SYZYGIES AND IMPLICITIZATION OF TENSOR PRODUCT SURFACES

BY

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DISSERTATION

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Abstract

A tensor product surface is the closure of the image of a rational map $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$. These surfaces arise in geometric modeling and in this context it is useful to know the implicit equation of λ in $\mathbb{P}^3$. Currently, syzygies and Rees algebras provide the fastest and most versatile method to find implicit equations of parameterized surfaces. Knowing the structure of the syzygies of the polynomials that define the map λ allows us to formulate faster algorithms for implicitization of these surfaces and also to understand their singularities. We show that for tensor product surfaces without basepoints, the existence of a linear syzygy imposes strong conditions on the structure of the syzygies that determine the implicit equation. For tensor product surfaces with basepoints we show that the syzygies that determine the implicit equation of λ are closely related to the geometry of the set of points at which λ is undefined.

Para mi familia

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List of Symbols

$\mathrm{Sym}_R(I)$	symmetric algebra of the ideal I over R
$\mathcal{R}_I$	Rees algebra of the ideal I
$\mathrm{Syz}(I)$	syzygy module of the ideal I
$\mathrm{Syz}(f_0, \ldots, f_n)$	syzygy module of $f_0, \ldots, f_n$
$\mathcal{Z}$	complex $\mathcal{Z}$

Chapter 1

Introduction

1.1 Introduction

Given a parameterized curve or surface in projective space, such as the image of $\mathbb{P}^1 \to \mathbb{P}^2$ or $\mathbb{P}^2 \to \mathbb{P}^3$, the implicitization problem consists of finding the equations whose vanishing locus defines the closed image of the given parameterization. The implicitization problem has been of increasing interest to commutative algebraists and algebraic geometers due to its applications in Computer Aided Geometric Design(CAGD). In this context, knowing the implicit equation of the curve or surface is important to perform elementary operations with these objects. For example, it is easier to describe the curve of intersection of two surfaces or the points of intersection of a curve and a surface by using the implicit equations of the surfaces instead of their parameterizations. In a similar way, the problem of testing wether a given point in the codomain lies in the image of the parameterization becomes easier by having the implicit equation of the closed image. Using the implicit equations of a curve or surface to perform these operations is computationally and theoretically more efficient than only working with the parameterizations. For this reason, there is great interest in finding faster algorithms that compute implicit equations of parameterized curves and surfaces.

There are several algebraic tools to solve the implicitization problem for parameterized curves and surfaces. These include Gröbner bases, resultants and syzygies. To summarize, Gröbner bases algorithms provide a straightforward theoretical approach that in practice tends to be very slow. Resultants provide a more convenient representation of the implicit equation as a determinant of a matrix but they fail for parameterizations with *basepoints*. In contrast, since their first appearance in the work of Sederberg and Chen [SC95] and Cox [Cox03], syzygies have provided faster methods to obtain implicit equations that work for more general parameterizations.

Throughout this thesis we focus on the implicitization problem for tensor product surfaces using syzygy techniques. The main goal is to describe the structure of the syzygies that determine the implicit equation of a tensor product surface in order to learn about their singularities and to set up faster algorithms that compute their implicit equations.

Given a 4-dimensional vector space $U = \{f_0, f_1, f_2, f_3\} \subset H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b))$ with $\gcd(f_0, f_1, f_2, f_3) = 1$, we obtain a rational map $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$. A *tensor product* surface is the closure of the image of λ_U and is denoted X_U. One of the main tools of implicitization with syzygies is the use of a complex $\mathcal{Z}$ of graded modules associated to $U = \{f_0, f_1, f_2, f_3\}$. In short, to find the implicit equation of X_U using $\mathcal{Z}$, we fix a basis and find the matrices representing the maps of $\mathcal{Z}$ in a suitable degree ν. Finally, the determinant of the complex $\mathcal{Z}_\nu$ is a power of the implicit equation of X_U.

The connection of $\mathcal{Z}$ with the syzygies of $U = \{f_0, f_1, f_2, f_3\}$ comes from the observation that $\mathcal{Z}_1 = \mathrm{Syz}(f_0, f_1, f_2, f_3) \otimes S$, where $S := k[X, Y, Z, W]$ is the coordinate ring of $\mathbb{P}^3$. Thus to obtain the matrix of $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ we are led to the computation of $\mathrm{Syz}(f_0, f_1, f_2, f_3)$. Let $I_U = \langle f_0, f_1, f_2, f_3 \rangle$ be the ideal generated by U inside the total coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. The vanishing locus $\mathbf{V}(I_U)$ inside $\mathbb{P}^1 \times \mathbb{P}^1$ is referred to as the set of basepoints of U and is denoted by $\mathcal{B}$. If $\mathcal{B} \neq \emptyset$ we say U has basepoints, otherwise we say U is basepoint free. We also refer to $\mathcal{B}$ as the base locus of λ_U.

It is important to point out that for most practical purposes in CAGD, knowing the matrix of the map $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ is sufficient to perform the aforementioned elementary operations with curves and surfaces, for example as shown by Busé and Luu Ba [BLB12] for the intersection of two surfaces. The matrix d_1 is known as a representation matrix for X_U. A representation matrix for X_U is generically of full rank and the gcd of its maximal minors is equal to a power of the implicit equation of X_U. These matrices have been studied by Botbol and Dickenstein [BD16] and Botbol, Busé and Chardin [BBC14] among others.

The main original results of this thesis are presented in Chapters 2 and 3. Chapter 2 is based on the paper [DS16] and deals with tensor product surfaces without basepoints. We describe the structure of the syzygies that determine the implicit equation of X_U under the assumption that I_U has a linear syzygy of bidegree $(1,0)$ or bidegree $(0,1)$ and $U \subset H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b))$ with $a, b \geq 2$ is basepoint-free. In this case we prove that the existence of a linear syzygy imposes very strong constraints, in particular the matrix of the map $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ has a very special shape which allows us to describe the codimension one singular locus of X_U. Chapter 3 is based on the paper [Dua16] and focuses on tensor product surfaces such that $U \subseteq H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,1))$ has a generic set of basepoints in $\mathbb{P}^1 \times \mathbb{P}^1$. We show that the syzygies that appear in the map $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ are determined by specific generators of the ideal associated to the generic set of points $\mathcal{B}$. All of the evidence for this work was provided by many computations in Macaulay 2.

The exposition of the results in this thesis is slightly modified from that in [DS16] and [Dua16]. In the remaining part of this chapter we provide the common background for implicitization of tensor product surfaces needed for [DS16] and [Dua16]. Therefore the content of Chapters 2 and 3 is focused on the theorems and examples from [DS16] and [Dua16]. In Chapter 4 we present several examples and algorithms based on the results obtained in Chapters 2 and 3. We also discuss future work on implicitization of tensor product surfaces of more general bidegree based on a more detailed understanding of ideals of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Throughout, k will denote an algebraically closed field of characteristic zero.

1.2 Implicitization with syzygies

The use of syzygies to find implicit equations of parameterized curves and surfaces was first introduced by Sederberg, Chen [SC95] and Cox [Cox03]. Using the syzygies of the defining polynomials of a curve or surface parameterization, Sederberg and Chen define a square matrix Q such that the determinant of Q is equal to a power of the irreducible implicit equation of the closure of the image of the parameterization. These first implicitization results using syzygies worked for curves and surfaces of the form $\mathbb{P}^1 \to \mathbb{P}^2$, $\mathbb{P}^2 \to \mathbb{P}^3$ under suitable assumptions on the polynomials defining the parameterizations and are contained in several papers such as [DÁ01], [BCD03] and [AHW05]. Later, in the work of Busé and Jouanoulou [BJ03], the use of syzygies to find implicit equations of parameterized hypersurfaces of the form $\mathbb{P}^{n-1} \to \mathbb{P}^n$ was formalized in a precise geometric and algebraic setting using Rees algebras. In this section we explain the connection of the implicitization problem with Rees algebras and syzygies by following the exposition of the ideas of [BJ03] given by Chardin in [Cha06]. The main algebraic tool that we use to find implicit equations and that we introduce in this section is the complex $\mathcal{Z}$.

Rees algebras

Let $R = k[X_0, \ldots, X_{n-1}]$ be the coordinate ring for $\mathbb{P}^{n-1}$ and take $f_0, \ldots, f_n \in R$ to be $n+1$ homogeneous polynomials of degree d defining a rational map

$$\lambda : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n, \quad P \mapsto [f_0(P) : \ldots : f_n(P)].$$

Let $\mathcal{H}$ denote the closure of the image of λ and assume $\mathcal{H}$ is of dimension $n-1$. Set $S = k[Y_0, \ldots, Y_n]$. Since $\mathcal{H}$ is of codimension one in $\mathbb{P}^n$, there exists an irreducible polynomial $H \in S$ such that $\mathcal{H} = \mathbf{V}(H)$. The equation $H = 0$ is the implicit equation of λ. Set $\mathcal{B} = \mathbf{V}(f_0, \ldots, f_n) \subset \mathbb{P}^{n-1}$ and suppose that $\mathcal{B}$ is a set of finitely many local complete intersection points. The set $\mathcal{B}$ is the set of basepoints of λ and will also be referred to as the base locus of λ. The reason we assume $\mathcal{B}$ is a local complete intersection is because this allows us to relate the blow-up of λ at $\mathcal{B}$ to the symmetric algebra of the ideal $I = \langle f_0, \ldots, f_n \rangle$ and ultimately to the syzygies of I. To study the parameterized hypersurface $\mathcal{H}$ and find its implicit equation, we blow up the base locus of λ and obtain the following commutative diagram of algebraic varieties

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d] & \mathbb{P}^{n-1} \times \mathbb{P}^n \arrow[d, "\pi"] \\ \mathbb{P}^{n-1} \arrow[r, dashed, "\lambda"] & \mathbb{P}^n. \end{tikzcd}$$

In the diagram above, Γ is the blow-up of $\mathbb{P}^{n-1}$ along $\mathcal{B}$ and it is also the closure of the graph of λ inside $\mathbb{P}^{n-1} \times \mathbb{P}^n$. In terms of commutative rings, we have that $\Gamma = \text{Proj}(\mathcal{R}_I)$ where $\mathcal{R}_I$ denotes the Rees algebra of the ideal I. The Rees algebra of an ideal $I = \langle f_0, \dots, f_n \rangle \subset R$ is defined by $\mathcal{R}_I = R \oplus I \oplus I^2 \oplus \cdots$. The embedding $\Gamma \hookrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^n$ corresponds to the surjection of commutative rings given by

$$\beta : R[Y_0, \dots, Y_n] \to \mathcal{R}_I, \quad Y_i \mapsto f_i.$$

The ideal $\ker \beta$ is known as the ideal of defining equations of $\mathcal{R}_I$ and we can obtain the polynomial H defining $\mathcal{H}$ from $\ker \beta$. Indeed the closed image of λ in $\mathbb{P}^n$ corresponds to $\pi(\Gamma)$ which is defined by $\mathcal{R}_I \cap k[Y_0, \dots, Y_n]$. Thus $\langle H \rangle = \ker \beta \cap k[Y_0, \dots, Y_n]$.

Example 1.2.1. Let $\lambda : \mathbb{P}^2 \to \mathbb{P}^3$ be defined by $[s,t,u] \mapsto [s^2t, t^2u, ust, su^2]$. Then $\ker \beta = \langle uY_2 - tY_3, sY_1 - tY_2, uY_0 - sY_2, Y_2^3 - Y_0Y_1Y_3 \rangle$. The implicit equation of λ is $Y_2^3 - Y_0Y_1Y_3 = 0$ and $\langle Y_2^3 - Y_0Y_1Y_3 \rangle = \ker \beta \cap k[Y_0, Y_1, Y_2, Y_3]$.

However, Rees algebras are hard to study and finding general explicit descriptions of their defining equations is a difficult open problem that has been studied by Huneke, Vasconcelos, Kustin, Polini, Ulrich ([Hun82],[Vas94], [UV93],[KPU15]) among others.

The symmetric algebra of an ideal I over R, denoted $\text{Sym}_R(I)$, is easier to understand than $\mathcal{R}_I$ and is closely related to it via a canonical surjection $\sigma : \text{Sym}_R(I) \to \mathcal{R}_I$. Let us recall the definition of the symmetric algebra of an ideal I in terms of the tensor algebra of I from [Eis95].

Definition 1.2.2. The tensor algebra of the ideal $I \subset R$ is the graded non-commutative algebra

$$T_R(I) = R \oplus I \oplus I^{\otimes 2} \oplus I^{\otimes 3} \oplus \cdots$$

where the product of $x_1 \otimes \cdots \otimes x_m$ and $y_1 \otimes \cdots \otimes y_n$ is $x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n$. The symmetric algebra of I is the algebra $\text{Sym}_R(I)$ obtained from $T_R(I)$ by imposing the commutative law, that is, by factoring out the two sided ideal generated by the relations $x \otimes y - y \otimes x$ for all $x, y \in M$.

