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On the Stability and Robustness of an Adaptive Nonlinear Control Scheme*

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Abstract

We study the stability and robustness properties of an adaptive nonlinear regulation scheme. For the case where the equilibrium of the nonlinear system depends on the unknown parameters, we prove the robustness of the adaptive scheme to unmodeled dynamics using converse Lyapunov arguments. We also show that under some additional conditions, the closed-loop adaptive system has an exponentially stable parameter-dependent equilibrium, and is robust not only to small bounded disturbances and unmodeled dynamics, but also to slow time variations of the unknown parameters.

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1 Introduction

In the rapidly growing literature on adaptive control of nonlinear systems (see [1] for a recent survey) the only results dealing with robustness issues are those found in [2,3,4], where the proposed adaptive nonlinear regulation schemes are shown to be robust to fast stable unmodeled dynamics. The robustness analyses there are carried out under the simplifying assumption that the vector fields multiplying the unknown parameters vanish at the origin, which implies in particular that the origin is an equilibrium of the nonlinear system for any value of the unknown constant parameters. However, in the absence of unmodeled dynamics, the extended direct adaptive scheme of [4] is shown to achieve stability and regulation even when the equilibrium depends on the unknown parameters.

In the present paper, we focus our attention on the case of parameter-dependent equilibrium. In Section 2 we show that, in the absence of unmodeled dynamics, the adaptive scheme of [3,4] does not only achieve stability of the closed-loop system and regulation of the original state to the equilibrium, but in fact renders a parameter-dependent linear variety of the state space **exponentially attractive**. Then, in Section 3, we combine these results with a converse Lyapunov argument to show that the stability properties of the closed-loop adaptive system are robust to fast stable unmodeled dynamics.

As a corollary of the main result of Section 2, we show that, when the number of unknown parameters is less than or equal to twice the number of independent control inputs, and, in addition, a rank condition is satisfied, the aforementioned linear variety collapses to a single point. This point is then an **exponentially stable** equilibrium of the closed-loop adaptive system, whose stability properties are therefore inherently robust to unmodeled dynamics and small bounded disturbances. In Section 4 we combine this result with yet another converse Lyapunov argument to show that, in the presence of slow time variations of the unknown parameters, and under the above conditions, the adaptive scheme of [3,4] achieves convergence of the closed-loop system state to a small residual set. This is the first available result on adaptive control of nonlinear systems with time-varying parameters.

2 Stability Properties Without Unmodeled Dynamics

The extended direct adaptive scheme developed by Kanellakopoulos, Kokotovic and Marino [3,4] is applicable to full-state feedback linearizable nonlinear systems that satisfy the so-called **extended matching condition (EMC)**. Consider the system

$$\dot{z} = f_0(z) + \sum_{i=1}^{p} \theta_i f_i(z) + \sum_{j=1}^{m} g_j(z) u_j \,, \tag{2.1}$$

where $z \in \mathbb{R}^n$ is the state, $u = [u_1, \ldots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$ is the input, $\theta = [\theta_1, \ldots, \theta_p]^{\mathrm{T}} \in \mathbb{R}^p$ is the vector of unknown constant parameters, and f_i, $0 \le i \le p$, g_i, $1 \le j \le m$, are smooth vector fields with $f_0(0) = 0$. The following result is proven in [3,4]:

Proposition 1. *There exist neighborhoods $B_z \subset \mathbb{R}^n$ and $B_\theta \subset \mathbb{R}^p$ of $z = 0$ and $\theta = 0$, a state-feedback control*

$$u = \alpha(z) + B(z) v \,, \tag{2.2}$$

with $B(z)$ an $m \times m$ matrix nonsingular in B_z, and a state diffeomorphism $x = \phi(z)$ with $\phi(0) = 0$, such that the system (2.1) with the control (2.2) becomes in the x-coordinates

$$\begin{aligned} \dot{x}_i^j &= x_{i+1}^j, \quad 1 \le i \le k_j - 2 \\ \dot{x}_{k_j-1}^j &= x_{k_j}^j + \sum_{\ell=1}^{p} \theta_\ell w_{1\ell}^j(x) = x_{k_j}^j + \theta^{\mathrm{T}} w_1^j(x) \qquad 1 \le j \le m \\ \dot{x}_{k_j}^j &= v_j + \sum_{\ell=1}^{p} \theta_\ell w_{2\ell}^j(x) = v_j + \theta^{\mathrm{T}} w_2^j(x) \end{aligned} \tag{2.3}$$

if and only if the following conditions are satisfied in a neighborhood U_z of the origin $z = 0$:

(i) **Feedback linearization condition** *[5,6]: The distributions*

$$\mathcal{G}_i = \operatorname{span}\left\{ g_1, \ldots, g_m, \operatorname{ad}_{f_0} g_1, \ldots, g_m, \ldots, \operatorname{ad}_{f_0}^i g_1, \ldots, \operatorname{ad}_{f_0}^i g_m \right\}, \quad 0 \le i \le n-1 \tag{2.4}$$

are involutive and of constant dimension m_i, with $m_0 = m$, $m_{n-1} = n$ ($k_j = \operatorname{card}\{r_i \ge j, i \ge 0\}$, $j = 1, \ldots, m$, $r_0 = m_0$, $r_i = m_i - m_{i+1}, i \ge 1$).

(ii) Extended matching condition (EMC)

$$f_i \in \mathcal{G}_1, \quad 1 \leq i \leq p. \tag{2.5}$$

□

The scheme of [3,4] employs parameter estimates $\hat{\theta}_i$ of the unknown parameters θ_i to design an implementable adaptive controller. The first step in this direction is to replace the states $x^j_{k_j}$, $1 \leq j \leq m$, with the new states

$$\hat{x}^j_{k_j} = x^j_{k_j} + \sum_{\ell=1}^{p} \hat{\theta}_\ell w^j_{1\ell}(x), \quad 1 \leq j \leq m. \tag{2.6}$$

In order to guarantee that the mapping $x \to \hat{x}$ is one-to-one, onto and continuous, where

$$\begin{aligned} x &= [x^1_1, \ldots, x^1_{k_1-1}, x^1_{k_1}, \ldots, x^m_1, \ldots, x^m_{k_m-1}, x^m_{k_m}]^{\mathrm{T}} \\ \hat{x} &= [x^1_1, \ldots, x^1_{k_1-1}, \hat{x}^1_{k_1}, \ldots, x^m_1, \ldots, x^m_{k_m-1}, \hat{x}^m_{k_m}]^{\mathrm{T}}, \end{aligned} \tag{2.7}$$

it is assumed that there exists a constant $\delta > 0$ such that

$$\det(I + J(x, \hat{\theta})) \geq \delta, \quad \forall x \in B_x, \quad \forall \hat{\theta} \in B_\theta, \tag{2.8}$$

where $B_x = \phi(B_z)$ and

$$J_{ij}(x, \hat{\theta}) = \sum_{\ell=1}^{p} \hat{\theta}_\ell \frac{\partial w^i_{1\ell}(x)}{\partial x^i_{k_j}}, \quad 1 \leq i, j \leq m. \tag{2.9}$$

Then, the last two equations of each of the m subsystems of (2.3) are rewritten as

$$\dot{x}^j_{k_j-1} = \hat{x}^j_{k_j} + (\theta - \hat{\theta})^{\mathrm{T}} w^j_1(x) \tag{2.10}$$

$$\begin{aligned} \dot{\hat{x}}^j_{k_j} &= [v_j + \theta^{\mathrm{T}} w^j_2(x)] + \sum_{i=1}^{m} J_{ij}(x, \hat{\theta})[v_i + \theta^{\mathrm{T}} w^i_2(x)] \\ &\quad + \hat{\theta}^{\mathrm{T}} \sum_{i=1}^{m} \left(\sum_{\ell=1}^{k_i-1} \frac{\partial w^j_1(x)}{\partial x^i_\ell} x^i_{\ell+1} + \frac{\partial w^j_1(x)}{\partial x^i_{k_i-1}} w^i_1(x)^{\mathrm{T}} \theta \right) + \dot{\hat{\theta}}^{\mathrm{T}} w^j_1(x) \\ &= [I + J(x, \hat{\theta})]_j [v + w^{\mathrm{T}}_2(x)\theta] + \hat{\theta}^{\mathrm{T}} w^j_3(x) + \hat{\theta}^{\mathrm{T}} w^j_4(x)\theta + \dot{\hat{\theta}}^{\mathrm{T}} w^j_1(x). \end{aligned} \tag{2.11}$$

