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ANALYZING VARIABLE CANCELLATIONS TO GENERALIZE SYMBOLIC MATHEMATICAL CALCULATIONS

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ANALYZING VARIABLE CANCELLATIONS TO GENERALIZE SYMBOLIC MATHEMATICAL CALCULATIONS *

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ABSTRACT

This is an extended version of a paper appearing in the *Proceedings of the Third Annual IEEE Conference on Artificial Intelligence Applications*, Orlando, FL, February 1987.

Mathematical reasoning provides the basis for problem solving and learning in many complex domains. A model for mathematical reasoning in support of explanation-based learning is presented, and an implemented learning system in the domain of classical physics is described. The system's mathematical reasoning processes are guided by the manner in which variables are cancelled in specific problem solutions. Attention focusses on how *obstacles* are eliminated from calculations. Obstacles are variables that preclude the direct evaluation of the problem's unknown. Analyzing the cancellation of obstacles leads to the generalization of the specific solution. An illustrative example highlights an important issue in explanation-based learning, namely generalizing number. It is argued that such generalization requires extension of the sample solution's explanation. This type of generalization cannot be performed by the standard explanation-based approach of propagating constraints. An approach that overcomes this shortcoming is presented.

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ANALYZING VARIABLE CANCELLATIONS TO GENERALIZE SYMBOLIC MATHEMATICAL CALCULATIONS

1. INTRODUCTION

We are applying the paradigm of *explanation-based learning* [1-3] to the generalization of symbolic mathematical calculations. The mathematical reasoning component of an implemented machine learning system is presented. A significant feature of this system is that it can verify and augment human-provided solutions to specific problems. Once understood, the technique used by the human is generalized, and the result is added to the system's knowledge base. This new knowledge is then available to help solve future problems and as a stepping stone toward acquiring more difficult concepts.

In explanation-based learning, a specific problem solution is generalized into a form that can be later used to solve conceptually similar problems. The generalization process is driven by the *explanation* of why the solution worked. Deep knowledge about the domain allows the explanation to be developed and then extended.

Our system is capable of performing many of the mathematical manipulations expected of a college freshman who has encountered the calculus. Mathematical reasoning provides the basis for problem solving and learning in many complex domains. We concentrate on the manner in which variables are cancelled in specific calculations. Reasoning about these cancellations leads to a general version of the specific solution. Currently we are focussing on generalizing solutions to college-level physics problems: hence the name of the complete system, **Physics 101**. (The overall operation of this system is described in [4].) The domain of classical physics provides realistic and complex problems with which our understanding and generalization algorithms can be tested. A physics problem concerning conservation of momentum is used as an illustrative example in this paper. Several other problems, including one involving conservation of energy, have also been solved using the techniques described here.

The sample momentum problem addresses the important issue in explanation-based learning of generalizing number. Generalizing number can involve generalizing such things as the number of entities involved in a situation or the number of times some action is performed. This issue has been largely ignored in previous explanation-based learning research. Instead, other research has focused on changing constants into variables and determining the general constraints on those variables. While we recognize the importance of doing this, in our system we have concentrated on augmenting the explanation. Once the explanation is augmented, a standard explanation-based learning algorithm [5] is applied.

We envision incorporating ideas such as ours into systems that perform symbolic mathematical computations [6-9]. In this vein, it can be viewed as a *learning apprentice* [10-14] for domains based on mathematical calculation. Learning apprentices have been defined [10] as

> *interactive* knowledge-based consultants that directly assimilate new knowledge by observing and analyzing the problem-solving steps contributed by their users through their *normal* use of the system.

Since our system constructs detailed explanations, it can explain its actions to naive users, point out faulty human solution steps, and fill in the gaps in sketchy calculations. In addition, it graphically illustrates its processing during generalization. For these reasons, this work also has implications for intelligent computer-aided instruction (ICAI) [15, 16]. Although we are currently working within the domain of physics, the results obtained are relevant to any mathematically-based domain.

2. INITIAL KNOWLEDGE

Physics 101 possesses a large number of mathematical problem-solving strategies. For example, it can symbolically integrate expressions, cancel variables, perform arithmetic, and replace terms by utilizing known formulae. Figure 1 contains the initial physics formulae provided to the system. These formulae are instantiated for each specific physical situation. Newton's second and third laws are included. (Newton's first law is a special case of his second law.) The second law states that the net force on an object equals its mass times its acceleration. The net force is decomposed into two components: the external force and the internal force. External forces result from any external fields that act upon objects. Object I's internal force is the sum of the forces the other objects exert on object I. These *inter-object* forces are constrained by Newton's third law, which says that every action has an equal and opposite reaction.

