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THE MIXED BOUNDARY-INITIAL VALVE PROBLEM FOR VLASOV EQUATION

KENNETH EVANS, JR

UNIVERSITY OF ILLINOIS - URBANA, ILLINOIS

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Kenneth Evans, Jr.

Coordinated Science Laboratory
University of Illinois
Urbana, Illinois

ABSTRACT

The mixed boundary initial value problem for the Vlasov equation is treated by double Laplace transformation. It is found that certain relations must be satisfied by the boundary and initial values in order to have a well-defined problem. The case of a cold beam in a cold plasma is treated as a simple example. An approach to the problem via discrete velocities is treated in an appendix.

INTRODUCTION

In this paper the combined boundary-initial value problem for the linearized Vlasov equation will be treated. The method used, double Laplace transformation, is naturally suited to such a problem. A similar treatment for systems of linear, partial differential equations has already been given,[1,2] and will be referred to as I. In plasma physics the treatment I is applicable to the hydrodynamic approximation, which consists of the linearized, lower-order, moment equations of the Vlasov equation, along with Maxwell's equations.[3] Although the validity of this approximation is not well established, it is often used because of its simplicity (relative to more rigorous formulations) when there is any hope it might apply. The Vlasov equation, on the other hand, is on better footing as the lowest order of the BBGKY hierarchy.[4] The present paper is an extension of the ideas of I to the Vlasov equation.

The classic treatment of the solutions to the Vlasov equation is that given by Landau[5] and later refined by Jackson.[6] In the first part of Landau's paper the Vlasov equation is treated as an initial value problem by treating each spatial Fourier component separately and Laplace transforming in the time. This is perhaps the most common treatment of the equation. In the second part of the paper the equation is treated as a boundary value problem by treating each temporal Fourier component separately, then obtaining an integral equation and an asymptotic expansion. In the treatment given in this paper, that is, for the mixed boundary-initial value problem, Laplace transformation is made in both the spatial and temporal variables. This treatment is most appropriate for initial disturbances in bounded systems, for example, the beam-plasma system, where there is a natural boundary where the beam enters the plasma.

FORMALISM

The equations to be considered in this paper are the Vlasov equation for an electron plasma in a fixed positive ion background and Maxwell's equations for the self-consistent electric field. The magnetic field will be neglected, and the one-dimensional, linearized version of these equations will be used. The equations are:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{df_o}{dv} = 0 \qquad (1)$$

$$\frac{\partial E}{\partial t} - 4\pi n_o e \int vfdv = 0 \qquad (2)$$

$$\frac{\partial E}{\partial x} + 4\pi n_o e \int fdv = 0, \qquad (3)$$

where $f_o = f_o(v)$ is the unperturbed distribution function, $f = f(x,t,v)$ is the perturbed distribution function, $E = E(x,t)$ is the perturbed electric field, and n_o is the unperturbed density. The electrons have mass m and charge (-e).

It can be seen immediately that there is a relation that must be satisfied by the initial conditions:

$$\frac{\partial E(x,0)}{\partial x} + 4\pi n_o e \int f(x,0,v')dv' = 0 \qquad (4)$$

This relation is analogous to the relation arising in I as a result of characteristics lying along the line t = 0. Similarly, there are two relations which must be satisfied by the boundary conditions:

$$\frac{\partial E(0,t)}{\partial t} - 4\pi n_o e \int v'f(0,t,v')dv' = 0 \qquad (5)$$

$$\frac{e}{m}E(0,t) = \left[\frac{\partial f(0,t)}{\partial t}\Big/\frac{\partial f_o}{\partial v}\right]_{v=0} \qquad (6)$$

These are analogous to relations arising in I as a result of characteristics lying on the line $x = 0$. Relations (4-6) are restrictions on the possible choices of boundary and initial values.

