}(y)| \langle y\rangle^{-1}\, \mathrm{d}y \lesssim \|k\|_{H^1} \|\langle \cdot \rangle^{-1}\|_{L^2}.$$

To this end, we consider the L_z^2 and the $\dot H_z^1$ norms separately. For the L_z^2 norm, split the integral into two regions, one where $|z|$ is small, and its complement:

$$\int \left| f\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)\right|^2 \mathrm{d}z$$
$$= \int_{|z|\leqslant 1} \left| f\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)\right|^2 \mathrm{d}z + \int_{|z|>1} \left| f\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)\right|^2 \mathrm{d}z.$$

The first term is bounded since f is bounded. Set $y = \mathrm{sgn}(x)\sqrt{z^2+x^2}$ in the second integral to obtain

$$\int_{|y^2-x^2|>1} |f(y)|^2 \frac{y}{\sqrt{y^2-x^2}}\, \mathrm{d}y,$$

which is bounded since f is a Schwarz function. The same argument serves to bound the derivative since

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[f\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)\right] = f'\left(\mathrm{sgn}(x)\sqrt{z^2+x^2}\right)\frac{\mathrm{sgn}(x)z}{\sqrt{z^2+x^2}} \quad \text{and} \quad \frac{|z|}{\sqrt{z^2+x^2}} \leqslant 1.$$

5.7.4   Proof of Lemma 5.4.4: Bilinear $X^{s,b}$ Estimate

By duality, it suffices to show that

$$\left| \iint \mathcal{M}(uv)_{xx}\overline{\phi}\, \mathrm{d}x\, \mathrm{d}t \right| \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|\phi\|_{X^{-(s+a),b}} \tag{5.11}$$

for any $\phi \in X^{-(s+a),b}$. The left-hand side of (5.11) is equal to

$$\left| \iint \frac{\xi^2\, \widehat{uv}(\xi,\tau)}{\sqrt{\xi^2+\xi^4}}\, \overline{\widehat{\phi}(\xi,\tau)}\, \mathrm{d}\xi\, \mathrm{d}\tau \right| = \left| \iiiint \frac{\xi^2\, \widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi-\xi_1,\tau-\tau_1)}{\sqrt{\xi^2+\xi^4}}\, \overline{\widehat{\phi}(\xi,\tau)}\, \mathrm{d}\xi_1\, \mathrm{d}\tau_1\, \mathrm{d}\xi\, \mathrm{d}\tau \right|.$$

Now define

$$p(\xi,\tau) = \langle\xi\rangle^s\langle|\tau| - \sqrt{\xi^2 + \xi^4}\rangle^b \, \hat{u}(\xi,\tau), \qquad q(\xi,\tau) = \langle\xi\rangle^s\langle|\tau| - \sqrt{\xi^2 + \xi^4}\rangle^b \, \hat{v}(\xi,\tau),$$

$$r(\xi,\tau) = \langle\xi\rangle^{-(s+a)}\langle|\tau| - \sqrt{\xi^2 + \xi^4}\rangle^b \, \overline{\hat{\phi}(\xi,\tau)}.$$

The desired bound (5.11) is equivalent to showing that

$$\left|\iiiint M(\xi,\xi_1,\tau,\tau_1)p(\xi_1,\tau_1)q(\xi - \xi_1,\tau - \tau_1)r(\xi,\tau)\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi\,\mathrm{d}\tau\right| \lesssim \|p\|_{L^2_{\xi,\tau}}\|q\|_{L^2_{\xi,\tau}}\|r\|_{L^2_{\xi,\tau}},$$

where the multiplier M is

$$M = \frac{\xi^2\langle\xi\rangle^{s+a}\langle\xi_1\rangle^{-s}\langle\xi - \xi_1\rangle^{-s}}{\sqrt{\xi^2 + \xi^4}\langle|\tau| - \xi^2\rangle^b\langle|\tau_1| - \xi_1^2\rangle^b\langle|\tau - \tau_1| - (\xi - \xi_1)^2\rangle^b}.$$

There are six possibilities for the signs of τ, τ_1, and $\tau - \tau_1$:

(a) $\tau_1 \geqslant 0$, $\tau - \tau_1 \geqslant 0$,

(b) $\tau_1 \geqslant 0$, $\tau - \tau_1 \leqslant 0$, and $\tau \geqslant 0$,

(c) $\tau_1 \geqslant 0$, $\tau - \tau_1 \leqslant 0$, and $\tau \leqslant 0$,

(d) $\tau_1 \leqslant 0$, $\tau - \tau_1 \leqslant 0$,

(e) $\tau_1 \leqslant 0$, $\tau - \tau_1 \geqslant 0$, and $\tau \leqslant 0$,

(f) $\tau_1 \leqslant 0$, $\tau - \tau_1 \geqslant 0$, and $\tau \geqslant 0$.

Since L^2 norms are invariant under reflections, we can use the substitution $(\tau,\tau_1) \mapsto -(\tau,\tau_1)$ to reduce (d), (e), and (f) to (a), (b), and (c) respectively.

Consider first (a). By Cauchy-Schwarz in the ξ-τ integral, it suffices to show that

$$\left\|\iint M(\xi,\xi_1,\tau,\tau_1)p(\xi_1,\tau_1)q(\xi - \xi_1,\tau - \tau_1)\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\right\|_{L^2_{\xi,\tau}} \lesssim \|p\|_{L^2_{\xi,\tau}}\|q\|_{L^2_{\xi,\tau}}.$$

Using Cauchy-Schwarz and Young's inequalities, the left-hand side of this is bounded by

$$\left\|\|M\|_{L^2_{\xi_1,\tau_1}}\|f(\xi_1,\tau_1)g(\xi - \xi_1,\tau - \tau_1)\|_{L^2_{\xi_1,\tau_1}}\right\|_{L^2_{\xi,\tau}}$$

$$\lesssim \left(\sup_{\xi,\tau}\|M\|_{L^2_{\xi_1,\tau_1}}\right)\|f(\xi_1,\tau_1)g(\xi - \xi_1,\tau - \tau_1)\|_{L^2_{\xi_1,\tau_1,\xi,\tau}}$$

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$$= \left(\sup_{\xi,\tau} \|M\|_{L^2_{\xi_1,\tau_1}}\right) \|f^2 * g^2\|_{L^1_{\xi,\tau}}^{1/2}$$

$$\lesssim \left(\sup_{\xi,\tau} \|M\|_{L^2_{\xi_1,\tau_1}}\right) \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}}.$$

Thus, in Case (a), it suffices to show that

$$\sup_{\xi,\tau} \iint \frac{\xi^4 \langle\xi\rangle^{2s+2a} \langle\xi_1\rangle^{-2s} \langle\xi-\xi_1\rangle^{-2s}}{(\xi^2+\xi^4)\langle\tau-\xi^2\rangle^{2b}\langle\tau_1-\xi_1^2\rangle^{2b}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b}} \, d\xi_1 \, d\tau_1$$

is finite. Using the fact that $\langle a\rangle\langle b\rangle \gtrsim \langle a+b\rangle$, we can eliminate the τ dependence to obtain

$$\sup_{\xi} \iint \frac{\xi^4 \langle\xi\rangle^{2s+2a} \langle\xi_1\rangle^{-2s} \langle\xi-\xi_1\rangle^{-2s}}{(\xi^2+\xi^4)\langle\tau_1-\xi_1^2\rangle^{2b}\langle\tau_1-2\xi\xi_1+\xi_1^2\rangle^{2b}} \, d\xi_1 \, d\tau_1$$

Applying Lemma 2.3.4 in τ_1 and observing that $\xi^4/(\xi^2+\xi^4) < 1$, we are reduced to bounding

$$\sup_{\xi} \int \frac{\langle\xi\rangle^{2s+2a} \langle\xi_1\rangle^{-2s} \langle\xi-\xi_1\rangle^{-2s}}{\langle\xi_1(\xi_1-\xi)\rangle^{1-}} \, d\xi_1.$$

We consider several cases.