The certainty-equivalence control

$$\begin{aligned} v &= -w^{\mathrm{T}}_2(x)\hat{\theta} \\ &\quad -[I + J(x, \hat{\theta})]^{-1} \begin{bmatrix} \gamma^1_1 x^1_1 + \ldots + \gamma^1_{k_1} \hat{x}^1_{k_1} + \hat{\theta}^{\mathrm{T}} w^1_3(x) + \hat{\theta}^{\mathrm{T}} w^1_4(x)\hat{\theta} + \dot{\hat{\theta}}^{\mathrm{T}} w^1_1(x) \\ \vdots \\ \gamma^m_1 x^m_1 + \ldots + \gamma^m_{k_m} \hat{x}^m_{k_m} + \hat{\theta}^{\mathrm{T}} w^m_3(x) + \hat{\theta}^{\mathrm{T}} w^m_4(x)\hat{\theta} + \dot{\hat{\theta}}^{\mathrm{T}} w^m_1(x) \end{bmatrix} \end{aligned} \tag{2.12}$$

results in the error system

$$\dot{\hat{x}} = A\hat{x} + W(x,\hat{\theta})(\theta - \hat{\theta}), \tag{2.13}$$

where

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}, \quad A_j = \begin{bmatrix} 0 & \\ \vdots & I_{k_j} \\ 0 & \\ -\gamma_1^j & \dots \quad -\gamma_{k_j}^j \end{bmatrix}, \tag{2.14}$$

$$W(x,\hat{\theta}) = \begin{bmatrix} W^1(x,\hat{\theta}) \\ \vdots \\ W^m(x,\hat{\theta}) \end{bmatrix}, \quad W^j(x,\hat{\theta}) = \begin{bmatrix} 0 \ \dots \ 0 \\ \vdots \qquad \vdots \\ 0 \ \dots \ 0 \\ w_1^j(x)^{\mathrm{T}} \\ \left[I + J(x,\hat{\theta})\right]_j w_2^{\mathrm{T}}(x) + \hat{\theta}^{\mathrm{T}} w_4^j(x) \end{bmatrix}, \tag{2.15}$$

and the gains γ_i^j, $1 \le i \le k_j$, $1 \le j \le m$, are chosen to place the eigenvalues of A_j at some desired stable locations.

The "error form" (2.13) suggests the following parameter update law:

$$\dot{\hat{\theta}} = \Gamma W^{\mathrm{T}}(x,\hat{\theta}) P \hat{x}, \tag{2.16}$$

with $\Gamma = \Gamma^{\mathrm{T}} > 0$ and

$$P = \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_m \end{bmatrix}, \quad P_j = P_j^{\mathrm{T}} > 0, \quad P_j A_j + A_j^{\mathrm{T}} P_j = -I_{k_j}, \quad 0 \le j \le m. \tag{2.17}$$

The stability of the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$ of the adaptive scheme (2.13)–(2.16) is then established using the quadratic Lyapunov function

$$V(\hat{x},\hat{\theta}) = \hat{x}^{\mathrm{T}} P \hat{x} + (\theta - \hat{\theta})^{\mathrm{T}} \Gamma^{-1} (\theta - \hat{\theta}), \tag{2.18}$$

whose derivative along the solutions of (2.13)–(2.16) is

$$\dot{V}(\hat{x},\hat{\theta}) = -\|\hat{x}\|^2 \le 0. \tag{2.19}$$

This implies that $\hat{x} = 0$, $\hat{\theta} = \theta$ is stable and its region of attraction contains the set

$$\hat{\Omega}_V = \{(\hat{x},\hat{\theta}) : V(\hat{x},\hat{\theta}) \le c\}, \tag{2.20}$$

where c is the largest constant such that

$$\Omega_V = \{(x,\hat{\theta}) : V(\hat{x},\hat{\theta}) \leq c\} \subset B_x \times B_\theta \,. \tag{2.21}$$

The invariance theorem of LaSalle now guarantees that $(\hat{x}(t),\hat{\theta}(t))$ tends to the largest invariant set of (2.13)–(2.16) contained in the set where $\dot{V} = 0$. An immediate consequence of this fact and (2.19) is that

$$\lim_{t\to\infty} \hat{x}(t) = 0 \,, \quad \lim_{t\to\infty} \dot{\hat{x}}(t) = 0 \,, \quad \lim_{t\to\infty} \dot{\hat{\theta}}(t) = 0 \,. \tag{2.22}$$

The equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, is expressed in the $(x,\hat{\theta})$-coordinates as

$$x = x^{\mathrm{e}},\ \hat{\theta} = \theta \,, \tag{2.23}$$

where the θ-dependent x^{e} is defined by

$$x^{\mathrm{e}} = [0,\ldots,0,x^{1\mathrm{e}}_{k_1},\ldots,0,\ldots,0,x^{m\mathrm{e}}_{k_m}] \tag{2.24}$$

$$x^{j\mathrm{e}}_{k_j} + \theta^{\mathrm{T}} w^j_1(x^{\mathrm{e}}) = 0 \,, \quad 1 \leq j \leq m \,. \tag{2.25}$$

Note that the system (2.25) has, because of (2.8), a unique solution for each $\theta \in B_\theta$. Since $x^j_{k_j-1}$, $1 \leq j \leq m$, can be expressed as smooth functions of $\hat{x}$ and $\hat{\theta}$, it follows from (2.22) that

$$\lim_{t\to\infty} \dot{x}^j_{k_j-1}(t) = \lim_{t\to\infty} [x^j_{k_j} + \theta^{\mathrm{T}} w^j_1(x)] = 0 \,, \quad 1 \leq j \leq m \,. \tag{2.26}$$

Combining (2.22), (2.25) and (2.26) with the fact that $w^j_1(x)$, $1 \leq j \leq m$, are smooth vector fields, we conclude that

$$\lim_{t\to\infty} x(t) = x^{\mathrm{e}}. \tag{2.27}$$

We are now ready to state and prove the main result of this section:

Theorem 1. *The equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$ of the adaptive scheme (2.13)–(2.16) is* stable *and a subset of its region of attraction is the set Ω_V defined in (2.21). Furthermore, its state* $(\hat{x}(t),\hat{\theta}(t))$ converges exponentially to the linear variety

$$M = \left\{(\hat{x},\hat{\theta}) : \hat{x} = 0 \,,\ W_{\mathrm{e}}(\theta - \hat{\theta}) = 0\right\}, \tag{2.28}$$

where

$$W_{\rm e} = \left[w_1^1(x^{\rm e}), \ldots, w_1^m(x^{\rm e})\,, w_2^1(x^{\rm e}), \ldots, w_2^m(x^{\rm e})\right]^{\rm T}. \tag{2.29}$$

