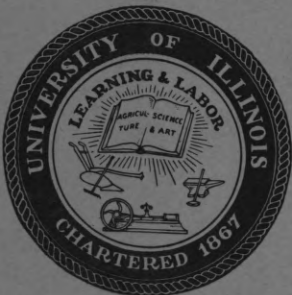


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AND SYNTAX-ANALYSIS ALGORITHM
FOR CONTEXT-FREE LANGUAGES

T. Kasami

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AN EFFICIENT RECOGNITION AND SYNTAX-ANALYSIS ALGORITHM
FOR CONTEXT-FREE LANGUAGES*

T. Kasami¹

Abstract

An efficient algorithm of recognition and syntax-analysis for the full class of context-free languages without the difficulty of exponential growth of computing time with the length n of input sequence is presented. This algorithm makes use of a fundamental algebraic property of a context-free language. It is shown in this paper that a context-free language is n^3 -recognizable in the sense of Hartmanis and Stearns by a double-tape or double-head single-tape Turing machine and it is n^4 -recognizable by a single-head single-tape Turing machine. The size of memory required for recognition is proportional to n^2 . If we use a random-access memory whose size is proportional to n^2 , the computing time required for syntax-analysis is upper-bounded by $C_1 n^3 + C_2 n^2 N$, where N denotes the number of non-equivalent valid derivation sequences for a given input sequence and C_i 's are constants independent of input sequences. If we use two tapes of length $C_3 n^2$ and two tapes of length $C_4 n$ as working memories, the computing time for syntax-analysis is upper-bounded by $n^3 (C_5 + C_6 N)$.

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¹University of Illinois, on leave from Osaka University.

1. Introduction and Preliminaries

Since the introduction of Chomsky the theory of context-free languages and its applications to natural or programming languages have been studied extensively (1 - 10). It is important in the theory and its application to find efficient algorithms for recognition or syntax-analysis of sequences of a context-free language (CFL). For some practically important but considerably restricted subclasses of CFL's, several highly efficient algorithms of syntax-analysis have been proposed in which the computing time is proportional to the length of an input sentence (9, 10). These algorithms have difficulty in pinpointing the locations of the errors for a syntactically incorrect input sequence. To the author's knowledge, however, there was no known general method of recognition or syntax-analysis of a CFL in which the time required for recognition or analysis does not increase exponentially with the length of input sequence (10).*

An efficient algorithm of recognition and syntax-analysis for the full class of CFL without the difficulty of exponential growth of computing time is presented in this paper. This algorithm may be modified to give some diagnostic information on errors (15).

For convenient reference, the relevant definitions and notions of context-free grammar are presented here briefly. The set of all finite sequences, including the null sequence Λ , over a finite alphabet Σ is denoted by Σ^* . A context-free grammar G is an ordered quadruple (V_N, V_T, P, S) in which

*The author became aware of an unpublished work by D. H. Younger, after he submitted the original manuscript.

(1) V_N and V_T are disjoint finite alphabets which are the nonterminal and terminal vocabularies of G , respectively. Let $V = V_N \cup V_T$.

(2) P is a finite set of rewriting rules of the form,

$$Y \rightarrow \phi$$

where $Y \in V_N$, $\phi \in V^*$ and $\phi \neq \Lambda$.

(3) $S \in V_N$. S is the initial symbol.

We shall mainly use a grammar in Greibach's standard 2-form (6).

Let us name rewriting rules in P as g_1, \dots, g_i, \dots , respectively. We adopt the left to right derivations without any loss of generality (5). We then write

$$\begin{array}{c} g_i \\ \varphi_1 \rightarrow \varphi_2 \end{array}$$

if $\varphi_1 = w_1 Y w_2$, $\varphi_2 = w_1 \phi w_2$, $w_1 \in V_T^*$, $w_2 \in V^*$ and rewriting rule $g_i: Y \rightarrow \phi$ is in P . The language generated by G is defined as the set $\{\varphi\}$ of sequences over V_T such that there exists sequence $\varphi_0, \dots, \varphi_m$ and rewriting rules in P g_1, \dots, g_m with $\varphi_0 = S$, $\varphi_m = \varphi$ and $\varphi_{i-1} \xrightarrow{g_i} \varphi_i$ ($1 \leq i \leq m$). Sequence g_1, \dots, g_m is said to be a valid derivation sequence (d.s.) of G for φ . The language generated by a context-free grammar is said to be a context-free language (CFL). Hereafter, let G denote a context-free grammar and let L denote the CFL generated by G over an alphabet $A (= V_T)$. By a recognition algorithm of L , we mean a procedure for testing whether for any sequence \bar{a} over A , \bar{a} is in L . By a syntax-analysis algorithm of L generated by G , we mean a procedure to find all valid derivation sequences of G for any given sequence in L . Since we adopt the left to right derivations, different valid derivations for a sequence in L are not equivalent to each other.

