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A CONSTRAINT MAPPING TECHNIQUE FOR SYSTEM OPTIMIZATION

Stephen J. Kahne

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ABSTRACT

Physical constraints on complex systems are often specified in both the state space and control space. A technique is presented in this paper for digital computation to determine if the specifications are mutually consistent with the system dynamics and, if so, for mapping all constraints into the control space. The method is applicable to nonlinear, non-stationary systems. However, closed form solutions are given for linear non-stationary systems. Some examples show application of the technique to linear and non-linear systems.

SUMMARY

When specifying the desired performance of a high dimensional control system, the designer usually has certain ideas in mind pertaining to the overall function (or mission) of the system. This notion of mission is the basis of any mathematical criterion which will be used to evaluate the system behavior. It is safe to assume that in most cases, this mission is a mission in the time domain, not in the frequency domain. With the advent and widespread use of the state space approach to system design, these time domain specifications can be directly used in the system design without referring to a subsidiary set of frequency domain specifications. The constraint mapping technique described in this paper is intended to be another step toward effective utilization of the state approach to system design. A tacit assumption of this method is the availability of digital computing facilities for the mapping and ultimately for the optimization procedure.

From a purely intuitive point of view, system optimization is an efficient method of scanning admissible regions in the control space. It is the task of optimal control theory to discover which subspaces contain the optimal control. If the physical constraints are in the control space at the outset, the designer has a subspace of admissible controls already established. If, in addition, physical constraints in the state space are specified, it is shown in this paper that this implicitly specifies a second subspace of controls. Therefore, the logical intersection of these subspaces is the space of admissible controls. The constraint mapping technique provides a numerical method for determining the space of admissible controls, if such a space exists. Moreover, it provides a closed form solution to this problem for certain classes of systems (i.e., linear, time varying systems).

In modern guidance and control theory, one important problem is the following:

Given the system dynamics

$$\underline{x}(k+1) = \underline{f}[\underline{x}(k), k] + \underline{u}(k) \tag{1}$$

where the underlined quantities represent n-vectors and $\underline{f}$ is a vector function of its vector argument. Find $\underline{u}$ (k) such that, for k=1,2,.....

a) $$\underline{x}^-(k) \leq \underline{x}(k) \leq \underline{x}^+(k) \tag{2a}$$

b) $$\underline{u}^-(k) \leq \underline{u}(k) \leq u^+(k) \tag{2b}$$

c) $$F[\underline{x}(k), \underline{u}(k), k] \text{ is minimized} \tag{2c}$$

This is the so-called "optimal control" problem with state space and control space constraints.

The constraint mapping procedure described in this paper automatically computes the relation between (2a)* and (2b) which either discloses an inconsistency in these specifications or, if they are consistent, determines $^{**}\underline{u}^+$ (k) and $^{**}\underline{u}^-$ (k) so the problem may equivalently be stated as follows: Given the system dynamics for k=1,2,.....

$$\underline{x}(k+1) = \underline{f}[\underline{x}(k), k] + \underline{u}(k) \tag{3}$$

Find $\underline{u}$ (k) such that

a) $$^{**}\underline{u}^-(k) \leq \underline{u}(k) \leq {}^{**}\underline{u}^+(k) \tag{4a}$$

b) $$F[\underline{x}(k), \underline{u}(k), k] \text{ is minimized} \tag{4b}$$

The transformed problem stated above is a variation of the "ordinary" optimal control problem with control space constraints which has been extensively studied. It is shown also that the region specified by $^{**}\underline{u}^-$ (k) and $^{**}\underline{u}^+$ (k) contains exactly the admissible controls. That is, in order to satisfy (2a) and (2b), it is necessary and sufficient for the optimal control

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* Numbers in parentheses refer to numbered equations, inequalities or other expressions; those in brackets refer to numbered reference documents.

to come from the region so defined. It is also shown that this region is a function of $\underline{x}(k)$.

Introduction

In specifying the performance of a high dimensional system, the designer usually has certain ideas in mind pertaining to the overall function (or mission) of the system. This notion of the mission is the key to any mathematical criterion which will be used to evaluate the system behavior. It is safe to say that in most cases, this mission is a mission in the time domain, not in the frequency domain. With the advent and widespread use of the state space approach to system design, these time domain specifications can be directly used in the system design without referring to a subsidiary set of frequency domain specifications. This study represents an attempt to utilize the time domain specifications in a manner which emphasizes their physical meaning to the system mission. A large scale digital computer is assumed to be the means for system control.

The seven references contain information relevant to the following discussion. [1,2,3,4,5,6,7]

Problem Statement

Consider the system

$$\underline{x}(k+1) = \underline{f}[\underline{x}(k), k] + \underline{u}(k) \tag{5}$$

where underlined quantities represent n-vectors. Find $\underline{u}(k)$ such that, for $k=1,2,\ldots$.

$$\underline{x}^-(k) \leq \underline{x}(k) \leq \underline{x}^+(k) \tag{6}$$

$$\underline{u}^-(k) \leq \underline{u}(k) \leq \underline{u}^+(k) \tag{7}$$

$$F[\underline{x}(k), \underline{u}(k), k] \text{ is minimized} \tag{8}$$

For the case where there is no physical bound on $u_j(k_i)$, for example, we impose artificial bounds $u_j^+(k_i) = +R$, $u_j^-(k_i) = -R$, for R sufficiently large.

Problems of this type are of great importance in current guidance and control systems for trajectory optimization. [6] The efficient solution of this problem should be useful in many currently important areas of space technology.

The work described in this paper presents the first step in the solution of this problem. It shows a method of transforming this problem into the one stated in (3), (4a), (4b).

Constraint Mapping

In describing the technique of constraint mapping which is introduced in this paper, it is convenient to define several sets in the space of real numbers. The notation follows.

In the state space, X(k) is the set of all state variables $x_j(k)$ at time k. X(k+1) is the set of all state variables $x_j(k+1)$ at time k+1.

