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**ROBUST ADAPTIVE
NONLINEAR CONTROL
UNDER EXTENDED
MATCHING CONDITIONS**

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**ROBUST ADAPTIVE NONLINEAR CONTROL
UNDER EXTENDED MATCHING CONDITIONS¹**

Ioannis Kanellakopoulos, Petar V. Kokotovic and Riccardo Marino.

ABSTRACT

A new direct adaptive regulation scheme is proposed for nonlinear systems satisfying extended matching conditions. Two sets of additional conditions are given: first, for the regulation to be global, and second, for asymptotic parameter convergence. It is also shown that the new scheme is robust with respect to unmodeled dynamics.

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1. INTRODUCTION

In the last couple of years an *adaptive nonlinear methodology* has emerged which combines basic concepts of *feedback linearization* (Jakubczyk and Respondek, 1980, Hunt *et al.*, 1983, Isidori, 1985) with the Lyapunov design of *adaptive linear control* (Narendra and Annaswamy, 1989, Sastry and Bodson, 1989). This methodology assumes, first, a linear parametrization (the unknown constant parameters appear linearly in the nonlinear model of the plant to be controlled), and, second, a form of feedback linearization (complete or partial) for all values of the unknown parameters. The results obtained thus far make two types of additional assumptions under which specific control laws are designed. The first type of assumptions further restricts the form of the dependence of plant models on unknown parameters: in Taylor *et al.* (1989) the model satisfies a *strict matching condition*, while in Nam and Arapostathis (1988) it appears in the so-called *pure-feedback form*. The second type of assumptions, made by Sastry and Isidori (1989) and Sastry and Kokotovic (1990), do not impose further restrictions on parameter dependence, but, instead, restrict the nonlinearities, assuming that they satisfy a "linear growth" condition. The adaptive control laws designed under these assumptions differ in their applicability and complexity. The restrictiveness of the control laws based on the first type of assumptions is compensated for by their simplicity. The price paid for wider applicability of the control laws based on the second type of assumptions is an overparametrization, i.e., the need to use the estimates of not only the actual parameters, but also of fictitious parameters representing products and powers of the actual ones.

This report extends the results of Taylor *et al.* (1989) by introducing less restrictive matching conditions. The new adaptive control law avoids any overparametrization and possesses the same robustness properties with respect to *unmodeled dynamics*.

We consider the class of nonlinear plants

$$\dot{x} = f_1(x, a) + F_1(x, a)z + G_1(x, a)u, \quad (1.1)$$

$$\mu \dot{z} = f_2(x, a) + F_2(x, a)z + G_2(x, a)u, \quad (1.2)$$

where $x \in R^n$, $z \in R^q$ are the states of the modeled and unmodeled dynamics, respectively, $u \in R^m$ is the control ($m \leq n$), $a \in R^p$ is the vector of unknown constant parameters and μ is a

small positive scalar determining the presence of the unmodeled dynamics. In the assumptions below, and throughout the paper, B_y denotes an open ball centered at $y=0$.

Assumption 1 (Equilibrium). The columns of F_1, F_2, G_1, G_2 , and f_1, f_2 , are real smooth vector fields on R^n such that

$$\begin{bmatrix} f_1(0,a) \\ f_2(0,a) \end{bmatrix} \in \text{Im} \left\{ \begin{bmatrix} G_1(0,a) \\ G_2(0,a) \end{bmatrix} \right\} \text{ for all } a \in B_a . \quad (1.3)$$

and hence, for every $a \in B_a$ there exists a control which places the equilibrium of the system (1.1)-(1.2) at $x=0, z=0$. ■

Assumption 2 (Unmodeled dynamics). The unmodeled dynamics are *asymptotically stable* for all fixed values of $x \in B_x$ and $a \in B_a$, that is, there exists a constant $\sigma > 0$ such that

$$\text{Re } \lambda \{ F_2(x,a) \} \leq -\sigma < 0 . \quad (1.4)$$

There may be several reasons for not including these dynamics in the design model : our inability to accurately model them; our intention to simplify the model and to reduce the controller complexity; the fact that their states are not available for feedback. While the smallness of μ implies that *the unmodeled dynamics are fast*, a singular perturbation from $\mu > 0$ to $\mu = 0$, that is,

$$0 = f_2(x,a) + F_2(x,a)z + G_2(x,a)u , \quad (1.5)$$

results in the *reduced-order model*

$$\dot{x} = f(x,a) + G(x,a)u = f(x,a) + \sum_{i=1}^m g_i(x,a)u_i , \quad (1.6)$$

where f and G are defined by

$$f(x,a) = f_1(x,a) - F_1(x,a)F_2^{-1}(x,a)f_2(x,a) , \quad (1.7)$$

$$G(x,a) = G_1(x,a) - F_1(x,a)F_2^{-1}(x,a)G_2(x,a) . \quad (1.8)$$

The columns of $G(x,a)$ are assumed to be linearly independent for all $x \in B_x, a \in B_a$. Note that $F_2^{-1}(x,a)$ exists by (1.4) and that, by (1.3),

$$f(0,a) \in \text{Im} \{G(0,a)\} \text{ for all } a \in B_a . \quad (1.9)$$

Design tasks. Given that *the x-part of the state is available for feedback*, our objective is to find a feedback control u that will *regulate* the state of the plant, i.e., that will result in

$$x(t), z(t) \text{ bounded, } \lim_{t \rightarrow \infty} x(t) = 0, \lim_{t \rightarrow \infty} z(t) = 0 . \quad (1.10)$$

The three separate design tasks are:

- (i) **Ideal design.** When the actual value of a is known and there are no unmodeled dynamics ($\mu=0$), design a controller in order to achieve regulation of (1.6).
- (ii) **Parametric uncertainty.** Find a method to cope with the fact that the parameter vector a is unknown.
- (iii) **Unmodeled dynamics.** Establish that the designed controller is robust in the presence of $\mu > 0$.

The first task is solved in Section 2 by *feedback linearization* of the plant (1.6). The second task is the topic of Section 3, where we develop an *adaptive control law*. The third task is the topic of Section 4, where the full-order closed-loop system is analyzed and *robustness* bounds are computed.