As has been stated above, the equations will be Laplace transformed in both x and t. Only solutions in the region $x \geq 0$ and $t \geq 0$ will be considered. The transforms will be defined as follows:

$$\begin{bmatrix} f(p,q,v) \\ E(p,q) \end{bmatrix} = \int_0^\infty dt \int_0^\infty dx\, e^{-pt-qx} \begin{bmatrix} f(x,t,v) \\ E(x,t) \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} f^I(p,v) \\ E^I(p) \end{bmatrix} = \int_0^\infty dt\, e^{-pt} \begin{bmatrix} f(0,t,v) \\ E(0,t) \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} f^{II}(q,v) \\ E^{II}(q) \end{bmatrix} = \int_0^\infty dx\, e^{-qx} \begin{bmatrix} f(x,0,v) \\ E(x,0) \end{bmatrix} \tag{9}$$

The inverses are given by:[7]

$$\begin{bmatrix} f(x,t,v) \\ E(x,t) \end{bmatrix} = \frac{1}{(2\pi i)^2} \int_{C_p} dp \int_{C_q} dq\, e^{pt+qx} \begin{bmatrix} f(p,q,v) \\ E(p,q) \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} f(0,t,v) \\ E(0,t) \end{bmatrix} = \frac{1}{2\pi i} \int_{C_p} dp\, e^{pt} \begin{bmatrix} f^I(p,v) \\ E^I(p) \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} f(x,0,v) \\ E(x,0) \end{bmatrix} = \frac{1}{2\pi i} \int_{C_q} dq\, e^{qx} \begin{bmatrix} f^{II}(q,v) \\ E^{II}(q) \end{bmatrix} \tag{12}$$

The contours C_p and C_q are straight lines $p = p_o + ip_i$ and $q = q_o + iq_i$, where p_i and q_i vary from $-\infty$ to $+\infty$, and p_o and q_o are chosen so that the contours lie to the right of all singularities in the p and q planes respectively. These contours are shown in Fig. 1.

To find the Laplace transformed solution to Eqs. (1-3), the equations are multiplied by $\exp(-pt - qx)$ and integrated on x and t from 0 to ∞. After integrating by parts, the equations become:

