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SERIES NASH SOLUTION OF TWO-PERSON NONZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAMES*

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ABSTRACT

It is well-known that the Nash equilibrium solution of a two-person nonzero-sum linear differential game with a quadratic cost function can be expressed in terms of the solution of coupled generalized Riccati-type matrix differential equations. For high order games the numerical determination of the solution of the nonlinear coupled equations may be difficult or even not possible when the application dictates the use of small memory computers. In this paper a series solution is suggested by means of a parameter imbedding method. Instead of solving a high order Riccati matrix equation, a lower order matrix Riccati equation corresponding to a zero-sum game is solved. In addition, lower order linear equations have to be solved. These solutions to lower order equations are the coefficients of the series solution for the nonzero-sum game. Cost functions corresponding to truncated solutions are compared with those for exact Nash equilibrium solutions.

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1. Introduction

Consider a two-person linear differential game described by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\,\underset{\sim}{x} + \underset{\sim}{B}_1\underset{\sim}{u}_1 + \underset{\sim}{B}_2\underset{\sim}{u}_2 \;, \tag{1}$$

$$\underset{\sim}{x}(t_o) = \underset{\sim}{x}_o, \tag{2}$$

where the n-vector $\underset{\sim}{x}$ is the state of the game, the m_1-vector $\underset{\sim}{u}_1$ is the strategy of player 1, the m_2-vector $\underset{\sim}{u}_2$ is the strategy of player 2, $\underset{\sim}{A}$, $\underset{\sim}{B}_1$, and $\underset{\sim}{B}_2$ are nxn, nxm_1, and nxm_2 matrices whose elements are piecewise continuous in time t, and t_o is a fixed instant of time. The cost function for player i is

$$J_i(\underset{\sim}{u}_1,\underset{\sim}{u}_2) = \tfrac{1}{2}\langle \underset{\sim}{x}_f, \underset{\sim}{S}_{if}\, \underset{\sim}{x}_f\rangle + \tfrac{1}{2}\int_{t_o}^{t_f}[\langle \underset{\sim}{x}, \underset{\sim}{Q}_i \underset{\sim}{x}\rangle + \langle \underset{\sim}{u}_i, \underset{\sim}{R}_{ii}\underset{\sim}{u}_i\rangle + \langle \underset{\sim}{u}_j, \underset{\sim}{R}_{ij}\underset{\sim}{u}_j\rangle dt \tag{3}$$

for i = 1,2, where j = 2 when i = 1, and j = 1 when i = 2. The terminal time t_f is fixed, and $\underset{\sim}{Q}_i$, $\underset{\sim}{R}_{ii}$ and $\underset{\sim}{R}_{ij}$ are symmetric matrices whose elements are piece-wise continuous in t. Furthermore $\underset{\sim}{R}_{11} > 0$, $\underset{\sim}{R}_{22} > 0$, $\underset{\sim}{R}_{12} < 0$, $\underset{\sim}{R}_{21} < 0$, $\underset{\sim}{S}_{1f} \geq 0$, $\underset{\sim}{S}_{2f} \leq 0$, $\underset{\sim}{Q}_1 \geq 0$, and $\underset{\sim}{Q}_2 \leq 0$. All of the above matrices are assumed to be known to both players.

Whereas minmax strategies are natural choices for zero-sum games, the latitude for "reasonable" strategies for nonzero-sum games is much wider. For example, depending on the nature of the information available to each player and the possibility of cooperation or noncooperation, different attributes for the strategies may be desirable. Minmax noninferior, and Nash equilibrium solutions have been investigated before [1,2]. Furthermore, open-loop and feedback Nash strategies generally lead to different values for the cost functions [2].

For nonzero sum linear games with quadratic cost functions the minmax strategies and noninferior strategies are obtained by solving decoupled Riccati-type matrix equations. However, the Nash strategies are obtained from <u>coupled</u> Riccati-type <u>matrix</u> equations.

A strategy set $(\underset{\sim}{u}_1^*, \underset{\sim}{u}_2^*)$ is called a Nash equilibrium strategy set if

$$J_1(\underset{\sim}{u}_1,\underset{\sim}{u}_2^*) \geq J_1(\underset{\sim}{u}_1^*,\underset{\sim}{u}_2^*), \tag{4}$$

$$J_2(\underset{\sim}{u}_1^*,\underset{\sim}{u}_2) \geq J_2(\underset{\sim}{u}_1^*,\underset{\sim}{u}_2^*). \tag{5}$$

If $J_1 + J_2 = 0$, the game is zero-sum. It has been shown [1] that if $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ are required to be feedback functions of $\underset{\sim}{x}$, the Nash equilibrium strategies are

$$\underset{\sim}{u}_i^* = -\underset{\sim}{R}_{ii}^{-1}\,\underset{\sim}{B}_i'\,\underset{\sim}{S}_i\,\underset{\sim}{x} \tag{6}$$

where $\underset{\sim}{S}_i$ satisfies the generalized matrix Riccati equations

$$\begin{aligned}\dot{\underset{\sim}{S}}_i = &-\underset{\sim}{A}'\underset{\sim}{S}_i - \underset{\sim}{S}_i\underset{\sim}{A} - \underset{\sim}{Q}_i + \underset{\sim}{S}_i\underset{\sim}{B}_i\underset{\sim}{R}_{ii}^{-1}\underset{\sim}{B}_i'\underset{\sim}{S}_i + \underset{\sim}{S}_i\underset{\sim}{B}_j\underset{\sim}{R}_{jj}^{-1}\underset{\sim}{B}_j'\underset{\sim}{S}_j \\ &+ \underset{\sim}{S}_j\underset{\sim}{B}_j\underset{\sim}{R}_{jj}^{-1}\underset{\sim}{B}_j'\underset{\sim}{S}_i - \underset{\sim}{S}_j\underset{\sim}{B}_j\underset{\sim}{R}_{jj}^{-1}\underset{\sim}{R}_{ij}\underset{\sim}{R}_{jj}^{-1}\underset{\sim}{B}_j'\underset{\sim}{S}_j\,, \quad \underset{\sim}{S}_i(t_f) = \underset{\sim}{S}_{if}\end{aligned} \tag{7}$$

for $i = 1,2$, where $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. Sufficient conditions which guarantee the existence of solutions $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ for $t_o \leq t \leq t_f$ are given by Rhodes [3].

Equation (7) for $i = 1,2$ represents coupled nonlinear matrix differential equations. $\tilde{S}_1$ and $\tilde{S}_2$ are nxn matrices but since they are symmetric, there are n(n+1) different variables. If n is large, the numerical determination of the solution of these n(n+1) coupled nonlinear differential equations could be quite formidable. Furthermore, in applications where the solutions have to be obtained "on-line" using limited computational capability, the solution of these n(n+1) nonlinear differential equations may not be feasible.

In this paper, a parameter imbedding method is employed to obtain series solutions for $\tilde{S}_1$ and $\tilde{S}_2$. Instead of n(n+1) nonlinear equations, only n(n+1)/2 nonlinear equations have to be solved. Higher order terms in the series require the solution of n(n+1)/2 linear equations for each term. However, the set of linear equations for each term has the same homogeneous part and only the forcing terms are different. The cost functions using the truncated strategies will be compared with the exact cost functions. As in all imbedding methods a bonus of the calculation is that a wider class of problems is solved. The imbedding parameter introduced in the paper allows an examination of a two-person game which is zero sum when the parameter is zero and when the parameter has a value of unity, the solution is an approximation for the original problem. However, since the solution is obtained as a power series in the parameter, a range of nonzero-sum games with varying degrees of asymmetry from the zero-sum condition $J_1 = -J_2$ is automatically studied.

Thus the sensitivity of the solution to a change in the asymmetry from the zero-sum condition is available.