2. FEEDBACK LINEARIZATION

The following assumption is crucial for our ability to solve Task (i).

Assumption 3 (Feedback linearization). For all $x \in B_x$, $a \in B_a$, there exist a state diffeomorphism $\bar{x} = \phi(x;a)$, with $\phi(0;a) = 0$, and a feedback control $u = \alpha(x;a) + B^{-1}(x;a)v$, with $B(x;a)$ a nonsingular $m \times m$ matrix, that transform (1.6) into a linear system in Brunovsky canonical form with controllability indices k_i , $i=1, \dots, m$ ($\sum_{i=1}^m k_i = n$)

$$\dot{\bar{x}} = A_L \bar{x} + B_L v . \quad (2.1)$$

Necessary and sufficient conditions for the *local* validity of Assumption 3 are well known. They are also *necessary* (but not sufficient) for its *global* validity, which may be difficult to verify

(Boothby, 1985). As shown in Jakubczyk and Respondek (1980) and Hunt *et al.* (1983), Assumption 3 will hold locally, i.e., in a neighborhood U_0 of the origin in R^n , if and only if the distributions

$$G^0 = sp \{g_1, \dots, g_m\}, \quad (2.2)$$

$$G^i = sp \{g_1, \dots, g_m, ad_f g_1, \dots, ad_f g_m, \dots, ad_f^i g_1, \dots, ad_f^i g_m\}, \quad i=1, \dots, n-1, \quad (2.3)$$

satisfy the following conditions for all $x \in U_0, a \in B_a$:

$$G^{n-1} = R^n, \quad (2.4)$$

$$G^i \text{ is involutive and of constant dimension } m_i, \quad i=0, \dots, n-2. \quad (2.5)$$

Then there exist smooth functions ϕ_i ($i=1, \dots, m$) of x , dependent on the parameter a , that is, $\phi_i(x; a)$, with $\phi_i(0; a)=0$ and linearly independent differentials $d\phi_i$, such that for all $x \in U_0, a \in B_a$

$$k_i = \text{card} \{r_j \geq i, j \geq 0\}, \quad i=1, \dots, m, \quad r_0 = m_0, \quad r_j = m_j - m_{j-1}, \quad j \geq 1, \quad (2.6)$$

$$\text{the } m \times m \text{ matrix with } \langle d\phi_i, ad_f^{k_i-1} g_j \rangle \text{ as its } (i,j)\text{-th element is nonsingular,} \quad (2.7)$$

$$\langle d\phi_i, ad_f^l g_l \rangle = 0, \quad l=1, \dots, m, \quad j=0, \dots, k_i-2, \quad i=1, \dots, m, \quad (2.8)$$

$$\{\phi_i, L_f \phi_i, \dots, L_f^{k_i-1} \phi_i, \quad i=1, \dots, m\} \text{ is a local diffeomorphism which preserves the origin.} \quad (2.9)$$

Note, in particular, that the controllability indices are required to be the same for all $a \in B_a$, and that $m_0 = m$ by the linear independence of the columns of G . The new coordinates \bar{x} are then defined as

$$\bar{x}_i^j = L_f^{j-1} \phi_i(x; a), \quad j=1, \dots, k_i, \quad i=1, \dots, m. \quad (2.10)$$

The system (1.6) is transformed into (2.1) by using the diffeomorphism (2.10) and the feedback control

$$u = \alpha(x; a) + B^{-1}(x; a)v, \quad (2.11)$$

where

$$\alpha(x; a) = -[L_f^{k_1} \phi_1, \dots, L_f^{k_m} \phi_m]^T, \quad (2.12)$$

$$B(x;a) = \begin{bmatrix} L_{g_1} L_f^{k_1-1} \phi_1 & \dots & L_{g_m} L_f^{k_1-1} \phi_1 \\ \vdots & & \vdots \\ L_{g_1} L_f^{k_m-1} \phi_m & \dots & L_{g_m} L_f^{k_m-1} \phi_m \end{bmatrix}. \quad (2.13)$$

The nonsingularity of the matrix $B(x;a)$ for all $x \in U_0$, $a \in B_a$, is a direct consequence of (2.7) and (2.8). The resulting linear system of the form (2.1) consists of m independent controllable subsystems in Brunovsky form, called "the \bar{x}_i -subsystems." For each of these subsystems we design a linear feedback control law

$$v_i = - \sum_{j=1}^{k_i} \gamma_i^j \bar{x}_i^j, \quad i=1, \dots, m, \quad (2.14)$$

so that the k_i -th degree polynomials $s^{k_i} + \gamma_i^{k_i} s^{k_i-1} + \dots + \gamma_i^2 s + \gamma_i^1$ are Hurwitz. This results in *exponentially stable* \bar{x}_i -subsystems

$$\dot{\bar{x}}_i = A_i \bar{x}_i = \begin{bmatrix} 0 & & & \\ \vdots & & I & \\ 0 & & & \\ -\gamma_i^1 & \dots & \dots & -\gamma_i^{k_i} \end{bmatrix} \bar{x}_i. \quad (2.15)$$

This design solves Task (i), that is, it achieves state regulation in the case where the parameter vector a is known and no unmodeled dynamics are present.

3. ADAPTIVE REGULATION

For the design under parametric uncertainty (Task (ii)) we now impose two restrictions on the class of reduced-order models (1.6).

Assumption 4 (Linear parametrization). For all $x \in B_x$, $a \in B_a$,

$$f(x,a) = f^0(x) + \sum_{j=1}^p f^j(x) a_j, \quad (3.1)$$

$$g_i(x,a) = g_i^0(x) + \sum_{j=1}^p g_i^j(x) a_j, \quad i=1, \dots, m, \quad (3.2)$$

where $f^j(x)$ and $g_i^j(x)$ ($j=0,\dots,p$, $i=1,\dots,m$) are real smooth vector fields on R^n , such that $f^0(0)=0$, $f^j(0) \in G^0(0,a)$, $j=1,\dots,p$ (note that these are a special case of (1.9)), and $g_1(x,a), \dots, g_m(x,a)$ are linearly independent.

