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**CONTROLLABILITY,
POLE PLACEMENT AND
STABILIZABILITY
IN LARGE SCALE
COMPOSITE SYSTEMS**

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CONTROLLABILITY, POLE PLACEMENT AND STABILIZABILITY
IN LARGE SCALE COMPOSITE SYSTEMS[†]

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Abstract

This work gives some preliminary results related to research on Large Scale Composite (LSC) systems. An interconnected system model is considered. Each subsystem is assumed to be linear, time invariant, and have both local inputs and interaction inputs from other subsystems. The main question is: "What can be done without lumping all subsystems into one large system model, and what can be done by each subsystem using local observations and local control inputs alone?" Results on controllability, pole placement and stabilizability are given for the general LSC system and the "chain" structure. Both state-space and transfer function concepts are used. LSC systems may model various physical interconnected structures, e.g. power systems, economic systems etc. and applications of the results to these areas are presently being considered.

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I. Introduction

The control of large scale systems is receiving increasing attention in research lately. Some results related to hierarchical control concepts [1], controllability, stabilizability and pole placement in the decentralized model [2,3,4], problems of restricted information [5,6] and stability of interconnected systems [7,8] have been obtained. In this paper we shall investigate a structure named the "Large Scale Composite (LSC) Systems" model, which may be used for large scale systems composed of interconnections of many subsystems. Some related work will be surveyed in Part II. In Part III we shall develop a canonic form suitable for showing any subsystem, in Part IV implications of this form in the areas of pole placement and local stabilizability will be investigated. We shall then turn to overall characteristics and give two necessary theorems on controllability of LSC systems. It will be seen that the identification of sets of subsystems forming "levels" is useful for analysis purposes. We will then investigate in Part VI a special multilevel structure, namely a "chain" of tandem connected subsystems. Necessary and sufficient conditions for controllability of chains with no mutual eigenvalues will be given, and in Part VII local stabilizability and local pole placement in chains will be investigated.

II. The LSC Systems Model

Let the system S be composed of m subsystems s_1, s_2, \dots, s_m . The j^{th} subsystem will be modeled by,

$$s_j: \quad \begin{aligned} \dot{x}^j &= A^j x^j + \sum_{i=1}^m B_i^j u_i^j \\ y^j &= C^j x^j + \sum_{i=1}^m D_i^j u_i^j \end{aligned} \quad (1)$$

where u_i^j , for $i \neq j$, are the interaction inputs, i.e. inputs from other subsystems, and are defined as,

$$u_i^j = H_{ij} y^i \quad \text{for } i \neq j \quad (2)$$

x^j , u_i^j , y^j are n_j , r_{ij} and q_j vectors respectively, and the matrices are real and of proper dimension u_j^j is the input of s_j which is not connected to any other subsystem; it shall also be called "the locally available input" (see Fig. 1).

The study of the LSC systems model is motivated by the fact that a large number of interconnected systems exist in the physical world, e.g. power systems, economic systems, neurological systems [7,9,10,11], however a generalized theory of their control is lacking.

Preliminary work on controllability and observability of composite systems was first reported in a paper by E. G. Gilbert [12]. This was later extended with papers by Chen and Desoer [13,14], Chen [15] and Panda [16] and was also summarized in a book by Chen [17]. In these, two simple system interconnection configurations were considered. These were the parallel and tandem arrangements as seen in Figures 2(a) and (b). Given that the two subsystems are controllable and observable, conditions

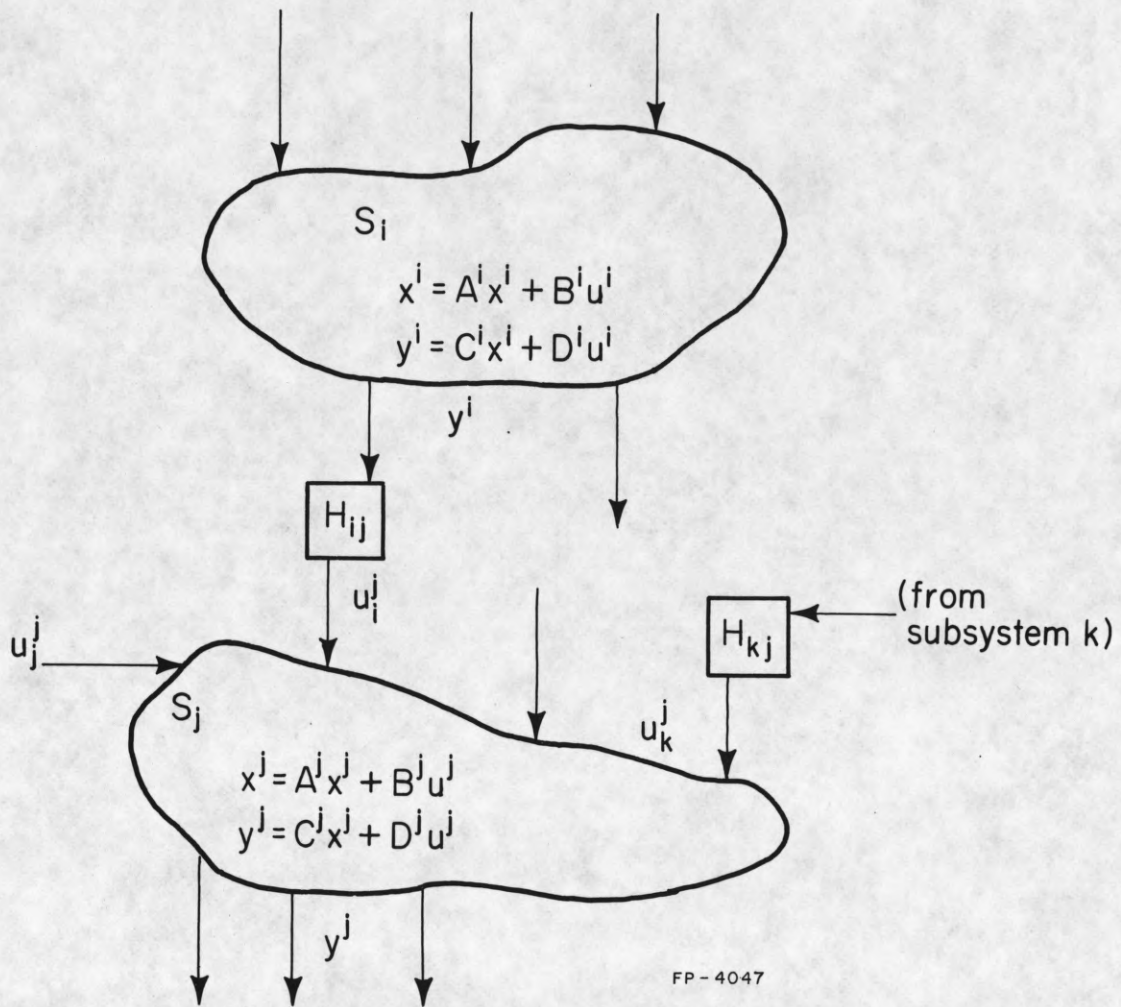
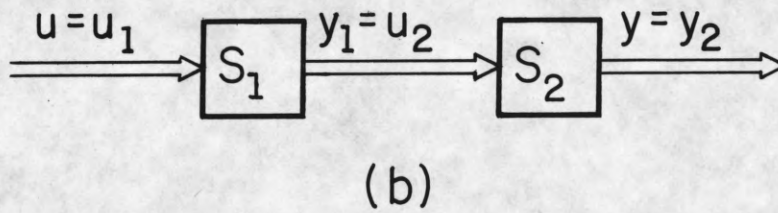
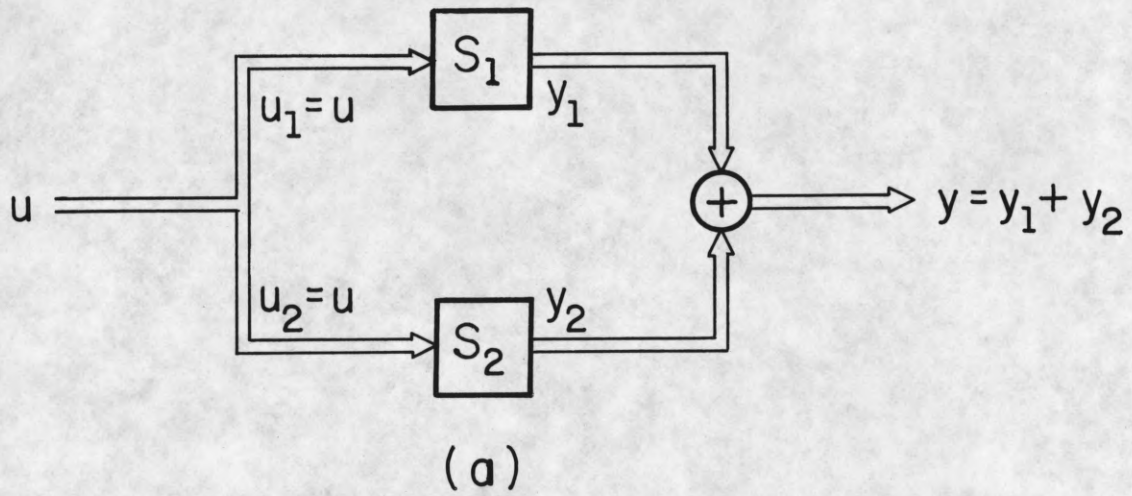