When the set of basepoints $\mathcal{B}$ of λ is a local complete intersection we obtain the equality $\text{Proj}(\text{Sym}_R(I)) =$

$\mathrm{Proj}(\mathcal{R}_I) = \Gamma$, see Theorem 1.2.3. This implies that under the assumption that $\mathcal{B}$ is a local complete intersection we can extract the implicit equation of $\mathcal{H}$ from $\mathrm{Sym}_R(I)$. Thus we shift our attention to the study of $\mathrm{Sym}_R(I)$. We have a canonical surjection

$$\alpha : R[Y_0, \ldots, Y_n] \to \mathrm{Sym}_R(I), \quad Y_i \mapsto f_i.$$

Then $\mathrm{Sym}_R(I) = R[Y_0, \ldots, Y_n]/\ker\alpha$, where $\ker\alpha = \{a_0Y_0 + \cdots + a_nY_n | a_0f_0 + \cdots + a_nf_n = 0, a_i \in R\}$. So elements in $\ker\alpha$ are exactly elements in the syzygy module of I which we denote by $\mathrm{Syz}(I)$ or by $\mathrm{Syz}(f_0, \ldots, f_n)$. In terms of the canonical surjection β for the Rees algebra, $\ker\alpha$ is the degree one piece of $\ker\beta$, i.e $\ker\alpha = (\ker\beta)_1$. To be precise, the relation between these to kernels is given by

Theorem 1.2.3 ([Cha06])**.** The prime ideal $\ker\beta$ is the saturation of $\ker\alpha$ with respect to the homogeneous maximal ideal $\mathfrak{m} = \langle X_0, \ldots, X_n\rangle$ if $\mathcal{B}$ is a local complete intersection. Equivalently, $\mathrm{Proj}(\mathrm{Sym}_R(I)) = \mathrm{Proj}(\mathcal{R}_I)$.

It is important to note that the previous theorem does not imply that $\mathcal{R}_I = \mathrm{Sym}_R(I)$ and the measure of when they become equal as graded algebras is given by the saturation index of $\mathrm{Sym}_R(I)$ with respect to the grading given by R. The saturation index η is the least integer such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_\nu = 0$ for all $\nu \geq \eta$. Here $\mathfrak{m} := \langle X_0, \ldots, X_{n-1}\rangle$ and if M is an R-module, $H^0_{\mathfrak{m}}(M) := \{m \in M | \exists l, \quad X_i^l m = 0 \ \forall i\}$. The next Lemma gives a bound for the saturation index of $\mathrm{Sym}_R(I)$. Recall that if M is an $\mathbb{N}$-graded ring then its initial degree is $\mathrm{indeg}(M) := \min\{\nu \in \mathbb{N} : M_\nu \neq 0\}$.

Lemma 1.2.4 ([Bus05])**.** Define the integer

$$\nu := (n-1)(d-1) - \mathrm{indeg}(I :_R \mathfrak{m}^\infty) \in \mathbb{N}.$$

Then, for all integers $\nu \geq \eta$ we have $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_\nu = 0$.

Now we relate $\langle H\rangle = \ker\beta \cap k[Y_0, \ldots, Y_n]$ with the annihilator of a graded piece of $\mathrm{Sym}_R(I)$.

Proposition 1.2.5 ([Bus05])**.** Suppose $\mathrm{Proj}(\mathrm{Sym}_R(I)) = \mathrm{Proj}(\mathcal{R}_I)$ and let ν be such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_\mu = 0$ for all $\mu \geq \nu$. Then

$$\mathrm{ann}_{k[Y_0,\ldots,Y_n]}(\mathrm{Sym}_R(I)_\mu) = \ker\beta \cap k[Y_0, \ldots, Y_n] = \langle H\rangle.$$

Thus the implicit equation is the generator of $\mathrm{ann}_{k[Y_0,\ldots,Y_n]}(\mathrm{Sym}_R(I)_\mu)$ for a suitable μ. Finally, we use the determinant of the complex $\mathcal{Z}$ in degree μ to find the generator of a principal ideal J such that $\sqrt{J} = \mathrm{ann}_{k[Y_0,\ldots,Y_n]}(\mathrm{Sym}_R(I)_\mu)$, i.e $J = \langle H^b\rangle$ for some positive integer b. In the next subsection, we define the complex $\mathcal{Z}$ and state some of its properties. Afterwards, we state the implicitization Theorem 1.2.9 to find H using the complex $\mathcal{Z}$.

Approximation Complexes

Approximation complexes are complexes of graded modules associated to the generators of an ideal $I = \langle f_0, \dots, f_n \rangle$ in a commutative ring R. These complexes were introduced by Herzog, Simis and Vasconcelos [HSV82] to study the canonical surjection $\sigma : \mathrm{Sym}_R(I) \to \mathcal{R}(I)$ where $I = \langle f_0, \dots, f_n \rangle$. The complex $\mathcal{Z}$ is one type of approximation complex. In this subsection, we define the complex $\mathcal{Z}$ and describe some of its properties. First, recall the definition of the Koszul complex of a sequence $x = (x_1, \dots, x_r)$ of elements in a commutative ring A.

Definition 1.2.6. The Koszul complex $K_\bullet(x; A)$ is the complex with modules $K_p(x; A) := \bigwedge^p A^r \cong A^{\binom{p}{r}}$ and maps

$$d_p : K_p(x; A) \to K_{p-1}(x; A)$$

defined by

$$g \cdot e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto g \cdot \sum_{j=1}^{p} (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \cdots \wedge e_{i_p}$$

where e_i are elements in a standard basis for A^r.

To define the complex $\mathcal{Z}$ we will use the rings $R = k[X_0, \dots, X_{n-1}]$ and $S = k[Y_0, \dots, Y_n]$ defined previously in the context of implicitization for the rational map $\lambda : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$. The sequence $(f) = (f_0, \dots, f_n)$ denotes the sequence of elements in R that define the map λ. Let $T = R \otimes S$ and consider the elements of the sequence (f) as elements in T. We can associate to the sequence (f) and the sequence $(Y) = (Y_0, \dots, Y_n)$ their corresponding Koszul complexes $K_\bullet(f; T)$ and $K_\bullet(Y; T)$. These complexes have the same modules at each degree and we denote their differentials by d_p^f and d_p^Y. The differentials of these two Koszul complexes satisfy the relation

$$d_{p-1}^f \circ d_p^Y + d_{p-1}^Y \circ d_p^f = 0.$$

This relation allows us to define the complex $\mathcal{Z}$ based on the two complexes $K_\bullet(f, T)$ and $K_\bullet(Y, T)$.

Definition 1.2.7. Let $Z_p(f; T) := \ker d_p^f$, then $d_p^Y(Z_{p-1}(f; T)) \subset Z_p(f; T)$. The complex $\mathcal{Z}$ associated to the sequence (f) is defined by $\mathcal{Z}_p = Z_p(f; T)$ and differential d_p^Y.

Since the sequence (f) is in R, we note that $Z_p(f; T) = Z_p(f; R) \otimes S$. We also observe that $Z_0(f; R) = R$. We will be most interested in the first map d_1^Y of the complex $\mathcal{Z}$. First, we see that an element in $\ker(d_1^f)$

is an element $s_0e_0 + \cdots + s_ne_n \in R^n$ such that $d_1^f(s_0e_0 + \cdots s_ne_n) = 0$. That is,

$$s_0f_0 + \cdots + s_nf_n = 0.$$

Thus $\mathcal{Z}_1 = \mathrm{Syz}(f_0, \ldots, f_n) \otimes S$. The map $d_1^Y : \mathrm{Syz}(f_0, \ldots, f_n) \otimes S \to R \otimes S$ is given by

$$(s_0, \ldots, s_n) \mapsto s_0Y_0 + \cdots + s_nY_n.$$

Since the complex $\mathcal{Z}$ has modules $\mathcal{Z}_p = Z_p(f; R) \otimes S$, given a nonnegative integer μ, we use the convention that $(\mathcal{Z}_p)_\mu$ denotes the graded piece $(Z_p(f; R))_\mu \otimes S$. Then the complex $\mathcal{Z}_\mu$ consists of the modules $(\mathcal{Z}_p)_\mu$ and the maps $d_p : (\mathcal{Z}_p)_\mu \to (\mathcal{Z}_p)_\mu$ are the restrictions of d_p^Y to the graded piece μ. We will often refer to the complex $\mathcal{Z}_\mu$ as the μ-strand of the complex $\mathcal{Z}$. Another important property of the complex $\mathcal{Z}$ is given by the following theorem.

Theorem 1.2.8 ([Cha06])**.** We have $H_0(\mathcal{Z}) \cong \mathrm{Sym}_R(I)$ and the homology modules $H_i(\mathcal{Z})$ are $\mathrm{Sym}_R(I)$-modules that only depend on the ideal $I = (f_0, \ldots, f_n) \subset R$, up to isomorphism.

Finally we arrive at the statement of the implicitization theorem for $\mathcal{H}$.

Theorem 1.2.9 ([BJ03])**.** Suppose that the ideal $I = \langle f_0, \ldots, f_n \rangle$ is a local complete intersection in $\mathbb{P}^{n-1}$ of codimension $n-1$ such that the map $\lambda : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$ defined by $f_0, \ldots, f_n$ is generically finite. Let ν be an integer such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_\nu = 0$ for all $\nu \geq \eta$. Then

$$\det(\mathcal{Z}_\nu) = H^{\deg \lambda}.$$

Although we did not define the determinant of a complex, it will often be the case that for any nonnegative integer μ, $\mathcal{Z}_\mu$ consists of a single map or two maps. If $\mathcal{Z}_\mu$ just has one nonzero map the determinant of the complex is the determinant of a matrix representing the map with respect to a chosen basis. If $\mathcal{Z}_\mu$ consists of two nonzero maps, then it has the form

$$\mathcal{Z}_\nu : \qquad 0 \longrightarrow (\mathcal{Z}_2)_\nu \xrightarrow{d_2} (\mathcal{Z}_1)_\nu \xrightarrow{d_1} (\mathcal{Z}_0)_\nu \longrightarrow 0.$$

The determinant the this complex is obtained by finding the determinant of a maximal nonzero minor of a matrix representation for d_1 and dividing it by the determinant of a complementary maximal nonzero minor for a matrix representation of d_2.

Example 1.2.10. Consider the parameterization $\lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by $[s, t, u] \mapsto [s^2t, t^2u, stu, su^2]$ from

Example 1.2.1. The set of basepoints of λ is $\mathcal{B} = \{[1:0:0],[0:1:0],[0:0:1]\}$ and they form a complete intersection. Note that the ideal $I = \langle s^2t, t^2u, stu, su^2\rangle$ is saturated, thus $\operatorname{indeg}(I :_R \mathfrak{m}^\infty) = \operatorname{indeg}(I) = 3$. Using the bound from Lemma 1.2.4, with $n = 3, d = 3$ and $\operatorname{indeg}(I :_R \mathfrak{m}^\infty) = 3$ gives $\nu = 1$. To set up the matrix for $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ in degree $\nu = 1$, we find a basis for $\operatorname{Syz}(I)$ in degree one. We have

$$\operatorname{Syz}(I) = \begin{pmatrix} -u & 0 & 0 \\ s & -t & -u \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix},$$

hence $\dim \operatorname{Syz}(I)_1 = 3$. The images of these three syzygies under d_1 are given by

$$-uX + sY, \quad -tY + sZ, \quad -uY + tW.$$

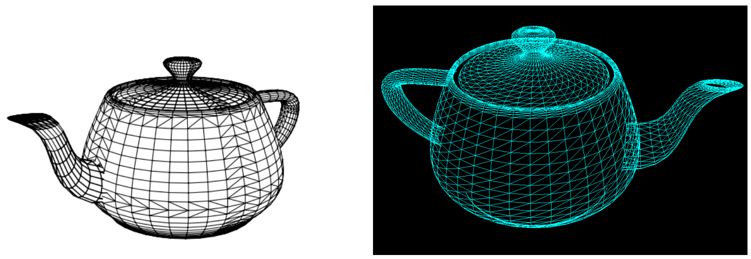
Fix the basis $\{s, t, u\}$ for R_1. Then the matrix M for $d_1 : (\mathcal{Z}_1)_1 \to (\mathcal{Z}_0)_1$ is given by

$$M = \begin{pmatrix} Y & Z & 0 \\ 0 & -Y & W \\ -X & 0 & -Y \end{pmatrix}.$$

The determinant $\det \mathcal{Z}_1 = \det M = Y^3 - ZWX = 0$ gives the implicit equation for λ. This equation agrees with the computation from Example 1.2.1 which involved the generators of $\ker \beta$.

1.3 Tensor product surfaces

Tensor product surfaces are very important in CAGD applications because they are useful to model intricate geometry. For example, Figure 1.2 shows two surface patch models of Newell's Teapot. The region around the intersection of the spout and the body of the teapot is smoother in the model with tensor product surfaces on the left than the same region of the teapot on the right modeled with triangular surfaces (of the form $\lambda : \mathbb{P}^2 \to \mathbb{P}^3$).



(a) Tensor product surface model, Sederberg [Sed12]

(b) Triangular surface model

Figure 1.1: Newell's Teapot modeled by parameterized surface patches.

In this section we define tensor product surfaces and present a theorem by Botbol to find their implicit equations. This theorem follows the formalism explained in Section 1.2 but in the more general case of a bigraded surface. In fact, although we state the theorem for $\mathbb{P}^1 \times \mathbb{P}^1$, this result holds more generally for implicitization of parameterized multigraded hypersurfaces in $\mathbb{P}^n$. We also include an example that illustrates how to set up the complex $\mathcal{Z}_\nu$ to compute the implicit equation of a tensor product surface of bidegree $(2, 1)$ with two basepoints.