Case 1. $\langle\xi\rangle^s \lesssim \langle\xi_1\rangle^s \langle\xi-\xi_1\rangle^s$.

This case reduces to bounding

$$\sup_{\xi} \langle\xi\rangle^{2a} \int \frac{1}{\langle\xi_1(\xi_1-\xi)\rangle^{1-}} \, d\xi_1.$$

Let

$$x = \xi_1(\xi-\xi_1) \quad \Rightarrow \quad 2\xi_1 = \xi \pm \sqrt{\xi^2+4x} \quad \text{and} \quad dx = \pm\sqrt{\xi^2+4x} \, d\xi_1.$$

Then the supremum above is bounded by

$$\sup_{\xi} \langle\xi\rangle^{2a} \int \frac{1}{\langle x\rangle^{1-}\sqrt{\xi^2+4x}} \, dx.$$

By [38, Lemma 6.3], this is bounded by

$$\sup_{\xi} \langle\xi\rangle^{2a-1+},$$

which is finite as long as $a < \frac{1}{2}$.

Case 2. $\langle\xi\rangle \ll \langle\xi_1\rangle\langle\xi-\xi_1\rangle$ and $s < 0$.

Case 2a. $|\xi| \lesssim 1$.

In this case, we must control $\int\langle\xi_1\rangle^{-4s-2+} \, d\xi_1$, which is possible when $s > -\frac{1}{4}$.

Case 2b. $|\xi_1| \gg |\xi| \gtrsim 1$.

In this case, we arrive at

$$\sup_{\xi} \langle \xi \rangle^{2s+2a} \int_{|\xi_1| \gg |\xi|} \langle \xi_1 \rangle^{-4s-2+} \, d\xi_1 \lesssim \sup_{\xi} \langle \xi \rangle^{2a-2s-1+}.$$

This is finite when $a < s + \frac{1}{2}$.

Case 2c. $|\xi| \gg |\xi_1| \gg 1$.

In this case, we arrive at

$$\sup_{\xi} \langle \xi \rangle^{2a-1+} \int_{|\xi_1| \ll |\xi|} \langle \xi_1 \rangle^{-2s-1+} \, d\xi_1.$$

Since $s < 0$, this converges if $a < s + \frac{1}{2}$.

Case 2d. $|\xi_1| \approx |\xi|$.

This is only possible if $|\xi - \xi_1| \gg 1$ and $|\xi_1| \gg 1$. Thus we need to bound

$$\sup_{\xi} \langle \xi \rangle^{2a} \int \langle \xi - \xi_1 \rangle^{-1-2s+} \langle \xi_1 \rangle^{-1+} \, d\xi_1.$$

Using Lemma 2.3.4, we see that this can be bounded for $s > -\frac{1}{2}$ as long as $a < s + \frac{1}{2}$. This completes the proof for the combination of τ signs described in (a).

For the combination of signs described in (b), we follow the same procedure of estimating using Cauchy-Schwarz and Young's inequalites, but exchange the role of (ξ, τ) and (ξ_1, τ_1). It then suffices to control

$$\sup_{\xi_1} \int \frac{\langle \xi \rangle^{2s+2a} \langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \xi(\xi_1 - \xi) \rangle^{1-}} \, d\xi. \tag{5.12}$$

Case (c) can be reduced to the same estimate by performing the change of variables $(\xi_1, \tau_1) \mapsto (\xi - \xi_1, \tau - \tau_1)$ and then carrying out the same series of estimates. To bound this supremum (5.12), we consider similar cases.

Case 1'. $\langle \xi \rangle^s \lesssim \langle \xi_1 \rangle^s \langle \xi - \xi_1 \rangle^s$.

The procedure here is precisely the same as in **Case 1**.

Case 2'. $\langle \xi \rangle \ll \langle \xi_1 \rangle \langle \xi - \xi_1 \rangle$ and $s < 0$.

Case 2a'. $|\xi_1| \lesssim 1$.

In this case, we must control $\int \langle \xi_1 \rangle^{2a-2+} \, d\xi_1$, which is possible when $a < \frac{1}{2}$.

Case 2b'. $|\xi_1| \gg |\xi| \gtrsim 1$.

In this case, we arrive at

$$\sup_{\xi_1} \langle \xi_1 \rangle^{-4s-1+} \int_{|\xi| \ll |\xi_1|} \langle \xi \rangle^{2s+2a-1+} \, d\xi \lesssim \sup_{\xi_1} \langle \xi_1 \rangle^{-4s-1+\max\{2s+2a,0\}+},$$

which is finite if $a < s + \frac{1}{2}$ and $s > -\frac{1}{4}$.

Case 2c'. $|\xi| \gg |\xi_1| \gg 1$.

In this case, we arrive at

$$\sup_{\xi_1} \langle \xi_1 \rangle^{-2s} \int_{|\xi| \gg |\xi_1|} \langle \xi \rangle^{2a-2+} \, d\xi \lesssim \sup_{\xi_1} \langle \xi_1 \rangle^{-2s+2a-1+},$$

which is finite if $a < s + \frac{1}{2}$.

Case 2d'. $|\xi_1| \approx |\xi|$.

This case only arises if $|\xi - \xi_1| \gg 1$ and $|\xi_1| \gg 1$. Thus we need to bound

$$\sup_{\xi_1} \langle \xi_1 \rangle^{-2s} \int \langle \xi - \xi_1 \rangle^{-2s-1+} \langle \xi \rangle^{2s+2a-1+} \, d\xi.$$

Using Lemma 2.3.4, we see that this can be bounded for $-\frac{1}{2} < s < 0$ and $a < \frac{1}{2}$ by

$$\sup_{\xi_1} \langle \xi_1 \rangle^{-2s+\max\{-2s,2s+2a\}-1+}.$$

This is finite for $a < \frac{1}{2}$ and $s > -\frac{1}{4}$. This completes the proof.

5.7.5 Proof of Lemma 5.4.5: Kato Smoothing for Duhamel Term

Again, it suffices to consider evaluation at $x = 0$ since a spatial translation of G does not affect the magnitude of $\widehat{\mathcal{M}(G)}$. At $x = 0$, we have

$$\int_0^t W_{R,2}^{t-t'} G \, dt' = \frac{1}{2i} \int \int_0^t \frac{e^{i(t-t')\sqrt{\xi^2+\xi^4}} - e^{-i(t-t')\sqrt{\xi^2+\xi^4}}}{\sqrt{\xi^2 + \xi^4}} \mathcal{F}_x(G)(\xi, t') \, dt' \, d\xi.$$

Also, note that

$$\mathcal{F}_x(G)(\xi, t') = \int e^{i\tau t'} \widehat{G}(\xi, \tau) \, d\tau$$

and

$$\int_0^t e^{it'(\tau \mp \sqrt{\xi^2+\xi^4})} \, dt' = \frac{e^{it(\tau \mp \sqrt{\xi^2+\xi^4})} - 1}{i(\tau \mp \sqrt{\xi^2 + \xi^4})}.$$

Thus we wish to bound

$$\iint \frac{e^{it\tau} - e^{\pm it\sqrt{\xi^2+\xi^4}}}{\sqrt{\xi^2+\xi^4}\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)} \widehat{G}(\xi,\tau)\, d\xi\, d\tau.$$