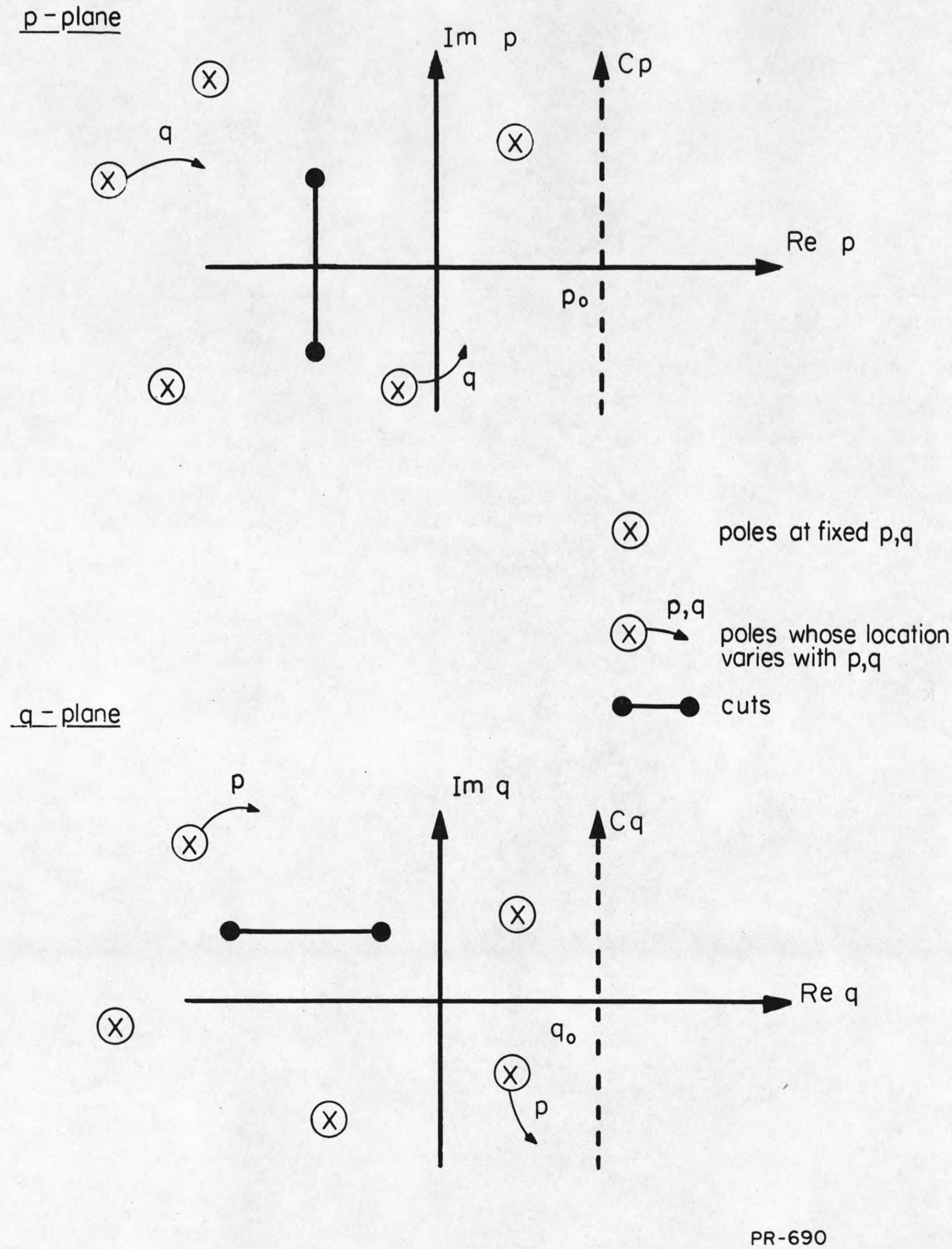

Figure 1. Typical singularities and contours of integration for inversion of Laplace transforms.

$$(p + vq)f(p,q,v) - \frac{e}{m}\frac{df_o}{dv}E(p,q) = f^{II}(q,v) + vf^{I}(p,v) \tag{13}$$

$$pE(p,q) - 4\pi n_o e \int v'f(p,q,v')dv' = E^{II}(q) \tag{14}$$

and:

$$pE^{I}(p) - 4\pi n_o e \int v'f^{I}(p,v')dv' = E(0,0) \tag{15}$$

$$qE^{II}(q) + 4\pi n_o e \int f^{II}(q,v')dv' = E(0,0) \tag{16}$$

The solution for the electric field is easily found to be:

$$E(p,q) = \frac{N(p,q)}{D(p,q)} \tag{17}$$

where:

$$N(p,q) = \frac{E^{I}(p)}{q} - \frac{4\pi n_o e}{q}\int dv' \frac{[f^{II}(q,v') + v'f^{I}(p,v')]}{p + v'q}$$

$$= \frac{E^{II}(q)}{p} + \frac{4\pi n_o e}{p}\int v'dv' \frac{[f^{II}(q,v') + v'f^{I}(p,v')]}{p + v'q} \tag{18}$$

$$D(p,q) = 1 + \frac{\omega_p^2}{q}\int \frac{df_o}{dv'}\frac{dv'}{p + v'q}$$

$$= 1 - \frac{\omega_p^2}{p}\int \frac{df_o}{dv'}\frac{v'dv'}{p + v'q} \tag{19}$$

and ω_p is the plasma frequency. The solution for $f(p,q,v)$ is then easily found from Eq. (13).