Proof. The first part of the theorem is already proven in (2.18)–(2.27). For the proof of the second part, we must first find the largest invariant set of (2.13)–(2.16) contained in the set where $\dot{V} = 0$, i.e., in the set $\hat{x} \equiv 0$, or equivalently, $x \equiv x^{\rm e}$. Setting $\hat{x} = 0$, $x = x^{\rm e}$, $\dot{\hat{x}} = 0$ in (2.13)–(2.16), we obtain $\dot{\hat{\theta}} = 0$ and

$$W(x^{\rm e}, \hat{\theta})(\theta - \hat{\theta}) = 0\,. \tag{2.30}$$

Substituting (2.15) into (2.30) we obtain the equivalent expression

$$w_1^j(x^{\rm e})^{\rm T}(\theta - \hat{\theta}) = 0\,, \quad 1 \leq j \leq m \tag{2.31}$$

$$\left[\left[I + J(x^{\rm e}, \hat{\theta})\right]_j w_2^{\rm T}(x^{\rm e}) + \hat{\theta}^{\rm T} w_4^j(x^{\rm e})\right](\theta - \hat{\theta}) = 0\,, \quad 1 \leq j \leq m\,. \tag{2.32}$$

But from (2.11) and (2.31) we have

$$w_4^j(x^{\rm e})(\theta - \hat{\theta}) = \sum_{i=1}^{m} \frac{\partial w_1^j(x^{\rm e})}{\partial x_{k_i-1}^i} w_1^i(x^{\rm e})^{\rm T}(\theta - \hat{\theta}) = 0\,. \tag{2.33}$$

Substituting (2.33) in (2.32) and using (2.8), we see that (2.32) is equivalent to

$$w_2^{\rm T}(x^{\rm e})(\theta - \hat{\theta}) = 0\,. \tag{2.34}$$

Thus, the largest invariant set of (2.13)–(2.16) contained in the set where $\dot{V} = 0$, is the set M defined by (2.28)–(2.29):

$$M = \left\{(\hat{x}, \hat{\theta}) : \hat{x} = 0\,,\ W_{\rm e}(\theta - \hat{\theta}) = 0\right\}.$$

Since $W_{\rm e}$ is a constant $2m \times p$ matrix with rank r, the set M is a linear variety of dimension $p - r$ that contains the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$. By LaSalle's invariance theorem, the state $(\hat{x}(t), \hat{\theta}(t))$ of (2.13)–(2.16) tends asymptotically to the set M. To complete the proof of the theorem, we now need to show that the rate of convergence to the set M is exponential, i.e., that for every compact set $S \subset \Omega_V$ there exist constants $K > 0$, $\lambda > 0$ such that

$$\begin{pmatrix} x(0) \\ \hat{\theta}(0) \end{pmatrix} \in S \Rightarrow\ \operatorname{dist}\left(\begin{pmatrix} \hat{x}(t) \\ \hat{\theta}(t) \end{pmatrix}, M\right) \leq K e^{-\lambda t} \operatorname{dist}\left(\begin{pmatrix} \hat{x}(0) \\ \hat{\theta}(0) \end{pmatrix}, M\right). \tag{2.35}$$

Towards this end, we first replace $\hat{\theta}$ by the new states φ and ψ of dimension r and $p-r$, respectively, which are defined as

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{bmatrix}\bar{W}_{\mathrm{e}}\\ T_{\mathrm{e}}\end{bmatrix}(\theta-\hat{\theta}) \triangleq T(\theta-\hat{\theta})\,. \tag{2.36}$$

In (2.36), $\bar{W}_{\mathrm{e}}$ is an $r\times p$ matrix that has the same nullspace as W_{e} (recall that r is the rank of W_{e}), so that

$$W_{\mathrm{e}}(\theta-\hat{\theta}) = 0 \Longleftrightarrow \bar{W}_{\mathrm{e}}(\theta-\hat{\theta}) = 0 \Longleftrightarrow \varphi = 0\,, \tag{2.37}$$

and T_{e} is a $(p-r)\times p$ matrix chosen so that the transformation matrix T in (2.36) is nonsingular. Due to the definitions (2.36)–(2.37), (2.35) can now be equivalently expressed as

$$\begin{pmatrix}x(0)\\ \hat{\theta}(0)\end{pmatrix} \in S \Rightarrow \left\| \begin{matrix}\hat{x}(t)\\ \varphi(t)\end{matrix} \right\| \le K_1 e^{-\lambda_1 t} \left\| \begin{matrix}\hat{x}(0)\\ \varphi(0)\end{matrix} \right\|\,, \tag{2.38}$$

with K_1, λ_1 some positive constants that depend on S. In the $(\hat{x},\varphi,\psi)$-coordinates, the adaptive scheme (2.13)–(2.16) becomes

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + U_\theta(x,\varphi,\psi)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\\ \begin{pmatrix}\dot{\varphi}\\ \dot{\psi}\end{pmatrix} &= \Gamma_1 U_\theta^{\mathrm{T}}(x,\varphi,\psi)P\hat{x}\,,\end{aligned} \tag{2.39}$$

where

$$U_\theta(x,\varphi,\psi) = W\left(x, \theta - T^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)T^{-1} \tag{2.40}$$

$$\Gamma_1 = T\Gamma T^{\mathrm{T}} = \Gamma_1^{\mathrm{T}} > 0\,. \tag{2.41}$$

We have already shown (cf. (2.30)–(2.34)) that $\forall\,\hat{\theta},\,\theta^0 \in B_\theta$:

$$W(x^{\mathrm{e}},\theta^0)(\theta-\hat{\theta}) = 0 \Longleftrightarrow W_{\mathrm{e}}(\theta-\hat{\theta}) = 0\,. \tag{2.42}$$

From (2.36) and (2.40) we conclude that for all (φ^0,ψ^0), (φ,ψ) corresponding to θ^0, $\hat{\theta}\in B_\theta$, we have

$$U_\theta(x^{\mathrm{e}},\varphi^0,\psi^0)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = 0 \Longleftrightarrow W(x^{\mathrm{e}},\theta^0)(\theta-\hat{\theta}) = 0\,, \tag{2.43}$$

which, combined with (2.37) and (2.42), results in

$$U_\theta(x^{\mathrm{e}},\varphi^0,\psi^0)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = 0 \Longleftrightarrow \varphi = 0\,. \tag{2.44}$$

In particular, (2.44) implies that

$$U_\theta(x^e, 0, \psi^0)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = 0 \Longleftrightarrow \varphi = 0\,. \tag{2.45}$$

Hence,

$$U_\theta(x^e, 0, \psi^0) = \begin{bmatrix} U_1 & 0 \\ U_2 & 0 \end{bmatrix} \triangleq U_e\,, \quad \text{rank}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = r\,. \tag{2.46}$$

Let us now examine the linearization of (2.39) around $\hat{x} = 0$, $\varphi = 0$, $\psi = \psi^0 = \text{const.}$ (equivalently, $x = x^e$, $\hat{\theta} = \text{const.}$, $W_e(\theta - \hat{\theta}) = 0$):

$$\begin{aligned} \delta\dot{\hat{x}} &= A\delta\hat{x} + U_e\begin{pmatrix}\delta\varphi\\ \delta\psi\end{pmatrix} \\ \begin{pmatrix}\delta\dot{\varphi}\\ \delta\dot{\psi}\end{pmatrix} &= \Gamma_1 U_e^{\mathrm{T}} P\delta\hat{x}\,. \end{aligned} \tag{2.47}$$

Using (2.46) and the decomposition

$$\Gamma_1 = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^{\mathrm{T}} & \Gamma_{22} \end{bmatrix}, \tag{2.48}$$

we can rewrite (2.47) as

$$\begin{aligned} \delta\dot{\hat{x}} &= A\delta\hat{x} + \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\delta\varphi \\ \delta\dot{\varphi} &= \Gamma_{11}[U_1^{\mathrm{T}}\, U_2^{\mathrm{T}}]P\delta\hat{x} \\ \delta\dot{\psi} &= \Gamma_{12}^{\mathrm{T}}[U_1^{\mathrm{T}}\, U_2^{\mathrm{T}}]P\delta\hat{x}\,. \end{aligned} \tag{2.49}$$