One reasonable method to estimate the efficiency of algorithm is to see how the computing time and the size of required memory grow with n , the length of input sequence. Hartmanis and Stearns have introduced the concept "T(n)-recognizable" to measure the complexity of a recognition problem (11). As a standard automaton, they considered a Turing machine TM with a one-way input tape using the symbols in a finite alphabet A and a one-way output tape using two symbols "1" and "0" besides working tapes. A TM is said to recognize L if and only if for any input sequence \bar{a} on A , the n -th output digit of T is "1" if the first n digits of \bar{a} is in L and is "0" otherwise. L is said to be T(n)-recognizable if and only if there is a TM which recognizes L and, for any input sequence \bar{a} , prints the n -th output digit in T(n) or fewer operations.

It is shown in this paper that any context-free language is n^3 -recognizable by a double-tape or a double-head single-tape Turing machine and n^4 -recognizable by a single-head single-tape Turing machine. The size of memory required for recognition is proportional to n^2 . The measures of efficiency of the syntax-analysis algorithm* presented here are as follows. We hereafter use notation C_i to designate constants independent of input sequences. If we use a random-access memory whose size is proportional to n^2 , the computing time is upper-bounded by $C_{11}n^3 + C_{12}n^2N$, N being the number of non-equivalent valid derivation sequences for a given input sequence. If we use two tapes of length $C_{20}n^2$ and two of length $C_{21}n$ as working memories, the computing time is upper-bounded by $n^3(C_{22} + C_{23}N)$.

*This algorithm also rejects input sequences which are not in L .

2. Derivation Sequences

Since we can effectively construct a grammar in standard 2-form strongly equivalent to a given grammar (Greibach (6)), we shall use a grammar G in standard 2-form for L . A grammar is in standard 2-form if all of the rules are of the forms:

Type I: $Y \rightarrow a Y_1 Y_2,$

Type II: $Y \rightarrow a Y_1,$

Type III: $Y \rightarrow a.$

We use notations $Y, Y_1, Y_2 \dots$ for nonterminal symbols and $a, a_1, a_2 \dots$ for terminal symbols. Let us name the rules of type I, the rules of type II and the rules of type III, $\ell_1, \dots, \ell_{m_1}, p_1, \dots, p_{m_2}$, and q_1, \dots, q_{m_3} , respectively. For nonterminal symbol Y and terminal symbol a , $R(Y, a)$, $R_1(Y, a)$, $R_2(Y, a)$ and $R_3(Y, a)$ denote the set of rules, the set of rules of type I, the set of rules of type II and the set of rules of type III respectively, in which the non-terminal symbol on the left side is Y and the terminal symbol is a . $R(Y, a)$ is possibly empty. $R_3(Y, a)$ is empty or consists of one rule. Let $N(p_i)$ denote the nonterminal symbol on the right-hand side of rule p_i and let $N_1(\ell_i)$ and $N_2(\ell_i)$ denote the first and second nonterminal symbols in the right-hand side of rule ℓ_i respectively. The following arguments are illustrated by a running example.

Example 1: Consider a grammar $G_o = (V_{To}, V_{No}, S, P_o)$ where $V_{To} = \{ (,), +, v \}$, $V_{No} = \{ S \}$ and P_o :

$S \rightarrow (S + S),$

$S \rightarrow SS,$

$S \rightarrow v.$

This grammar generates a simple class of familiar algebraic forms. By Greibach's procedure, we can easily construct a grammar G_1 in standard form strongly equivalent to G_0 as follows:

$$G_1 = \{V_{To}, V_{No}, S, P_1\} ,$$

where P_1 consists of the rules:

$$S \rightarrow (S + S) ,$$

$$S \rightarrow (S + S)S ,$$

$$S \rightarrow vS ,$$

$$S \rightarrow v .$$

We can further construct a grammar G_{ex} in 2-standard form strongly equivalent to G_1 as follows:

$$G_{ex} = \{V_{To}, V_N, S, P_2\} ,$$

where $V_N = \{S, E, U, V, W\}$ and P_2 consists of the rules:

$$l_1: S \rightarrow (SU ,$$

$$l_2: S \rightarrow (SV ,$$

$$l_3: U \rightarrow +SE ,$$

$$l_4: V \rightarrow +SW ,$$

$$P_1: S \rightarrow vS ,$$

$$P_2: W \rightarrow)S ,$$

$$q_1: S \rightarrow v ,$$

$$q_2: E \rightarrow) .$$

Let L_{ex} denote the language generated by G_{ex} . Although for this grammar there exists a much simpler and more efficient algorithm than the general method in this paper, we have chosen this running example because of its

simplicity and familiarity. For G_{ex} , $N_1(l_1) = S$, $N_2(l_1) = U$, $N(p_1) = S$,
 $R(S, () = \{l_1, l_2\}$, $R(S, v) = \{p_1, q_1\}$, $R(U, +) = \{l_3\}$, $R(V, +) = \{l_4\}$,
 $R(W,)) = \{p_2\}$, $R(E,)) = \{q_2\}$ and all other $R(Y, a)$ are empty.

We shall introduce a "dummy" symbol r_i for each l_i and rewrite the right-hand side of rule l_i by inserting r_i between $N_1(l_i)$ and $N_2(l_i)$, i.e.,

$$l_i: Y \rightarrow aY_1Y_2$$

$$l_i: Y \rightarrow aY_1r_iY_2$$

No sequence should be substituted for r_i ; its function is to indicate the relation between $N_1(l_i)$ and $N_2(l_i)$ explicitly, as will be made clear in the following sections.

Let $\bar{a} (= a_1, a_2, \dots, a_n)$ be an input sequence on A . A Y -derivation sequence (Y -d.s.) of G for \bar{a} is defined as follows:

- 1) If $R(Y, a_1)$ is empty, there is no Y -d.s. Otherwise, as the first step choose any one rule, say l_i (or p_i or q_i), in $R(Y, a_1)$.

Then write

$$l_i, N_1(l_i), r_i N_2(l_i)$$

(or $p_i, N(p_i)$ or q_i), which is called a partial Y -d.s.

- 2) Suppose that $j-1$ steps have been done. If the partial Y -d.s. ξ contains no nonterminal symbol* or $j-1 = n$, terminate the procedure. Then, the sequence is a Y -d.s. Otherwise, let the first non-terminal symbol of ξ be Y_h . If $R(Y_h, a_j)$ is empty, terminate the procedure. The sequence ξ is a Y -d.s. Otherwise, choose any rule of $R(Y_h, a_j)$ and

* l_i, p_i, q_i , and r_i are not regarded as nonterminal symbols.

substitute for Y_h the sequence consisting of the name of the chosen rule and the left-hand side of the rule in the same manner as in step 1). For example, assume that the chosen rule is l_i and the symbol just preceding the " Y_h " is r_k . The substitution for " Y_h " is " $r_k l_i, N_1(l_i), r_1 N_2(l_i)$." Note that we do not write a comma between an r symbol and the symbol following it.

If a Y-d.s. is obtained through exactly n steps (substitutions) and contains no nonterminal symbol, this Y-d.s. is said to be valid.

Remark 1: The last symbol in a valid Y-d.s. is a q symbol.

Remark 2: There exists a one to one correspondence between the set of valid S-d.s. for \bar{a} and the set of valid d.s. for \bar{a} . A valid d.s. is obtained from a valid S-d.s. by deleting all r symbols. Conversely, it follows immediately from the definition of S-d.s. that for a given valid d.s. \bar{g} there is a unique valid S-d.s. from which \bar{g} is obtained by deleting all r symbols. Therefore, \bar{a} is in L if and only if there exists a valid S-d.s. for \bar{a} .

Example 2: Let $\bar{a} = (v + (v + v))^*$. A valid S-d.s. is derived as follows:

$$\begin{aligned} \bar{a} &= (v + (v + v)) \\ &\quad l_1 \quad S \quad r_1 U \\ &\quad \quad q_1 \quad r_1 U \\ &\quad \quad \quad r_1 l_3 \quad S \quad r_3 E \\ &\quad \quad \quad \quad l_1 \quad S \quad r_1 U \quad r_3 E \\ &\quad \quad \quad \quad \quad q_1 \quad r_1 U \quad r_3 E \\ &\quad \quad \quad \quad \quad \quad r_1 l_3 \quad S \quad r_3 E \quad r_3 E \\ &\quad \quad \quad \quad \quad \quad \quad q_1 \quad r_3 E \quad r_3 E \\ &\quad \quad \quad \quad \quad \quad \quad \quad r_3 q_2 \quad r_3 E \\ &\quad \quad \quad \quad \quad \quad \quad \quad \quad r_3 q_2 \end{aligned}$$

*We shall omit the commas between symbols.