The use of parameter imbedding for obtaining series solutions for almost zero-sum games has been investigated before [4]. The method in [4] when applied to the linear-quadratic differential game considered in this paper will yield the same zeroth order term in the series as the imbedding method below. The remaining terms in the series are different for the two imbeddings but the two series will yield the same solution for unity value of the imbedding parameter. The method discussed below leads to simpler calculation of the series.

In optimal control, parameter imbedding has been used to achieve computation reduction for the design of large scale systems [5]. The method is particularly useful for "weakly coupled" systems because a few terms in the series yield a performance index which is close to the optimal one [5].

2. Series Solution by Parameter Imbedding

Consider the equations

$$\dot{S}_i = -A'S_i - S_iA - \bar{Q}_i + S_iB_i\bar{R}_{ii}^{-1}B_i'S_i + S_iB_j\bar{R}_{jj}^{-1}B_j'S_j + S_jB_j\bar{R}_{jj}^{-1}B_j'S_i - S_jB_j\bar{R}_{jj}^{-1}\bar{R}_{ij}\bar{R}_{jj}^{-1}B_j'S_j\,, \quad S_i(\epsilon,t_f) = \bar{S}_{if} \tag{8}$$

$i = 1,2$, where $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$, where

$$\bar{\underset{\sim}{S}}_{if} = \left(\frac{\underset{\sim}{S}_{if} - \underset{\sim}{S}_{jf}}{2}\right) + \epsilon\left(\frac{\underset{\sim}{S}_{if} + \underset{\sim}{S}_{jf}}{2}\right) \tag{9}$$

$$\bar{\underset{\sim}{Q}}_{i} = \left(\frac{\underset{\sim}{Q}_{i} - \underset{\sim}{Q}_{j}}{2}\right) + \epsilon\left(\frac{\underset{\sim}{Q}_{i} + \underset{\sim}{Q}_{j}}{2}\right) \tag{10}$$

$$\bar{\underset{\sim}{R}}_{ii}^{-1} = \underset{\sim}{R}_{ii}^{-1}\left[\left(\frac{\underset{\sim}{R}_{ii} - \underset{\sim}{R}_{ji}}{2}\right) + \epsilon\left(\frac{\underset{\sim}{R}_{ii} + \underset{\sim}{R}_{ji}}{2}\right)\right]\underset{\sim}{R}_{ii}^{-1} \tag{11}$$

$$\bar{\underset{\sim}{R}}_{ij} = \bar{\underset{\sim}{R}}_{jj}\underset{\sim}{R}_{jj}^{-1}\left[\left(-\frac{\underset{\sim}{R}_{jj} - \underset{\sim}{R}_{ij}}{2}\right) + \epsilon\left(\frac{\underset{\sim}{R}_{jj} + \underset{\sim}{R}_{ij}}{2}\right)\right]\underset{\sim}{R}_{jj}^{-1}\bar{\underset{\sim}{R}}_{jj}, \tag{12}$$

and ϵ is a scalar parameter. Clearly (8) reduces to (7) when $\epsilon = 1$. Assume the existence and uniqueness of the solution of (8) for all t in $[t_o, t_f]$ and for all ϵ in an interval I which includes [0,1]. Since the right hand side of (8) is a polynomial in $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$, and ϵ, it follows that $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are infinitely differentiable with respect to ϵ for all t in $[t_o, t_f]$ and all ϵ in I [6]. Hence $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are analytic with respect to ϵ in I.

Let the solutions be expanded about $\epsilon = 0$,

$$\underset{\sim}{S}_1(\epsilon, t) = \sum_{i=0}^{\infty}\left[\left.\frac{\partial^i \underset{\sim}{S}_1(\epsilon, t)}{\partial \epsilon^i}\right|_{\epsilon=0} \frac{\epsilon^i}{i!}\right] \tag{13}$$

$$\underset{\sim}{S}_2(\epsilon, t) = \sum_{i=0}^{\infty}\left[\left.\frac{\partial^i \underset{\sim}{S}_2(\epsilon, t)}{\partial \epsilon^i}\right|_{\epsilon=0} \frac{\epsilon^i}{i!}\right] \tag{14}$$

In this section, the equations that must be satisfied by the coefficients in (13) and (14) are presented. The convergence of these series in I is guaranteed by the analyticity of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$. When (13) and (14) are evaluated at $\epsilon = 1$, the solution for the original nonzero-sum game is obtained.

Calculation of $\underset{\sim}{S}_1(0,t)$ and $\underset{\sim}{S}_2(0,t)$

The zeroth order terms are the solutions of (8) for $i = 1,2$, with ϵ set equal to zero:

$$\dot{\underset{\sim}{S}}_i = -\underset{\sim}{A}'\underset{\sim}{S}_i - \underset{\sim}{S}_i\underset{\sim}{A} - \left(\frac{\underset{\sim}{Q}_i - \underset{\sim}{Q}_j}{2}\right) + \underset{\sim}{S}_i\left(\frac{\underset{\sim}{E}_i - \underset{\sim}{F}_i}{2}\right)\underset{\sim}{S}_i + \underset{\sim}{S}_i\left(\frac{\underset{\sim}{E}_j - \underset{\sim}{F}_j}{2}\right)\underset{\sim}{S}_j + \underset{\sim}{S}_j\left(\frac{\underset{\sim}{E}_i - \underset{\sim}{F}_i}{2}\right)\underset{\sim}{S}_i + \underset{\sim}{S}_j\left(\frac{\underset{\sim}{E}_j - \underset{\sim}{F}_j}{2}\right)\underset{\sim}{S}_j\ , \quad \underset{\sim}{S}_i(0,t_f) = \frac{\underset{\sim}{S}_{if} - \underset{\sim}{S}_{jf}}{2} \tag{15}$$

where

$$\underset{\sim}{E}_i = \underset{\sim}{B}_i\underset{\sim}{R}_{ii}^{-1}\underset{\sim}{B}_i' \tag{16}$$

$$\underset{\sim}{F}_i = \underset{\sim}{B}_i\underset{\sim}{R}_{ii}^{-1}\underset{\sim}{R}_{ji}\underset{\sim}{R}_{ii}^{-1}\underset{\sim}{B}_i', \tag{17}$$

$j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. The dependence on t has been left out for convenience in writing. Although (15) for $i = 1,2$ are coupled equations, $\underset{\sim}{S}_1(0,t)$ and $\underset{\sim}{S}_2(0,t)$ can be obtained by solving only the lower order matrix equation given below. Sufficient conditions which guarantee the existence and uniqueness of the solution of the reduced are discussed in Section 3.

Consider the equation

$$\dot{\hat{\underset{\sim}{S}}} = -\underset{\sim}{A}'\hat{\underset{\sim}{S}} - \hat{\underset{\sim}{S}}\underset{\sim}{A} - \left(\frac{\underset{\sim}{Q}_1 - \underset{\sim}{Q}_2}{2}\right) + \hat{\underset{\sim}{S}}\left(\frac{\underset{\sim}{E}_1 - \underset{\sim}{F}_1 - \underset{\sim}{E}_2 + \underset{\sim}{F}_2}{2}\right)\hat{\underset{\sim}{S}},\ \hat{\underset{\sim}{S}}(t_f) = \frac{\underset{\sim}{S}_{1f} - \underset{\sim}{S}_{2f}}{2}\ , \tag{18}$$

where $\underset{\sim}{E}_i$ and $\underset{\sim}{F}_i$ are defined in (16) and (17). It is easily verified that

$$\underset{\sim}{S}_1(0,t) = \hat{\underset{\sim}{S}} \tag{19}$$

and

$$\underset{\sim}{S}_2(0,t) = -\underset{\sim}{\hat{S}} \tag{20}$$

satisfy (15) for i = 1,2. Thus $\underset{\sim}{S}_1(0,t)$ and $\underset{\sim}{S}_2(0,t)$ can be obtained by solving (18), and since $\hat{S}$ is symmetric, this entails solving $n(n+1)/2$ coupled nonlinear scalar equations.