Assumption 5 (Extended matching). For all $x \in B_x$, $a \in B_a$,

$$f^j \in G^1, \quad j=1,\dots,p, \quad (3.3)$$

$$g_i^j \in G^0, \quad j=1,\dots,p, \quad i=1,\dots,m. \quad (3.4)$$

This *extended matching* condition is less restrictive than the following well-known condition (Barmish *et al.*, 1983, Marino and Nicosia, 1984), which was used for adaptive control in Taylor *et al.* (1989):

Assumption 6 (Strict matching). For all $x \in B_x$, $a \in B_a$,

$$f^j \in G^0, \quad g_i^j \in G^0, \quad j=1,\dots,p, \quad i=1,\dots,m. \quad (3.5)$$

The meaning of the strict matching condition is that parametric uncertainty is allowed to enter both the f -vector and the G -matrix of (1.6) only in equations that also include a control-driven term. Under the extended matching condition, the parametric uncertainty in the f -vector of (1.6) can precede the next control term by one integrator, when the system is transformed in the form (2.1).

Example 3.1. To illustrate the difference between the two matching conditions, let us consider the following simple but realistic dimensionless second-order model of a DC-motor:

$$\frac{d\omega}{dt} = i + \lambda(\omega, a), \quad (3.6)$$

$$\frac{T_e}{T_m} \frac{di}{dt} = -\omega - i + u_a, \quad (3.7)$$

where, in normalized units, ω is the motor speed, i is the armature current, u_a is the armature voltage (control input), and T_e and T_m are the electrical and mechanical constants of the motor. The parametric uncertainty in this case is due to an unknown parameter a in the torque characteristic

$\lambda(\omega, a)$ of the motor load. The system (3.6)-(3.7) does not satisfy the strict matching condition. However, the extended matching condition is satisfied, since the parametric uncertainty precedes the control input by only one integrator. The adaptive approach of Taylor *et al.* (1989) cannot be applied without first treating the ratio T_e/T_m as a singular perturbation parameter and replacing (3.6)-(3.7) by the reduced-order model

$$\frac{d\omega}{dt} = -\omega + u_a + \lambda(\omega, a), \quad (3.8)$$

which clearly satisfies Assumption 6. This order reduction is not necessary for the design approach of the present paper.

Coordinate transformation. As shown in Taylor *et al.* (1989), under the strict matching condition the diffeomorphism (2.10) is independent of the parameters in a , that is, $\phi_i(x; a)$ can be written simply as $\phi_i(x)$. However, under the extended matching condition the diffeomorphism depends on a in the following way:

$$\bar{x}_i^j = L_{f_0}^{j-1} \phi_i(x), \quad j=1, \dots, k_i-1, \quad i=1, \dots, m, \quad (3.9)$$

$$\bar{x}_i^{k_i} = L_{f_0}^{k_i-1} \phi_i(x) + \sum_{j=1}^p a_j L_{f_j} L_{f_0}^{k_i-2} \phi_i(x), \quad i=1, \dots, m. \quad (3.10)$$

Since the parameter vector a is unknown, the variables defined by (3.9)-(3.10) are not available for feedback. To overcome this difficulty, we replace the unknown a by an estimate $\hat{a}(t)$. Then, instead of (3.9)-(3.10), we use the nonlinear time-varying change of coordinates

$$\tilde{x}_i^j = L_{f_0}^{j-1} \phi_i(x), \quad j=1, \dots, k_i-1, \quad i=1, \dots, m, \quad (3.11)$$

$$\tilde{x}_i^{k_i} = L_{f_0}^{k_i-2} \phi_i(x, \hat{a}) = L_{f_0}^{k_i-1} \phi_i(x) + \sum_{j=1}^p \hat{a}_j(t) L_{f_j} L_{f_0}^{k_i-2} \phi_i(x), \quad i=1, \dots, m. \quad (3.12)$$

If we augment this change of coordinates by the equations $\hat{a} = \hat{a}$, we get a time-invariant mapping $\bar{\phi}$ that maps (x, \hat{a}) to (\tilde{x}, \hat{a}) . Since (3.9)-(3.10) defines a diffeomorphism for every $a \in B_a$ and (3.12) depends only linearly on \hat{a} , the mapping $\bar{\phi}$ is one-to-one, onto and C^∞ . The Inverse Function Theorem (cf. Boothby, 1986) establishes that $\bar{\phi}$ is a diffeomorphism on $B_x \times B_a$. Finally, the fact that for every $a \in B_a$ the diffeomorphism (3.9)-(3.10) preserves the origin, guarantees that $x = 0$ implies $\tilde{x} = 0$ and vice versa.

Because of the dependence of (3.12) on the parameter estimates, the new adaptive control law in this paper is more complex than the one proposed by Taylor *et al.* (1989). We now develop this new control law by incorporating not only the parameter estimate \hat{a} , but also its rate of change $\dot{\hat{a}}$, into a "certainty-equivalence" form of the feedback-linearizing control (2.11).

Certainty equivalence control. Using Assumption 5, (2.8), and (3.11), it can be shown that

$$\begin{aligned} L_f \tilde{x}_i^j &= L_{f0} \tilde{x}_i^j, & j=1, \dots, k_i-2, & i=1, \dots, m, \\ L_{g_r} \tilde{x}_i^j &= 0, & j=1, \dots, k_i-1, & i=1, \dots, m, & r=1, \dots, m. \end{aligned} \quad (3.13)$$

