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20. methods for the determination of 3-D rigid body motion from an image sequence are described. For the pure translation case, 2 new methods are shown for determining object structure. For a general rigid motion consisting of rotation plus translation, the equations used in existing methods for objects with curved surfaces are easily derived. A result is obtained for the uniqueness of the motion parameters from the essential parameters - it conflicts with the 2 previously obtained results. An efficient method is also described. Object structure can be determined by the method for the pure translation case. For planar surfaced objects, an efficient method is shown for computing the pure parameters.

DETERMINING 3-D MOTION AND STRUCTURE OF A RIGID BODY USING THE SPHERICAL PROJECTION

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ABSTRACT

Projection on the unit sphere is proposed as a fundamental analytical operation in determining 3-D motion and structure of a rigid body from an image sequence. Points on the image plane are represented by their central projections on the unit sphere, using a homogeneous coordinate parameterization. Based on the simple geometry of corresponding points on the unit sphere, methods for the determination of 3-D rigid body motion from an image sequence are described. For the pure translation case, 2 new methods are shown for determining object structure. For a general rigid motion consisting of rotation plus translation, the equations used in existing methods for objects with curved surfaces are easily derived. Object structure can be determined by the method for the pure translation case. For planar surfaced objects, an efficient method is shown for computing the pure parameters.

I. INTRODUCTION

The determination of 3-D motion of a rigid body from an image sequence is important in many applications such as robotic vision. Any 3-D rigid body motion can be decomposed into a rotation about an axis through the origin, followed by a translation. The motion parameters to be found are the 3 rotational and 3 translational components. Also, the object structure and location relative to the camera must be determined. This can be represented by a map of relative depths of object surface points.

Consider a (central projective) image sequence of a moving rigid object from a single camera. Various image space data can be used to compute the motion parameters. For small interframe object image displacements, optical flow [1-2] or image point shifts [3] are found. In general, point correspondences (PCs) in the image sequence are found [4-11]. This approach is analyzed in this paper, where points on the image plane are represented by their central projections on the unit sphere.

3 types of motion are analyzed - pure rotation (about an axis (CO)), pure translation, and a general rigid motion (consisting of a rotation plus translation). For the case of pure rotation, 2 and 3 frame methods are shown. For the case of pure translation, 2 methods are described for determining object structure.

For the case of a general rigid motion, past methods for determining the motion parameters necessitated the solution of non-linear equations. They are solved using numerical methods. The important question of uniqueness of solution was not answered even partially until recently by Tsai/Huang [4-5]. They have shown that given 4(8) point correspondences, linear methods can be used to find a set of pure (essential) parameters for objects with planar (curved) surfaces, from which the motion parameters are computed. For the curved-surface case, the 8-point linear method has also been presented independently by Longuet-Higgins [10]. He discussed briefly the uniqueness of the 8 essential parameters, but not that of the actual motion parameters. Along the line of obtaining nonlinear equations, recently Nagel/Neumann [6-7] have derived a vector equation involving only the rotation parameters, which is a generalized version of Ullman's polar equation [11]. The method required 5 PCs over 2 frames to generate 3 equations in 3 unknowns.

The above 2 principal PC methods have the same basic origins using the model of central projection on the unit sphere (spherical projection). It is seen that PCs on the unit sphere obey a simple geometry with respect to the basic motion vectors - the rotational axis and the unit translation. Various sets of equations can be written, including those used by Tsai/Huang and Nagel/Neumann. For objects with planar surfaces, an efficient method for computing the pure parameters is also described.

II. CENTRAL PROJECTION ON THE UNIT SPHERE

The central projection on the unit sphere of a point p is the point $\hat{p}$ (fig. 1). Given $\bar{p}$, the central projection of p on the image plane (z=F), $\hat{p}$ is found from

$$\hat{p} = \frac{\bar{p}}{\|\bar{p}\|} \quad \text{where } p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \bar{p} = \begin{bmatrix} X \\ Y \\ F \end{bmatrix} \quad p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

Only $\hat{p}$ (the unit vector representing the direction of vector p where $p = \| p \| \hat{p}$) can be determined from $\bar{p}$; $\| p \|$ (the depth of point p) is lost. The image point $\bar{p}$ is represented by $\hat{p}$, the re-projection of $\bar{p}$ on the unit sphere.

There is a 1-to-1 correspondence between a point on the image plane and some point on the hemisphere $\hat{z} \geq 0$ of the unit sphere. The correspondents of finite image space points lie on the open hemisphere $\hat{z} > 0$; the correspondents of directed points of infinity on the image plane lie on the great circle $\hat{z} = 0$. Note that the geometrical extensions of the corresponding areas differ in one characteristic. The closed hemisphere on the unit sphere is finite, but the open plane on the image plane is infinite. It will be shown that the unit sphere is preferable as the surface of central projection from a theoretical aspect (in terms of geometry).

The parameterization of points on a 2-D surface (e.g. image plane and unit sphere) is a crucial step that affects the subsequent analysis. Most of the past work uses a non-homogeneous coordinate representation - XY cartesian coordinates in the plane and (r Θ ϕ) [r fixed] spherical coordinates on a sphere. The alternative parameterization is homogeneous coordinates, which will be seen to be advantageous from a computational standpoint.

$\hat{p}$ is parameterized by its 3-D xyz coordinates (not by spherical coordinates). Note that this parameterization of $\hat{p}$ is equivalently the homogeneous coordinate representation of $\overset{o}{p}$, the central projection of p on the unit image plane (z = 1).

A word on notation. For the 2 frame analysis, a point p (at time τ in frame 1) moves to the (corresponding) point p′ (at time τ' in frame 2). Associated with the 3-D PC pair (p,p′) are the PC pairs $(\bar{p},\bar{p}')$ and $(\hat{p},\hat{p}')$ on the image plane and unit sphere (respectively). Over 2 frames the term PC refers implicitly to the latter 2 types, unless otherwise noted. For the 3 frame analysis, a point p^1 (at time τ^1 in frame 1) moves to a (corresponding) point p^2 (at time τ^2 in frame 2) and to a (corresponding) point p^3 (at time τ^3 in frame 3). Associated with the 3-D PC pair (p^1,p^2,p^3) are the PC triples $(\bar{p}^1,\bar{p}^2,\bar{p}^3)$ and $(\hat{p}^1,\hat{p}^2,\hat{p}^3)$ on the image plane and unit sphere (respectively). Over 3 frames, the term PC refers implicitly to the latter 2 types, unless otherwise noted.

The spherical projection model arises naturally in optical flow based methods [1-2]. It has also been used in the analysis of the pure translation case [9]. Note that in these methods, points on a sphere are parameterized exclusively by (2-D) spherical coordinates.

III. PURE ROTATION ABOUT AN AXIS (CO) (DEGENERATE MOTION CASE 1)

It is assumed that the 3-D object motion (relative to the camera) is a rotation R by θ about an axis $\hat{n}$ (CO) (fig. 2). The 3-D point transformation p -> p′ is given by

$$p' = Rp \tag{2}$$

where

R is a 3 x 3 orthogonal matrix and det(R)=1

In 3-D, a point and its images under the rotation lie on a circle, contained in a plane perpendicular to $\hat{n}$. This circle defines a cone with longitudinal axis $\hat{n}$ (CO).

Though pure rotation about an axis (CO) rarely occurs, it is of use in special cases, e.g. where the camera pans within a static environment where the axis of rotation pierces the focal point. It is also useful in the determination of the rotational component of the displacement between 2 coordinate systems, a basic operation in computer graphics. Theoretically, the following analysis is useful later for a general motion consisting of a rotation plus translation.

A. <u>Determination of R - 2 Frame Case</u>

The following is an analysis on the image plane. The image space projections of a 3-D point and its images lie on a conic section, the type determined by the orientation of $\hat{n}$. The image space point transformation (X,Y) -> (X',Y') is given by

$$X' = F \frac{r_{11}X + r_{12}Y + r_{13}F}{r_{31}X + r_{32}Y + r_{33}F}$$

$$Y' = F \frac{r_{21}X + r_{22}Y + r_{23}F}{r_{31}X + r_{32}Y + r_{33}F}$$

where $$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{3a}$$

or

$$\rho\, \bar{p}' = R\bar{p} \tag{3b}$$

The mapping given by (3a,b) is an orthogonal collineation R, in terms of planar XY (non-homogeneous) coordinates (3a) homogeneous coordinates (3b). Immediately, there follows a method for determining R from 4 PCs over 2 frames.

<u>Theorem R1</u> (4 PC 2 Frame Method)

The rotation R is uniquely determined from 4 (image space) PCs $(\bar{p}_i, \bar{p}_i{}')$ i=1,...,4 where

no 3 of p_i i=1,...,4 are not contained in a plane (CO)

<u>Proof</u>

There is a 1-to-1 correspondence between a (non-singular) collineation mapping and a 3 x 3 parameter matrix, such as in (4) [15]. A parameter matrix determines a single mapping, and conversely, a mapping there is a unique parameter matrix. As shown in [15-16], any non-singular collineation parameter matrix can be determined to a scale factor (hence, the mapping is determined) from 4 image space PCs i.e. $(\bar{p}_i, \bar{p}_i')$ i=1,...,4 where no 3 of $\bar{p}_i$ i=1,...,4 are collinear (i.e., no 3 of p_i i=1,...,4 are coplanar). One of the (non-zero) elements of R is chosen as 1, from which R is determined to a scale factor by solving a system of 8 linear equations in 8 unknowns. An alternative, more efficient method is described later [Appendix 2]. R is obtained by normalizing rows (or columns) to unit vectors and insuring det(R)=1. Then the rotational axis $\hat{n}$ and angle of rotation θ are found [Appendix 1].

QED

The non-singularity condition is a function of the configuration of points in frame 1 - it requires that the 4 points in frame 1 do not all lie on a plane (CO). Theoretically, R can always be determined for objects occupying 3-space. It is now seen that fewer than 4 PCs are actually required to determine R.

The following is an analysis on the unit sphere. The projections on the unit sphere of a 3-D point and its images lie on a circle, contained in a plane perpendicular to $\hat{n}$ (just as in 3-D). Note that there are not several types of loci, as on the image plane. In fact, the point transformation $\hat{p} \rightarrow \hat{p}'$ (on the unit sphere) is also a rotation R

$$\hat{p}' = R\hat{p}$$

where

$$R = \begin{bmatrix} \hat{r}_1^T \\ \hat{r}_2^T \\ \hat{r}_3^T \end{bmatrix} \qquad \hat{r}_i = R^{-1}\hat{e}_i \quad i = 1,2,3 \tag{4}$$

From (A1.2),

$$\hat{p}' = \cos\theta\hat{p} + (1 - \cos\theta)(\hat{n}\cdot\hat{p})\hat{n} + \sin\theta(\hat{n} \times \hat{p}) \tag{5}$$

Immediately, there follows a method for determining R from 3 PCs over 2 frames.

<u>Theorem R2</u> (3 PC 2 Frame Method)

The rotation R is uniquely determined from 3 PCs $(\hat{p}_i, \hat{p}_i')$ i=1,2,3 where

p_i i=1,2,3 are not contained in a plane (CO)

<u>Proof</u>

3 PCs $(\hat{p}_i, \hat{p}_i')$ i=1,2,3 generates 9 linear equations in the 9 elements of R. They are decoupled into 3 sets of 3 linear equations each.

$$A\hat{r}_1 = x'$$
$$A\hat{r}_2 = y' \qquad \text{where}$$
$$A\hat{r}_3 = z'$$

$$A = \begin{bmatrix} \hat{p}_1^T \\ \hat{p}_2^T \\ \hat{p}_3^T \end{bmatrix} \quad x' = \begin{bmatrix} \hat{x}_1' \\ \hat{x}_2' \\ \hat{x}_3' \end{bmatrix} \quad y' = \begin{bmatrix} \hat{y}_1' \\ \hat{y}_2' \\ \hat{y}_3' \end{bmatrix} \quad z' = \begin{bmatrix} \hat{z}_1' \\ \hat{z}_2' \\ \hat{z}_3' \end{bmatrix} \tag{6}$$

Only 2 of the 3 rows of R - $\hat{r}_1$ $\hat{r}_2$ $\hat{r}_3$ - need be computed. The 3rd row follows from the fact that the $\hat{r}_1$, $\hat{r}_2$, $\hat{r}_3$ form a right handed orthonormal system (since an element of R is equal to its cofactor). R is uniquely determined if A is non-singular - that is $\det(A)=\hat{p}_1 \cdot (\hat{p}_2 \times \hat{p}_3) \neq 0$, i.e. $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ do not lie on a great circle or p_1, p_2, p_3 are not contained in a plane (CO)).

QED

From (6), R is determined as

$$R = \begin{bmatrix} x'^T \\ y'^T \\ z'^T \end{bmatrix} (A^{-1})^T$$

$$= \frac{1}{\hat{p}_1 \cdot (\hat{p}_2 \times \hat{p}_3)} [\hat{p}_1' \quad \hat{p}_2' \quad \hat{p}_3'] \begin{bmatrix} (\hat{p}_2 x \hat{p}_3)^T \\ (\hat{p}_3 x \hat{p}_1)^T \\ (\hat{p}_1 x \hat{p}_2)^T \end{bmatrix} \tag{7a}$$

From (7a) and Appendix 1, $(\hat{n},\theta)$ are determined from

<u>case 1</u> $\mathrm{mod}_{2\pi}\theta = 0$ (rotation is a full turn)

$\hat{n}$ = any unit vector $\qquad \theta = 0$

<u>case 2</u>

$\mathrm{mod}_{2\pi}\theta = \pi$ (rotation is a half turn) (7b)

$$\hat{n}\hat{n}^T = \frac{1}{2} \begin{bmatrix} (\hat{p}_1 \cdot \hat{e}_1)(\hat{p}_2 x \hat{p}_3) + (\hat{p}_2 \cdot \hat{e}_1)(\hat{p}_3 x \hat{p}_1) + (\hat{p}_3 \cdot \hat{e}_1)(\hat{p}_1 x \hat{p}_2) + \hat{e}_1 \\ (\hat{p}_1 \cdot \hat{e}_2)(\hat{p}_2 x \hat{p}_3) + (\hat{p}_2 \cdot \hat{e}_2)(\hat{p}_3 x \hat{p}_1) + (\hat{p}_3 \cdot \hat{e}_2)(\hat{p}_1 x \hat{p}_2) + \hat{e}_2 \\ (\hat{p}_1 \cdot \hat{e}_3)(\hat{p}_2 x \hat{p}_3) + (\hat{p}_2 \cdot \hat{e}_3)(\hat{p}_3 x \hat{p}_1) + (\hat{p}_3 \cdot \hat{e}_3)(\hat{p}_1 x \hat{p}_2) + \hat{e}_3 \end{bmatrix}$$

$$\theta = \pi$$

<u>case 3</u> $\mathrm{mod}_{2\pi}\theta \neq 0, \pi$ (rotation is not a full or half turn)

$$\sin \theta\hat{n} = \frac{1}{2} \frac{1}{\hat{p}_1 \cdot (\hat{p}_2 x \hat{p}_3)} \left[(\hat{p}_1 x \hat{p}_2) + (\hat{p}_2 x \hat{p}_3) + (\hat{p}_3 x \hat{p}_1) \right]$$

$$\cos \theta = \frac{1}{2} \frac{1}{\hat{p}_1 \cdot (\hat{p}_2 x \hat{p}_3)} [(\hat{p}_2 x \hat{p}_3) \cdot \hat{p}_1' + (\hat{p}_3 x \hat{p}_1) \cdot \hat{p}_2' + (\hat{p}_1 x \hat{p}_2) \cdot \hat{p}_3'] - 1$$

QED

The non-singularity condition is a function of the configuration of points in frame 1 - it requires that the 3 3-D points are not contained in a plane (CO). Theoretically, R can always be determined for objects occupying 3-space. In actuality <u>any</u> 2 PCs uniquely determines the rotation. The following result contains 2 proofs and methods for determining R from (any) 2 PCs over 2 frames (fig. 3).

