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CONTROL OF FLEXIBLE JOINT ROBOTS: A SURVEY

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CONTROL OF FLEXIBLE JOINT ROBOTS: A SURVEY

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Abstract

The robotics literature of the 1980's contains important advances in our understanding of the dynamics and control of joint flexibility in robotic manipulators. This paper contains a survey of the applications of geometric control theory, singular perturbation theory, robust and adaptive control theory to the control of flexible joint robots.

1 Introduction

The desire for higher performance from robot manipulators has spurred research in a number of areas: mechanics, sensors and actuators, control, computer architectures and artificial intelligence, to name a few. In the area of control, the trend of the past decade has been to apply increasingly sophisticated tools from nonlinear control theory with the goal of developing controllers that achieve accurate, high speed tracking with low sensitivity to modeling errors, disturbances, payloads, sensor noise, and the like. Concurrent advances in microprocessor technology have made the implementation of complicated nonlinear control algorithms feasible from a practical standpoint.

At the same time, achieving high performance by applying advanced control techniques requires an increased understanding of the dynamics of robot manipulators. To borrow from the language of linear control theory, as the bandwidth of the control system increases, dynamic effects which previously were beyond the frequency range of interest, now must be considered in the compensator design.

For robot manipulators this means that a dynamic model of a robot as an ideal chain of coupled rigid bodies is frequently not adequate for the design of controllers to achieve high performance. Beginning in the early 1980's the problem of joint flexibility began to be recognized as an important factor limiting robot cycle time. In this paper we survey the progress that has been made during the past decade in our understanding of the dynamics and control of flexible joint robots.

2 Background

The bibliography at the end of this paper, while not complete, nevertheless indicates a considerable research effort directed toward the control problem for flexible joint robots. The earliest reference of which the author is aware that included joint compliance in the modeling and control design of a robot manipulator was (Liegeois, et.al.[44]). The first work on the development of general Lagrangian dynamic models for flexible joint robots was contained in (Nicosia, et.al.[52]). From the same group of researchers also came studies of controllability properties, including feedback linearization (Cesareo and Marino[11]), (Marino and Nicosia [48]), (Marino [45]), and singular perturbation methods (Ficolo, et.al. [22]). Dynamic feedback linearization results were presented in (DeLuca, et. al. [19] and [20]), (DeLuca [17] and [18]), and (DeSimone and Nicolo [21]).

In (Marino and Spong [49]), (Forrest–Barlach and Babcock [24]) feedback linearization results using nonlinear *static* (as opposed to dynamic) state feedback were presented for special classes of robots. In (Spong [71]) these static feedback linearization results were extended to general n link manipulators based on a simplified dynamic model. The importance of modeling on the controllability properties of flexible joint robots was brought out in this paper.

Robustness results for flexible joint robots have been appeared in a number of references including (Spong [73]), (Sira–Ramirez and Spong [81]), (Widman and Ahmad [97]), (Pfeiffer, et.al. [61]), (Troch and Kopacek [94]), (Grimm, et.al. [31]), (Kuntze and Jacubasch [42]), (Kuntze, et.al. [43]), and (Canudas and Lys [10]). Most of these robust control results are based on feedback linearization properties which guarantee matching conditions. So–called *outer loop* compensators have been designed using Lyapunov techniques (Spong [73]), sliding mode theory (Sira–Ramirez and Spong [81], Slotine and Hong [68]), and functional analytic methods (Grimm et.al. [31]).

The problem of observer design for flexible joint robots is of great practical importance. Some results have been reported in (Albert and Spong [2]), (Nicosia, et.al. [55]), (Bortoff, et.al. [8]), and (Nicosia and Tomei [53]).

The first application of integral manifold techniques, which are based on earlier singular perturbation models, was in (Khorasani and Kokotović[39]), (Khorasani and Spong [40]), (Spong, et.al. [78] and [79]).

Adaptive control of flexible joint robots has been investigated in (Marino and Nicosia [47]), (Tomei, et.al. [92]), (Spong [76]), (Ghorbel, et.al. [26] and [27]), and (Hung and Spong [36]). Force control has recently been investigated in (Spong [75]), and (Mills [50]).

Experimental results have to date been sparse. Early experimental work of (Sweet and Good [87]) and (Good, et.al.[29]) was influential in motivating researchers to study the problem of joint flexibility. Experimental tests of feedback linearization and adaptive control methods have been reported in (Hung [35]), (Ghorbel, et.al. [26] and [27]).

3 Modeling

In this section we discuss the similarities and differences between the dynamic and control properties of flexible joint robots and those of rigid robots. A robot manipulator consists of n links interconnected at n joints into an open kinematic chain. Each link is driven by an actuator, which may be electric, hydraulic, or pneumatic. The actuator may be located directly at the joint that it actuates or it may drive the link through a remote transmission of some sort, such as with pulleys, chains, torque tubes, belts, or tendons.

A rigid robot model assumes that both the links and the coupling between the actuators and links are perfectly rigid. The dynamics can thus be described by n generalized coordinates representing the degrees–of–freedom of the n joints.

If we denote the kinetic(KE) and potential(PE) energies of the robot as

$$KE = 1/2\dot{\mathbf{q}}^T M(\mathbf{q})\dot{\mathbf{q}} \qquad ; \qquad PE = V(\mathbf{q}) \tag{1}$$

where $\mathbf{q}$ is the n–vector of joint variables, $M(\mathbf{q})$ is the symmetric, positive definite $n \times n$ inertia matrix , and $V(\mathbf{q})$ denotes the potential energy due to gravity, a straightforward application of the Euler–Lagrange equations results in the dynamic model (Spong and Vidyasagar [82]),

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \tag{2}$$

where $C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$ represents Coriolis and centrifugal generalized forces, $\mathbf{g}(\mathbf{q}) = \nabla V^T(\mathbf{q})$ represents gravitational generalized forces, and $\mathbf{u}$ represents the input generalized force from the actuators.

Remark: It is common to express the inertia matrix $M(\mathbf{q})$ in the above equation as the sum $M(\mathbf{q}) = D(\mathbf{q}) + J$, where $D(\mathbf{q})$ represents the inertia of the n rigid links and J is a constant diagonal matrix representing the inertia of the actuators [82]. In the case of electrically driven robots, for example, J represents the inertia of the rotor **about the axis of rotation induced by the magnetic field of the stator** but ignores the effect of link motion on the actuator. The link motion may, in fact, be changing the direction of this axis is $\Re^3$. This means that gyroscopic coupling between actuators and links is neglected in the above model formulation. This fact, which is generally unrecognized in the literature, becomes important when joint flexibility is included in the model formulation. (See [73] for further discussions of this.)

The structural properties of (2) that are most relevant for control purposes are by now well–known ([60]) and can be summarized as follows:

- **Property 1: For the rigid robot model (2) there is an independent control input for each degree of freedom.**
- **Property 2: The dynamic equations (2) define a passive mapping $\mathbf{u} \rightarrow \dot{\mathbf{q}}$ between input torque and link velocity.**

Property 2 follows from a straightforward application of Hamilton's equations. Let H denote the Hamiltonian, i.e., the sum of the kinetic and potential energies of the robot. Then, it is easy to show ([60]) that

$$\frac{dH}{dt} = \dot{\mathbf{q}}^T\mathbf{u} \tag{3}$$

from which it follows that

$$\int_0^t \dot{\mathbf{q}}(\tau)^T\mathbf{u}(\tau)d\tau = H(t) - H(0) \geq -H(0) \tag{4}$$

and passivity of the mapping $\mathbf{u} \rightarrow \dot{\mathbf{q}}$ follows. Related to the above passivity property is the following

- **Property 3: For suitable choice of the matrix $C(\mathbf{q}, \dot{\mathbf{q}})$ in (2) the matrix**

$$\dot{M}(\mathbf{q}) - 2C(\mathbf{q}, \dot{\mathbf{q}}) \tag{5}$$

is skew symmetric [82].