1.3.1 Definition of tensor product surfaces

Let $R = \bigoplus_{0 \le a, 0 \le b} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b))$ be the total coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. For shorter notation we write $H^0(a, b)$ for $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b))$. The ring R is a bigraded k-algebra by taking $R_{(a,b)} = H^0(a, b)$ and R is generated as a k-algebra by $H^0(1, 0)$ and $H^0(0, 1)$. Note that $\dim H^0(1, 0) = \dim H^0(0, 1) = 2$. If $\{s, t\}$ is a basis for $H^0(1, 0)$ and $\{u, v\}$ a basis for $H^0(0, 1)$, then $R \cong k[s, t] \otimes k[u, v]$ with grading given by $\deg s, t = (1, 0)$ and $\deg u, v = (0, 1)$. Let $R_{(i,j)}$ denote the (i, j) graded piece of R. An element $F \in R$ is bihomogeneous if $F \in R_{(i,j)}$ for some $(i, j) \in \mathbb{N}^2$. If $F \in R_{(i,j)}$, we say that its bidegree is $\deg F = (i, j)$. Suppose that $I = (F_1, \ldots, F_n) \subset R$ is an ideal. If each F_i is bihomogeneous, then we say that I is a bihomogeneous ideal.

A *tensor product* surface is the closed image of a rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$. We describe a tensor product surface concretely in terms of polynomials by using the bigraded ring R. To define a rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ we take $U = \{f_0, f_1, f_2, f_3\} \subset R_{(a,b)}$ to be a 4-dimensional k-vector subspace with $\gcd(f_0, f_1, f_2, f_3) = 1$. Then U defines a rational map

$$\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3, \ \ [s, t; u, v] \mapsto [f_0, f_1, f_2, f_3].$$

The vanishing locus $\mathbb{V}(U)$ inside $\mathbb{P}^1 \times \mathbb{P}^1$ is the set of points at which λ_U is undefined. Set $\mathcal{B} := \mathbb{V}(U) \subset \mathbb{P}^1 \times \mathbb{P}^1$, if $\mathcal{B} = \emptyset$ we say U is basepoint-free, otherwise we say U has basepoints. We also refer to $\mathcal{B}$ as the base locus of the map λ_U. The assumption $\gcd(f_0, f_1, f_2, f_3) = 1$ implies $\mathcal{B}$ is at most a finite set of isolated points. In what follows, we denote the closure of the image of λ_U in $\mathbb{P}^3$ by X_U and its implicit equation by H. We let $S := k[X, Y, Z, W]$ be the coordinate ring of $\mathbb{P}^3$ and let $I_U := \langle f_0, f_1, f_2, f_3 \rangle$ be the bihomogeneous ideal of R generated by the elements of U.

Example 1.3.1. The vector space $U = \{t^2u, t^2v, s^2v, stu\} \subset R_{(2,1)}$ defines a tensor product surface of bidegree $(2, 1)$,

$$\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3; \ \ [s, t; u, v] \mapsto [t^2u, t^2v, s^2v, stu].$$

The implicit equation of X_U is $X^2Z - YW^2 = 0$. This is a tensor product surface with one basepoint given by $\mathcal{B} = \{[1 : 0; 1 : 0]\}$.

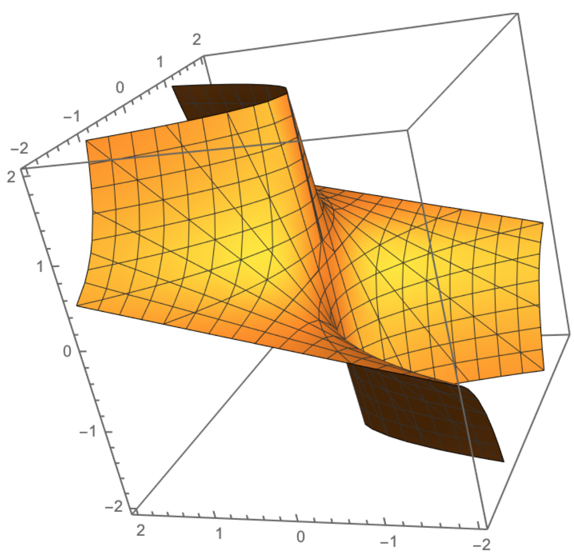


Figure 1.2: Surface X_U from Example 1.3.1 restricted to the affine patch $W = 1$.

1.3.2 Implicitization of tensor product surfaces using syzygies

In what follows, given $U = \{f_0, f_1, f_2, f_3\}$, we will use the complex $\mathcal{Z}$ to obtain the implicit equation for X_U. Throughout this section and the rest of this thesis, $\mathcal{Z}$ will be the complex associated to the sequences (f_0, f_1, f_2, f_3) and (X, Y, Z, W). Given any bidegree ν, $\mathcal{Z}_\nu$ denotes the bigraded piece of $\mathcal{Z}$ in bidegree ν.

Theorem 1.3.2 (Botbol 2011)**.** Let $U = \{f_0, f_1, f_2, f_3\} \subset R_{(a,b)}$, $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ be the rational map defined by U and suppose $\mathcal{B} = \mathbf{V}(I_U)$ is empty or a local complete intersection. Set $\nu = (2a-1, b-1)$(equivalently $\nu = (a-1, 2b-1)$), then

$$\det \mathcal{Z}_\nu = H^{\deg \lambda_U}$$

where H denotes the irreducible implicit equation of X_U.

The next two observations describe the complex $\mathcal{Z}$ when $\mathcal{B}$ is empty and when it is not. These descriptions follow from the lemmas contained in the next subsection.

- If $\mathcal{B} = \emptyset$, the complex $\mathcal{Z}_\nu$ is given by

$$\mathcal{Z}_\nu : \qquad 0 \longrightarrow (\mathcal{Z}_1)_\nu \xrightarrow{d_1} (\mathcal{Z}_0)_\nu \longrightarrow 0.$$

  In this case, d_1 is represented by a square matrix of size $2ab$ and $\det \mathcal{Z}_\nu = \det d_1$.

- If $\mathcal{B} \neq \emptyset$, the complex $\mathcal{Z}_\nu$ is given by

$$\mathcal{Z}_\nu : \qquad 0 \longrightarrow (\mathcal{Z}_2)_\nu \xrightarrow{d_2} (\mathcal{Z}_1)_\nu \xrightarrow{d_1} (\mathcal{Z}_0)_\nu \longrightarrow 0.$$

  In this case $\mathcal{Z}_\nu$ is exact. The determinant of the complex $\mathcal{Z}_\nu$ is calculated by taking the determinant of a maximal non vanishing minor of d_1 and dividing it by the determinant of the complementary maximal minor of d_2. The size of d_1 is $2ab \times (2ab + E)$ where E is the sum of the multiplicities of the basepoints of U.

The computation of the implicit equation of a tensor product surface using Botbol's theorem is illustrated in the following example.

Example 1.3.3. Let $U = \{s^2v, stv, stu, t^2u\} \subset R_{(2,1)}$. Then $(a, b) = (2, 1)$ and $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ has two basepoints $\mathcal{B} = \{[1:0;1:0], [0:1;0:1]\}$ that form a local complete intersection. Following Botbol's result, we set up the complex $\mathcal{Z}_\nu$ based on the syzygies of I_U in bidegree $\nu = (2a-1, b-1) = (3, 0)$ and then compute $\det(\mathcal{Z}_\nu)$. We start by finding a basis for $(\mathcal{Z}_1)_\nu$, this basis consists of the syzygies of I_U in bidegree $(3,0)$. Using `Macaulay2`, we get

$$\mathrm{Syz}(I_U) = \begin{pmatrix} -t & 0 & 0 \\ s & 0 & -u \\ 0 & t & v \\ 0 & -s & 0 \end{pmatrix}.$$

The first two linear syzygies span a free module. We have $(\mathcal{Z}_1)_\nu = \ker(d_1^f)_\nu \subseteq \bigwedge^1 R^4 = R^4$ and let $e_0, \ldots, e_3$ be a basis for R^4. A basis for the syzygies in bidegree $(3,0)$ is obtained by multiplying the two syzygies of bidegree $(1,0)$ by the monomials $\{s^2, st, t^2\}$. That is, we multiply the syzygies in bidegree $(1,0)$ times elements in a basis for $R_{(2,0)}$. Let $S_1 = -te_0 + se_1$ and $S_2 = te_2 - se_3$ be the two syzygies in bidegree $(1,0)$. Then a basis for $\mathrm{Syz}(I_U)_\nu$ is given by

$$\{s^2S_1, stS_1, t^2S_1, s^2S_2, stS_2, t^2S_2\}.$$

Then we apply the Koszul differential on the sequence (X, Y, Z, W) to the basis of $\mathrm{Syz}(I_U)_\nu$. For example

$$s^2(-te_0 + se_1) \mapsto -ts^2X + s^3Y.$$

Next, we fix a basis for $R_{(3,0)} = \{s^3, s^2t, st^2, t^3\}$ and write the coefficient vectors of the images of the basis for $\mathrm{Syz}(I_U)_\nu$ in a matrix,

$$d_1 = \begin{pmatrix} Y & 0 & 0 & W & 0 & 0 \\ -X & Y & 0 & -Z & W & 0 \\ 0 & -X & Y & 0 & -Z & W \\ 0 & 0 & -X & 0 & 0 & -Z \end{pmatrix}.$$

We could proceed in a similar fashion for $(\mathcal{Z}_2)_\nu$ and obtain a matrix for d_2, but in this case Lemma 1.3.7 implies $d_2 = \ker d_1$. Hence we obtain the complex

$$\mathcal{Z}_\nu : 0 \longrightarrow (\mathcal{Z}_2)_\nu \xrightarrow{\begin{pmatrix} W & 0 \\ -Z & W \\ 0 & -Z \\ -Y & 0 \\ X & -Y \\ 0 & X \end{pmatrix}} (\mathcal{Z}_1)_\nu \xrightarrow{\begin{pmatrix} Y & 0 & 0 & W & 0 & 0 \\ -X & Y & 0 & -Z & W & 0 \\ 0 & -X & Y & 0 & -Z & W \\ 0 & 0 & -X & 0 & 0 & -Z \end{pmatrix}} (\mathcal{Z}_0)_\nu .$$

We compute the determinant of the complex by finding the alternating quotient of two complementary maximal minors

$$\det \mathcal{Z}_\nu = \frac{\begin{vmatrix} Y & 0 & 0 & W \\ -X & Y & 0 & -Z \\ 0 & -X & Y & 0 \\ 0 & 0 & -X & 0 \end{vmatrix}}{\begin{vmatrix} X & -Y \\ 0 & X \end{vmatrix}} = \frac{-X^2YZ + X^3W}{X^2} = XW - YZ$$

1.3.3 Technical Lemmas

The following Lemmas are presented in [Bot11], they give the degree of the determinant of $\mathcal{Z}_\nu$ in terms of a, b where $\nu = (2a-1, b-1)$ and (a, b) is the bidegree of the tensor product surface X_U. These will be used in Chapter 2 and Chapter 3 to prove Theorem 2.0.1 and Theorem 3.0.1 .

Lemma 1.3.4 ([Bot11])**.** Let $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ be a finite rational map with finitely many local complete intersection basepoints(or none), given by four bihomogeneous polynomials f_0, f_1, f_2, f_3 of bidegree (a, b). Take $\nu = (2a-1, b-1)$(equivalently $\nu = (a-1, 2b-1)$) and write $\Delta_\nu = \det \mathcal{Z}_\nu$. Then

$$\deg(\Delta_\nu) = 2ab - \dim(H_2)_{(4a-1,3b-1)},$$

and $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ is represented by a square matrix of size $2ab$ iff $(H_2)_{(4a-1,3b-1)} = 0$.

In the previous Lemma, $(H_2)_{(4a-1,3b-1)}$ denotes the homology of the complex $\mathcal{Z}$ associated to the sequence (f) in bidegree $(4a-1, 3b-1)$.

Lemma 1.3.5 ([Bot11])**.** Let $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ be a finite rational map given by 4 homogeneous polynomials $f_0, \ldots, f_3$ defining an ideal I, where $f_i \in R_{(a,b)}$. Assume that $\mathcal{P} = \mathrm{Proj}(R/I)$ is finite and a local complete intersection. Then,

$$\deg(\lambda)\deg(\mathcal{H}) = 2ab - \sum_{x \in \mathcal{P}} e_x,$$

where e_x is the multiplicity at x and $\mathcal{H}$ denotes the closure of the image of λ.

Remark 1.3.6 ([Bot11])**.** Under the hypotheses of Theorem 1.3.2 we have that $\Delta_\nu = H^{\deg \lambda}$. From Lemma 1.3.4 and Lemma 1.3.5 we obtain

$$\deg(\Delta_\nu) = \deg(\mathcal{H}^{\deg \lambda}) = \deg(\lambda)\deg(\mathcal{H}) = 2ab - \sum_{x \in \mathcal{P}} e_x.$$

The following Lemma is not stated in [Bot11] but it follows easily from the previous results.

Lemma 1.3.7. Let $\mathcal{Z}$ be the approximation complex associated to $(f_0, \ldots, f_3)$, with differential on the variables (X, Y, Z, W). If $U = \{f_0, \ldots, f_3\}$ has r basepoints, each of multiplicity one, then

$$\dim_k(\mathcal{Z}_1)_\nu = 2ab + r$$

where $\nu = (2a-1, b-1)$.