<u>Theorem R3</u> (2 PC 2 Frame Method 1,2)

The rotation R is uniquely determined by any 2 PCs $(\hat{p}_i, \hat{p}'_i)$ i=1,2.

<u>Proof 1</u>

Since the object lies in front of the camera, a point projection $\hat{p}_i$ on the unit sphere lies on the open hemisphere $z > 0$. Thus, any 2 distinct $\hat{p}_1$, $\hat{p}_2$ are not aligned. A 3rd non-trivial PC $(\hat{p}_{12}, \hat{p}'_{12})$ can be determined (fig. 3a) from

$$\hat{p}_{12} = \hat{p}_1 \times \hat{p}_2$$

$$\hat{p}'_{12} = \hat{p}'_1 \times \hat{p}'_2 = R\hat{p}_1 \times R\hat{p}_2 = R(\hat{p}_1 \times \hat{p}_2) = R\hat{p}_{12} \tag{8}$$

$\hat{p}_{12}$ is a pole of the great circle (considered as an equator) containing $\hat{p}_1$, $\hat{p}_2$ (i.e., $\hat{p}_{12}$, $\hat{p}_1$, $\hat{p}_2$ are not contained in a plane (CO)). By Theorem R2, R is uniquely determined.

QED1

Proof 2

The following is an alternative, geometrically based proof, leading to a different method. There are 2 types of PCs on the unit sphere. If $p = \hat{n}$ or $\theta = m(2\pi)$ then $\hat{p}' = \hat{p}$, i.e., $\hat{p}$ is a fixed point under R (case A). If $\hat{p} \neq \hat{n}$, then $\hat{p}$ rotates by θ on a circle contained in a plane perpendicular to $\hat{n}$ (case B). The following expressions involving only $\hat{n}$, $(\hat{p},\hat{p}')$, $\cos\theta$ are derived from (5).

Case A (fixed PC) $\quad \hat{n}\cdot\hat{p} = \hat{n}\cdot\hat{p}' = 1 \quad \hat{n}_3 \geq 0$ (9a)

Case B (non-fixed PC) $\quad (\hat{n}\cdot\hat{p})^2 = (\hat{n}\cdot\hat{p}')^2 = \left[\frac{\hat{p}\cdot\hat{p}' - \cos\theta}{1 - \cos\theta}\right] \quad n_3 \geq 0$ (9b)

Note that given $\hat{n}$ and non-fixed PC $(\hat{p},\hat{p}')$ - $\cos\theta$ is determined. From (5), analogous expressions to (9a,b) involving only $(\hat{p},\hat{p}')$, $\hat{n}$, $\sin\theta$ are given by

Case A (fixed PC) $\quad \hat{n}\cdot(\hat{p} \times \hat{p}') = 0$

(10)

Case B (non-fixed PC) $\quad \hat{n}\cdot(\hat{p} \times \hat{p}') = \sin\theta\ (\hat{n} \times \hat{p})\cdot(\hat{n} \times \hat{p})$

Note that given $\hat{n}$ and non-fixed PC $(\hat{p},\hat{p}')$ - $\sin\theta$ is determined.

The following is a procedure for determining θ given $\hat{n}$ and a non-fixed PC $(\hat{p},\hat{p}')$. Given $\hat{n}$ and a non-fixed PC $(\hat{p},\hat{p}')$, $\cos\theta$ is found from (9b) and $\sin\theta$ is found from (10) - this determines $-\theta$. The solution sets $(\hat{n},\theta)$ and $(-\hat{n},-\theta)$ are equivalent - they represent the same physical rotation. Given $\cos\theta$, this is an alternative method for determining $\sin\theta$

$$\sin\theta = k[1 - \cos^2\theta]^{1/2}$$

(11)

$$k = \mathrm{sgn}(\sin\theta) = \mathrm{sgn}[\hat{n}\cdot(\hat{p} \times \hat{p}')]$$

This is more efficient than (10).

The 3 cases involving 2 PCs are now considered. Consider Case 1 where $(\hat{p}_1,\hat{p}_1')$ is fixed and $(\hat{p}_2,\hat{p}_2')$ is non-fixed. Of the 2 possibilities for Case A, only $\hat{p} = \hat{n}$ is possible (otherwise both PCs are fixed under the identity transformation). Thus, $\hat{n}$ is determined, and θ can be found from (9b) and (10) or (11) using non-fixed $(\hat{p}_2,\hat{p}_2')$. Note that Case 1 does not have a singular configuration.

Consider the remaining 2 cases, Case 2 and Case 3. From (9a,b)

$$\begin{bmatrix} (\hat{p}_1' - \hat{p}_1)^T \\ \\ (\hat{p}_2' - \hat{p}_2)^T \end{bmatrix} \hat{n} = 0 \qquad (12)$$

The vectors $\hat{p}_1'-\hat{p}_1$ and $\hat{p}_2'-\hat{p}_2$ (zero or non-zero) are orthogonal (trivially or non-trivially) to the vector $\hat{n}$ (fig. 3b). In the non-trivial case, $\pm\hat{n}$ can be determined as the normalized cross product of $\hat{p}_1'-\hat{p}_1$ with $\hat{p}_2'-\hat{p}_2$, provided they are not aligned.

The non-singularity condition required is now considered. The cross product is given by

$$\begin{aligned}(\hat{p}_1'-\hat{p}_1)\times(\hat{p}_2'-\hat{p}_2) &= (\hat{p}_1\times\hat{p}_2) + (\hat{p}_1'\times\hat{p}_2') - (\hat{p}_1'\times\hat{p}_2) - (\hat{p}_1\times\hat{p}_2') \\ &= (\hat{p}_1\times\hat{p}_2) + R(\hat{p}_1\times\hat{p}_2) - R\hat{p}_1\times\hat{p}_2 - \hat{p}_1 \times R\hat{p}_2 \qquad (13)\end{aligned}$$

Now, $\hat{n}\cdot[(\hat{p}_1'-\hat{p}_1) \times (\hat{p}_2'-\hat{p}_2)] = 0 \iff (\hat{p}_1-\hat{p}_1') \times (\hat{p}_2-\hat{p}_2') = 0$

$\iff \hat{p}_1'-\hat{p}_1$, $\hat{p}_2'-\hat{p}_2$ are aligned, or one of $\hat{p}_1'-\hat{p}_1$ and $\hat{p}_2'-\hat{p}_2$ is 0. From (13),

$$\hat{n}\cdot[(\hat{p}_1'-\hat{p}_1)\times(\hat{p}_2'-\hat{p}_2)] = \hat{n}\cdot(\hat{p}_1\times\hat{p}_2) + \hat{n}\cdot R(\hat{p}_1\times\hat{p}_2) - \hat{n}\cdot(R\hat{p}_1\times\hat{p}_2) - \hat{n}\cdot(\hat{p}_1\times R\hat{p}_2)$$

$$= \hat{n}\cdot(\hat{p}_1 x \hat{p}_2) + R^T\hat{n}\cdot(\hat{p}_1 x \hat{p}_2) - \hat{n}\cdot[\hat{p}_1 x R^T\hat{p}_2 + \hat{p}_1 x R\hat{p}_2]$$

$$= 2\hat{n}\cdot(\hat{p}_1 x \hat{p}_2) - (R + R^T)\hat{p}_2 - (\hat{n} x \hat{p}_1) \tag{14}$$

Consider $R + R^T$. From the decomposition of R by cos Θ, sin Θ, $\hat{n}$ in (A1.2),

$$R + R^T = (S + S^T) + (K + K^T) = 2S + 0 \text{ (S symmetric, K skew symmetric)}$$

$$= 2[\cos\Theta\, I + (1 - \cos\Theta)\, \hat{n}\, \hat{n}^T] \tag{15}$$

Thus,

$$(R + R^T)\hat{p}_2\cdot(\hat{n} x \hat{p}_1) = 2[\cos\Theta\, I + (1 - \cos\Theta)\hat{n}\hat{n}^T]\hat{p}_2\cdot(\hat{n} x \hat{p}_1)$$

$$= 2[\cos\Theta\, \hat{p}_2 + (1 - \cos\Theta)(\hat{n}\cdot\hat{p}_2)\hat{n}]\cdot(\hat{n} x \hat{p}_1)$$

$$= 2\cos\Theta\, \hat{n}\cdot(\hat{p}_1 x \hat{p}_2) \tag{16}$$

Substituting into (14) and simplifying,

$$\hat{n}\cdot[(\hat{p}_1'-\hat{p}_1) \text{ x } (\hat{p}_2'-\hat{p}_2)] = 2(1 - \cos\Theta)\hat{n}\cdot(\hat{p}_1 x \hat{p}_2) \tag{17}$$

There are 2 cases where (17) is 0. The 1st case accounts itself as Case 2, $\cos\Theta = 1$, i.e. $\Theta = m(2\pi)$ for integral m. That is, the rotation leaves the body in a fixed position, whether it be by a trivial rotation by 0 degrees (this sub-case can be ruled out) or a non-trivial full turn rotation modulo 2π. Here, all PCs are fixed and both $\hat{p}_1'-\hat{p}_1$ and $\hat{p}_2'-\hat{p}_2$ are 0. The solution is $(\hat{n},\Theta) = (*,m(2\pi))$ or R = I.

As with case 1, this case does not have a singular configuration. The 2nd case accounts itself as a singular configuration of Case 3 (general case), where both PCs are non fixed. There, $\hat{n}\cdot(\hat{p}_1 x \hat{p}_2) = 0$, i.e. $\hat{n}$, $\hat{p}_1$, $\hat{p}_2$ lie on a great circle. Here, $\hat{p}_1'-\hat{p}_1$ and $\hat{p}_2'-\hat{p}_2$ are aligned. The non-singular configuration of Case 3 gives $\hat{n}$. Then θ is determined from (9b) and (10) or (11).

Hence, of the 3 cases for 2 PCs, only 1 (Case 3) has a singular configuration where $\hat{n}$ cannot be determined. It is now shown that the singular condition can be eliminated.

The same procedure in the 1st proof can be used to determine a 3rd PC $(\hat{p}_{12}, \hat{p}'_{12})$ from the 2 PCs $(\hat{p}_i, \hat{p}'_i)$ $i = 1,2$. Then, $\hat{p}_1, \hat{p}_2, \hat{p}_{12}$ are linearly independent. Consider the singular configuration of Case 3. Suppose that $\hat{n}\cdot(\hat{p}_1 x \hat{p}_2) = \hat{n}\cdot(\hat{p}_2 x \hat{p}_3) = \hat{n}\cdot(\hat{p}_3 x \hat{p}_1) = 0$, i.e., all 3 possible PC pairs among the 3 PCs are in a singular configuration.

$$\begin{bmatrix} (\hat{p}_1 \text{ x } \hat{p}_2)^T \\ (\hat{p}_2 \text{ x } \hat{p}_3)^T \\ (\hat{p}_3 \text{ x } \hat{p}_1)^T \end{bmatrix} \quad \hat{n} = 0 \tag{18}$$

This implies that $\det = (\hat{p}_1 x \hat{p}_2)\cdot[(\hat{p}_2 x \hat{p}_3) x (\hat{p}_3 x \hat{p}_1)] = [(\hat{p}_1\cdot(\hat{p}_2 x \hat{p}_3)]^2 = 0$. This is a contradiction, since $\hat{p}_1\cdot(\hat{p}_2 x \hat{p}_3) \neq 0$ if $\hat{p}_1, \hat{p}_2, \hat{p}_3$ are independent. Thus, there is at least 1 (maybe 2, or even 3) PC pair which is in a non-singular configuration from which $\hat{n}$ can be determined from above. Then θ is determined from (9b) and (10) or (11). In summary, all 3 cases have been accounted for, in terms of determining R without having any restrictions on singular configurations.

QED2

The absence of a non-singularity condition is peculiar only to the case of pure rotation about an axis (CO). The 2nd proof was more tedious than the 1st, in that various cases had to be considered individually. However, the geometric based mechanics of the procedure for the 2nd proof were considerably more intuitive than for the 1st proof. Note that the 2nd proof arrives at the solution for $(\hat{n},\theta)$ directly, whereas the 1st proof arrives at the solution for $(\hat{n},\theta)$ indirectly via R.