Sequence l_2, p_1, S, r_2V is also a S-d.s. for \bar{a} , but this sequence is not valid.

Let us define $\bar{R}(\alpha, a)$ as follows:

- (1) If α is l_i or $r_h l_i$, $\bar{R}(\alpha, a) = R(N_1(l_i), a)$.
- (2) If α is p_i or $r_h p_i$, $\bar{R}(\alpha, a) = R(N(p_i), a)$.
- (3) If α is q_i or $r_h q_i$, $\bar{R}(\alpha, a) = \bigcup_{\text{all } m} \{r_m x \mid x \in R(N_2(l_m), a)\}$

Example 3: For G_{ex} , note that $N_1(l_i) = N(p_{i'}) = S$

($1 \leq i \leq 4, 1 \leq i' \leq 2$). $\bar{R}(l_i, ()) = \bar{R}(r_h l_i, ()) = \bar{R}(p_{i'}, ()) = \bar{R}(r_h p_{i'}, ()) = R(S, ()) = \{l_1, l_2\}$, $\bar{R}(l_i, v) = \bar{R}(r_h l_i, v) = \bar{R}(p_{i'}, v) = \bar{R}(r_h p_{i'}, v) = R(S, v) = \{p_1, q_1\}$, $\bar{R}(q_{i'}, ()) = \bar{R}(r_h q_{i'}, ()) = \{r_3 q_2, r_4 p_2\}$, $\bar{R}(q_{i'}, +) = \bar{R}(r_h q_{i'}, +) = \{r_1 l_3, r_2 l_4\}$ and all other $\bar{R}(Y, a)$ are empty.

Hereafter let $\alpha, \alpha_1, \alpha_2, \dots, \beta, \beta_1, \dots$ designate either $l_1, p_i, q_i, r_h l_i, r_h p_i, r_h q_i, \mathcal{E}_h^{(k)}$ or $\lambda_h^{(k)*}$ and let $x, y, z \dots$ designate either l_i, p_i or q_i . If $\bar{\alpha} (= \alpha_1, \alpha_2, \dots, \alpha_n)$ satisfies the following conditions:

- (1) $\alpha_1 \in R(Y, a_1)$
- (2) $\alpha_j \in \bar{R}(\alpha_{j-1}, a_j) \quad (1 < j \leq n)$
- (3) α_n is q_i or $r_h q_i \quad (1 \leq h \leq m_1, 1 \leq i \leq m_3)$,

$\bar{\alpha}$ is said to be a quasi-valid Y-s. for input sequence $\bar{a} = a_1, a_2, \dots, a_n$. If $\bar{\alpha}$ is a valid Y-d.s., it is also a quasi-valid Y-s.

If a sequence can be reduced to null sequence \wedge by eliminating all the symbols other than l_h and r_h ($1 \leq h \leq m_1$) and by applying the rules

$$l_h r_h \rightarrow \wedge \quad (1 \leq h \leq m_1),$$

we say that the sequence satisfies the D-condition.

Example 4: The first sequence in Example 2 is a quasi-valid S-s. satisfying the D-condition.

* $\mathcal{E}_h^{(k)}$ and $\lambda_h^{(k)}$ are defined in Section 3.

Theorem 1: A quasi-valid Y-s. is a valid Y-d.s., if and only if it satisfies the D-condition.