Equation (18) is the matrix Riccati equation corresponding to the zero-sum differential game (1) with cost functions

$$J_a = \frac{J_1 - J_2}{2} \tag{21}$$

$$J_b = \frac{J_2 - J_1}{2} \tag{22}$$

when J_1 and J_2 are given in (3). This is identical to the zeroth order term that would be obtained if the imbedding in [4] is applied.

Calculation of First Order Coefficients

Differentiating (8) with respect to ϵ, setting $\epsilon = 0$, and substituting $\underset{\sim}{S}_2(0,t) = -\underset{\sim}{S}_1(0,t)$:

$$\begin{aligned} \frac{\partial \underset{\sim}{S}_i}{\partial \epsilon} = &-\left[\underset{\sim}{A}' - \underset{\sim}{S}_1 \frac{(\underset{\sim}{E}_1 - \underset{\sim}{F}_1 - \underset{\sim}{E}_2 + \underset{\sim}{F}_2)}{2}\right] \frac{\partial \underset{\sim}{S}_i}{\partial \epsilon} - \frac{\partial \underset{\sim}{S}_i}{\partial \epsilon}\left[\underset{\sim}{A} - \frac{(\underset{\sim}{E}_1 - \underset{\sim}{F}_1 - \underset{\sim}{E}_2 + \underset{\sim}{F}_2)}{2} \underset{\sim}{S}_1\right] \\ &- \frac{(\underset{\sim}{Q}_1 + \underset{\sim}{Q}_2)}{2} + \underset{\sim}{S}_1 \frac{(\underset{\sim}{E}_i + \underset{\sim}{F}_i - 3\underset{\sim}{E}_j - 3\underset{\sim}{F}_j)}{2} \underset{\sim}{S}_1 \, , \quad \left.\frac{\partial \underset{\sim}{S}_i}{\partial \epsilon}\right|_{t_f} = \frac{\underset{\sim}{S}_{1f} + \underset{\sim}{S}_{2f}}{2} \, . \end{aligned} \tag{23}$$

$\underset{\sim}{S}_1(0,t)$ can be obtained from the solution of (18), and then substituted in (23). Thus $\partial \underset{\sim}{S}_i/\partial \epsilon$ can be obtained by solving the linear equation (23). $\partial \underset{\sim}{S}_1/\partial \epsilon$ is symmetric so that the solution of (23) involves $n(n+1)/2$ linear scalar equations. Notice that for $i = 1,2$, the homogeneous portion of (23) remains the same. Only the forcing terms change.

The coefficients $\partial S_1/\partial\epsilon$ and $\partial S_2/\partial\epsilon$ described above are different from those that would be obtained if the imbedding in [4] were to be applied. The equations resulting from (23) for $i = 1,2$, are much simpler because ϵ enters linearly in the coefficients of the matrix equations in (8) whereas in the corresponding matrix equations using Starr's imbedding, ϵ would enter nonlinearly.

Calculation of kth Order Coefficients

By repeated differentiation of (8) and setting $\epsilon = 0$,

$$
\begin{aligned}
\frac{\partial^k \dot{S}_i}{\partial\epsilon^k} = &- \left[A' - S_1 \frac{(E_1 - F_1 - E_2 + F_2)}{2}\right] \frac{\partial^k S_i}{\partial\epsilon^k} - \frac{\partial^k S_i}{\partial\epsilon^k}\left[A - \frac{(E_1 - F_1 - E_2 + F_2)}{2} S_1\right] \\
&+ \frac{1}{2}\sum_{p=1}^{k-1} \frac{k!}{p!(k-p)!}\left[\frac{\partial^p S_i}{\partial\epsilon^p}(E_i - F_i)\frac{\partial^{k-p} S_i}{\partial\epsilon^{k-p}} + \frac{\partial^p S_i}{\partial\epsilon^p}(E_j - F_j)\frac{\partial^{k-p} S_j}{\partial\epsilon^{k-p}}\right. \\
&\left. + \frac{\partial^p S_j}{\partial\epsilon^p}(E_j - F_j)\frac{\partial^{k-p} S_i}{\partial\epsilon^{k-p}} + \frac{\partial^p S_j}{\partial\epsilon^p}(E_j - F_j)\frac{\partial^{k-p} S_j}{\partial\epsilon^{k-p}}\right] \\
&+ \frac{1}{2}\sum_{p=0}^{k-1} \frac{k!}{p!(k-p-1)!}\left[\frac{\partial^p S_i}{\partial\epsilon^p}(E_i + F_i)\frac{\partial^{k-p-1} S_i}{\partial\epsilon^{k-p-1}} + \frac{\partial^p S_i}{\partial\epsilon^p}(E_j + F_j)\frac{\partial^{k-p-1} S_j}{\partial\epsilon^{k-p-1}}\right. \\
&\left. + \frac{\partial^p S_j}{\partial\epsilon^p}(E_j + F_j)\frac{\partial^{k-p-1} S_i}{\partial\epsilon^{k-p-1}} - \frac{\partial^p S_j}{\partial\epsilon^p}(E_j + F_j)\frac{\partial^{k-p-1} S_j}{\partial\epsilon^{k-p-1}}\right], \quad \left.\frac{\partial^k S_i}{\partial\epsilon^k}\right|_{t_f} = 0
\end{aligned}
\tag{24}
$$

for $k \geq 2$, $i = 1,2$.

Equation (24), for $i = 1,2$, $j=2$ if $i=1$, $j=1$ if $i=2$, provides an algorithm for solving for the kth order partial derivatives of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$, based on prior calculations of the partial derivatives of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ up to order k-1. Notice that (24), for $i = 1,2$ are not coupled, so far as the calculation of the kth order partial derivatives is concerned. Furthermore, the homogeneous part of (24) does not change with i nor k. In fact the homogeneous part is the same for all $k \geq 1$. Only the forcing terms change for the calculation of the various partial derivatives.

Approximate Nash feedback strategies for players 1 and 2 are obtained by truncating the series in (13) and (14), setting $\epsilon=1$, and substituting these approximate values of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ in (16). The zeroth order terms of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are the exact solutions when the game is zero-sum, i.e., $\underset{\sim}{Q}_1 = -\underset{\sim}{Q}_2$, $\underset{\sim}{E}_1 = \underset{\sim}{F}_1$, $\underset{\sim}{E}_2 = -\underset{\sim}{F}_2$ and $\underset{\sim}{S}_{1f} = -\underset{\sim}{S}_{2f}$. If the game is not zero-sum, but if the norms of $\underset{\sim}{E}_1 + \underset{\sim}{F}_1$, $\underset{\sim}{E}_2 + \underset{\sim}{F}_2$, $\underset{\sim}{Q}_1 + \underset{\sim}{Q}_2$, and $\underset{\sim}{S}_{1f} + \underset{\sim}{S}_{2f}$ are much smaller than the corresponding norms of $\underset{\sim}{E}_1 - \underset{\sim}{F}_1$, $\underset{\sim}{E}_2 - \underset{\sim}{F}_2$, $\underset{\sim}{Q}_1 - \underset{\sim}{Q}_2$, and $\underset{\sim}{S}_{1f} - \underset{\sim}{S}_{2f}$, the game is called an almost zero-sum game. One would intuitively expect that for almost zero-sum games, an approximation for $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ using only a small number of terms would yield cost functions close to the exact Nash equilibrium cost functions. The degree of approximation of the Nash equilibrium cost functions is discussed in Section 4.