Suppose R is computed by (6) from 3 PCs $(\hat{p}_i,\hat{p}_i')$ i = 1,2,3 for any rigid motion. If it is not orthogonal, then the motion is not a pure rotation. However, the converse is not necessarily true. The question is: Could there be non-pure rotational (about an axis (CO)) rigid motions which produce the 3 observed PCs $(\hat{p}_i,\hat{p}_i')$ i = 1,2,3, from which an orthogonal R was computed? Arbitrarily fix the 3-D points $p_i = k_i\hat{p}_i$ $(k_i > 0)$ i = 1,2,3. Applying the rigid body constraint that the lengths $\overline{p_1p_2}$, $\overline{p_2p_3}$, $\overline{p_3p_1}$ are invariant, the possible configurations of p_i' are given by

$$
\begin{aligned}
p_1' &= t_1\hat{p}_1' \\
p_2' &= t_2\hat{p}_2' \qquad \text{where } t_1,t_2,t_3 \text{ are solutions to} \\
p_3' &= t_3\hat{p}_3' \qquad\qquad\qquad\qquad\qquad\qquad (19) \\
\overline{p_1p_2}^2 &= t_1^2 + t_2^2 - 2t_1t_2\ \hat{p}_1\cdot\hat{p}_2 \\
\overline{p_2p_3}^2 &= t_2^2 + t_3^2 - 2t_2t_3\ \hat{p}_2\cdot\hat{p}_3 \\
\overline{p_3p_1}^2 &= t_3^2 + t_1^2 - 2t_3t_1\ \hat{p}_3\cdot\hat{p}_1
\end{aligned}
$$

It has been shown in [13] that there are at most 4 positive solutions. Thus, there are up to 3 other possible rigid motions for an arbitrary configuration of $p_i = k_i\hat{p}_i$ i = 1,2,3.

There is the question of what component of the rotation can be determined from a single PC. The answer is given immediately by the following result using a geometrical approach.

<u>Theorem R4</u> (1 PC 2 Frame Method)

The solutions for the rotation obtained from a single PC are given by

Case 1: PC is fixed

- R is a full turn modulo 2π ($\theta = m(2\pi)$)

- $\hat{n}$ is the fixed PC (θ is undetermined)

Case 2: PC is non-fixed

- $\hat{n}$ lies on a great circle with pole $\hat{p}'-\hat{p}$

(for each case θ is determined)

<u>Proof</u>

Consider Case 1, where the PC is fixed. 1st, this could be the result of a multiple of a full turn rotation. 2nd, a single homogeneous equation in $\hat{n}$ can be determined (9b). For each such $\hat{n}$, a corresponding θ is determined from (9b) and (10) or (11). Thus, the solutions for R are such that the axis $\hat{n}$ lies on a great circle (θ determined for each case), i.e. the solution vectors $\bar{n} = \tan(\theta/2)\hat{n}$ are contained in a plane (CO) with normal $\hat{p}'-\hat{p}$.

QED

This fact is useful later on in the solution of non-linear equations in the case of a general rigid motion. It is impossible to obtain a map of relative depths among object surface points - object structure cannot be determined.

B. <u>Determination of R - 3 Frame Case</u>

The success of the 2-frame analysis on the unit sphere (both theoretically and computationally) directs the 3-frame analysis onto the unit sphere. Specifically, a geometrical approach is taken.

The rotation over 3 frames is assumed to have the same axis $\hat{n}$ (CO). The 3-D interframe point transformations $p^1 \rightarrow p^2$ and $p^2 \rightarrow p^3$ are given by

$$
\begin{aligned}
p^2 &= R_{12}\, p^1 \qquad \text{axis } \hat{n}, \text{ angle } \theta_{12} \\
p^3 &= R_{23}\, p^2 \qquad \text{axis } \hat{n}, \text{ angle } \theta_{23}
\end{aligned}
\tag{20}
$$

The following result is a method for determining the interframe rotations from a single PC (fig. 4).

<u>Theorem R5</u> (1 PC 3 Frame Method)

The solutions for the interframe rotations (R_{12}, R_{23}) over 3 frames from a single PC are given by

Case 1: PC is non-fixed (R_{12}, R_{23}) are uniquely determined where

- $\mathrm{mod}_{2\pi}\theta_{12} \neq 0$, $\mathrm{mod}_{2\pi}\theta_{23} \neq 0$, $2\pi - \mathrm{mod}_{2\pi}\theta_{12}$

Case 2: PC is fixed (R_{12}, R_{23}) are not uniquely determined

- $\hat{n}$ is determined and $(\theta_{12}, \theta_{23})$ are indeterminate
- $\mathrm{mod}_{2\pi}\theta_{12} = \mathrm{mod}_{2\pi}\theta_{23} = 0$ and $\hat{n}$ is indeterminate (full turn rotation modulo 2π)

<u>Proof</u>:

Applying (9b) over all 3 interframe pairs,

$$
\begin{bmatrix}
(\hat{p}^2 - \hat{p}^1)^T \\
(\hat{p}^3 - \hat{p}^2)^T \\
(\hat{p}^1 - \hat{p}^3)
\end{bmatrix}
\hat{n} = 0
\tag{21}
$$

The 3 vectors $(\hat{p}^2-\hat{p}^1)$, $(\hat{p}^3-\hat{p}^2)$, $(\hat{p}^1-\hat{p}^3)$ (zero or non-zero) are orthogonal to $\hat{n}$ (trivially or non-trivially). There are 2 cases to consider. Consider Case 1, where all 3 PCs over 3 frames are identical. Then (21) is trivial. This means that this fixed PC is in fact $\pm\hat{n}$ (the interframe rotation angles cannot be determined) or that $\mathrm{mod}_{2\pi}\theta_{12} = \mathrm{mod}_{2\pi}\theta_{23} = 0$ (the axis $\hat{n}$ cannot be determined).

Consider Case 2, where some 2 interframe PCs are distinct. Then (21) is non-trivial. Some 2 of the 3 vectors $(\hat{p}^2-\hat{p}^1)$, $\hat{p}^3-\hat{p}^2)$, $(\hat{p}^1-\hat{p}^3)$ are non-zero and perpendicular to $\hat{n}$ (fig. 4). $\pm\hat{n}$ can be found as the normalized cross product of any 2 non-zero, non-aligned vectors. $\hat{p}^1$, $\hat{p}^2$ cannot be identical by means of a full turn multiple, otherwise $\hat{p}^2-\hat{p}^1$ is 0 and $\hat{p}^3-\hat{p}^2$ and $\hat{p}^1-\hat{p}^3$ are aligned. Similarly, $\hat{p}^3$ cannot be identical with $\hat{p}^1$ $(\mathrm{mod}_{2\pi}\theta_{23} \neq (2\pi - \mathrm{mod}_{2\pi}\theta_{12}))$ nor can it be identical with itself by means of a full turn multiple $(\mathrm{mod}_{2\pi}\theta_{23} \neq 0)$. It is geometrically obvious that in the remaining cases, all 3 of $\hat{p}^1$, $\hat{p}^2$, $\hat{p}^3$ are distinct. In fact, the 3 vectors $\hat{p}^2-\hat{p}^1$, $\hat{p}^3-\hat{p}^2$, $\hat{p}^1-\hat{p}^3$ are <u>all</u> non-zero and not aligned. Given $\pm\hat{n}$, $(\theta_{12}, \theta_{23})$ are found from (9b) and (10) or (11).

QED

There are singular cases where one or both of (R_{12},R_{23}) cannot be determined. For the degenerate case (interframe PCs fixed), the non-singularity condition is a function only of the location of p relative to $\hat{n}$. For the non-degenerate case (not all interframe PCs fixed) the non-singularity condition is a function only of the interframe angles of rotation. The following result is a method for determining the interframe rotations from (any) 2 PCs over 3 frames.

Theorem R6 (2 PC 3 Frame Method)

The interframe rotations R_{12}, R_{23} over 3 frames are uniquely determined from any 2 PCs.

Proof

The result follows from applying Theorem T3 over the 2 interframe pairs $f_1 \rightarrow f_2$ and $f_2 \rightarrow f_3$.

QED

As in the 2 frame case, a relative depth map of points cannot be determined - object structure cannot be determined.

IV. PURE TRANSLATION (DEGENERATE MOTION CASE 2)

It is assumed that the 3-D object motion (relative to the camera) over a sequence of images is a translation in a constant direction (fig. 5). The 3-D point transformation $p \rightarrow p'$ is given by

$$p' = p + t \quad \text{where } t = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \tag{22}$$

It is not uncommon to find objects (e.g. wheeled vehicles) which move by pure translation. The following analysis can be applied as methods for stereopsis.

A. Determination of $\hat{t}$ - 2 Frame Case

Given 2 image space PCs $(\bar{p}_i, \bar{p}_i')$ $i = 1,2$ where $\bar{p}_1, \bar{p}_2, \bar{t}$ are not collinear, $\bar{t}$ (FOC/FOE) can be found as the intersection of the lines $\overleftrightarrow{\bar{p}_1\bar{p}_1'}$ and $\overleftrightarrow{\bar{p}_2\bar{p}_2'}$. Then the direction $\hat{t}$ of t can be found. This is a well known result.

The point transformation $\hat{p} \rightarrow \hat{p}'$ (on the unit sphere) is given by

$$\hat{p}' = \frac{\|p\|}{\|p'\|}\hat{p} + \frac{\|t\|}{\|p'\|}\hat{t} \tag{23}$$

It follows that $\hat{t}\cdot(\hat{p}\times\hat{p}') = 0$, i.e. $\hat{t},\hat{p},\hat{p}'$ lie on a great circle (fig. 6). From (23),

$$\hat{p}_i' \times \hat{t} = \frac{\|p_i\|}{\|p_i'\|}(\hat{p}_i \times \hat{t}) \tag{24a}$$

$$\hat{p}_i' \times \hat{p}_i = \frac{\|t\|}{\|p_i'\|}(\hat{t} \times \hat{p}_i) \tag{24b}$$

From (24a,b), it follows that $\hat{p}_i'$ lies on the arc with end points $\hat{p}_i$, $\hat{t}$ (fig. 5). This is easy to see since $\hat{p}_i'$ must lie on the half of the plane containing the great circle, on the same side with t. The following result is a method for the determination of $\hat{t}$ using 2 PCs over 2 frames (fig. 6).

<u>Theorem T1</u> (2 PC 2 Frame Method)

$\hat{t}$ is uniquely determined from 2 PCs $(\hat{p}_i,\hat{p}_i')$ $i = 1,2$ where

$\hat{p}_1,\hat{p}_2,\hat{t}$ are not contained in a plane (CO).

<u>Proof</u>

From (23), $\pm\hat{t}$ can be determined from

$$(\hat{p}_1' \times \hat{p}_1) \times (\hat{p}_2' \times \hat{p}_2) = \frac{\|t\|^2}{\|p_1'\|\|p_2'\|}\hat{t}\cdot(\hat{p}_1 \times \hat{p}_2)\hat{t} \tag{25}$$

provided that $\hat{t}\cdot(\hat{p}_1\times\hat{p}_2) \neq 0$, i.e. $\hat{t}$, $\hat{p}_1$, $\hat{p}_2$ do not lie on a great circle or t, p_1, p_2 are not contained in a plane (CO).

From (24b),

$$(\hat{p}_i' \times \hat{p}_i)\cdot(\hat{t} \times \hat{p}_i) > 0 \quad \text{where } (\hat{p}_i,\ \hat{p}_i') \text{ is non-fixed PC} \qquad (26)$$

The sign of $\hat{t}$ is resolved using (26) for a non-fixed PC $(\hat{p}_i, \hat{p}_i')$.

QED

The non-singularity condition is a function of the configuration of points in frame 1 p_1, p_2 with respect to t - it requires that t, p_1, p_2 are not contained in a plane (CO). Theoretically, $\hat{t}$ can always be determined if the object occupies 3-space. The following result is a method for the determination of $\hat{t}$ using 3 PCs over 2 frames.

<u>Theorem T2</u> (3 PC 2 Frame Method)

$\hat{t}$ is uniquely determined from 3 PCs $(\hat{p}_i, \hat{p}_i')$ $i = 1,2,3$ where

p_1, p_2, p_3 are not contained in a plane (CO)

<u>Proof</u>

By the argument in Theorem R3, if p_1, p_2, p_3 are linearly independent (not coplanar) then at last 1 pair (maybe 2, even 3) is not contained in a plane (CO) with a non-zero vector t. The result follows from Theorem T1.

QED

The non-singularity condition is a function only of the configuration of the points in frame 1 (independent of t) - it requires that p_1, p_2, p_3 are not contained in a plane (CO). Theoretically, $\hat{t}$ can always be determined if the object occupies 3-space.

B. Determination of $\hat{t}$ - 3 Frame Case

It is assumed the translation over 3 frames is in the same direction $\hat{t}$. The 3-D interframe point transformations $p^1 \rightarrow p^2$, $p^2 \rightarrow p^3$ are given by

$$p^2 = p^1 + t_{12} \qquad t_{12} = \|t_{12}\|\,\hat{t}$$
$$\text{where} \qquad (27)$$
$$p^3 = p^2 + t_{23} \qquad t_{23} = \|t_{23}\|\,\hat{t}$$

On the unit sphere, the interframe point transformations $\hat{p}^1 \rightarrow \hat{p}^2$, $\hat{p}^2 \rightarrow \hat{p}^3$ are given by

$$\hat{p}^2 = \frac{\|p^1\|}{\|p^2\|}\hat{p}^1 + \frac{\|t_{12}\|}{\|p^2\|}\hat{t} \qquad (28a)$$

$$\hat{p}^3 = \frac{\|p^2\|}{\|p^3\|}\hat{p}^2 + \frac{\|t_{23}\|}{\|p^3\|}\hat{t} \qquad (28b)$$

From the 2 frame case, $\hat{t}\cdot(\hat{p}^1 x \hat{p}^2) = 0$ and $\hat{t}\cdot(\hat{p}^2 x \hat{p}^3) = 0$, i.e. $(\hat{p}^1, \hat{p}^2, \hat{p}^3, \hat{t})$ lie on a great circle (fig. 9). The following result is a general method for the determination of $\hat{t}$ using 2 PCs over 2 to 3 frames (fig. 7).

<u>Theorem T3</u> (2 PC 2 to 3 Frame Method)

$\hat{t}$ is uniquely determined from a total of 2 PCs $(\hat{p}^i_1, \hat{p}^j_1)$ $(\hat{p}^i_2, \hat{p}^k_2)$ over 2 to 3 frames where

$\hat{t}$, $\hat{p}^i_1, \hat{p}^i_2$ are not contained in a plane (CO)

<u>Proof</u>

The case where $j = k$ (2 frame case) follows from Theorem T1 (2 PC Frame method). The case where $j \neq k$ (different interframe pairs) is now considered. From the fact the $\hat{p}^1_i, \hat{p}^2_i, \hat{p}^3_i, \hat{t}$ are contained on a great circle,

$$(\hat{p}_1^i \times \hat{p}_1^j) \propto (\hat{p}_1^j \times \hat{p}_1^k) \propto (\hat{p}_1^k \times \hat{p}_1^i)$$

$$(\hat{p}_2^i \times \hat{p}_2^j) \propto (\hat{p}_2^j \times \hat{p}_2^k) \propto (\hat{p}_2^k \times \hat{p}_2^i) \tag{29}$$

Since $(\hat{p}_2^k \times \hat{p}_2^i)$ is aligned with $(\hat{p}_2^i \times \hat{p}_2^j)$, the result follows from Theorem T1.