Finally, a crucial property for the design of adaptive control algorithms is

- **Property 4: The equations of motion (2) are linear in a suitably defined set of parameters.**

In other words (2) can be written

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\theta = \mathbf{u} \tag{6}$$

where θ is a vector of parameters (masses, moments of inertia, etc.) and Y is a matrix of appropriate dimension known as the regressor.

A flexible joint robot model, by contrast, assumes that the links are rigid but that the actuators are elastically coupled to the links. This elastic coupling introduces an additional degree of freedom at each joint so that the $2n$ generalized coordinates are required to describe the configuration of the robot.

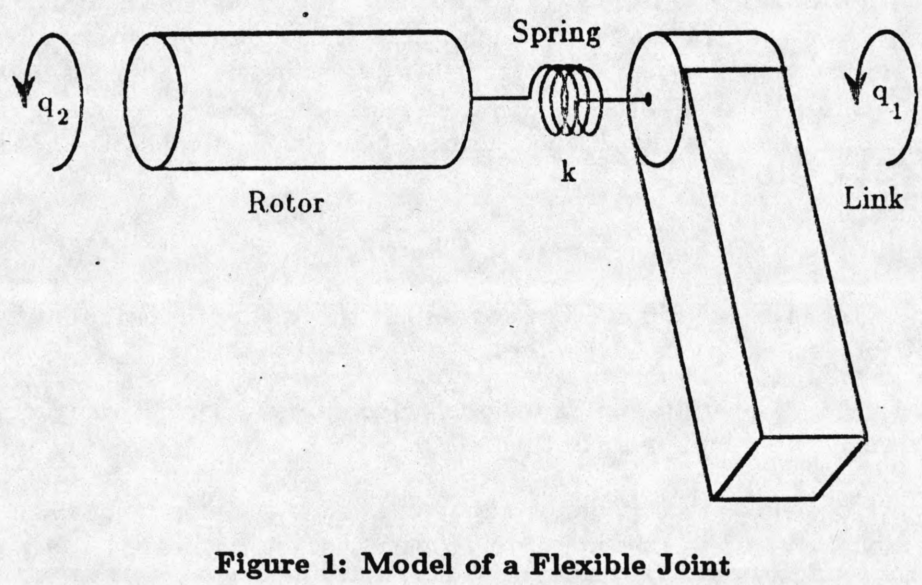

Figure 1: Model of a Flexible Joint

For simplicity of exposition assume that all joints are revolute and are directly actuated by a DC–electric motor. This means that the stator of the i–th motor is rigidly mounted to the i–1–st link, while its rotor is elastically coupled to the i–th link. Referring to Figure 1, let q^{2i} represent the angle of the i–th link and q^{2i-1} represent the angle of the i–th actuator, both angles being measured relative to the previous link. Therefore the link angles are precisely the usual Denavit–Hartenberg joint variables (Spong and Vidyasagar [82]) so that the kinematic description of the manipulator is identical to that of a rigid joint robot. We note that the difference $q^{2i} - q^{2i-1}$ represents the elastic displacement of the joint. Define the vectors $\mathbf{q}_1 = (q^2, q^4, \ldots, q^{2n})^T$ and $\mathbf{q}_2 = (q^1, q^3, \ldots, q^{2n-1})^T$ of joint angles and actuator angles, respectively. Then

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \tag{7}$$

is the $2n$–vector of generalized coordinates for the system. We model the joint flexibility by a linear torsional spring at each joint and denote by K the diagonal matrix of joint stiffness coefficients.

We further assume that the rotors can be modeled as uniform cylinders so that, for example, the gravitational potential energy of the system is independent of the rotor position and is therefore a function only of $\mathbf{q}_1$. If, as in the case of the rigid model (2), we model the actuator inertia by a constant diagonal $n \times n$ matrix J then the inertial and reaction forces will be independent of rotor velocities and accelerations. In this case, as shown in (Spong [73]), the dynamic equations of motion of an elastic joint robot can be expressed in terms of the partition of the generalized coordinates (7) as

$$\begin{bmatrix} D(\mathbf{q}_1) & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} + \begin{bmatrix} C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{g}(\mathbf{q}_1) \\ 0 \end{bmatrix} + \begin{bmatrix} K(\mathbf{q}_1 - \mathbf{q}_2) \\ -K(\mathbf{q}_1 - \mathbf{q}_2) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{u} \end{bmatrix} \tag{8}$$

or, writing these out,

$$\begin{aligned} D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) + K(\mathbf{q}_1 - \mathbf{q}_2) &= 0 \\ J\ddot{\mathbf{q}}_2 - K(\mathbf{q}_1 - \mathbf{q}_2) &= \mathbf{u}. \end{aligned} \tag{9}$$

It is easy to see that (9) reduces to (2) in the limit as the joint stiffness tends to infinity, and that (9) has roughly the same computational complexity as an n–link rigid robot.

In some references, for example, [11], [18], [19], [40], [48], [55], the inertia of the (cylindrical) actuators about three independent axes is modeled. This, of course, leads to a more detailed and more complex model. In this case the inertial forces, reaction forces, etc., will depend on the rotor velocities and accelerations, and the equations of motion become

$$\begin{bmatrix} M_{11}(\mathbf{q}_1) & M_{12}(\mathbf{q}_1) \\ M_{21}(\mathbf{q}_1) & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} + \begin{bmatrix} C_1(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}} \\ C_2(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{g}(\mathbf{q}_1) \\ 0 \end{bmatrix} + \begin{bmatrix} K(\mathbf{q}_1 - \mathbf{q}_2) \\ -K(\mathbf{q}_1 - \mathbf{q}_2) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{u} \end{bmatrix} \tag{10}$$

One should note in the above that, because the rotors are modeled as uniform cylinders, the inertia matrix is a function only of the joint variables $\mathbf{q}_1$. This also means that the Christoffel symbols (Spong and Vidyasagar [82]) which define the matrices C_1 and C_2 will be independent of the rotor variables, as will the gravitational forces acting on the system. Note, however, that the rotor velocity does enter into the Coriolis and centrifugal terms since C_1 and C_2 are multiplied by $\dot{\mathbf{q}}$.

Note that we have not explicitly included damping in our models. While a certain amount of damping is always present, it was demonstrated experimentally in (Sweet and Good [87]) that the problems of joint flexibility are most evident primarily in the case that the joint damping is small. We idealize this by ignoring the damping altogether.

It is important to understand both flexible joint models (9) and (10) because, as we will see they possess different properties with respect to feedback linearization. In the model (10) there are two sources of interaction between the degrees–of–freedom:

- 1) the torque transmitted through the spring and
- 2) the inertial coupling represented by the off–diagonal terms of the inertia matrix and the associated Coriolis forces.

In the simplified model (9) the inertial coupling is ignored and only the torque transmitted through the joints serves to couple the actuators to the links. In practice the inertial coupling between actuators and links is considerably weaker and can most often be ignored in the control system design (see Spong [73]).