3. Dependence of Nash Cost Functions on Imbedding Parameter

In Section 2, parameter imbedding was introduced in the generalized matrix Riccati equations in such a way that when $\epsilon = 1$, the imbedded generalized matrix Riccati equation (8) reduces to the original generalized matrix Riccati equation (7). Hence the series in (13) and (14) when $\epsilon = 1$ are the Nash solutions for the cost functions in (3). One might ask if the series in (13) and (14) for any $\epsilon \neq 1$ could correspond to Nash solutions for some other cost functions. It is readily verified that indeed (13) and (14) are Nash solutions for the differential game (1) and cost functions

$$\bar{J}_i = \tfrac{1}{2}\, \underset{\sim}{x}_f' \, \bar{\underset{\sim}{S}}_{if} \, \underset{\sim}{x}_f + \tfrac{1}{2} \int_{t_o}^{t_f} (\underset{\sim}{x}' \bar{\underset{\sim}{Q}}_i \underset{\sim}{x} + \underset{\sim}{u}_i' \bar{\underset{\sim}{R}}_{ii} \underset{\sim}{u}_i + \underset{\sim}{u}_j' \bar{\underset{\sim}{R}}_{ij} \underset{\sim}{u}_j) dt \tag{25}$$

where $\bar{\underset{\sim}{S}}_{if}$, $\bar{\underset{\sim}{Q}}_i$, $\bar{\underset{\sim}{R}}_{ii}$, and $\bar{\underset{\sim}{R}}_{ij}$ are defined in (9), (10), (11), and (12), for $i = 1,2$, $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. One way of verifying that (13) and (14) are the Nash solutions for (25) for the differential game (1) is to apply the result in [1] for obtaining the generalized matrix Riccati equations. Furthermore the cost functions $\bar{J}_1$ and $\bar{J}_2$ reduce to J_1 and J_2 of (3) when $\epsilon = 1$, and satisfy the zero-sum condition $\bar{J}_1 + \bar{J}_2 = 0$ when $\epsilon = 0$.

Although ϵ enters linearly in $\bar{\underset{\sim}{S}}_{1f}$, $\bar{\underset{\sim}{S}}_{2f}$, $\bar{\underset{\sim}{Q}}_1$, and $\bar{\underset{\sim}{Q}}_2$, the dependence on ϵ of $\bar{\underset{\sim}{R}}_{11}$, $\bar{\underset{\sim}{R}}_{22}$, $\bar{\underset{\sim}{R}}_{12}$, and $\bar{\underset{\sim}{R}}_{21}$ are much more complicated. However, the matrices in the imbedded generalized matrix Riccati equation are linear in ϵ. This linearity simplifies not only the calculation of the power series for $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$, but also the calculation of $\bar{J}_1$ and $\bar{J}_2$. If the imbedding in [4] were to be applied, the imbedded weighting matrices in $\bar{J}_1$ and $\bar{J}_2$ would be linear in ϵ but the matrices in the imbedded generalized matrix Riccati equations would have complex dependence on ϵ.

The Nash strategies for players 1 and 2 for the cost function in (25) are

$$\underset{\sim}{u}_i = -\underset{\sim}{\bar{R}}_{ii}^{-1} \underset{\sim}{B}_i' \underset{\sim}{S}_i(\epsilon,t)\underset{\sim}{x} \tag{26}$$

where $\underset{\sim}{S}_1(\epsilon,t)$ and $\underset{\sim}{S}_2(\epsilon,t)$ are solutions of (8) for $i = 1,2$. The solutions cover a class of games ranging from a zero-sum game when $\epsilon = 0$ to the original nonzero-sum game when $\epsilon = 1$.

In Section 2, it has been assumed that there exists a unique solution for the imbedded generalized matrix Riccati equations, for all t in $[t_o,t_f]$ and for all ϵ in an interval that includes [0,1]. This insures the existence and uniqueness of a Nash solution for the imbedded nonzero-sum game with cost functions in (25). Sufficient conditions which guarantee existence of the generalized matrix Riccati equations for the nonzero-sum game were derived by Rhodes [3]. In terms of the notation of this paper, these conditions are: $\underset{\sim}{\bar{S}}_{1f} + \underset{\sim}{\bar{S}}_{2f} \geq 0$, $\underset{\sim}{\bar{Q}}_1 + \underset{\sim}{\bar{Q}}_2 \geq 0$, $\underset{\sim}{\bar{R}}_{11} + \underset{\sim}{\bar{R}}_{21} \geq 0$, and $\underset{\sim}{\bar{R}}_{22} + \underset{\sim}{\bar{R}}_{12} \geq 0$, for all t in $[t_o,t_f]$ and all ϵ in I; and the existence and uniqueness of the solutions of the matrix Riccati equations for two zero-sum games, one with cost $\bar{J}_1$ and another with cost $-\bar{J}_2$. Furthermore a sufficient condition for the existence and uniqueness of the solution of the matrix Riccati equation for a zero sum game with a cost function $\bar{J}_1$ is that the relative controllability matrix

$$\int_t^{t_f} \underset{\sim}{\phi}(t_f,s)[\underset{\sim}{B}_1 \underset{\sim}{\bar{R}}_{11}^{-1} \underset{\sim}{B}_1' + \underset{\sim}{B}_2 \underset{\sim}{\bar{R}}_{12}^{-1} \underset{\sim}{B}_2']\underset{\sim}{\phi}'(t_f,s)ds \geq 0 \tag{27}$$

is positive semi-definite for all t in $[t_o,t_f]$ and for all ϵ in I [3]. Similarly, for a zero-sum game with cost function $-\bar{J}_2$ a sufficient condition is that the relative controllability matrix

$$-\int_t^{t_f} \underset{\sim}{\phi}(t_f,s)[\underset{\sim}{B}_1 \underset{\sim}{\bar{R}}_{21}^{-1} \underset{\sim}{B}_1' + \underset{\sim}{B}_2 \underset{\sim}{\bar{R}}_{22}^{-1} \underset{\sim}{B}_2']\underset{\sim}{\phi}'(t_f,s)ds \geq 0 \tag{28}$$

is positive semi-definite for all t in $[t_o, t_f]$ and for all ϵ in I. The matrix $\underset{\sim}{\phi}(t,\tau)$ is the transition matrix for the system in (1). From the results in [3] a sufficient condition for the existence and uniqueness of the solution for the resulting zero-sum game when ϵ is set to zero in (25) is that

$$\int_t^{t_f} \underset{\sim}{\phi}(t_f,s)[\underset{\sim}{B}_1 \underset{\sim}{R}_{11}^{-1}(\underset{\sim}{R}_{11}-\underset{\sim}{R}_{21})\underset{\sim}{R}_{11}^{-1} \underset{\sim}{B}_1' - \underset{\sim}{B}_2 \underset{\sim}{R}_{22}^{-1}(\underset{\sim}{R}_{22}-\underset{\sim}{R}_{12})\underset{\sim}{R}_{22}^{-1} \underset{\sim}{B}_2']\underset{\sim}{\phi}'(t_f,s)ds \geq 0 \quad (29)$$

is positive semi-definite for all t in $[t_o, t_f]$. But if (27) and (28) are satisfied for all ϵ in I (including $\epsilon = 0$), it follows that (29) is also satisfied. Hence (27) and (28) guarantee the existence and uniqueness of the solution of (18) (and hence of $\underset{\sim}{S}_1(0,t)$ and $\underset{\sim}{S}_2(0,t)$. The higher order terms in the series of (13) and (14) are solutions of (23) and (24) which are linear differential equations with continuous coefficients and continuous forcing functions. Hence there exist unique solutions to these equations.

4. Degree of Approximation of Cost Functions

In this section the effects of truncation of the strategies of the players on the cost functions are examined. The power series of $\bar{J}_1$ and $\bar{J}_2$ in ϵ define the functions for all ϵ, and when these are evaluated at $\epsilon = 1$ the values are the Nash cost functions for the original nonzero-sum game. The cost functions for truncated strategies will be investigated by comparing the coefficients of their McLaurin's series with those for Nash cost functions.