The stability properties of this linear system are established using the quadratic Lyapunov function

$$V_\ell(\delta\hat{x}, \delta\varphi, \delta\psi) = \delta\hat{x}^{\mathrm{T}}P\delta\hat{x} + (\delta\varphi^{\mathrm{T}}\delta\psi^{\mathrm{T}})\Gamma_1^{-1}\begin{pmatrix}\delta\varphi\\ \delta\psi\end{pmatrix}, \tag{2.50}$$

whose derivative along the solutions of (2.47) is

$$\dot{V}_\ell(\delta\hat{x}, \delta\varphi, \delta\psi) = -\|\delta\hat{x}\|^2 \le 0\,. \tag{2.51}$$

Thus, the equilibrium $\delta\hat{x} = 0$, $\delta\varphi = 0$, $\delta\psi = 0$ of (2.49) is **stable**. Furthermore, the largest invariant set of (2.47) contained in the set where $\dot{V}_\ell = 0$ is, by (2.45), the set

$$M_\ell = \{(\delta\hat{x}, \delta\varphi, \delta\psi) : \delta\hat{x} = 0, \delta\varphi = 0\}\,. \tag{2.52}$$

Hence, the linear system (2.49) is stable and its state converges to the $(p-r)$-dimensional linear subspace M_ℓ. Moreover, by (2.45) again, every point of M_ℓ is an equilibrium of (2.47):

$$\delta\hat{x}(0)=0\,,\ \delta\varphi(0)=0 \Rightarrow \delta\hat{x}(t)\equiv\delta\hat{x}(0)\,,\ \delta\varphi(t)\equiv\delta\varphi(0)\,,\ \delta\psi(t)\equiv\delta\psi(0)\,. \tag{2.53}$$

We can now conclude that the state matrix of (2.47)

$$A_\ell = \begin{bmatrix} A & U_e \\ \Gamma_1 U_e^{\mathrm{T}} P & 0 \end{bmatrix} = \begin{bmatrix} A & U_1 & 0 \\ & U_2 & 0 \\ \Gamma_{11}[U_1^{\mathrm{T}}\, U_2^{\mathrm{T}}]P & 0 & 0 \\ \Gamma_{12}[U_1^{\mathrm{T}}\, U_2^{\mathrm{T}}]P & 0 & 0 \end{bmatrix} \tag{2.54}$$

has $n+r$ of its $n+p$ eigenvalues in the open left half plane and the remaining $p-r$ at the origin $s=0$. The LHP eigenvalues are the eigenvalues of the submatrix

$$A_{\mathrm{LHP}} = \begin{bmatrix} A & U_1 \\ & U_2 \\ \Gamma_{11}[U_1^{\mathrm{T}}\, U_2^{\mathrm{T}}]P & 0 \end{bmatrix}, \tag{2.55}$$

which corresponds to the $\begin{pmatrix}\delta\hat{x}\\ \delta\varphi\end{pmatrix}$-subsystem. From (2.52) we see that this subsystem is not affected by $\delta\psi$:

$$\begin{pmatrix}\delta\dot{\hat{x}}\\ \delta\dot{\varphi}\end{pmatrix} = A_{\mathrm{LHP}}\begin{pmatrix}\delta\hat{x}\\ \delta\varphi\end{pmatrix}. \tag{2.56}$$

Hence, there exist constants $K_0>0$, $\lambda_0>0$ such that

$$\left\| \begin{matrix}\delta\hat{x}(t)\\ \delta\varphi(t)\end{matrix} \right\| \le K_0 e^{-\lambda_0 t} \left\| \begin{matrix}\delta\hat{x}(0)\\ \delta\varphi(0)\end{matrix} \right\|. \tag{2.57}$$

This proves that the linear variety M defined by (2.28)–(2.29) is an equilibrium manifold of (2.13)–(2.16) that is not only **stable** and **attractive** with a region of attraction that contains the set Ω_V defined by (2.21), but also **exponentially attractive**. Hence, for every compact set $S\subset\Omega_V$ there exist constants $K>0$, $\lambda>0$ such that (2.35) is satisfied. □

Corollary 1.1. *For every compact set $S\subset\Omega_V$ there exist constants $K_1>0$, $\lambda_1>0$ and a class-$\mathcal{K}$ function $\sigma(\cdot)$ such that for every $(x(0),\hat{\theta}(0))\in S$, the solutions of the transformed adaptive scheme (2.39)*

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + U_\theta(x,\varphi,\psi)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \\ \begin{pmatrix}\dot{\varphi}\\ \dot{\psi}\end{pmatrix} &= \Gamma_1 U_\theta^{\mathrm{T}}(x,\varphi,\psi)P\hat{x} \end{aligned}$$

starting from the corresponding $(\hat{x}(0), \varphi(0), \psi(0))$ *satisfy the following inequalities* $\forall t \geq 0$*:*

$$\left\| \begin{matrix} \hat{x}(t) \\ \varphi(t) \end{matrix} \right\| \leq K_1 e^{-\lambda_1 t} \left\| \begin{matrix} \hat{x}(0) \\ \varphi(0) \end{matrix} \right\| \tag{2.58}$$

$$\|\psi(t)\| \leq \sigma \left(\left\| \begin{matrix} \hat{x}(0) \\ \varphi(0) \\ \psi(0) \end{matrix} \right\| \right). \tag{2.59}$$

□

Corollary 1.2. *If* $r = p$*, that is, if the rank of* W_e *is equal to the number of unknown parameters, the equilibrium* $\hat{x} = 0$*,* $\hat{\theta} = \theta$ *of the adaptive scheme (2.13)–(2.16) is* exponentially stable *with a region of attraction that contains the set* Ω_V*. Thus, the stability of* $\hat{x} = 0$*,* $\hat{\theta} = \theta$ *is* robust *with respect to both fast stable unmodeled dynamics and small disturbances.*

Proof. If $\text{rank}(W_e) = p$, then

$$W_e(\theta - \hat{\theta}) = 0 \Longleftrightarrow \hat{\theta} = \theta\,, \tag{2.60}$$

which implies that the linear variety M defined by (2.28) collapses to the single point $\hat{x} = 0$, $\hat{\theta} = \theta$. By Theorem 1, this point is an exponentially stable equilibrium of (2.13)–(2.16). (In the coordinates of (2.39), $r = p$ implies that the $(p - r)$-dimensional ψ vanishes, and thus the state $(\hat{x}(t), \varphi(t))$ satisfies (2.58).) The robustness with respect to fast stable unmodeled dynamics and small disturbances follows from standard results on singular perturbations (see, e.g., Corollary 7.2.3 in Kokotovic, Khalil and O'Reilly [7]) and total stability (see, e.g., Theorems 56.1–56.3 in Hahn [8]). □

Remark 1.1. Since W_e is a matrix of dimension $2m \times p$, its rank can be equal to p only if $p \leq 2m$. This means that the adaptive scheme (2.13)–(2.16) can have $\hat{x} = 0$, $\hat{\theta} = \theta$ as an exponentially stable equilibrium only if the number of unknown parameters is less than or equal to twice the number of independent control inputs. □

Remark 1.2. It is of interest to compare the stability properties established in this section for the adaptive scheme of Kanellakopoulos, Kokotovic and Marino [3,4] with those established by Isidori and Byrnes [9] and Huang and Rugh [10] for their nonadaptive schemes.

Rewriting the system (2.3) as

$$\begin{aligned}\dot{x} &= A_0x + B_0v + W(x)\theta \\ \dot{\theta} &= 0\,,\end{aligned} \tag{2.61}$$

with the obvious definitions for A_0, B_0 and $W(x)$, one may try to use the schemes of [9] and [10] to regulate the measured state x to the equilibrium $x = x^e$ and reject the unmeasured constant disturbance θ. It is straightforward to verify that for this particular problem, Assumption (H3) of [9] and condition (3.11) of [10] are both equivalent to

$$r = \operatorname{rank}(W(x^e)) = p\,, \tag{2.62}$$

which is exactly the assumption of Corollary 1.2. Furthermore, the system (2.61) satisfies all the remaining assumptions of [9] and [10] if $w_1^j(0) = 0$, $j = 1, \dots, m$, in which case $x^e = 0$.