QED

The non-singularity condition for $\hat{t}$ is a function of the configuration of points in frame 1 with respect to t - it requires that t, p_1, p_2 are not contained in a plane (CO). Theoretically, $\hat{t}$ can always be determined if the object occupies 3-space. The following result is a general method for the determination of $\hat{t}$ using 3 PCs over 2 to 3 frames.

<u>Theorem T4</u> (3 PC 2 to 3 Frame Method)

$\hat{t}$ is uniquely determined from a total of 3 PCs given 2-3 frames $(\hat{p}_1^i, \hat{p}_1^j)$ $(\hat{p}_2^i, \hat{p}_2^k)$ $(\hat{p}_3^l, \hat{p}_3^m)$ where

p_1^i, p_2^i, p_3^i are not contained in a plane (CO).

<u>Proof</u>

By the argument in Theorem R3, if p_1^i, p_2^i, p_3^i are not contained in a plane (CO) (i.e., they are linearly independent) then at least 1 vector pair (maybe 2, even 3) is not contained in a plane (CO) with a nonzero vector t. The result follows from Theorem T3.

QED

The method in Theorem T4 is applicable to the n frame case. The non-singularity condition is a function only of the configuration of points in frame 1 (independent of t) - it requires that p_1^i, p_2^i, p_3^i are not contained in a plane (CO). Theoretically, $\hat{t}$ can always be determined if the object occupies 3-space.

C. Determination of Object Structure - 2 Frame Case

The determination of object structure for the pure translation case is a well known fact. A method based on the spherical coordinate representation of points on a sphere was given in [9]. Based on a homogeneous coordinate parameterization of points on the unit sphere, 2 new alternative methods are now described.

The following result contains 2 proofs and methods for the determination of object structure over 2 frames (fig. 8).

Theorem T5 (2 Frame Object Structure Method)

Given

- $\hat{t}$
- set of PCs $(\hat{p}_i, \hat{p}_i')$ $i = 1, \ldots, n$

a map of relative depths among t p_i, p_i' $i = 1, \ldots, n$ can be determined (excluding points on the line (CO) $k\hat{t}$).

Proof 1

From (20)

$$\begin{bmatrix} 1 & \hat{p}_i \cdot \hat{t} \\ \hat{p}_i \cdot \hat{t} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{\|p_i\|}{\|p_i'\|} \\ \dfrac{\|t\|}{\|p_i'\|} \end{bmatrix} = \begin{bmatrix} \hat{p}_i' \cdot \hat{p}_i \\ \hat{p}_i' \cdot \hat{t} \end{bmatrix} \qquad (30)$$

Given $\hat{t}$ and the PC $(\hat{p}_i, \hat{p}_i')$ the ratios $(\|p_i\|/\|p_i'\|)$ (relative depth of p_i to p_i') and $(\|t\|/\|p_i'\|)$ (amount of translation t relative to the depth of p_i') can be determined from (30) if det $= 1 - (\hat{p}_i \cdot \hat{t})^2 \neq 0$, i.e., $\hat{p}_i \cdot \hat{t} \neq \pm 1$ ($\hat{p}_i$ does not lie on a line (CO) containing the vector $\hat{t}$). A solution is impossible for the singular fixed PC $(\hat{p}_i, \hat{p}_i')$, corresponding to the singular 3-D PC $(k\hat{t}, k'\hat{t})$. A map of relative depths among $\hat{p}_i'$ $i = 1,\ldots,n$ is obtained from $(\|t\|/\|p_i'\|)$ $i = 1,\ldots,n$, where points on the line (CO) $k\hat{t}$ are excluded. This immediately gives the object structure. From $(\|p_i\|/\|p_i'\|)$ $i = 1,\ldots,n$ a complete map of relative depths among t, p_i, p_i' $i = 1,\ldots,n$ are obtained (excluding points on the line (CO) $k\hat{t}$).

QED1

<u>Proof 2</u>

Under the translation, a 3-D line ℓ_i maps to a parallel line ℓ_i'. Define q_i and q_i' as the normals to the planes (CO) containing ℓ_i (NCO) and ℓ_i' (NCO) (respectively) (fig. 9). From the correspondence $(\hat{q}_i, \hat{q}_i')$, the direction $\hat{m}_i = \hat{m}_i'$ of ℓ_i and ℓ_i' are found from

$$\hat{q}_i \times \hat{q}_i' = k_i(\hat{q}_i \cdot \hat{t})\hat{m}_i \qquad k_i > 0 \tag{31}$$

provided $\hat{q}_i \cdot \hat{t} \neq 0$ (t does not lie in the plane (CO) containing ℓ_i), i.e. $(\hat{q}_i, \hat{q}_i')$ is non-fixed [12].

Consider an arbitrary set of 3 non-collinear 3-D points p_1, p_2, p_3 which are contained in a plane π (NCO). In fact, non-collinear p_1', p_2', p_3' are contained in a plane π' (NCO) (parallel to π). Define ℓ_{ij} as the line determined by p_i, p_j and ℓ_{ij}' as the line determined by p_i', p_j'. Given the 3 PCs $(\hat{p}_i, \hat{p}_i')$ $i = 1,2,3$ the sensed correspondences $(\hat{q}_{12}, \hat{q}_{12}')$, $(\hat{q}_{23}, \hat{q}_{23}')$, $(\hat{q}_{31}, \hat{q}_{31}')$ can be determined.

There are 4 possible cases for the 3 correspondences $(\hat{q}_{12},\hat{q}'_{12})$, $(\hat{q}_{23},\hat{q}'_{23})$, $(\hat{q}_{31},\hat{q}'_{31})$ in terms of being fixed or non-fixed. It is now shown that the correspondences cannot all be fixed. Assume the 3 correspondences are fixed, i.e. $\hat{q}_{12}\cdot\hat{t} = 0$, $\hat{q}_{23}\cdot\hat{t} = 0$, $\hat{q}_{31}\cdot\hat{t} = 0$. In matrix form,

$$\begin{bmatrix} \hat{q}_{12} \\ \hat{q}_{23} \\ \hat{q}_{31} \end{bmatrix} \cdot \hat{t} = 0 \tag{32}$$

Since t is non-zero, this requires that det $= \hat{q}_{12}\cdot(\hat{q}_{23}\times\hat{q}_{31}) = 0$, i.e. $\hat{q}_{12},\hat{q}_{23},\hat{q}_{31}$ are contained in a plane (CO). It is now shown that this leads to a contradiction.

Consider the non-trivial case, where all 3 $\hat{q}_{12},\hat{q}_{23},\hat{q}_{31}$ are distinct. This implies that the line segments $(\overline{p_1p_2}, \overline{p_2p_3}, \overline{p_3p_1})$ lie in concurrent planes (CO). This is clearly impossible for p_1,p_2,p_3 to be coplanar on a plane π (NCO).

Consider the trivial case, where a subset of $\hat{q}_{12},\hat{q}_{23},\hat{q}_{31}$ are aligned. Since p_1,p_2,p_3 are coplanar on π, π must contain O and all 3 vectors must be aligned (the case where only 2 vectors are aligned is excluded). Thus, (32) implies that t,p_1,p_2,p_3 are coplanar on a plane (CO), contrary to assumption. Since $\hat{t}$ is assumed to be given and obtained from $(\hat{p}_i,\hat{p}'_i)$ $i = 1,\ldots,n$, some 2 of p_i $i = 1,\ldots,n$, are not contained in a plane (CO) with t. A point exists which is not contained in π (CO). This point can be substituted for one of p_1,p_2,p_3, in which case t,p_1,p_2,p_3 are not coplanar on a plane (CO). By contradiction, all 3 correspondences are not all fixed (with the proper choice of p_1,p_2,p_3).

There are 3 remaining cases. In Case 1, all 3 correspondences are non-fixed. Here, t does not lie in any 3 of the planes (CO) containing the 3 line segments in frame 1. In Case 2, 1 correspondence is fixed and 2 correspondences are non-fixed. Here, t lies in the plane (CO) of one of the line segments in frame 1. In Case 3, 2 correspondences are fixed and 1 correspondence is non-fixed. Here, t lies in the intersection of planes (CO) containing 2 line segments in frame 1. That is, t is contained in a line (CO) containing one of p_1, p_2, p_3.

For Cases 1 and 2, some 2 of $\hat{m}_{12}, \hat{m}_{23}, \hat{m}_{31}$ can be determined, from which the common normal $\hat{e}$ of π, π' can be found (fig. 9).

$$\pi : \hat{e} \cdot p = k \qquad\qquad \pi' : \hat{e} \cdot p' = k'$$

where

$$\hat{e} = \widehat{\hat{m}_a \times \hat{m}_b} \qquad\qquad \hat{m}_a, \hat{m}_b \in \{\hat{m}_{12}, \hat{m}_{23}, \hat{m}_{31}\} \tag{33}$$

For Case 3, $\hat{e}$ cannot be determined. Eliminating k, k' in (28),

$$\frac{\| p_i \|}{\| p_j \|} = \frac{\hat{e} \cdot p_j}{\hat{e} \cdot p_i} \qquad\qquad \frac{\| p_i' \|}{\| p_j' \|} = \frac{\hat{e} \cdot p_j'}{\hat{e} \cdot p_i'} \tag{34}$$

Thus, the relative depths among p_i $i = 1,2,3$ and p_i' $i = 1,2,3$ (object structure) are determined given $\hat{e}$ (excluding points on the line (CO) $k\hat{t}$).

This procedure is applied to triples of points to cover all points p_i $i = 1,\ldots,n$ and p_i' $i = 1,\ldots,n$. The relative depths among p_i, p_i' $i = 1,\ldots,n$ are determined (excluding points on the line (CO) $k\hat{t}$). For some non-fixed $(\hat{p}_i, \hat{p}_i')$ $(\| p_i \| / \| p_i' \|)$ and $(\| t \| / \| p_i' \|)$ are solved for from (30). Thus, a complete map of relative depths among t p_i, p_i' $i = 1,\ldots,n$ are obtained (excluding points on the line (CO) $k\hat{t}$).

There are 2 points to note. 1st, the analytical methods based on the homogeneous coordinate representation are free of any unnecessary trigonometric functions inherent in the methods based on the spherical coordinate representation [9]. The manipulations are less complex, and involve only elementary vector operations. 2nd, 2 methods are given - a purely analytical method and a geometrically based method. Both are equivalent from a theoretical standpoint - the difference is purely computational.

An application is stereopsis, where the 2 cameras are shifted by a known amount. By choosing the shift vector with 0 z-component (the optical axes are parallel and image planes are contained in the same plane), the 3-D locations of <u>all</u> points in front of the camera can always be obtained.

D. <u>Determination of Object Structure - 3 Frame Case</u>

The approach is to apply the 2 methods for the 2 frame case, over all 3 interframe pairs. The following result is a method for determining object structure over 3 frames (fig. 10).

<u>Theorem T6</u> (3 Frame Object Structure Method)

Given

- $\hat{t}$

- PC sets over 3 interframe pairs

$(\hat{p}_i^1, \hat{p}_i^2)$ $\quad i = 1,\ldots,n_{12} \quad f_1 \to f_2$

$(\hat{p}_j^2, \hat{p}_j^3)$ $\quad j = 1,\ldots,n_{23} \quad f_2 \to f_3$

$(\hat{p}^3_k, \hat{p}^1_k)$ $k = 1,\ldots,n_{31}$ $f_3 \rightarrow f_1$

a map of relative depths among points in all 3 frames can be determined among

$\hat{t}$

$(\hat{p}^1_i, \hat{p}^2_i)$ $i = 1,\ldots,n_{12}$ $f_1 \rightarrow f_2$

$(\hat{p}^2_j, \hat{p}^3_j)$ $j = 1,\ldots,n_{23}$ $f_2 \rightarrow f_3$

$(\hat{p}^3_k, \hat{p}^1_k)$ $k = 1,\ldots,n_{31}$ $f_3 \rightarrow f_1$

(excluding points on the line (CO) $k\hat{t}$)

Proof

Applying Theorem T5 (2 Frame Object Structure Method) over all 3 interframe pairs $f_1 \rightarrow f_2$, $f_2 \rightarrow f_3$, $f_3 \rightarrow f_1$ determines a map of relative depths of 3 sets of points:

$\hat{t}$ $(\hat{p}^1_i, \hat{p}^2_i)$ $i = 1,\ldots,n_{12}$ $f_1 \rightarrow f_2$

$\hat{t}$ $(\hat{p}^2_j, \hat{p}^3_j)$ $j = 1,\ldots,n_{23}$ $f_2 \rightarrow f_3$

$\hat{t}$ $(\hat{p}^3_k, \hat{p}^1_k)$ $k = 1,\ldots,n_{31}$ $f_3 \rightarrow f_1$

(excluding points on line (CO) $k\hat{t}$)

Since t is common to all 3 sets, the stated result follows.

QED

Note that a map of relative depths of object surface points is obtained for each frame. A map of $n_{12} + n_{31}$, $n_{12} + n_{23}$, $n_{23} + n_{31}$ object surface points is obtained for frames f_1, f_2, f_3 (respectively). There is an analogous result to Theorem T6 for the n frame case $-\binom{n}{2}$ applications of the 2 frame object structure methods are required.

V. ROTATION PLUS TRANSLATION - (GENERAL RIGID MOTION))

It is assumed that the 3-D object motion (relative to the camera) is not a pure rotation about an axis (CO) R or a pure translation t, but a general rigid motion consisting of a rotation R followed by a translation t ($\neq 0$)(fig. 11).

$$p' = Rp + t \tag{35}$$

The 2 degenerate motions are applicable in specific (and often rare) situations. In general, the rigid motion has both a rotational and a translational component.