$$\forall objI\ [\overrightarrow{velocity}_{objI}(t) = \frac{d}{dt}\overrightarrow{position}_{objI}(t)]$$

$$\forall objI\ [\overrightarrow{acceleration}_{objI}(t) = \frac{d}{dt}\overrightarrow{velocity}_{objI}(t)]$$

$$\forall objI\ [\overrightarrow{force}_{net,objI}(t) = mass_{objI} * \overrightarrow{acceleration}_{objI}(t)]$$

$$\forall objI\ [\overrightarrow{force}_{net,objI}(t) = \overrightarrow{force}_{external,objI}(t) + \overrightarrow{force}_{internal,objI}(t)]$$

$$\forall objI\ [\overrightarrow{force}_{internal,objI}(t) = \sum_{\substack{objJ = obj1 \\ objJ \neq objI}}^{objN} \overrightarrow{force}_{objJ,objI}(t)]$$

$$\forall objI\ [\forall objJ \neq objI\ [\overrightarrow{force}_{objJ,objI}(t) = - \overrightarrow{force}_{objI,objJ}(t)]]$$

Figure 1. The Initial Physics Formulae of the System

3. UNDERSTANDING SOLUTIONS

Understanding a human-provided solution involves two phases. First, the system attempts to verify that each of the human's solution steps mathematically follows. If successful, in the second phase the mathematical reasoning component of **Physics 101** builds an explanation of *why* the solution works. A sample collision problem illustrates these two phases.

Verifying Solutions

Assume a human uses equation 1 while solving a two-object[1], one-dimensional collision problem. (See [4] for more details about this sample problem.) The goal is to determine $velocity_{obj\,1,X}(t\,2)$ - the velocity, in the x-direction, of object 1 at time 2. The value of the other seven variables are known.

[1] For clarity, a two-object collision problem is assumed here. However, the current implementation requires an example involving at least three objects to properly motivate the general version of equation 1. Later sections of this paper describe the reasons for this.

$$mass_{obj1}\,velocity_{obj1,x(t1)} + mass_{obj2}\,velocity_{obj2,x(t1)} = mass_{obj1}\,velocity_{obj1,x(t2)} + mass_{obj2}\,velocity_{obj2,x(t2)} \quad (1)$$

Without being explicitly stated, the principle of conservation of momentum is being invoked, as the momentum (*mass* × *velocity*) of the objects at two different times is equated. This equation is not a variation of any formula known to the system (figure 1). A physically-consistent mathematical derivation is needed if **Physics 101** is to accept equation 1. Since the two sides of this equation only differ as to the time at which they are evaluated, an attempt is made to determine a time-dependent expression describing the general form of one side of the equation.

The actual calculations of the system appear in figure 2. The goal is to convert, via a series of equality-preserving transformations, the top expression in figure 2 into an equivalent expression whose time dependence is explicit. Once this is done, the system can determine if equation 1 is valid. (The top expression is called the *left-hand side* of the calculation, while the other expressions are termed *right-hand sides*.)

Figure 3 illustrates three possible forms of the underlying time-dependent expression. The expression could be periodic, and hence could be equated at times separated by some number of periods. Alternatively, the expression could be parabolic. Here there would be some quadratic relationship between times where it is valid to equate the expression. A third possibility is that the expression is constant with respect to time. In this last case, it is valid to equate the expression at *any* two times.

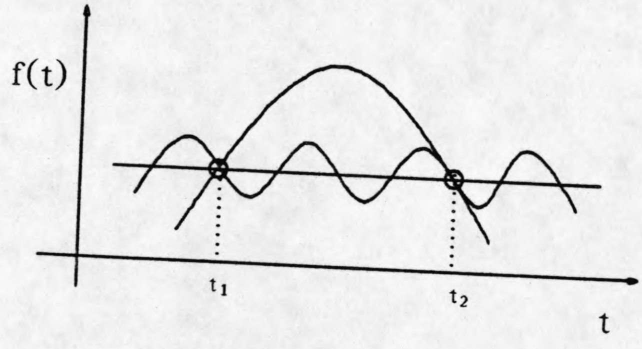

Figure 3. Equating an Expression at Two Times

$$\text{mass}_{obj1}\,\text{velocity}_{obj1,x}(t) + \text{mass}_{obj2}\,\text{velocity}_{obj2,x}(t)$$

$$\text{SubstFormulae} \quad = \text{mass}_{obj1} \int \text{acceleration}_{obj1,x}(t)\,dt + \text{mass}_{obj2} \int \text{acceleration}_{obj2,x}(t)\,dt$$

$$\text{SubstToMergeAlgebra} \quad = \text{mass}_{obj1} \int (\text{force}_{net,obj1,x}(t) / \text{mass}_{obj1})\,dt + \text{mass}_{obj2} \int (\text{force}_{net,obj2,x}(t) / \text{mass}_{obj2})\,dt$$

$$\text{ConstantsOutOfCalculus} \quad = (\text{mass}_{obj1} / \text{mass}_{obj1}) \int \text{force}_{net,obj1,x}(t)\,dt + (\text{mass}_{obj2} / \text{mass}_{obj2}) \int \text{force}_{net,obj2,x}(t)\,dt$$

$$\text{CancelAlgebra} \quad = 1 \int \text{force}_{net,obj1,x}(t)\,dt + 1 \int \text{force}_{net,obj2,x}(t)\,dt$$