Assuming that all these conditions are met, we are in the case where all three schemes are applicable. While the nonadaptive schemes of [9,10] employ a full-order linear observer based on the linearization of (2.61) around the point $x = 0$, $\theta = 0$ [9] or $x = 0$, $\theta =$ const.[10], the update law (2.16) of the adaptive scheme of [3,4] can be interpreted as a reduced-order nonlinear observer. Finally, while the schemes of [9,10] are based on linearized designs and hence provide only local stability properties, the scheme of [3,4] results, in the case of the system (2.61), in a closed-loop system with a globally asymptotically stable equilibrium. □

3 Stability in the Presence of Unmodeled Dynamics

In the case where $r = p$, Corollary 1.2 states that the stability of the equilibrium of (2.13)–(2.16) is robust with respect to fast stable unmodeled dynamics. In this section, we prove that this statement is true even when $r < p$. We first prove the following converse Lyapunov result:

Lemma 1. *Consider the composite system*

$$\begin{aligned}\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y)\,,\end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, and the vector fields $f(\cdot)$, $g(\cdot)$ have continuous first-order partial derivatives that are bounded on a compact set S_0. Assume that this system has a stable *equilibrium at $x = 0$, $y = 0$ and that the subspace $x = 0$ is an equilibrium manifold of (3.1), that is, $f(0, y) \equiv 0$, $g(0, y) \equiv 0$. Moreover, assume that there exists a compact set $S_1 \subset S_0$ such that*

$$\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in S_1 \Rightarrow \begin{cases} \|x(t)\| \leq k e^{-\alpha(t-t_0)} \|x_0\| \\ \|y(t)\| \leq \sigma\left(\left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| \right) \end{cases} , \quad \forall t \geq t_0 \tag{3.2}$$

with $k \geq 1$, $\alpha > 0$ constants, and $\sigma(\cdot)$ a class-$\mathcal{K}$ function. Then there exists a Lyapunov function $V(x, y)$ that provides $\Omega_V \subset S_1$ as an estimate of the region of attraction and satisfies

$$\dot{V}(x, y) \leq -\alpha_1 \|x\|^2 \tag{3.3}$$

$$\left\| \frac{\partial V(x, y)}{\partial x} \right\| \leq \alpha_2 \|x\| \tag{3.4}$$

for some positive constants α_1, α_2, and for all $(x, y) \in \Omega_V$.

Proof. Let $p_x(t - t_0, x_0, y_0)$ denote the x-part of the solution of (3.1) which passes through (x_0, y_0) at time t_0. Then the function

$$V(x, y) = \int_t^{t+T} \|p_x(\tau - t, x, y)\|^2 d\tau + \left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\|, \tag{3.5}$$

with $T > 0$ a finite constant to be determined later, is a Lyapunov function for (3.1) which satisfies (3.3) and (3.4). To prove this claim, we first note that because of the boundedness of their first-order partial derivatives and the fact that $f(0, y) \equiv 0$, $g(0, y) \equiv 0$, the vector fields $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are Lipschitz continuous with respect to x uniformly in y on S_1; that is, there exists a constant $b > 0$, such that for all $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in S_1$:

$$\left\| \begin{pmatrix} f(x_1, y_1) \\ g(x_1, y_1) \end{pmatrix} - \begin{pmatrix} f(x_2, y_2) \\ g(x_2, y_2) \end{pmatrix} \right\| \leq b \|x_1 - x_2\|. \tag{3.6}$$

In particular, this implies that

$$\|x\| e^{-b(\tau - t)} \leq \|p_x(\tau - t, x, y)\| \leq \|x\| e^{b(\tau - t)}. \tag{3.7}$$

Also, from (3.2) we have

$$\left\| \begin{matrix} x_0 \\ y_0 \end{matrix} \right\| \geq \sigma^{-1}(\|y(t)\|). \tag{3.8}$$

Combining (3.7) and (3.8) we see that (3.5) is a locally positive definite function:

$$V(x,y) \geq \int_t^{t+T} \|x\|^2 e^{-2b(\tau-t)} d\tau + \sigma^{-1}(\|y\|) = \alpha_0 \|x\|^2 + \sigma^{-1}(\|y\|) \tag{3.9}$$

with $\alpha_0 = \dfrac{1-e^{-2bT}}{2b} > 0$. The rest of the proof is almost identical to the proof of Theorem 56.1 in [8]. The derivative of (3.5) along the solutions of (3.1) is

$$\dot{V}(x,y) = -\|p_x(0,x,y)\|^2 + \|p_x(T,x,y)\|^2. \tag{3.10}$$

Using (3.2) and the identity $p_x(0,x,y) = x$, we obtain from (3.10)

$$\dot{V}(x,y) \leq -\|x\|^2 + k^2 e^{-2\alpha T} \|x\|^2. \tag{3.11}$$

Thus, if we choose

$$T = \frac{2\ln k + \ln 2}{2\alpha} \tag{3.12}$$

in (3.11), we obtain (3.3) with $\alpha_1 = \frac{1}{2}$:

$$\dot{V}(x,y) \leq -\frac{1}{2}\|x\|^2. \tag{3.13}$$

The inequalities (3.9) and (3.13) prove that (3.5) is a Lyapunov function which satisfies (3.3) and provides the following estimate of the region of attraction:

$$\Omega_V = \{(x,y) : V(x,y) \leq c\}, \tag{3.14}$$

where c is the largest positive constant such that $\Omega_V \subset S_1$. Finally, in order to obtain (3.4), we note that

$$\frac{d}{dt} p_x(t-t_0, x_0, y_0) = f\left(p_x(t-t_0,x_0,y_0), y\right). \tag{3.15}$$

Since $f(\cdot)$ has continuous and bounded first partial derivatives on S, we can differentiate (3.15) to obtain

$$\dot{q}_{ij} = \sum_{k=1}^{n_1} \frac{\partial f_i}{\partial x_k}(p_x(t-t_0,x_0,y_0),y)\, q_{kj}\,, \quad 1 \leq i,j \leq n_1\,, \tag{3.16}$$

where q_{ij} denotes the partial derivative $\partial p_{x_i} / \partial x_{0j}$. From (3.16), the compactness of S_0 and the boundedness of $\partial f_i / \partial x_k$ on S_0, we obtain the following estimate, with $k_1 > 0$ a constant and $b > 0$ as defined in (3.6):

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega_V \Rightarrow \left| \frac{\partial p_{x_i}(\tau-t,x,y)}{\partial x_j} \right| \leq k_1 e^{b(\tau-t)}, \quad 1 \leq i,j \leq n_1. \tag{3.17}$$

From (3.5) we have

$$\frac{\partial V(x,y)}{\partial x_i} = 2\int_t^{t+T} \sum_{j=1}^{n_1} p_{x_j}(\tau - t, x, y) \frac{\partial p_{x_j}(\tau - t, x, y)}{\partial x_i} d\tau \,. \tag{3.18}$$