A. Objects with Planar Surfaces

It is assumed that the object has a planar surface - polyhedra for example. The motion of a single planar surface is isolated. Consider the corresponding planes π, π' (NCO) containing the corresponding planar surfaces.

$$\pi: e \cdot p = 1 \qquad \pi': e' \cdot p' = 1$$

where

$$e' = \frac{1}{1 + e \cdot t} Re \qquad e = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{36}$$

As demonstrated in [4], the mapping (X,Y) -> (X'Y') induced on the image plane (z = F) is a non-singular collineation A. The mapping p -> p' induced on the unit sphere is given by

$$\hat{p}' = e' \cdot \hat{p}' \left[\frac{1}{e \cdot \hat{p}} R\hat{p} + t \right] \tag{37}$$

This is equivalent to

$\rho\hat{p}' = A\hat{p}$ where

$$A = [\hat{c}_1 + at \quad \hat{c}_2 + bt \quad \hat{c}_3 + ct] \tag{38}$$

$$R = [\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3] \quad \hat{c}_i = R\hat{e}_i \quad i = 1,2,3$$

(38) is a collineation of image space ($z = 1$ or $z = F$) points as obtained in [4] (derived by an alternative procedure). It is important to note that a homogeneous coordinate representation of points is used in (38). A direct consequence is a superior method for the solution of A [Appendix 2].

This method has 2 advantages over the method given in [4]. Number 1, this method is more efficient. Only the solution to 5 sets of 3 linear equations in 3 variables is required rather than the solution of a set of 8 linear equations in 8 variables. Number 2, this method is immune to singular cases where some element of A (e.g. a_{33}) is 0. Given A, the motion parameters are uniquely determined in 2 or 3 frames [4]. Note that R may be solved in the same way by the 4 point 2 Frame Method for the pure rotation case.

B. <u>Objects with Surfaces of Arbitrary Geometry</u>

There are no constraints on the object surface. The range of applicable objects now includes those with higher order, non-planar (curved) surfaces. Note that the set of applicable objects still includes those with planar surfaces.

In 3-D, the rigid motion $p \rightarrow p'$ can be decomposed into the 2 basic degenerate motions. 1st, p transforms by a pure rotation about an axis (CO) R to $\hat{p}'' = R\hat{p}$. 2nd, $\hat{p}''$ transforms by a pure translatiion t to $p' = p'' + t$.

On the unit sphere, the mapping $\hat{p} \rightarrow \hat{p}'$ is decomposed into 2 motions. 1st, $\hat{p}$ rotates by R to $\hat{p}'' = R\hat{p}$. Geometrically, $\hat{p}$ and $\hat{p}''$ lie on a circle contained in a plane perpendicular to the vector $\hat{n}$. 2nd, $\hat{p}''$ rotates on a great circle (containing $\hat{t}$) to $\hat{p}'$. Geometrically, $\hat{p}''$ and $\hat{p}'$ lie on a great circle T containing $\hat{t}$ (fig. 12). These geometrical facts are the basis for the following derivation.

From the fact that the points $R\hat{p}_i, \hat{p}_i', \hat{t}$ lie on a great circle T_i, a set of n scalar, homogeneous equations can be written.

$$\hat{t} \cdot (\hat{p}_i' \times R\hat{p}_i) = 0 \qquad i = 1, \ldots, n \tag{39}$$

This is equivalent to

$$\hat{p}_i'^{T} E \, \hat{p}_i = 0 \qquad i = 1, \ldots, n \tag{40}$$

where

$$E = [\hat{t}\times\hat{c}_1 \quad \hat{t}\times\hat{c}_2 \quad \hat{t}\times\hat{c}_3]$$

$$R = [\hat{c}_1 \quad \hat{c}_2 \quad \hat{c}_3] \qquad \hat{c}_i = R\hat{e}_i \quad i = 1,2,3$$

The analytical solution and conditions for uniqueness to (40) have been obtained [5]. 8 PCs are required to determine E to a scale. The SVD of E is the basis for the determination of the motion parameters. Alternatively, a method using vector operations has also been shown [10].

The great circles T_i $i = 1,\ldots,n$ have a common diameter $\pm\hat{t}$, i.e. T_i $i = 1,\ldots,n$ are contained in concurrent planes (CO) (fig. 12). Define g_i as the vector which is aligned with one of the poles of T_i

$$g_i = R\hat{p}_i \times \hat{p}_i' \tag{41}$$

Since g_i $i = 1,\ldots,n$ are contained in a plane (CO), a set of $n - 2$ can be written.

$$g_i \cdot (g_1 \times g_2) = 0 \qquad i = 3, \ldots, n \tag{42}$$

This is equivalent to

$$[R^{-1}\hat{p}_i' \cdot (\hat{p}_1 x \hat{p}_i)][R\hat{p}_2 \cdot (\hat{p}_1' x \hat{p}_2')] - [R\hat{p}_i \cdot (\hat{p}_1' x \hat{p}_i')][R^{-1}\hat{p}_2' \cdot (\hat{p}_1 x \hat{p}_2)] = 0$$

$$i = 3, \ldots, n \tag{43}$$

This is equivalent to the equation set derived in [6-7], which is a generalized form of the polar equation in [11]. 5 PCs generate 3 4th order equations in the 3 rotational parameters.

Whereas the analytical solution and conditions for uniqueness to (40) have been obtained, a similar theoretical analysis for the solution to (43) remains to be done. This problem is currently under investigation. Numerical solutions by local, iterative search indicate the possibility of a unique solution using as few as 5 points. But it may be that the conditions for uniqueness are complex.

The point to note is the fact that the equation sets of the 2 principal methods are easily derived from a basic geometrical fact on the unit sphere. The spherical projection model was not used in the original derivations of these equation sets. The derivation of (40) in [5] was made by elimination of "z" in the image space mapping (X,Y) -> (X′,Y′). The derivation of (43) in [6-7] was made using a lengthy, complex sequence of vector operations. In each case, fundamental origins of the equation sets were not isolated and their derivations were not intuitively obvious. As a result the 2 methods appeared to be unrelated - they in fact have the same origins.

C. Determination of Object Structure

The approach is to apply the 2 Frame Object Structure Method for the pure translation case. Given the solution to R, $\hat{p}_i$ $i = 1,\ldots,n$ are mapped by R to $\hat{p}''_i = R\hat{p}_i$ $i = 1,\ldots,n$. From $(\hat{p}''_i,\hat{p}'_i)$ $i = 1,\ldots,n$ a map of relative depths among t (p_i,p'_i) $i = 1,\ldots,n$ (excluding points on the line (CO) $k\hat{t}$) is obtained. Given the depth of a single point or $\|t\|$, the 3-D object points can be found.

An application is stereopsis, where the translational component describing the relative configurations of the 2 cameras are known. If the translation component vector is chosen with 0 z-component, then 3-D locations of <u>all</u> points in front of the camera can always be obtained.

VI. SUMMARY

For the degenerate motion case of pure rotation about an axis (CO) R, R can be determined but object structure cannot. Over 2 frames, methods using 4, 3, 2, 1 PC(s) were shown. 2 procedures (purely analytical and geometrically based) were given for the 2 PC method. The 3,4 PC methods can be used to detect non-pure rotational motion. Over 3 frames, (R_{12}, R_{23}) are uniquely determined from a total of 2 PCs over different interframe pairs.

For the degenerate motion case of pure translation t, both $\hat{t}$ and object structure can be determined. Over 2 frames, 2 methods (based on the homogeneous coordinate representation of points) are shown for determining object structure. An application is stereopsis, where the 2 cameras are shifted by a known amount. Over 3 frames, object structure can be determined by applying the 2 frame methods.

For the general rigid motion case involving a rotation R followed by a translation t, $(R,\hat{t})$ and object structure can be uniquely determined. For the case of objects with planar surfaces, an efficient method (based on the homogeneous coordinate representation of points) for the determination of the collineation matrix A is shown. For the case of objects with surfaces of arbitrary geometry, a simple geometric fact on the unit sphere is shown to be the basic origin of the 2 principal PC methods.

Overall, the use of the unit sphere as the surface of central projection and homogeneous coordinate representation of points is advantageous, both theoretically and computationally. A fundamental theory based on the simple geometry of PCs on the unit sphere is shown for the 2 degenerate motion cases and a general rigid motion. The resultant methods are efficient, more than the previously developed methods.

APPENDIX 1

Determination of $(\hat{n},\theta)$

Note that a clw (cclw) rotation θ about $\hat{n}$ is equivalently a clw (cclw) rotation $-\theta$ about $-\hat{n}$, where $\theta \in [-\pi,\pi)$. The sign of the angle θ (clw/cclw) of rotational axis $\hat{n}$ can be arbitrarily chosen.

The approach is to use the invariant quantities of R. The trace of R is a function only of cos θ

$$Tr(R) = 1 + 2\cos\theta \tag{A1.1}$$

Thus, cos θ is determined by the diagonal elements of R in terms of its trace (scalar invariant).

The decomposition of R as the sum of a symmetric and skew symmetric matrix in terms of (n, θ) is given by

$$R = S + K \tag{A1.2}$$

where

$$S = (s_{ij}) = \cos\theta\, I + (1 - \cos\theta)\hat{n}\hat{n}^T \qquad \text{(symmetric)}$$

$$K = (k_{ij}) = \sin\theta\, N \qquad \text{(skew symmetric)}$$

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

From (A1.2),

$$r_{ij} - r_{ji} = (s_{ij} + k_{ij}) - (s_{ji} + k_{ji})$$

$$= (s_{ij} - s_{ji}) + (k_{ij} - k_{ji}) \qquad \text{(A1.3)}$$
$$= k_{ij} - k_{ji}$$

Explicitly,

$$c = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = 2 \sin \theta \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad \text{(A1.4)}$$

Thus, $\sin \theta\hat{n}$ is determined by the off diagonal elements of R, in terms of c (vector invariant of R). $\hat{n}$ can only be determined from (A1.4) where $\sin \theta \neq 0$, i.e. when $\theta \neq 0$ and $\theta \neq -\pi$ (Case 3). There are singular cases where $\theta = 0$ (Case 1) and $\theta = -\pi$ (Case 2).

Consider Case 1 (rotation is a full turn) where $\theta = 0$. This is detected from (A1.1) where $\cos \theta = 1$ (i.e. $R = I$). $\hat{n}$ can be any unit vector.

Consider Case 2 (rotation is a half turn) where $\theta = 0$. This is detected from (A1.1) where $\cos \theta = -1$. From (A1.2), $\hat{n}\hat{n}^T$ can be found from symmetric R.

$$\hat{n}\hat{n}^T = [n_1\hat{n} \quad n_2\hat{n} \quad n_3\hat{n}] = \frac{1}{2} (R + I)$$

Since $\hat{n} \neq 0$, at least 1 of the columns (maybe 2, even 3) of $\hat{n}\hat{n}^T$ are non-zero. Thus, $\pm\hat{n}$ can be determined by normalizing a non-zero column of $\hat{n}\hat{n}^T$. Note that there is not a problem for the ambiguity in the direction $\hat{n}$, since $(\hat{n}, \pi)$ and $(-\hat{n}, \pi)$ represent the same physical rotation.

Consider Case 3 (general case). (A1.1) gives $\cos \theta$. Normalizing c to $\pm\hat{c}$ fixes $\sin \theta$ and its sign. Let $\hat{n}_+ = \hat{c}$ and $\sin \theta_+ > 0$ satisfy (A1.4). Clearly, $\hat{n}_- = -\hat{c}$ and $\sin \theta_- < 0$ is also a solution. There are 2 solution cases.

<u>case 1</u> $\hat{n}_+ = \hat{c} \qquad \sin \theta_+ \ (>0) \qquad \cos \theta_+ \qquad \theta_+ > 0$

(A1.5)

<u>case 2</u> $\hat{n}_- = -\hat{c}$ $\quad \sin\theta_-$ (<0) $\quad \cos\theta_-$ $\quad \theta_- < 0$

$$\theta_+, \theta_- \in [-\pi, \pi)$$

where

$$\begin{aligned} \sin\theta_+ &= \sin\theta_- \\ \cos\theta_+ &= \cos\theta_- \end{aligned} \quad \Rightarrow \quad \theta_+ = -\theta_-$$

It is clear that the angles of rotation θ_+, θ_- for cases 1, 2 (respectively) differ only by a sign, i.e., $\theta_+ = -\theta_-$. That is, the 2 solutions in (A1.5) are equivalent - they represent the same physical rotation.

APPENDIX 2

Determination of a Non-Singular Collineation Matrix A from 4 PCs

The xyz coordinates of $\hat{p}$ (on the unit sphere) are in fact the homogeneous coordinates of $\bar{p}$ and $\mathring{p}$ (the central projection of p on the unit image plane z = 1). Without loss of generality, the following manipulations will be confined on the unit image plane.

Consider the solution of A given 4 PCs $(\hat{p}_i, \hat{p}_i')$ i = 1,...,4. Given the 4 quadrangular points $\mathring{p}_1$, $\mathring{p}_2$, $\mathring{p}_3$, $\mathring{p}_4$ (no 3 of which are collinear), they are assigned the fundamental points with relative coordinates

$$\mathring{p}_{1r} = [1 \quad 0 \quad 0]^T$$
$$\mathring{p}_{2r} = [0 \quad 1 \quad 0]^T \qquad\qquad \mathring{p}_{4r} = [1 \quad 1 \quad 1]^T$$
$$\mathring{p}_{3r} = [0 \quad 0 \quad 1]^T$$

triangle of reference points — unit point

With respect to this basis, the relative coordinates of $\mathring{p}_1'$, $\mathring{p}_2'$, $\mathring{p}_3'$, $\mathring{p}_4'$ are found [18]. The absolute coordinates of $\mathring{p}_1, \mathring{p}_2, \mathring{p}_3$ fixed by $\mathring{p}_4$ are found from solving for s_1, s_2, s_3 in

$$\mathring{p}_4 = [\mathring{p}_1 \quad \mathring{p}_2 \quad \mathring{p}_3] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{A2.1}$$

Then the absolute coordinates $\mathring{p}_{1f}, \mathring{p}_{2f}, \mathring{p}_{3f}, \mathring{p}_{4f}$ of $\mathring{p}_1, \mathring{p}_2, \mathring{p}_3, \mathring{p}_4$ (respectively) are given by

$$\overset{o}{p}_{1f} = s_1\overset{o}{p}_1$$

$$\overset{o}{p}_{2f} = s_2\overset{o}{p}_2 \qquad \overset{o}{p}_{4f} = \overset{o}{p}_4 \tag{A2.2}$$

$$\overset{o}{p}_{3f} = s_3\overset{o}{p}_3$$

The coordinates of $\overset{o}{p}'_1, \overset{o}{p}'_2, \overset{o}{p}'_3, \overset{o}{p}'_4$ relative to the established basis are found from solving d_i, e_i, f_i $i = 1,\ldots,4$ in

$$\overset{o}{p}'_i = [\overset{o}{p}_{1f} \quad \overset{o}{p}_{2f} \quad \overset{o}{p}_{3f}] \begin{bmatrix} d_i \\ e_i \\ f_i \end{bmatrix} \qquad i = 1,\ldots,4 \tag{A2.3}$$

The relative coordinates $\overset{o}{p}'_{1r}, \overset{o}{p}'_{2r}, \overset{o}{p}'_{3r}, \overset{o}{p}'_{4r}$ of $\overset{o}{p}'_1, \overset{o}{p}'_2, \overset{o}{p}'_3, \overset{o}{p}'_4$ are given by

$$\overset{o}{p}'_{ir} = [d_i \quad e_i \quad f_i]^T \qquad i = 1,\ldots,4 \tag{A2.4}$$

Nonsingular solutions to (A2.1), (A2.3) are obtained, since $\overset{o}{p}_1, \overset{o}{p}_2, \overset{o}{p}_3$ are non-collinear. Then A is computed to a scale factor by conventional methods [14], from solving for t_1, t_2, t_3 in

$$\overset{o}{p}_{4r} = [\overset{o}{p}_{1r} \quad \overset{o}{p}_{2r} \quad \overset{o}{p}_{3r}] \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \tag{A2.5}$$

The solution to (A2.5) is non-singular, since $\overset{o}{p}'_1, \overset{o}{p}'_2, \overset{o}{p}'_3$ are also non-collinear. Then A is given by

$$A = [t_1\overset{o}{p}'_{1r} \quad t_2\overset{o}{p}'_{2r} \quad t_3\overset{o}{p}'_{3r}] \tag{A2.6}$$

There are 2 points to note. 1st, in terms of operations, only the solutions to 5 sets of 3 linear equations in 3 variables are required. 2nd, there are no singularities when an element of A is zero. These points make this method superior to the one given in [4].