RELATIONS AMONG THE BOUNDARY AND INITIAL CONDITIONS

In order for the inverse Laplace transforms, Eqs. (10-12), to be defined, it must be possible to find contours C_p and C_q such that $E(p,q)$ and

$f(p,q,v)$ are analytic to the right of these contours in the p and q planes. It was found in I, on the other hand, that unless the boundary and initial conditions were suitably restricted, that such contours could not in general be found. The lack of the required analyticity occurred when there were characteristics associated with negative characteristic velocities. In that case information specified at $t = 0$ was propagated to $x = 0$. The result was that the boundary conditions were determined by the initial conditions and could not be specified independently. Figure 2 shows how information specified at $t = 0$ can affect the boundary at $x = 0$ in this case. The Vlasov equation can be thought of as a system of equations of the type considered in I. (See the appendix at the end of this paper.) There is one equation for each value of the velocity v, and the characteristic velocities are these values of v. A number of these characteristic velocities are, then, of course, negative. One would hence expect relations among the boundary and initial conditions to occur for the Vlasov equation in much the same way as they occurred in I when there were negative characteristic velocities. It will be seen that this is, in fact, the case.

Consider the singularities of $E(p,q)$ in the p-plane for a given q. There is, first of all, a cut along the line $p = - vq$ for $- \infty < v < \infty$. See Fig. 3. This is a generalization of the cut which appears along the imaginary p-axis in the Landau treatment for the value $q = ik$. To investigate the discontinuity of $D(p,q)$ across this cut, it is convenient to define:

$$D^{\pm}(v,q) = \lim_{\frac{-p}{q} \to v \pm i\epsilon} D(p,q) \tag{20}$$

Using the relation:

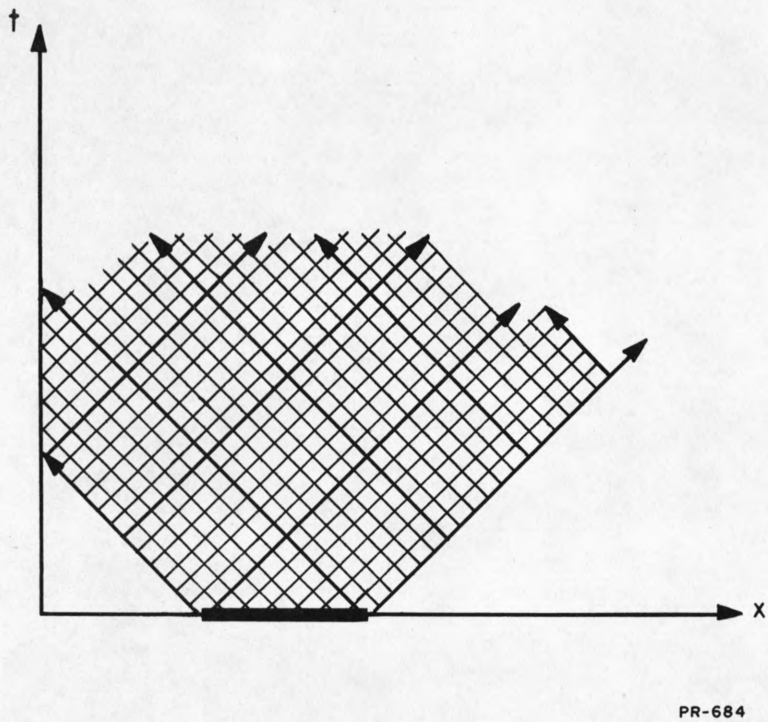

Figure 2. Propagation of information specified at t = 0 to the boundary x = 0.