Case a. Both players use the Nash strategies of (26) for i = 1,2. Denoting the Nash cost function for player 1 by J_{1a} and that for player 2 by J_{2a}, the cost functions are [1]:

$$J_{ia} = \frac{1}{2}\, \underset{\sim}{x}'\, \underset{\sim}{S}_i\, \underset{\sim}{x}\ , \quad i = 1,2, \tag{30}$$

where $\underset{\sim}{S}_i$ is given by (8). The next three cases are compared with this exact Nash equilibrium situation.

Case b. Player 1 uses the strategy

$$\underset{\sim}{u}_1 = -\bar{\underset{\sim}{R}}_{11}^{-1}\, \underset{\sim}{B}_1'\, \underset{\sim}{M}_1\, \underset{\sim}{x} \tag{31}$$

where

$$\underset{\sim}{M}_1 = \sum_{i=0}^{m-1} \frac{\epsilon^i}{i!}\, \frac{\partial^i \underset{\sim}{S}_i}{\partial \epsilon}\,(0,t) \tag{32}$$

but player 2 uses the exact Nash strategy in (26). Denote the cost function for player 1 by J_{1b} and that for player 2 by J_{2b}.

Since the cost functions for linear-quadratic differential games with linear feedback strategies are always quadratic in $\underset{\sim}{x}$, J_{1b} and J_{2b} are of the form

$$J_{ib} = \frac{1}{2}\, \underset{\sim}{x}'\, \underset{\sim}{P}_i \underset{\sim}{x} \tag{33}$$

where $\underset{\sim}{P}_i$ satisfies [1]

$$\dot{\underset{\sim}{P}}_i = -(A-\bar{\underset{\sim}{E}}_i\underset{\sim}{M}_i - \bar{\underset{\sim}{E}}_2\underset{\sim}{S}_2)'\underset{\sim}{P}_i - \underset{\sim}{P}_i(\underset{\sim}{A}-\bar{\underset{\sim}{E}}_1\underset{\sim}{M}_1 - \bar{\underset{\sim}{E}}_2\underset{\sim}{S}_2) - \bar{\underset{\sim}{Q}}_i - \underset{\sim}{G}_i\bar{\underset{\sim}{E}}_i\underset{\sim}{G}_i - \underset{\sim}{G}_j\bar{\underset{\sim}{F}}_j\underset{\sim}{G}_j,\ \underset{\sim}{P}_i(\epsilon,t_f) = \bar{\underset{\sim}{S}}_{if}, \tag{34}$$

where

$$\bar{\underset{\sim}{E}}_i = \frac{\underset{\sim}{E}_i - \underset{\sim}{F}_i}{2} + \epsilon\, \frac{\underset{\sim}{E}_i + \underset{\sim}{F}_i}{2}\ ;\ \bar{\underset{\sim}{F}}_i = -\frac{\underset{\sim}{E}_i - \underset{\sim}{F}_i}{2} + \epsilon\, \frac{\underset{\sim}{E}_i + \underset{\sim}{F}_i}{2}\ , \tag{35}$$

$$\underset{\sim}{G}_1 = \underset{\sim}{M}_1,\quad \underset{\sim}{G}_2 = \underset{\sim}{S}_2, \tag{36}$$

and as before $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$. $\underset{\sim}{E}_i$ and $\underset{\sim}{F}_i$ are defined in (16) and (17), and $\bar{\underset{\sim}{S}}_{if}$ is defined in (9).

It can be shown that

$$\left.\frac{\partial^i J_{1b}}{\partial\epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{1a}}{\partial\epsilon^i}\right|_{\epsilon=0}, \quad i = 0,1,\ldots,2m-1 \tag{37}$$

and

$$\left.\frac{\partial^i J_{2b}}{\partial\epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{2a}}{\partial\epsilon^i}\right|_{\epsilon=0}, \quad i = 0,1,\ldots,m. \tag{38}$$

The approximation property in (37) is the same as for optimal control [7,8]. Thus if the first m terms of the McLaurin's series for $\underset{\sim}{u}_1$ are equal to the first m terms of the Nash solution (26) with $i = 1$ for player 1, and if player 2 uses the Nash strategy in (26) with $i = 2$, then the first 2m terms of the McLaurin's series of J_{1b} will be equal to the first 2m terms of the McLaurin's series of the exact Nash cost function J_{1a}. Player 2 will not achieve the exact Nash pay-off function J_{2a} but (38) will be satisfied. Relation (37) is proved using the same arguments as in [8]. That is, $\underset{\sim}{P}_i - \underset{\sim}{S}_i \equiv \underset{\sim}{\Gamma}_i$ is formed. Using the differential equations for $\underset{\sim}{P}_i$ and $\underset{\sim}{S}_i$, it is straightforward to show that $\partial^k \Gamma_i / \partial\epsilon^k = 0$ at $\epsilon = 0$, for $k = 0,\ldots,2m-1$ when $i = 1$ and for $k = 0,\ldots,m$ when $i = 2$.

<u>Case c</u>. Player 1 uses the strategy in (31) and (32) and player 2 uses the strategy

$$\underset{\sim}{u}_2 = -\bar{\underset{\sim}{R}}_{22}^{-1} \underset{\sim}{B}_2' \underset{\sim}{M}_2 \underset{\sim}{x}, \tag{39}$$

$$\underset{\sim}{M}_2 = \sum_{i=0}^{k-1} \frac{\epsilon^i}{i!} \frac{\partial^i \underset{\sim}{S}_2}{\partial\epsilon^i}(0,t). \tag{40}$$

Denoting the cost functions for players 1 and 2 by J_{1c} and J_{2c}, then

$$J_{ic} = \frac{1}{2}\, \underset{\sim}{x}' \underset{\sim}{L}_i\, \underset{\sim}{x}\ , \quad i = 1,2 \tag{41}$$

where $\underset{\sim}{L}_i$ satisfies

$$\dot{\underset{\sim}{L}}_i = -(\underset{\sim}{A} - \bar{\underset{\sim}{E}}_1 \underset{\sim}{M}_1 - \bar{\underset{\sim}{E}}_2 \underset{\sim}{M}_2)' \underset{\sim}{L}_i - \underset{\sim}{L}_i (\underset{\sim}{A} - \bar{\underset{\sim}{E}}_1 \underset{\sim}{M}_1 - \bar{\underset{\sim}{E}}_2 \underset{\sim}{M}_2) - \bar{\underset{\sim}{Q}}_i - \underset{\sim}{M}_i \bar{\underset{\sim}{E}}_i \underset{\sim}{M}_i - \underset{\sim}{M}_j \bar{\underset{\sim}{F}}_j \underset{\sim}{M}_j \ ,$$

$$L_i(\epsilon, t_f) = \bar{\underset{\sim}{S}}_{if} \ . \tag{42}$$

It can be shown that

$$\left.\frac{\partial^i J_{1c}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{1a}}{\partial \epsilon^i}\right|_{\epsilon=0} , \quad i = 0,1,\ldots,\min\left\{ {2m-1 \atop k} \right\} \tag{43}$$

and

$$\left.\frac{\partial^i J_{2c}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{2a}}{\partial \epsilon^i}\right|_{\epsilon=0} , \quad i = 0,1,\ldots,\min\left\{ {2k-1 \atop m} \right\} . \tag{44}$$

The derivations of (43) and (44) are analogous to those of (37) and (38). Thus if both players use first order corrections on nominally zero-sum strategies, the resulting cost functions match the exact Nash cost functions to second order.

<u>Case d</u>. Player 1 uses the truncated strategy in (31) and (32) and player 2 uses the optimal strategy which minimizes $\bar{J}_2$ knowing that player 1 is using the strategy in (31) and (32).