Proof. From the proof of Lemma 1.3.4 in [Bot11], we know that $\mathcal{Z}_\nu$ is acyclic and that $(\mathcal{Z}_3)_\nu = 0$. Hence $\mathcal{Z}_\nu$ has the form

$$\mathcal{Z}_\nu : \qquad 0 \longrightarrow (\mathcal{Z}_2)_\nu \longrightarrow (\mathcal{Z}_1)_\nu \longrightarrow (\mathcal{Z}_0)_\nu \longrightarrow 0.$$

Using Lemma 1.3.4 and Lemma 1.3.5, and the fact that I_U is a local complete intersection, Botbol observes that

$$\dim_k(\mathcal{Z}_2)_\nu = \sum_{x \in X} e_x$$

where e_x denotes the multiplicity of the basepoint x of U and X is the set of basepoints of U. Since U has r basepoints each of multiplicity one, $\dim_k(\mathcal{Z}_2)_\nu = r$. Also, $(\mathcal{Z}_0)_\nu = R_{(2a-1,b-1)}$, thus $\dim_k(\mathcal{Z}_0)_\nu = 2ab$. It follows that $\dim_k(\mathcal{Z}_1)_\nu = 2ab + r$. $\square$

1.4 Generic sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$

The idea to understand tensor product surfaces with basepoints is to study the geometry of the base locus of U in $\mathbb{P}^1 \times \mathbb{P}^1$. To do this we focus on the ideals of R that correspond to points in $\mathbb{P}^1 \times \mathbb{P}^1$. In this section we describe such ideals and state a theorem of Van Tuyl that describes their bigraded Hilbert functions.

1.4.1 Description of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$

Throughout this section, we follow the notation and definitions from Giuffrida, Maggioni and Ragusa [GMR92] and from Guardo and Van Tuyl [GVT15] to describe sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. A point in $\mathbb{P}^1 \times \mathbb{P}^1$ is denoted $A \times B$, where $A \in \mathbb{P}^1$ and $B \in \mathbb{P}^1$. Let h be a non-zero linear form in the variables s, t that vanishes at A and let l be a non-zero linear form in the variables u, v that vanishes at B. The ideal of R that corresponds to P is denoted by I_P and $I_P = \langle h, l \rangle$. The form h is a $(1,0)$ line and the form l as a $(0,1)$ line in $\mathbb{P}^1 \times \mathbb{P}^1$. These lines are members of the two different rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and a point in $\mathbb{P}^1 \times \mathbb{P}^1$ is uniquely determined by their intersection. If $X = \{P_1, \ldots, P_r\}$ is a set of r distinct points and $I_{P_i} = \langle h_i, l_i \rangle$, then I_X, the ideal corresponding to X, is given by $I_X = \bigcap_{i=1}^r I_{P_i}$.

There are two projections $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by $\pi_1(A \times B) = A$ and $\pi_2(A \times B) = B$. Throughout, X will denote a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and I_X will denote its corresponding defining ideal in R.

As with sets of points in $\mathbb{P}^n$, we use Hilbert functions to study sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Since the ring R is bigraded, the Hilbert function of X takes the shape of a matrix.

Definition 1.4.1. Let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. The bigraded Hilbert function of X, $H_X : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ is defined by

$$H_X(i,j) = \dim_k R_{(i,j)} - \dim_k (I_X)_{(i,j)}.$$

The bigraded Hilbert function of X has similar properties to the Hilbert function of a set of points in projective space. When $\mathbb{P}^1 \times \mathbb{P}^1$ is considered as a subvariety of $\mathbb{P}^3$ by the Segre embedding, X becomes a subscheme of $\mathbb{P}^3$, in this case, $H_X(i) = H_X(i,i) = \deg X$ for all $i \gg 0$. The work of Guardo and Van Tuyl [GVT15] provides a thorough introduction to the study of bigraded Hilbert functions of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

1.4.2 Combinatorial description of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$

Using the description of a single point in $\mathbb{P}^1 \times \mathbb{P}^1$ as the intersection of a $(1,0)$ line and a $(0,1)$ line, a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ can be visualized as markings of some intersection points inside a rectangular grid as in Example 1.4.4. Following this way of visualizing points in $\mathbb{P}^1 \times \mathbb{P}^1$, we describe sets of points in a combinatorial way using partitions. Let $h_1, \ldots, h_q$ be the horizontal lines in $\mathbb{P}^1 \times \mathbb{P}^1$ that contain points of X. The ordering of these lines doesn't play any role so we may arrange them in such a way that

$$|X \cap h_1| \geq |X \cap h_2| \geq \ldots \geq |X \cap h_q|.$$

Let $\alpha_i = |X \cap h_i|$ and associate to X the tuple $\alpha_X = (\alpha_1, \ldots, \alpha_q)$. Note that α_X is a partition of $|X|$. An analogous process yields a partition β_X of $|X|$, which depends on the points on the vertical rulings of X. For the next theorem, we use the notion of the conjugate of a partition.

Definition 1.4.2. The conjugate of a partition λ is the tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*$ where $\lambda_i^* = \#\{\lambda_j \in \lambda | \lambda_j \geq i\}$.

For a set X of points in $\mathbb{P}^1 \times \mathbb{P}^1$ we usually have $\alpha_X^* \neq \beta_X$. Theorem 1.4.3 describes the Hilbert function $H_X(i,j)$ for $(i,j) \gg (0,0)$ in terms of the partitions α, β. It was first formulated by Van Tuyl [VT02], here we use the statement from Guardo and Van Tuyl [GVT15].

Theorem 1.4.3 ([GVT15])**.** Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be any set of points with associated tuples $\alpha_X = (\alpha_1, \ldots, \alpha_h)$ and $\beta_X = (\beta_1, \ldots, \beta_\nu)$ and let $h = |\pi_1(X)|$ and $\nu = |\pi_2(X)|$.

1. For all $j \in \mathbb{N}$, if $i \geq h-1$, then

$$H_X(i,j) = \alpha_1^* + \cdots + \alpha_{j+1}^*$$

where $\alpha_X^* = (\alpha_1^*, \ldots, \alpha_{\alpha_1}^*)$ is the conjugate of α_X and where we make the convention that $\alpha_l^* = 0$ if $l > \alpha_1$.

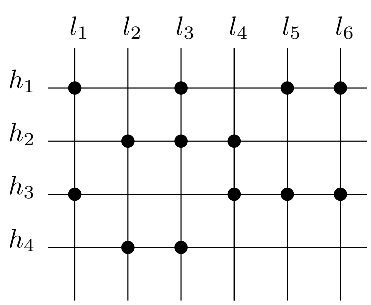
2. For all $i \in \mathbb{N}$, if $j \geq \nu - 1$, then

$$H_X(i,j) = \beta_1^* + \cdots + \beta_{j+1}^*$$

where $\beta_X^* = (\beta_1^*, \ldots, \beta_{\beta_1}^*)$ is the conjugate of β_X and where we make the convention that $\beta_l^* = 0$ if $l > \beta_1$.

Thus if we know the values of $H_X(i,j)$ for $(i,j) \gg 0$ we are able to determine α and β. This in turn gives us information about the vertical an horizontal rulings that contain X. We will use this theorem to give a geometric description of a generic set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$ in subsection 3.1.

Example 1.4.4. Let X be the set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ on the left below. Then $\alpha_X = (4,4,3,2)$, $\beta_X = (3,2,2,2,2,2)$ and $\alpha_X^* = (4,4,3,2)$, $\beta_X^* = (6,6,1)$. The previous theorem implies that H_X has the following form.



$H_X =$

	0	1	2	3	4	5	6
0						6	6
1						12	12
2						13	13
3	4	8	11	13	13	13	13
4	4	8	11	13	13	13	13

The blank entries cannot in general be deduced from the information contained in α_X and β_X. In Examples 3.1.2 and 3.1.3 we illustrate this fact with two sets of points X_1, X_2 in $\mathbb{P}^1 \times \mathbb{P}^1$ whose Hilbert function is different but for which $\alpha_{X_1} = \alpha_{X_2}$ and $\beta_{X_1} = \beta_{X_2}$.

Chapter 2

Tensor product surfaces and linear syzygies

In this chapter we focus on describing the structure of $\mathrm{Syz}(I_U)$ in degree $\nu = (2a-1, b-1)$ for the case when U is basepoint free and I_U has a linear syzygy. We assume that $U = \{p_0, p_1, p_2, p_3\} \subset R_{(a,b)}$. The condition that U is basepoint free can also be stated algebraically by requiring that $\sqrt{I_U} = (s,t) \cap (u,v)$.

Our study of tensor product surfaces when I_U has a linear syzygy is motivated by the work of Schenck, Seceleanu and Validashti [SSV14] for surfaces of bidegree $(2,1)$. In [SSV14], they classify all possible numerical types of resolutions of I_U when $U \subset R_{(2,1)}$ is basepoint free. The results in this chapter indicate that for higher bidegree (a,b) with $a, b \geq 2$, a linear syzygy also determines the structure of a submodule of $\mathrm{Syz}(I_U)$. In particular, the linear syzygy allows us to write an explicit basis for $\mathrm{Syz}(I_U)_\nu$ and to describe part of the codimension one singular locus of X_U. The main result is:

Theorem 2.0.1. If $a, b \geq 2$ and U is basepoint free, then there is at most one linear first syzygy on I_U. A linear first syzygy explicitly determines a pair of additional first syzygies. These three syzygies determine the first map d_1 of the complex $\mathcal{Z}$ in degree $\nu = (2a-1, b-1)$.

The statements of the theorems, examples and proofs in this chapter are the same as in [DS16] with minor changes in notation. The changes in notation were incorporated to be consistent with the notation used throughout this thesis.

Example 2.0.2. Suppose $(a,b) = (2,2)$, and

$$U = \mathrm{Span}\{t^2u^2 + s^2uv, t^2uv + s^2v^2, t^2v^2, s^2u^2\} \subseteq R_{(2,2)}.$$

A computation in `Macaulay2` shows that I_U has seven minimal first syzygies, in bidegrees

$$(0,1), (2,1), (2,1), (0,3), (2,2), (4,1), (6,0).$$

A basis for $\mathrm{Syz}(I_U)_\nu$, $\nu = (3,1)$ is determined by the three syzygies in bidegrees $(0,1), (2,1), (2,1)$. A priori these three syzygies could span a module that is not free and this needs to be taken into account to find a basis for $\mathrm{Syz}(I_U)_\nu$. However, the proof of Theorem 2.0.1 reveals that these three syzygies are free and the two syzygies in bidegree $(2,1)$ can be written explicitly using the syzygy in bidegree $(1,0)$. The three syzygies of bidegree $(0,1),(2,1),(2,1)$ are generated by the columns of

$$\begin{bmatrix} v & 0 & s^2u \\ -u & -t^2v & 0 \\ 0 & t^2u+s^2v & 0 \\ 0 & 0 & -t^2u-s^2v \end{bmatrix}$$

and the map d_1 in bidegree $\nu = (3,1)$ is

$$d_1 = \begin{bmatrix} X & 0 & 0 & 0 & Z & 0 & -W & 0 \\ -Y & 0 & 0 & 0 & 0 & 0 & X & 0 \\ 0 & X & 0 & 0 & 0 & Z & 0 & -W \\ 0 & -Y & 0 & 0 & 0 & 0 & 0 & X \\ 0 & 0 & X & 0 & -Y & 0 & 0 & 0 \\ 0 & 0 & -Y & 0 & Z & 0 & -W & 0 \\ 0 & 0 & 0 & X & 0 & -Y & 0 & 0 \\ 0 & 0 & 0 & -Y & 0 & Z & 0 & -W \end{bmatrix}$$

The determinant of this matrix is

$$(X^3Z + Y^3W - X^2Y^2)^2.$$

The implicit equation defining X_U is $X^3Z + Y^3W - X^2Y^2 = 0$, and λ_U is $2:1$. The codimension one singular locus of X_U contains a line and we will see in Corollary 2.1.5 that this is always the case.

2.1 Proof of main theorems

To prove the main theorem we start by stating a simple result from [SSV14] about the structure of the ideal I_U when I_U has a linear syzygy. We include the proof of this observation since it is also a key step in the algorithm we formulate to compute the implicit equation of X_U when U is basepoint free and has a linear syzygy.

Lemma 2.1.1 (Lemma 3.1 [SSV14]). If I_U has a linear first syzygy of bidegree $(0,1)$, then

$$I_U = \langle pu, pv, p_2, p_3 \rangle,$$

where p is bihomogeneous of bidegree $(a, b-1)$.

Proof. Rewrite the linear syzygy

$$\sum_{i=0}^{3}(a_i u + b_i v)p_i = 0 = u \cdot \sum_{i=0}^{3} a_i p_i + v \cdot \sum_{i=0}^{3} b_i p_i,$$

and let $g_0 = \sum_{i=0}^{3} a_i p_i$, $g_1 = \sum_{i=0}^{3} b_i p_i$. The relation above implies that (g_0, g_1) is a syzygy on (u, v). Since the syzygy module of (u, v) is generated by the Koszul syzygy, this means

$$\begin{bmatrix} g_0 \\ g_1 \end{bmatrix} = p \cdot \begin{bmatrix} -v \\ u \end{bmatrix}.$$

A similar argument applies if I_U has a first syzygy of degree $(1,0)$. □

The proof of the Theorem 2.0.1 is divided into the proof of several theorems. We start by using the previous Lemma 2.1.1 to show uniqueness of the linear syzygy.

Theorem 2.1.2. If $a, b \geq 2$ and U is basepoint free, then there can be at most one linear first syzygy on I_U.