In the control space we have the following sets to be defined below: ${}^{1}U_k^j$, ${}^{1'}U_k^j$, ${}^{2}U_k^j$, ${}^{3}U_k^j$, ${}^{*}U_k^j$, ${}^{**}U_k^j$. In general the subscript indicates the time at which the set is defined (i.e., ${}^{*}U_1^j$ is the set ${}^{*}U^j$ at time k=1). Thus, for example, ${}^{*}u_j(k) \in {}^{*}U_k^j$ is the j^{th} element of a particular control vector ${}^{*}\underline{u}(k)$ at time k. ϕ is the null set. R is the real line. A bar over a set (e.g., $\overline{{}^{*}U_k^j}$) denotes its compliment. The sets are defined in terms of the problem statement of (5), (6), (7) and (8) for j = 1,2,..., n.

The technique described below shows how to construct a set ${}^{1}U_k^j$ so that ${}^{1}U_k^j = \overline{{}^{3}U_k^j}$.

$u_j(k) \in {}^{3}U_k^j$ implies violation of (5), (6). A control variable selected from ${}^{3}U_k^j$ is sufficient to drive the system outside the allowable region in the state space (at time k+1).

$u_j(k) \in {}^{2}U_k^j$ implies satisfaction of (7). The set so defined is the connected set on R whose boundaries are $\underline{u}^{+}(k)$ and $\underline{u}^{-}(k)$ in (7).

$^{1'}U_k^j \subseteq {}^{1}U_k^j$

$u_j(k) \in {}^{1'}U_k^j$ implies satisfaction of (5), (6). A control variable selected from ${}^{1'}U_k^j$ necessarily drives the system into an allowable state at time k+1, where the allowable states are defined by (6).

$^{*}U_k^j = {}^{1}U_k^j \cap {}^{2}U_k^j$

$u_j(k) \notin {}^{*}U_k^j$ implies violation of (5), (6), and/or (7). In order to drive the system into an allowable (defined by (6)) state at time k+1, it is necessary to select a control from ${}^{*}U_k^j$.

$^{**}U_k^j = {}^{1'}U_k^j \cap {}^{2}U_k^j$

$u_j(k) \in {}^{**}U_k^j$ implies satisfaction of (5), (6), (7). In order to drive the system into an allowable state at time k+1, it is necessary and sufficient to select a control from ${}^{**}U_k^j$.

It is clear that all these sets are compact. A graphical presentation of these definitions is given in Figure 1.

The set ${}^{**}U_k^j$ is the set of all controls which are candidates for the "optimal" control. It is this set which we would like to calculate at the outset. However, this set is a function of the present state of the system (see Appendix). That is, ${}^{**}U_k^j = {}^{**}U_k^j\,(X_i(k))$. For this reason, we have found it desirable to introduce ${}^{*}U_k^j$, not a function of $X_i(k)$. However, this set contains some inadmissible controls. The technique maps the connected set X(k) into the connected set ${}^{1}U_k^j$ by determining the boundaries of ${}^{1}U_k^j$, called ${}^{1}u_j^+(k)$ and ${}^{1}u_j^-(k)$. From this we obtain ${}^{*}U_k^j$ directly.