In state space we set $\mathbf{x} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix} \in \Re^{4n}$. Invertibility of the inertia matrices means that both (9) and (10) are of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u} \tag{11}$$

with $\mathbf{f}(\mathbf{x}) \in \Re^{4n}, G(\mathbf{x}) \in \Re^{4n\times n}$. If we take as output the vector $\mathbf{q}_1$ of link angles then we have an output equation

$$\mathbf{y} = \mathbf{q}_1 = H\mathbf{x} \tag{12}$$

where $H = [I\ 0\ 0\ 0] \in \Re^{n\times 4n}$. Note that (11)–(12) is a so-called square system, i.e., the numbers of inputs and outputs are the same.

It is straightforward now to show that, of the four properties of rigid robot dynamics discussed previously, Property 3, skew symmetry of $\dot{M} - 2C$, and Property 4, linearity in the parameters, hold also in the case of flexible joint robots. However, for flexible joint robots the number of control inputs is n but there are $2n$ degrees–of–freedom; thus, Property 1 fails. Since Property 1 is crucial for all inverse dynamics[1] based control results for rigid robots, it is not obvious how to extend most of the available robust control techniques to the case of flexible joint robots.

Also, the mapping from input torque to link velocity is not passive in the flexible joint case. This is the familiar situation of non–collocation of input and output variables and has fundamental consequences for stability of control laws. For example, recent passivity based adaptive control approaches for rigid robots (Slotine and Li [69], Sadegh and Horowitz [67]) cannot be directly applied to flexible joint robots without further assumptions.

[1] also called computed torque.

It is apparent that the four properties above are rather special and have resulted in very strong robust and adaptive control results for rigid robots. The fact that the first two properties above fail to hold for flexible joint robots greatly complicates the control problem.

4 Feedback Linearization

The idea of linearizing and decoupling the nonlinear dynamics of robots by suitable control goes back to the idea of computed torque (feedforward control) and inverse dynamics (feedback control). For rigid robots the feedback linearization is conceptually simple since there is an independent control for each generalized coordinates. This means that the nonlinearities in the system (2) can be directly cancelled by the input,

$$\mathbf{u} = M(\mathbf{q})\mathbf{v} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}), \tag{13}$$

where $\mathbf{v} = (v_1, \ldots, v_n)$ is a new input to the system, leading to a decoupled set of double integrators,

$$\ddot{q}_j = v_j. \tag{14}$$

The analogous result for flexible joint robots is considerably more complex. The simplest way to understand input/output linearization is to ask, "What is the minimum number of times the output must be differentiated so that the input appears?" Specifically, given a square nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{p} g_i(x)u_i \tag{15}$$

$$y_i = h_i(x) \quad ; \quad i = 1, ..., p \tag{16}$$

with $x \in \Re^n$, $u_i \in \Re$, $y_i \in \Re$, we differentiate the j-th output with respect to time to get

$$\begin{aligned} \dot{y}_j &= \frac{\partial h_j}{\partial x} f(x) + \sum_{i=1}^{p} \frac{\partial h_j}{\partial x} g_i(x) u_i \\ &:= L_f h_j + \sum_{i=1}^{p} (L_{g_i} h_j) u_i \end{aligned} \tag{17}$$

where $L_f h_j$, $L_{g_i} h_j$ stand for the Lie derivative of h_j with respect to f, g_i, respectively. Note that if each $(L_{g_i} h_j)(x) \equiv 0$ then the inputs do not appear in (17). Define γ_j to be the smallest integer such that at least one of the inputs appears in $y_j^{\gamma_j}$, i.e.,

$$y_j^{\gamma_j} = L_f^{\gamma_j} h_j + \sum_{i=1}^{p} L_{g_i}(L_f^{\gamma_j - 1} h_j) u_i \tag{18}$$

with at least one $L_{g_i}(L_f^{\gamma_j-1} h_j) \neq 0$. The number γ_j is called the *relative degree* of y_j. We define a matrix $A(x)$, called the decoupling matrix as the $p \times p$ matrix with entries

$$a_{ji} = L_{g_i} L_f^{\gamma_j - 1} h_j. \tag{19}$$

In this case the input/output relation defined by (15)–(16) may be written as

$$\begin{bmatrix} y_1^{\gamma_1} \\ \vdots \\ y_p^{\gamma_p} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} + A(x) \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \tag{20}$$

If $A(x)$ is nonsingular and if $\sum_{j=1}^{p} \gamma_j = n$ then the system is fully linearizable (locally) and the control law

$$u = A^{-1}(x) \left\{ v - \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} \right\} \tag{21}$$

yields the linear decoupled system

$$\begin{bmatrix} y_1^{\gamma_1} \\ \vdots \\ y_p^{\gamma_p} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix}. \tag{22}$$

The condition $\sum \gamma_j = n$ means that the system has no *zero dynamics* and the control law (21) is known as a *static state feedback linearizing control law*. In the case of the rigid robot model (2) it is easy to show that the relative degree of each output is 2 and the matrix A^{-1} in (21) is just the inertia matrix $M(\mathbf{q}) = D(\mathbf{q}) + J$ of the robot. Thus

- **all rigid robot configurations, are globally feedback linearizable with static state feedback.**

The analogous result for flexible joint robots depends, of course, on whether the model (9) or the model (10) is being considered. It has been shown in (Spong[71]) that

- **The simplified model (9) is also globally feedback linearizable with static state feedback.**

Taking as output $\mathbf{y}$ the link angle $\mathbf{q}_1$, it is straightforward to show that the input $\mathbf{u}$ does not appear until the fourth derivative of the output. In this case the relative degree of each output is 4 and, in fact,

$$\mathbf{y}^{(4)} = \mathbf{F}(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2) + A(\mathbf{q}_1)\mathbf{u} \tag{23}$$

where the decoupling matrix $A = D^{-1}KJ^{-1}$, is invertible. Thus the control

$$\mathbf{u} = A^{-1}(\mathbf{q}_1)\left(\mathbf{v} - \mathbf{F}(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2)\right) \tag{24}$$

results in the linear system

$$\mathbf{y}^{(4)} = \mathbf{v}. \tag{25}$$

In this case, the input/output linearization is equivalent to full state linearization. Choosing as state variables

$$\mathbf{y}_1 = \mathbf{y} = \text{link position} \tag{26}$$

$$\mathbf{y}_2 = \dot{\mathbf{y}} = \text{link velocity} \tag{27}$$

$$\mathbf{y}_3 = \ddot{\mathbf{y}} = \text{link acceleration} \tag{28}$$

$$\mathbf{y}_4 = \dddot{\mathbf{y}} = \text{link jerk} \tag{29}$$

yields the linear system (Spong [71])

$$\begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \\ \dot{\mathbf{y}}_3 \\ \dot{\mathbf{y}}_4 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \mathbf{v}. \tag{30}$$

The situation regarding the model (10) is considerably different. In this case, for many types of robots, the model (10) leads to a system with a singular decoupling matrix. Hence, feedback linearization of (10) is not always possible with static state feedback, although some special configurations are linearizable. For example three–link cylindrical robots are statically linearizable even when modeled by (10) whereas 2 and 3–link elbow manipulators are not, in general.[2]

In general, if the decoupling matrix $A(x)$ is singular then we may still be able to linearize the input/output map by use of *dynamic feedback*. This approach amounts to adding integrators, i.e., increasing the dimension of the state space, until a system is formed that has a nonsingular decoupling matrix. It was shown by (DeLuca, [18]) that

- **For any configuration, the flexible joint model (10) is linearizable with dynamic feedback.**