Figure 1. The LSC system configuration. [For simplicity $B^j \triangleq (B_1^j : B_2^j \dots : B_m^j)$, $u^{j'} \triangleq (u_2^{j'} : u_2^{j'} \dots u_m^{j'})$ and $D^j \triangleq (D_1^j : D_2^j \dots : D_m^j)$.]



FP-4044

Figure 2. The parallel and tandem arrangements for two systems.

under which the composite system would be, were found. We shall be extending these results to chains of m subsystems, each one of which will also have local inputs.

Apart from some hierarchical optimization considerations [1], research has been done so far on only one aspect of LSC systems, and that is composite stability. Some results have been obtained by Bailey [18], Thompson [19], Grujic and Siljak [7], Siljak [20,21] and Michel and Porter [8]. In this paper we shall not be dealing with the question of stability, apart from stabilizability of the LTI model we are using. However, it is interesting to note that in the papers above on stability, the subsystems themselves and the interconnections considered have been of a much more general nature than any considered in research of other problems related to composite systems.

We should also point out that in research on large scale systems some interesting results have been found for the so-called Decentralized Systems structure [2,3,4]. The fact that these are not directly applicable to LSC systems will become apparent in the following where we shall be introducing some concepts, tools and results related to questions of their controllability, stabilizability and pole placement.

III. Controllability of the Subsystem

For notational simplicity we shall drop the index j in (1) and define $B_2 = (B_1^j, B_2^j, \dots, B_{j-1}^j, B_{j+1}^j, \dots, B_m^j)$ and $B_1 = B_j^j$. D_2 , D_1 , u_2 and u_1 are formed similarly to give

$$\begin{aligned}\dot{x} &= Ax + B_1 u_1 + B_2 u_2 \\ y &= Cx + D_1 u_1 + D_2 u_2\end{aligned}\tag{3}$$

as the equations for any one subsystem. Here u_1 shows the locally available inputs, u_2 the interaction inputs, i.e. all inputs from other subsystems. Some further notation will be necessary in developing a canonical form related to the problem of controllability using only u_1 (local controllability).

β = range space of B

$$C_1 = \{A|\beta_1\} = \beta_1 + A\beta_1 + A^2\beta_1 + \dots + A^{n-1}\beta_1$$

$$C_2 = \{A|\beta_2\}$$

$$C_3 = C_1 \cap C_2$$

$$C_1 = C_1^* \oplus C_3 \quad C_1^* \perp C_3$$

$$C_2 = C_2^* \oplus C_3 \quad C_2^* \perp C_3$$

$$Q_1 = [B_1 \ AB_1 \ A^2 B_1 \ \dots \ A^{n-1} B_1]$$

$$Q_2 = [B_2 \ AB_2 \ A^2 B_2 \ \dots \ A^{n-1} B_2].$$

C_1 is the locally controllable subspace, C_2 is the subspace controllable by interaction inputs, C_3 is the subspace controllable by both.

Note that, \mathcal{C}_1^* and \mathcal{C}_2^* represent the subspaces controllable by local inputs alone, and interaction inputs alone, respectively. Since we have not so far assumed complete controllability, we shall also define \mathcal{C}_4 , the non-controllable subspace, so that

$$\mathcal{C}_1^* \oplus \mathcal{C}_2^* \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 = \mathbb{R}^n.$$

The dimensions of the subspaces are denoted as $k_i = \dim \mathcal{C}_i$, $i = 1, \dots, 4$; $k_i^* = \dim \mathcal{C}_i^*$ $i = 1, 2$ are equal to the ranks of their controllability matrices, if defined. We are assuming that the reader is familiar with terminology and some of the results related to controllability subspaces. (General references are [22,23].) Therefore, the following fact will be given without proof.

Lemma 1: The controllable subspaces \mathcal{C}_1 and \mathcal{C}_2 are A invariant.

Lemma 2: \mathcal{C}_3 is A invariant.

Proof: Let $e \in \mathcal{C}_3$. Since $\mathcal{C}_3 \subset \mathcal{C}_1$, $e \in \mathcal{C}_1$. From A invariance of \mathcal{C}_1 , $Ae \in \mathcal{C}_1$. On the other hand, since $\mathcal{C}_3 \subset \mathcal{C}_2$ $e \in \mathcal{C}_2$. From A invariance of \mathcal{C}_2 $Ae \in \mathcal{C}_2$. Therefore $Ae \in \mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C}_3$. Therefore \mathcal{C}_3 is A invariant.

We shall now give the "controllability-decomposition canonic form" for a subsystem.

Theorem 1: There exists a linear transformation $x(t) = Tx'(t)$ such that (3) is transformed into the controllability decomposition canonic form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \\ B_{31} \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ B_{22} \\ B_{32} \\ 0 \end{bmatrix} u_2 \quad (4)$$

where

$$[A_{11}, B_{11}], [A_{22}, B_{22}], \left[\begin{bmatrix} A_{11} & 0 \\ A_{31} & A_{33} \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{31} \end{bmatrix} \right] \text{ and } \left[\begin{bmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{bmatrix}, \begin{bmatrix} B_{22} \\ B_{32} \end{bmatrix} \right]$$

are controllable pairs.

Proof: We shall prove the theorem by actually constructing a T matrix to satisfy the transformation. Define T as,

$$T = (T_1 \ T_2 \ T_3 \ T_4). \quad (5)$$

Let T_1 's columns be the k_1^* basis vectors of \mathcal{C}_1^* , T_2 's columns be the k_2^* basis vectors of \mathcal{C}_2^* and T_3 's columns be the k_3 basis vectors of \mathcal{C}_3 . Form T_4 with any choice of k_4 columns linearly independent of the rest and among themselves. T^{-1} exists since all columns of T are independent. Partition T^{-1} to be

$$T^{-1} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \quad (6)$$

such that

$$T^{-1}T = \begin{bmatrix} P_1T_1 & P_1T_2 & P_1T_3 & P_1T_4 \\ P_2T_1 & P_2T_2 & P_2T_3 & P_2T_4 \\ P_3T_1 & P_3T_2 & P_3T_3 & P_3T_4 \\ P_4T_1 & P_4T_2 & P_4T_3 & P_4T_4 \end{bmatrix} = \begin{bmatrix} I_{k_1^*} & & & \\ & I_{k_2^*} & & 0 \\ & & I_{k_3} & \\ 0 & & & I_{k_4} \end{bmatrix} \quad (7)$$

Since $P_2T_1 = 0$ and T_1 spans \mathcal{C}_1^* , $P_2e = 0$ for every $e \in \mathcal{C}_1^*$. Since $P_2T_3 = 0$ and T_3 spans \mathcal{C}_3 ; $P_2e = 0$ for every $e \in \mathcal{C}_3$. Therefore $P_2e = 0$ for every $e \in \mathcal{C}_1$. Columns of T_1 are in \mathcal{C}_1 and owing to A invariance properties AT_1 will also be in \mathcal{C}_1 . Therefore $P_2(AT_1) = 0$.