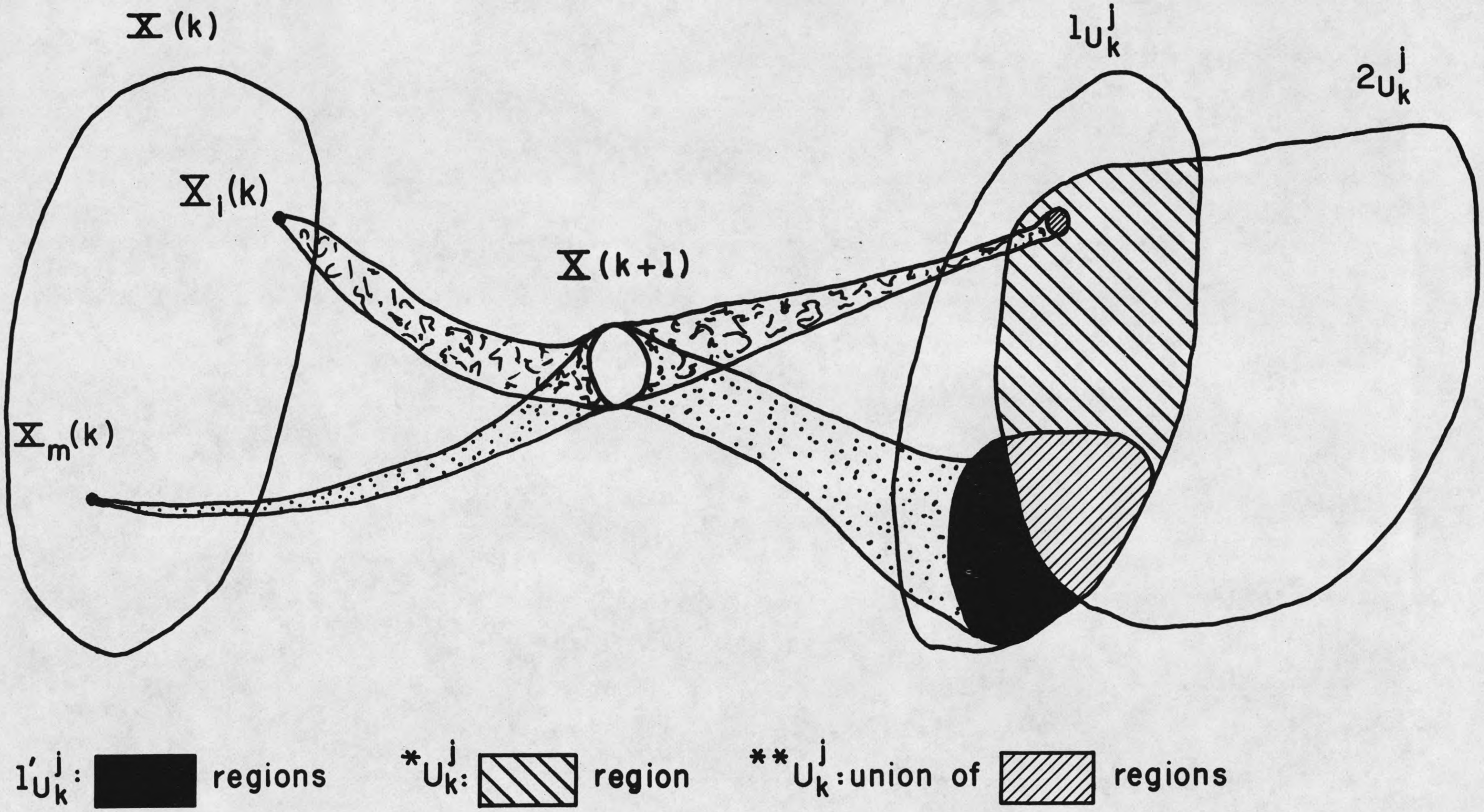

Figure 1. Graphical Presentation of Constraint Mapping Procedure

Consider the nature of the mapping. In general, the boundaries of ${}^{1}U_k^j$ must be determined by an iterative numerical procedure. For cases where f_j is a monotonic function*, it is sufficient to consider only $\underline{x}^{\pm}(k)$ to locate the boundaries of ${}^{1}U_k^j$. Of course, linear systems are an example of such a situation. Where f_j is not monotonic, it may not be sufficient to examine only $\underline{x}^{\pm}(k)$ to find ${}^{1}U_k^j$. For the general case, a numerical procedure has been developed to perform the necessary boundary mapping. A description of this procedure, the method of adaptive constrained descent is reported in [7].

Another interesting feature of the technique is the way in which it handles the problem of inconsistent specifications. It is quite possible, as previously mentioned, that (5), (6), and (7) are mutually inconsistent. In case they are, one of two results occurs. In some cases it will turn out that ${}^{*}U_k^j = \phi$. If this happens, obviously ${}^{**}U_k^j = \phi$. A second possibility is simply ${}^{**}U_k^j = \phi$. The latter case is more difficult to detect and will undoubtedly prove to be more troublesome when (8) is considered. For the purposes of this paper, however, the result of the two cases is the same, namely, that the problem specified by (3), (4a), (4b), is meaningless and (6) and/or (7) must be altered.

A Detailed Example

It was pointed out in the previous section that in the case of linear systems, the selection of ${}^{1}U_k^j$ is a straight forward task, requiring only an examination of $\underline{x}^{\pm}(k)$. We will now discuss a specific example to demonstrate this fact.

Consider the system

$$\underline{x}(k+1) = A(k)\,\underline{x}(k) + \underline{u}(k) \tag{9}$$

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*We define $f_j(\underline{x}(k))$ to be a monotonic function if $\forall\, \epsilon > 0$, $\forall\, x_i(k)$, $f_j[x_1(k)\, x_2(k), \ldots, x_i(k) + \epsilon, x_{i+1}(k), \ldots, x_n(k)] \geq f_j[\underline{x}(k)]$.

where A(k) is an n by n matrix. From (9) one has

$$u_j(k) = x_j(k+1) - \sum_{i=1}^{n} a_{ji}(k)\, x_i(k) \qquad (10)$$

The set ${}^{1}U_k^j$ will be determined by obtaining its end points, ${}^{1}u_j^{\pm}(k)$. (10) yields $u_j^+(k)$ when the first term on the right is greatest and each of the others are least. This will occur if $x_j^+(k+1)$ is used and, if $a_{ji}(k)$ is positive, using $x_i^-(k)$. If $a_{ji}(k)$ is negative, then $x_i^+(k)$ is used. This may be stated as follows. Define

$$S_{ji}(k) = -\mathrm{sgn}\,[a_{ji}(k)] \qquad (11)$$

if $S_{ji}(k) = +1$, to find ${}^{1}u_j^{\pm}(k)$, use $x_i^{\pm}(k)$ in (14). If $S_{ji}(k) = -1$ to find ${}^{1}u_j^{\pm}(k)$, use $x_i^{\mp}(k)$. The $u_j(k)$ which results is the boundary of ${}^{1}U_k^j$.

As a particular example of this process, consider the system shown in Figure 2. Then (9) becomes

$$\underline{x}(k+1) = A(k)\,\underline{x}(k) + \underline{u}(k)$$

where

$$A(k) = \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix}$$

For this problem, the constraints of (6) and (7) are shown graphically in Figure 3. We have assumed for illustration that three instants of time are sufficient to convey the significant system behavior.