In this case the model (10) is linearizable with dynamic feedback to n chains of 6–th order integrators. To see how the dynamic feedback necessarily comes about consider the system (10) with output equal to $\mathbf{q}_1$, the vector of link angles. Because of the inertial coupling between $\mathbf{q}_1$ and $\mathbf{q}_2$ in (10) it is evident that some or all of the components of $\ddot{\mathbf{q}}_1$ may already contain the control input $\mathbf{u}$. Specifically, eliminating $\ddot{\mathbf{q}}_2$ in the first equation in (10) yields an expression of the form

$$\ddot{\mathbf{q}}_1 = \mathbf{f}_1(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2) + G_1(\mathbf{q}_1)\mathbf{u}. \tag{31}$$

In fact, it is easy to see that $G_1(\mathbf{q}_1)$ is given by

$$G_1 = -M_{11}^{-1} M_{12} M_{22}^{-1}. \tag{32}$$

Both M_{11} and M_{22} are invertible since the inertia matrix is positive definite. In general, however, G_1 is singular (DeLuca [18]), and the function $\mathbf{f}_1$ does not lie in the range space of G_1, which is to say that (31) is not linearizable with static feedback. We further see that differentiating the output $\mathbf{q}_1$ again would introduce the derivative of some of the inputs u_i. In this case we can (temporarily) think of those components u_i of $\mathbf{u}$ appearing in (31) as new state variables, say w_{1i}. If the output is differentiated again, the definition

[2] Of course, cartesian robot dynamics are already linear.

$$\dot{w}_{1i} := w_{2i} \tag{33}$$

defines a new state equation and state variable w_{2i} while simultaneously eliminating the derivative of the input from output derivative. In this fashion, the output can be repeatedly differentiated, and additional state variables can likewise be defined until a nonsingular decoupling matrix results. This process, which increases the dimension of the state space through the introduction of (33) is equivalent to adding integrators to the input channels. The new states are thus states of a dynamic compensator. It is shown in (DeLuca [18]) that the above process always terminates after at most four steps and an algorithm for computing the required dynamic feedback law is given.

It is interesting to note, as pointed out in (Grimm, et.al. [31]), that if damping terms of the form $B(\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2)$ are included in equation (9), then the input appears after the third derivative of $\mathbf{y}$ with a nonsingular decoupling matrix. In this case the system has non trivial zero dynamics. The nature of these zero dynamics are not known at present. However, even if these zero dynamics are stable, it appears likely that a feedback linearizing control leading to a third order input/output map and resulting from the inclusion of such damping terms would be ill–conditioned, especially since these damping terms are quite small in practice. Nevertheless, it is quite interesting that, unlike rigid robots, the feedback linearization properties of the dynamic equations for a flexible joint robot can change considerable depending on which terms are included in the model.

5 Robust Control

For simplicity we will henceforth consider only the simplified model (9), i.e.,

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) + K(\mathbf{q}_1 - \mathbf{q}_2) = 0 \tag{34}$$

$$J\ddot{\mathbf{q}}_2 - K(\mathbf{q}_1 - \mathbf{q}_2) = \mathbf{u} \tag{35}$$

with output

$$\mathbf{y} = \mathbf{q}_1 \tag{36}$$

which is globally feedback linearizable to the fourth order system

$$\mathbf{y}^{(4)} = \mathbf{v} \tag{37}$$

with static feedback of the form

$$\mathbf{u} = \alpha(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2) + \beta(\mathbf{q}_1)\mathbf{v}. \tag{38}$$

In this section we investigate the robustness of the feedback linearization approach. More specifically we investigate the design of additional feedback compensation to guarantee robust tracking.

Feedback linearization is a structural property of nonlinear systems and does not, by itself, solve the control design problem. Since feedback linearization relies on exact cancellation of nonlinearities, the issue of parameter uncertainty is important to consider. The importance of feedback linearizability is that it permits satisfaction of the so-called **matching conditions**, wherein all uncertainty lies in the range space of the input. When this matching condition holds there are several available techniques to design robust controllers. Specifically, suppose instead of (38) that the control law is given by

$$\mathbf{u} = \hat{\alpha} + \hat{\beta}\mathbf{v}. \tag{39}$$

where $\hat{\alpha}$ and $\hat{\beta}$ represent nominal or estimated values of α and β, respectively. The deviations $\Delta\alpha = \hat{\alpha} - \alpha$, $\Delta\beta = \hat{\beta} - \beta$ are then due to (and are a measure of) the parametric uncertainty. This uncertainty arises from several sources: unknown mass payloads, computational round off, intentional model simplification, etc.

With the approximate feedback linearizing control law (39) the system (37) becomes

$$\mathbf{y}^{(4)} = \mathbf{v} + \eta(\mathbf{v}, \mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \dddot{\mathbf{y}}) \tag{40}$$

where

$$\eta = (\beta^{-1}\hat{\beta} - I)\mathbf{v} + \beta^{-1}\Delta\alpha \tag{41}$$

is refered to as the "uncertainty". Given a suitably smooth desired reference trajectory $t \rightarrow \mathbf{y}_d(t)$, we set

$$\mathbf{v} = \mathbf{y}_d^{(4)} - \sum_{i=0}^{3} K_i \left(\mathbf{y}_d^{(i)} - \mathbf{y}^{(i)}\right) + \Delta\mathbf{v} = \mathbf{y}_d^{(4)} - \mathbf{K}\mathbf{e} + \Delta\mathbf{v} \tag{42}$$

where we define the tracking error vector $\mathbf{e}$ and gain $\mathbf{K}$ as

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{y} - \mathbf{y}_d \\ \dot{\mathbf{y}} - \dot{\mathbf{y}}_d \\ \ddot{\mathbf{y}} - \ddot{\mathbf{y}}_d \\ \dddot{\mathbf{y}} - \dddot{\mathbf{y}}_d \end{bmatrix} \quad ; \quad \mathbf{K} = [K_0 \cdots K_3]. \tag{43}$$

If we substitute the outer loop control law (42) into (40) we can write the system in "error space" as

$$\dot{\mathbf{e}} = A\mathbf{e} + B\{\Delta\mathbf{v} + \Psi\} \tag{44}$$

where Ψ is the nonlinear function (also referred to as the "uncertainty")

$$\Psi = (\beta^{-1}\hat{\beta} - I)(\mathbf{y}_4^{(d)} + \mathbf{K}\mathbf{e} + \Delta\mathbf{v}) + \beta^{-1}\Delta\alpha \tag{45}$$

and the matrices A and B are given by

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -K_1 & -K_2 & -K_3 & -K_4 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \tag{46}$$

where the K_i's are chosen so that A is a Hurwitz matrix.

The problem of robust trajectory tracking now reduces to the problem of stabilizing the nonlinear, time-varying uncertain system (44) by suitable choice of the input $\Delta\mathbf{v}$. However, the problem of stabilizing (44) is nontrivial since Ψ is a function of both $\mathbf{e}$ and $\Delta\mathbf{v}$ and hence Ψ cannot be treated merely as a disturbance to be rejected by $\Delta\mathbf{v}$. Approaches that can be used to design $\Delta\mathbf{v}$ to guarantee robust tracking include Lyapunov (Spong [73]) and sliding mode designs (Sira–Ramirez and Spong [66]), high gain, and other approaches.