The optimal strategy for player 2 is

$$\underset{\sim}{u}_2 = -\bar{\underset{\sim}{R}}_{22}^{-1} \underset{\sim}{B}_2' \underset{\sim}{K}_2 \underset{\sim}{x} \tag{45}$$

where

$$\dot{\underset{\sim}{K}}_2 = -\underset{\sim}{K}_2(\underset{\sim}{A}-\bar{\underset{\sim}{E}}_1\underset{\sim}{M}_1) - (\underset{\sim}{A}-\bar{\underset{\sim}{E}}_1\underset{\sim}{M}_1)'\underset{\sim}{K}_2 + \underset{\sim}{K}_2\,\bar{\underset{\sim}{E}}_2\underset{\sim}{K}_2 - \bar{\underset{\sim}{Q}}_2 - \underset{\sim}{M}_1\bar{\underset{\sim}{F}}_1\underset{\sim}{M}_1,\ \underset{\sim}{K}_2(\epsilon,t_f) = \bar{\underset{\sim}{S}}_{2f}. \tag{46}$$

The cost functions are

$$J_{id} = \tfrac{1}{2}\,\underset{\sim}{x}'\underset{\sim}{K}_i\,\underset{\sim}{x}\ ,\quad i = 1,2, \tag{47}$$

where $\underset{\sim}{K}_1$ satisfies

$$\dot{\underset{\sim}{K}}_1 = -\underset{\sim}{K}_1(\underset{\sim}{A}-\bar{\underset{\sim}{E}}_1\underset{\sim}{M}_1 - \bar{\underset{\sim}{E}}_2\underset{\sim}{K}_2) - (\underset{\sim}{A}-\bar{\underset{\sim}{E}}_1\underset{\sim}{M}_1 - \bar{\underset{\sim}{E}}_2\underset{\sim}{K}_2)'\underset{\sim}{K}_1 - \underset{\sim}{M}_1\bar{\underset{\sim}{E}}_1\underset{\sim}{M}_1 - \bar{\underset{\sim}{Q}}_1 - \underset{\sim}{K}_2\bar{\underset{\sim}{F}}_2\underset{\sim}{K}_2,\ \underset{\sim}{K}_1(\epsilon,t_f) = \bar{\underset{\sim}{S}}_{1f}. \tag{48}$$

The cost functions J_{1d} and J_{2d} have the property that

$$\left.\frac{\partial^i J_{1d}}{\partial\epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{1a}}{\partial\epsilon^i}\right|_{\epsilon=0}\ ,\ i = 0,1,\ldots,m, \tag{49}$$

and

$$\left.\frac{\partial^i J_{2d}}{\partial\epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{2a}}{\partial\epsilon^i}\right|_{\epsilon=0}\ ,\ i = 0,1,\ldots,m. \tag{50}$$

Cases b, c, and d are for player 1 using a truncated strategy and player 2 using either Nash, truncated, or optimal strategy knowing that player 1 uses a truncated strategy. By symmetry the roles of players 1 and 2 may be interchanged to investigate two other possible strategies for player 1 when player 2 uses truncated strategy. The comparisons in the above cases are with respect to the exact Nash strategy of case a. Cost functions for cases b, c, and d may be compared among themselves instead of the one for case a by deriving differential equations for

$\underset{\sim}{P}_i - \underset{\sim}{L}_i$, $\underset{\sim}{P}_i - \underset{\sim}{K}_i$, and $\underset{\sim}{L}_i - \underset{\sim}{K}_i$, for i = 1,2, and investigating to what order their partial derivatives with respect to ϵ are zero identically. The results are

$$\left.\frac{\partial^i J_{1b}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{1c}}{\partial \epsilon^i}\right|_{\epsilon=0}, \; i = 0,1,\ldots,k \tag{51}$$

$$\left.\frac{\partial^i J_{2b}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{2c}}{\partial \epsilon^i}\right|_{\epsilon=0}, \; i = 0,1,\ldots,k \tag{52}$$

$$\left.\frac{\partial^i J_{1b}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{1d}}{\partial \epsilon^i}\right|_{\epsilon=0}, \; i = 0,1,\ldots,m \tag{53}$$

$$\left.\frac{\partial^i J_{2b}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{2d}}{\partial \epsilon^i}\right|_{\epsilon=0}, \; i = o,1,\ldots,2m-1 \tag{54}$$

$$\left.\frac{\partial^i J_{1c}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{1d}}{\partial \epsilon^i}\right|_{\epsilon=0}, \; i = 0,1,\ldots,\min\left\{\begin{matrix} m \\ k \end{matrix}\right\} \tag{55}$$

and

$$\left.\frac{\partial^i J_{2c}}{\partial \epsilon^i}\right|_{\epsilon=0} = \left.\frac{\partial^i J_{2d}}{\partial \epsilon^i}\right|_{\epsilon=0}, \; i = 0,1,\ldots,\min\left\{\begin{matrix} m \\ 2k-1 \end{matrix}\right\}. \tag{56}$$

For a given initial state, the functions may be plotted against the scalar parameter ϵ. $J_{1a}(\epsilon)$ and $J_{2a}(\epsilon)$ are the exact Nash cost functions for a class of almost zero sum games, including a zero-sum game when $\epsilon = 0$, and the original nonzero-sum game when $\epsilon = 1$. Equations (37), (38), (43), (44), (49), and (50) indicate the closeness of the curves for J_{1b}, J_{1c}, and J_{1d} to J_{1a}, and the closeness of the curves for J_{2b}, J_{2c}, and J_{2d} to J_{2a}.

Equations (51) to (56) show that the different truncated strategies compared among themselves yield cost functions which are close to each other. For example, in (51) and (52), it is seen that the truncation order for player 1 has no effect on the comparison of the cost functions between Cases b and c. In fact when $k \to \infty$, Case c becomes Case b so that J_{1c} becomes identical to J_{1b} and J_{2c} becomes identical to J_{2b}. Similarly, from (53) and (54), it is seen that when $m \to \infty$, Cases b, d, and a become the same. For a comparison of Cases c and d, (55) and (56) show that the cost function differences depend on m and k because the strategy of player 2 in Case d depends on the truncation m of player 1 and the strategies are both truncated for Case c and these depend on both m and k.

From the above four cases, it is seen that if player 1 uses a truncated strategy, and if player 2 uses a Nash strategy, the cost function of player 2 is not as close to the exact Nash value as that of player 1, in the sense of (37) and (38). Player 2 will achieve the same degree of approximation to mth order in the sense of (44) by truncating his strategy to kth order where k is more than half as large as m. If player 2 uses the optimal strategy which minimizes $\bar{J}_2$ knowing that player 1 uses a truncated strategy his cost function would not be too different from what he would obtain had he used a truncated Nash strategy, in the sense of (44) and (50).

In solving for the cost functions in McLaurin's series form, various partial derivatives with respect to ϵ of $\underset{\sim}{S}_i$, $\underset{\sim}{P}_i$, $\underset{\sim}{L}_i$, $\underset{\sim}{K}_i$ are needed. However, the homogeneous parts of the differential equations for these partial derivatives evaluated at $\epsilon = 0$ are all the same. The forcing terms are easily formed because of the simple manner in which ϵ appears in the weighting matrices of the equations.