Let Ψ be a smooth cut-off function such that $\Psi = 1$ on $[-1,1]$ and $\Psi = 0$ outside $[-2,2]$. Let $\Psi^C = 1 - \Psi$. Then write

$$2\eta(t)\int_0^t F_2^{t-t'} G\, dt' = \eta(t)\iint \frac{\left(e^{it\tau} - e^{\pm it\sqrt{\xi^2+\xi^4}}\right)\Psi\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)}{\sqrt{\xi^2+\xi^4}\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)}\widehat{G}(\xi,\tau)\, d\xi\, d\tau$$

$$+ \eta(t)\iint \frac{e^{it\tau}\,\Psi^C\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)}{\sqrt{\xi^2+\xi^4}\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)}\widehat{G}(\xi,\tau)\, d\xi\, d\tau$$

$$- \eta(t)\iint \frac{e^{\pm it\sqrt{\xi^2+\xi^4}}\Psi^C\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)}{\sqrt{\xi^2+\xi^4}\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)}\widehat{G}(\xi,\tau)\, d\xi\, d\tau$$

$$= \mathrm{I} + \mathrm{II} - \mathrm{III}\,.$$

By Taylor expanding, we have

$$\frac{e^{it\tau} - e^{\pm it\sqrt{\xi^2+\xi^4}}}{\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)} = -e^{it\tau}\sum_{k=1}^{\infty} \frac{(-it)^k}{k!}(\tau \mp \sqrt{\xi^2+\xi^4})^{k-1}.$$

Therefore $\|\mathrm{I}\|_{H_t^{\frac{2s+1}{4}}}$ is bounded by

$$\sum_{k=1}^{\infty} \frac{\|\eta(t)t^k\|_{H^1}}{k!}\left\|\iint e^{it\tau}\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)^{k-1}\Psi\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)\frac{\widehat{G}(\xi,\tau)}{\sqrt{\xi^2+\xi^4}}\, d\xi\, d\tau\right\|_{H_t^{\frac{2s+1}{4}}}$$

$$\lesssim \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\left\|\langle\tau\rangle^{\frac{2s+1}{4}}\int \left(\tau \mp \sqrt{\xi^2+\xi^4}\right)^{k-1}\Psi\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)\frac{\widehat{G}(\xi,\tau)}{\sqrt{\xi^2+\xi^4}}\, d\xi\right\|_{L_\tau^2}$$

$$\lesssim \left\|\langle\tau\rangle^{\frac{2s+1}{4}}\int \Psi\left(\tau \mp \sqrt{\xi^2+\xi^4}\right)\frac{\widehat{G}(\xi,\tau)}{\sqrt{\xi^2+\xi^4}}\, d\xi\right\|_{L_\tau^2}.$$

Using the Cauchy-Schwarz inequality in τ, this can be bounded by

$$\left[\int \langle\tau\rangle^{\frac{2s+1}{2}}\left(\int_{\left|\tau\mp\sqrt{\xi^2+\xi^4}\right|<1} \langle\xi\rangle^{-2s}\, d\xi\right)\left(\int_{\left|\tau\mp\sqrt{\xi^2+\xi^4}\right|<1} \langle\xi\rangle^{2s}\frac{|\widehat{G}(\xi,\tau)|^2}{\xi^2+\xi^4}\, d\xi\right) d\tau\right]^{1/2}$$

$$\lesssim \sup_\tau \left(\langle\tau\rangle^{\frac{2s+1}{2}}\int_{\left|\tau\mp\sqrt{\xi^2+\xi^4}\right|<1} \langle\xi\rangle^{-2s}\, d\xi\right)^{1/2}\|\mathcal{M}(G)\|_{X^{s,-b}}$$

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$$\lesssim \|\mathcal{M}(G)\|_{X^{s,-b}}.$$

The first inequality holds since on the region of interest in I, we have

$$1 \approx \frac{1}{\left\langle \tau \mp \sqrt{\xi^2 + \xi^4} \right\rangle^b} \leqslant \frac{1}{\left\langle |\tau| - \sqrt{\xi^2 + \xi^4} \right\rangle^b}.$$

The supremum bound holds since

$$\langle \tau \rangle^{\frac{2s+1}{2}} \int_{\left| \tau \mp \sqrt{\xi^2+\xi^4} \right|<1} \langle \xi \rangle^{-2s} \,\mathrm{d}\xi \;\lesssim\; \begin{cases} 1 & \text{if } |\tau| \lesssim 1 \\[2mm] \langle \tau \rangle^{\frac{2s+1}{2}} \int_{\left| \tau \mp \sqrt{|z|+z^2} \right|<1} \langle z \rangle^{-s-1/2} \,\mathrm{d}z & \text{if } |\tau| \gg 1. \end{cases}$$

The latter bound comes from changing variables $\xi^2 \mapsto z$. The right-hand side is finite since the integrand is of order $|\tau|^{-s-1/2}$ over an interval of length ≈ 1.

Next consider III. When $|\xi| \leqslant 1$, we have, using $b < \frac{1}{2}$, the bound

$$\left\| \eta(t) \iint \frac{e^{\pm it\sqrt{\xi^2+\xi^4}} \Psi^C \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)}{\sqrt{\xi^2 + \xi^4} \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)} \widehat{G}(\xi, \tau) \,\mathrm{d}\xi \,\mathrm{d}\tau \right\|_{H_t^{\frac{2s+1}{4}}}$$

$$\lesssim \iint_{|\xi| \leqslant 1} \frac{\left\| \eta(t) e^{\pm it\sqrt{\xi^2+\xi^4}} \right\|_{H^{\frac{2s+1}{4}}}}{\left| \tau \mp \sqrt{\xi^2 + \xi^4} \right|} \Psi^C \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right) \left| \widehat{\mathcal{M}(G)}(\xi, \tau) \right| \,\mathrm{d}\xi \,\mathrm{d}\tau$$

$$\lesssim \iint \frac{\chi_{[-1,1]}(\xi)}{\left\langle \tau \mp \sqrt{\xi^2 + \xi^4} \right\rangle} \left| \widehat{\mathcal{M}(G)}(\xi, \tau) \right| \,\mathrm{d}\xi \,\mathrm{d}\tau$$

$$\lesssim \|\mathcal{M}(G)\|_{X^{s,-b}} \left\| \frac{\chi_{[-1,1]}(\xi)}{\left\langle \tau \mp \sqrt{\xi^2 + \xi^4} \right\rangle^{1-b}} \right\|_{L^2_{\tau,\xi}} \lesssim \|\mathcal{M}(G)\|_{X^{s,-b}}.$$

To control the part of III where $|\xi| \geqslant 1$, change variables in the ξ integral by setting $z = \pm\sqrt{\xi^2 + \xi^4} \approx \pm\xi^2$. Then, noticing that the integral is an inverse Fourier transform, we obtain the bound

$$\left\| \eta(t) \iint_{|\xi| \geqslant 1} \frac{e^{\pm it\sqrt{\xi^2+\xi^4}} \Psi^C \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)}{\sqrt{\xi^2 + \xi^4} \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)} \widehat{G}(\xi, \tau) \,\mathrm{d}\xi \,\mathrm{d}\tau \right\|_{H_t^{\frac{2s+1}{4}}}$$

$$\lesssim \left\| \langle z \rangle^{\frac{2s+1}{4}} \int \frac{|\widehat{G}(\xi(z), \tau)|}{\langle \tau - z \rangle |4\xi(z)^3 + 2\xi(z)|} \,\mathrm{d}\tau \right\|_{L^2_{|z| \geqslant 2}}$$

$$\lesssim \left\| \langle z \rangle^{\frac{2s-1}{4}} \int \frac{|\widehat{G}(\xi(z),\tau)|}{\langle \tau - z \rangle \sqrt{\xi(z)^2 + \xi(z)^4}} \, d\tau \right\|_{L^2_{|z| \geqslant 2}}.$$

By Cauchy-Schwarz in the τ integral, using the fact that $b < \frac{1}{2}$, this is bounded by

$$\left\| \langle z \rangle^{\frac{2s-1}{4}} \frac{|\widehat{G}(\xi(z),\tau)|}{\langle \tau - z \rangle^b (\xi(z)^2 + \xi(z)^4)^{1/2}} \right\|_{L^2_{|z| \geqslant 2} L^2_\tau}.$$

Changing variables back to ξ, this is bounded by $\|\mathcal{M}(G)\|_{X^{s,-b}}$, as desired. It remains to bound II.