Thus, in the new coordinates \tilde{x} , the plant (1.6) consists of m subsystems, each described by a set of equations of the form

$$\begin{aligned} \dot{\tilde{x}}_i^1 &= \tilde{x}_i^2, \\ &\vdots \\ \dot{\tilde{x}}_i^{k_i-2} &= \tilde{x}_i^{k_i-1}, \\ \dot{\tilde{x}}_i^{k_i-1} &= \tilde{x}_i^{k_i} + w_1^i(x) \tilde{a}, \\ \dot{\tilde{x}}_i^{k_i} &= \alpha_1^i(x) + \alpha_2^i(x) \hat{a} + \alpha_3^i(x) a + \hat{a}^T \alpha_4^i(x) a + w_1^i(x) \dot{\hat{a}} \\ &\quad + \sum_{r=1}^m [\beta_{1,r}^i(x) + \beta_{2,r}^i(x) \hat{a} + \beta_{3,r}^i(x) a + \hat{a}^T \beta_{4,r}^i(x) a] u_r, \end{aligned} \quad (3.14)$$

where $\tilde{a} = a - \hat{a}$ is the parameter error, and the expressions for $w_1^i, \alpha_j^i, \beta_{j,r}^i, j=1, \dots, 4, i, r=1, \dots, m$ are given in the Appendix. The design goal now is to find a control which will render the \tilde{x}_i -subsystems in (3.14) linear in the parameter error \tilde{a} and make them otherwise independent of the unknown parameter vector a . To achieve this goal, we calculate the control u from the equation

$$B(x, \hat{a}) u = v - \alpha_1(x) - [\alpha_2(x) + \alpha_3(x) + \alpha_4(x, \hat{a})] \hat{a} - w_1(x) \dot{\hat{a}}, \quad (3.15)$$

with the obvious definitions for $\alpha_j, j=1, \dots, 4, w_1$ and $B(x, \hat{a})$ (consider (3.14) for $i=1, \dots, m$). Note that $B(x, \hat{a})$ can be expressed as

$$B(x, \hat{a}) = \begin{bmatrix} L_{g_1} L_f L_{f0}^{k_1-2} \phi_1 & \dots & L_{g_m} L_f L_{f0}^{k_1-2} \phi_1 \\ \vdots & & \vdots \\ L_{g_1} L_f L_{f0}^{k_m-2} \phi_m & \dots & L_{g_m} L_f L_{f0}^{k_m-2} \phi_m \end{bmatrix}. \quad (3.16)$$

This form is directly obtained from (2.13) when the diffeomorphism (2.10) is replaced by the coordinate transformation (3.11)-(3.12).

An important feature of the feedback control defined by (3.15) is that it depends not only on the parameter estimate \hat{a} , but also on its rate of change $\dot{\hat{a}}$, which appears in (3.14) and in (3.15) because of the dependence of (3.12) on \hat{a} . The rate of change $\dot{\hat{a}}$ is yet to be determined by our design of a parameter update law. This update law will assure that the parameter estimate \hat{a} will remain in B_a . This, in turn, will guarantee the invertibility of the matrix $B(x, \hat{a})$ in (3.15) for the reason mentioned after (2.13). However, as $\dot{\hat{a}}$ will in general be a function of the control u , our adaptive scheme will be implicit, and, hence, the invertibility of $B(x, \hat{a})$ will no longer be sufficient to guarantee the solvability of (3.15). In the case where a control u satisfying (3.15) *does* exist, its substitution in (3.14) gives

$$\begin{aligned} \dot{\tilde{x}}_i^1 &= \tilde{x}_i^2 \\ &\vdots \\ \dot{\tilde{x}}_i^{k_i-2} &= \tilde{x}_i^{k_i-1} \\ \dot{\tilde{x}}_i^{k_i-1} &= \tilde{x}_i^{k_i} + w_1^i(x) \tilde{a} \\ \dot{\tilde{x}}_i^{k_i} &= v_i + w_2^i(x, \hat{a}, u) \tilde{a} , \end{aligned} \quad (3.17)$$

with $w_2^i(x, \hat{a}, u)$ as defined in the Appendix and v_i as in (2.14), but with \bar{x} replaced by \tilde{x} . A further substitution of v_i into (3.17) shows that each of the \tilde{x}_i -subsystems can be put in the form of an *error system*

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i + W_i(x, \hat{a}, u) \tilde{a} , \quad (3.18)$$

with A_i as defined in (2.15). This form, familiar in the linear adaptive control literature (Narendra and Annaswamy, 1989, Sastry and Bodson, 1989), will be instrumental in the stability proof. The quantity multiplying the parameter error \tilde{a} , often referred to as "the regressor", is

$$W_i(x, \hat{a}, u) = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ w_1^i(x) \\ w_2^i(x, \hat{a}, u) \end{bmatrix} . \quad (3.19)$$

Parameter update law. The familiar form of (3.18) suggests the parameter update law

$$\dot{\hat{a}} = -\dot{\tilde{a}} = \Gamma \sum_{i=1}^m W_i^T(x, \hat{a}, u) P_i \tilde{x}_i, \quad (3.20)$$

where $\Gamma > 0$ is the *adaptation-gain* matrix and $P_i = P_i^T > 0$ is chosen to satisfy the Lyapunov equation

$$P_i A_i + A_i^T P_i = -I_{k_i}, \quad i=1, \dots, m. \quad (3.21)$$

Note that there exists a control that places the equilibrium of the system consisting of (1.6) and (3.20) at $x=0$ and some $\hat{a} = \text{const.} \in B_a$. This follows from (1.9) and the fact that $x=0 \Leftrightarrow \tilde{x}=0 \Rightarrow \dot{\hat{a}}=0$.

Starting from (3.20) and with some algebraic manipulations, we can show that the parameter update law is affine in the control u , that is,

$$\dot{\hat{a}} = \Gamma K_0(x, \hat{a}) + \Gamma K_1(x, \hat{a}) u, \quad (3.22)$$

where K_0 ($p \times 1$) and K_1 ($p \times m$) are defined in the Appendix. The matrix K_1 has the property

$$K_1(0, \hat{a}) = 0 \quad \text{for all } \hat{a} \in B_a. \quad (3.23)$$

Substituting (3.22) into (3.15) we get

$$[B(x, \hat{a}) + L_1(x, \hat{a})] u = v - L_0(x, \hat{a}), \quad (3.24)$$

with the obvious definitions for L_0, L_1 . From (3.23) it follows that $L_1(0, \hat{a})=0$ for all $\hat{a} \in B_a$, and, since $B(x, \hat{a})$ is invertible for all $x \in B_x, \hat{a} \in B_a$, we have the following result:

Proposition 1 (Local solvability). For every $\hat{a} \in B_a$ there exists a ball B_0 such that the matrix $B(x, \hat{a}) + L_1(x, \hat{a})$ in (3.24) is invertible for all $x \in B_0$. Hence, in B_0 there exists a *unique* solution of (3.22) and (3.24) (with v as defined after (3.17)):

$$u = \hat{u}(x, \hat{a}), \quad (3.25)$$

$$\dot{\hat{a}} = \psi(x, \hat{a}). \quad (3.26)$$

Use of this solution in (1.6) results in the adaptive closed-loop system

$$\begin{aligned}\dot{x} &= f(x,a) + G(x,a)\hat{u}(x,\hat{a}), \\ \dot{\hat{a}} &= \psi(x,\hat{a}).\end{aligned}\tag{3.27}$$

The difficulty is that the existence of such a solution *cannot* be guaranteed for all $x \in B_x$, $a \in B_a$, since in general there exist values of x and \hat{a} for which the matrix $B(x,\hat{a}) + L_1(x,\hat{a})$ is singular. One may be able to handle this singularity by a modification of the update law. For example, the parameter estimates can be kept constant at their most recent values while the state is on a manifold where $B + L_1$ is singular. For such a modification to be effective, no complete trajectory of the closed-loop system (3.27) is to belong to such a singularity manifold, that is, no singularity manifold is to contain an invariant set of (3.27). We stress that (3.24) is solvable in the whole region bounded by the singularity manifolds, and *not* in some infinitesimally small region, as the term "local" may suggest. An illustration of this fact is given in Example 3.2 at the end of this section.

Special cases. There are two important special cases where the global solvability of (3.24) can be guaranteed. First, when the strict matching condition (Assumption 6) is satisfied. In this case, (3.15) does not contain the \hat{a} -dependent term, and has a unique solution due to the invertibility of $B(x,\hat{a})$. The second special case is when the following assumption is satisfied:

Assumption 7 (Extended matching with parameter-independent G). For all $x \in B_x$, $a \in B_a$,

$$f^j \in G^1, \quad j=1,\dots,p, \quad g_i^j=0, \quad j=1,\dots,p, \quad i=1,\dots,m.\tag{3.28}$$

In this case, the update law (3.20) is independent of the control u , i.e., $K_1(x,\hat{a})=0$ in (3.22). Hence, $L_1(x,\hat{a})=0$ and (3.24) becomes uniquely solvable due to the invertibility of $B(x,\hat{a})$.

Stability. The stability properties of the closed-loop system (3.27) are now established using the Lyapunov function

$$V(\tilde{x},\tilde{a}) = \sum_{i=1}^m \tilde{x}_i^T P_i \tilde{x}_i + \tilde{a}^T \Gamma^{-1} \tilde{a},\tag{3.29}$$

with P_i , $i=1,\dots,m$, and Γ as defined in (3.21). The time derivative of V along the solutions of (3.27) is

$$\dot{V}(\tilde{x}, \tilde{a}) = - \sum_{i=1}^m \tilde{x}_i^T \tilde{x}_i + 2 \sum_{i=1}^m \tilde{x}_i^T P_i W_i \tilde{a} + 2 \tilde{a}^T \Gamma^{-1} \tilde{a} . \quad (3.30)$$

It is now clear that the update law (3.20) is chosen to guarantee that

$$\dot{V}(\tilde{x}, \tilde{a}) = - \sum_{i=1}^m \|\tilde{x}_i\|^2 = - \|\tilde{x}\|^2 \leq 0 , \quad (3.31)$$

and, hence, that $\tilde{x}(t)$, $\tilde{a}(t)$ are bounded. Therefore, $x(t)$, $\hat{a}(t)$ are bounded, and u is also bounded, since the u_i , $i=1, \dots, m$, as defined by (3.25) and (2.14), are continuous functions of x and \hat{a} . By LaSalle's invariance theorem (cf. Hale, 1980) the state of the system (3.27) will converge to the largest invariant set contained in the set on which $\dot{V}(\tilde{x}, \tilde{a})=0$, that is, $\tilde{x}=0$. We conclude that as $t \rightarrow \infty$, $\tilde{x}(t) \rightarrow 0$. Since $\bar{\phi}$, defined after (3.12), is a diffeomorphism on $B_x \times B_a$ with $\bar{\phi}(0, \hat{a}) = (0, \hat{a})$, the fact that $\tilde{x}(t) \rightarrow 0$ as $t \rightarrow \infty$ implies that $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

We have thus proved the following result:

Theorem 1 (Adaptive regulation). Under Assumptions 1, 3 and 4, the state-feedback control (3.25) and the update law (3.26) guarantee that the solution $(x(t), \hat{a}(t))$ of the closed-loop system (3.27) has the properties

$$\begin{aligned} \lim_{t \rightarrow \infty} x(t) &= 0 , \\ x(t), \hat{a}(t) &- a \text{ bounded,} \end{aligned}$$

for every initial condition (x_0, \hat{a}_0) that belongs to Ω , where

(i) if Assumption 5 holds, Ω is a sufficiently small neighborhood of the point $(0, a)$ in R^{n+p} ,

or,

(ii) if *either* Assumption 6 *or* 7 holds, Ω is the set

$$\Omega = \{(x, \hat{a}) : V(x, \hat{a}) \leq c\} , \quad (3.32)$$

with $V(x, \hat{a})$ as defined in (3.29) and c the largest constant for which the set Ω is contained in $B_x \times B_a$.

We examine now two important special cases of Theorem 1.

Corollary 1 (Global adaptive regulation). When Assumptions 1, 3, 4 and either 6 or 7 hold for all $(x, a) \in R^{n+p}$, the adaptive regulation is global, that is, $\Omega = R^{n+p}$.

■

Corollary 2 (Parameter convergence). When, in addition to the Assumptions of Theorem 1, the following conditions are valid

- (i) $g_i^j = 0, j=1, \dots, p, i=1, \dots, m$ (parameter-independent G),
- (ii) $p \leq m$,
- (iii) the vectors $f^j(0), j=1, \dots, p$ are linearly independent,

the parameter estimates converge to their true values, that is, the point $(0, a)$ is an asymptotically stable equilibrium of the closed-loop system (3.27), with a region of attraction Ω as defined in Theorem 1.

Proof. It is sufficient to prove that the *only* invariant set contained in the set $\tilde{x} = 0$ is the point $\tilde{x} = 0, \tilde{a} = 0$.