5. Examples

Example 1: A velocity-controlled pursuit-evasion game as in [1] is considered here to illustrate the procedure

$$\dot{r} = v_p - v_e \tag{57}$$

$$J_p = \tfrac{1}{2}\,\sigma_p^2 r_f' r_f + \tfrac{1}{2}\int_o^{t_f} (v_p' v_p/c_p + v_e' v_e/c_{pe})dt \tag{58}$$

$$J_e = -\tfrac{1}{2}\,\sigma_e^2 r_f' r_f + \tfrac{1}{2}\int_o^{t_f} (v_e' v_e/c_e + v_p' v_p/c_{ep})dt. \tag{59}$$

Consider the case $\sigma_p^2 = \sigma_e^2 = c_p c_e = 1$, $a = c_p/c_{pe} = 0$, $b = c_e/c_{ep} = 0$, $\omega^2 = c_p/c_e = 4$, and $t_f = 0.5$. The 0th order term is found from (18)

$$\dot{\hat{\underset{\sim}{S}}} = 0.75\ \hat{\underset{\sim}{S}}^2\ ,\quad \hat{\underset{\sim}{S}}(0.5) = 1. \tag{60}$$

Thus

$$S_1(0,t) = \hat{\underset{\sim}{S}}(t)\ ,\quad S_2(0,t) = -\hat{\underset{\sim}{S}}(t). \tag{61}$$

The first order terms are found from (23)

$$\frac{\partial \dot{S}_1}{\partial \epsilon} = 1.5\ S_1(0,t)\,\frac{\partial S_1}{\partial \epsilon} + 0.25(S_1(0,t))^2\ ,\quad \left.\frac{\partial S_1}{\partial \epsilon}\right|_{t=0.5} = 0 \tag{62}$$

$$\frac{\partial \dot{S}_2}{\partial \epsilon} = 1.5\ S_1(0,t)\,\frac{\partial S_2}{\partial \epsilon} - 2.75(S_1(0,t))^2\ ,\quad \left.\frac{\partial S_2}{\partial \epsilon}\right|_{t=0.5} = 0. \tag{63}$$

The 2nd order terms are found from (24)

$$\frac{\partial^2 \dot{S}_1}{\partial\epsilon^2} = 1.5\ S_1(0,t)\ \frac{\partial^2 S_1}{\partial\epsilon^2} + \left[2\left(\frac{\partial S_1}{\partial\epsilon}\right)^2 + \left(\frac{\partial S_1}{\partial\epsilon}\right)\left(\frac{\partial S_2}{\partial\epsilon}\right) + 0.5\left(\frac{\partial S_2}{\partial\epsilon}\right)^2\right]$$
$$+ \left[4S_1\ \frac{\partial S_1}{\partial\epsilon} + S_1\ \frac{\partial S_2}{\partial\epsilon} + S_2\ \frac{\partial S_1}{\partial\epsilon} - S_2\ \frac{\partial S_2}{\partial\epsilon}\right],\ \frac{\partial^2 S_1}{\partial\epsilon^2}\Bigg|_{t=0.5} = 0 \tag{64}$$

$$\frac{\partial^2 \dot{S}_2}{\partial\epsilon^2} = 1.5\ S_1(0,t)\ \frac{\partial^2 S_2}{\partial\epsilon^2} + \left[0.5\left(\frac{\partial S_2}{\partial\epsilon}\right)^2 + 4\left(\frac{\partial S_1}{\partial\epsilon}\right)\left(\frac{\partial S_2}{\partial\epsilon}\right) + 2\left(\frac{\partial S_1}{\partial\epsilon}\right)^2\right]$$
$$+ \left[S_2\ \frac{\partial S_2}{\partial\epsilon} + 4\ S_2\ \frac{\partial S_1}{\partial\epsilon} + 4S_1\ \frac{\partial S_2}{\partial\epsilon} - 4S_1\ \frac{\partial S_1}{\partial\epsilon}\right],\ \frac{\partial^2 S_2}{\partial\epsilon^2}\Bigg|_{t=0.5} = 0. \tag{65}$$

Similarly 3rd and higher order terms can be found by solving (24) for that order.

Approximate strategies are then found from truncated series of (13) and (14) with $\epsilon = 1$. The costs to pursuer and evader when 4th order truncations are used are found to be 0.591 $(\frac{r^2(0)}{2})$ and -0.241 $(\frac{r^2(0)}{2})$ while the exact Nash strategies*yields $J_1 = 0.593\ (\frac{r^2(0)}{2})$ and $J_2 = -\ 0.241\ (\frac{r^2(0)}{2})$.

Example 2: An acceleration controlled pursuit-evasion game closely related to that given in [1] is considered

$$\dot{r} = v \qquad r(0) = 1 \tag{66}$$

$$\dot{v} = a_p - a_e \qquad v(0) = 0 \tag{67}$$

$$J_p = r_f' r_f + \tfrac{1}{2}\int_0^{0.5} (r'r + a_p' a_p)\,dt \tag{68}$$

$$J_e = -0.6\ r_f' r_f + \tfrac{1}{2}\int_0^{0.5} (-1.2r'r + 0.4\ a_e' a_e)\,dt\ . \tag{69}$$

---

*In the two examples of this section, $R_{12} = R_{21} = 0$ so that the sufficient conditions for the existence of solutions for S_1 and S_2 in [3] are not satisfied. However in these examples, it can be shown that the solutions for S_1 and S_2 exist.

Presence of quadratic terms of the states in the integral has the following meaning: evader (pursuer) wants to maximize (minimize) the distance not only at the final instant but also during the course of pursuit. In general this problem may not be transformed into one of velocity-controlled pursuit-evasion game. Thus to obtain exact Nash strategies, it will be necessary to solve coupled Riccati-type matrix equations. Using 0th order truncations, one obtains $J_1 = 1.615$, $J_2 = -0.884$. Using first order truncations, one obtains $J_1 = 1.490$, $J_2 = -0.856$. When 2nd order truncations are used, $J_1 = 1.476$, $J_2 = -0.860$ are obtained. The exact Nash strategies yield $J_1 = 1.478$, $J_2 = -0.862$.

At t = 0.25	$S_1(11)$	$S_1(12)$	$S_1(22)$	$S_2(11)$	$S_2(12)$	$S_2(22)$
Exact Nash Strategy	2.30	0.543	0.133	-1.49	-0.336	-0.081
0th Order truncation	1.89	0.437	0.106	-1.89	-0.437	-0.106
1st Order truncation	2.32	0.548	0.134	-1.50	-0.338	-0.081
2nd Order truncation	2.30	0.543	0.133	-1.49	-0.335	-0.081

At t = 0	$S_1(11)$	$S_1(12)$	$S_1(22)$	$S_2(11)$	$S_2(12)$	$S_2(22)$
Exact Nash Strategy	2.96	1.34	0.647	-1.72	-0.712	-0.331
0th Order truncation	2.26	0.990	0.471	-2.26	-0.990	-0.471
1st Order truncation	3.15	1.44	0.694	-1.82	-0.760	-0.355
2nd Order truncation	3.00	1.37	0.658	-1.70	-0.700	-0.325

Table 1. Comparison of S for Example 2.

6. Concluding Remarks

By a simple linear parameter imbedding of the generalized matrix Riccati equations, strategies for two-person linear quadratic nonzero-sum differential games are obtained in series form. For almost zero-sum games approximate Nash cost functions may be obtained with much less computation using lower order equations in contrast to the exact solution which involves solving higher order equations. The method requires solving a matrix Riccati equation for a zero-sum game with half as many variables as the original problem, and a set of linear equations with the same homogeneous part. The linear equations are also of lower dimensionality. Finally, the effect of truncating the series on the cost functions are discussed.

The numerical examples show that low order truncations yield reasonably accurate results. In general the method would be most useful when the cost functionscorrespond to almost zero-sum games, and when the dimensionality is high. Although realistic games are usually not linear with quadratic cost functions, one possible method of obtaining an approximation to Nash strategies is to expand the cost function and differential equations in Taylor series and consider strategies which are also Taylor's series. The first order approximations in the control strategies are based on a linearized game with quadratic cost functions. This approach is analogous to that of Albrekht and Lukes as applied to control problems [9,10].