For II, let R denote the set $\{|\tau| \gg |\xi|^2\} \cup \{|\xi| \lesssim 1\}$ and notice that

$$\langle \tau \rangle \lesssim \chi_R(\xi,\tau) \langle \tau - |\xi|^2 \rangle + |\xi|^2$$

and $(2s+1)/4 \geqslant 0$, so we have the bounds

$$\| \mathrm{II} \|_{H_t^{\frac{2s+1}{4}}} \lesssim \left\| \langle \tau \rangle^{\frac{2s+1}{4}} \int \frac{1}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, d\xi \right\|_{L^2_\tau}$$

$$\lesssim \left\| \int \chi_R(\xi,\tau) \langle |\tau| - \xi^2 \rangle^{\frac{2s-3}{4}} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, d\xi \right\|_{L^2_\tau} + \left\| \int \frac{|\xi|^{s+1/2}}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, d\xi \right\|_{L^2_\tau}$$

The second term on the last line can be bounded by $\|\mathcal{M}(G)\|_{X^{s,b}}$ using Cauchy-Schwarz in ξ provided that

$$\sup_\tau \int \frac{|\xi|}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle^{2-2b}} \, d\xi \; < \; \infty,$$

which holds since $b < 1/2$ and

$$\sup_\tau \int \frac{|\xi|}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle^{2-2b}} \, d\xi \lesssim \sup_\tau \int \frac{|\xi|}{\langle |\tau| - \xi^2 \rangle^{2-2b}} \, d\xi \approx \sup_\tau \int \frac{1}{\langle |\tau| - z \rangle^{2-2b}} \, dz.$$

For $s \leqslant \frac{1}{2}$, we go back to

$$\left\| \langle \tau \rangle^{\frac{2s+1}{4}} \int \frac{1}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, d\xi \right\|_{L^2_\tau}$$

and estimate using Cauchy-Schwarz in ξ to obtain the bound

$$\left[\int \langle \tau \rangle^{\frac{2s+1}{2}} \left(\int \frac{1}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle^{2-2b} \langle \xi \rangle^{2s}} \, d\xi \right) \left(\int \frac{\langle \xi \rangle^{2s}}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle^{2b}} \frac{|\widehat{G}(\xi,\tau)|^2}{\xi^2 + \xi^4} \, d\xi \right) d\tau \right]^{1/2},$$

which can be bounded by $\|\mathcal{M}(G)\|_{X^{s,-b}}$ as long as

$$\sup_\tau \langle \tau \rangle^{\frac{2s+1}{2}} \int \frac{1}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle^{2-2b} \langle \xi \rangle^{2s}} \, d\xi$$

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is finite. To see that this holds for $s \leq \frac{1}{2}$, recall that $\langle \tau \pm \sqrt{\xi^2 + \xi^4} \rangle \approx \langle \tau \pm \xi^2 \rangle$. Consider $|\xi| \ll 1$ first. In this case, change variables in the integral by $\xi^2 \mapsto z$. Then apply Lemma 2.3.4 to bound the integral as follows:

$$\sup_\tau \langle \tau \rangle^{\frac{2s+1}{2}} \int \frac{1}{\langle \tau \mp z \rangle^{2-2b} \langle z \rangle^{s+\frac{1}{2}}} \, \mathrm{d}z \lesssim \sup_\tau \langle \tau \rangle^{s + \frac{1}{2} - \min\{2-2b, s+\frac{1}{2}\}} < \infty,$$

assuming that $b < \frac{1}{2}$ and $-\frac{1}{2} \leq s \leq \frac{1}{2}$. Similarly, if $|\xi| \gtrsim 1$, again change variables by setting $z = \xi^2$ and apply Lemma 2.3.4 to obtain $\langle \tau \rangle^{s + \frac{1}{2} - 2 + 2b}$. This is finite for $b \leq \frac{1}{2}$ and $s \leq \frac{1}{2}$.

For the estimate on the derivative term, the procedure is similar. We break the Duhamel integral down into three pieces $\widetilde{\mathrm{I}} + \widetilde{\mathrm{II}} - \widetilde{\mathrm{III}}$:

$$\eta(t) \int_0^t W_{R,2}^{t-t'} G_x \, \mathrm{d}t' = \eta(t) \iint \frac{i\xi \left(e^{it\tau} - e^{\pm it\sqrt{\xi^2 + \xi^4}} \right) \Psi \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)}{\sqrt{\xi^2 + \xi^4} \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)} \widehat{G}(\xi, \tau) \, \mathrm{d}\xi \, \mathrm{d}\tau$$

$$+ \eta(t) \iint \frac{i\xi e^{it\tau} \Psi^C \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)}{\sqrt{\xi^2 + \xi^4} \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)} \widehat{G}(\xi, \tau) \, \mathrm{d}\xi \, \mathrm{d}\tau$$

$$- \eta(t) \iint \frac{i\xi e^{\pm it\sqrt{\xi^2 + \xi^4}} \Psi^C \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)}{\sqrt{\xi^2 + \xi^4} \left(\tau \mp \sqrt{\xi^2 + \xi^4} \right)} \widehat{G}(\xi, \tau) \, \mathrm{d}\xi \, \mathrm{d}\tau$$

$$= \widetilde{\mathrm{I}} + \widetilde{\mathrm{II}} - \widetilde{\mathrm{III}}.$$

The only difference from the previous case is that each term now has a factor of $i\xi$ from the spatial derivative and we will take fewer time derivatives: $\frac{2s-1}{4}$ instead of $\frac{2s+1}{4}$.

To estimate $\widetilde{\mathrm{I}}$, notice that on the region of integration $\tau \approx \sqrt{\xi^2 + \xi^4}$, so the additional $|\xi|$ factor is equivalent to $|\tau|^{1/2}$. This brings us exactly back to the situation addressed above for I.

To estimate $\widetilde{\mathrm{III}}$, when $|\xi| \lesssim 1$, the bounds are identical to those for III. When $|\xi| \gg 1$, we change variables as we did for III. The additional factor of ξ is equivalent to a factor of $|z|^{1/2}$, which exactly replaces the lost time derivative, and we are again back to the situation addressed in bounding III.

Estimating $\widetilde{\mathrm{II}}$ is a bit more complex. If $s \geq \frac{1}{2}$, we have

$$\|\widetilde{\mathrm{II}}\|_{H_t^{\frac{2s-1}{4}}} \lesssim \left\| \langle \tau \rangle^{\frac{2s-1}{4}} \int \frac{|\xi|}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle \sqrt{\xi^2 + \xi^4}} \frac{|\widehat{G}(\xi, \tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L_\tau^2}$$

$$\lesssim \left\| \int \chi_R(\xi, \tau) \langle |\tau| - \xi^2 \rangle^{\frac{2s-3}{4}} \frac{|\widehat{G}(\xi, \tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L_\tau^2} + \left\| \int \frac{|\xi|^{s+1/2}}{\langle \tau \mp \sqrt{\xi^2 + \xi^4} \rangle} \frac{|\widehat{G}(\xi, \tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L_\tau^2}.$$

Thus when $s \geqslant \frac{1}{2}$, we have the bound

$$\|\mathcal{M}(G)\|_{X^{s,-b}} + \left\| \int \chi_R(\xi,\tau)\langle|\tau| - \xi^2\rangle^{\frac{2s-3}{4}} |\widehat{\mathcal{M}(G)}(\xi,\tau)| \, \mathrm{d}\xi \right\|_{L^2_\tau}$$

as before. For $s \leqslant \frac{1}{2}$, go back to

$$\left\| \langle\tau\rangle^{\frac{2s-1}{4}} \int \frac{|\xi|}{\langle\tau \mp \sqrt{\xi^2 + \xi^4}\rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L^2_\tau}.$$

On the region where $|\tau| \ll \xi^2$ and $|\xi| \gtrsim 1$, we obtain the bound

$$\left\| \langle\tau\rangle^{\frac{2s-1}{4}} \int \chi_Q(\xi,\tau) \frac{1}{\langle\xi\rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L^2_\tau},$$

where $Q = \{|\tau| \ll |\xi|^2\} \cap \{|\xi| \gtrsim 1\}$.