Proof. Suppose L is a linear syzygy of bidegree $(0,1)$ on I_U. By Lemma 2.1.1 we may assume

$$I_U = \langle pu, pv, p_2, p_3 \rangle = \langle p_0, p_1, p_2, p_3 \rangle$$

where p is homogeneous of bidegree $(a, b-1)$. Suppose there is another minimal first linear syzygy of bidegree $(0,1)$,

$$\sum_{i=0}^{3} p_i(a_i u + b_i v) = 0.$$

Let

$$\begin{aligned} \widetilde{p_2} &= \sum_{i=0}^{3} a_i p_i, \\ \widetilde{p_3} &= \sum_{i=0}^{3} b_i p_i, \end{aligned}$$

so $\widetilde{p_2}u+\widetilde{p_3}v = 0$. But the syzygy module on $[u, v]$ is generated by $[v, -u]$ so we must have $\widetilde{p_2} = qv, \widetilde{p_3} = -qu$ for some q of bidegree $(a, b-1)$. If in addition

$$D = \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix}$$

is nonzero, then

$$I_U = \langle pu, pv, \widetilde{p_2}, \widetilde{p_3} \rangle = \langle pu, pv, qu, qv \rangle.$$

Example V.1.4.3 of [Har77] shows that curves $\mathbf{V}(f)$ of bidegree (a, b) and $\mathbf{V}(g)$ of bidegree (c, d) on $\mathbb{P}^1 \times \mathbb{P}^1$, sharing no common component, meet in $ad + bc$ points. If p and q share a common factor, then clearly I_U is not basepoint free; if they do not share a common factor, then $\mathbf{V}(p, q)$ consists of $2ab - 2a$ points; since $a, b \geq 2$, this again forces I_U to have basepoints. The same argument works if the additional syzygy is of bidegree $(1, 0)$, save that in this case since q is of bidegree $(a-1, b)$, $\mathbf{V}(p, q)$ consists of $2ab-a-b+1$ points, and again I_U is not basepoint free.

Next, suppose $D = 0$. If $a_2 = a_3 = b_2 = b_3 = 0$, then the second minimal first syzygy involves only pu and pv. If the syzygy is of bidegree $(0, 1)$ then by Lemmma 2.1.1, $(pu, pv) = (qv, qu)$. Thus

$$pu = qv \implies p = fv, \;\; q = fu \implies fv^2 = fu^2$$

a contradiction. If the syzygy is of bidegree $(1, 0)$, then $(pu, pv) = (qs, qt)$, and

$$pu = qs \implies p = fs, \;\; q = fu \implies fsv = fut,$$

again a contradiction.

Finally, if $D = 0$ and a_2, a_3, b_2, b_3 are not all zero, then $c \cdot [a_2, b_2] = [a_3, b_3]$ for some $c \neq 0$, so letting $\widetilde{p_2} = p_2 + cp_3$, we may assume the syzygy involves only $pu, pv, \widetilde{p_2}$. If the syzygy is of degree $(0, 1)$, letting $l_i = a_i u + b_i v$ for $i = 0, 1, 2$ we have

$$pul_0 + pvl_1 + \widetilde{p_2}l_2 = 0.$$

Since $\langle l_2 \rangle$ is prime, either $l_2 | ul_0 + vl_1$ or $l_2 | p$. In the former case, $ul_0 + vl_1 = l_2 l_3$ for some $l_3 \in k[u, v]_1$, hence $pl_3 + \widetilde{p_2} = 0$. In particular $p | \widetilde{p_2}$, so $\mathbf{V}(p, p_3)$ contains $2ab - a$ points and I_U is not basepoint free.

In the latter case, $p' l_2 = p$ for some $p' \in R_{(a-2,b)}$, so $p' l_2 ul_0 + p' l_2 vl_1 + \widetilde{p_2} l_2 = 0$. Hence $p' ul_0 + p' vl_1 + \widetilde{p_2} = 0$, so p' is a common factor of p and $\widetilde{p_2}$ of bidegree $(a, b-2)$, so $\mathbf{V}(p', p_3)$ contains $2ab - 2a$ points and I_U is not basepoint free. A similar argument works if the additional syzygy is of bidegree $(1, 0)$. $\square$

Theorem 2.1.3. If U is basepoint free, $a, b \geq 2$, and there is a linear syzygy L of bidegree $(0, 1)$ on I_U, then there are two additional first syzygies S_1, S_2 of bidegree $(a, b-1)$, such that

$$\dim\langle L, S_1, S_2\rangle_{(2a-1,b-1)} = 2ab.$$

Proof. By Lemma 2.1.1, we may assume $(p_0, p_1) = (pu, pv)$. Write $p_2 = g_2v+f_2u$. Then $f_2p_0+g_2p_1-pp_2 = 0$, so the kernel of $[pu, pv, p_2]$ contains the columns of the matrix

$$M = \begin{bmatrix} v & f_2 \\ -u & g_2 \\ 0 & -p \end{bmatrix}.$$

In fact, M is the syzygy matrix of $[pu, pv, p_2]$; the sequence $\{pu, p_2\}$ is not regular iff the two polynomials share a common factor. If $u|p_2$, then let $p_2' = p_2 + pv$; $u|p_2'$ or $p|p_2'$ imply I_U is not basepoint free. So the depth of the ideal of 2×2 minors of M is two and exactness follows from the Buchsbaum-Eisenbud criterion [Eis95]. Writing $p_3 = f_3u + g_3v$, the syzygy module of I_U contains the columns of $N = \mathrm{Span}\{L, S_1, S_2\}$, where

$$N = \begin{bmatrix} v & f_2 & f_3 \\ -u & g_2 & g_3 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}.$$

As the bottom 3×3 submatrix of N is upper triangular, $\{L, S_1, S_2\}$ span a free R-module. The linear syzygy L is of bidegree $(0, 1)$, so in the ν-strand of the complex $\mathcal{Z}$, it gives rise to

$$h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a-1, b-2)) = 2a(b-1)$$

columns of the matrix of the map d_1. The two syzygies S_1, S_2 of bidegree $(a, b-1)$ each give rise to

$$h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a-1, 0)) = a$$

columns of the matrix of d^1. That the columns are independent follows from the fact that $\{L, S_1, S_2\}$ span a free R-module. Hence, these syzygies yield $2ab$ columns of the degree ν component of the matrix d_1. $\square$

For Theorem 2.1.2 and Theorem 2.1.3 to hold, we need $a, b \geq 2$, even if U is basepoint free. If either a or b is at most one, there can be additional linear syzygies for example if $(a, b) = (1, 1)$, then there are four minimal first syzygies. However, it is easy to see that the theorems both hold if L is of bidegree $(1, 0)$.

Corollary 2.1.4. If $a, b \geq 2$, U is basepoint free, and I_U has a linear first syzygy, then the first matrix of the approximation complex in degree $\nu = (2a-1, b-1)$ is determined by the syzygies $\{L, S_1, S_2\}$.

Proof. From Theorem 2.1.3 we know that the dimension of the module spanned by $\{L, S_1, S_2\}$ in degree ν is $2ab$. From Lemma 1.3.7 we know that $\dim_k(\mathcal{Z}_1)_\nu = 2ab$. Hence a basis for $\mathrm{Syz}(I_U)_\nu$ is obtained from $\{L, S_1, S_2\}$ and this basis determines $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$. □

Corollary 2.1.5. If $a, b \geq 2$, U is basepoint free, and I_U has a linear first syzygy, then the singular locus of X_U contains a line.

Proof. Let $I_U = \langle pu, pv, p_2, p_3 \rangle$. By Corollary 2.1.4, the matrix representing d_1 in bidegree ν has as its leftmost $2a(b-1)$ columns a block matrix P. For each monomial $m_c = s^{2a-1-c}t^c$ with $c \in \{0, \ldots, 2a-1\}$, there is a $b \times b-1$ block B corresponding to elements $m_c \cdot \{v^{b-2}, \ldots, u^{b-2}\} \cdot L$, with $L = vx_0 - ux_1$, hence

$$B = \begin{bmatrix} x_0 & 0 & \cdots & \cdots & 0 \\ -x_1 & x_0 & 0 & \vdots & 0 \\ \vdots & -x_1 & \ddots & \vdots & 0 \\ \vdots & 0 & x_0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & -x_1 & x_0 \\ 0 & 0 & 0 & 0 & -x_1 \end{bmatrix} \text{ and } P = \begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & \ddots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & B \end{bmatrix}$$

Computing the Laplace expansion of the determinant of the matrix d_1 using the $2ab-2a$ minors of P shows the implicit equation for X_U takes the form

$$x_0^{2ab-2a} \cdot f_0 + x_0^{2ab-2a-1} x_1 \cdot f_1 + \cdots + x_1^{2ab-2a} \cdot f_{2ab-2a} = 0.$$

So X_U is singular along $\mathbf{V}(x_0, x_1)$, with multiplicity at least $2ab - 2a$. □

2.2 Applications to the bidegree $(2,2)$ case

We close this chapter with some examples in the bidegree $(2,2)$ case; without loss of generality we assume that I_U has a linear first syzygy of bidegree $(0,1)$, so $I_U = \langle pu, pv, p_2, p_3\rangle$. Hence p is of bidegree $(2,1)$. There are three possible factorizations for p:

(1) p is irreducible.

(2) p is the product of an irreducible form of bidegree $(1,1)$ and a form of bidegree $(1,0)$. So $p = ql$, where $q = a_0su + a_1tu + a_2tv$ and $l = b_0s + b_1t$. The locus of such forms is the image of the map

$$\mathbb{P}(H^0(1,1)) \times \mathbb{P}(H^0(1,0)) = \mathbb{P}^3 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^5,$$

$(a_0 : a_1 : a_2 : a_3) \times (b_0 : b_1) \mapsto (a_0b_0 : a_0b_1 + a_2b_0 : a_1b_0 : a_1b_1 + a_3b_0 : a_3b_1)$, which is a quartic hypersurface.

$$Q = \mathbf{V}(x_2^2x_3^2 - x_1x_2x_3x_4 + x_0x_2x_4^2 + x_1^2x_3x_5 - 2x_0x_2x_3x_5 - x_0x_1x_4x_5 + x_0^2x_5^2)$$

Note that $\Sigma_{2,1} \subseteq \mathbf{V}(Q)$.

(3) p is a product of three linear forms, two of bidegree $(1,0)$ and one of bidegree $(0,1)$. Then identifying the coefficients of $p = a_0s^2u + a_1stu + a_2t^2u + a_3s^2v + a_4stv + a_5t^2v$ with a point of $\mathbb{P}^5$, such a decomposition corresponds to a point on the Segre variety $\Sigma_{2,1}$, whose ideal is defined by the 2×2 minors of

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}.$$

Examples of possible bigraded free resolutions for these three cases are shown below, where p_2 and p_3 are chosen generically. If p_2 and p_3 are not chosen generically, there are many additional possible types bigraded free resolutions. It would be interesting to prove that the free resolution below are always the bigraded resolutions for generic choices of p_2 and p_3. For brevity, we denote $R(a,b)$ by (a,b). In all three cases, X_U has degree $2ab = 8$, in contrast to Example 2.0.2.

Example 2.2.1. Suppose $p \notin \mathbf{V}(Q)$. Changing coordinates, we may assume p is the point $(1:0:0:0:0:1)$, which corresponds to $p = s^2u + t^2v$. In this case, I_U has minimal free resolution

$$0 \longleftarrow I_U \longleftarrow (-2,-2)^4 \longleftarrow \begin{array}{c} (-2,-3) \\ \oplus \\ (-4,-3)^2 \\ \oplus \\ (-4,-4) \\ \oplus \\ (-3,-5)^2 \\ \oplus \\ (-6,-3) \\ \oplus \\ (-8,-2) \end{array} \longleftarrow \begin{array}{c} (-4,-5)^3 \\ \oplus \\ (-6,-4)^2 \\ \oplus \\ (-8,-3)^2 \end{array} \longleftarrow \begin{array}{c} (-6,-5) \\ \oplus \\ (-8,-4) \end{array} \longleftarrow 0\,.$$

The reduced singular locus of X_U consists of curves of degrees 1, 2, and 3.

Example 2.2.2. Suppose $p \in \mathbf{V}(Q) \setminus \Sigma_{2,1}$. After a change of coordinates, we may assume p is the point $(1:2:1:1:1:0)$, which corresponds to $s^2u + 2stu + t^2u + s^2v + stv$. In this case, I_U has minimal free resolution

$$0 \longleftarrow I_U \longleftarrow (-2,-2)^4 \longleftarrow \begin{array}{c} (-2,-3) \\ \oplus \\ (-4,-3)^2 \\ \oplus \\ (-4,-4)^3 \\ \oplus \\ (-3,-5)^2 \\ \oplus \\ (-6,-3) \\ \oplus \\ (-7,-2) \end{array} \longleftarrow \begin{array}{c} (-4,-5)^3 \\ \oplus \\ (-6,-4)^2 \\ \oplus \\ (-7,-3)^2 \end{array} \longleftarrow \begin{array}{c} (-6,-5) \\ \oplus \\ (-7,-4) \end{array} \longleftarrow 0\,.$$

The reduced singular locus of X_U consists of curves of degrees 1, 1 and 4.

Example 2.2.3. Now suppose $p \in \Sigma_{2,1}$. After a change of coordinates, we may assume p is the point $(1:1:1:1:1:1)$, which corresponds to $s^2u + stu + t^2u + s^2v + stv + t^2v$. In this case, I_U has minimal free resolution

$$0 \longleftarrow I_U \longleftarrow (-2,-2)^4 \longleftarrow \begin{array}{c} (-2,-3) \\ \oplus \\ (-4,-3)^2 \\ \oplus \\ (-4,-4) \\ \oplus \\ (-3,-5)^2 \\ \oplus \\ (-6,-2) \end{array} \longleftarrow \begin{array}{c} (-4,-5)^3 \\ \oplus \\ (-6,-4) \\ \oplus \\ (-6,-3)^2 \end{array} \longleftarrow (-6,-5) \longleftarrow 0\,.$$

The reduced singular locus of X_U consists of curves of degrees 1 and 4.