Similar reasoning will give,

$$T^{-1}AT = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \quad T^{-1}B_1 = \begin{bmatrix} B_{11} \\ 0 \\ B_{31} \\ 0 \end{bmatrix}, \quad T^{-1}B_2 = \begin{bmatrix} 0 \\ B_{22} \\ B_{32} \\ 0 \end{bmatrix} \quad (8)$$

Controllability properties may easily be shown.

We shall now give an algorithm for picking the necessary basis vectors. Basis vectors for \mathcal{C}_1 and \mathcal{C}_2 can be found easily enough, however we have to make sure that the k_3 vectors in \mathcal{C}_1 associated with \mathcal{C}_3 are the same as k_3 vectors in \mathcal{C}_2 associated with \mathcal{C}_3 .

Step 1. Using the Gram-Schmidt procedure find a set of orthonormal basis vectors from k_1 independent columns of Q_1 .

Step 2. Continue the Gram-Schmidt procedure with k_2 independent columns from Q_2 . The basis vectors found in this step alone will span \mathcal{C}_2^* . This assertion can be proved in the following way: The basis vectors found in Step 1 will span \mathcal{C}_1 . The basis vectors found by the completion of

Step 2 will span $C_1 + C_2$. But $C_1 + C_2 = C_1 \oplus C_2^*$. Therefore the basis vectors found in Step 2 alone will span C_2^* .

Step 3. Starting with the (now available) k_2^* basis vectors that span C_2^* go through the Gram-Schmidt procedure with the k_2 independent columns from Q_2 . The new basis vectors will span C_3 .

Step 4. Starting with the (now available) k_3 basis vectors that span C_3 to through the Gram-Schmidt procedure with the k_1 independent columns from Q_1 . The new basis vectors will span C_1^* .

Step 5. Generate a vector orthogonal to all those that have been found so far. Repeat this step until T_4 is filled.

IV. Implications of the Controllability Decomposition Canonic Form

The controllability decomposition canonic form will be of great help in analyzing LSC systems. Considering each subsystem as controllable we shall use as the model of each subsystem[†]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \\ B_{31} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ B_{22} \\ B_{32} \end{bmatrix} u_2. \quad (9)$$

[†]We have to point out that Aoki has previously used this model in an entirely different context [2,3]. He has considered the above as the complete model for a large scale system with two "control agents". An extension of the decomposition for m control agents would be fairly complicated requiring all pairwise controllable subspaces, triply controllable subspaces etc. Note also that Aoki's results are not directly applicable since the inputs u_2 are not locally available in our model.

Consider the local linear state feedback:

$$u_1 = (K_1 \ K_2 \ K_3) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (10)$$

which will give

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} + B_{11}K_1 & B_{11}K_2 & B_{11}K_3 \\ 0 & A_{22} & 0 \\ A_{31} + B_{31}K_1 & A_{32} + B_{31}K_2 & A_{33} + B_{31}K_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{22} \\ B_{32} \end{bmatrix} u_2. \quad (11)$$

Denoting the new A matrix as \bar{A} , we observe that

$$\det(sI - \bar{A}) = \det(sI - A_{22}) \det \begin{bmatrix} sI - A_{11} - B_{11}K_1 & -B_{11}K_3 \\ -A_{31} - B_{31}K_1 & sI - A_{33} - B_{31}K_3 \end{bmatrix} \quad (12)$$

(11) and (12) will give,

Theorem 2: Let Eq. (9) show any subsystem in a LSC system S.

- a) Eigenvalues related to A_{11} can be arbitrarily place by local linear state feedback, without any constraints because of interconnections.
- b) Eigenvalues related to A_{22} cannot be shifted by local feedback.
- c) If there is no feedback through the interconnections the LSC system is locally stabilizable by local linear state feedback if and only if eigenvalues of A_{22} are in the left half complex plane (for every subsystem).

This last fact warrants some further comments. If no feedback exists through the interconnections all subsystems preceding s_j can be lumped together as $\dot{x} = Ax + Bu$, $Y = Cx + Du$ so that $u_2 = Hy$. After local state feedback $U = Kx$ the equations for the composite system are,

$$\begin{bmatrix} \dot{x} \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A+BK & & & 0 \\ & \dots & & \\ & 0 & & \\ & B_{22}H(C+DK) & & \bar{A} \\ & B_{32}H(C+DK) & & \end{bmatrix} \begin{bmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (13)$$

i.e. eigenvalues of A_{22} have not been changed at all. Also note that eigenvalues related to A_{33} , can be placed at will by local feedback.

V. The Effect of Predecessors in LSC Systems

Having looked at the subsystems at the local level, we can now address the question of overall characteristics in more detail. For this, some definitions and some new concepts will be used.

Definition 1: Let s_1, s_2, \dots, s_m be the component subsystems of a LSC system S . s_i is a predecessor of s_j if the output of s_i is used as part of the input to s_j . In other words, s_i is a predecessor of s_j if $H_{ij} \neq 0$.

In this section we shall consider the restricted case where $D_i^j \equiv 0$ for all i, j . The concepts may be extended to the nonzero case but it is doubtful whether this will bring any more insight.

Consider s_j 's interaction inputs

$$B_2 u_2 = \sum_{\substack{i=1 \\ i \neq j}}^m B_{iH_{ij}}^j y^i = \sum_{\substack{i=1 \\ i \neq j}}^m B_{iH_{ij}}^j C_{x^i}^i. \quad (14)$$

Using only the existing predecessors that enter the sum in (14), form the matrix

$$C^{P(j)} = \text{block diag } (H_{1j} C^1 \ H_{2j} C^2 \ \dots \ H_{mj} C^m) \quad (15)$$

and the related predecessor states vector,

$$x^{P(j)} = \begin{bmatrix} 1 \\ x^1 \\ x^2 \\ \cdot \\ \cdot \\ x^m \end{bmatrix} \quad (16)$$

where $P(j)$ denotes "predecessors of s_j ." Note that this could be used in Eq. (13). It can be seen now, that a "predecessor subsystem's observability" can be defined. If $A^{P(j)}$ is the block diagonal matrix formed from the A matrices of predecessor subsystems, the predecessor subsystem's observable subspace is spanned by independent rows of

$$\begin{bmatrix} C^{P(j)} \\ C^{P(j)} A^{P(j)} \\ C^{P(j)} [A^{P(j)}]^2 \\ \cdot \\ \cdot \\ C^{P(j)} [A^{P(j)}]^{n(j)-1} \end{bmatrix} \quad (17)$$

where $n(j)$ is the total number of states in s_j 's predecessors (see Fig. 3).