Using condition (i), it is readily shown that with $\tilde{x} = \dot{\tilde{x}} = 0$ the error form (3.18) reduces to

$$W_s(0, \hat{a}) \tilde{a} = 0, \quad (3.33)$$

where

$$W_s(0, \hat{a}) = \begin{bmatrix} L_{f_1} L_{\hat{f}} L_{f_0}^{k_1-2} \phi_1(0, \hat{a}) & \dots & L_{f_p} L_{\hat{f}} L_{f_0}^{k_1-2} \phi_1(0, \hat{a}) \\ \vdots & & \vdots \\ L_{f_1} L_{\hat{f}} L_{f_0}^{k_m-2} \phi_m(0, \hat{a}) & \dots & L_{f_p} L_{\hat{f}} L_{f_0}^{k_m-2} \phi_m(0, \hat{a}) \end{bmatrix}. \quad (3.34)$$

This is to be compared with (3.16) evaluated at $\tilde{x} = 0$:

$$B(0, \hat{a}) = \begin{bmatrix} L_{g_1} L_{\hat{f}} L_{f_0}^{k_1-2} \phi_1(0, \hat{a}) & \dots & L_{g_m} L_{\hat{f}} L_{f_0}^{k_1-2} \phi_1(0, \hat{a}) \\ \vdots & & \vdots \\ L_{g_1} L_{\hat{f}} L_{f_0}^{k_m-2} \phi_m(0, \hat{a}) & \dots & L_{g_m} L_{\hat{f}} L_{f_0}^{k_m-2} \phi_m(0, \hat{a}) \end{bmatrix}. \quad (3.35)$$

Condition (ii), combined with $f^j(0) \in \mathbf{G}^0(0)$ from Assumption 4 (with \mathbf{G}^0 independent of a because of condition (i)), implies that the vectors $f^j(0)$ can be expressed as linear combinations of the vectors $g_i(0)$, i.e.,

$$F(0) = G(0)D, \quad F = [f^1 \dots f^p], \quad (3.36)$$

with D a constant $m \times p$ matrix. Since $F(0)$ is of rank p by condition (iii) and $G(0)$ is of rank m , D is of rank p . Comparing (3.34) with (3.35) and using (3.36) we obtain

$$W_s(0, \hat{a}) = B(0, \hat{a})D. \quad (3.37)$$

Since $B(0, \hat{a})$ is nonsingular for all $\hat{a} \in B_a$ by Assumption 3, and D is of rank p , (3.33) and (3.37) imply that

$$\tilde{a} = 0, \quad (3.38)$$

and the proof of the Corollary is complete. ■

Remark 3.1. Under the conditions of Corollary 2, our adaptive scheme achieves *exact* feedback linearization of the nonlinear system (1.6) as $t \rightarrow \infty$ and *asymptotic* stability of the equilibrium $x = 0$, $\hat{a} = a$ of the adaptive system (3.27).

Remark 3.2. In the special case where $m=p=1$ and Assumption 6 (strict matching) is satisfied, condition (i) can be removed, that is, G can be allowed to depend on a .

Example 3.2. The system

$$\dot{x}_1 = (10+a)x_2, \quad (3.39)$$

$$\dot{x}_2 = -x_2^3 + (10+a)u, \quad (3.40)$$

demonstrates the problems that can arise with the solvability of (3.24). In this case, we have

$$f(x) = \begin{bmatrix} (10+a)x_2 \\ -x_2^3 \end{bmatrix}, f^0(x) = \begin{bmatrix} 10x_2 \\ -x_2^3 \end{bmatrix}, f^1(x) = \begin{bmatrix} x_2 \\ 0 \end{bmatrix},$$

$$g(x) = \begin{bmatrix} 0 \\ (10+a) \end{bmatrix}, g^0(x) = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, g^1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$ad_f g = -\frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & 10+a \\ 0 & -3x_2^2 \end{bmatrix} \begin{bmatrix} 0 \\ 10+a \end{bmatrix} = \begin{bmatrix} -(10+a)^2 \\ 3(10+a)x_2^2 \end{bmatrix},$$

$$G^0 = sp \left\{ \begin{bmatrix} 0 \\ 10+a \end{bmatrix} \right\}, \dim G^0 = 1,$$

$$G^1 = sp \left\{ \begin{bmatrix} 0 \\ 10+a \end{bmatrix}, \begin{bmatrix} -(10+a)^2 \\ 3(10+a)x_2^2 \end{bmatrix} \right\} = R^2 \text{ for } |a| < 10.$$

Thus, all the assumptions of Theorem 1 are satisfied for $a \in B_a = \{a : |a| < 10\}$ and for all $x \in R^2$.

Using

$$\phi_1(x) = x_1, L_f \phi_1 = [1 \ 0] \begin{bmatrix} (10+\hat{a})x_2 \\ -x_2^3 \end{bmatrix} = (10+\hat{a})x_2,$$

the new coordinates are

$$\tilde{x}_1 = x_1, \quad (3.41)$$

$$\tilde{x}_2 = (10+\hat{a})x_2. \quad (3.42)$$

When in the transformed system

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 + \tilde{a} x_2, \\ \dot{\tilde{x}}_2 &= -(10+\hat{a})x_2^3 + (10+\hat{a})(10+a)u + \hat{a} x_2, \end{aligned}$$

the certainty-equivalence control is defined as

$$u = \frac{1}{(10+\hat{a})^2} [-x_1 - (10+\hat{a})x_2 - \hat{a} x_2 + (10+\hat{a})x_2^3], \quad (3.43)$$

the system is reduced to the error form (3.18)

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 + \tilde{a} x_2, \\ \dot{\tilde{x}}_2 &= -\tilde{x}_1 - \tilde{x}_2 + \tilde{a}(10+\hat{a})u, \end{aligned}$$

which, for $\tilde{a} = 0$, is exponentially stable. The Lyapunov function

$$V(x, \hat{a}) = \tilde{x}^T \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} \tilde{x} + \tilde{a}^2,$$

results in the update law

$$\dot{\hat{a}} = 1.5x_1x_2 + (10 + \hat{a})[0.5x_1 + (10 + \hat{a})x_2]u, \quad (3.44)$$

which, combined with (3.43), implicitly defines the adaptive feedback control as

$$(10 + \hat{a})[(10 + \hat{a}) + x_2(0.5x_1 + (10 + \hat{a})x_2)]u = -x_1 - (10 + \hat{a})x_2 - 1.5x_1x_2^2 + 0.5(10 + \hat{a})x_2^3. \quad (3.45)$$