For k = 1,

$$u_j(1) = x_j(2) - \sum_{i=1}^{n} a_{ji}(1)\, x_i(1) \qquad (12)$$

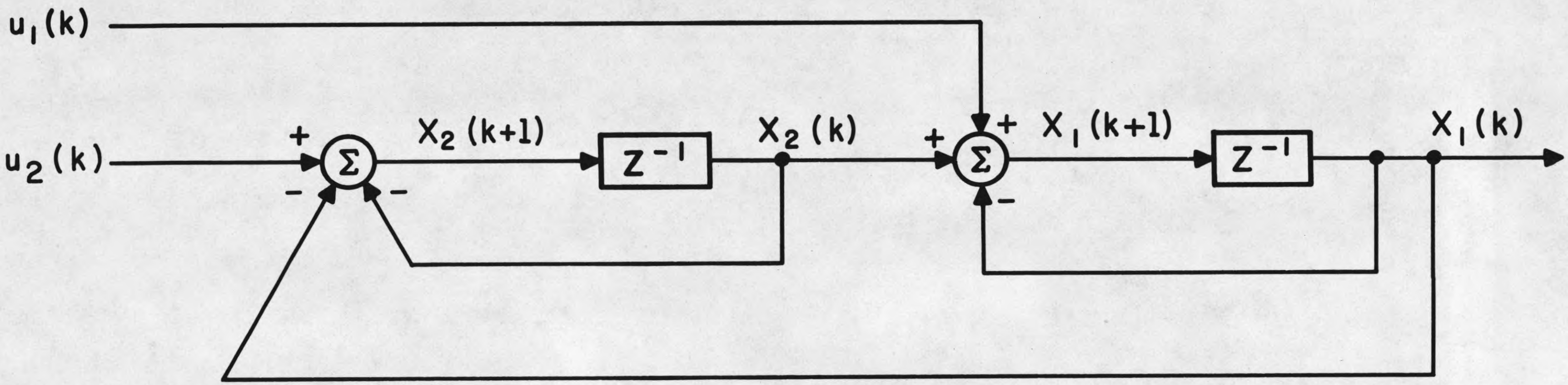

Figure 2. System for First Example

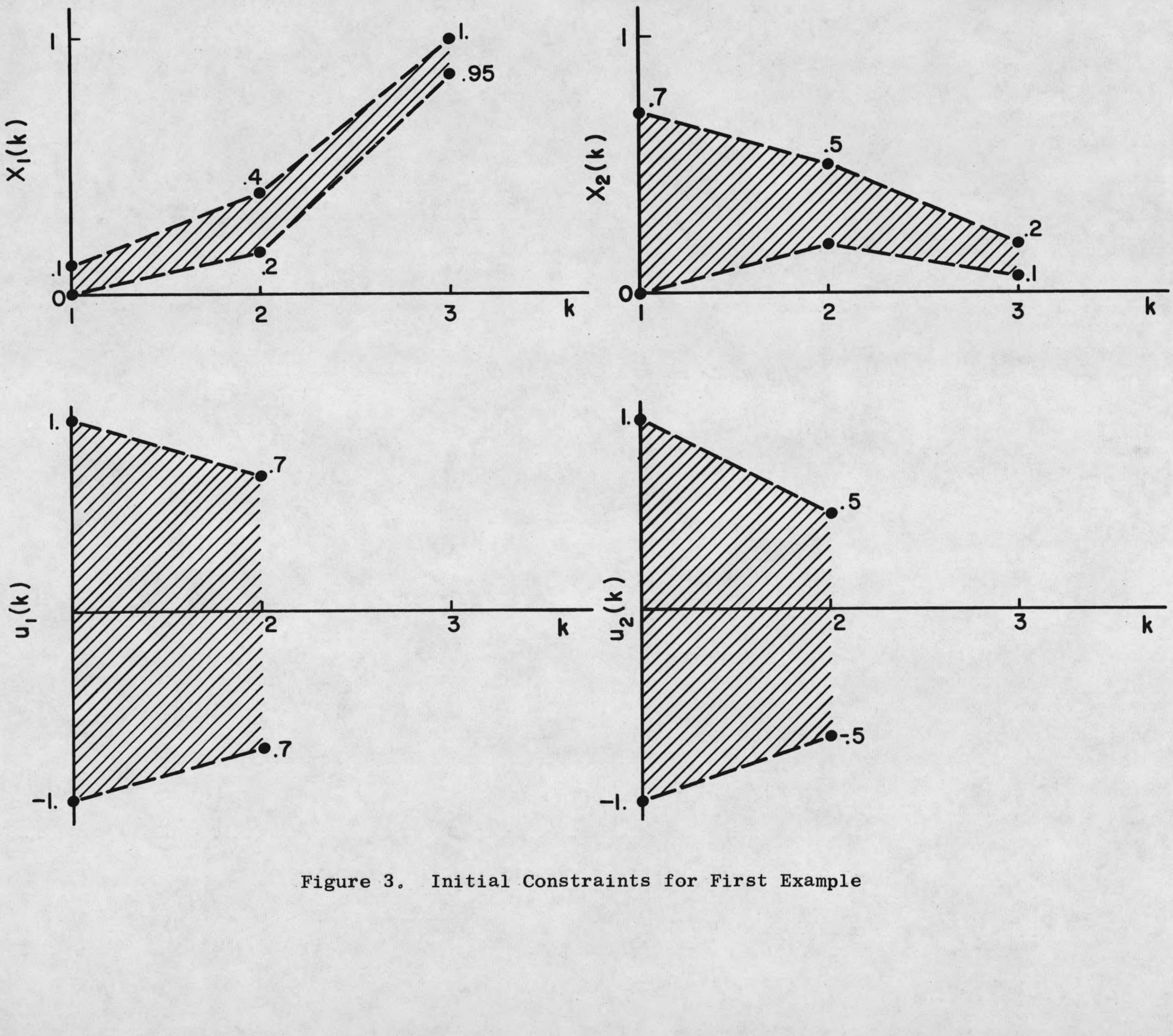

Figure 3. Initial Constraints for First Example

and

$$S_{ji}(1) = -\text{sgn}\,(a_{ji})$$

In particular

$$u_1(1) = x_1(2) - \sum_{i=1}^{n} a_{ji}\, x_i(1)$$

and

$$S_{11}(1) = +1 \quad , \quad S_{12}(1) = -1$$

Thus,

$$^{1}u_1^{+}(1) = x_1^{+}(2) - a_{11}\, x_1^{+}(1) - a_{12}\, x_2^{-}(1)$$

$$= .4 - (-1)\,(.1) - (1)\,(0)$$

$$= .4 + .1 = 0.5$$

and

$$^{1}u_1^{-}(1) = x_1^{-}(2) - a_{11}\, x_1^{-}(1) - a_{12}\, x_2^{+}(1)$$

$$= .2 - (-1)\,(0) - (1)\,(.7)$$

$$= .2 - .7 = -0.5$$

Also from Fig. 3, one has

$$^{2}u_1^{+}(1) = 1.0 \quad , \quad ^{2}u_1^{-}(1) = -1.0$$

and therefore

$$^{*}u_1^{+}(1) = 0.5 \quad , \quad ^{*}u_1^{-}(1) = -0.5$$

In a similar manner, Figure 4 is established completing the determination of $\underline{u}^{+}(k)$ and $\underline{u}^{-}(k)$ for this problem.