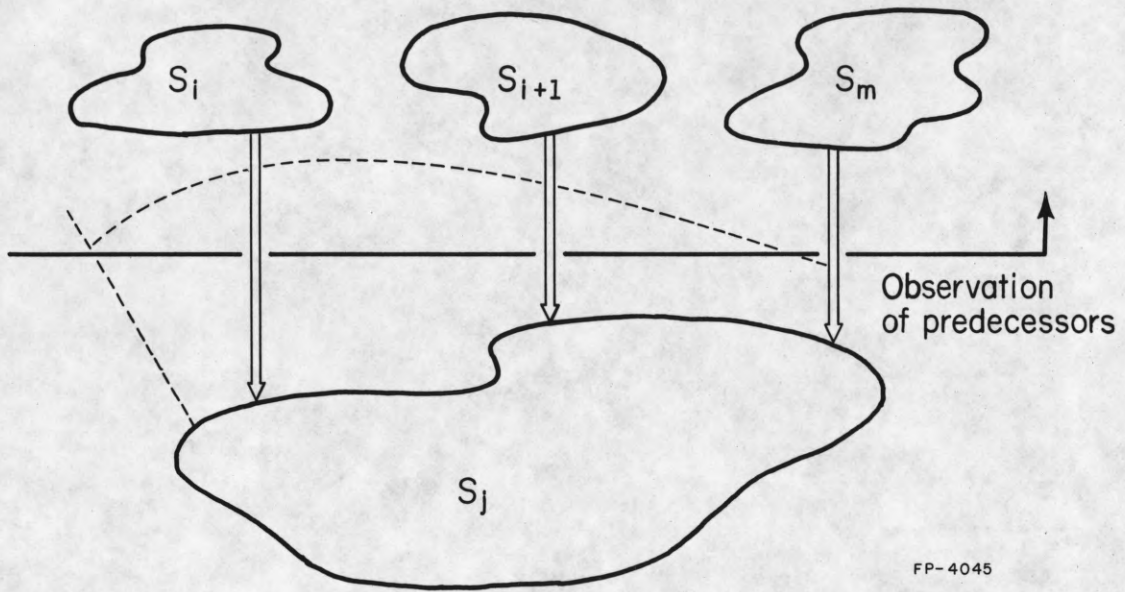


Figure 3. Subsystem j and its predecessors.

If H_{ij} 's are unity matrices for all existing connected system pairs, it can be shown that observability of each subsystem implies predecessor subsystem observability. On the other hand if $C_2 \neq \phi$, controllability of s_j is going to depend on whether its predecessor subsystem observable subspace is controllable by its predecessors. Before formalizing we shall repeat some more related concepts.

Definition 2: The number of input degrees of freedom δ of a system is the minimum number of inputs required for state controllability of that system.

Referring to (9) and considering δ only for interaction inputs, we can state:

Lemma 3: Min δ_2 is the maximum number of Jordan blocks associated with any eigenvalue of A_{22} ; max δ_2 is equal to the rank of B_{22} .

Proof: According to a theorem by Kalman [24] a necessary and sufficient condition for controllability, assuming the Jordan form, is linear independence of rows in B_{22} related to the last rows of Jordan blocks of the same eigenvalue. Since we need at least the same number of independent columns for linear independence, this will give the lower bound. For the upper bound consider the controllability matrix. The linearly dependent columns of B_{22} will not span any subspace that cannot also be spanned by using the linearly independent columns of B_{22} .

In LSC systems we are interested in subsystems controlling other subsystems, rather than any particular input. Therefore we shall now address the problem of identifying the class of necessary predecessor sets. Two methods for this are given in Appendix I. Note that because of the interconnection structure a mode in s_j may have to be controlled by more

than one predecessor. Also note that the minimum number of predecessors controlling \mathcal{C}_2 might be turning part of \mathcal{C}_3 over to the local inputs.

Denote the class of necessary predecessors by L . Each $\ell \in L$ shows a set of numbers identifying the predecessors.

Theorem 3: A necessary condition for controllability of a subsystem is that for at least one $\ell \in L$, its predecessor subsystems observable subspace is within the controllable subspaces of its predecessors.

Example 1: Let s_i : $\dot{x}^i = A^i x^i + B^i u^i$
 $y^i = x^i$ for $i = 1, 2, 3, 4$ (18)

$$A^1 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad A^3 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad A^4 = \begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix} \quad (19)$$

$$B^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad B^i = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for } i = 2, 4 \quad B^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let these be the predecessor subsystems of s_j , s_j being the system given in Appendix I. It was found there that $L = \{(1,3), (1,4), (2,3)\}$. Let the H_{ij} be unity matrices. It is seen that only for $\ell = (1,4)$ is the predecessor observable subspace within the controllable subspaces of the predecessors.

We shall now show by another example that Theorem 3 does not also give a sufficient condition.

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We shall now show by another example that Theorem 3 does not also give a sufficient condition.

Example 2: Consider the three interconnected subsystems, (see Fig. 4)

$$\begin{aligned}\dot{x}^1 &= A_1^1 x^1 + B_1^1 u_1^1 + B_2^1 u_2^1 + B_4^1 u_4^1 \\ \dot{x}^2 &= A_2^2 x^2 + B_2^2 u_2^2 + B_3^2 u_3^2 \\ \dot{x}^3 &= A_3^3 x^3 + B_3^3 u_3^3 + B_1^3 u_1^1\end{aligned}\quad (20)$$

where

$$A^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \quad A^3 = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}\quad (21)$$

$$B_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad B_2^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad B_4^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad B_2^2 = B_3^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_3^2 = B_1^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y^1 = x_2^1 \quad y^2 = x_2^2 \quad y^3 = x_2^3 \quad H_{21} = H_{32} = H_{13} = 1$$

and u_4^1 shows a connection with other subsystems. Each subsystem is completely state controllable. Taken in pairs they would also be controllable. However since the controllability-observability relations form a loop the composite system is not controllable.

We have used the term loop in a literal sense though an exact definition using graph theoretic concepts is also possible. This is done in Appendix II to which we shall have more reason to refer in the next section. The example given makes the following theorem, with important implications in LSC systems, obvious.

Theorem 4: A necessary condition for controllability of a LSC system is the nonexistence of a loop of interaction input controllable subspace-

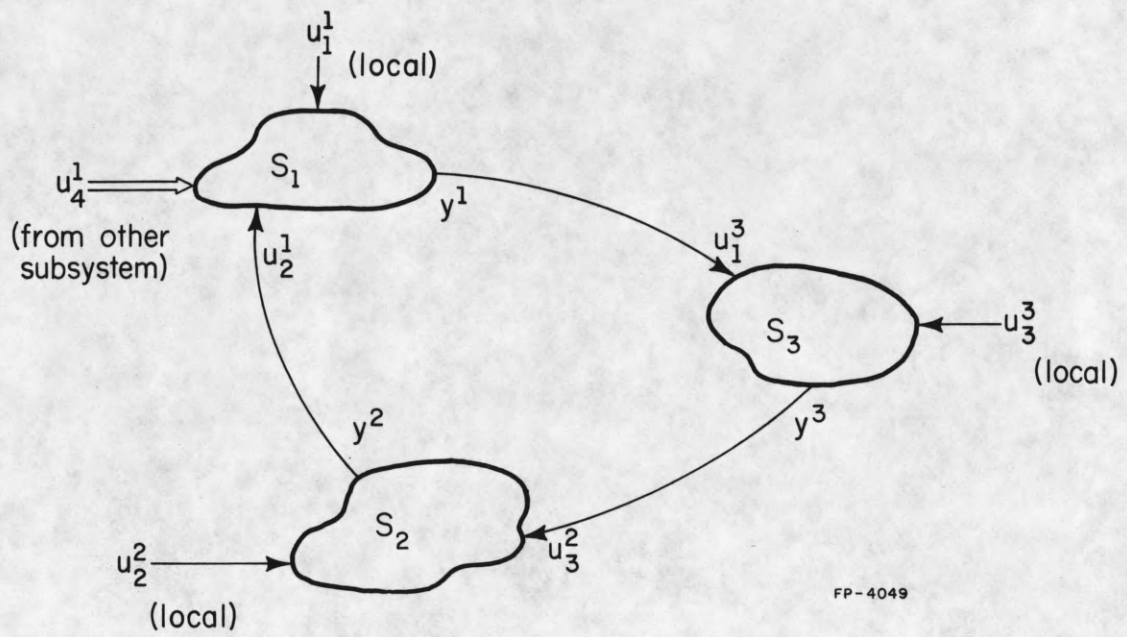


Figure 4. System configuration for Example 2.

predecessor observable subspace pairs.