This method is applicable to a general rigid motion of a single planar surface and pure rotation about an axis (CO) of a surface with arbitrary geometry.

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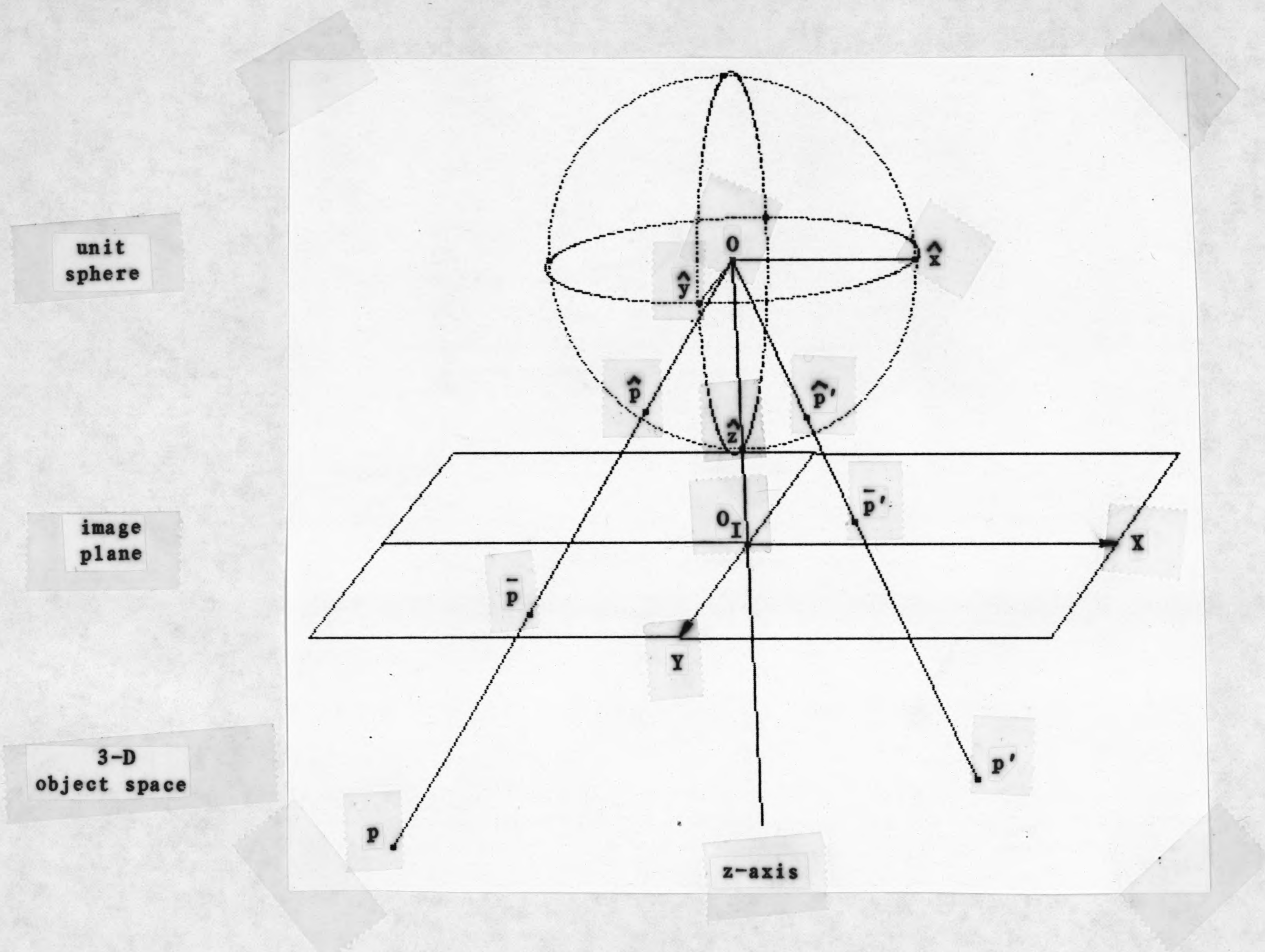

fig. 1 central projection of 3-D PC (p,p') as $(\bar{p},\bar{p}')$ (on image plane) and $(\hat{p},\hat{p}')$ (on unit sphere)

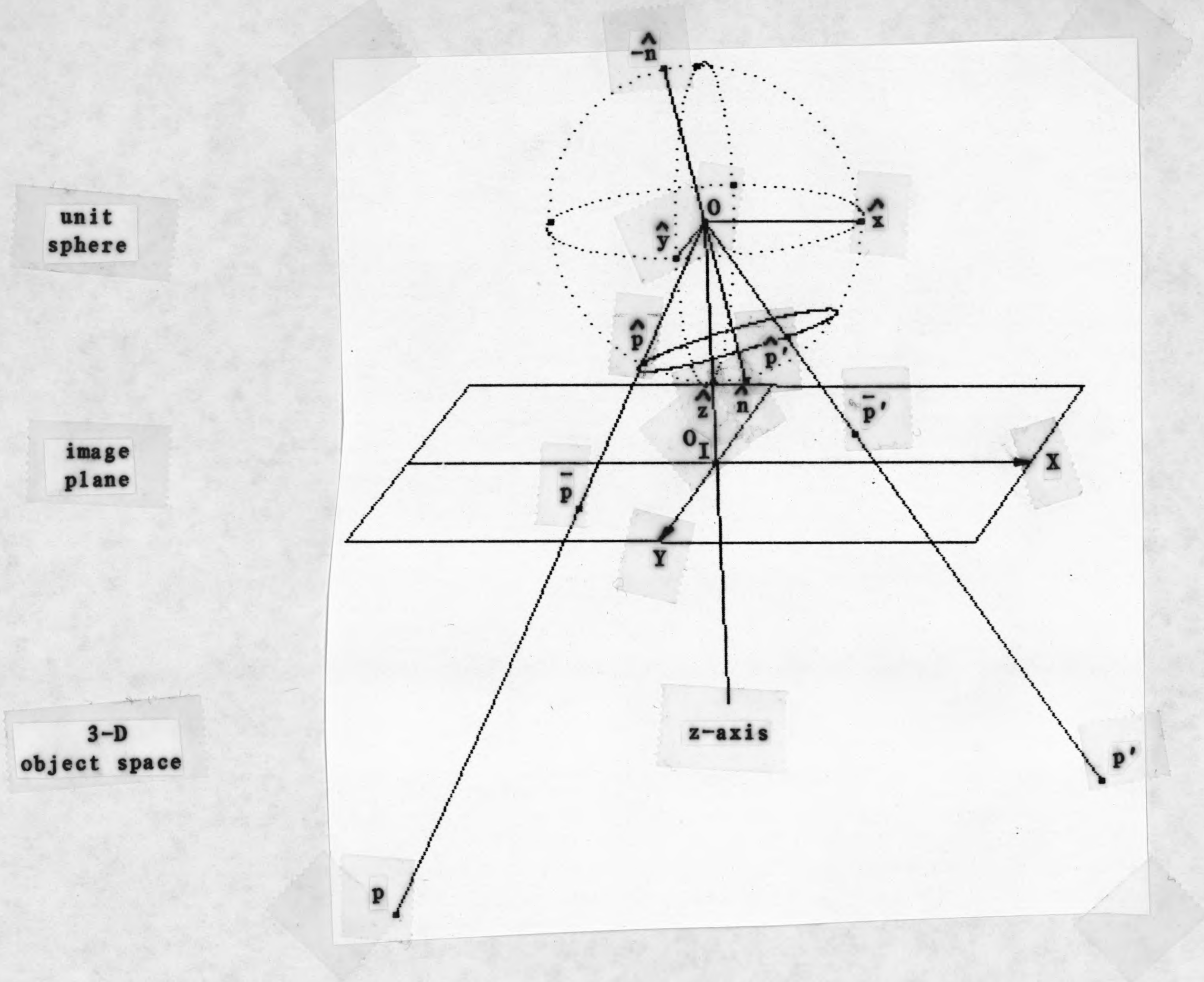

fig. 2 (3-D rotation R over 2 frames) 3-D: $p \rightarrow p'$ image plane: $\bar{p} \rightarrow \bar{p}'$ unit sphere: $\hat{p} \rightarrow \hat{p}'$

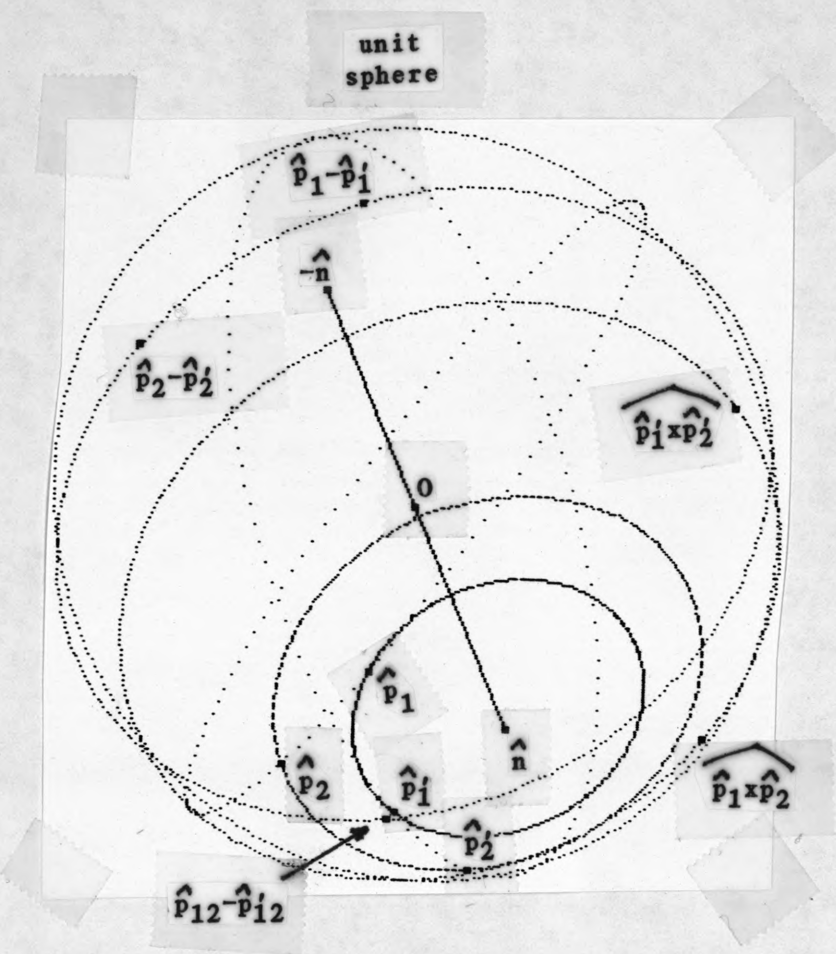

fig. 3 (3-D rotation R over 2 frames)

determination of R from 2 PCs - Methods 1 and 2

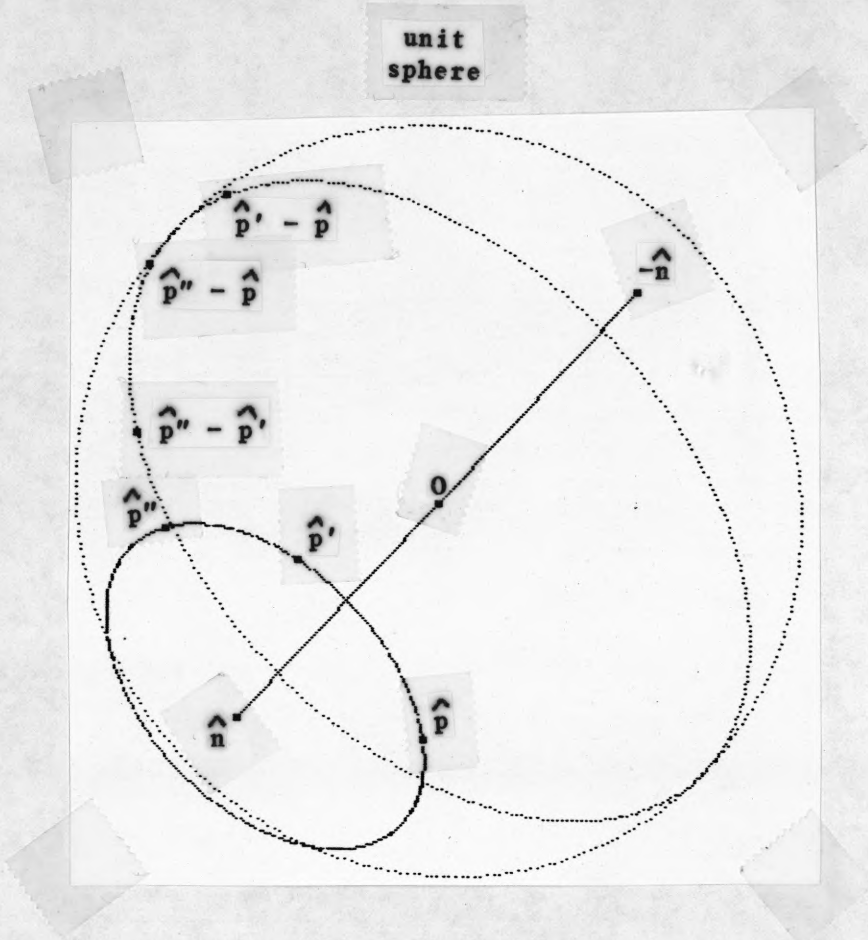

fig. 4 (3-D rotations R_{12}, R_{23} over 3 frames)