On the region where $|\tau| \gtrsim |\xi|^2$ or $|\xi| \lesssim 1$, we have

$$\|\widetilde{\Pi}\|_{H_t^{\frac{2s-1}{4}}} \lesssim \left\| \langle\tau\rangle^{\frac{2s-1}{4}} \int \frac{|\xi|}{\langle\tau \mp \sqrt{\xi^2 + \xi^4}\rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L^2_\tau}$$

$$\lesssim \left\| \int \frac{|\xi|^{s+1/2}}{\langle\tau \mp \sqrt{\xi^2 + \xi^4}\rangle} \frac{|\widehat{G}(\xi,\tau)|}{\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L^2_\tau},$$

which can be bounded by $\|\mathcal{M}G\|_{X^{s,-b}}$ as we have already seen. This completes the proof.

5.7.6 Proof of Lemma 5.4.6

We want to show that

$$\left\| \langle\tau\rangle^{\frac{2(s+a)-1}{4}} \int_{\substack{|\xi| \gtrsim 1 \\ |\tau| \ll \xi^2}} \frac{\xi^2 \, \widehat{uv}(\xi,\tau)}{\langle\xi\rangle\sqrt{\xi^2 + \xi^4}} \, \mathrm{d}\xi \right\|_{L^2_\tau} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}$$

for $\frac{1}{2} - b > 0$ sufficiently small.

Writing the Fourier transform of uv as a convolution, we have

$$\widehat{uv}(\xi,\tau) = \iint \widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi - \xi_1, \tau - \tau_1) \, \mathrm{d}\xi_1 \, \mathrm{d}\tau_1.$$

Let $f(\xi,\tau) = \widehat{u}(\xi,\tau)\langle\xi\rangle^s\langle|\tau| - \xi^2\rangle^b$ and $g(\xi,\tau) = \widehat{v}(\xi,\tau)\langle\xi\rangle^s\langle|\tau| - \xi^2\rangle^b$. Using this and dropping the factor $\xi^2/\sqrt{\xi^2 + \xi^4}$, the desired bound becomes

$$\left\| \langle\tau\rangle^{\frac{2(s+a)-1}{4}} \iiint_{\substack{|\xi| \gtrsim 1 \\ |\tau| \ll \xi^2}} \frac{f(\xi_1,\tau_1)g(\xi - \xi_1, \tau - \tau_1)\langle\xi_1\rangle^{-s}\langle\xi - \xi_1\rangle^{-s}}{\langle\xi\rangle\langle|\tau_1| - \xi_1^2\rangle^b\langle|\tau - \tau_1| - (\xi - \xi_1)^2\rangle^b} \, \mathrm{d}\xi \, \mathrm{d}\xi_1 \, \mathrm{d}\tau_1 \right\|_{L^2_\tau} \tag{5.13}$$

$$\lesssim \|f\|_{L^2_\xi L^2_\tau} \|g\|_{L^2_\xi L^2_\tau}.$$

Case 1. $\operatorname{sgn}(\tau_1) = \operatorname{sgn}(\tau - \tau_1)$.

Using the Cauchy-Schwarz inequality in the ξ-ξ_1-τ_1 integral of (5.13), then Cauchy-Schwarz in τ, and finally Young's inequality, we obtain the bounds

$$\left\| \|M\|_{L^2_{\xi,\xi_1,\tau_1}\left(|\xi|\gtrsim 1, |\tau|\ll \xi^2\right)} \|f(\xi_1,\tau_1)g(\xi-\xi_1,\tau-\tau_1)\|_{L^2_{\xi,\xi_1,\tau_1}} \right\|_{L^2_\tau}$$

$$\lesssim \left(\sup_\tau \iiint_{\substack{|\xi|\gtrsim 1 \\ |\tau|\ll\xi^2}} M^2 \, \mathrm{d}\xi \, \mathrm{d}\xi_1 \, \mathrm{d}\tau_1 \right)^{1/2} \|f^2 * g^2\|_{L^1_{\xi,\tau}}^{1/2}$$

$$\lesssim \left(\sup_\tau \iiint_{\substack{|\xi|\gtrsim 1 \\ |\tau|\ll\xi^2}} M^2 \, \mathrm{d}\xi \, \mathrm{d}\xi_1 \, \mathrm{d}\tau_1 \right)^{1/2} \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}}.$$

where

$$M = M(\xi_1, \xi, \tau, \tau_1) = \frac{\langle\tau\rangle^{\frac{2(s+a)-1}{4}}}{\langle\xi\rangle\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s\langle|\tau_1|-\xi_1^2\rangle^b\langle|\tau-\tau_1|-(\xi-\xi_1)^2\rangle^b}.$$

Thus, it suffices to show that the supremum above is finite. Using Lemma 2.3.4 in the τ_1 integral, along with the assumption that τ_1 and $\tau-\tau_1$ have the same sign, we arrive at

$$\sup_\tau \iint_{\substack{|\xi|\gtrsim 1 \\ |\tau|\ll\xi^2}} \frac{\langle\tau\rangle^{\frac{2(s+a)-1}{2}}\langle\xi\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\xi\rangle^2\langle\tau\pm(\xi_1^2+(\xi-\xi_1)^2)\rangle^{4b-1}} \, \mathrm{d}\xi \, \mathrm{d}\xi_1.$$

Since $|\tau| \ll \xi^2$ and $\xi_1^2 + (\xi-\xi_1)^2 \gtrsim \max\{\xi_1^2, \xi^2\}$, we have

$$\langle\tau\pm(\xi_1^2+(\xi-\xi_1)^2)\rangle^{4b-1} \approx \langle\max\{|\xi|,|\xi_1|\}\rangle^{8b-2}.$$

Using this, and dropping the $\langle\tau\rangle$ term, it suffices to bound

$$\iint \frac{\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\xi\rangle^2\langle\max\{|\xi|,|\xi_1|\}\rangle^{8b-2}} \, \mathrm{d}\xi \, \mathrm{d}\xi_1.$$

If $|\xi| \gtrsim |\xi_1|$, this is bounded by

$$\iint_{|\xi_1|\lesssim|\xi|} \langle\xi_1\rangle^{-2s}\langle\xi\rangle^{-8b}\langle\xi-\xi_1\rangle^{-2s} \, \mathrm{d}\xi \, \mathrm{d}\xi_1 \lesssim \iint_{|\xi_1|\lesssim|\xi|} \langle\xi_1\rangle^{-2s}\langle\xi\rangle^{-8b}\langle\xi\rangle^{\max\{-2s,0\}} \, \mathrm{d}\xi \, \mathrm{d}\xi_1$$

$$\lesssim \int \langle\xi\rangle^{1-2s}\langle\xi\rangle^{-8b}\langle\xi\rangle^{\max\{-2s,0\}} \, \mathrm{d}\xi,$$

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which is finite if $1 - 2s - 8b + \max\{-2s, 0\} < -1$, which holds for $s > \frac{1}{2} - 2b$, i.e. $s > -\frac{1}{2}$ for $\frac{1}{2} - b > 0$ sufficiently small. If $|\xi| \ll |\xi_1|$, we need to bound

$$\iint \langle \xi_1 \rangle^{2-8b-4s} \langle \xi \rangle^{-2} \, d\xi \, d\xi_1 \lesssim \int \langle \xi_1 \rangle^{2-8b-4s} \, d\xi_1,$$

which is finite if $s > -\frac{1}{4}$ and $\frac{1}{2} - b > 0$ is sufficiently small.