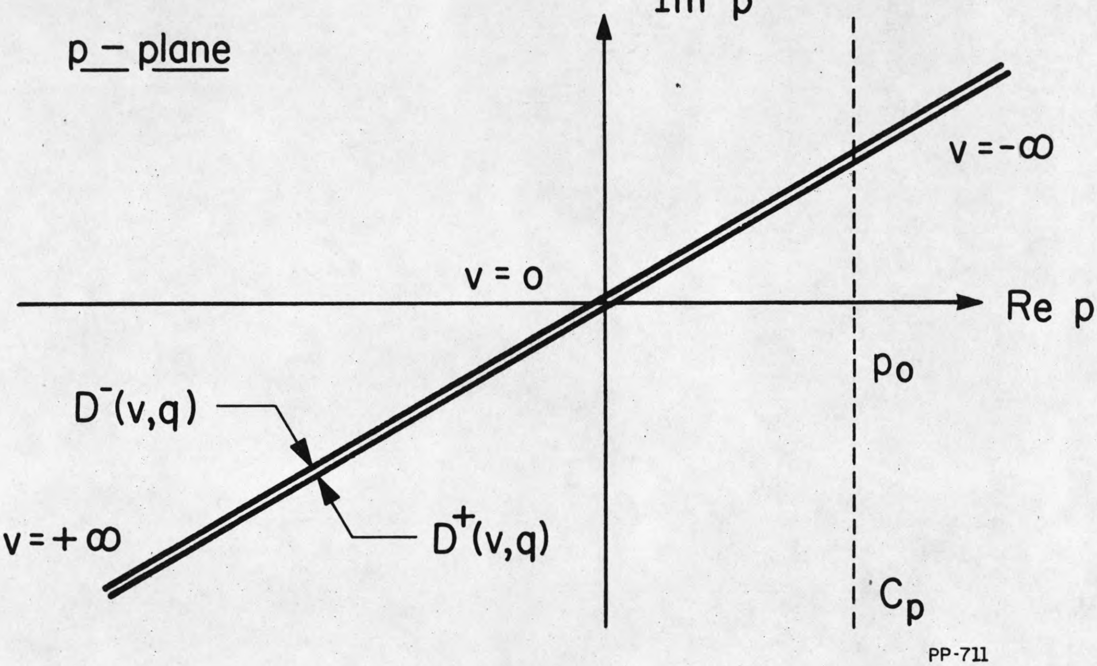

Figure 3. Cut of E(p,q) in the p-plane for a given q. (Re q > 0)

$$\lim_{s \to u \pm i\epsilon} \left(\frac{1}{t - s}\right) = P\left(\frac{1}{t - u}\right) \pm i\pi\delta(t-u) \tag{21}$$

where P indicates the principal value, it is easy to calculate:

$$D^{\pm}(v,q) = 1 + \left(\frac{\omega_p}{q}\right)^2 P \int \frac{df_o}{dv'} \frac{dv'}{v' - v} \pm i\pi\left(\frac{\omega_p}{q}\right)^2 \frac{df_o}{dv} \tag{22}$$

or that:

$$D^+ = D^- + \Delta D \tag{23}$$

with:

$$\Delta D = 2\pi i \left(\frac{\omega_p}{q}\right)^2 \frac{df_o}{dv} \tag{24}$$

Similarly if:

$$N^{\pm}(v,q) = \lim_{\frac{-p}{q} \to v \pm i\epsilon} N(p,q), \tag{25}$$

then:

$$N^+ = N^- + \Delta N \tag{26}$$

$$\Delta N = - \frac{8\pi^2 i n_o e}{q^2} \left[f^{II}(q,v) + vf^I(-vq,v)\right] \tag{27}$$

It can be seen from Fig. 3 that this cut makes it impossible to draw a contour C_p to the right of all singularities. To have a well-defined Laplace transform, then, it is necessary to choose the boundary and initial conditions so as to eliminate this cut, or more exactly, to eliminate that part of it in the right half-plane. As could be expected, the part of the cut in the right half-plane is that due to negative values of v. To eliminate the cut, it is sufficient to determine the boundary and initial conditions for negative v so that:

$$\frac{N^+}{D^+} = \frac{N^-}{D^-} = \frac{N^- + \Delta N}{D^- + \Delta D} \tag{28}$$