$$\text{RemoveIdentities} \quad = \int \text{force}_{net,obj1,x}(t)\,dt + \int \text{force}_{net,obj2,x}(t)\,dt$$

$$\text{SubstFormulae} \quad = \int (\text{force}_{external,obj1,x}(t) + \text{force}_{internal,obj1,x}(t))\,dt + \int (\text{force}_{external,obj2,x}(t) + \text{force}_{internal,obj2,x}(t))\,dt$$

$$\text{SubstFormulae} \quad = \int (\text{force}_{external,obj1,x}(t) + \text{force}_{obj2,obj1,x}(t))\,dt + \int (\text{force}_{external,obj2,x}(t) + \text{force}_{obj1,obj2,x}(t))\,dt$$

$$\text{SubstToMergeAlgebra} \quad = \int (\text{force}_{external,obj1,x}(t) - \text{force}_{obj1,obj2,x}(t))\,dt + \int (\text{force}_{external,obj2,x}(t) + \text{force}_{obj1,obj2,x}(t))\,dt$$

$$\text{CombineCalculus} \quad = \int (\text{force}_{external,obj1,x}(t) - \text{force}_{obj1,obj2,x}(t) + \text{force}_{external,obj2,x}(t) + \text{force}_{obj1,obj2,x}(t))\,dt$$

$$\text{CancelAlgebra} \quad = \int (\text{force}_{external,obj1,x}(t) + 0\,\text{kg m} / s^2 + \text{force}_{external,obj2,x}(t))\,dt$$

$$\text{RemoveIdentities} \quad = \int (\text{force}_{external,obj1,x}(t) + \text{force}_{external,obj2,x}(t))\,dt$$

$$\text{SubstValues} \quad = \int (0\,\text{kg m} / s^2 + 0\,\text{kg m} / s^2)\,dt$$

$$\text{AddNumbers} \quad = \int 0\,\text{kg m} / s^2\,dt$$

$$\text{SolveCalculus} \quad = \text{constant}_1$$

Figure 2. Verifying Equation 1

The annotations to the left of the expressions in figure 2 are produced by the system. These annotations indicate how **Physics 101**'s mathematical component explains each calculation step. In the first step, the formulae substitutions are chosen as a last resort.[2] This means that they are not chosen in support of a variable cancellation. In the next step, the formulae substitutions are chosen because the mass terms can be cancelled. Before this cancellation can take place, however, the cancelling terms must be brought together. The calculation continues in a like manner until all the unknown variables are

[2] Initially, the system chose to replace the velocities by the derivative of the positions. This led nowhere and the system backtracked. No other backtracking occurred during the calculation of figure 2. The system is guided by the goal of cancelling variables, which greatly reduces the amount of unnecessary substitutions during problem solving. See [17] for further discussion of **Physics 101**'s problem solver.

eliminated. Then the known values are substituted and the ensuing arithmetic and calculus is solved. Since the initial expression is constant, it can be equated at any two times. Equation 1 is valid.

Explaining Solutions

At this point the system has ascertained that the human's use of a new equation is indeed valid. In the next step, the system must determine a reason for including each variable in this equation. This will determine which variables are required in its general form.

In the explanation process, **Physics 101** determines how the value of the current problem's *unknown* is obtained. The problem's unknown is the variable whose value is being sought; in the sample problem, $velocity_{obj\,1}$. During this process, the system determines the role of each variable in the initial expression of the calculation.

During a calculation one of three things can happen to a variable: (1) its value can be substituted, (2) it can be symbolically replaced during a formulae substitution, or (3) it can be *cancelled*. Understanding and generalizing variable cancellation drives **Physics 101**'s mathematical reasoning component. The system can identify the following five types of variable cancellations:

additive identity
: These are algebraic cancellations of the form $x - x = 0$. The second *CancelAlgebra* step in figure 2 contains two additive cancellations.

multiplicative identity
: These are algebraic cancellations of the form $x / x = 1$. The first *CancelAlgebra* step in figure 2 involves two multiplicative cancellations.

multiplication by zero
: These are cancellations that result from an expression (which may contain several variables) being multiplied by zero. None appear in figure 2.

integration (to a number)
: This type of cancellation occurs when variables disappear during symbolic integration. When integration produces *new* variables (other than the integration constant), this calculation is viewed as a substitution involving the original terms. No cancellations of this type appear in figure 2.

differentiation (to a number)
This is analogous to cancellation during integration.

Obstacles are variables appearing in a calculation but whose values are not known. *Primary obstacles* are obstacle variables descended from the unknown. In the momentum problem the only primary obstacle is $force_{internal,obj\,1}$.[3] If the value of each of the primary obstacles were known, the value of the unknown would be specified. The system ascertains how these obstacles are eliminated from the calculation. Cancelling obstacles is seen as the essence of the solution strategy, because when all the obstacles have been cancelled the value of the unknown can be easily calculated.