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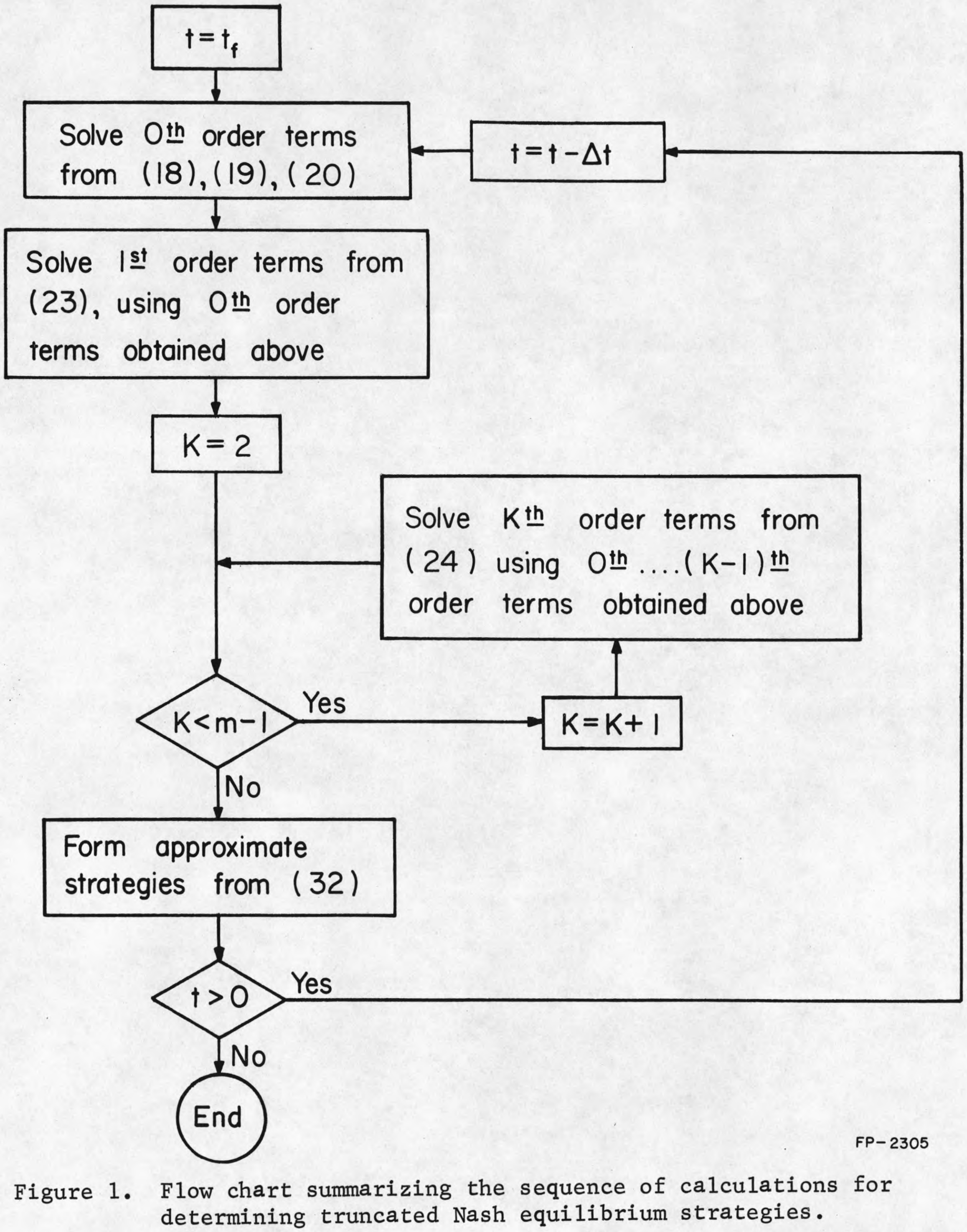

Figure 1. Flow chart summarizing the sequence of calculations for determining truncated Nash equilibrium strategies.

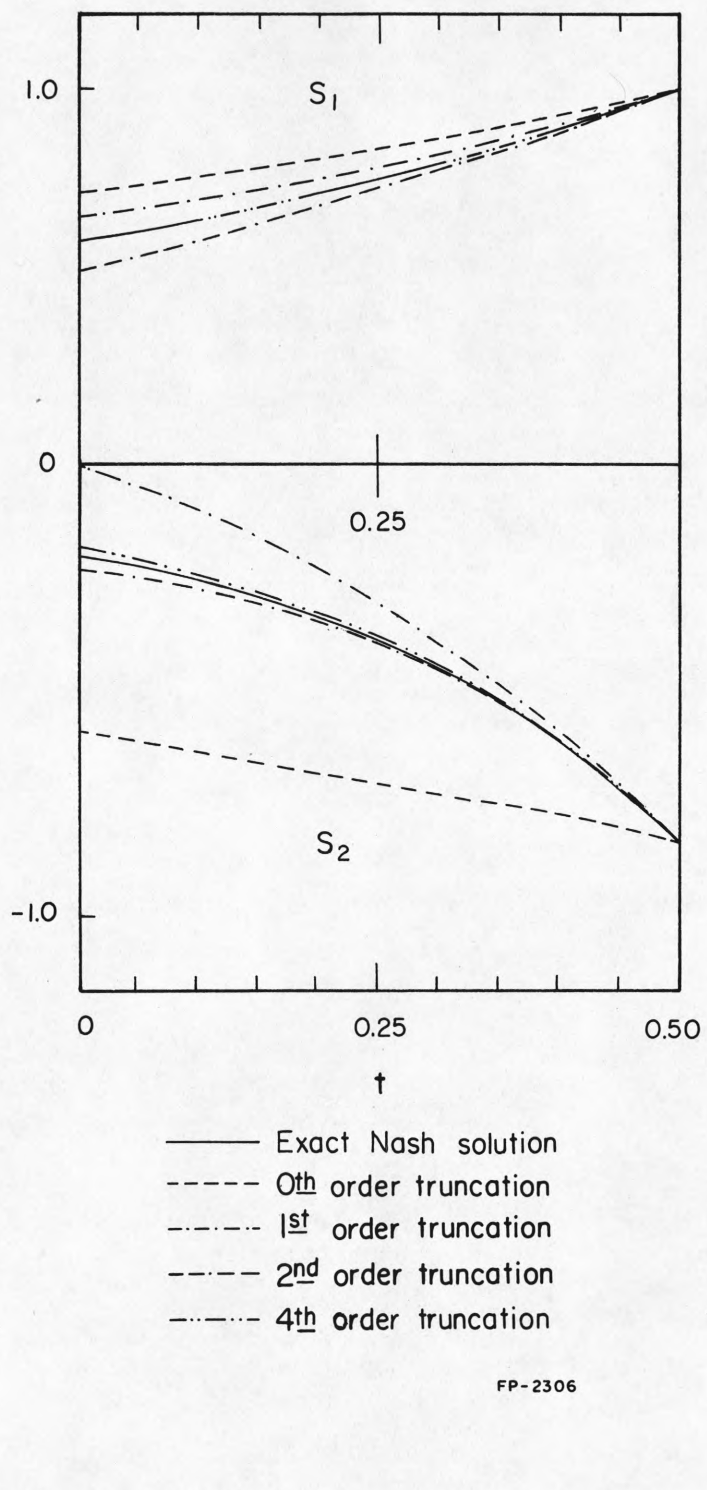

Figure 2. $S_1(t)$ and $S_2(t)$ for various degrees of truncation for Example 1.

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It is well-known that the Nash equilibrium solution of a two-person nonzero-sum linear differential game with a quadratic cost function can be expressed in terms of the solution of coupled generalized Riccati-type matrix differential equations. For high order games the numerical determination of the solution of the nonlinear coupled equations may be difficult or even not possible when the application dictates the use of small memory computers. In this paper a series solution is suggested by means of a parameter imbedding method. Instead of solving a high order Riccati matrix equation, a lower order matrix Riccati equation corresponding to a zero-sum game is solved. In addition, lower order linear equations have to be solved. These solutions to lower order equations are the coefficients of the series solution for the nonzero-sum game. Cost functions corresponding to truncated solutions are compared with those for exact Nash equilibrium solutions.

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