As one example of a robust design procedure we will outline the approach to robust control of uncertain systems based on Lyapunov's second method. In order to design $\Delta\mathbf{v}$ we make the following assumptions:

- (A1) There exist positive constants $\overline{\beta}$ and $\underline{\beta}$ such that

$$\underline{\beta} \leq \|\beta^{-1}(x)\| \leq \overline{\beta} \tag{47}$$

- (A2) There is a positive constant $\varsigma < 1$ such that

$$\|\beta^{-1}\hat{\beta} - I\| \leq \varsigma \tag{48}$$

- (A3) There is a known function $\phi(x,t)$ such that

$$\|\hat{\alpha} - \alpha\| \leq \phi < \infty \tag{49}$$

We note that (A2) can always be satisfied by suitable choice of $\hat{\beta}$. For example, the choice $\hat{\beta} = 1/cI$, where I is the identity matrix and the constant c is $1/2(\bar{\beta} + \underline{\beta})$ results in

$$\|\beta^{-1}\hat{\beta} - I\| \leq \frac{\bar{\beta} - \underline{\beta}}{\bar{\beta} + \underline{\beta}} < 1. \tag{50}$$

Next, we note that from our assumptions on the uncertainty we have

$$\|\Psi\| \leq \varsigma(\|\mathbf{y}_d^{(4)}\| + \|\mathbf{Ke}\| + \|\Delta\mathbf{v}\|) + \bar{\beta}\phi \leq \bar{\phi} + \varsigma\|\Delta\mathbf{v}\| \tag{51}$$

where $\bar{\phi} := \varsigma(\|\mathbf{y}_d^{(4)}\| + \|\mathbf{Ke}\|) + \bar{\beta}\phi$. Suppose that we can simultaneously satisfy the inequalities

$$\|\Psi\| \leq \rho(\mathbf{e},t) \tag{52}$$
$$\|\Delta\mathbf{v}\| \leq \rho(\mathbf{e},t) \tag{53}$$

for a known function $\rho(\mathbf{e},t)$. The function ρ can be determined as follows. First suppose that $\Delta\mathbf{v}$ satisfies (53). Then from (51) we have

$$\|\Psi\| \leq \bar{\phi} + \varsigma\rho := \rho \tag{54}$$

This definition of ρ is well-defined since $\varsigma < 1$ and we have

$$\rho = \frac{1}{1-\varsigma}\bar{\phi} \tag{55}$$

It now follows that the null solution of (44) is uniformly asymptotically stable (in a generalized sense) if $\Delta\mathbf{v}$ is chosen as

$$\Delta\mathbf{v} = \begin{cases} -\rho\dfrac{B^T P\mathbf{e}}{\|B^T P\mathbf{e}\|} & ; \text{ if } \|B^T P\mathbf{e}\| \neq 0 \\ 0 & ; \text{ if } \|B^T P\mathbf{e}\| = 0 \end{cases} \tag{56}$$

where P is the unique positive definite solution to the Lyapunov equation

$$A^T P + PA = -Q \tag{57}$$

for a given symmetric, positive definite Q. The argument is completed by noting that indeed $\|\Delta\mathbf{v}\| \leq \rho$.

6 Observers

The results in the previous section are obtained under the assumption of full state feedback. In fact, robustness to parametric uncertainty is achieved only if both the original state variables $\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{q}_2, \dot{\mathbf{q}}_2$ and the transformed state variables $\mathbf{y}, \dot{\mathbf{y}}, \ddot{\mathbf{y}}, \dddot{\mathbf{y}}$ are available for feedback. This is a crucial difference from the case of rigid robots results which achieve robustness using only the link position and velocity.

At this point it is important to consider how these variables are to be obtained so that they may be used for feedback. The original state variables, position and velocity of the link and rotor are easily obtainable. However the added expense of instrumenting both the links and the actuators may be prohibitive. Most present robot designs are such that either the joint variables or the actuator variables are measured but not both.

The robust outer loop approaches require, in addition, the link acceleration and jerk, which are difficult or impossible to obtain with present technology. These problems have motivated the investigation of nonlinear observers to estimate the needed state variables. These observer results can be classified according to whether they estimate position and velocity or acceleration and jerk.

6.1 Exact and Approximate Observers

Given a nonlinear system of the form

$$\dot{x} = f(x) + G(x)u \tag{58}$$
$$y = h(x) \tag{59}$$

with $x \in \Re^n, u \in \Re^m, y \in \Re^p$, an observer is another dynamic system

$$\dot{\hat{x}} = \hat{f}(\hat{x}, y, u) \tag{60}$$

such that for any initial conditions $x(t_0) = x_0,\ \hat{x}(t_0) = \hat{x}_0$

$$\lim_{t\to\infty}(\hat{x}(t) - x(t)) = 0 \tag{61}$$

One approach which allows us to exploit the well developed theory of observers for linear systems is to seek state space and output transformations of (58) and (59), respectively,

$$z = z(x) \tag{62}$$
$$w = w(y) \tag{63}$$

such that, in the new coordinates, the system can be expressed

$$\dot{z} = Az + \phi(w, u) \tag{64}$$
$$w = Cz \tag{65}$$

where (C, A) is an observable plair. If this possible then an observer can be designed as a linear observer plus an "output injection" term to cancel the nonlinearity ϕ. With $\hat{z}$ denoting the estimate of z, the observer takes the form

$$\dot{\hat{z}} = A\hat{z} + \phi(w,u) + L(w - C\hat{z}) \tag{66}$$

The estimation error $\tilde{z} = \hat{z} - z$ then satisfies

$$\dot{\tilde{z}} = (A - LC)\tilde{z} \tag{67}$$

By observability of (C, A) the observer gain matrix L can be chosen so that $\tilde{z} \to 0$ with arbitrary decay rate. Since z and x are related by the diffeomorphism (61) it follows that

$$\hat{x}(t) := z^{-1}(\hat{z}(t)) \to x(t). \tag{68}$$

Unfortunately, neither of the dynamic models (9) or (10) satisfies the necessary and sufficient conditions for the existence of such transformations (61), (62), except, as shown in (Albert and Spong [2]), for the simple case of a one link robot. In (Nicosia, et.al. [55]) an approximate observer was derived using the above approach by neglecting terms quadratic and higher relative to a particular operating point set.

Exploiting the skew–symmetry property of robot dynamics, an observer which estimates the full state from measurements of the link position and velocity (Tomei[91]) and the link position alone(Nicosia and Tomei([53]) has recently been derived. Robustness properties of the above observers are not known. However, computer simulation studies performed in the above references indicate a measure of robustness to parametric uncertainty.

Finally, in (Nicosia and Tornambé [53]) the use of high–gain to estimate both states and parameters is investigated.

6.2 Observer to Estimate Acceleration and Jerk

In order to design a robust feedback linearizing control law we require the link accelerations and jerks in addition to the link and rotor positions and velocities. The robust observer design problem for acceleration and jerk can be stated as the problem of designing an observer for the uncertain system

$$\dot{\mathbf{y}} = A\mathbf{y} + B\{\mathbf{v} + \eta\} \tag{69}$$
$$\mathbf{w} = C\mathbf{y}. \tag{70}$$

In this case, one can design a "high gain" observer of the form

$$\dot{\hat{\mathbf{y}}} = A\hat{\mathbf{y}} + B\mathbf{v} + L(\mathbf{w} - C\hat{\mathbf{y}}) \tag{71}$$

The estimation error is then governed by

$$\dot{\mathbf{e}} = (A - LC)\mathbf{e} + B\eta \tag{72}$$

Given known bounds on the uncertainty η, it can be shown that the estimation error is uniformly ultimately bounded for "sufficiently large" observer gain L (Bortoff, et.al. [8]).