Figure 4 illustrates the concept of primary obstacles. The goal of the sample problem is to determine the value of a velocity. Since this is not known, the problem is transformed to that of finding acceleration (for simplicity, the integral sign is ignored here). However, the value of acceleration is not known either. This leads to the substitution of the net force divided by mass. The mass is known, but the net force is not. The net force is then decomposed into two components - a known external force and an unknown internal force. The internal force is the lone obstacle to knowing the value of the velocity. **Physics 101** needs to determine how the solution in figure 2 circumvents the need to know the value of this variable.

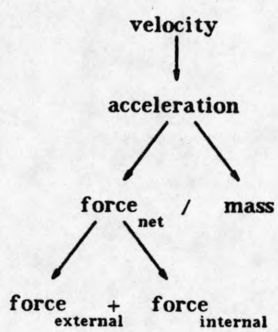

Figure 4. Decomposing the Unknown

[3] Actually, the right-most occurrence of $force_{obj\,2,obj\,1}$ in figure 2 satisfies the above definition of a primary obstacle. However, since this variable is descended from a primary obstacle ($force_{internal,obj\,1}$), it is not considered a primary obstacle. Cancelling a primary obstacle means that the values of its descendants need not be known.

First, the system determines that $force_{internal,obj\,1}$ is eliminated by being *additively cancelled*. Although cancelled additively, this variable originally appears in a multiplicative expression ($a = \frac{f}{m}$). Hence, the system must determine how it is *additively isolated*. It discovers that multiplication by $mass_{obj\,1}$ performed this task. So an explanation of the $mass_{obj\,1}$ term in the left-hand side expression is obtained.

The next thing to do is to determine how the terms that additively cancel $force_{internal,obj\,1}$ are introduced into the calculation. $Force_{internal,obj\,1}$ is replaced by the equivalent $force_{obj\,2,obj\,1}$, which is cancelled by the equal-and-opposite $force_{obj\,1,obj\,2}$ descended from $velocity_{obj\,2}$. The $force_{obj\,1,obj\,2}$, too, must first be additively isolated. **Physics 101** discovers that the left-hand side's $mass_{obj\,2}$ performs this isolation. The system now has explanations for the $mass_{obj\,2}$ and the $velocity_{obj\,2}$ terms in the left-hand side.

Cancellation of the primary obstacles requires the presence of additional variables on the left-hand side of the equation. These extra terms may themselves contain obstacle variables. These are called *secondary obstacles*. **Physics 101** must also determine how these obstacles are eliminated from the calculation. The elimination of the secondary obstacles may in turn require the presence of additional variables in the left-hand side expression, which may introduce additional secondary obstacles. This recursion must terminate, however, as the calculation is known to have eliminated all of the unknown terms.

Once the system determines how all of the obstacles in the calculation are cancelled, generalization can occur. At this time, **Physics 101** can also report any variables in the left-hand side of a calculation that are irrelevant to the determination of the value of the unknown.

4. GENERALIZING SOLUTIONS

Physics 101 performs generalization by using its explanation of the specific solution to guide the determination of the problem's unknown in the general case. This process is illustrated in the following figures.[4] The system starts with the generalized unknown, $velocity_{objI}$. It then performs the general versions of the specific formulae substitutions that produced the first of the primary obstacles. This can be seen in figure 5.

$$velocity_{objI,X}(t)$$

$$= \int acceleration_{objI,X}(t)\, dt$$

$$= \int (force_{net,objI,X}(t) \;/\; mass_{objI})\, dt$$

$$= (1 \;/\; mass_{objI}) \int force_{net,objI,X}(t)\, dt$$

$$= (1 \;/\; mass_{objI}) \int (force_{external,objI,X}(t) + force_{internal,objI,X}(t))\, dt$$

Figure 5. Introduction of the Primary Obstacle

Recall that the internal force is additively cancelled in the specific case. Hence, the next generalization step is to additively isolate $force_{internal,objI}$. The variable $mass_{objI}$ is introduced into the left-hand side of the general calculation in order to accomplish this isolation. Figure 6 presents this generalization step.

[4] During generalization, **Physics 101** produces a graphical description of its processing. The figures that follow (except figure 9) are actual outputs of the implemented system. It is expected that these outputs will prove useful in an ICAI system.

$$
\begin{aligned}
& mass_{objI}\ velocity_{objI,X}(t) \\
& = mass_{objI} \int acceleration_{objI,X}(t)\ dt \\
& = mass_{objI} \int (force_{net,objI,X}(t)\ /\ mass_{objI})\ dt \\
& = (mass_{objI}\ /\ mass_{objI}) \int force_{net,objI,X}(t)\ dt \\
& = 1 \int force_{net,objI,X}(t)\ dt \\
& = \int force_{net,objI,X}(t)\ dt \\
& = \int (force_{external,objI,X}(t) + force_{internal,objI,X}(t))\ dt
\end{aligned}
$$

Figure 6. Introduction of $Mass_{objI}$ to Additively Isolate the Primary Obstacle

At this point the general version of the primary obstacle is isolated for an additive cancellation. To perform this cancellation, those terms that will cancel the internal force must be introduced into the general calculation. The system determines that in the specific solution the net internal force acting on object 1 is indirectly cancelled because each of the inter-object forces acting upon object 1 is individually directly cancelled. (Recall that in figure 2, the formula $force_{internal,obj1} = force_{obj2,obj1}$ is used. The second from last formula in figure 1 is the general version of this specific formula.)