Proof: In the derivation of a Y-d.s., the partial Y-d.s. of the first step satisfies the D-condition. If the partial Y-d.s. of the (j-1)-th step satisfies the D-condition, it is easily verified that it still does after one substitution. By induction on j, we see that a Y-d.s. satisfies the D-condition. Since a valid Y-d.s. is a quasi-valid Y-s., we have the "only if part." The "if part" also is proved by induction on n, the length of input sequence. If $n = 1$, then a quasi-valid Y-s. consists of only one q symbol. It is obvious that the sequence consisting of this q-symbol also is a valid Y-d.s. Assume that for $n < m$, the "if part" holds. Suppose that $n = m$ and $\bar{\alpha}$ is a quasi-valid Y-s. satisfying the D-condition. If $\bar{\alpha}$ contains no l symbols, $\bar{\alpha}$ is obviously a valid Y-d.s. Suppose that α_j is the first l symbol and $\alpha_j = l_h$. Then there exists an $\alpha_{j'}$ which contains the r-symbol paired with α_j . Let $\alpha_{j'} = r_h x$. Subsequence $\alpha_1, \dots, \alpha_{j-1}$ contains neither l nor r-symbols. Subsequence $\alpha_{j+1}, \dots, \alpha_{j'-1}$ is a quasi-valid $N_1(\alpha_j)$ -s. for input sequence $a_{j+1}, \dots, a_{j'-1}$ satisfying the D-condition, and subsequence $x, \alpha_{j'+1}, \dots, \alpha_n$ is a quasi-valid $N_2(\alpha_j)$ -s. for input sequence $a_{j'}, \dots, a_n$ satisfying the D-condition. It follows from the induction hypothesis that both subsequences are valid. Therefore $\bar{\alpha}$ is also valid by the definition of valid d.s.

Corollary 1: Sequence \bar{a} is in L if and only if there exists a quasi-valid S-s. for \bar{a} which satisfies the D-condition.*

* This corollary may be considered a version of the fundamental theorem due to Chomsky and Schutzenberger (2, 6).

This corollary is the fundamental basis of our recognition algorithm.

3. Main Theorem

Let $\bar{a} = (a_1, a_2, \dots, a_n)$ be an input sequence on A. Let us define $F_j^{(k)}$ ($1 \leq j \leq n$, $0 \leq k < j$) as follows:

$$(1) \quad F_1^{(0)} = \{(q_s, \alpha_1)_1 | \alpha_1 \in R_3(S, a_1)\} \cup \{(q_s, \alpha_1)_0 | \alpha_1 \in R_1(S, a_1) \cup R_2(S, a_1)\},$$

where q_s is a special symbol indicating the beginning of the input sequence.

$$F_j^{(0)} = \{(\alpha_{j-1}, \alpha_j)_v | \exists \alpha [(\alpha, \alpha_{j-1}) \in F_{j-1}^{(0)}]; \alpha_j \in \bar{R}(\alpha_{j-1}, a_j); \\ v = 1 \text{ if } \alpha_j \text{ is a } q \text{ symbol or a combination of an } r \text{ symbol} \\ \text{and a } q \text{ symbol, } v = 0 \text{ otherwise}\},$$

where (α, β) means $(\alpha, \beta)_1$ or $(\alpha, \beta)_0$,

(2) For each $(\alpha_{j-1}, \alpha_j)_v \in F_j^{(k-1)}$, let $T_{k-1}(\alpha_{j-1}, \alpha_j)$ and $\pi((\alpha_{j-1}, \alpha_j)_v)$ be defined as follows:

Case I: $\alpha_j = r_h$ or $\alpha_j = r_h x$ ($1 \leq h \leq m_1$).

If α_{j-1} is q_s or l_h' ($h \neq h'$), T_{k-1} and π are not defined. If $\alpha_{j-1} = l_h$, $T_{k-1}(\alpha_{j-1}, \alpha_j) = \mathcal{E}_h^{(k-1)}$. Otherwise, $T_{k-1}(\alpha_{j-1}, \alpha_j) = r_h$.

Case II: α_j is neither r_h nor $r_h x$ ($1 \leq h \leq m_1$). If α_{j-1} is r_h , $T_{k-1}(\alpha_{j-1}, \alpha_j) = \lambda_h^{(k-1)}$. If α_{j-1} is $r_h x$, $T_{k-1}(\alpha_{j-1}, \alpha_j) = x$. Otherwise, $T_{k-1}(\alpha_{j-1}, \alpha_j) = \alpha_{j-1}$.

The definition of $T_{k-1}(\alpha, \beta)$ is summarized in Table 1.

If $v = 1$ and α_j is not an l symbol, $\pi((\alpha_{j-1}, \alpha_j)_v)$ is defined to be 1. Otherwise, $\pi((\alpha_{j-1}, \alpha_j)_v) = 0$.