The expression resulting from substitution of the appropriate quantities into Eq. (28) is:

$$[f^{II}(q,v) + vf^{I}(-vq,v)]\left[1 + \left(\frac{\omega_p}{q}\right)^2 P\int \frac{df_o}{dv'}\frac{dv'}{v' - v}\right] =$$

$$\frac{df_o}{dv}\left\{\left(\frac{\omega_p}{q}\right)^2 P\int dv' \frac{[f^{II}(q,v') + v'f^{I}(-vq,v')]}{v' - v} - \frac{eE^{I}(-vq)}{mq}\right\} \qquad (29)$$

provided $v < 0$. Eq. (29) is the relation that must be satisfied by the boundary and initial conditions for a well-defined solution $E(p,q)$.

It is still necessary to determine if $f(p,q,v)$ is analytic to the right of the contour C_p. The function $f(p,q,v)$ is given by:

$$f(p,q,v) = \frac{1}{p + vq}\left\{\frac{e}{m}\frac{df_o}{dv} E(p,q) + f^{II}(q,v) + vf^{I}(p,v)\right\} \qquad (30)$$

Provided $E(p,q)$ is analytic to the right of C_p, $f(p,q,v)$ has only a pole at $p = - vq$ for negative v that could be a problem. Explicit calculation can verify, however, that the residue of this pole is zero in virtue of Eq. (29), so that there is, in fact, no pole of $f(p,q,v)$ at $p = - vq$ for negative v.

It has now been demonstrated that in order to have a well-formulated mixed boundary initial value problem for the equations given, the boundary and initial conditions must be chosen to satisfy Eqs. (4-6) and (29). It has been assumed that the boundary and initial conditions are sufficiently well-behaved that the preceding analysis can be carried out. By picking sufficiently singular boundary and initial conditions one can produce arbitrary steady-state plasma oscillations,[8,6] in which case the analysis given in this paper is not relevant.

SATISFACTION OF THE BOUNDARY AND INITIAL CONDITIONS

It should be determined if the solution, subject to the restrictions on the boundary and initial conditions, does indeed satisfy the specified boundary and initial conditions. In order to do this it will be assumed that the boundary and initial conditions can be expanded as follows:

$$E^{I}(p) = \frac{1}{p} E(0,0) + \frac{1}{p^2} \frac{\partial E}{\partial t} (0,0) + --- \qquad (31)$$

$$E^{II}(q) = \frac{1}{q} E(0,0) + \frac{1}{q^2} \frac{\partial E}{\partial x} (0,0) + --- \qquad (32)$$

with similar expressions for $f^{I}(p,v)$ and $f^{II}(q,v)$. To see if the initial conditions are satisfied, one can move the contour C_p to $p_o = \infty$, keeping t sufficiently small so that $e^{pt} \simeq 1$. The inversion integral for $E(x,t\to 0)$ then becomes:

$$E(x,t\to 0) = (2\pi i)^{-2} \int dpdqE(p\to\infty,q)e^{pt + qx} \qquad (33)$$

An analogous expression for $x\to 0$ is obtained. This method is described in detail in I.

Examination of the solution, Eqs. (17-19), indicates:

$$E(p\to\infty,q) = \frac{E^{II}(q)}{p} + --- \qquad (34)$$

so that:

$$\begin{aligned} E(x,t\to 0) &= (2\pi i)^{-2} \int dpdq\frac{E^{II}(q)}{p}e^{pt + qx} \\ &= (2\pi i)^{-1} \int dqE^{II}(q)e^{qx} \\ &= E(x,0) \qquad (35) \end{aligned}$$

(The higher order terms in Eq. (34) do not contribute in the first integration in Eq. (35). Eq. (35) indicates that the solution $E(x,t)$ satisfies the initial condition. Similarly one can see that:

$$f(p\to\infty,q,v) = \frac{f^{II}(q,v)}{p} \tag{36}$$

$$E(p,q\to\infty) = \frac{E^{I}(p)}{q} \tag{37}$$

$$f(p,q\to\infty,v) = \frac{f^{I}(p,v)}{q} \tag{38}$$

and, hence, that $E(x,t)$ and $f(x,t,v)$ satisfy all the boundary and initial conditions.