determination of R_{12}, R_{23} from correspondences of 1 point

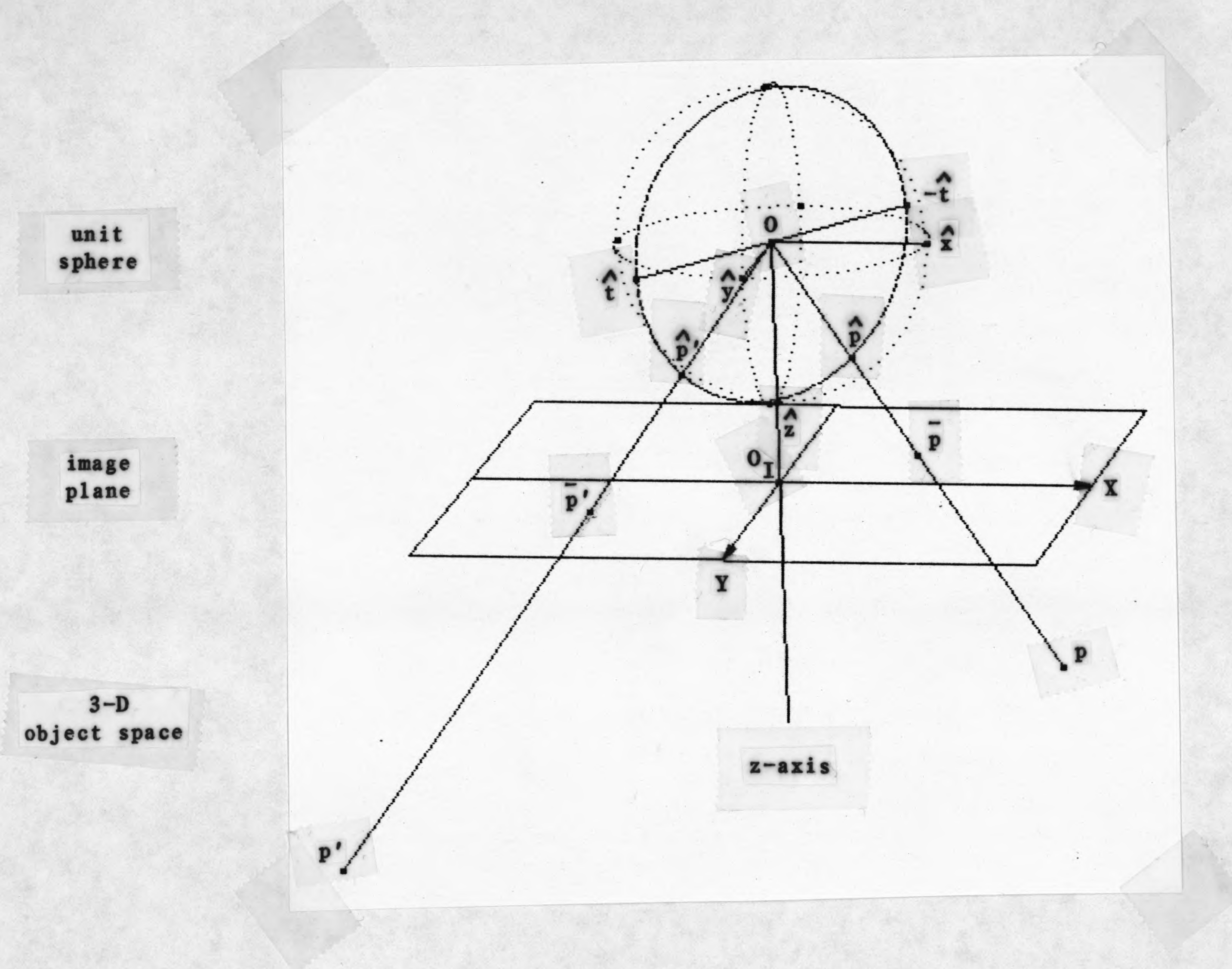

fig. 5 (3-D translation t over 2 frames) 3-D: p -> p' image plane: $\bar{p}$ -> $\bar{p}'$ unit sphere: $\hat{p}$ -> $\hat{p}'$

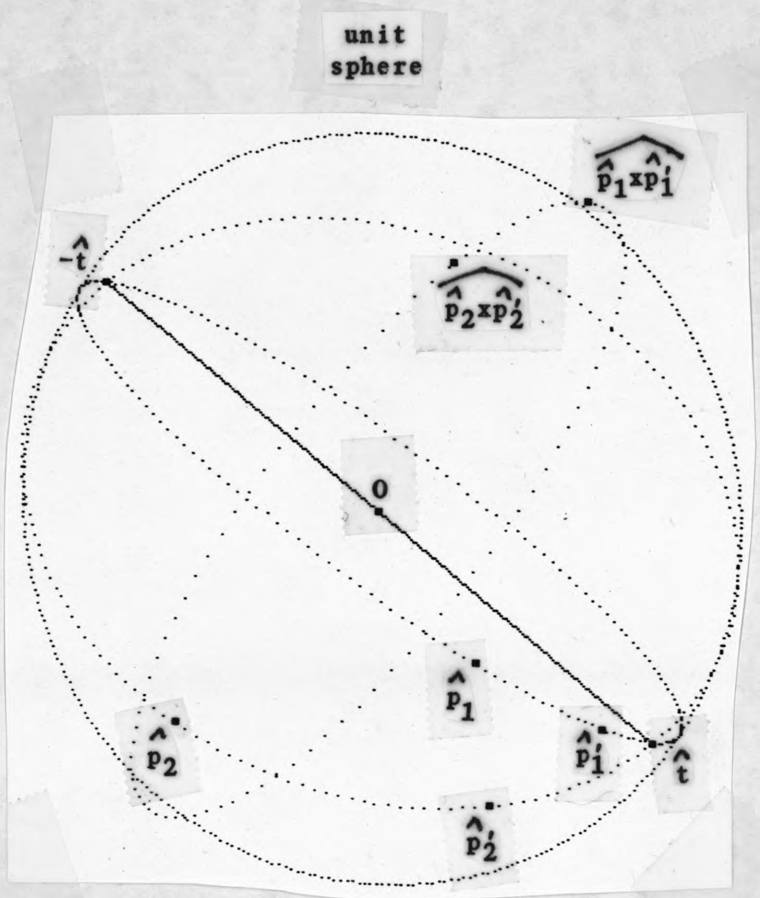

fig. 6 (3-D translation t over 2 frames)

determination of $\hat{t}$ from 2 PCs

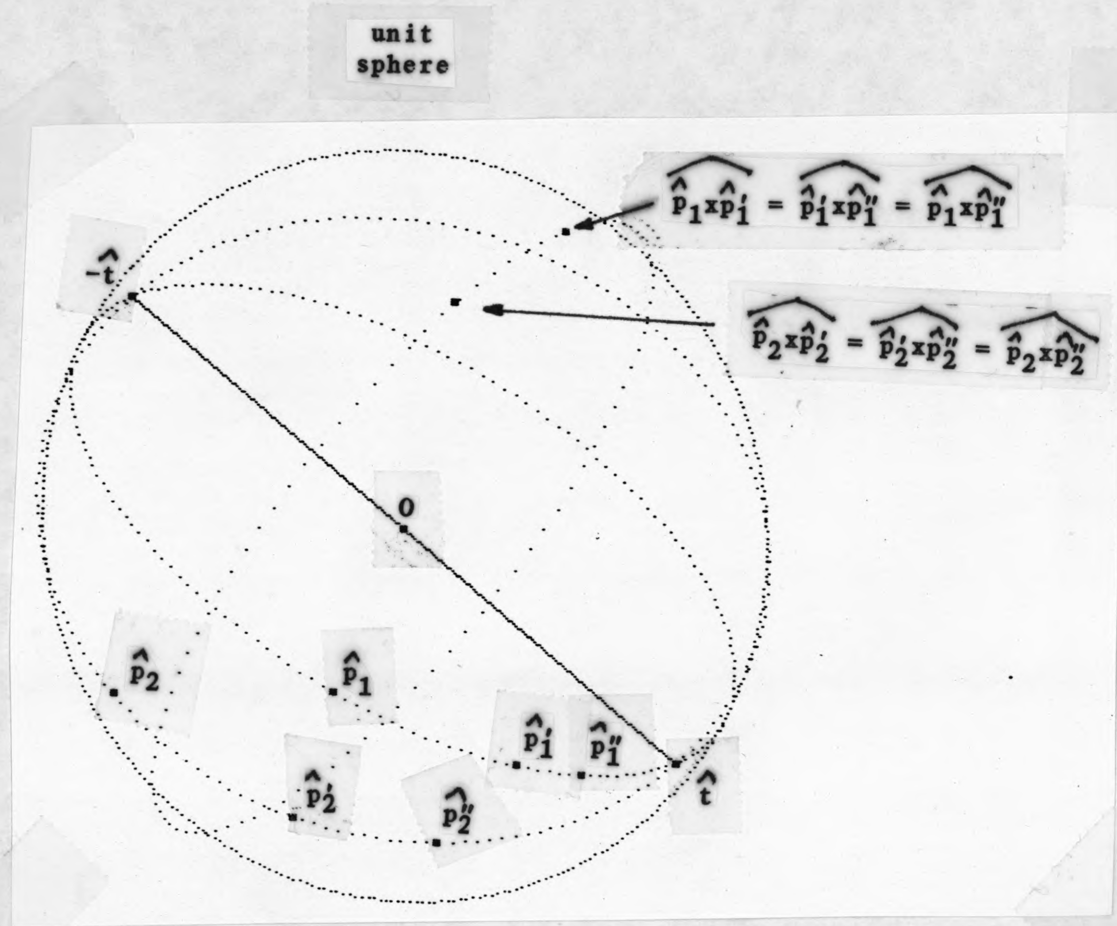

fig. 7 (3-D translations t_{12}, t_{23} over 3 frames)