$\alpha \backslash \beta$	r_h or r_{hx} (only for $k=1$)	$l_{i'}, p_{i'}, q_{i'}, e_{i'}^{(j'-1)}$ or $\lambda_{i'}^{(j'-1)}$ ($j' < k-1$)
r_i	r_h	$\lambda_i^{(k-1)}$
r_{ix} (only for $k=1$)	r_h	x
l_h	$e_h^{(k-1)}$	l_h
l_i ($i \neq h$)	not defined	l_i
q_s	not defined	q_s
$p_i, q_i,$ $e_i^{(j)}$ or $\lambda_i^{(j)}$ ($j < k-1$)	r_h	α

Table 1. Definition Table of $T_{k-1}(\alpha, \beta)$.

$$(3) F_j^{(k)} = \{ (T_{k-1}(\alpha_{j-2}, \alpha_{j-1}), T_{k-1}(\alpha_{j-1}, \alpha_j)) \pi((\alpha_{j-1}, \alpha_j)_v) \mid (\alpha_{j-2}, \alpha_{j-1}) \in F_{j-1}^{(k-1)}; (\alpha_{j-1}, \alpha_j)_v \in F_j^{(k-1)} \}.$$

Example 5: For $\bar{a} = (v + (v + v))$, $F_j^{(0)}, F_j^{(1)}, \dots, F_j^{(8)}$ are listed in Table 2.

For a sequence $\bar{\alpha} = (\alpha_j, \dots, \alpha_m)$ ($k-1 \leq j$), let $T_{k-1}\bar{\alpha} = T_{k-1}(\alpha_j, \alpha_{j+1}), \dots, T_{k-1}(\alpha_{m-1}, \alpha_m)$. If $\bar{\alpha}$ contains any adjacent incompatible pair $(\ell_{h'}, r_h)$ ($h' \neq h$) or (q_s, r_h) , $T_{k-1}\bar{\alpha}$ is not defined. Otherwise, T_{k-1} deletes the last non-r symbol and each non-r symbol in $\bar{\alpha}$ preceding an r symbol, moves the remaining non-r symbols to the right by one place, replaces each r_h by $\mathcal{E}_h^{(k-1)}$ if the preceding symbol is ℓ_h , leaves the remaining r symbols unchanged and writes $\lambda_h^{(k-1)}$ in each empty place following an r symbol. It will be shown that we can easily test whether $\bar{\alpha}$ satisfies the D-condition by applying T_{k-1}, T_k, \dots successively to $\bar{\alpha}$. The following arguments are based on this simple idea. But the problem is that we must consider a set of sequences instead of a single sequence and test whether the set contains a sequence satisfying the D-condition. For this purpose, we need some device and have introduced the extra symbols $\mathcal{E}_h^{(k-1)}$ and $\lambda_h^{(k-1)}$. We shall show some elementary properties of $F_j^{(m)}$ in Lemmas 1 and 2.

Lemma 1: (a) If (x, α) or $(r_h x, \alpha) \in F_j^{(0)}$ and $(r_h, x, \gamma) \in F_j^{(0)}$, then $(r_h, x, \alpha) \in F_j^{(0)}$.

(b) If $(r_h x, \alpha)$ and $(x, \beta) \in F_j^{(0)}$, then $(x, \alpha) \in F_j^{(0)}$.

(c) If $(\alpha, r_h x) \in F_j^{(0)}$, then $\alpha = q_i$ or r_h, q_i .

(d) If $(\alpha, r_h y) \in F_j^{(0)}$ and $(r_h x, \beta) \in F_{j+1}^{(0)}$, then $(\alpha, r_h x) \in F_j^{(0)}$.