In the general case, *all* of the other objects in a situation exert an inter-object force on object I. *All* of these inter-object forces need to be cancelled. In the specific case, $velocity_{obj2}$ produced the canceller of object 1's internal force. The $mass_{obj2}$ term is needed to isolate the canceller for the additive cancellation. So to cancel $force_{internal,objI}$ in the general case, a *mass* × *velocity* term must come from every other object in the situation. Figure 7 presents the introduction of the summation that produces the variables that cancel $force_{internal,objI}$. Notice how the goal of cancellation motivates generalizing the number of objects involved in this expression.

Once all the cancellers of the generalized primary obstacle are present, the primary obstacle itself can be cancelled. This is shown in figure 8, which is a continuation of figure 7 (the last line of figure 7 is repeated in figure 8).

$$
\begin{aligned}
& mass_{objI}\, velocity_{objI,X}(t) + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} mass_{objJ}\, velocity_{objJ,X}(t) \\
&= mass_{objI} \int acceleration_{objI,X}(t)\, dt + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} mass_{objJ} \int acceleration_{objJ,X}(t)\, dt \\
&= mass_{objI} \int (force_{net,objI,X}(t) \;/\; mass_{objI})\, dt + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} mass_{objJ} \int (force_{net,objJ,X}(t) \;/\; mass_{objJ})\, dt \\
&= (mass_{objI} \;/\; mass_{objI}) \int force_{net,objI,X}(t)\, dt + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} (mass_{objJ} \;/\; mass_{objJ}) \int force_{net,objJ,X}(t)\, dt \\
&= 1 \int force_{net,objI,X}(t)\, dt + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} 1 \int force_{net,objJ,X}(t)\, dt \\
&= \int force_{net,objI,X}(t)\, dt + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} \int force_{net,objJ,X}(t)\, dt \\
&= \int (force_{external,objI,X}(t) + force_{internal,objI,X}(t))\, dt + \sum_{\substack{objJ=obj1 \\ objJ \neq objI}}^{objN} \int (force_{external,objJ,X}(t) + force_{internal,objJ,X}(t))\, dt
\end{aligned}
$$

Figure 7. Introduction of the Cancellers of the Primary Obstacle

Now that the primary obstacle is cancelled, the system checks to see if any secondary obstacles have been introduced. As can be seen in figure 8, the inter-object forces *not* involving object I still remain in the expression. Figure 9 graphically illustrates these remaining forces. All of the forces acting on object I have been cancelled, while a force between objects J and K still appears whenever neither J nor K equal I. This highlights an important aspect of generalizing number. Introducing more entities may create interactions that do not appear in the specific example.

$$\vdots$$

$$= \int \left(\text{force}_{\text{external},\text{objI},X}(t) + \text{force}_{\text{internal},\text{objI},X}(t)\right)dt + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \int \left(\text{force}_{\text{external},\text{objJ},X}(t) + \text{force}_{\text{internal},\text{objJ},X}(t)\right)dt$$

$$= \int \left(\text{force}_{\text{external},\text{objI},X}(t) + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \text{force}_{\text{objJ},\text{objI},X}(t)\right)dt + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \int \left(\text{force}_{\text{external},\text{objJ},X}(t) + \sum_{\substack{\text{objK} = \text{obj1} \\ \text{objK} \neq \text{objJ}}}^{\text{objN}} \text{force}_{\text{objK},\text{objJ},X}(t)\right)dt$$

$$= \int \left(\text{force}_{\text{external},\text{objI},X}(t) + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \left[-\,\text{force}_{\text{objI},\text{objJ},X}(t)\right]\right)dt + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \int \left(\text{force}_{\text{external},\text{objJ},X}(t) + \sum_{\substack{\text{objK} = \text{obj1} \\ \text{objK} \neq \text{objJ}}}^{\text{objN}} \text{force}_{\text{objK},\text{objJ},X}(t)\right)dt$$

$$= \int \left(\text{force}_{\text{external},\text{objI},X}(t) + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \left[-\,\text{force}_{\text{objI},\text{objJ},X}(t)\right] + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \text{force}_{\text{external},\text{objJ},X}(t) + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \sum_{\substack{\text{objK} = \text{obj1} \\ \text{objK} \neq \text{objJ}}}^{\text{objN}} \text{force}_{\text{objK},\text{objJ},X}(t)\right)dt$$

$$= \int \left(\text{force}_{\text{external},\text{objI},X}(t) + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} 0\,\text{kg}\,\text{m}/\text{s}^2 + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \text{force}_{\text{external},\text{objJ},X}(t) + \sum_{\substack{\text{objJ} = \text{obj1} \\ \text{objJ} \neq \text{objI}}}^{\text{objN}} \sum_{\substack{\text{objK} = \text{obj1} \\ \text{objK} \neq \text{objJ}\ \text{objI}}}^{\text{objN}} \text{force}_{\text{objK},\text{objJ},X}(t)\right)dt$$