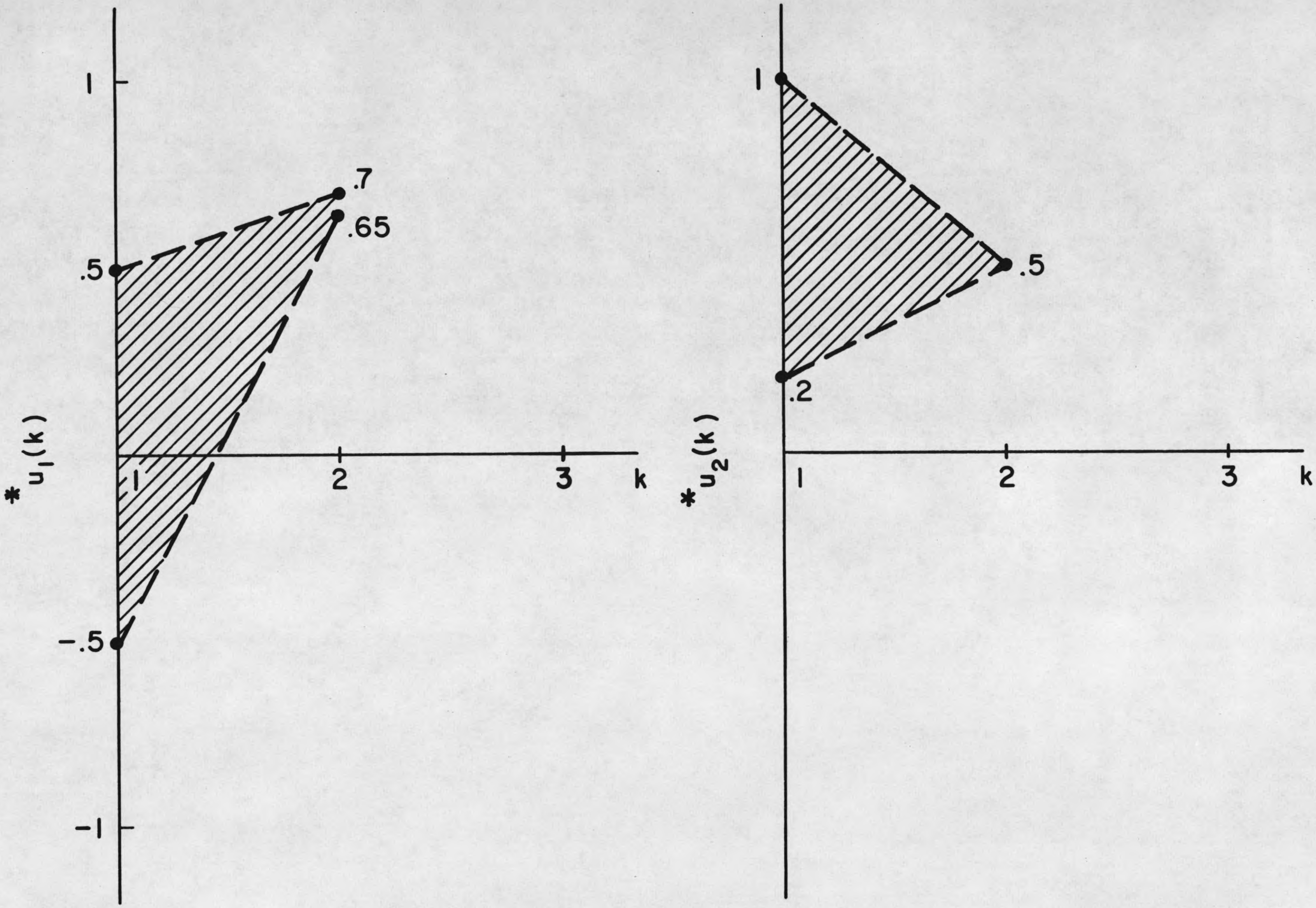

Figure 4. Bounds on $^*U_k^j$ for First Example

Although this example does not explicitly demonstrate how an inconsistency in the specifications is detected, the calculation required to obtain $u_2^{\pm}(2)$ shows how this may arise. The calculations required are tedious but straightforward. Moreover, they are ideally suited for automatic computation because of their "table-lookup" nature.

It should be emphasized that the extreme values of $u_j(k)$ occur at extreme values of $x_j(k)$ only in special cases (in the case of linear systems, for example). For the general case an iteration procedure is required.

Further Examples

Another example of a linear, stationary system is shown in the sequence of Figures 5 to 7.

A non-linear relay system with hysteresis and dead-zone is shown in the last sequence of figures (Figs. 8 to 10).