Table 2. Examples of $F_j^{(k)}$

j	1	2	3	4	5	6	7	8	9
\bar{a}	(v	+	(v	+	v))
$F_j^{(0)}$	$(q_s, l_1)_0$ $(q_s, l_2)_0$	$(l_1, p_1)_0$ $(l_1, q_1)_1$ $(l_2, p_1)_0$ $(l_2, q_1)_1$	$(q_1, r_1 l_3)_0$ $(q_1, r_2 l_4)_0$	$(r_1 l_3, l_1)_0$ $(r_1 l_3, l_2)_0$ $(r_2 l_4, l_1)_0$ $(r_2 l_4, l_2)_0$	$(l_1, p_1)_0$ $(l_1, q_1)_1$ $(l_2, p_1)_0$ $(l_2, q_1)_1$	$(q_1, r_1 l_3)_0$ $(q_1, r_2 l_4)_0$	$(r_1 l_3, p_1)_0$ $(r_1 l_3, q_1)_1$ $(r_2 l_4, p_1)_0$ $(r_2 l_4, q_1)_1$	$(q_1, r_3 q_2)_1$ $(q_1, r_4 p_2)_0$	$(r_3 q_2, r_3 q_2)_1$ $(r_3 q_2, r_4 p_2)_0$
$F_j^{(1)}$		$(q_s, l_1)_0$ $(q_s, l_2)_0$	$(l_1, r_1)_0$ $(l_1, r_2)_0$ $(l_2, r_1)_0$ $(l_2, r_2)_0$	$(r_1, l_3)_0$ $(r_2, l_4)_0$	(l_3, l_1) (l_3, l_2) (l_4, l_1) (l_4, l_2)	$(l_1, r_1)_0$ $(l_1, r_2)_0$ $(l_2, r_1)_0$ $(l_2, r_2)_0$	(r_1, l_3) (r_2, l_4)	$(l_3, r_3)_1$ $(l_3, r_4)_0$ $(l_4, r_3)_1$ $(l_4, r_4)_0$	$(r_3, r_3)_1$ $(r_3, r_4)_0$
$F_j^{(2)}$			$(q_s, \epsilon_1^{(1)})_0$ $(q_s, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_0$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, l_3)_0$ $(\lambda_2^{(1)}, l_4)_0$	$(l_3, \epsilon_1^{(1)})_0$ $(l_3, \epsilon_2^{(1)})_0$ $(l_4, \epsilon_1^{(1)})_0$ $(l_4, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_0$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, \epsilon_3^{(1)})_1$ $(\lambda_2^{(1)}, \epsilon_4^{(1)})_0$	$(\epsilon_3^{(1)}, r_3)_1$ $(\epsilon_3^{(1)}, r_4)_0$

$F_j^{(3)}$				$(q_s, \epsilon_1^{(1)})_0$ $(q_s, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_0$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, l_3)_0$ $(\lambda_2^{(1)}, l_4)_0$	$(l_3, \epsilon_1^{(1)})_0$ $(l_3, \epsilon_2^{(1)})_0$ $(l_4, \epsilon_1^{(1)})_0$ $(l_4, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_1$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, r_3)_1$
$F_j^{(4)}$					$(q_s, \epsilon_1^{(1)})_0$ $(q_s, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_0$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, l_3)_0$ $(\lambda_2^{(1)}, l_4)_0$	$(l_3, \epsilon_1^{(1)})_1$ $(l_3, \epsilon_2^{(1)})_0$ $(l_4, \epsilon_1^{(1)})_1$ $(l_4, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, r_3)_1$
$F_j^{(5)}$						$(q_s, \epsilon_1^{(1)})_0$ $(q_s, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_0$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, l_3)$ $(\lambda_2^{(1)}, l_4)$	$(l_3, r_3)_1$ $(l_4, r_3)_1$
$F_j^{(6)}$							$(q_s, \epsilon_1^{(1)})_0$ $(q_s, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_0$ $(\epsilon_2^{(1)}, \lambda_2^{(1)})_0$	$(\lambda_1^{(1)}, \epsilon_3^{(5)})_1$
$F_j^{(7)}$								$(q_s, \epsilon_1^{(1)})_0$ $(q_s, \epsilon_2^{(1)})_0$	$(\epsilon_1^{(1)}, \lambda_1^{(1)})_1$
$F_j^{(8)}$									$(q_s, \epsilon_1^{(1)})_1$

(e) If $(r_h x, r_i y) \in F_j^{(0)}$, then $x \in R_3(N_2(\ell_h), a_{j-1})$ and x is determined uniquely by h and a_{j-1} .

(f) There exists neither $\mathcal{E}_h^{(0)}$ nor $\lambda_h^{(0)}$.

Proof: (a) Note that $\bar{R}(x, a_j) = \bar{R}(r_h x, a_j) = \bar{R}(r_h, x, a_j)$. Since (x, α) or $(r_h x, \alpha) \in F_j^{(0)}$, $\alpha \in \bar{R}(r_h, x, a_j)$. Since $(r_h, x, \gamma) \in F_j^{(0)}$, $j \geq 2$ and there exists β such that $(\beta, r_h, x) \in F_{j-1}^{(0)}$ by the definition of $F_j^{(0)}$. Therefore, $(r_h, x, \alpha) \in F_j^{(0)}$.