7 Singular Perturbation Approach

In this section we discuss an alternative approach to the control of (34)–(35) based on a singular perturbation formulation of the dynamic equations. These techniques are useful when the joint stiffness is large relative to other parameters in the system. To begin we define $\phi = K(\mathbf{q}_2 - \mathbf{q}_1)$. The function ϕ thus represents the torque transmitted through the joint. By rescaling ϕ if necessary we may take K as a scalar and we assume K is $O(1/\epsilon^2)$ for some (small) parameter $\epsilon > 0$. Equation (35) can then be expressed in terms of ϕ as

$$\epsilon^2 J\ddot{\phi} + K_1\phi = K_1(\mathbf{u} - J\ddot{\mathbf{q}}_1). \tag{73}$$

Using (35) the combined system becomes

$$\begin{aligned} D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) &= \phi && (74)\\ \epsilon^2 J\ddot{\phi} + K_1\phi &= K_1(\mathbf{u} - J\ddot{\mathbf{q}}_1) && (75)\\ &= K_1\mathbf{u} \\ &\quad -K_1 J D^{-1}(\mathbf{q}_1)\{\phi - C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 - \mathbf{g}(\mathbf{q}_1)\} \end{aligned}$$

In state space we take

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{q}_1 \\ \dot{\mathbf{q}}_1 \end{pmatrix}; \mathbf{z} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \epsilon\dot{\phi} \end{pmatrix} \tag{76}$$

and express (74)–(75) as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{a}_1(\mathbf{x}) + A_1(\mathbf{x})\mathbf{z} && (77)\\ \epsilon\dot{\mathbf{z}} &= \mathbf{a}_2(\mathbf{x}) + A_2(\mathbf{x})\mathbf{z} + B\mathbf{u} && (78) \end{aligned}$$

where

$$\mathbf{a}_1(\mathbf{x}) = \begin{pmatrix} \mathbf{x}_2 \\ -D^{-1}(C\mathbf{x}_2 + \mathbf{g}) \end{pmatrix} \;\; ; \;\; \mathbf{a}_2(\mathbf{x}) = \begin{pmatrix} 0 \\ K_1 D^{-1}(C\mathbf{x}_2 + \mathbf{g}) \end{pmatrix} \tag{79}$$

$$A_1(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ D^{-1} & 0 \end{bmatrix} \;\; ; \; A_2(\mathbf{x}) = \begin{bmatrix} 0 & I \\ -K_1(D^{-1} + J^{-1}) & 0 \end{bmatrix} \tag{80}$$

$$B = \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix}. \tag{81}$$

The system (77) – (78) is a singular perturbation of the rigid robot model (2). The link positions and velocities are the "slow" variables while the joint torques and torque rates are the "fast" variables. At $\epsilon = 0$, corresponding to infinite joint stiffness, (77)–(78) becomes

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{a}_1(\bar{\mathbf{x}}) + A_1(\bar{\mathbf{x}})\bar{\mathbf{z}} && (82)\\ 0 &= \mathbf{a}_2(\bar{\mathbf{x}}) + A_2(\bar{\mathbf{x}})\bar{\mathbf{z}} + B\bar{\mathbf{u}} && (83) \end{aligned}$$

where the overbar denotes that all variables are computed at $\epsilon = 0$. Since $A_2(\bar{\mathbf{x}})$ is invertible from (80) we have

$$\bar{\mathbf{z}} = -A_2^{-1}(\bar{\mathbf{x}})(\mathbf{a}_2(\bar{\mathbf{x}}) + B\bar{\mathbf{u}}). \tag{84}$$

Substituting (84) into (82) yields the slow reduced order system

$$\dot{\bar{\mathbf{x}}} = \mathbf{a}_s(\bar{\mathbf{x}}) + B_s(\bar{\mathbf{x}})\mathbf{u} \tag{85}$$

where

$$\mathbf{a}_s = \mathbf{a}_1 - A_1 A_2^{-1}\mathbf{a}_2 \tag{86}$$
$$B_s = -A_1 A_2^{-1} B. \tag{87}$$

It can be shown (Spong [73]) that the reduced order model (85) is precisely the rigid robot model in terms of $\bar{\mathbf{x}}$. The easiest way to observe this is to notice from (75) that at $\epsilon = 0$,

$$\phi = \mathbf{u} - J\ddot{\mathbf{q}}_1 \tag{88}$$

Substituting this expression for ϕ into (74) yields the rigid model (2).

7.1 Composite Control

Inspecting (80), we see that the resonant modes at the joints are purely oscillatory with resonant frequency determined by $K_1(D^{-1} + J^{-1})$ for fixed $\mathbf{q}_1$. This agrees with the experimental observations of (Sweet and Good [87]) that the joint natural frequencies are configuration dependent. The idea of *composite* control is to set

$$\mathbf{u} = \mathbf{u}_s(\mathbf{x}, t) + \mathbf{u}_f(\mathbf{z}). \tag{89}$$

The term $\mathbf{u}_s$ is the so-called *slow control* and $\mathbf{u}_f$ is the *fast control*. As an example we see from the definition of ϕ that the choice

$$\mathbf{u}_f = K_v(\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2) \tag{90}$$

is proportional to $\dot{\phi}$ and hence qualifies as a fast feedback term. If we take $K_v = K_2/\epsilon$ and substitute (89) and (90) into (78) we obtain

$$\epsilon^2 J\ddot{\phi} + \epsilon K_2\dot{\phi} + K_1\phi = K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). \tag{91}$$

Carrying this through the various steps we see that (78) becomes

$$\epsilon\dot{\mathbf{z}} = a_2(\mathbf{x}) + \tilde{A}_2(\mathbf{x})\mathbf{z} + B\mathbf{u}_s \tag{92}$$

where

$$\tilde{A}_2 = \begin{bmatrix} 0 & I \\ -K_1(D^{-1} + J^{-1}) & -\epsilon K_2 \end{bmatrix} \tag{93}$$

where now $\tilde{A}_2$ is stable for all $\mathbf{q}_1$. The gain K_2 can be a function of $\mathbf{q}_1$ if desired. Note also that the addition of this particular $\mathbf{u}_f$ given by (90) does not alter the reduced order slow system (85). The design of the slow control $\mathbf{u}_s$ in (89) is the subject of the next three sections.

7.2 Integral Manifold

Definition: In the $4n$–dimensional state space of (77)–(78) a $2n$–dimensional manifold M_ϵ can be defined by the expression

$$\mathbf{z} = \mathbf{h}(\mathbf{x}, \mathbf{u}_s, \epsilon). \tag{94}$$

The manifold M_ϵ is said to be an **integral manifold** of (77) – (78) if it is invariant under solutions of the system. In other words, given an admissible input $t \rightarrow \mathbf{u}_s(t)$, if $\mathbf{x}(t), z(t)$ are solutions of (77) – (78) for $t > t_0$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$; $\mathbf{z}(t_0) = \mathbf{z}_0$, then

$$\mathbf{z}^0 = \mathbf{h}(\mathbf{x}^0, \mathbf{u}_s(t_0), \epsilon) \tag{95}$$

implies that, for $t > t_0$

$$\mathbf{z}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}_s(t), \epsilon). \tag{96}$$

Thus if the system starts initially on M_ϵ at time $t = t_0$, then the system trajectory evolves on M_ϵ for all $t > t_0$ The integral manifold M_ϵ is characterized by the following partial differential equation, formed by substituting the expression (94) into (92) and using (77),

$$\epsilon \left\{ \frac{\partial \mathbf{h}}{\partial \mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}_s} \frac{\partial \mathbf{u}_s}{\partial \mathbf{x}} \right\} \{a_1 + A_1 \mathbf{h}\} = a_2 + A_2 \mathbf{h} + B \mathbf{u}_s \tag{97}$$

where we have assumed $\mathbf{u}_s = \mathbf{u}_s(\mathbf{x})$.