Figure 8. Cancellation of the Primary Obstacle

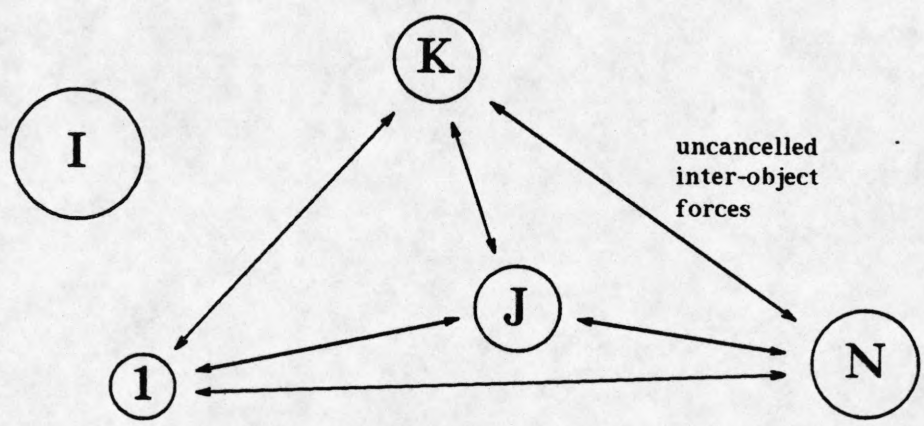

Figure 9. The Remaining Inter-Object Forces

Physics 101 cannot eliminate the remaining inter-object forces if the specific example only involves a two-object collision. It does not detect that the remaining forces all cancel one another, since in the two-object example there is no hint of how to deal with these secondary obstacles. A three-body collision must be analyzed by the system to properly motivate this cancellation. (In a three-body collision, $force_{obj\,3,obj\,2}$ cancels $force_{obj\,2,obj\,3}$; neither of these variables are descendants of $velocity_{obj\,1}$.) When the specific example involves three objects, the system continues as in figure 10. It ascertains that the remaining inter-object forces cancel.

$$mass_{objI}\ velocity_{objI,X}(t) + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} mass_{objJ}\ velocity_{objJ,X}(t)$$

$$\vdots$$

$$= \int \Big(force_{external,objI,X}(t) + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} 0\,kg\,m/s^2 + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} force_{external,objJ,X}(t) + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} \sum_{\substack{objK=obj1\\ objK \neq objJ\ objI}}^{objN} force_{objK,objJ,X}(t)\Big)\,dt$$

$$= \int \Big(force_{external,objI,X}(t) + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} 0\,kg\,m/s^2 + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} force_{external,objJ,X}(t) + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} \sum_{\substack{objK=obj1\\ objK \neq objJ\ objI}}^{objN} 0\,kg\,m/s^2\Big)\,dt$$

$$= \int \Big(force_{external,objI,X}(t) + \sum_{\substack{objJ=obj1\\ objJ \neq objI}}^{objN} force_{external,objJ,X}(t)\Big)\,dt$$

Figure 10. Cancellation of the Secondary Obstacles

Figure 11 contains the final result of generalization. A constraint propagation algorithm [5] is applied to the calculation structure of figure 10. This algorithm determines that there is no constraint that restricts this formula to the x-direction. It applies equally well to the y- and z-components of velocity. Hence, the acquired formula is a vector law. This process also derives the constraints that the masses of the objects be constant over time (since each was factored out of a temporal integral - see figure 2), and that the objects cannot have zero mass (since their masses appear in the denominator of expressions).

The Result of Generalization

$$\sum_{objI = obj1}^{objN} \overrightarrow{force}_{external,objI}(t) = \int \sum_{objI = obj1}^{objN} mass_{objI}\ \overrightarrow{velocity}_{objI}(t)\ dt$$

The Preferred Formulation

$$\frac{d}{dt} \sum_{objI = obj1}^{objN} \overrightarrow{force}_{external,objI}(t) = \sum_{objI = obj1}^{objN} mass_{objI}\ \overrightarrow{velocity}_{objI}(t)$$

Preconditions

(AND (IndependentOf? $mass_{objI}$ t) (NOT (ZeroValued? $mass_{objI}$)))

Figure 11. The Final Result

Notice that those variables whose values are used in the specific solution remain in the general formula. Only symbolic *cancellation* can fully eliminate a variable. The differential form of the simplified final equation is produced and added to **Physics 101**'s collection of general formulae. (Generalizing the two-body collision results in an expression still containing those inter-object forces that do not involve object I.) The new formula says: *The rate of change of the total momentum of a collection of objects is determined by the sum of the external forces on those objects*. Other problems, which involve any number of bodies under the influence of external forces, can be solved by the system using this generalized result.