Case 2. $\mathrm{sgn}(\tau_1) \neq \mathrm{sgn}(\tau - \tau_1)$ and $|\xi_1| \lesssim |\xi|$.

Using Cauchy-Schwarz and Young's inequalities just as in Case 1 and dropping the $\langle \tau \rangle$ term, it suffices to show that

$$\sup_{\tau} \iint_{\substack{|\xi| \gtrsim 1 \\ |\xi| \gtrsim |\xi_1| \\ |\tau| \ll \xi^2}} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2s}}{\langle \xi \rangle^2 \langle \tau \pm \xi(\xi - 2\xi_1) \rangle^{2b}} \, d\xi_1 \, d\xi$$

is finite. Using the change of variables $z = \xi(\xi - 2\xi_1)$ in the ξ_1 integral, we arrive at

$$\sup_{\tau} \iint_{\substack{|z| \lesssim \xi^2 \\ |\tau| \ll \xi^2}} \frac{\langle \xi - z/\xi \rangle^{-2s} \langle \xi + z/\xi \rangle^{-2s}}{\langle \xi \rangle^3 \langle \tau \pm z \rangle^{2b}} \, dz \, d\xi.$$

Notice that since $|z| \lesssim \xi^2$, we have $|\xi \pm z/\xi| \lesssim |\xi|$. This yields

$$\sup_{\tau} \iint_{\substack{|z| \lesssim \xi^2 \\ |\tau| \ll \xi^2}} \frac{\langle \xi \rangle^{\max\{-4s, 0\}}}{\langle \xi \rangle^3 \langle \tau \pm z \rangle^{2b}} \, dz \, d\xi \lesssim \int \langle \xi \rangle^{\max\{-4s, 0\}} \langle \xi \rangle^{-3} \langle \xi \rangle^{2(1-2b)} \, d\xi,$$

which is finite for $s > -\frac{1}{2}$ and $\frac{1}{2} - b > 0$ sufficiently small.

Case 3. $\mathrm{sgn}(\tau_1) \neq \mathrm{sgn}(\tau - \tau_1)$ and $|\xi| \ll |\xi_1|$.

By duality, to establish (5.13), it suffices to show that

$$\left| \iiiint_{\substack{|\tau| \ll \xi^2 \\ 1 \lesssim |\xi| \ll |\xi_1|}} M \, f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) \, \phi(\tau) \, d\xi \, d\tau \, d\xi_1 \, d\tau_1 \right| \lesssim \|\phi\|_{L_\tau^2} \|f\|_{L_{\xi,\tau}^2} \|g\|_{L_{\xi,\tau}^2},$$

where $M = M(\xi, \xi_1, \tau, \tau_1)$ is defined as in Case 1. Using Cauchy-Schwarz in the ξ_1-τ_1 integrals, it suffices to show that

$$\left\| \iint_{\substack{|\tau| \ll \xi^2 \\ 1 \lesssim |\xi| \ll |\xi_1|}} M \, g(\xi - \xi_1, \tau - \tau_1) \, \phi(\tau) \, d\xi \, d\tau \right\|_{L_{\xi_1, \tau_1}^2} \lesssim \|\phi\|_{L_\tau^2} \|g\|_{L_{\xi,\tau}^2}.$$

Using Cauchy-Schwarz in ξ-τ and then Young's inequality, the left-hand side of this quantity is bounded by

$$\left\| \|M\langle\xi\rangle^{\frac{1}{2}+}\|_{L^2_{\xi,\tau}\left(|\tau|\ll\xi^2, 1\lesssim|\xi|\ll|\xi_1|\right)} \|g(\xi-\xi_1,\tau-\tau_1)\phi(\tau)\langle\xi\rangle^{-\frac{1}{2}-}\|_{L^2_{\xi,\tau}} \right\|_{L^2_{\xi_1,\tau_1}}$$

$$\lesssim \left(\sup_{\xi_1,\tau_1} \iint\limits_{\substack{|\tau|\ll\xi^2 \\ 1\lesssim|\xi|\ll|\xi_1|}} M^2\langle\xi\rangle^{1+} \,d\xi\,d\tau \right)^{1/2} \|g\|_{L^2_{\xi,\tau}}\|\langle\xi\rangle^{-\frac{1}{2}-}\|_{L^2_\xi}\|\phi\|_{L^2_\tau}.$$

Thus it suffices to show the following supremum is finite:

$$\sup_{\xi_1} \iint\limits_{\substack{|\tau|\ll\xi^2 \\ 1\lesssim|\xi|\ll|\xi_1|}} \frac{\langle\tau\rangle^{\frac{2(s+a)-1}{2}}\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\xi\rangle^{1-}\langle\tau\pm\xi(\xi-2\xi_1)\rangle^{2b}} \,d\xi\,d\tau$$

$$\lesssim \sup_{\xi_1} \langle\xi_1\rangle^{-4s} \iint\limits_{\substack{|\tau|\ll\xi^2 \\ |\xi|\ll|\xi_1|}} \frac{\langle\tau\rangle^{\frac{2(s+a)-1}{2}}}{\langle\xi\rangle^{1-}\langle\xi\rangle^{2b}\langle\xi_1\rangle^{2b}} \,d\xi\,d\tau$$

$$\lesssim \sup_{\xi_1} \langle\xi_1\rangle^{-2b-4s} \int_{|\xi|\ll|\xi_1|} \frac{\langle\xi\rangle^{2(s+a)+1}}{\langle\xi\rangle^{1-}\langle\xi\rangle^{2b}} \,d\xi$$

$$\lesssim \sup_{\xi_1} \langle\xi_1\rangle^{-2b-4s}\langle\xi_1\rangle^{\max\{0, 2(s+a)-2b+1+\}}.$$

If the maximum in the last line is zero, we have a finite bound for $s > -\frac{1}{4}$ if $\frac{1}{2}-b>0$ is sufficiently small. Otherwise we require $-4b-4s+2(s+a)+1 < 0$, which holds for $a < s + \frac{1}{2}$ as long as $\frac{1}{2}-b>0$ is sufficiently small.