For (3.45) to be solvable, the term multiplying the control u must be nonzero, which cannot be achieved globally. Indeed, the system (3.39)-(3.40) satisfies neither Assumption 6 nor Assumption 7 and thus does not fall under case (ii), when Theorem 1 provides a global result. However, Proposition 1 is satisfied and (3.45) is solvable locally. Suppose it is known that a belongs to the interval $[-8, -2]$, and \hat{a}_0 is chosen inside that interval. Then, if x_1, x_2 remain close to zero, the term $(10 + \hat{a})$ dominates the x -dependent term, thus guaranteeing the solvability of (3.45). Furthermore, (3.43) and (3.44) result in $\dot{V}(x, \hat{a}) = -\|\tilde{x}\|^2$, thus yielding $x(t), \hat{a}(t)$ bounded and

$$\lim_{t \rightarrow \infty} \tilde{x}(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = 0.$$

On the other hand, if x_1, x_2 are arbitrary, (3.45) may not have a solution. In particular, if $\hat{a} = -5, x_1 = 15, x_2 = -2$, the term multiplying u is zero. Nevertheless, the region of attraction Ω defined in case (i) of Theorem 1 is not some infinitesimally small neighborhood of the origin. For example, a subset of this region is

$$\{(x_1, x_2, \hat{a}) : |x_1| < 2, |x_2| < 2, |5 + \hat{a}| < 3\}. \quad (3.46)$$

For case (i) of Theorem 1, such (conservative) estimates of the region of attraction can always be found.

4. ROBUSTNESS OF ADAPTIVE REGULATION

We now return to the full-order system (1.1)-(1.2). The results of Section 2 apply to this system when $\mu = 0$. To deal with the case $\mu > 0$, we view the z -part of the state as consisting of two terms

$$z = \xi(x, a, \hat{a}) + \eta. \quad (4.1)$$

Here $\xi(x, a, \hat{a})$ is the so-called "manifold function"

$$\xi(x, a, \hat{a}) = -F_2^{-1}(x, a) [f_2(x, a) + G_2(x, a)u(x, \hat{a})], \quad (4.2)$$

which is exactly equal to z when $\mu=0$, as we can see by comparing (4.2) with (1.5). The new variable η describes the "off-manifold" behavior of z . Using (4.1), our plant (1.1)-(1.2), along with the adaptive controller defined by (3.25) and (3.26), can be expressed as

$$\dot{x} = f(x, a) + G(x, a)\hat{a}(x, \hat{a}) + F_1(x, a)\eta, \quad (4.3)$$

$$\mu\dot{\eta} = F_2(x, a)\eta - \mu\dot{\xi}(x, a, \hat{a}), \quad (4.4)$$

$$\dot{\hat{a}} = \psi(x, \hat{a}). \quad (4.5)$$

Our robustness result is:

Theorem 2 (Robustness). Suppose Assumptions 1, 2, 3, 4 and either 6 or 7 are satisfied. Then there exists a scalar $\mu^* > 0$ such that for all $\mu \in [0, \mu^*]$ a Lyapunov function $V(x, \hat{a}, \eta)$ can be found which guarantees that $(0, 0, a)$ is a stable equilibrium of the full-order closed-loop system (4.3)-(4.5) and that the state regulation

$$\lim_{t \rightarrow \infty} x(t) = 0, \quad \lim_{t \rightarrow \infty} \eta(t) = 0, \quad (4.6)$$

is achieved for all initial conditions (x_0, η_0, \hat{a}_0) in the set

$$\Omega = \{x, \eta, \hat{a} : V(x, \hat{a}, \eta) \leq c\}, \quad (4.7)$$

where c is the largest constant for which the set Ω is contained in $B_x \times B_\eta \times B_a$.

Proof (outline). The choice of the Lyapunov function V is made as in Taylor *et al.* (1989). The requirement that the time derivative of V for (4.3)-(4.5) be negative semidefinite in $B_x \times B_\eta \times B_a$ is then satisfied by choice of positive bounding constants c_1 , c_2 and c_3 . Once these constants are determined, it is shown that for Theorem 2 to hold, μ can be as large as

$$\mu^* = \frac{1}{c_1 c_2 + c_3}. \quad (4.8)$$

The exact form of V and the details of the proof are given in Kanellakopoulos (1989). ■

Remark 4.1. In the presence of unmodeled dynamics global regulation cannot be achieved, that is, the results of Theorem 2 *cannot* be extended beyond $B_x \times B_\eta \times B_a$ to R^{n+q+p} even if its assumptions hold globally. Such an extension would render the constants c_1, c_2, c_3 infinitely large, thus reducing the allowed μ -interval in (4.8) to the point 0. The conflict is obvious: the larger we want to make the region of attraction, the more we have to restrict the allowed dynamic uncertainty. In the limiting case $\mu=0$ we return to the setting of Section 2 and Corollary 1 becomes valid again.

Remark 4.2. The nonadaptive-adaptive comparison in Taylor *et al.* (1989) applies *mutatis mutandis* to the adaptive control scheme proposed here.

Remark 4.3. All the assumptions of Theorem 2, except for Assumptions 1 and 2, are required to hold *only for the reduced-order model* (1.6) and not for the full order plant (1.1)-(1.2). The benefit of this is that the assumptions can be satisfied by a broader class of plants.