If $\mathbf{h}$ satisfies (97), then the dynamics of the flexible joint system on the integral manifold is formed by replacing $\mathbf{z}$ by $\mathbf{h}$ in (77) to yield

$$\dot{\mathbf{x}} = a_1(\mathbf{x}) + A_1(\mathbf{x}) \mathbf{h}(\mathbf{x}, \mathbf{u}_s, \epsilon) \tag{98}$$

Equation (98), called the *reduced flexible system*, is of the same order as the rigid system (85) but is a more accurate approximation of the flexible joint system. In fact, on the integral manifold, equation (98) is not an approximation at all, but rather represents the exact dynamics of the system restricted to M_ϵ.

If we let

$$\eta = \mathbf{z} - \mathbf{h}(\mathbf{x}, \mathbf{u}_s, \epsilon) \tag{99}$$

represent the deviation of the fast variable $\mathbf{z}$ from the integral manifold, then from (92) and (97), it follows that η satisfies

$$\epsilon \dot{\eta} = \tilde{A}_2(\mathbf{x}) \eta \tag{100}$$

The complete system (on and off the manifold) is now described by

$$\begin{aligned} \dot{\mathbf{x}} &= a_1(\mathbf{x}) + A_1(\mathbf{x}) \mathbf{h}(\mathbf{x}, \mathbf{u}_s, \epsilon) + A_1(\mathbf{x}) \eta && (101) \\ \epsilon \dot{\eta} &= \tilde{A}_2(\mathbf{x}) \eta && (102) \end{aligned}$$

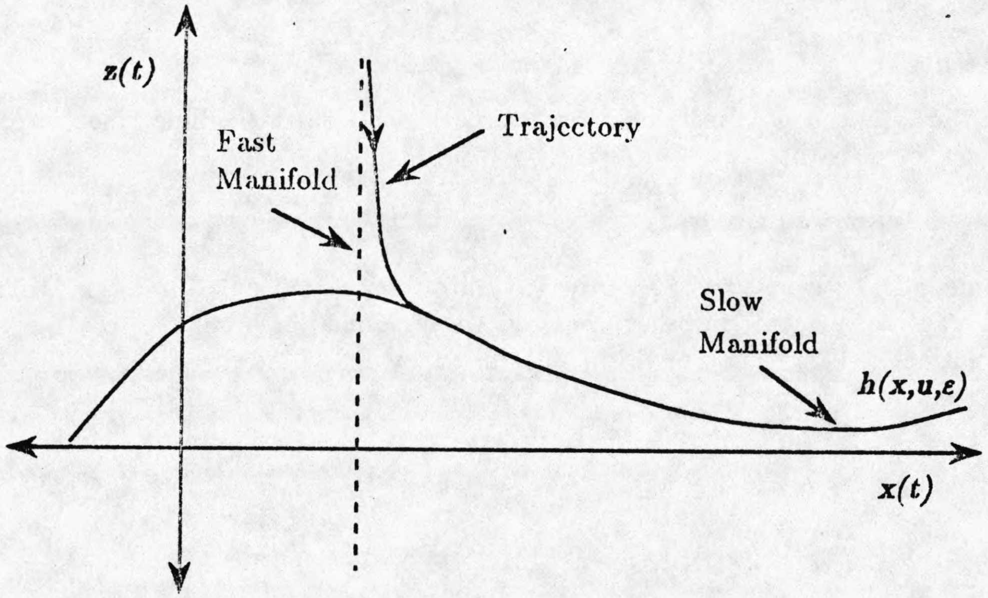

Figure 2 Slow and Fast Manifolds

Since we have used a composite control law to stabilize the fast dynamics, the system (102) is asymptotically stable in the fast time scale. It follows that for any initial condition the system trajectory will rapidly converge to the integral manifold M_ϵ along a *fast* manifold after which the trajectory will flow along the integral (*slow*) manifold M_ϵ. Thus, after the decay of the initial fast transients, the system will "nearly" be governed by the reduced order model (98).

7.3 Corrective Control

We note from (94) that the integral manifold depends on the control input $\mathbf{u}_s$. This is highly significant and means that the integral manifold M_ϵ can be "shaped" by appropriate choice of the control input. To see how this is done we compute formal power series expansions of the function $\mathbf{h}$ and the control $\mathbf{u}_s$ in terms of the perturbation parameter ϵ as

$$\mathbf{h}(\mathbf{x},\mathbf{u}_s,\epsilon) = \mathbf{h}_0(\mathbf{x},\mathbf{u}_0) + \epsilon\mathbf{h}_1(\mathbf{x},\mathbf{u}_0+\epsilon\mathbf{u}_1) + \cdots \tag{103}$$

$$\mathbf{u}_s = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \cdots \tag{104}$$

The expression (104) defines a so–called *corrective control* since it consists of correction terms $\epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots$ to the control $\mathbf{u}_0$. The control $\mathbf{u}_0$ is designed for the system at $\epsilon = 0$. Substituting these expansions into the manifold condition (97) gives

$$\epsilon\{\dot{\mathbf{h}}_0 + \epsilon\dot{\mathbf{h}}_1 + \cdots\} = a_2 + \tilde{A}_2\{\mathbf{h}_0 + \epsilon\mathbf{h}_1 + \cdots\} + B\{\mathbf{u}_0 + \epsilon\mathbf{u}_1 + \cdots\} \tag{105}$$

Equating coefficients of powers of ϵ gives

$$0 = a_2 + \tilde{A}_2\mathbf{h}_0 + B\mathbf{u}_0 \tag{106}$$

$$\dot{\mathbf{h}}_0 = \tilde{A}_2\mathbf{h}_1 + B\mathbf{u}_1 \tag{107}$$

$$\vdots \tag{108}$$

where $\dot{\mathbf{h}}_0$ denotes the total derivative

$$\dot{\mathbf{h}}_0 = \left\{ \frac{\partial \mathbf{h}_0}{\partial \mathbf{x}} + \frac{\partial \mathbf{h}_0}{\partial \mathbf{u}_s} \frac{\partial \mathbf{u}_s}{\partial \mathbf{x}} \right\} \{a_1 + A_1 \mathbf{h}_0\} \tag{109}$$

In general, for $k > 1$ we have

$$\dot{\mathbf{h}}_{k-1} = \tilde{A}_2 \mathbf{h}_k + B\mathbf{u}_k. \tag{110}$$

Solving (106) for $\mathbf{h}_0$ gives

$$\mathbf{h}_0 = -\tilde{A}_2^{-1}(a_2 + B\mathbf{u}_0). \tag{111}$$

It follows that up to $O(\epsilon)$ the reduced system

$$\begin{aligned} \dot{x} &= a_1 + A_1 \mathbf{h}_0 \\ &= a_1 - A_1 \tilde{A}_2^{-1} a_2 - A_1 \tilde{A}_2^{-1} B \mathbf{u}_0 \\ &= a_s + B_s \mathbf{u}_0 \end{aligned} \tag{112}$$

is just the rigid system (85). Thus the control $\mathbf{u}_0$ can be any suitable control designed on the basis of the rigid system (85). Once $\mathbf{u}_0$ is chosen then $\dot{\mathbf{h}}_0$ is known in principle and $\mathbf{h}_1$ can likewise be found as

$$\mathbf{h}_1 = \tilde{A}_2^{-1}\{\dot{\mathbf{h}}_0 - B\mathbf{u}_1\}. \tag{113}$$