Chapter 3

Tensor product surfaces with basepoints

In this chapter we focus on tensor product surfaces X_U such that $U \subset H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b))$ has $b = 1$ and such that $\mathcal{B}$ is a generic finite set of points. Tensor product surfaces for which $b = 1$ are also known as rational ruled surfaces in the literature. These surfaces have been studied before by Chen, Zheng and Sederberg [CZS01] and by Dohm [Doh09]. Chen, Zheng and Sederberg show that the implicit equation of a generically injective parameterization can be obtained from a submodule of the module of syzygies of I_U and show that this module is of rank two. Dohm generalizes these results for a parameterization of any degree and provides an algorithm to reparameterize the given rational ruled surface. In this work we study the syzygies of I_U based on the geometry of its base locus $\mathcal{B}$. This approach gives the exact degrees of the syzygies that determine $\mathcal{Z}_\nu$ and answers the question raised by Chen, Cox and Liu [CCL05] of what can be said about the degrees of the syzygies that determine the implicit equation. The statement of the main result is the following:

Theorem 3.0.1. Let $(I_{\mathcal{B}})_{(a,1)} \subset H^0(a,1)$ be the k-vector space of forms of bidegree $(a,1)$ that vanish at a generic set $\mathcal{B}$ of r points in $\mathbb{P}^1 \times \mathbb{P}^1$. Take $U = \{f_0, f_1, f_2, f_3\}$ to be a general 4-dimensional vector subspace of $(I_{\mathcal{B}})_{(a,1)}$ and $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ the rational map determined by U. Then the map $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ where $\nu = (2a-1, 0)$, is determined by the syzygies of (f_0, f_1, f_2, f_3) in bidegrees $\left(a - \left\lfloor \frac{r}{2} \right\rfloor, 0\right)$, $\left(a - \left\lceil \frac{r}{2} \right\rceil, 0\right)$. The map d_1 completely determines the complex $\mathcal{Z}_\nu$ from which we compute the implicit equation of X_U.

The statements of the theorems, examples and proofs in this chapter are the same as in [Dua16] with minor changes in notation. The changes in notation were incorporated to be consistent with the notation used throughout this thesis.

Example 3.0.2. Take U to be a 4–dimensional vector subspace of $R_{(3,1)}$ given by

$$U = \{s^3v - st^2u + st^2v, t^3u + st^2u + st^2v, s^2tu + st^2u - 3st^2v, s^2tv - 5st^2u + st^2v\}.$$

Then $\mathcal{B} = \{[1,0;1,0],[0,1;0,1]\}$ so U has two basepoints. We use `Macaulay2` to check that $\mathcal{B}$ is a generic set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. To compute the implicit equation of X_U, we need to set up the complex $\mathcal{Z}_\nu$ and then find $\det \mathcal{Z}_\nu$. The complex $\mathcal{Z}_\nu$,

$$\mathcal{Z}_\nu : \qquad 0 \longrightarrow (\mathcal{Z}_2)_\nu \xrightarrow{d_2} (\mathcal{Z}_1)_\nu \xrightarrow{d_1} (\mathcal{Z}_0)_\nu \longrightarrow 0$$

only has three nonzero terms and it is exact. Following Theorem 1.3.2, $\nu = (5,0)$. The ideal I_U has two syzygies in degree $(2,0)$ and three syzygies in bidegree $(1,1)$. The syzygies in degree $(2,0)$ span a free R-module and determine a basis for the syzygies of bidegree $(5,0)$ by multiplying by $\{s^3, s^2t, st^2, t^3\}$. We proceed to apply the map corresponding to the Koszul differential on (X,Y,Z,W) to this basis and obtain the matrix d_1. Using `Macaulay2` and $d_2 = \ker d_1$ we obtain $d_1 =$

$$\begin{pmatrix}
107Y & 0 & 0 & 0 & 107W & 0 & 0 & 0 \\
-228Y-107Z-52W & 107Y & 0 & 0 & -107X+1082Y+535Z+335W & 107W & 0 & 0 \\
-55X-32Z-41W & -228Y-107Z-52W & 107Y & 0 & -442X-49Z+295W & -107X+1082Y+535Z+335W & 107W & 0 \\
0 & -55X-32Z-41W & -228Y-107Z-52W & 107Y & 0 & -442X-49Z+295W & -107X+1082Y+535Z+335W & 107W \\
0 & 0 & -55X-32Z-41W & -228Y-107Z-52W & 0 & 0 & -442X-49Z+295W & -107X+1082Y+535Z+335W \\
0 & 0 & 0 & -55X-32Z-41W & 0 & 0 & 0 & -442X-49Z+295W
\end{pmatrix},$$

$$d_2 = \begin{pmatrix}
-1/107W & 0 \\
1/107X-1082/11449Y-5/107Z-335/11449W & -1/107W \\
442/11449X+49/11449Z-295/11449W & 1/107X-1082/11449Y-5/107Z-335/11449W \\
0 & 442/11449X+49/11449Z-295/11449W \\
1/107Y & 0 \\
-228/11449Y-1/107Z-52/11449W & 1/107Y \\
-55/11449X-32/11449Z-41/11449W & -228/11449Y-1/107Z-52/11449W \\
0 & -55/11449X-32/11449Z-41/11449W
\end{pmatrix}.$$

The determinant of $\mathcal{Z}_\nu$ is computed by $\det \mathcal{Z}_\nu = \det M_1 / \det M_2$, where M_1 is a maximal nonzero minor of d_1 and M_2 is the complementary maximal nonzero minor of d_2,

$$\begin{aligned}
\det \mathcal{Z}_\nu = \; & -8831798120631365X^3Y + 623043212873630840X^2Y^2 - 2432437780569525764XY^3 \\
& +154155021741929280X^2YZ - 2181293557648299312XY^2Z - 694982223019864504Y^3Z- \\
& 516419322835463088XYZ^2 - 679406142698023733Y^2Z^2 - 167804164291995935YZ^3 \\
& +29064644724259583X^2YW - 2253232567794532976XY^2W + 347491111509932252Y^3W+ \\
& 8831798120631365X^2ZW - 1142192364219107259XYZW - 288398353175526028Y^2ZW- \\
& 109996031138772455XZ^2W - 277318460987824861YZ^2W - 33560832858399187Z^3W \\
& +8831798120631365X^2W^2 - 417342605736743957XYW^2 + 335768906731639713Y^2W^2 \\
& -103733483380506578XZW^2 + 6583704053561563YZW^2 - 20232846603628218Z^2W^2- \\
& 20232846603628218XW^3 + 65676462387967787YW^3 + 3693297395900389ZW^3+ \\
& 3693297395900389W^4.
\end{aligned}$$

3.1 Ideal of a generic set of points in $\mathbb{P}^1 \times \mathbb{P}^1$

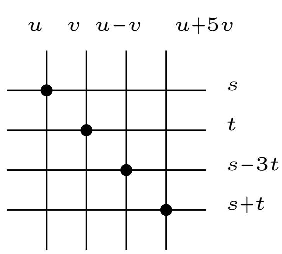
Using the description of some of the minimal generators of the ideal $I_{\mathcal{B}}$ associated to a generic set of points $\mathcal{B}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by Van Tuyl [VT05] we are able to understand exactly the degrees of the syzygies of (f_0, f_1, f_2, f_3) that determine the implicit equation of X_U. Using this description in Section 3.2 will allow us to find syzygies of $I_U = \langle f_0, f_1, f_2, f_3 \rangle$.

Definition 3.1.1. Let X be a set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$. The set X is said to be generic if its bigraded Hilbert function, is determined by

$$H_X(i,j) = \min\{(i+1)(j+1), r\}.$$

Being a generic set of points is a property of the Hilbert function.

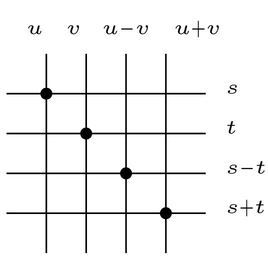
Example 3.1.2. Let $I_{X_1} = (s,u) \cap (t,v) \cap (s-3t, u-v) \cap (s+t, u+5v)$. Using `Macaulay2` we compute the bigraded Hilbert function H_{X_1} on the right and conclude X_1 is generic. The distribution of X_1 on the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ is illustrated on the left.



$H_{X_1} =$

	0	1	2	3	4
0	1	2	3	4	4
1	2	4	4	4	4
2	3	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4

Example 3.1.3. Let $I_{X_2} = (s,u) \cap (t,v) \cap (s-t, u-v) \cap (s+t, u+v)$. Using `Macaulay2` we compute the bigraded Hilbert function H_{X_2} on the right and conclude that X_2 is not generic.



$H_{X_2} =$

	0	1	2	3	4
0	1	2	3	4	4
1	2	3	4	4	4
2	3	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4

The ideal I_{X_2} has a minimal generator of bidegree $(1,1)$ whereas the generic set of points X_1 from Example 3.1.2 has no generator of such bidegreee. However we have $\alpha_{X_1} = \alpha_{X_2}$ and $\beta_{X_1} = \beta_{X_2}$.

Note that from the definition of a generic set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ we cannot immediately describe the bidegrees of the minimal generators of I_X nor the distribution of the points of X in families of $(1,0)$ and $(0,1)$. We now focus on describing these two aspects for the rest of this section.

Geometric description of X in terms of rulings

Let X be a set of generic points in $\mathbb{P}^1 \times \mathbb{P}^1$. Using the Hilbert function of X, we see that for (i,j) with $0 \leq j$ and $i \geq r-1$, $H_X(i,j) = r$ and for (i,j) with $0 \leq i$ and $j \geq r-1$, $H_X(i,j) = r$. Using Theorem 1.4.3, we know that for $j = 0$ and $i \gg 0$, $H_X(i,0) = \alpha_1^* = r$. Since X consists exactly of r points, we conclude that $\alpha_i^* = 0$ for all $i > 0$ and $\alpha_X = (1, \ldots, 1)$ has r parts. Similarly, $\beta_X = (1, \ldots, 1)$ has r parts. Using the definition of α_X, we know that the number of parts of α_X is equal to the number of horizontal lines in $\mathbb{P}^1 \times \mathbb{P}^1$ that contain points of X. Moreover, each part α_i of α_X corresponds to the number of points in X that lie on the horizontal line h_i. From $\alpha_X = (1, \ldots, 1)$, we conclude that there are r distinct horizontal lines in $\mathbb{P}^1 \times \mathbb{P}^1$ each containing exactly one point of X. We have a similar statement for β_X and vertical lines. After reordering of the vertical and horizontal lines that contain points of X, we can describe X combinatorially as the set of diagonal points inside an $r \times r$ grid of lines.

Generators of I_X

If $\mathcal{E}$ is any subset of $\mathbb{N}^2$ and $\underline{a} = (a_0, a_1) \in \mathbb{N}^2$ is any pair, then $\mathcal{E} + \underline{a}$ denotes the set $\{\underline{e} + \underline{a} : \underline{e} \in \mathcal{E}\}$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$.

For every $i \geq 0$, let

$$j(i) := \min\{t \in \mathbb{N} | H_X(i,t) = H_X(i,t+1)\},$$

and for every $j \geq 0$, let

$$i(j) := \min\{t \in \mathbb{N} | H_X(t,j) = H_X(t+1,j)\}.$$

Theorem 3.1.4 ([VT05])**.** Let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Fix $e_1 = (1,0)$ and $j \in \mathbb{N}$. Set $\underline{i} = (i(j), j)$. Then

$$(I_X)_{\underline{i}+(q+1)e_1} = R_{e_1}(I_X)_{\underline{i}+qe_1} \quad \forall q \in \mathbb{N}.$$

In particular, if there exists $\underline{l} \in \mathbb{N}^2$ and $t \in \{1,2\}$ such that $H_X(\underline{l}) = H_X(\underline{l} - e_t) = H_X(\underline{l} - 2e_t)$, then I_X has no minimal generators of bidegree $\underline{l}$.

Theorem 3.1.4 gives a way to exclude bidegrees that will not show up as bidegrees of minimal generators for I_X. It also gives a description of the higher degree pieces of I_X once the bigraded Hilbert function has stabilized in one of the coordinates.

Define the $k[s,t]$-module M by

$$M = \bigoplus_{i=0}^{\infty} (I_X)_{(i,1)}.$$

Following the notation of Theorem 3.1.4, fix $j = 1$. Then $i(1)$ is the value of i at which the bigraded Hilbert function H_X of X stabilizes for the column $j = 1$. So for all $q \in \mathbb{N}$,

$$H_X(i(1), 1) = H_X(i(1) + q, 1).$$

Recall that r is the number of points in X. We want to find $i(1)$ and understand the generators of M as a $k[s,t]$ module.

Proposition 3.1.5. Let X be a generic set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$ with associated bihomogeneous ideal I_X. Then the $k[s,t]$-module M has two minimal generators in bidegrees $(k,1),(k,1)$ if $r = 2k$ and two minimal generators in bidegrees $(k,1),(k+1,1)$ if $r = 2k+1$.