Combining (3.18) with (3.17) and (3.2), we obtain

$$\left| \frac{\partial V(x,y)}{\partial x_i} \right| \leq 2\int_t^{t+T} n_1 k k_1 e^{(-\alpha+b)(\tau-t)} d\tau \|x\| = k_2 \|x\| \,, \tag{3.19}$$

which implies (3.4) with $\alpha_2 = n_1 k_2$. □

Let us now assume that the system (2.1) is actually the reduced-order system obtained by neglecting the fast unmodeled dynamics of the composite system

$$\begin{aligned} \dot{z} &= f_z(z,\theta) + F_1(z)\xi + G_1(z)u \\ \mu\dot{\xi} &= f_\xi(z,\theta) + F_2(z)\xi + G_2(z)u \,, \end{aligned} \tag{3.20}$$

where $\xi \in \mathbb{R}^\nu$ is the state of the unmodeled dynamics, $\mu > 0$ is a small constant, and $F_2(z)$ is such that

$$\text{Re}\,\{\lambda\,(F_2(z))\} \leq -\sigma_1\,, \quad \forall z \in B_z\,, \tag{3.21}$$

for some constant $\sigma_1 > 0$. The change of variables

$$\xi = \ell(z,u,\theta) + \eta\,, \tag{3.22}$$

which exhibits the function

$$\ell(z,u,\theta) = -F_2^{-1}(z)[f_\xi(z,\theta) + G(z)u] \tag{3.23}$$

as the **quasi-steady-state** of ξ, and η as its **fast transient**, transforms (3.20) into the standard singular perturbation form

$$\begin{aligned} \dot{z} &= f(z,\theta) + G(z)u + F_1(z)\eta \\ \mu\dot{\eta} &= F_2(z)\eta - \mu\dot{\ell} \end{aligned} \tag{3.24}$$

with

$$f(z,\theta) = f_z(z,\theta) - F_1(z)F_2^{-1}(z)f_\xi(z,\theta) \tag{3.25}$$

$$G(z) = G_1(z) - F_1(z)F_2^{-1}(z)G_2(z)\,. \tag{3.26}$$

Assuming that

$$f(z,\theta) = f_0(z) + \sum_{i=1}^{p} \theta_i f_i(z)\,, \tag{3.27}$$

and neglecting the fast unmodeled dynamics of (3.24) by setting $\eta \equiv 0$, we obtain the reduced-order system (2.1). Furthermore, assuming that the conditions (2.4), (2.5) and (2.8) are satisfied, we can apply the procedure of Section 2 to design an adaptive scheme for the reduced-order system. Using the diffeomorphism $x = \phi(z)$ of Proposition 1 followed by the change of coordinates (2.6), and applying the control defined by (2.2), (2.12) together with the update law (2.16), we obtain the perturbed adaptive scheme

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + W(x,\hat{\theta})(\theta - \hat{\theta}) + R(x)\eta \\ \dot{\hat{\theta}} &= \Gamma W^{\mathrm{T}}(x,\hat{\theta}) P \hat{x} \\ \mu\dot{\eta} &= Q(x)\eta - \mu\dot{\hat{h}}(\hat{x},\hat{\theta},\eta,\theta), \end{aligned} \tag{3.28}$$

where

$$R(x) = \phi_z\left(\phi^{-1}(x)\right) F_1(\phi^{-1}(x))\,,\; Q(x) = F_2(\phi^{-1}(x)) \tag{3.29}$$

$$h(x,\hat{\theta},\theta) = \ell\left(\phi^{-1}(x), \alpha(\phi^{-1}(x)) + B(\phi^{-1}(x))v(x,\hat{\theta}),\theta\right) \tag{3.30}$$

$$\hat{h}(\hat{x},\hat{\theta},\theta) = h(\omega(\hat{x},\hat{\theta}),\hat{\theta},\theta) \tag{3.31}$$

$$\dot{\hat{h}}(\hat{x},\hat{\theta},\eta,\theta) = \hat{h}_{\hat{x}}\dot{\hat{x}} + \hat{h}_{\hat{\theta}}\dot{\hat{\theta}}\,, \tag{3.32}$$

where ω is such that $x = \omega(\hat{x},\hat{\theta})$. In order to investigate the stability of the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, $\eta = 0$ of (3.28), it is convenient to first apply the change of coordinates (2.36), which transforms (3.28) into

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + U_\theta(x,\varphi,\psi)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + R(x)\eta \\ \begin{pmatrix}\dot{\varphi}\\ \dot{\psi}\end{pmatrix} &= \Gamma_1 U_\theta^{\mathrm{T}}(x,\varphi,\psi) P\hat{x} \\ \mu\dot{\eta} &= Q(x)\eta - \mu\dot{\zeta}(\hat{x},\varphi,\psi,\eta,\theta) \end{aligned} \tag{3.33}$$

with the obvious definition for $\zeta(\hat{x},\varphi,\psi,\theta)$, and then investigate the stability of the equilibrium $\hat{x} = 0$, $\varphi = 0$, $\psi = 0$, $\eta = 0$ of (3.33).

From Corollary 1.1 we know that the adaptive scheme (2.39), which is the reduced-order system we obtain if we set $\eta \equiv 0$, $\mu = 0$ in (3.33), satisfies the hypotheses of Lemma 1, with $(\hat{x}, \varphi)$ and ψ playing the roles of x and y, respectively, in (3.1). Thus, from Lemma 1, there exists a Lyapunov function $V(\hat{x}, \varphi, \psi)$ that provides Ω_V as an estimate of the region of attraction of the equilibrium $\hat{x} = 0$, $\varphi = 0$, $\psi = 0$, and satisfies the inequalities

$$\dot{V}_{(2.39)}(\hat{x}, \varphi, \psi) \leq -\alpha_1 \left\| \begin{matrix} \hat{x} \\ \varphi \end{matrix} \right\|^2 \tag{3.34}$$

$$\left\| \frac{\partial V(\hat{x}, \varphi, \psi)}{\partial(\hat{x}, \varphi)} \right\| \leq \alpha_2 \left\| \begin{matrix} \hat{x} \\ \varphi \end{matrix} \right\| \tag{3.35}$$

for some positive constants α_1, α_2 and for all $(\hat{x}, \varphi, \psi) \in \Omega_V$.

The stability properties of the perturbed adaptive scheme (3.33) can then be investigated using the composite Lyapunov function

$$V_c(\hat{x}, \varphi, \psi, \eta) = c_1 V(\hat{x}, \varphi, \psi) + c_2 \eta^{\mathrm{T}} P_f(x) \eta \,, \tag{3.36}$$

where c_1 and c_2 are positive constants and $P_f(x)$ is the positive definite solution of

$$P_f(x) Q(x) + Q^{\mathrm{T}}(x) P_f(x) = -I \,. \tag{3.37}$$

The time derivative of V_c along the solutions of (3.33) is

$$\begin{aligned} \dot{V}_c \;=\; & c_1 \left[\dot{V}_{(2.39)}(\hat{x}, \varphi, \psi) + \frac{\partial V(\hat{x}, \varphi, \psi)}{\partial \hat{x}} R(x) \eta \right] \\ & + c_2 \left[-\frac{1}{\mu} \eta^{\mathrm{T}} \eta + \eta^{\mathrm{T}} \dot{P}_f(x, \varphi, \psi, \eta, \theta) \eta - 2 \eta^{\mathrm{T}} P_f(x) \dot{\zeta}(\hat{x}, \varphi, \psi, \eta, \theta) \right] . \end{aligned} \tag{3.38}$$

The function $\mathring{h}$, as defined in (3.32), vanishes on the linear variety

$$M_c = \left\{ (\hat{x}, \hat{\theta}, \eta) : \hat{x} = 0 \,,\; W_e(\theta - \hat{\theta}) = 0 \,,\; \eta = 0 \right\} . \tag{3.39}$$