5.7.7   Proof of Lemma 5.4.7

Recall that we want to show that for $\frac{1}{2} < s+a \leq \frac{5}{2}$ and $a < \min\{1, s+\frac{1}{2}\}$, we have

$$\left\| \int \chi_R(\xi,\tau)\langle|\tau|-\xi^2\rangle^{\frac{2(s+a)-3}{4}} \frac{\xi^2}{\sqrt{\xi^2+\xi^4}}|\widehat{uv}(\xi,\tau)| \,d\xi \right\|_{L^2_\tau} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

Writing the Fourier transform as a convolution and canceling the $\xi^2/\sqrt{\xi^2+\xi^4}$ factor, we need to bound

$$\left\| \iiint \chi_R(\xi,\tau)\langle|\tau|-\xi^2\rangle^{\frac{2(s+a)-3}{4}} |\widehat{u}(\xi_1,\tau_1)||\widehat{v}(\xi-\xi_1,\tau-\tau_1)| \,d\xi_1\,d\tau_1\,d\xi \right\|_{L^2_\tau}$$

$$= \left\| \iiint \frac{\chi_R(\xi,\tau)\,\langle|\tau|-\xi^2\rangle^{\frac{2(s+a)-3}{4}}|f(\xi_1,\tau_1)||g(\xi-\xi_1,\tau-\tau_1)|\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi}{\langle\xi_1\rangle^s\langle\xi-\xi_1\rangle^s\langle|\tau_1|-\xi_1^2\rangle^b\langle|\tau-\tau_1|-(\xi-\xi_1)^2\rangle^b} \right\|_{L^2_\tau}$$

where

$$f(\xi,\tau) = \widehat{u}(\xi,\tau)\langle\xi\rangle^s\langle|\tau|-\xi^2\rangle^b \qquad g(\xi,\tau) = \widehat{v}(\xi,\tau)\langle\xi\rangle^s\langle|\tau|-\xi^2\rangle^b.$$

Using the Cauchy-Schwarz inequality in the ξ_1-τ_1-ξ integral, followed by Young's inequality as in the proof of Lemma 5.4.6, it suffices to show that

$$\sup_\tau \iiint \chi_R(\xi,\tau)\frac{\langle|\tau|-\xi^2\rangle^{\frac{2(s+a)-3}{2}}\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle|\tau_1|-\xi_1^2\rangle^{2b}\langle|\tau-\tau_1|-(\xi-\xi_1)^2\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi$$

is finite. If τ_1 and $\tau-\tau_1$ have the same sign, we apply Lemma 2.3.4 in the τ_1 integral and obtain the bound

$$\sup_\tau \iint \chi_R(\xi,\tau)\frac{\langle|\tau|-\xi^2\rangle^{\frac{2(s+a)-3}{2}}\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle|\tau|-\xi^2+2\xi_1(\xi-\xi_1)\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi. \tag{5.14}$$

If τ_1 and $\tau-\tau_1$ have different signs, it's bounded by

$$\sup_\tau \iint \chi_R(\xi,\tau)\frac{\langle\lambda\tau-\xi^2\rangle^{\frac{2(s+a)-3}{2}}\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2s}}{\langle\lambda\tau-\xi^2+2\xi\xi_1\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi,$$

where $\lambda = \mathrm{sgn}(\tau-\tau_1) = \pm1$. Here we have taken advantage of the fact that we're confined to the set R to conclude that $\langle|\tau|-\xi^2\rangle \approx \langle\lambda\tau-\xi^2\rangle$. Changing variables in the ξ_1 integral by $\xi_1 \mapsto \xi-\xi_1$, and dropping the λ, we obtain

$$\sup_\tau \iint \chi_R(\xi,\tau)\frac{\langle\tau-\xi^2\rangle^{\frac{2(s+a)-3}{2}}\langle\xi-\xi_1\rangle^{-2s}\langle\xi_1\rangle^{-2s}}{\langle\tau-\xi^2+2\xi(\xi-\xi_1)\rangle^{2b}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi. \tag{5.15}$$

When $\frac{3}{2} \leqslant s+a < \frac{5}{2}$, we use the inequalities

$$\langle|\tau|-\xi^2\rangle \lesssim \langle|\tau|-\xi^2+2\xi_1(\xi-\xi_1)\rangle\langle\xi_1\rangle\langle\xi-\xi_1\rangle \quad \text{and}$$

$$\langle\tau-\xi^2\rangle \lesssim \langle\tau-\xi^2+2\xi(\xi-\xi_1)\rangle\langle\xi\rangle\langle\xi-\xi_1\rangle$$

in (5.14) and (5.15) respectively. They yield the following bounds for (5.14) and (5.15):

$$\sup_\tau \iint_R \frac{1}{\langle\xi_1\rangle^{s+\frac{3}{2}-a}\langle\xi-\xi_1\rangle^{s+\frac{3}{2}-a}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi \qquad \text{for (5.14) and}$$

$$\sup_\tau \iint_R \frac{\langle\xi\rangle^{s+a-\frac{3}{2}}}{\langle\xi_1\rangle^{2s}\langle\xi-\xi_1\rangle^{s+\frac{3}{2}-a}}\,\mathrm{d}\xi_1\,\mathrm{d}\xi \qquad \text{for (5.15).}$$

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Using Lemma 2.3.4, we see that former is finite as long as $s + \frac{3}{2} - a > 1$, which holds when $a < s + \frac{1}{2}$.

For the latter, we use Lemma 2.3.4 in the ξ_1 integral, using the assumption that $a < s + \frac{1}{2}$, to obtain $\int \langle \xi_1 \rangle^{s + a - \frac{3}{2} - (s + \frac{3}{2} - a)} \, d\xi_1$, which is convergent for $a < 1$.

When $\frac{1}{2} < s + a < \frac{3}{2}$, we use the inequality $\langle \tau - a \rangle \langle \tau - b \rangle \gtrsim \langle a - b \rangle$ to obtain

$$\iint \frac{1}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s} \langle \xi_1 (\xi - \xi_1) \rangle^{\frac{3}{2} - s - a}} \, d\xi_1 \, d\xi \qquad \text{from (5.14) and}$$

$$\iint \frac{1}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s} \langle \xi (\xi_1 - \xi) \rangle^{\frac{3}{2} - s - a}} \, d\xi_1 \, d\xi \qquad \text{from (5.15).}$$

In the nonresonant cases, i.e. when $|\xi_1|, |\xi - \xi_1| \gtrsim 1$ for the first equation and when $|\xi|, |\xi - \xi_1| \gtrsim 1$ for the second equation, we have $\langle \xi_1 (\xi - \xi_1) \rangle \approx \langle \xi_1 \rangle \langle \xi - \xi_1 \rangle$ and $\langle \xi (\xi_1 - \xi) \rangle \approx \langle \xi \rangle \langle \xi_1 - \xi \rangle$ respectively. Thus we have convergence if $a < s + \frac{1}{2}$ for the first equation. In the second equation, we use Lemma 2.3.4 to the estimate the ξ_1 integral. This yields a bound of $\int \langle \xi \rangle^{\frac{3}{2} + s - a} \, d\xi$, which is convergent if $a < s + \frac{1}{2}$.

The resonances can be treated simply. In the first equation, when $|\xi_1| \lesssim 1$, we have

$$\iint \frac{1}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2s} \langle \xi_1 (\xi - \xi_1) \rangle^{\frac{3}{2} - s - a}} \, d\xi_1 \, d\xi \lesssim \int_{-1}^{1} \int \langle \xi - \xi_1 \rangle^{-2s} \, d\xi \, d\xi_1.$$

This converges since $s > \frac{1}{2}$. The remaining resonances can be handled in exactly the same way – drop two of the three factors, and integrate, using the fact that we're integrating over a finite interval in one of the dimensions and that $s > \frac{1}{2}$ to obtain convergence.

5.8 Proof of Lemma 5.3.1: Explicit Linear Solution Formula

We begin by recalling a few definitions and properties related to the Laplace transform. For further information and proofs, see [89]. First, the Laplace transform of a function $u(t)$ on $[0, \infty)$ is defined by

$$\tilde{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) \, dt.$$

Furthermore, by an integration by parts argument, if u is twice differentiable for $t \in (0, \infty)$ and $\widetilde{u''}$ converges for some $\lambda_0 > 0$, we have

$$\widetilde{u''}(\lambda) = \lambda^2 \tilde{u}(\lambda) - \lambda \lim_{t \to 0^+} u(t) - \lim_{t \to 0^+} u'(t),$$

for $\lambda = \lambda_0$ or $\operatorname{Re}\lambda > \lambda_0$.