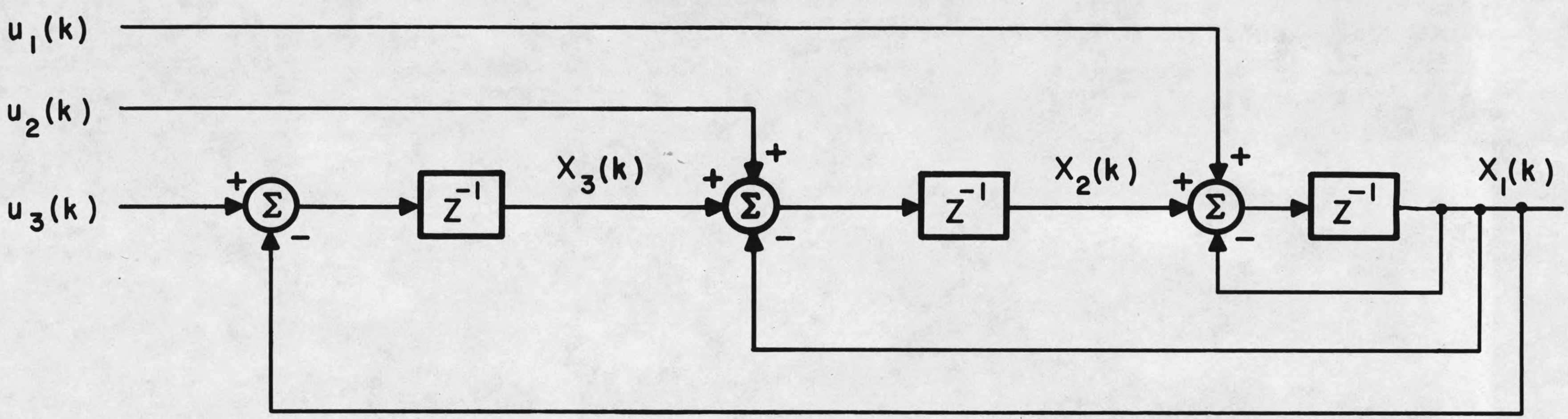

Figure 5. System for Second Example

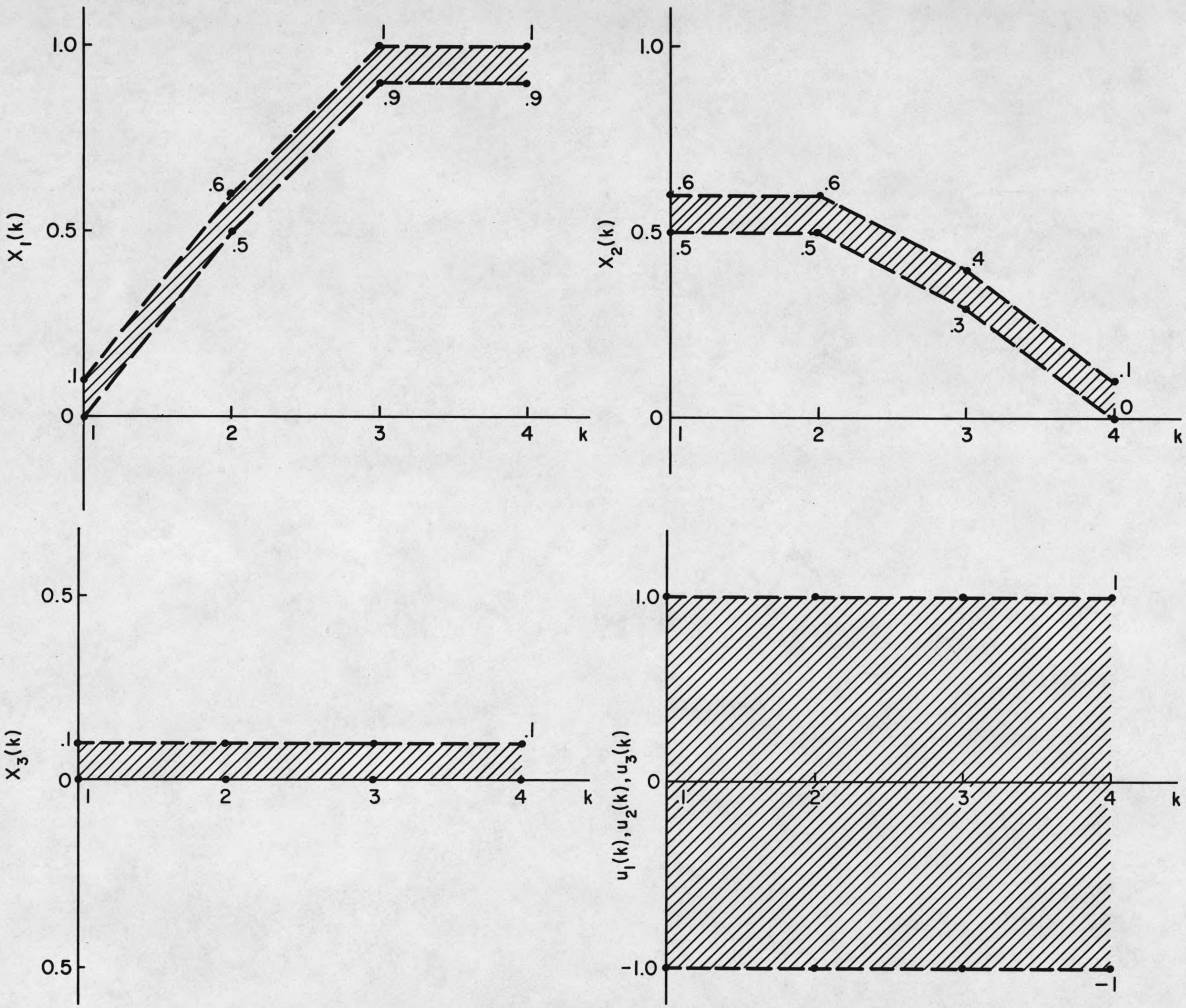

Figure 6. Initial Constraints for Second Example

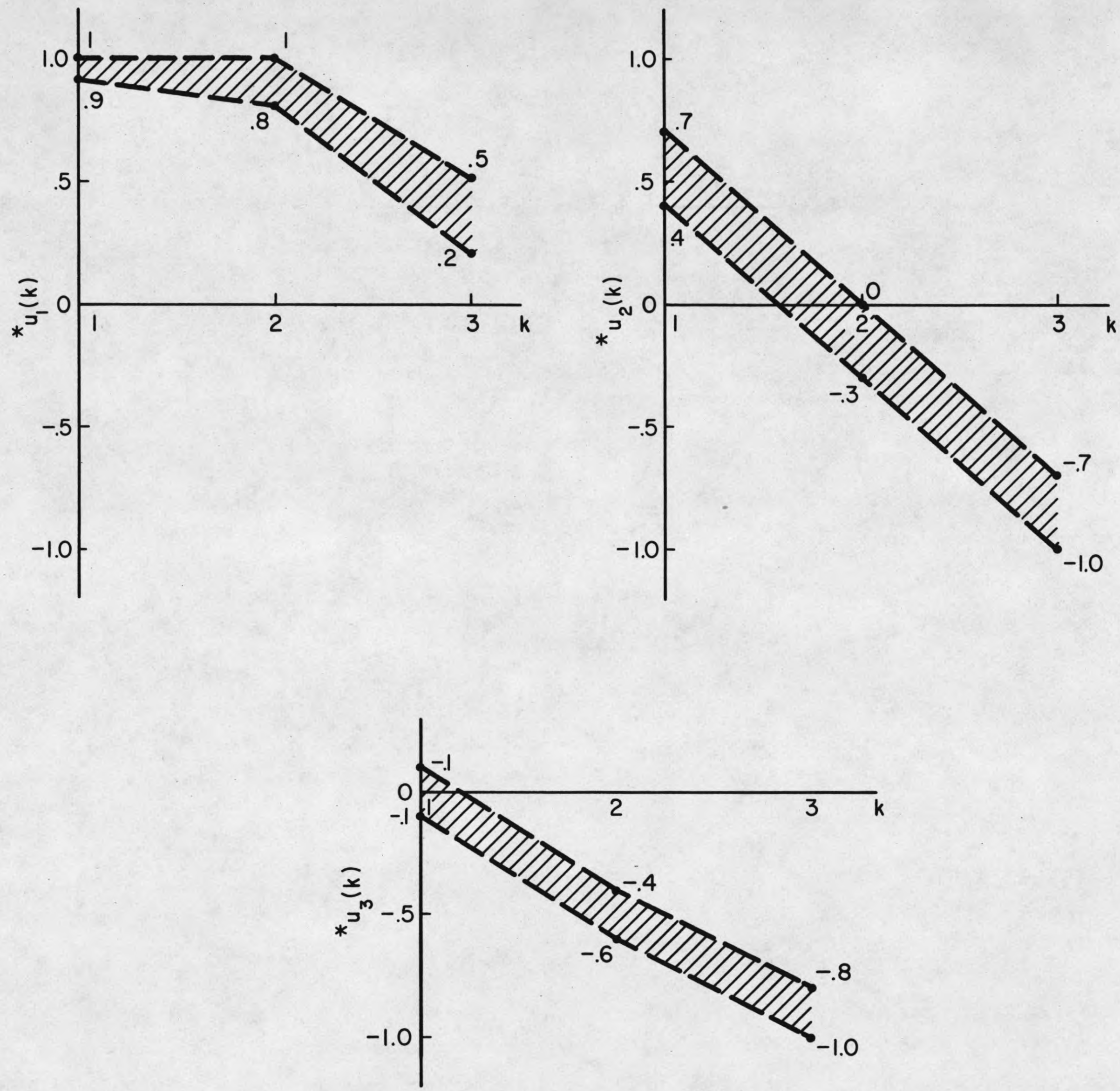

Figure 7. Bounds on $^{*}U_k^j$ for Second Example

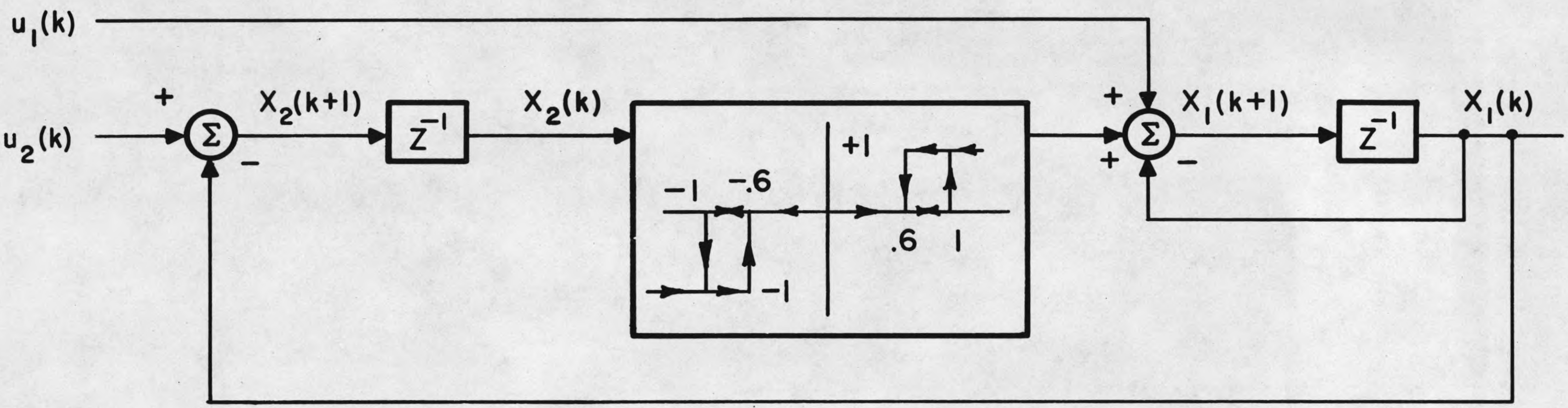

Figure 8. System for Third Example

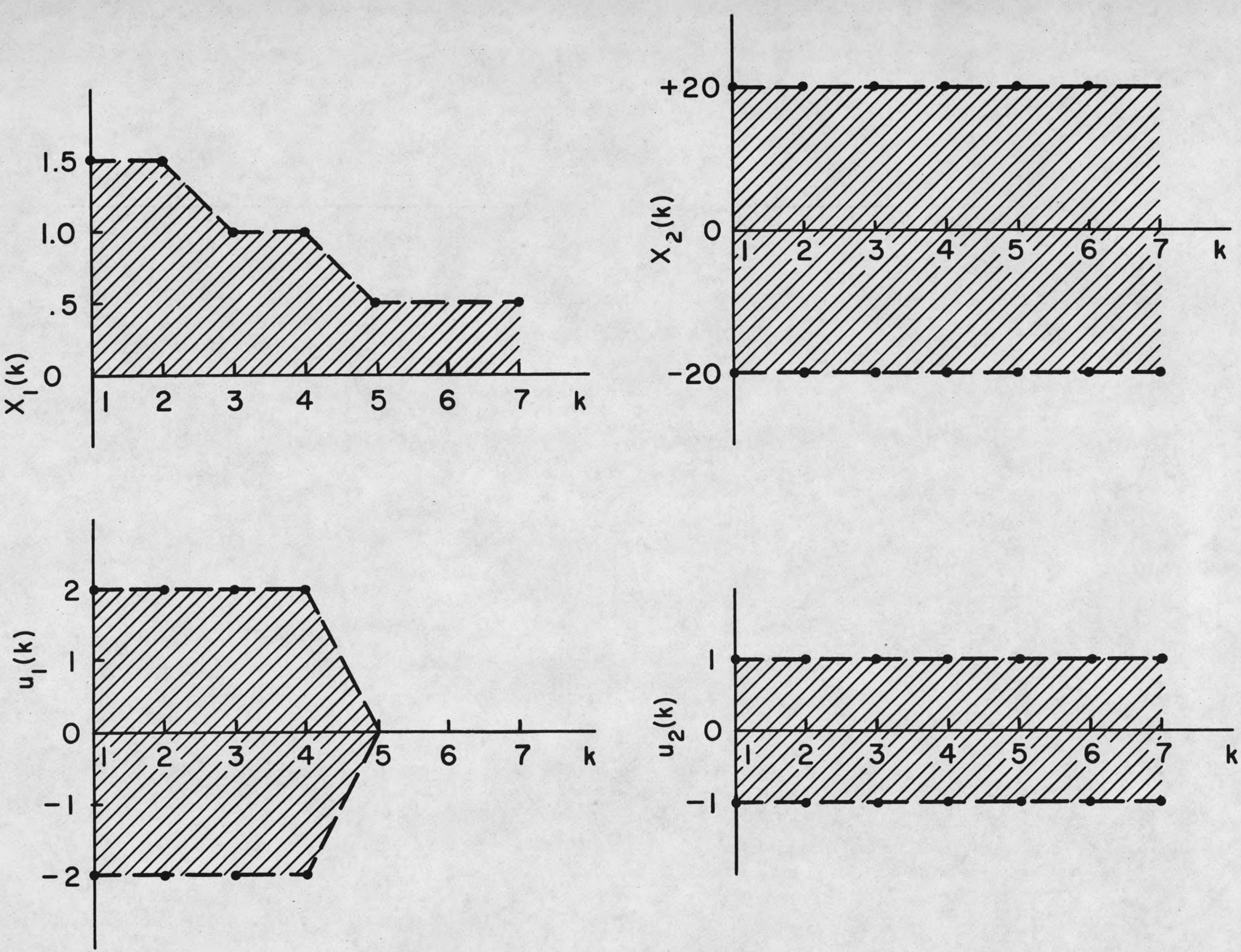

Figure 9. Initial Constraints for Third Example

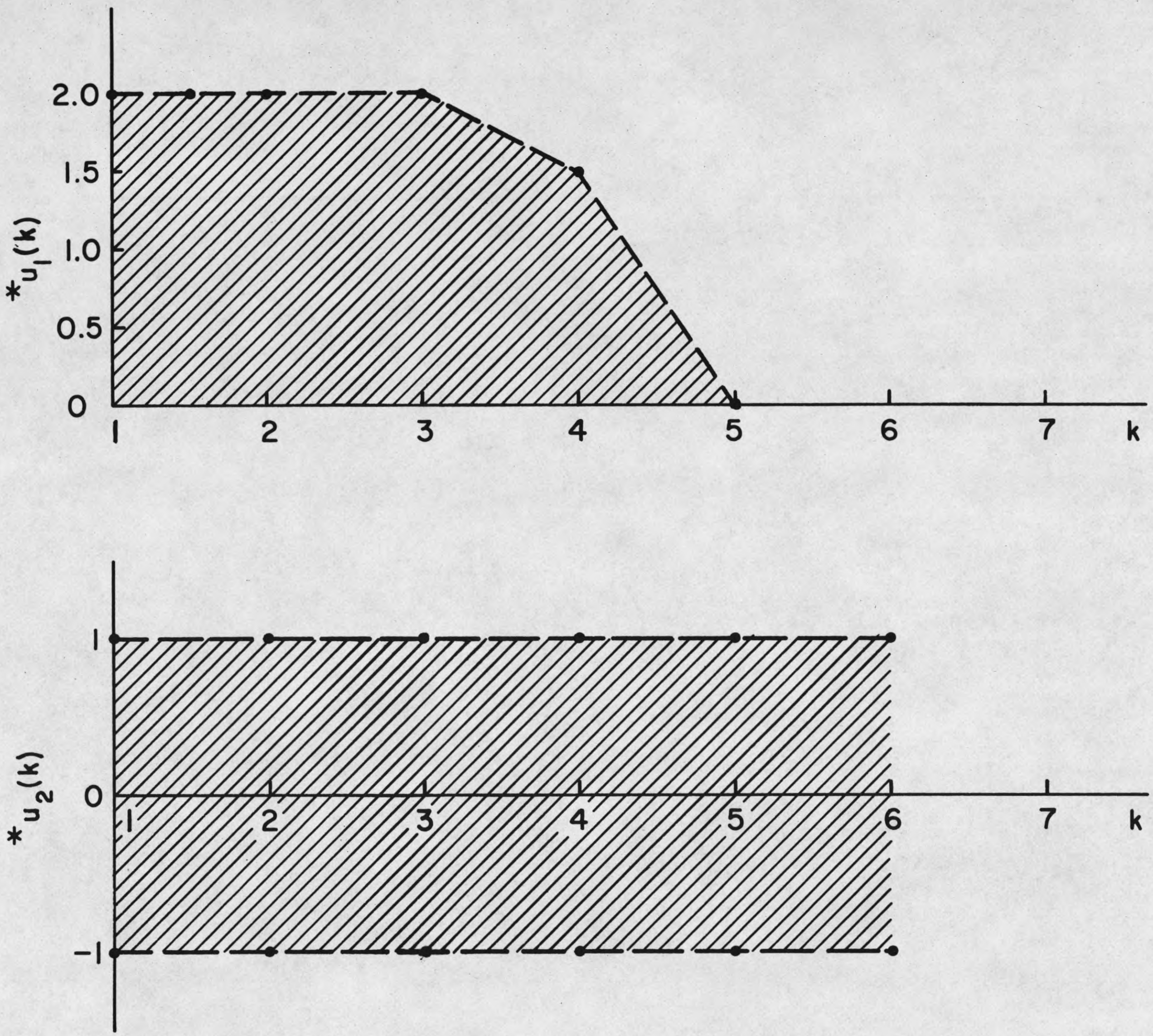

Figure 10. Bounds on $^{*}U_k^j$ for Third Example

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APPENDIX

It is instructive to consider a particular example to illustrate certain of the more general results described in the text of the paper. Let us consider a linear, stationary system for which