The Cancellation Graph

Figure 12 contains the *cancellation graph* for a three-body collision problem. This data structure is built by the system during the understanding of the specific solution. It holds the information that explains how the specific example's obstacles are eliminated from the calculation. This information is used to guide the generalization process illustrated above. This graph and its relation to the preceding figures are summarized below.

The graph in figure 12 records that the only obstacle of the unknown ($velocity_{obj\,1}$) is object 1's internal force. This primary obstacle is blocked from an additive cancellation because it is divided by mass (figure 5). Another mass term cancels the additive blocker (figure 6). Once object 1's internal force is

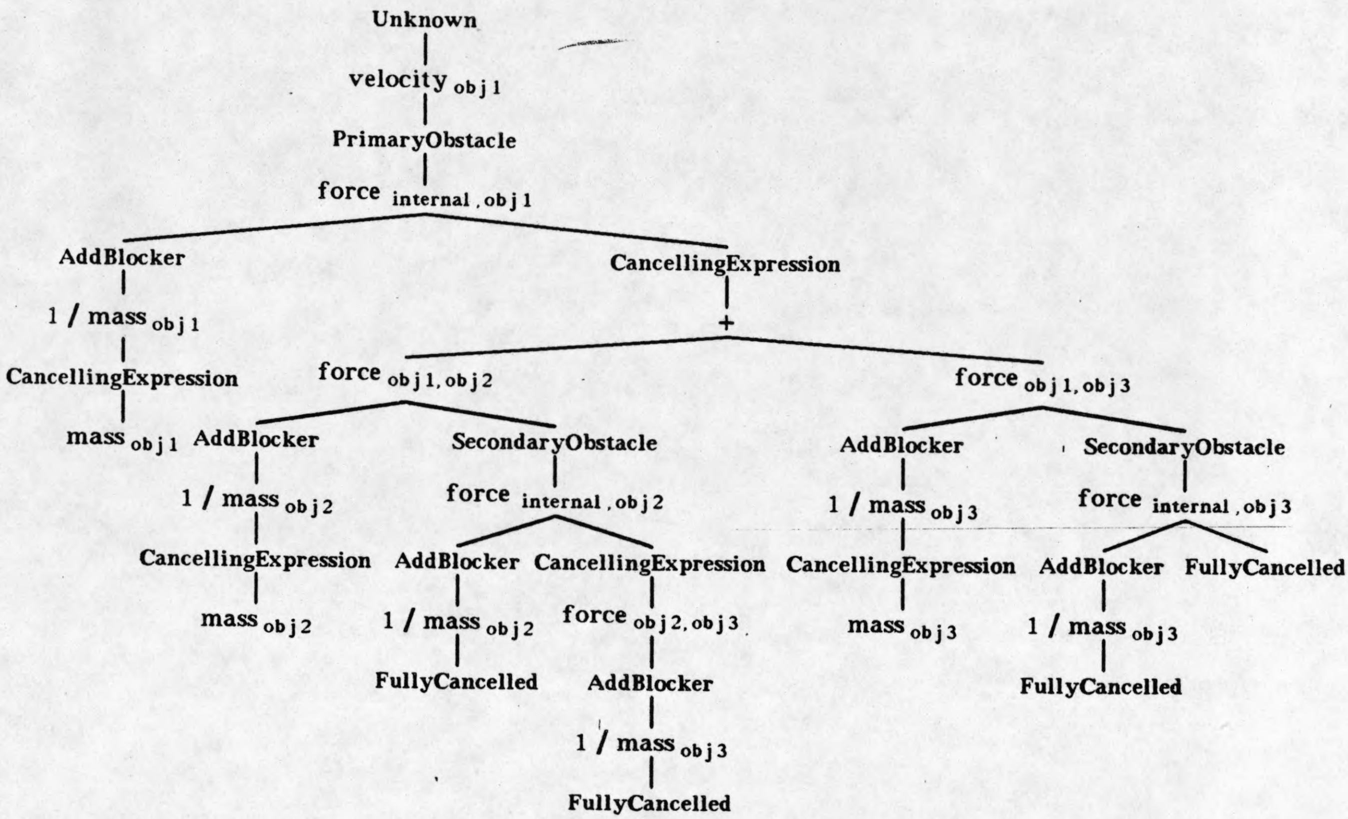

Figure 12. The Cancellation Graph

isolated, it is additively cancelled by $force_{obj\,1,obj\,2}$ and $force_{obj\,1,obj\,3}$ (figures 7 and 8). However, before cancellation can occur the additive blockers of both of these terms must be cancelled. Introducing these two inter-object forces results in the introduction of two secondary obstacles; the internal forces of objects 2 and 3. Both of these can be additively cancelled, since their additive blockers are already cancelled. The remainder of object 2's internal force is cancelled by the inter-object force between object's 2 and 3 (recall that a portion of this internal force cancelled part of object 1's internal force). Cancelling the internal force of object 2 also fully cancels the other secondary obstacle; the internal force of object 3 (figure 10). In a two-body problem the only secondary obstacle ($force_{internal\,,obj\,2}$) is fully cancelled when the primary obstacle ($force_{internal\,,obj\,1}$) is cancelled. In that case, there is no information to motivate the cancellation of the portions of the other internal forces that remain once the unknown's internal force is cancelled (figure 9).