Proof. Suppose $r = 2k$. Then

$$H_X(i,1) = \min\{r, (i+1)2\}.$$

For all $0 \le i < k-1$, $H_X(i,1) = (i+1)2$ and for $i \ge k-1$, $H_X(i,1) = r$. Thus $i(1) = k-1$ and we have

$$r = H_X(k-1,1) = H_X(k,1) = H_X(k+1,1).$$

Using Theorem 3.1.4 we know that I_X has no minimal generators of degree $(k+1,1)$ or higher. Thus possible minimal generators of I_X are in bidegrees $(k-1,1),(k,1)$. But we have

$$H_X(k-1,1) = r = \dim R_{(k-1,1)} - \dim(I_X)_{(k-1,1)}, \tag{3.1}$$

so $\dim(I_X)_{(k-1,1)} = 0$ and I_X does not have generators in bidegree $(k-1,1)$. A similar equation to (3.1) shows that $\dim(I_X)_{(k,1)} = 2$. Therefore M has two generators of bidegree $(k,1)$. Moreover, using the last part of Theorem 3.1.4 we have

$$(I_X)_{i(1)+(q+1)e_1} = R_{e_1}(I_X)_{i(1)+qe_1} \quad \forall q \in \mathbb{N}. \tag{3.2}$$

Since $R_{e_1} = \{s,t\}$ this shows M has two generators in bidegree $(k,1)$ as a $k[s,t]$ module. The proof for the case when r is odd is done in a similar way. $\square$

3.2 Syzygies of the ideal I_U

As was highlighted in the Section 1.3.2, the syzygies of I_U determine the complex $\mathcal{Z}$ which in turn determine the implicit equation of X_U. Let $\mathrm{Syz}(I_U)_{(-,0)}$ denote the $k[s,t]$-module of syzygies of $I_U = \langle f_0, f_1, f_2, f_3\rangle \subset R$ of degree zero in u, v. The following proposition states that $\mathrm{Syz}(I_U)_{(-,0)}$ has rank two and gives the bidegrees of its minimal generators.

Proposition 3.2.1. Let $\mathcal{B}$ be a generic set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $U = \{f_0, f_1, f_2, f_3\}$ be a general 4-dimensional vector space of $(I_{\mathcal{B}})_{(a,1)}$ with $a > k$. Then the $k[s,t]$-module $\mathrm{Syz}(I_U)_{(-,0)}$ has rank two. Moreover, $\mathrm{Syz}(I_U)_{(-,0)}$ has minimal generators of bidegree $(a-k, 0)$ if $r = 2k$ and of bidegrees $(a-k, 0), (a-k+1, 0)$ if $r = 2k+1$.

Remark 3.2.2. If $\mathbf{b} = \{b_1, \ldots, b_q\}$ is a basis for $(I_X)_{(a,1)}$, we make the convention that a general choice of U is given by a matrix C of size $4 \times q$ with $\mathbf{f} = C\mathbf{b}$ all of whose maximal minors are nonzero.

The proof of Proposition 3.2.1 will be done in four steps. The steps that will follow are done for the case $r = 2k$. If $r = 2k+1$, similar steps work by changing to $(a-k, 0)$ and $(a-k+1, 0)$.

Step 1. Use the description of a basis for $(I_{\mathcal{B}})_{(a,1)}$ to show that the generators of the ideal I_U may be written in a simpler and more convenient form.

From Proposition 3.1.5 we know $(I_{\mathcal{B}})_{(k,1)}$ is generated by two elements g_1, g_2. Thus by equation (3.2) in the proof of Proposition 3.1.5, a basis for $(I_{\mathcal{B}})_{(a,1)}$ is given by

$$\mathbf{b} = \{s^j g_1, s^{j-1}tg_1, \ldots, t^j g_1, s^j g_2, \ldots, st^{j-1}g_2, t^j g_2\},$$

where $j = a-k$. Following our convention from Remark 3.2.2, a general choice of U is a coefficient matrix C such that every maximal minor of C is nonzero with $\mathbf{f} = C\mathbf{b}$. Using the basis $\mathbf{b}$ for $(I_{\mathcal{B}})_{(a,1)}$

$$(f_0 \;\; f_1 \;\; f_2 \;\; f_3)^T = (s^j g_1 \;\; s^{j-1}tg_1 \;\; \cdots \;\; t^j g_1 \;\; s^j g_2 \;\; \cdots \;\; t^j g_2)C^T.$$

The first $j+1$ elements of the basis share the common factor g_1 and the last $j+1$ elements share the factor g_2, thus we may write

$$(f_0 \quad f_1 \quad f_2 \quad f_3) \;=\; (g_1 \quad g_2) \begin{pmatrix} Q_0 & Q_1 & Q_2 & Q_3 \\ P_0 & P_1 & P_2 & P_3 \end{pmatrix} \tag{3.3}$$

where $\deg Q_i = \deg P_i = (j, 0)$. This means Q_i, P_i are forms in the variables s, t. Denote the rightmost matrix in (3.3) by QP.

Using the notation in the first step we let

$$(Q_0\ \ Q_1\ \ Q_2\ \ Q_3) = (s^j\ \ s^{j-1}t\ \cdots\ t^j)A^T, \tag{3.4}$$

and

$$(P_0\ \ P_1\ \ P_2\ \ P_3) = (s^j\ \ s^{j-1}t\ \cdots\ t^j)B^T. \tag{3.5}$$

Denote the matrix of coefficients of Q_i's by A and the matrix of coefficients of P_i's by B. Observe that C is the block matrix formed by A and B, so $C = (A|B)$.

Step 2. Elements in $\mathrm{Syz}(I_U)_{(-,0)}$ are in one-to-one correspondence with elements in $\ker QP$.

Lemma 3.2.3. Let α be a positive integer. The ideal I_U has a syzygy L of bidegree $(\alpha, 0)$ if and only if L is an element in the kernel of the matrix

$$QP = \begin{pmatrix} Q_0 & Q_1 & Q_2 & Q_3 \\ P_0 & P_1 & P_2 & P_3 \end{pmatrix}$$

over the ring $k[s,t]$.

Proof. Suppose that $L = (s_0, s_1, s_2, s_3)$ is a syzygy of (f_0, f_1, f_2, f_3) of degreee $(\alpha, 0)$. Then

$$s_0f_0 + s_1f_1 + s_2f_2 + s_3f_3 = 0.$$

Using Step 1, $f_i = Q_ig_1 + P_ig_2$. We may substitute this in the previous equation and factor g_1 and g_2 as follows,

$$(s_0Q_0 + s_1Q_1 + s_2Q_2 + s_3Q_3)\cdot g_1 + (s_0P_0 + s_1P_1 + s_2P_2 + s_3P_3)\cdot g_2 = 0$$

The elements g_1, g_2 are minimal generators of $I_{\mathcal{B}}$ and thus form a complete intersection. Using the fact that the only syzygies of a complete intersection are Koszul we obtain

$$\begin{pmatrix} s_0Q_0 + s_1Q_1 + s_2Q_2 + s_3Q_3 \\ s_0P_0 + s_1P_1 + s_2P_2 + s_3P_3 \end{pmatrix} = q \begin{pmatrix} -g_2 \\ g_1 \end{pmatrix}.$$

Note that $\deg s_0Q_0 + s_1Q_1 + s_2Q_2 + s_3Q_3 = \deg s_0P_0 + s_1P_1 + s_2P_2 + s_3P_3 = (\alpha + j, 0)$. But $\deg g_i = (k, 1)$, hence $q = 0$ and

$$\begin{pmatrix} s_0Q_0 + s_1Q_1 + s_2Q_2 + s_3Q_3 \\ s_0P_0 + s_1P_1 + s_2P_2 + s_3P_3 \end{pmatrix} = 0$$

so $L \in \ker QP$. Verifying the other direction is straightforward from the previous calculations. □

Step 3. When QP is a matrix whose entries are polynomials in $k[s,t]$ of degree j, with coefficients from a generic matrix C as in equations (3.4) and (3.5), the kernel of QP has rank two and its minimal generators are of degree j.

A quick check reveals that the assumption $\gcd(f_0, f_1, f_2, f_3) = 1$ implies the matrix QP has rank two. Then we can prove that if the coefficients of QP are chosen generically following Remark 3.2.2, then the two minimal generators of $\ker QP$ have degree $j = a - k$. If the coefficients of QP are not chosen generically, we can expect the degrees of the two minimal generators of $\ker QP$ to be less than or equal to j.

Step 4. Using Step 2, we know that the syzygies of I_U in bidegree $(\alpha, 0)$ are in one-to-one correspondence with elements in the kernel of QP. Following Step 3, the kernel of QP is generated by two elements $K1, K2$ of degree j in s, t, hence I_U has two minimal syzygies of bidegree $(j, 0)$, $j = a - k$. This completes the proof of Proposition 3.2.1. □

Remark 3.2.4. Note that the two elements $K1, K2$ in the kernel of QP generate a free module. Indeed, QP fits into a sequence

$$0 \longrightarrow S^2 \longrightarrow S^4 \xrightarrow{QP} S^2 \longrightarrow \operatorname{coker} QP \longrightarrow 0 \tag{3.6}$$

where the leftmost nonzero map sends S^2 to the two syzygies $K1, K2$ and $S = k[s,t]$. Then by Hilbert's syzygy theorem over S, the leftmost map in the sequence (3.6) is injective.

3.3 Applications to implicitization of tensor product surfaces

We now focus on obtaining the implicit equation of X_U, when $U = \{f_0, f_1, f_2, f_3\} \subseteq (I_{\mathcal{B}})_{(a,1)}$ is a general 4-dimensional subspace and $\mathcal{B}$ is a generic set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $U \subseteq (I_{\mathcal{B}})_{(a,1)}$, $\mathcal{B}$ is the set of basepoints of X_U and we are in the context of Theorem 1.3.2. Recall that $\mathcal{Z}$ denotes the complex associated to the sequences (f_0, f_1, f_2, f_3) and (X, Y, Z, W).

3.3.1 Proof of the main theorem

To prove Theorem 3.0.1 we need two lemmas, the first one is Lemma 1.3.7 that gives the dimension of $(\mathcal{Z}_1)_\nu$. The second one provides a basis for $(\mathcal{Z}_1)_\nu$ and is stated below.

Lemma 3.3.1. Let $U = \{f_0, f_1, f_2, f_3\}$ be a general 4-dimensional vector subspace of $(I_\mathcal{B})_{(a,1)}$. A basis for $(\mathcal{Z}_1)_\nu$, $\nu = (2a-1, b-1)$, is obtained by multiplying the two syzygies of I_U in bidegree $(a-k, 0)$ times the elements in a monomial basis of $R_{(a+k-1,0)}$ if $r = 2k$. If $r = 2k+1$ then a basis for $(\mathcal{Z}_1)_\nu$ is obtained by multiplying the two syzygies in bidegrees $(a-k, 0)$, $(a-k+1, 0)$ times the elements in monomial bases for $R_{(a+k-1,0)}$ and $R_{(a+k-2,0)}$ respectively.

Proof. We have $(\mathcal{Z}_1)_\nu = \mathrm{Syz}(I_U)_\nu$. If r is even then I_U has two syzygies S_1, S_2 of bidegree $(a-k, 0)$ by Proposition 3.2.1. Using Remark 3.2.4 at the end of Section 3.2, we know that S_1, S_2 span a free module. Hence the set of syzygies $\mathcal{S} = \{m \cdot S_i | m \in R_{(a+k-1,0)}, i = 1, 2\}$ is an independent subset of $(\mathrm{Syz}(I_U))_\nu$. Note that $\dim_k R_{(a+k-1,0)} = a + k$, thus $\dim_k \mathrm{Span}\, \mathcal{S} = 2a + r$. From Lemma 1.3.7 $\dim(\mathcal{Z}_1)_\nu = 2a + r$, hence a basis for $(\mathcal{Z}_1)_\nu$ is determined by multiplying S_1, S_2 times a monomial basis for $R_{(a+k-1,0)}$. An analogous argument works in the case r is odd. $\square$

We are now ready to prove Theorem 3.0.1.

Proof. We consider the case $r = 2k$; the case $r = 2k+1$ is done similarly. Let S_1, S_2 be the two minimal syzygies of I_U in bidegree $(a-k, 0)$. By Lemma 3.3.1, $(\mathcal{Z}_1)_\nu$ has a basis given by the elements of the form $m \cdot S_i$ where $i = 1, 2$ and $m \in R_{(a+k-1,0)}$. Denote the matrix of the first map of the complex $\mathcal{Z}_\nu$ by d_1. Then d_1 is obtained by applying the Koszul differential on the sequence (X, Y, Z, W) to all the elements $\{m \cdot S_i\}$. Using the proof of Lemma 1.3.7, we know $\mathcal{Z}_\nu$ is exact, hence $d_2 = \ker d_1$. Therefore the complex $\mathcal{Z}_\nu$ that determines the implicit equation of X_U only depends on the syzygies of I_U in degree $(a-k, 0)$. $\square$

Corollary 3.3.2. Let $\mathcal{B}$ be a set of r points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $r = 2k$. If U is any 4-dimensional vector subspace of $(I_\mathcal{B})_{(k+1,1)}$, then X_U is projectively equivalent to $\mathbf{V}(XW - YZ)$.

Proof. From Theorem 3.1.5, $(I_\mathcal{B})_{(k+1,1)} = \{sg_1, tg_1, sg_2, tg_2\}$. Then any choice U is equivalent to $U = \{sg_1, tg_1, sg_2, tg_2\}$. It follows that $X_U = \mathbf{V}(XW - YZ)$. $\square$

The results in Theorem 3.0.1 do not generalize immediately to tensor product surfaces of more general bidegree (a, b) with $b > 1$. One of the advantages of the condition $b = 1$ is that the calculation of the syzygies of I_U is reduced to finding the kernel of the matrix QP over the polynomial ring $k[s,t]$ which has two fewer variables than R. This allows us to show in Remark 3.2.4 that the syzygies of I_U are free. For more general bidegree, the syzygies of I_U are not free and computing a basis for $\mathrm{Syz}(I_U)_\nu$ is more difficult because of the possible relations between the generators of $\mathrm{Syz}(I_U)_\nu$.