$$\underline{x}(k+1) = A\,\underline{x}(k) + \underline{u}(k)$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The equations for $\underline{u}(k)$ for this second order system are

$$u_1(k) = x_1(k+1) - x_1(k) - x_2(k)$$

$$u_2(k) = x_2(k+1) - x_1(k) - x_2(k)$$

$$\underline{x}^+(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \underline{x}^-(k) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\underline{u}^+(k) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \qquad \underline{u}^-(k) = \begin{bmatrix} -10 \\ -10 \end{bmatrix}$$

It is clear that $u_1^+(k) = +3.0$ and $u_1^-(k) = -3.0$. However, if $x_1(k) = x_2(k) = 0$ and we try $u_1(k) = +2$ we get $x_1(k+1) = +2$ which violates the state space constraints. This shows that being in ${}^1U_k^j$ is not sufficient for (6) to be satisfied.

In addition, there appears to be no reason to suspect that ${}^1U_k^j = \bigcup_t {}^{1'}U_k^j(X_t)$ except in special cases. Further, it has not been shown that ${}^1U_k^j$ could be so constructed in the general case. Since X(k) is a connected set, ${}^1U_k^j$ will also be connected provided $\underline{f}$ is a continuous function of its argument. If $\underline{f}$ is not continuous this may not be true leading us to believe ${}^1U_k^j \neq \bigcup_t {}^{1'}U_k^j(X_t)$ in general.