It has to be pointed out that not all loops of interconnections will cause uncontrollability. Again, the use of modal control concepts would make the above easier to check, as the example has shown.

VI. The Multilevel Structure

Refer to Theorem 4 and consider the set of subsystems $\{s_1 \dots s_m\}$. It is apparent that if this can be divided into disjoint sets such that the interconnections do not form any loops between the sets, controllability conditions will be simplified and Theorem 4 will only have to be applied within each set. The sets will show a multilevel structure (or a hierarchy) within the LSC system. A simple algorithm, using graph theory, to identify these sets has been given in Appendix II.

We shall now analyze the controllability of the multilevel structure considering each level as a single subsystem. This special structure shall be called a "chain." We shall first consider a chain of m controllable subsystems with no local inputs (Fig. 5a) then extend the results to the case with local inputs (Fig. 5b). Let the systems be:

$$s_1: \begin{aligned} \dot{x}^1 &= A^1 x^1 + B^1 u^1 \\ y^1 &= C^1 x^1 + D^1 u^1 \end{aligned} \quad (22)$$

$$s_j: \begin{aligned} \dot{x}^j &= A^j x^j + B^j u^j \\ y^j &= C^j x^j + D^j u^j \end{aligned} \quad (23)$$

$$u^j = H_{j-1,j} y^{j-1} \quad \text{for } j=2,3,\dots,m.$$

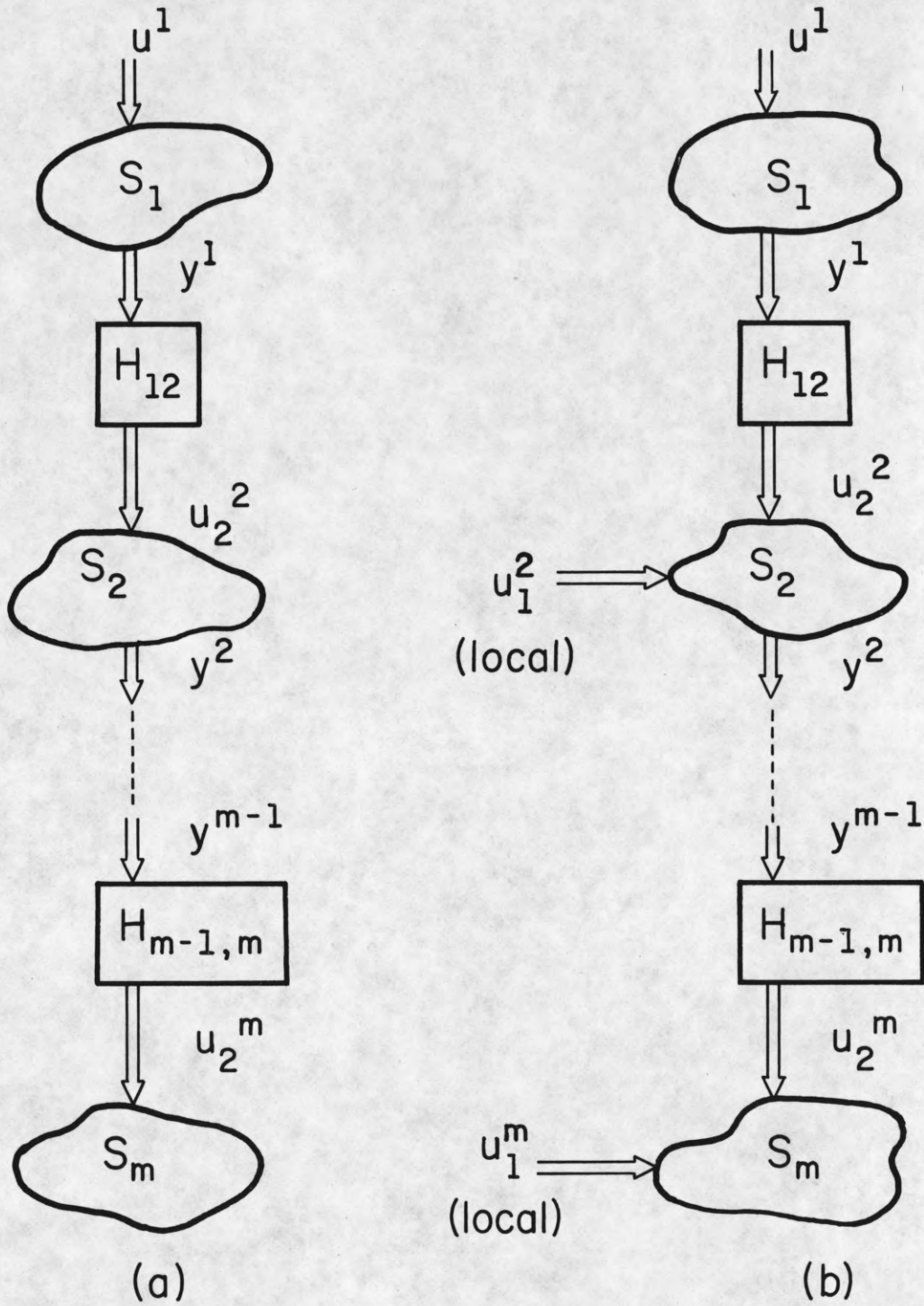


Figure 5. A chain of subsystems, with and without local inputs.

For simplicity of notation and presentation we shall consider the case where the subsystems have no eigenvalues in common. We shall find conditions under which the composite system is controllable using the inputs of the first subsystem in the chain. Form the state equations of the first j subsystems:

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \vdots \\ \dot{x}^{j-1} \\ \dot{x}^j \end{bmatrix} = \begin{bmatrix} A^1 & 0 & 0 \dots 0 \\ B^2 H_{12} C^1 & A^2 & 0 \dots 0 \\ B^3 H_{23} D^2 H_{12} C^1 & B^3 H_{23} C^2 & A^3 \dots 0 \\ \vdots & \vdots & \vdots \\ B^{j-1} (\prod_{k=3}^{j-1} H_{k-1,k} D^{k-1}) H_{12} C^1 & \dots & A^{j-1} \\ B^j (\prod_{k=3}^j H_{k-1,k} D^{k-1}) H_{12} C^1 & B^j (\prod_{k=4}^j H_{k-1,k} D^{k-1}) H_{23} C^2 & \dots & A^j \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ 0 \\ \vdots \\ x^j \end{bmatrix} + \begin{bmatrix} B^1 \\ B^2 H_{12} D^1 \\ B^3 H_{23} D^2 H_{12} D^1 \\ \vdots \\ B^j (\prod_{k=2}^j H_{k-1,k} D) \end{bmatrix} u^1 \tag{24}$$

where the matrix product π is assumed to be with index increasing leftwards. Assume that all subsystems were originally in Jordan form and consider the transformation:

$$x^j = Px \tag{25}$$

where

$$P = \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ P^{21} & I & 0 & \dots & 0 \\ P^{31} & P^{32} & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P^{j1} & P^{j2} & \dots & P^{j(j-1)} & I \end{bmatrix} \quad (26)$$

to change (24) into Jordan form. (This method is an extension of a proof due to Panda [16].) The new matrix has to be block diag(A¹, A², A³, ..., A^j), from which we get the conditions,

$$\begin{aligned} A^2 P^{21} - P^{21} A^1 &= B^2 H_{12} C^1 \\ A^3 P^{32} - P^{32} A^2 &= B^3 H_{23} C^2 \\ A^3 P^{31} - P^{31} A^1 &= P^{32} B^2 H_{12} C^1 + B^3 H_{23} D^2 H_{12} C^1 \\ \dots & \\ \dots & \end{aligned} \quad (27)$$

The new B matrix is

$$\begin{bmatrix} B^1 \\ P^{21} B^1 + B^2 H_{12} D^1 \\ P^{31} B^1 + P^{32} B^2 H_{12} D^2 + B^3 H_{23} D^2 H_{12} D^1 \\ \dots \\ P^{j1} B^1 + P^{j2} B^2 H_{12} D^2 + \dots + P^{j(j-1)} B^{j-1} \left(\prod_{k=2}^{j-1} H_{k-1,k} D^{k-1} \right) + B^j \left(\prod_{k=2}^{j-1} H_{k-1,k} D^{k-1} \right) \end{bmatrix} \quad (28)$$

For controllability we need linear independence of rows in the B matrix related to the last rows of Jordan blocks for the same eigenvalue.