3.3.2 Tensor product surfaces without basepoints

The main theorem in this paper allows us to describe the syzygies that determine the implicit equation of a map given by 4 generically chosen forms of bidegree $(a,1)$ that vanish at a generic set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. The techniques that we used to prove the main theorem depend on understanding the generators of the $k[s,t]$-module $M = \bigoplus_{i=0}^{\infty}(I_{\mathcal{B}})_{(i,1)}$ and using the description of $(I_{\mathcal{B}})_{(a,1)}$ given by (3.2) from Section 3.1. For the case of basepoints, $M = \langle g_1, g_2\rangle$. If $\mathcal{B} = \emptyset$ then $M = \langle u, v\rangle$. In the proof of the main theorem, we may substitute g_1, g_2 for the complete intersection u, v and the proof will still be valid. The statement of the theorem is the following:

Theorem 3.3.3. Let $U = \{f_0, f_1, f_2, f_3\}$ be a basepoint-free, general, 4-dimensional vector subspace of $R_{(a,1)} = H^0(a,1)$ and $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ the regular map determined by U. Then the first map of the approximation complex $\mathcal{Z}$ in bidegree $\nu = (2a-1, 0)$, is determined by two syzygies of $(f_0, \ldots, f_3)$ in bidegree $(a, 0)$.

A careful study of the implicitization of basepoint free tensor product surfaces of bidegree $(2,1)$ using syzygies was done by Schenck, Seceleanu and Validashti [SSV14]. Theorem 3.3.3 recovers their results for the case that I_U has no linear syzygies and extends them to bidegree $(a,1)$.

Chapter 4

Algorithms and closing remarks

In this chapter we propose two modifications to syzygy algorithms to obtain the matrix $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ o based on Theorem 2.0.1 and Theorem 3.0.1. We compare the performance of these algorithms to other algorithms using syzygies and to an algorithm using Gröbner bases and elimination ideals. We also provide additional examples and discuss some directions for future work.

4.1 Algorithms for implicitization of tensor product surfaces

We may use other methods, e.g Gröbner bases or resultants, to find implicit equations of parameterized surfaces. One way to use Gröbner bases is to compute the elimination ideal

$$J = \langle X - f_0, Y - f_1, Z - f_2, W - f_3 \rangle \cap S.$$

The ideal J is a principal ideal whose generator is the irreducible implicit equation of the parameterized surface defined by the polynomials f_0, f_1, f_2, f_3. Resultants are used for basepoint free parameterizations of the form $\lambda : \mathbb{P}^2 \to \mathbb{P}^3$. In this case H is computed by

$$\mathrm{Res}(f_0 - Xf_3, f_1 - Yf_3, f_2 - Zf_3) = H(X, Y, Z, 1)^{\deg \lambda}.$$

In the following examples we will compute H using Gröbner bases and compare timings to several algorithms using syzygies. With syzygies we compute the matrix $d_1 : (\mathcal{Z}_1)_\nu \to (\mathcal{Z}_0)_\nu$ using three approaches. The first one, which we refer to as Algorithm 1, uses the command `super basis({3*a-1,2*b-1}, image syz Iu)` to find a basis of the syzygies in bidegree $\nu = (2a - 1, b - 1)$. Then it applies d_1 to this basis and sets up the matrix for d_1. The second and third approaches which we refer to as Algorithm 2 and Algorithm 3 take into account the results in Theorem 2.0.1 and Theorem 3.0.1 respectively, these will be described in the next two subsections. Further examples that show the advantages of implicitization using syzygies over other methods are presented in the work of Botbol and Dickenstein [BD16](Section 5). All of the examples in this chapter were performed in `Macaulay2` running on macOS V.10.12.3, with 1.1GHz Intel Core m3 processor and 8GB RAM. The `Macaulay2` code to perform the examples that follow is available at `https://github.com/emduart2`. We begin with a simple example in bidegree $(8,1)$ which shows the advantages of using syzygies for implicitization over the use of Gröbner bases. In the timing tables that we present, the Elimination algorithm corresponds to the algorithm using Gröbner bases to compute the implicit equation.

Example 4.1.1. We let X be the set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ in Example 3.0.2, where $I_X = (s,u) \cap (t,v)$. Fix the bidegree $(a,b) = (8,1)$ and let $U \subset (I_X)_{(8,1)}$ be given by

$$\begin{aligned} U = \{ \quad & 3s^6t^2u + s^8v + 7s^4t^4v - s^3t^5v, \;\; t^8u + 5s^6t^2v + s^2t^6v, \\ & s^7tu + 11s^5t^3u + s^2t^6u - st^7u, \;\; -s^4t^4u + s^3t^5u + st^7u + s^7tv + 19s^5t^3v + st^7v\}. \end{aligned}$$

The table below summarizes the timings to compute d_1 with syzygy algorithms and the implicit equation with Gröbner bases.

Method	Timing (seconds)
Elimination	149.178
Algorithm 1	1.67
Algorithm 3	0.133

The speed boost of Algorithm 3 with respect to Algorithm 1 depends on knowing the structure of the syzygies of I_U. For this example, d_1 is a 16×18 matrix, the implicit equation H of X_U has degree 14 and it contains 115 terms.

4.1.1 Basepoint free tensor product surfaces

In this section we use Algorithm 2 to compute the matrix d_1 under the hypotheses of Theorem 2.0.1. We now describe Algorithm 2. The input is the set $U = \{p_0, p_1, p_2, p_3\}$ of bihomogeneous forms of bidegree (a,b). We check the conditions $a, b \geq 2$, $\sqrt{I_U} = (s,t) \cap (u,v)$ and that I_U has a linear syzygy to make sure U satisfies the hypotheses of Theorem 2.0.1. Next we rewrite $I_U = \langle pu, pv, p_2, p_3 \rangle$. This can be easily done in `Macaulay2` following the proof of Lemma 2.1.1. Writing $p_2 = g_2v + f_2u$, $p_3 = f_3u + g_2v$ we obtain the syzygy matrix

$$N = \begin{bmatrix} v & f_2 & f_3 \\ -u & g_2 & g_3 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}.$$

Now take each column of N and bump it up to bidegree $\nu = (2a-1, b-1)$. For the first column we do this by using a monomial basis for $R_{(2a-1,b-2)}$ and for the remaining columns we use a basis for $R_{(a-1,0)}$. By Theorem 2.0.1 this gives us exactly the $2ab$ independent syzygies that form a basis for $\mathrm{Syz}(I_U)_\nu$. Finally we apply d_1 to this basis and obtain the matrix for d_1. Note that using Algorithm 2 we only have to compute the syzygies of I_U in total degree one. Moreover, knowing that the columns of N span a free module also simplifies the computation of a basis for $\mathrm{Syz}(I_U)_\nu$.

Example 4.1.2. We consider Example 2.0.2. In this case U is of bidegree $(2,2)$ and basepoint free. The timings for this example are given in the table below.

Method	Timing (seconds)
Elimination	0.11
Algorithm 1	0.03
Algorithm 2	0.008

Now let $U = \{p_0, \ldots, p_3\}$ be basepoint free of bidegree $(3,2)$ and chose the coefficients of each p_i with respect to the basis $R_{(3,2)}$ randomly in `Macaulay2`. The timings are:

Method	Timing (seconds)
Elimination	67.64
Algorithm 1	9.85
Algorithm 2	0.013

4.1.2 Tensor product surfaces with basepoints

We now use and describe Algorithm 3 to compute d_1 for tensor product surfaces of bidegree $(a,1)$. The input of Algorithm 3 is a set $U = \{f_0, f_1, f_2, f_3\}$ of bihomogeneous forms of bidegree (a,b). We use the command `syz Iu` to compute all the minimal syzygies of I_U. Then we select the syzygies of bidegree zero in u, v, by Theorem 3.0.1 there are two such syzygies S_1, S_2. Then we use the fact that these syzygies are free to bump them up to bidegree $\nu = (2a-1, b-1)$ and obtain a basis for $\mathrm{Syz}(I_U)_\nu$. Finally apply d_1 to the basis for $\mathrm{Syz}(I_U)_\nu$ and obtain a matrix for d_1.

Example 4.1.3. For U as in Example 3.0.2 of bidegree $(3,1)$ and with two baspeoints, the timings are:

Method	Timing (seconds)
Elimination	0.02
Algorithm 1	0.29
Algorithm 2	0.0007

Example 4.1.4. Take U to be a random generic 4-dimensional vector subspace of $R_{(20,1)}$, then U is basepoint free. This example can be generated in `Macaualay2` by finding a basis of $R_{(20,1)}$ using `super basis({20,1},R)` and then multiplying this basis times a random matrix C of coefficients of the correct size. For this choice of U the timings are

Method	Timing (seconds)
Elimination	–
Algorithm 1	180
Algorithm 3	0.751

.

The `eliminate` ideal command to find the implicit equation using Gröbner bases did not finish the computation in at least 120 minutes and therefore was aborted. In this example, d_1 is a 40×40 matrix and H has degree 40.

4.2 Future work

Tensor product surfaces of general bidegree

In the following example we fix the base locus $\mathcal{B} \subset \mathbb{P}^1 \times \mathbb{P}^1$ and consider tensor product surfaces defined by $U \subset (I_{\mathcal{B}})_{(a,b)}$ for different values of (a,b). We discuss some of the difficulties of extending the results in Theorem 3.0.1 for higher bidegree.

Example 4.2.1. Let $\mathcal{B}$ be the set of points defined by $I_B = (s,u) \cap (t,v) \cap (s-3t, u-v) \cap (s+t, u+5v)$ given in Example 3.1.2. The ideal $I_{\mathcal{B}}$ has minimal generators in degrees

$$(4,0), (2,1), (2,1), (1,2), (1,2), (0,4).$$

We summarize this information in the following table whose (i,j) entry is $\dim(I_{\mathcal{B}})_{(i,j)}$,

$$\dim(I_B)_{(i,j)} = \begin{array}{c||c|c|c|c|c} & 0 & 1 & 2 & 3 & 4 \\ \hline\hline 0 & - & - & - & - & \mathbf{1} \\ \hline 1 & - & - & \mathbf{2} & 4 & * \\ \hline 2 & - & \mathbf{2} & 5 & 8 & * \\ \hline 3 & - & 4 & 8 & * & * \\ \hline 4 & \mathbf{1} & * & * & * & * \end{array}$$

Following the notation from Proposition 3.1.5 we know that the module M is spanned elements g_1, g_2 of bidegree $(2,1)$. For this choice of $\mathcal{B}$, we have

$$g_1 = 6t^2u + 7s^2v - 23stv, \quad g_2 = 2stu - 3s^2v + 7stv.$$

Case 1: If $U \subset (I_{\mathcal{B}})_{(3,1)}$, up to a change of coordinates, $U = \{sg_1, tg_1, sg_2, tg_2\}$. Then $\nu = (5,0)$ and we saw in Corolary 3.3.2 that X_U is projectively equivalent to $X_U = \mathbf{V}(XW - YZ)$. In this case $\mathrm{Syz}(I_U)$ is generated by syzygies in bidegrees

$$(1,0), (1,0), (1,1).$$

Case 2: Let U be a general 4-dimensional vector space of $(I_{\mathcal{B}})_{(2,2)}$. Then $\nu = (3,1)$ and $\mathrm{Syz}(I_U)$ is generated in bidegrees

$$(1,1), (1,1), (1,1), (1,1), (2,0), (0,2).$$

From Lemma 1.3.7, we know $\dim \mathrm{Syz}(I_U)_\nu = 2ab + r = 12$ since $r = 4$. This implies that the module generated by the five syzygies of degrees $(1,1), (2,1)$ is not free. A `Macaulay2` computation shows the kernel of the syzygy matrix of I_U has rank two.

Case 3: Let U be a general 4-dimensional vector subspace of $(I_{\mathcal{B}})_{(3,2)}$. Then $\nu = (5,1)$ and $\mathrm{Syz}(I_U)$ is generated in bidegrees

$$(2,1), (2,1), (2,1), (2,1), (5,0), (1,2), (1,2), (1,2).$$

As in the case of bidegree $(2,2)$, the dimension $\dim \mathrm{Syz}(I_U)_\nu = 16$ implies the module generated by the five syzygies of degrees $(2,1)$ and $(5,0)$ is not free. A `Macaulay2` computation shows the kernel of the syzygy matrix of I_U has rank two.

A glance at the table for the dimension of $(I_{\mathcal{B}})_{(i,j)}$ shows that the $k[s,t]$-module $N = \oplus_{j=0}^{\infty} (I_{\mathcal{B}})_{(j,2)}$ has more than two minimal generators. It would be interesting to know if the structure of the module N can be related to the syzygyes if I_U in degree $\nu = (2a-1, b-1)$ where $b = 2$ as was done for the proof of Theorem 3.0.1. Notice that one of the difficulties of generalizing the results in bidegree $(a,1)$ is that for bidegree $(a,2)$ the syzygies of I_U in degree ν are not free. Although the example for bidegree $(3,1)$ is the simplest possible, the behavior in bidegrees $(2,2), (3,2)$ is already more involved.

Residual resultants for tensor product surfaces

One of the main ideas for the study of tensor product surfaces in this thesis is to connect the structure of $\mathrm{Syz}(I_U)$ with the geometry of the base locus of λ_U via the ideal $I_{\mathcal{B}}$. The machinery behind this path to obtain the complex $\mathcal{Z}_\nu$ is based on the study of Rees algebras and its connection to syzygies. In [Bus01], Busé studied surfaces of the form $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ and proposed the use of residual resultants as an alternative to the use of syzygies to obtain the implicit equation of λ. Geometrically, residual resultants have the effect of erasing the base locus of λ. Using resolutions of ideals of points in $\mathbb{P}^2$ and a matrix of homogeneous polynomials associated to λ, it is possible to set up a complex $\mathcal{D}$ different from $\mathcal{Z}$ from which the implicit equation of λ can be obtained. The results in Chapter 4 suggest that this approach is also plausible for tensor product surfaces with basepoints.

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