EXAMPLE

As a simple example to illustrate the method, the case of a cold electron beam in a cold plasma will be taken. The unperturbed distribution function is:

$$f_o = \frac{1}{n_o}\left[n_p\delta(v) + n_b\delta(v-v_b)\right]$$

The subscript p refers to plasma quantities, and the subscript b refers to beam quantities. The following set of boundary and initial conditions will be chosen:

$$\begin{aligned} &f(0,t,v) = 0 \qquad && f(x,0,v) = \delta(v)\sin kx \\ &E(0,t,v) = 0 \qquad && E(x,0) = -\frac{m\omega_p^2}{ek}(1-\cos kx) \end{aligned} \tag{39}$$

It can be explicitly verified that this set satisfies Eqs. (4-6) and (29). The relations:

$$x\delta(x) = 0 \tag{40}$$

$$\delta(x) = -x\delta'(x) \tag{41}$$

are useful in showing that Eq. (29) is satisfied.

The transform of the electric field can then be written:

$$E(p,q) = \frac{\dfrac{-4\pi n_o e}{q} \displaystyle\int \frac{dv' f^{II}(q,v')}{p + v'q}}{1 + \dfrac{\omega_p^2}{q} \displaystyle\int \frac{df_o}{dv'} \frac{dv'}{p + v'q}} \tag{42}$$

After the velocity integrations are performed, $E(p,q)$ becomes:

$$E(p,q) = \frac{\dfrac{-4\pi n_o ek}{pq(q^2 + k^2)}}{1 + \dfrac{\omega_p^2}{p^2} + \dfrac{\omega_b^2}{(p + v_b q)^2}} \tag{43}$$

This is the same result as was obtained in a similar calculation using the hydrodynamic approximation.[9] The solution is extensively discussed in that paper.

SUMMARY

The mixed boundary-initial value problem for the Vlasov equation has been treated by double Laplace transformation. It has been shown that a well-formulated problem can be obtained only if the boundary and initial conditions are suitably restricted. The restrictions consist of three fairly simple relations which can be considered to determine the electric field boundary and initial conditions if the distribution function boundary and initial conditions are given. Another, more complicated, integral relation between the boundary and initial conditions associated with negative velocities must also be satisfied. It has also been shown that the given solution does satisfy the boundary and initial conditions in general. A simple example for a cold beam in a cold plasma has been shown to give results consistent with previous calculations.

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APPENDIX

It is possible to formulate Eqs. (1-3) into a set of linear, first order, hyperbolic, partial differential equations of the type considered in I, provided the velocity v is assumed to be a discrete parameter. The values of the velocity will be labeled v_i: $-N \le i \le N$. The number N will be very large. Let:

$$f_i(x,t) = f(x,t,v_i) \tag{44}$$

$$\Delta v_i = v_{i+1} - v_i \tag{45}$$

Eqs. (1-3) become:

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} - \frac{e}{m} E \frac{df_o}{dv_i} = 0 \tag{46}$$

$$\frac{\partial E}{\partial t} - 4\pi n_o e \sum_i v_i f_i \Delta v_i = 0 \tag{47}$$

with the subsidiary condition:

$$\frac{\partial E}{\partial x} + 4\pi n_o e \sum_i f_i \Delta v_i = 0 \tag{48}$$

These are a system of 2N+2 equations in 2N+2 unknowns, with a subsidiary condition which can be shown to be satisfied in virtue of the other equations, provided it is satisfied at t = 0.[1] It can easily be seen that relations analogous to Eqs. (4-5) are obtained:

$$\frac{\partial E(x,0)}{\partial x} + 4\pi n_o e \sum_i f_i(x,0) \Delta v_i = 0 \tag{49}$$

$$\frac{\partial E(0,t)}{\partial t} + 4\pi n_o e \sum_i v_i f_i(0,t)\Delta v_i = 0 \tag{50}$$

$$\left[\frac{\partial f_i}{\partial t} - \frac{e}{m} E \frac{df_o}{dv_i}\right]_{i=0} = 0 \tag{51}$$

These conditions are a result of characteristics lying on the boundaries $x = 0$ and $t = 0$.