Generalizing Number

Much research in explanation-based learning involves relaxing constraints on the entities in a situation, rather than generalizing the number of entities themselves. Nonetheless, many important concepts require generalizing number. Explanation-based learning provides a solution to a major problem, namely, how to know when it is valid and proper to generalize the number of entities.

A learning system's world knowledge may dictate that the general situation corresponding to a specific example may involve an indefinite number of entities. The system must recognize which parts of the explanation describe how the example's success depends on the specific number of entities. Once this is done, the explanation must be extended by replicating the appropriate portions. This replication process may impose constraints on allowable numbers. The system must ensure that the replication process does not itself introduce ill-effects. Notice that this process is guided by the explanation, and therefore does not require extensive problem-solving search.

To illustrate this point, compare the situation where a learning system observes someone clearing all four pyramids off of a box before moving the box, to a second situation where four wheels in a bin are used to build a wagon. Both scenarios involve four components, but require different generalizations. The explanation-based approach provides the foundation for the correct analysis of each situation. The general clearing operation will involve clearing *all* objects on a box (in this example, four), as the box must be cleared to be moved. The wagon-building plan, however, will allow the use of only four wheels regardless of how many are in the bin. The explanation of a component's functionality dictates the constraints on the number of components allowable in the general situation.

In the sample momentum problem, information about number, localized in a single physics formula, leads to a global restructuring of the solution. There are a number of valid generalizations of equation 1, some of which do not require generalizing number. For instance, the identity of the objects and the initial velocities could be generalized to produce: *the momentum of any two-object system is conserved, provided there are no external forces*. Unfortunately this is not broadly applicable. The system would need to learn separate rules when it encountered a four-object system, a five-object system, etc.

5. RELATED WORK

Our explanation-based approach to learning has much in common with [5, 18-34]. See [35] for a full discussion. Other work in the area of computer understanding and generalization of mathematical calculations appears in [36-40]. Additional research on the computer solution of physics problems is presented in [41-46].

Comparison to Other Explanation-Based Generalization Methods

A number of explanation-based generalization algorithms have been developed. Most [2, 5, 20, 21, 23, 29, 47] do not alter the structure of the explanation; no additional objects nor inference rules can be incorporated into the explanation. They work by changing constants in the observed example to variables with constraints. Another algorithm [3] allows for the elimination of easily-reconstructed details. However, as we have seen in generalizing number, extensive augmentation of the explanation can be required to produce the appropriate generalization. Only after properly augmenting the explanation should one of the constraint-propagation algorithms be applied.

Consider the **LEAP** system [10]. The system is shown an example of using NOR gates to compute the boolean AND of two OR's. It discovers that the technique generalizes to computing the boolean AND of any two inverted boolean functions. However, **LEAP** cannot generalize this technique to allow constructing the AND of an arbitrary number of inverted boolean functions using a multi-input NOR gate. This is the case even if **LEAP**'s initial background knowledge were to include the general version of DeMorgan's Law and the concept of multi-input NOR gates. Generalizing the number of functions requires alteration of the original example's explanation. This generalization cannot be performed using their *goal regression* algorithm alone.

Ellman's system [20] also illustrates the need for generalizing number. From an example of a four-bit circular shift register, his system constructs a generalized design for an arbitrary four-bit permutation register. A design for an N-bit circular shift register cannot be produced. As Ellman points out, such generalization, though desirable, cannot be done using the technique of changing constants to variables.

Other Approaches to Generalizing Number

Two other approaches to generalizing number have been recently proposed. In the **FERMI** system [48], cyclic patterns are recognized using empirical methods and the detected repeated pattern is generalized using explanation-based learning techniques. Another approach [49] notices when an operator is used repeatedly in a solution sequence and then determines, from the explanation structure, the constraints on two consecutive applications of the operator. **Physics 101** differs from these approaches in that the need for generalizing number is motivated in **Physics 101** by the analytical understanding of the specific solution and the knowledge of how the underlying technique extends to arbitrary situations.

6. CONCLUSION

We have designed and implemented a reasoning system that performs explanation-based learning in mathematically-oriented domains. The system's understanding and generalization processes are guided by the manner in which variables are cancelled in a specific problem. Attention focusses on how *obstacles* are eliminated in the specific problem. Obstacles are variables that preclude the direct evaluation of the unknown. Cancelling these variables allows the determination of the value of the unknown.

One important feature of analyzing variable cancellation is that the generalization of number is properly motivated. This feature is illustrated in the sample momentum problem presented in this paper. Generalizing the number of entities in a situation is ignored in most research in explanation-based learning. Instead, the focus is on determining the general constraints on the entities provided. Extending the structure of the explanation is necessary in order to generalize number. In our system, restructuring of the explanation is motivated by the need to cancel variables in the general case. A formal domain-independent account of generalizing number can be given only after detailed investigations of this phenomenon in many disparate domains.

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