Let Λ^k show the set of eigenvalues of subsystem k. Let $P_{li}^{21} B^1 + b_{li}^2 H_{12} D^2$ be

the rows related to $\lambda_i \in \Lambda^2$; $P_{\ell i}^{31} B^1 + P_{\ell i}^{32} B^2 H_{12} D^2 + b_{\ell i}^3 H_{23} D^2 H_{12} D^1$ be the rows related to $\lambda_i \in \Lambda^3$. The rows $P_{\ell i}^k$ can be obtained from equations (27). Defining $G_k(\lambda)$ $k = 1, 2, \dots, j$ to be the transfer function matrices of the subsystems, from the above we will get as a condition of controllability: for each $\lambda_i \in \Lambda^2$ linear independence of $b_{\ell i}^2 H_{12} G_1(\lambda_i)$; for each $\lambda_i \in \Lambda^3$ linear independence of $b_{\ell i}^3 H_{23} G_2(\lambda_i) H_{12} G_1(\lambda_i)$, etc. We can generalize to get the following theorem:

Theorem 5: A necessary and sufficient condition for controllability of a chain of j subsystems where $\Lambda^k \cap \Lambda^r = \emptyset$ for $k \neq r$ $k, r \in \{1, 2, \dots, j\}$ is that: for $r = 1, 2, \dots, j$ and for each $\lambda_i \in \Lambda^r$, the rows

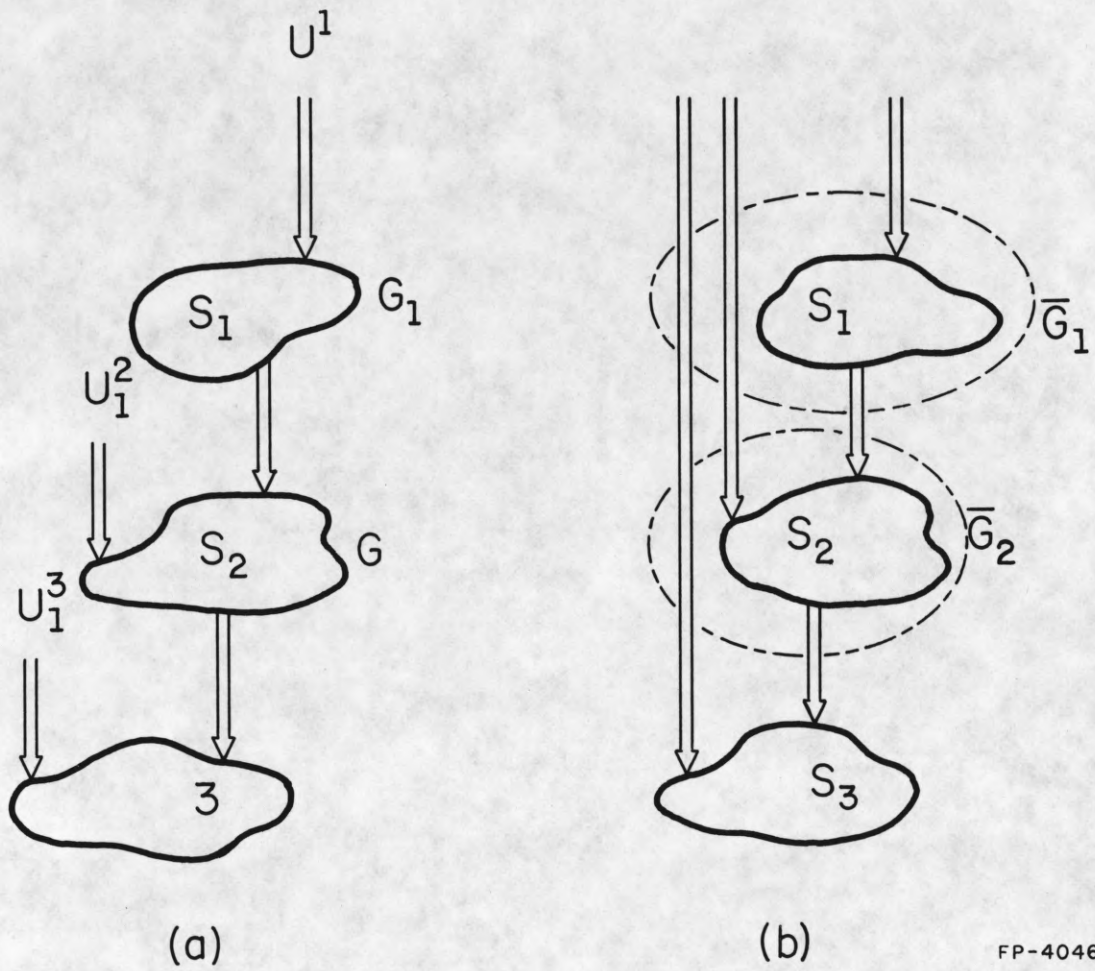
$$b_{\ell i}^r \prod_{k=2}^r H_{k-1, k} G_{k-1}(\lambda_i) \quad (29)$$

have to be linearly independent.

Note that $\prod_{k=2}^r H_{k-1, k} G_{k-1}(\lambda)$ is the overall transfer function matrix from input to s_r .

We shall now consider the case with local inputs. To simplify notation, assume that multiplication by the H matrices have already been made when defining the transfer function matrices. Consider the chain of three systems in Fig. 6(a). Theorem 5 could be applied if this configuration is represented as in Fig. 6(b). If the augmented subsystems have transfer-function matrices G_1 and G_2 the controllability condition would be:

For each $\lambda_i \in \Lambda^2$, linear independence of $\bar{b}_{\ell i}^2 \bar{G}_1(\lambda_i)$ and for each $\lambda_i \in \Lambda^3$, linear independence of $b_{\ell i}^3 \bar{G}_2(\lambda_i) \bar{G}_1(\lambda_i)$, where $\bar{b}_{\ell i}^2 = (0 : b_{\ell i}^2)$.



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Figure 6. A chain of subsystems with local inputs considered as a chain of augmented subsystems with no local inputs.