determination of $\hat{t}$ from 2 interframe PCs

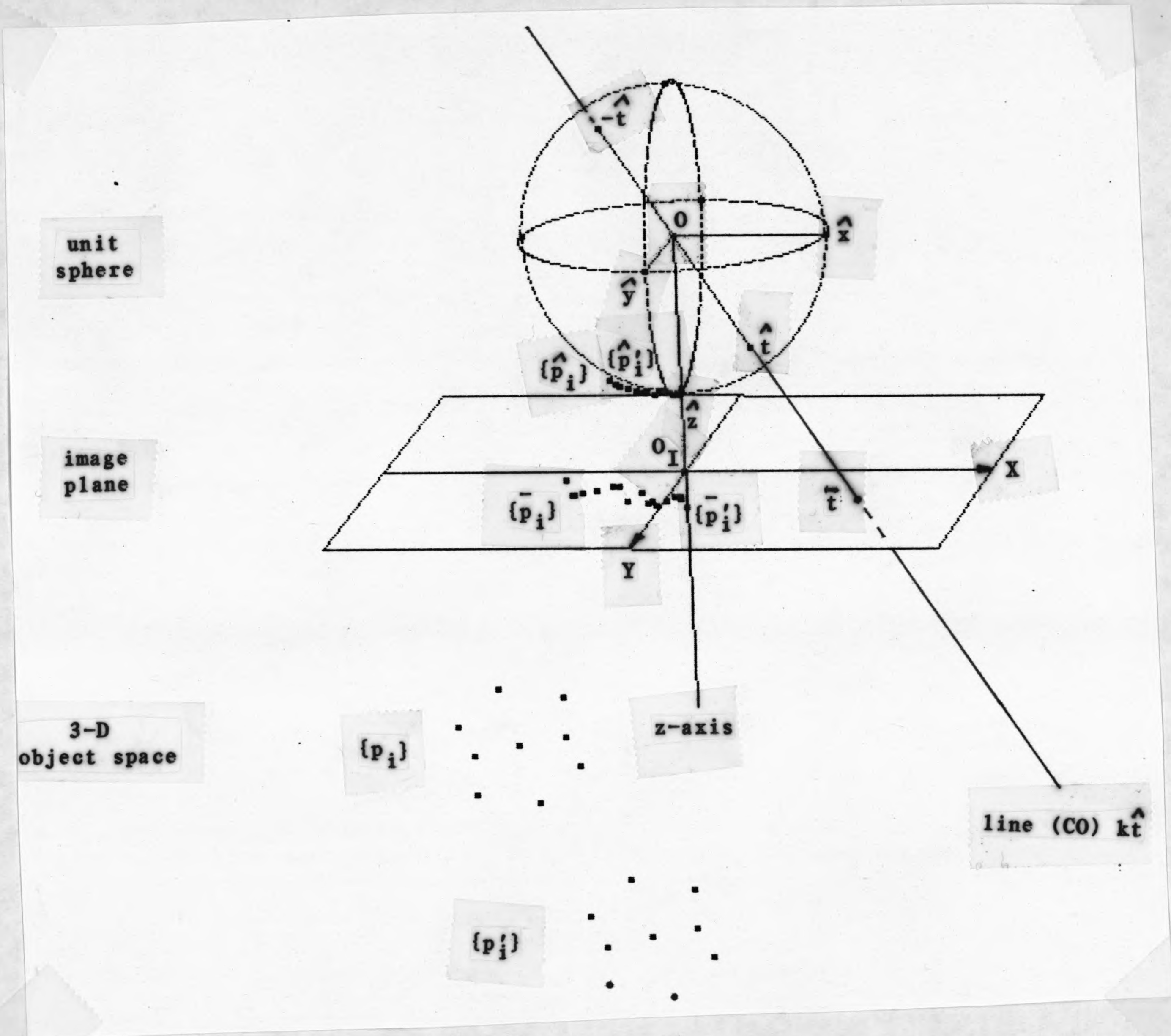

fig. 8 (3-D translation t over 2 frames) determination of object structure

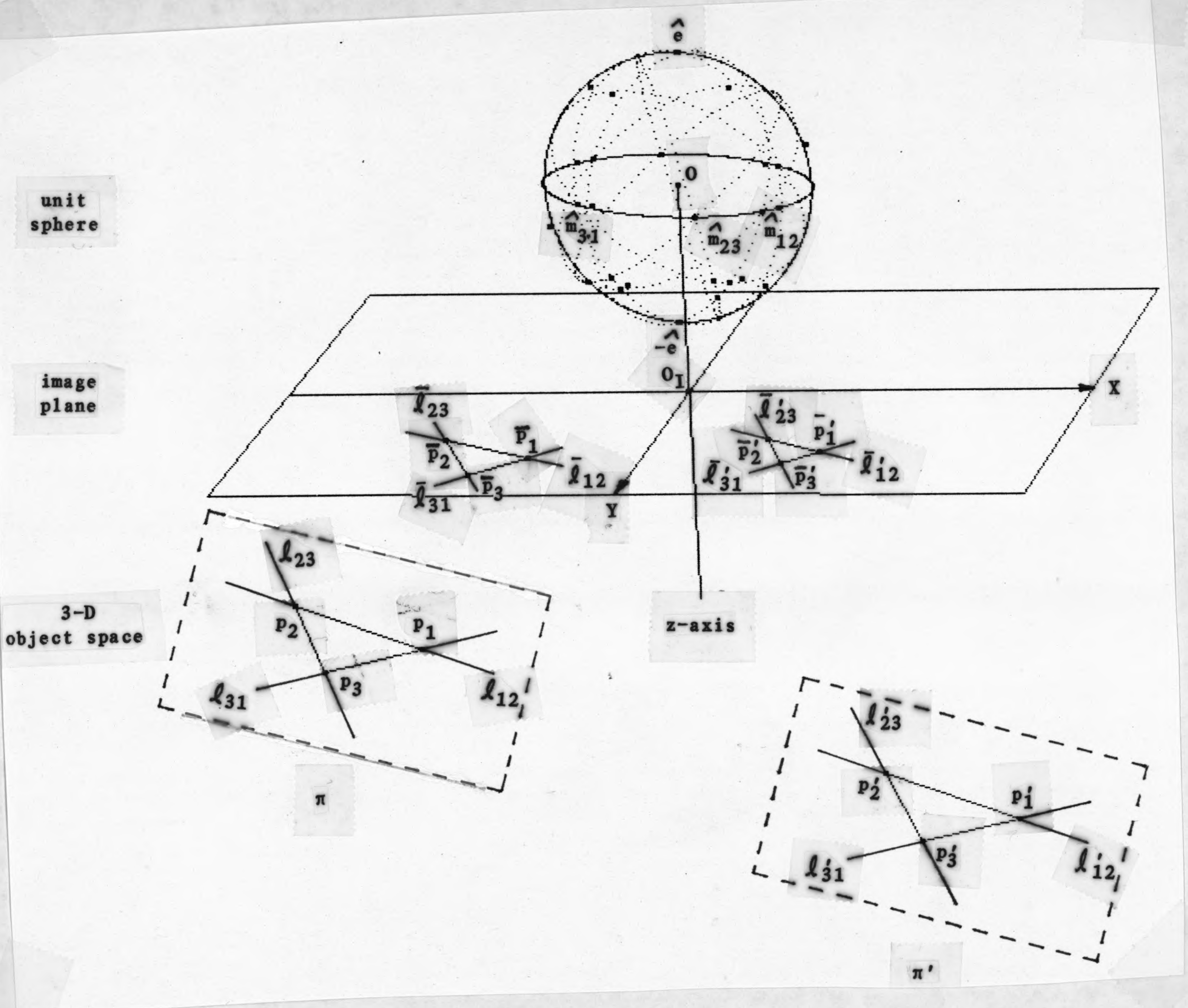

fig. 9 (3-D translation t over 2 frames) determination of relative depths for 3-point sets (p_1, p_2, p_3), (p'_1, p'_2, p'_3)

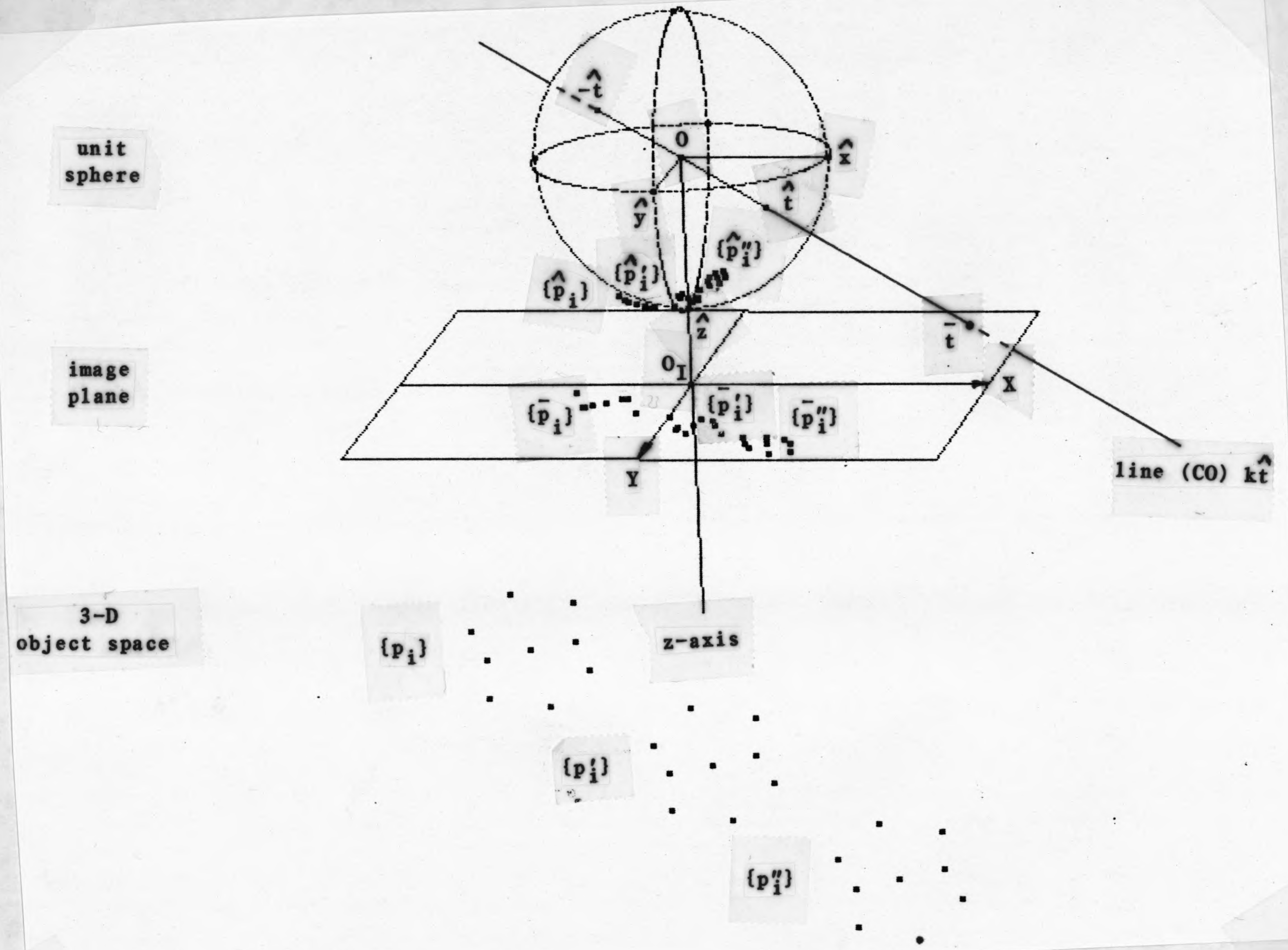

fig. 10 (3-D translations t_{12}, t_{23} over 3 frames) determination of object structure

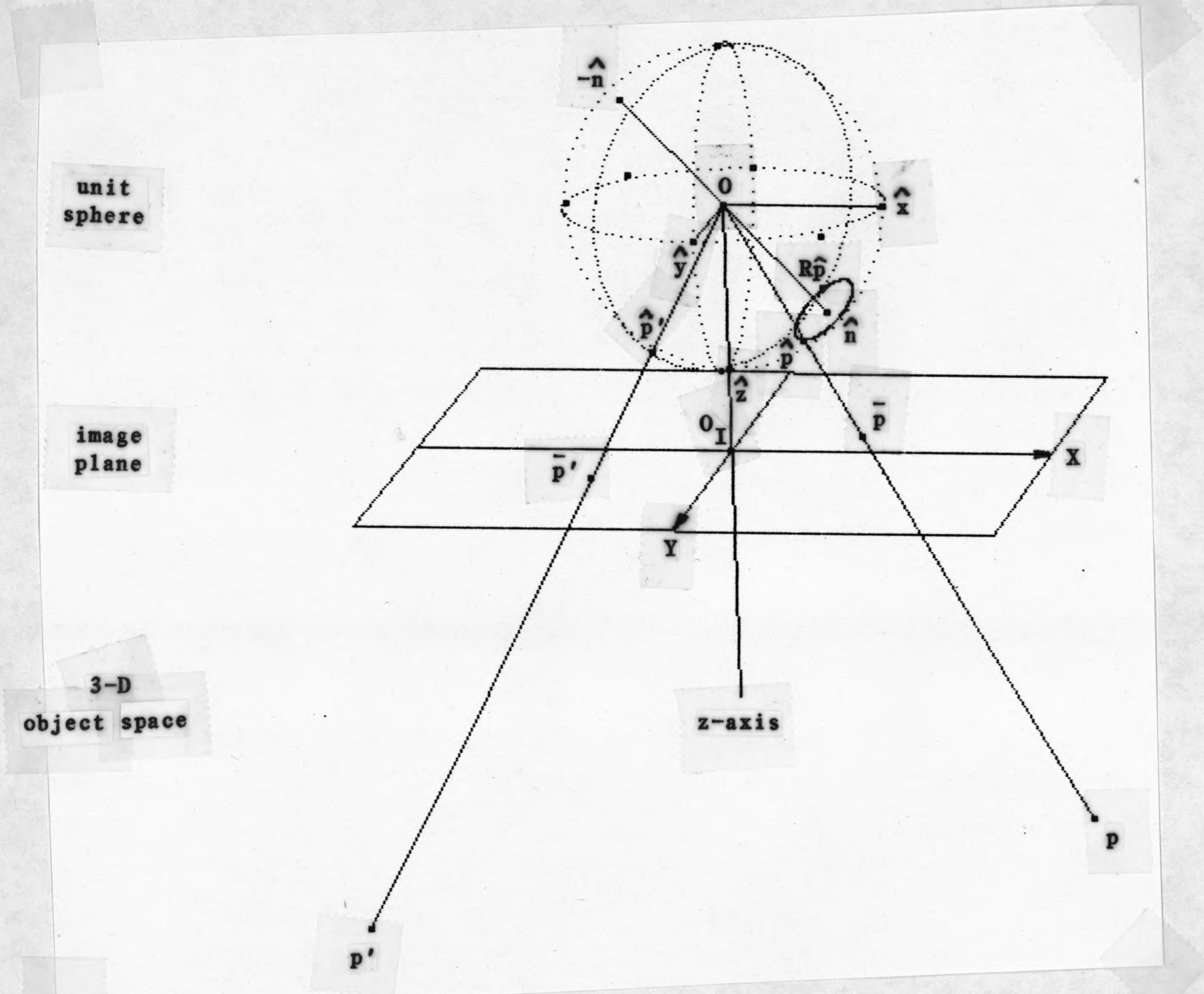

fig.11 (rotation R plus translation t over 2 frames) 3-D: p -> p' image plane: $\bar{p} \rightarrow \bar{p}'$ unit sphere: $\hat{p} \rightarrow \hat{p}'$

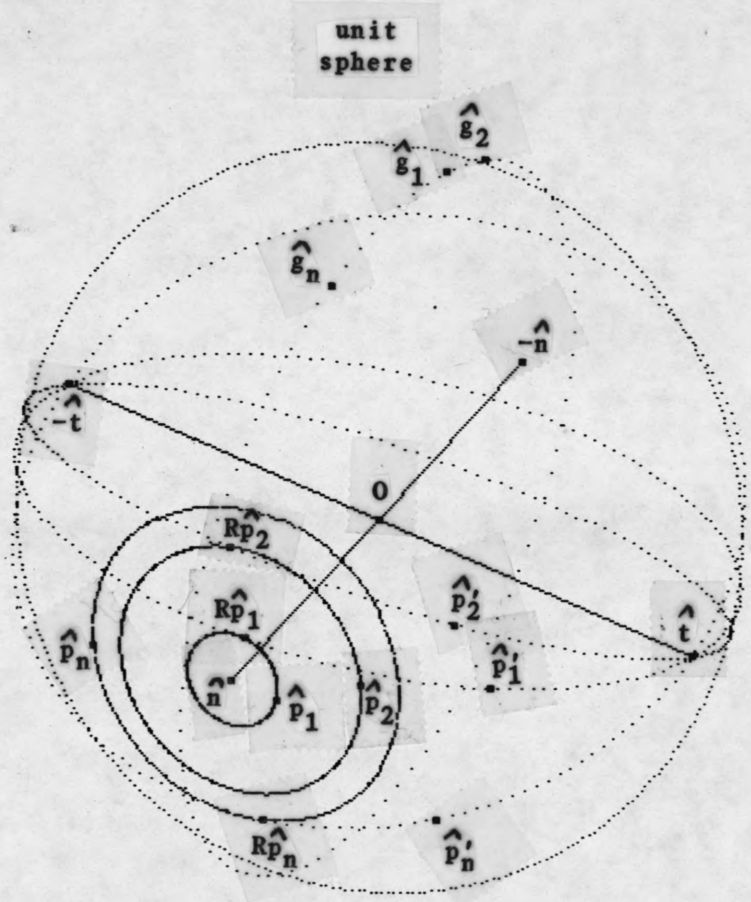

fig.12 (rotation R plus translation t over 2 frames) geometry of PCs $(\hat{p}_i, \hat{p}'_i)$ i = 1,...,n wrt $\hat{n}$ and $\hat{t}$