If Eqs. (49-50) are Laplace transformed, the result is:

$$(p + v_i q) f_i - \frac{e}{m} \frac{df_o}{dv_i} E = f_i^{II}(q) + v_i f_i^{I}(p) \tag{52}$$

$$pE - 4\pi e n_o \sum_i v_i f_i \Delta v_i = E^{II}(q) \tag{53}$$

In Eqs. (52-53) $f_i = f_i(p,q)$ and $E = E(p,q)$ are the double transforms of $f_i(x,t)$ and $E(x,t)$, $f_i^{I}(p)$ and $E^{I}(p)$ are the single transforms of $f_i(0,t)$ and $E(0,t)$, and $f_i^{II}(q)$ and $E^{II}(q)$ are the single transforms of $f_i(x,0)$ and $E(x,0)$. If we take f_{N+1} to be the electric field E, the solution is:

$$f_i(p,q) = \frac{[a_{jk} f_k^{II}(q) + b_{jk} f_k^{I}(p)] d_{ji}(p,q)}{\Delta(p,q)} \tag{54}$$

where:

$$a_{jk} = \delta_{jk} \tag{55}$$

$$b_{jk} = v_j \delta_{jk} \tag{56}$$

$$c_{jk} = \begin{cases} 0 & j,k \neq N+1 \\ 0 & j,k = N+1 \\ -\dfrac{e}{m} \dfrac{dfo}{dv_j} & k = N+1 \\ -4\pi n_o e v_k \Delta v_k & j = N+1 \end{cases} \tag{57}$$

$$\Delta_{jk} = a_{jk} p + b_{jk} q + c_{jk} \tag{58}$$

$$d_{jk} = \text{cofactor of } \Delta_{jk} \tag{59}$$

$$\Delta(p,q) = \det|\Delta_{jk}| \tag{60}$$

The function $\Delta(p,q)$ can be calculated to be:

$$\Delta(p,q) = p \prod_i (p + v_i q) - \sum_i \omega_p^2 \frac{df_o}{dv_i} v_i \Delta v_i \prod_{j \neq i} (p + v_j q) \qquad (61)$$

The N+1 terms are not included in the sums and products. It can be seen that $\Delta(p,q)$ is related to $D(p,q)$:

$$D(p,q) \longleftrightarrow \frac{\Delta(p,q)}{p \prod_i (p + v_i q)} \qquad (62)$$

The characteristic velocities c_i are found from the asymptotic values $p_i(q) = - c_i q$ of the roots of $\Delta(p,q) = 0$. The characteristic velocities can be seen from Eq. (61) to be the v_i. The fact that relations among the boundary and initial conditions only occur for negative characteristic velocities is consistent with the relation derived in Eq. (29).

The singularities of $E(p,q)$, apart from the singularities of the boundary and initial conditions, come from the zeros of $\Delta(p,q)$. It can be seen then that in the limit $N \to \infty$, $D(p,q)$ will have an infinite number of zeros. Evidently some of these are responsible for the cut which appears when v is treated as a continuous variable.

This approach of treating v as a discrete parameter has its main value as a bridge from the methods used in I to those treated in this paper. It provides some intuitive understanding of the problem; however, it will not be investigated further.

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13. ABSTRACT

The mixed boundary initial value problem for the Vlasov equation is treated by double Laplace transformation. It is found that certain relations must be satisified by the boundary and initial values in order to have a well-defined problem. The case of a cold beam in a cold plasma is treated as a simple example. An approach to the problem via discrete velocities is treated in an appendix.

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