We will also need to invert Laplace transforms. This is possible since if two functions have the same Laplace transform, they are equal almost everywhere. We have the following Mellin inversion formula. Suppose that the transform $\tilde{u}(\lambda)$ converges absolutely for some $\lambda_0 > 0$. Then we have the equality

$$u(t) = \frac{1}{2\pi i} \int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} e^{t\lambda} F(\lambda)\, d\lambda.$$

Taking the Laplace transform in time of (5.3), bearing in mind the zero initial conditions, yields the equation

$$\begin{cases} \lambda^2 \tilde{v}(x, \lambda) - \tilde{v}_{xx}(x, \lambda) + \tilde{v}_{xxxx}(x, \lambda) = 0, \\ \tilde{v}(0, \lambda) = \widetilde{h_1}(\lambda) \quad \tilde{v}_x(0, \lambda) = \widetilde{h_2}(\lambda). \end{cases}$$

The characteristic equation of this ordinary differential equation is $\lambda^2 - w^2 + w^4 = 0$, which has roots satisfying

$$w^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda^2}.$$

Notice that $\sqrt{\frac{1}{4} - \lambda^2}$ can be defined analytically on $\mathbb{C}\backslash[-1/2, 1/2]$ by

$$\left| \frac{1}{4} - \lambda^2 \right|^{1/2} e^{i(\theta_1 + \theta_2 + \pi)/2},$$

where $\theta_1 = \arg(\lambda + \frac{1}{2})$ and $\theta_2 = \arg(\lambda - \frac{1}{2})$. This map sends

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant 0, \ \lambda \notin [0, 1/2]\} \mapsto \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \geqslant 0, \ \lambda \notin [-1/2, 1/2]\}.$$

Let

$$a = -\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda^2} \right)^{1/2} \qquad b = -\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda^2} \right)^{1/2},$$

where the outermost root in a is defined with a branch cut in the bottom half-plane and the outermost root in b is defined with a branch cut in the top half-plane.

Then a and b are analytic for λ in the closed right half-plane except for the branch cut $[-1/2, 1/2]$. We also have $\operatorname{Re}a, \operatorname{Re}b \leqslant 0$ for all λ in the closed right half-plane. Since we're interested in solutions which decay at infinity, we only need concern ourselves with these two roots of the characteristic equation. Thus, supressing the λ dependence of a and b, we have

$$\tilde{u}(x, \lambda) = \frac{1}{a - b} \left[\left(a\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{bx} - \left(b\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{ax} \right].$$
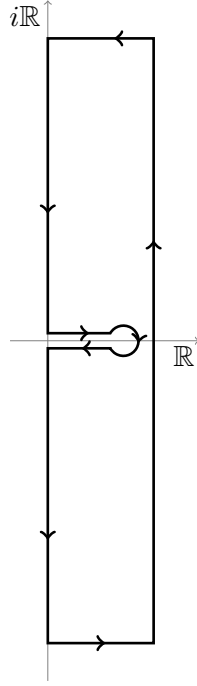
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Figure 5.1: The contour of integration

By Mellin inversion, we have, for any $c > \frac{1}{2}$, the equality

$$v(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda t}}{a-b} \left[\left(a\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{bx} - \left(b\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{ax} \right] \mathrm{d}\lambda$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\lambda t}}{a^2 - b^2} (a+b) \left[\left(a\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{bx} - \left(b\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{ax} \right] \mathrm{d}\lambda.$$

We can write this as an integral along the imaginary axis plus integrals along a keyhole contour about the branch cut and integrals along $s \pm iR$ for $s \in [0, c]$ with $R \to \infty$, as shown in Figure 5.1. The loop of radius ϵ about the singularity at $\lambda = 1/2$ can be disregarded since the integrand is at most order $1/(a^2 - b^2) \approx |\lambda - 1/2|^{-1/2} \approx \epsilon^{-1/2}$ there, while the length of the contour is order ϵ.

The integration along the lines $s \pm i\epsilon$ for $s \in [0, 1/2 - \epsilon]$ vanishes in the limit as $\epsilon \to 0$ – the integrals along the two lines cancel one another. This happens because $a(\bar{\lambda}) = \overline{a(\lambda)}$ and $b(\bar{\lambda}) = \overline{b(\lambda)}$. Thus integration over the two lines $s \pm i\epsilon$ for $s \in [0, 1/2 - \epsilon]$ is equal to twice the imaginary part of the integral over one of the lines. But the imaginary part of the integrand vanishes as $\epsilon \to 0$.

The decay of the integrals along $s \pm iR$ for $s \in [0, c]$ as $R \to \infty$ is justified as follows. By

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integration by parts, we have the bound

$$\left| \widetilde{h_i}(s \pm iR) \right| \lesssim R^{-1} \left(\|h_i\|_{L^\infty} + \|h_i\|_{L^1} + \|h_i'\|_{L^1} \right)$$

for $s \in [0, c]$. We also have

$$|a|, |b| \lesssim R^{1/2} \quad \text{and} \quad |a^2 - b^2| \approx R$$

for $\lambda = s \pm iR$ with R large. Thus, on these segments the integrand is order at most R^{-1}. Since the intervals are of finite length, we obtain decay as $R \to \infty$.

Thus we change the contour of integration to the imaginary axis, and arrive at

$$
\begin{aligned}
v(x, t) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\lambda t}}{a - b} \left[\left(a\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{bx} - \left(b\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{ax} \right] d\lambda \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\lambda t}}{a^2 - b^2} (a + b) \left[\left(a\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{bx} - \left(b\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{ax} \right] d\lambda \\
&= 2 \operatorname{Re} \frac{1}{2\pi i} \int_0^{i\infty} \frac{e^{\lambda t}}{a^2 - b^2} (a + b) \left[\left(a\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{bx} - \left(b\widetilde{h_1}(\lambda) - \widetilde{h_2}(\lambda) \right) e^{ax} \right] d\lambda.
\end{aligned}
$$

Make the change of variables $\lambda = i\mu\sqrt{\mu^2 + 1}$. Then $d\lambda = i\frac{2\mu^2 + 1}{\sqrt{\mu^2 + 1}} d\mu$. On the positive imaginary axis

$$a = -i\mu \quad \text{and} \quad b = -\sqrt{\mu^2 + 1},$$

so $v = \frac{1}{\pi} \operatorname{Re}(A_0 + B_0 + C_0 + D_0)$, where

$$
\begin{aligned}
A_0 &= -\int_0^\infty \frac{e^{it\mu\sqrt{\mu^2+1} - x\sqrt{\mu^2+1}}}{\sqrt{1 + \mu^2}} i\mu \left(i\mu + \sqrt{1 + \mu^2} \right) \widehat{h_1} \left(\mu\sqrt{\mu^2 + 1} \right) d\mu \\
B_0 &= -\int_0^\infty \frac{e^{it\mu\sqrt{\mu^2+1} - x\sqrt{\mu^2+1}}}{\sqrt{1 + \mu^2}} \left(i\mu + \sqrt{1 + \mu^2} \right) \widehat{h_2} \left(\mu\sqrt{\mu^2 + 1} \right) d\mu \\
C_0 &= \int_0^\infty e^{it\mu\sqrt{\mu^2+1} - ix\mu} \left(i\mu + \sqrt{1 + \mu^2} \right) \widehat{h_1} \left(\mu\sqrt{\mu^2 + 1} \right) d\mu \\
D_0 &= \int_0^\infty \frac{e^{it\mu\sqrt{\mu^2+1} - ix\mu}}{\sqrt{1 + \mu^2}} \left(i\mu + \sqrt{1 + \mu^2} \right) \widehat{h_2} \left(\mu\sqrt{\mu^2 + 1} \right) d\mu.
\end{aligned}
$$

For $x \geq 0$, this is equivalent to $2\pi v(x, t) = -A - B + C + D$. Here we used the formula $2\operatorname{Re} z = z + \bar{z}$ to rewrite the real parts of A_0, B_0, C_0, and D_0, and added the cut-off function ρ in A and B so that the integrals converge for all x.

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