Example 4.1. The claim made in the last remark is now illustrated by the system consisting of the DC-motor of Example 3.1 driven by an amplifier:

$$\frac{d\omega}{dt} = i + \lambda(\omega, a), \quad (4.9)$$

$$\frac{T_e}{T_m} \frac{di}{dt} = -\omega - i + u_a, \quad (4.10)$$

$$\mu \frac{du_a}{dt} = -u_a + k u, \quad (4.11)$$

where u_a, k, u , are the amplifier output, gain and input (control voltage), respectively. The full-order system (4.9)-(4.11) satisfies none of the Assumptions 5, 6, 7. However, observing that the time constant μ of the amplifier is very small, we can treat (ω, i) as x and u_a as z . Performing the singular perturbation from $\mu > 0$ to $\mu = 0$ we obtain the reduced-order model

$$\frac{d\omega}{dt} = i + \lambda(\omega, a), \quad (4.12)$$

$$\frac{T_e}{T_m} \frac{di}{dt} = -\omega - i + k u, \quad (4.13)$$

which clearly satisfies Assumption 7. If the load torque $\lambda(\omega, a)$ is linear in a , the system (4.12)-

(4.13) satisfies all the assumptions of Theorem 2, and can thus be regulated with our approach. In order to apply the approach of Taylor *et al.* (1989), one would have to treat ω as x and (i, u_a) as z , neglecting the dynamics of both (4.10) and (4.11). This would, however, result in either more restrictive robustness bounds or a smaller region of attraction.

Let us now return to the exact setting of Example 3.1 by removing the amplifier (4.11). We see that the system (3.6)-(3.7) falls under case (ii) of Theorem 1, and thus can be *globally* regulated using the adaptive controller (3.25)-(3.26). In contrast, the approach of Taylor *et al.* (1989) provides only a restricted region of attraction due to the presence of unmodeled dynamics. Furthermore, this region reduces in size as the ratio T_e/T_m increases.

5. CONCLUSIONS

In the direct adaptive regulation scheme proposed in this paper, the number of parameter estimates is equal to the number of unknown parameters. This is not the case with the general direct adaptive scheme of Sastry and Isidori (1989), which requires that the number of parameter estimates be augmented. The price paid for the simplification achieved here is a restriction on the class of plants. This restriction (Assumption 5) is less severe than the strict matching condition (Assumption 6) used in Taylor *et al.* (1989). It has not yet been determined whether a further relaxation of the matching condition is possible without an increase in the parameter estimates.

Instead of broadening the class of plants by a more relaxed matching condition, a similar effect is achieved here by allowing the presence of unmodeled dynamics. The same restriction is imposed only on a reduced-order model, while the adaptive regulation is still guaranteed for the actual plant, provided that the unmodeled dynamics are sufficiently fast in the sense of singular perturbations. As illustrated by the DC-motor example, nonlinear plants with such unmodeled dynamics are common in practice.

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APPENDIX

The terms appearing in (3.14) and (3.17) are defined as follows:

$$\alpha_1^i(x) = L_{f_0}^{k_i} \phi_i, \quad (\text{A.1})$$

$$\beta_{1,r}^i(x) = L_{g_r} L_{f_0}^{k_i-1} \phi_i, \quad (\text{A.2})$$

are scalars,

$$w_{1,j}^i = L_{f_j} L_{f_0}^{k_i-2} \phi_i, \quad (\text{A.3})$$

$$\alpha_{2,j}^i(x) = L_{f_0} L_{f_j} L_{f_0}^{k_i-2} \phi_i, \quad (\text{A.4})$$

$$\beta_{2,r,j}^i(x) = L_{g_r} L_{f_j} L_{f_0}^{k_i-2} \phi_i, \quad (\text{A.5})$$

$$\alpha_{3,j}^i(x) = L_{f_j} L_{f_0}^{k_i-1} \phi_i, \quad (\text{A.6})$$

$$\beta_{3,r,j}^i(x) = L_{g_r} L_{f_0}^{k_i-1} \phi_i, \quad (\text{A.7})$$

all for $j=1,\dots,p$, are the j -th elements of the $1 \times p$ row vectors $w_1^i(x)$, $\alpha_2^i(x)$, $\beta_{2,r}^i$, $\alpha_3^i(x)$, $\beta_{3,r}^i(x)$, $r=1,\dots,m$, respectively,

$$\alpha_{4,l,j}^i(x) = L_{f_l} L_{f_j} L_{f_0}^{k_i-2} \phi_i, \quad (\text{A.8})$$

$$\beta_{4,r,l,j}^i(x) = L_{g_r} L_{f_l} L_{f_j} L_{f_0}^{k_i-2} \phi_i, \quad (\text{A.9})$$

both for $l,j=1,\dots,p$, are the (l,j) -th elements of the $p \times p$ matrices $\alpha_4^i(x)$, $\beta_{4,r}^i(x)$, $r=1,\dots,m$, respectively, and

$$\begin{aligned} w_2^i(x, \hat{a}, u) &= \alpha_3^i(x) + \hat{a}^T \alpha_4^i(x) + \sum_{r=1}^m [\beta_{3,r}^i(x) + \hat{a}^T \beta_{4,r}^i(x)] u_r \\ &= \Delta_1^{iT}(x, \hat{a}) + u^T \Delta_2^{iT}(x, \hat{a}), \end{aligned} \quad (\text{A.10})$$

$$\Delta_1^i(x, \hat{a}) = [L_{f_1} \tilde{x}_i^{k_i}, \dots, L_{f_p} \tilde{x}_i^{k_i}]^T \quad (\text{A.11})$$

$$\Delta_2^i(x, \hat{a}) = [G^{1T} \nabla_x \tilde{x}_i^{k_i}, \dots, G^{pT} \nabla_x \tilde{x}_i^{k_i}]^T, \quad (\text{A.12})$$

$$G^j = [g_1^j, \dots, g_m^j], \quad j=0,\dots,p. \quad (\text{A.13})$$

Finally, the terms in (3.22) are defined as

$$K_0(x, \hat{a}) = \sum_{i=1}^m \left[\begin{array}{c} 0 \dots 0 \\ \vdots \\ 0 \dots 0 \end{array} , w_1^{iT}(x), \Delta_1^i(x, \hat{a}) \right] P_i \tilde{x}_i , \quad (\text{A.14})$$

$$K_1(x, \hat{a}) = \sum_{i=1}^m (P_i \tilde{x}_i)_{k_i} \Delta_2^i(x, \hat{a}) , \quad (\text{A.15})$$

where $(\cdot)_{k_i}$ denotes the k_i -th element of the $k_i \times 1$ vector.