Partition G_2 to show its local and interconnection inputs, such that $G_2 = (G_1^2 \ G_2^2)$ from which

$$\bar{G}_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & G_1^2 & G_2^2 \end{bmatrix} \quad (30)$$

and

$$\bar{G}_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & G^1 \end{bmatrix} \quad (31)$$

Therefore for $\lambda_i \in \Lambda^2$

$$\bar{b}_{li}^2 \bar{G}_1(\lambda_i) = b_{li}^2 \begin{bmatrix} I & 0 \\ 0 & G^1 \end{bmatrix} \quad (32)$$

should be linearly independent. For $\lambda_i \in \Lambda^3$

$$b_{li}^3 \bar{G}_2 \bar{G}_1 = b_{li}^3 \begin{bmatrix} I & 0 & 0 \\ 0 & G_1^2 & G_2^2 G^1 \end{bmatrix} \quad (33)$$

should be linearly independent. Now for each subsystem use the canonic form (9) where we shall assume that local and interconnection inputs do not share any eigenvalues. The Jordan form will give $A_{31} = 0$, $A_{32} = 0$. Noticing that $b_{li}^2 = (b_{1li}^2 : b_{2li}^2)$ (32) will be changed to $(b_{1li}^2 : b_{2li}^2 G^1)$. For λ_i an eigenvalue of A_{11}^2 (to be denoted as $\lambda_i \in \Lambda_{11}^2$), $b_{2li}^2 = 0$ and b_{1li}^2 are by assumption independent. For $\lambda_i \in \Lambda_{33}^2$, since mode λ_i is controllable by both local and interaction inputs b_{-1li}^2 are again independent. For $\lambda_i \in \Lambda_{22}^2$, $b_{1li}^2 = 0$ and linear independence of $b_{2li}^2 G^1(\lambda_i)$ is still required. A similar argument will show that (33) is reduced to linear dependence for

$\lambda_i \in \Lambda_{22}^3$ of $b_{2\ell_i}^3 [G_1^2(\lambda_i) : G_2^2(\lambda_i)G_1^1(\lambda_i)]$. Define $G_{kj}(\lambda)$ to be the transfer function matrix from the local input of subsystem k to the related interconnections input of subsystem j in the chain ($j > k$ by assumption). The above conditions would be: for $\lambda_i \in \Lambda_{22}^2$, linear independence of $b_{2\ell_i}^2 G_{12}(\lambda_i)$; for $\lambda_i \in \Lambda_{22}^3$, linear independence of $b_{2\ell_i}^3 [G_{23}(\lambda_i) : G_{13}(\lambda_i)]$. Generalizing we get the following theorem.

Theorem 6: A LSC system, composed of a chain of m controllable subsystems with $\Lambda^j \cap \Lambda^k = \emptyset$ $j \neq k$, $j, k = 1 \dots m$ is controllable if and only if for $j = 2, \dots, m$, and for each $\lambda_i \in \Lambda_{22}^j$ the rows

$$b_{2\ell_i}^j [G_{j-1,j}(\lambda_i) : G_{j-2,j}(\lambda_i) : \dots : G_{1,j}(\lambda_i)] \quad (34)$$

are linearly independent.

Theorem 6 leads to various interesting observations. Note that we need only A_{22} in Jordan form in each subsystem and not the whole A matrix. It is seen that controllability of its predecessor is neither necessary nor sufficient for the controllability of any subsystem. The Jordan form will identify the uncontrollable modes. These may occur in any one subsystem along the chain and since the remaining modes will still be controllable subsystems further down the chain may also be. Another point is that given the necessary number of inputs, any subsystems local inputs may be sufficient for controllability of a subsystem. Or a number of "higher level" subsystems may cooperate in controlling one lower in the hierarchy. It has to be pointed out here that the concept of controllability in LSC systems plays a more important role than usual, in that it is also

concerned with the effect of the control action through the interconnections. By analyzing controllability insight into local pole placement, stabilizability etc. is gained. We shall consider this question in Part VII.

VII. Stabilizability and Pole Placement in Chains

We shall now apply our remarks related to (9) and Theorem 6 to the stabilizability and pole placement properties of the chain structure.

Theorem 7: A LSC system, composed of a chain of m controllable subsystems with $\Lambda^j \cap \Lambda^k = \emptyset$ $j \neq k$ $j, k = 1 \dots m$, and $\Lambda_{ii}^j \cap \Lambda_{\ell\ell}^j = \emptyset$ $i \neq \ell$ $i, \ell = 1, 2, 3$ is locally stabilizable with linear state feedback if and only if

$\bigcup_{j=2}^m \Lambda_{22}^j \subseteq C^-$, where C^- is the left-half complex plane.

Proof: Theorem 7 is actually an application of Theorem 2c. Apply the local feedback

$$U_1^j = k_1^j x_1^j + k_2^j x_2^j + k_3^j x_3^j \quad \text{to } s_j, \quad (j = 2, \dots, m) \text{ and}$$

$$U^1 = k^1 x^1 \quad \text{to } s_1.$$

Denoting the new A matrix for the composite system as \tilde{A} , we get, after some manipulation,

$$\det(\lambda I - \tilde{A}) = \det(\lambda I - A^1 - B^1 K^1) \prod_{j=2}^m \det(\lambda I - A_{22}^j) \cdot \det \begin{bmatrix} \lambda I - A_{11}^j - B_{11}^j K_1^j & -B_{11}^j K_3^j \\ -A_{21}^j - B_{32}^j K_1^j & \lambda I - A_{33}^j - B_{33}^j K_3^j \end{bmatrix} \quad (35)$$

from which two facts are seen: a) eigenvalues of the A_{22}^j have not been affected, b) the local feedback of the x_2^j have been of no use. The following corollary can therefore be stated:

Corollary: In the LSC system described in Theorem 7, the local linear feedback of any $x \in C_2^*$ will have no effect on pole placement.[†]

The chain structure admits some more comments to be made, and that is in answering the questions: If the system cannot be locally stabilized, then which states should be fed back to which subsystems; and how many levels higher in the hierarchy must feedback be made to shift a certain pole that cannot be shifted by local feedback?

Assuming Jordan form, the answer to the first part is "those states related to the mode in question." For an answer to the second question consider Theorem 6 and equation (34). Check linear independence of $b_{2li}^j G_{j-1,j}(\lambda_i)$. If they are not linearly independent go to $b_{2li}^j [G_{j-1,j}(\lambda_j) : G_{j-2,j}(\lambda_i)]$ then $b_{2li}^j [G_{j-1,j}(\lambda_i) : G_{j-2,j}(\lambda_i) : G_{j-2,j}(\lambda_i)]$ etc. until a linearly independent set is found. Other combinations can be tested to see if a subset of these systems are also sufficient. If a single subsystem is required through which feedback is to be made, a certain independent set $b_{2li}^j G_{k,j}(\lambda_i)$ has to be found. Note that some of the eigenvalues $\bigcup_{p=k}^j \Lambda_{22}^p$ may also be changed in the process.

[†]Note that $\bigcup_{j=2}^m \Lambda_{22}^j$ would be the "fixed modes" defined by Wang and Davison [4]. However, owing to the LSC system structure, their determination is very much simplified.

Example: Consider the following three systems:

$$\begin{aligned}
 s_1: \quad \dot{x}_1^1 &= -3x_1^1 + u_1^1 & y^1 &= x_1^1 + u_1^1 \\
 s_2: \quad \dot{x}_1^2 &= -2x_1^2 + u_1^2 & u_2^2 &= y^1 \\
 & \dot{x}_2^2 &= -x_2^2 + u_2^2 & y^2 &= x_1^2 + 3x_2^2 - u_2^2 \\
 s_3: \quad \dot{x}_1^3 &= x_1^3 + u_1^3 & u_2^3 &= y^2 \\
 & \dot{x}_2^3 &= 2x_2^3 + u_2^3 \\
 & \dot{x}_3^3 &= 3x_3^3 + u_1^3 + u_2^3
 \end{aligned} \tag{36}$$

$\Lambda_{11}^3 = \{1\}$, $\Lambda_{33}^3 = \{3\}$ and these can be shifted to the left half complex plane by local linear state feedback. However, $\Lambda_{22}^3 = \{2\}$ and this pole must be shifted by the local inputs of s_1 or s_2 . The controllability condition will give:

$$\begin{aligned}
 [G_1^2(\lambda) : G_2^2(\lambda)G_1(\lambda)]_{\lambda=2} &= \left[\frac{1}{\lambda+2} : \left(\frac{3}{\lambda+1} - 1 \right) \left(\frac{1}{\lambda+3} + 1 \right) \right]_{\lambda=2} \\
 &= \left[\frac{1}{4} : 0 \right]
 \end{aligned} \tag{37}$$

which means $\lambda=2$ is not effected by the input of s_1 , i.e. feedback has to be made to u_1^2 , and the state to be fed back is x_2^3 .