Thus, from the definition of $\dot{\zeta}$ and (2.37), we have for all admissible ψ:

$$\dot{\zeta}(0, 0, \psi, 0, \theta) = 0 \,, \tag{3.40}$$

and, hence,

$$\dot{\zeta}_{\psi}(0, 0, \psi, 0, \theta) = 0 \,. \tag{3.41}$$

We conclude that $\dot{\zeta}$ is bounded by

$$\left\|\dot{\zeta}(\hat{x},\varphi,\psi,\eta,\theta)\right\| \leq \rho_1 \left\| \begin{matrix} \hat{x} \\ \varphi \end{matrix} \right\| + \rho_2\|\eta\|\,, \tag{3.42}$$

provided that for all $x \in B_x$, $\hat{\theta} \in B_\theta$, $\eta \in B_\eta$, $\theta \in B_\theta$ the following inequalities hold:

$$\left\|\zeta_{\hat{x}}(\hat{x},\varphi,\psi,\theta)\left[A\hat{x} + U_\theta(x,\varphi,\psi)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right] + \zeta_{\left(\substack{\varphi \\ \psi}\right)}(\hat{x},\varphi,\psi,\theta)\Gamma_1 U_\theta^{\mathrm{T}}(x,\varphi,\psi)P\hat{x}\right\| \leq \rho_1 \left\| \begin{matrix} \hat{x} \\ \varphi \end{matrix} \right\| \tag{3.43}$$

$$\|\zeta_{\hat{x}}(\hat{x},\varphi,\psi,\theta)R(x)\| \leq \rho_2\,. \tag{3.44}$$

Using (3.42) and, in addition, requiring that for all $x \in B_x$, $\hat{\theta} \in B_\theta$, $\eta \in B_\eta$, $\theta \in B_\theta$

$$\frac{2}{\alpha_1}\|P_{\mathrm{f}}(x)\|\rho_1 \leq c_1 \tag{3.45}$$

$$\frac{\alpha_2}{\alpha_1}\|R(x)\| \leq c_2 \tag{3.46}$$

$$2\|P_{\mathrm{f}}(x)\|\rho_2 + \|\dot{P}_{\mathrm{f}}(x,\varphi,\psi,\eta,\theta)\| \leq c_3\,, \tag{3.47}$$

we obtain from (3.34), (3.35) and (3.38):

$$\dot{V}_{\mathrm{c}} \leq -\alpha_1 \left[\left\| \begin{matrix} \hat{x} \\ \varphi \end{matrix} \right\| \; \|\eta\|\right]^{\mathrm{T}} \begin{bmatrix} c_1 & -c_1c_2 \\ -c_1c_2 & \left(\frac{1}{\mu} - c_3\right)c_2 \end{bmatrix} \begin{bmatrix} \left\| \begin{matrix} \hat{x} \\ \varphi \end{matrix} \right\| \\ \|\eta\| \end{bmatrix}. \tag{3.48}$$

From (3.48) we see that for every μ satisfying

$$0 < \mu < \mu^* = \frac{1}{c_1c_2 + c_3}\,, \tag{3.49}$$

the matrix in (3.48) is positive definite, and hence $\dot{V}_{\mathrm{c}}$ is negative semidefinite. This implies that for every μ satisfying (3.49) the equilibrium $\hat{x} = 0$, $\varphi = 0$, $\psi = 0$, $\eta = 0$ of (3.33), or, equivalently, the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, $\eta = 0$ of (3.28), is **stable**. Furthermore, a subset of its region of attraction is the set

$$\Omega_{\mathrm{c}} = \{(x,\hat{\theta},\eta) : V_{\mathrm{c}}(\hat{x},\varphi,\psi,\eta) \leq c\}\,, \tag{3.50}$$

with c the largest constant such that

$$\Omega_{\mathrm{c}} \subset \Omega_1 \times B_\eta\,, \quad \Omega_1 = \{(x,\hat{\theta}) : (\hat{x},\varphi,\psi) \in \Omega_V\} \cap B_x \times B_\theta\,. \tag{3.51}$$

From LaSalle's theorem we know that the state of (3.33) converges to the largest invariant set contained in the set where $\dot{V}_c = 0$, that is, in the set $\{(\hat{x}, \varphi, \psi, \eta) : \hat{x} = 0, \varphi = 0, \eta = 0\}$, which is itself an invariant set of (3.33) by virtue of (2.45). Hence, the linear variety (3.39) is an **attractive equilibrium manifold** of (3.28). Furthermore, this equilibrium manifold is **exponentially attractive** for small enough μ. To prove this claim, we linearize (3.33) around $\hat{x} = 0$, $\varphi = 0$, $\psi = \psi^0$, $\eta = 0$, using (3.41) (cf. (2.47)–(2.50)):

$$\begin{aligned}
\delta\dot{\hat{x}} &= A\delta\hat{x} + \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \delta\varphi + R_e\delta\eta \\
\delta\dot{\varphi} &= \Gamma_{11}[U_1^{\mathrm{T}} \;\; U_2^{\mathrm{T}}]P\delta\hat{x} \\
\delta\dot{\psi} &= \Gamma_{12}^{\mathrm{T}}[U_1^{\mathrm{T}} \;\; U_2^{\mathrm{T}}]P\delta\hat{x} \\
\mu\delta\dot{\eta} &= Q_e\delta\eta - \mu[Z_1\delta\hat{x} + Z_2\delta\varphi + Z_3\delta\eta],
\end{aligned} \tag{3.52}$$

where

$$R_e = R(x^e), \; Q_e = Q(x^e) \tag{3.53}$$

$$Z_1 = \zeta_{\hat{x}}(0,0,\psi^0,\theta)A + \zeta_{\binom{\varphi}{\psi}}(0,0,\psi^0,\theta)\Gamma_1 U_e^{\mathrm{T}} P \tag{3.54}$$

$$Z_2 = \zeta_{\hat{x}}(0,0,\psi^0,\theta)\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{3.55}$$

$$Z_3 = \zeta_{\hat{x}}(0,0,\psi^0,\theta)R_e. \tag{3.56}$$

Since Q_e is a Hurwitz matrix, by Theorem 3.1 of [7] the first $n+p$ eigenvalues of the system matrix of (3.52) are within $O(\mu)$ of the eigenvalues of the reduced-order-system matrix A_ℓ (cf. (2.54)), while the remaining ν are open-left-half-plane eigenvalues that are within $O(1)$ of the eigenvalues of $\frac{1}{\mu}Q_e$. This implies that there exists a $\mu^{**} > 0$ such that for all $\mu \in (0, \mu^{**})$ the $n+r+\nu$ eigenvalues corresponding to the $(\delta\hat{x}, \delta\varphi, \delta\eta)$-subsystem of (3.53) have negative real parts, while the remaining $p-r$ eigenvalues are within $O(\mu)$ of the origin $s = 0$. However, in this case we can actually show that these eigenvalues are at the origin, since

$$\det \begin{bmatrix} sI_n - A & \begin{matrix} -U_1 \\ -U_2 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & R_e \\ -\Gamma_{11}[U_1^{\mathrm{T}} \; U_2^{\mathrm{T}}]P & sI_r & 0 & 0 \\ -\Gamma_{12}^{\mathrm{T}}[U_1^{\mathrm{T}} \; U_2^{\mathrm{T}}]P & 0 & sI_{p-r} & 0 \\ Z_1 & Z_2 & 0 & sI_\nu - \frac{1}{\mu}Q_e + Z_3 \end{bmatrix} =$$

$$= s^{p-r} \det \begin{bmatrix} sI_n - A & \begin{matrix} -U_1 \\ -U_2 \end{matrix} & R_e \\ -\Gamma_{11}[U_1^{\mathrm{T}}\ U_2^{\mathrm{T}}]P & sI_r & 0 \\ Z_1 & Z_2 & sI_\nu - \frac{1}{\mu}Q_e + Z_3 \end{bmatrix} . \tag{3.57}$$

Thus, the exponential attractivity of the manifold M_c is established for every μ satisfying

$$0 < \mu < \mu_0 = \min\{\mu^*, \mu^{**}\}. \tag{3.58}$$

We summarize these results in the following:

Theorem 2. *For every μ satisfying (3.58), the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, $\eta = 0$ of the perturbed adaptive scheme (3.28) is* stable *and a subset of its region of attraction is the set Ω_c defined in (3.50). Furthermore, its state $(\hat{x}(t), \hat{\theta}(t), \eta(t))$* converges exponentially to the linear variety *M_c defined in (3.39) for all $(\hat{x}(0), \hat{\theta}(0), \eta(0)) \in \Omega_c$.* □

Corollary 2.1. *If $r = p$, that is, if the rank of W_e is equal to the number of unknown parameters, the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, $\eta = 0$ of the perturbed adaptive scheme (3.28) is* exponentially stable *for every μ satisfying (3.58), with a region of attraction that contains the set Ω_c.* □

4 Stability with Slowly-Time-Varying Parameters

The result established in Corollary 2.1 can be used to show that in the case where $r = p$, the adaptive scheme of [3,4] is robust not only to unmodeled dynamics, but also to slow time variations of the unknown parameters.