The design of $\mathbf{u}_1$ is now based on the expression (113). Once u_1 is designed then h_1 is determined, etc. This leads to an iterative procedure to generate $\mathbf{h}_i$ and $\mathbf{u}_i$. It was shown in (Spong [73]) that there exists $\mathbf{u}_1$, $\mathbf{u}_2$ such that the corrective control law

$$\mathbf{u}_s = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 \tag{114}$$

results in

$$\mathbf{h} \equiv \mathbf{h}_0. \tag{115}$$

In other words, the integral manifold $\mathbf{h}$ can be made to coincide identically with the "rigid manifold", i.e., the manifold describing the dynamics of the rigid joint robot. To state this another way, the rigid manifold can be made invariant under solutions of the flexible joint system. The fact that the above corrective control procedure terminates after finitely many steps (two steps, in fact!) is a stronger result than one would expect using standard results in the theory of singular perturbations and asymptotic expansions. Indeed, for the more complex model (10), the above result is no longer true. In case the model (10) is used for the corrective control design, the integral manifold can only be made to coincide with the rigid manifold up to a prescribed order $O(\epsilon^\ell)$ (see [79], [73]).

8 Adaptive Control

It has only been in the last few years that adaptive controllers for rigid robots have been derived which satisfy the property of global convergence, i.e., convergence of the tracking error to zero with all signals remaining bounded. See (Ortega and Spong [60]) for a recent survey of the subject. These results rely either on the existence of an inverse dynamics for rigid robots or on the passivity of the mapping $\mathbf{u} \rightarrow \dot{\mathbf{q}}$ from input torque to link velocity.

In the case of flexible joint robots the mapping $\mathbf{u} \rightarrow \dot{\mathbf{q}}_1$ is not passive. Moreover the analog of inverse dynamics control is the more complicated feedback linearization control which requires link accelerations and jerks. Thus the extension to the flexible joint case of these recent adaptive schemes is not obvious.

In this section we outline the result of (Spong [76]) which is based on the composite control strategy of the previous section. The slow control is designed using the algorithm of (Slotine and Li [69]) for the rigid model and the fast control is designed to damp out the joint oscillations in the fast time scale.

Slow Control

Given a twice continuously differentiable reference trajectory $\mathbf{q}^d(t)$ in joint space, the slow control $\mathbf{u}_s$ is given, with $\mathbf{q} = \mathbf{q}_1$ as:

$$\mathbf{u}_s = (\hat{D}(\mathbf{q}) + \hat{J})\mathbf{a} + \hat{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}) - K_D\mathbf{r} \tag{116}$$

where $\hat{D}, \hat{C}$ and $\hat{\mathbf{g}}$ represent the terms in the rigid model with estimated values of the parameters, K_D is a diagonal matrix of positive gains,

$$\mathbf{v} = \dot{\mathbf{q}}^d - \Lambda\tilde{\mathbf{q}} \quad ; \quad \mathbf{a} = \dot{\mathbf{v}}, \tag{117}$$

$$\mathbf{r} = \dot{\mathbf{q}} - \mathbf{v} = \dot{\tilde{\mathbf{q}}} + \Lambda\tilde{\mathbf{q}} \quad ; \quad \tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^d, \tag{118}$$

and Λ is a constant diagonal matrix. Substituting (116) into (2), since $\ddot{\mathbf{q}} = \dot{\mathbf{r}} + \mathbf{a}$ and $\dot{\mathbf{q}} = \mathbf{r} + \mathbf{v}$, we can write the combined rigid system as

$$\begin{aligned} (D + J)\dot{\mathbf{r}} + C\mathbf{r} + K_D\mathbf{r} &= (\tilde{D} + \tilde{J})\mathbf{a} + \tilde{C}\mathbf{v} + \tilde{\mathbf{g}} \\ &= Y(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}, \mathbf{a})\tilde{\theta} \end{aligned} \tag{119}$$

where $\tilde{\theta} = \tilde{\theta} - \theta$ is the parameter error. Note that the regressor function Y in (119) does not depend on the manipulator acceleration. It is shown in (Slotine and Li [69]) that, with the parameter update law

$$\dot{\tilde{\theta}} = -\Gamma^{-1}Y^T\mathbf{r}, \tag{120}$$

the system is globally convergent, i.e., the position and velocity tracking errors converge to zero with all signals remaining bounded. Using the composite control law

$$\mathbf{u} = \mathbf{u}_s(\mathbf{q}_1, \dot{\mathbf{q}}_1, t) + K_v(\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2) \tag{121}$$

where $\mathbf{u}_s$ is given by (116) and K_v is a constant diagonal matrix whose diagonal elements are $0(1/\epsilon)$ the overall system consisting of the slow subsystem and the boundary layer system can be written, up to $O(\epsilon)$, as

$$(D(\mathbf{q}_1) + J)\dot{\mathbf{r}} + C(\mathbf{q}, \dot{\mathbf{q}})\mathbf{r} + K_D\mathbf{r} = Y\tilde{\theta} + \eta(t/\epsilon) \tag{122}$$

$$\dot{\tilde{\theta}} = \Gamma^{-1}Y^T\mathbf{r}. \tag{123}$$

$$J\frac{d^2\eta}{d\tau^2} + K_2\frac{d\eta}{d\tau} + K_1\eta = 0 \tag{124}$$

where $\tau = t/\epsilon$ represents the fast time scale.

We note that only the rigid body parameters of the system are updated in this scheme. The joint stiffness and motor inertia need only be known with sufficient precision to determine K_2 in to stabilize the boundary layer system (124). Typically these parameters can be identified with sufficient accuracy off-line and will not change with varying payloads.

This approach to adaptive control of flexible joint robots appears in [76], [26], [27]. Experimental results are given showing the performance of the above approach on a real system, although a stability analysis is not given. The first stability analysis of this approach appears in [28]. This analysis shows that, if the bandwidth of the reference trajectory is too fast or if there are external disturbances acting on the system, then instability can result. Such instability phenomena are well–known in the adaptive control literature (see [37]) for a discussion of the causes of these instabilities.

There are several approaches that can be used at this point to maintain stability in the above system. First, the bandwidth of the reference trajectory must be suitably restricted. Second, the parameter update law can be modified using, for example the σ–modification scheme of (Ioannou and Kokotović[37]), or a persistently exciting reference signal can be added to ensure parameter convergence. Dead–zones can be incorporated to overcome instabilities due to parameter drift.

9 Conclusions

In this paper we have given a brief survey of some results on the modeling and control of flexible joint robots. For reasons of space we have not surveyed all of the results in the field. However, the list of references at the end is reasonably complete at the time of this writing (January, 1990). However, much remains to be done in this area. First, the problem of observer design is largely open. Second, computational issues, especially for the complicated feedback linearization strategies have not been adequately addressed in the literature.

Force control of flexible joint robots has only been addressed in ([75]) and ([50]). Further work, including experimental results are needed to understand the implications of joint flexibility for such tasks as assembly. This area is particularly important in light of the fact that most work on robotic force control to date relies heavily on passivity assumptions which break down in the case of joint flexibility.

Adaptive control is another area that is ripe for investigation. Adaptive feedback linearization is currently being actively pursued in the control literature. These results are based on state space systems and typically assume the property of linearity in the parameters. Therefore, these results cannot be applied to robots since the property of linearity in the parameter holds in configuration space but not in state space, unless the inertia matrix of the robot is constant. Finally, the adaptive version of the integral manifold approach of section 7 is a promising area for further investigation.

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