VIII. Conclusions

The LSC systems structure has been proposed as a model for certain large scale systems. Certain properties of this structure have been considered. A constructive proof for the existence of the controllability decomposition canonic form and an algorithm to get a system into this form was given. The canonic form was shown to be very useful in drawing

conclusions about controllability, pole placement and stabilizability of LSC systems. It was shown that necessary conditions for controllability of LSC systems are that; for any subsystem, for at least one set of necessary predecessors, the predecessor subsystems observable subspace is within the controllable subspaces of the subsystems predecessors; and also loops of interaction input controllable subspace-predecessor observable subspace pairs should not exist. Necessary and sufficient conditions for controllability of the chain structure were given and conclusions were reached relating to the pole placement and stabilizability properties of chains.

Algorithms for determining necessary predecessors for controllability of a subsystem, and for determining the existence of a multilevel structure in a LSC structure, using graph theory are given in the two appendices.

Appendix I. Identifying the Class of Necessary Predecessor Sets

The problem of identifying the class of necessary predecessor sets is an extension of the problem of finding the class of necessary inputs to a system such that controllability is preserved. This has been considered previously [25] but the results obtained seem to be erroneous. A detailed description and related problems may be found in [26]. We shall now give two different algorithms to find the necessary predecessor sets. The second assumes availability of the Jordan form for the system.

In (9) partition B_{22} into $\bar{B}_1, \bar{B}_2, \dots, \bar{B}_m$ such that each \bar{B}_i is associated with interaction inputs from subsystem i . (Note that \bar{B}_i is not equal to B_i^j of (1) since a transformation has been made to get (1) into (9).) Start with \bar{B}_1 . Find $A_{22}\bar{B}_1$. If this gives at least one column independent of those in \bar{B}_1 go on premultiplying by A_{22} . Each time check independence with previously found columns. When, at any step, no new column is found, further multiplication will give nothing new, since the $(A_{22}\bar{B}_i)$ pairs are A_{22} invariant. Then go to \bar{B}_2 and repeat procedure. With all new \bar{B}_i 's check columns generated by previous \bar{B} 's. Each step with a new \bar{B}_i will span a part of the subspace \mathcal{C}_2^* controlled by predecessor i . (It may also span a part already spanned by previously considered predecessors.) Terminate the procedure when k_2^* independent columns are obtained. If this termination occurs before there are k_2 independent columns, this means a part of the subspace \mathcal{C}_3 is being turned over to the local inputs. So far one set of necessary predecessors have been found. Repeat procedure until all possible combinations with \bar{B}_1 are exhausted, then start again with \bar{B}_2 . There exists various algorithms to accomplish this enumeration in a systematic manner, e.g. see [27].

For a different method, assume availability of the Jordan form. We are interested only in those rows in B_{22} that are related to the last rows of Jordan blocks of equal eigenvalues. If for λ_i p blocks exist, we need p linearly independent rows. It is a simple matter to find for each λ_i the combinations of inputs that would be sufficient to control it.

Example 3: Consider the subsystem s_j with interaction input only,

$$\dot{x}_2^j = A_{22}^j x_2^j + \sum_{\substack{i=1 \\ i \neq j}}^4 B_i^j u_i^j \quad (38)$$

where

$$A_{22}^j = \text{block diag} \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 2, 3, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right] \quad (39)$$

$$B_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_2^j = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_3^j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B_4^j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For controllability of $\lambda_{1,2,3} = 1$ it is seen that predecessor 1 is enough. Or predecessors 2 and 3 could also cooperate in controlling the mode. For $\lambda_4 = 2$ predecessor 1 or 2 or 4 would be necessary, etc. Denote the predecessors by a, b, c, d respectively. Writing as a Boolean function, for $\lambda_{1,2,3}$: $a \cup (b \cap c)$, for λ_4 : $a \cup b \cup d$, for λ_5 : $b \cup c \cup d$ and for $\lambda_{6,7}$: $c \cup d$. To find complete controllability minimization of the expression

$$[a \cup (b \cap c)] \cap [a \cup b \cup d] \cap [b \cup c \cup d] \cap [c \cup d] \quad (40)$$

is required. This gives

$$(a \cap c) \cup (a \cap d) \cup (b \cap c) \quad (41)$$

which means (1,3), (1,4) and (2,3) are the sets of predecessors that can control s_j .

This is an extension of the so-called "minimal covering problem" in switching theory [28].

Appendix II. Identifying a Multilevel Structure

We shall first give some concepts related to graph theory which we believe to be useful in the analysis of LSC systems. We assume the reader is familiar with the basic terms node and edge. Showing subsystems as nodes and interconnections as edges the graph will show the connection structure of the LSC system. There are various books that can be referenced in graph theory e.g. see [29]. A paper by Warfield [30] also exposes related concepts.

If a sequence of edges $\{(i,j), (j,k), (k,l) \dots (p,q)\}$ exist, with no repetition in the nodes (i,j,k etc.) this is called "a path from i to q." A node q is said to be reachable from i if there exists at least one path from i to q. If a node is reachable from itself, along a path the path is called a loop.

The main cause to refer to graph theory in LSC systems arises from the fact that the existence of a path in the graph, would imply control actions being transmitted through all subsystems along the path. Another reason would be considerations of "information flow" (see Ref. [5]).

Definition: The $m \times m$ matrix E is named the adjacency matrix and shows all interconnections among subsystems, such that $e_{ij} = 1$ if $H_{ij} \neq 0$, $e_{ij} = 0$ if $H_{ij} = 0$. We shall further assume $e_{ii} = 0$.

For the system in the examples given by equations (20) and (21) the adjacency matrix is:

$$E = \begin{matrix} & s_1 & s_2 & s_3 \\ \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix} \quad (42)$$

and the loop is obvious. In more complicated systems loops may not be easy to detect just by inspection of the matrix E , but there are various algorithms for path and loop enumeration that one can refer to [e.g. 31, 32, 27]. In LSC systems, however, the identification of levels is a slightly different problem, in that two loops sharing a node, are assumed to be in the same level. We give the following algorithm to identify levels in a LSC system, given its adjacency matrix.

It is well known in graph theory that E^k , where all operations are Boolean, will show the nodes that are reachable in k steps, i.e. if the element (i,j) is 1, there exists a path from node i to node j which passes through $k-1$ nodes. Define

$$M_k = \bigcup_{l=1}^k E^l. \quad (43)$$

This will show all nodes reachable in k steps or less.

1. Set $l = 1$, $M_l = E$ and $r_l =$ number of 1's in M_l .
2. Set $l = l+1$ and find E^l , M_l and r_l .
3. Check $r_l - r_{l-1}$. If not zero go back to step 2. If zero stop.

Note that the end might come before $l = m$ (number of nodes).

4. Consider each element on the main diagonal of the final M , the reachability matrix. If it is 1 the related node is within a loop. For each such nonzero element take its column and row and perform a bit by bit AND operation on them. Form a matrix L whose rows are the results of the

above operation. The i th row will show the nodes from which node i is reachable and which are reachable from node i .

5. Redundant rows of L will show the required sets of nodes that correspond to the levels in the LSC system.

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| 13. ABSTRACT <p>This work gives some preliminary results related to research on Large Scale Composite (LSC) systems. An interconnected system model is considered. Each subsystem is assumed to be linear, time invariant, and have both local inputs and interaction inputs from other subsystems. The main question is: "What can be done without lumping all subsystems into one large system model, and what can be done by each subsystem using local observations and local control inputs alone?" Results on controllability, pole placement and stabilizability are given for the general LSC system and the "chain" structure. Both state-space and transfer function concepts are used. LSC systems may model various physical interconnected structures, e.g. power systems, economic systems etc. and applications of the results to these areas are presently being considered.</p> | | | |

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