Let us consider the case where the unknown parameters θ are not constant, as was assumed in the previous sections, but evolve slowly with time according to the differential equation

$$\dot{\theta}(t) = \varepsilon F(\theta, t), \tag{4.1}$$

where $F(\cdot, \cdot)$ is continuous and ε is a small positive constant. We now make the following assumptions:

Assumption 1. *There exist compact sets S_θ^0, S_θ^1 with $S_\theta^1 \subset S_\theta^0 \subset B_\theta$ such that for every $t_0 \geq 0$, the solutions of (4.1) starting from any $\theta(0) \in S_\theta^1$, remain in S_θ^0 for all $t \geq t_0$.*

Assumption 2. *For every $\theta \in S_\theta^0$, the system (2.25) has a twice continuously differentiable isolated root, that is, the mapping $\theta \to x^e$ is C^2 on S_θ^0.*

Assumption 3. *For every $\theta \in S_\theta^0$, the corresponding $W(x^e) = W_e$ has rank p.*

The main result of this section is:

Theorem 3. *Under Assumptions 1–3 and for every μ satisfying (3.58), the solutions of the perturbed adaptive scheme with slowly-varying parameters*

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + W(x,\hat{\theta})(\theta - \hat{\theta}) + R(x)\eta \\ \dot{\hat{\theta}} &= \Gamma W^{\mathrm{T}}(x,\hat{\theta})P\hat{x} \\ \mu\dot{\eta} &= Q(x)\eta - \mu\dot{\hat{h}}(\hat{x},\hat{\theta},\eta,\theta) \\ \dot{\theta} &= \varepsilon F(\theta,t)\,, \end{aligned} \tag{4.2}$$

starting from any $(x(0),\hat{\theta}(0),\eta(0),\theta(0)) \in \Omega_c \times S_\theta^1$, remain bounded. Furthermore, for any $\delta > 0$ there exists an $\varepsilon_0(\delta) > 0$ such that for all $\varepsilon \in [0,\varepsilon_0]$:

$$\limsup_{t\to\infty} \left\| \begin{matrix} \hat{x}(t) \\ \hat{\theta}(t) - \theta(t) \\ \eta(t) \end{matrix} \right\| \leq \delta\,. \tag{4.3}$$

Proof. From Corollary 1.2, we know that the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, $\eta = 0$ of the system

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + W(x,\hat{\theta})(\theta - \hat{\theta}) \\ \dot{\hat{\theta}} &= \Gamma W^{\mathrm{T}}(x,\hat{\theta})P\hat{x} \\ \mu\dot{\eta} &= Q(x)\eta - \mu\dot{\hat{h}}(\hat{x},\hat{\theta},\eta,\theta) \end{aligned} \tag{4.4}$$

is exponentially stable for every fixed $\theta \in B_\theta$. Since S_θ^0 is a compact subset of B_θ, the equilibrium $\hat{x} = 0$, $\hat{\theta} = \theta$, $\eta = 0$ of (4.4) is exponentially stable uniformly in $\theta \in S_\theta^0$. It is then straightforward to prove that there exist a constant k_1 and a Lyapunov function $V(\hat{x},\hat{\theta},\eta,\theta)$ for (4.4) with the following properties for all $\theta \in S_\theta^0$ and for all $(\hat{x},\hat{\theta},\eta)$ such that $\left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\| \leq k_1$:

$$\alpha_1 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\|^2 \leq V(\hat{x},\hat{\theta},\eta,\theta) \leq \alpha_2 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\|^2 \tag{4.5}$$

$$\dot{V}_{(4.4)}(\hat{x},\hat{\theta},\eta,\theta) \leq -\alpha_3 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\|^2 \tag{4.6}$$

$$\left\| \frac{\partial V}{\partial(\hat{x},\hat{\theta},\eta)}(\hat{x},\hat{\theta},\eta,\theta) \right\| \leq \alpha_4 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\| \tag{4.7}$$

$$\left\| \frac{\partial V}{\partial\theta}(\hat{x},\hat{\theta},\eta,\theta) \right\| \leq \alpha_5 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\| \tag{4.8}$$

for some positive constants α_i, $1 \leq i \leq 5$. The proof of (4.5)–(4.7) is identical to the proof of Theorem 56.1 in [8], while (4.8) can be proven in the same way as (4.7), using the fact that the partial derivative with respect to θ of the right-hand side of (4.4) is bounded for all $\theta \in S_\theta^0$ and $(\hat{x},\hat{\theta},\eta)$ such that $\left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\| \leq k_1$. The time derivative of $V(\hat{x},\hat{\theta},\eta,\theta)$ along the solutions of (4.2) is

$$\dot{V}_{(4.2)}(\hat{x},\hat{\theta},\eta,\theta) = \dot{V}_{(4.4)}(\hat{x},\hat{\theta},\eta,\theta) + \frac{\partial V}{\partial\theta}(\hat{x},\hat{\theta},\eta,\theta)\varepsilon F(\theta,t)\,. \tag{4.9}$$

Using (4.6), (4.8) and Assumption 1, (4.9) leads to

$$\dot{V}_{(4.2)}(\hat{x},\hat{\theta},\eta,\theta) \leq -\alpha_3 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\|^2 + \varepsilon k_0 \alpha_5 \left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\| . \tag{4.10}$$

Thus

$$\left\| \begin{matrix} \hat{x} \\ \hat{\theta}-\theta \\ \eta \end{matrix} \right\| \geq \frac{\varepsilon\alpha_5 k_0}{\alpha_3} \Rightarrow \dot{V}_{(4.2)} \leq 0\,, \tag{4.11}$$

which, combined with (4.5), proves the boundedness of $(\hat{x}(t),\hat{\theta}(t),\eta(t))$, and, moreover, shows that

$$\limsup_{t\to\infty} \left\| \begin{matrix} \hat{x}(t) \\ \hat{\theta}(t)-\theta(t) \\ \eta(t) \end{matrix} \right\| \leq \varepsilon \frac{\alpha_5 k_0}{\alpha_3} \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}} . \tag{4.12}$$

Hence, (4.3) is proven with $\varepsilon_0(\delta) = \dfrac{\alpha_3}{\alpha_5 k_0}\left(\dfrac{\alpha_1}{\alpha_2}\right)^{\frac{1}{2}}\delta.$ □

5 Concluding Remarks

We have shown that when the vector fields multiplying the unknown parameters do not vanish at the origin, the adaptive scheme of [3,4] achieves more than stability and regulation: the parameter estimate errors also converge to a linear subspace. Moreover, the state of the closed-loop adaptive system converges exponentially to the resulting linear variety.

Even though the robustness analysis becomes now much more complicated than in the case where the above vector fields vanish at the origin, these additional properties enhance the robustness of the adaptive scheme. This is obvious in the special case where the linear variety collapses to a single point: the closed-loop adaptive system has then an exponentially stable equilibrium, and is thus inherently robust not only to unmodeled dynamics, but also to small bounded disturbances and, as shown in Section 4, to slow time variations of the unknown parameters.

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