(b) It is shown similarly that $\alpha \in \bar{R}(r_h x, a_j) = \bar{R}(x, a_j)$ and there exists α' such that $(\alpha', x) \in F_{j-1}^{(0)}$. By the definition of $F_j^{(0)}$, $(x, \alpha) \in F_j^{(0)}$.

(c) Since $(\alpha, r_h x) \in F_j^{(0)}$, $r_h x \in \bar{R}(\alpha, a_j)$. By the definition of $\bar{R}(\alpha, a_j)$, $\alpha = q_i$ or r_h, q_i .

(d) Since $(r_h x, \beta) \in F_{j+1}^{(0)}$, there exists α' such that $(\alpha', r_h x) \in F_j^{(0)}$. Therefore, $r_h x \in \bar{R}(\alpha', a_j)$. According to (c), α and α' are q symbols or combinations of an r symbol and a q symbol. Consequently, $r_h x \in \bar{R}(\alpha, a_j) (= \bar{R}(\alpha', a_j))$. This implies that $(\alpha, r_h x) \in F_j^{(0)}$.

(e) Since $(r_h x, r_i y) \in F_j^{(0)}$, $j \geq 2$ and there exists α such that $(\alpha, r_h x) \in F_{j-1}^{(0)}$ and $r_h x \in \bar{R}(\alpha, a_{j-1})$. By (c), x is a q symbol and α is a q symbol or a combination of an r symbol and a q symbol. By the definition of $\bar{R}(\alpha, a_{j-1})$,

$x \in R_3(N_2(\ell_h), a_{j-1})$. $R_3(Y, a)$ contains at most one element.

(f) (c) implies that $\mathcal{E}_h^{(0)}$ can not exist. It follows from the definition of $T_0(\alpha, \beta)$ that $\lambda_h^{(0)}$ does not exist.

Lemma 2: Suppose that $(\alpha_1, \alpha_2) \in F_j^{(k)}$ ($k \geq 2$).

(a) If $\alpha_1 = r_h$, then α_2 is $\lambda_h^{(k-1)}$ or an r symbol.

(b) If $\alpha_1 = \mathcal{E}_h^{(k')}$, then $k > k'$ and α_2 is $\lambda_h^{(k')}$ or an r symbol.

- (c) If $\alpha_1 = \lambda_h^{(k')}$ ($k' \geq 2$), then $k > k'$ and α_2 is $\lambda_h^{(k'-1)}$ or an r symbol.
- (d) If $\alpha_2 = \lambda_h^{(k-1)}$, then α_1 is r_h or $\mathcal{E}_h^{(k-1)}$.
- (e) If $\alpha_2 = \lambda_h^{(k')}$ ($k' < k-1$), α_1 is $\lambda_h^{(k'+1)}$ or $\mathcal{E}_h^{(k')}$.
- (f) If $\alpha_2 = \mathcal{E}_h^{(k')}$, then α_1 is neither $r_{h'}$, $\mathcal{E}_h^{(k'')}$ nor $\lambda_h^{(k''')}$ ($k'' > 1$).

Proof: Since $(\alpha_1, \alpha_2) \in F_j^{(k)}$ ($k \geq 2$), there exist β_1, β_2 and β_3 such that $(\beta_1, \beta_2) \in F_{j-1}^{(k-1)}$, $(\beta_2, \beta_3) \in F_j^{(k-1)}$, $\alpha_1 = T_{k-1}(\beta_1, \beta_2)$ and $\alpha_2 = T_{k-1}(\beta_2, \beta_3)$.

(a) Since $r_h = T_{k-1}(\beta_1, \beta_2)$, β_2 must be r_h from Table 1. Then $\alpha_2 (= T_{k-1}(r_h, \beta_3))$ is $\lambda_h^{(k-1)}$ or an r symbol.

(b) From Table 1, $k > k'$. Note that $\mathcal{E}_h^{(k')} = T_{k-1}(\beta_1, \beta_2)$. If $k' = k-1$, $\beta_2 = r_h$ according to Table 1. Thus, (b) is valid. Assume that (b) holds if $k-k' < i$. Consider the case where $k' = k-i$ ($i > 1$). Then, β_1 must be $\mathcal{E}_h^{(k')}$ from Table 1. Since $(\mathcal{E}_h^{(k')}, \beta_2) \in F_{j-1}^{(k-1)}$, β_2 is $\lambda_h^{(k')}$ or an r symbol by the induction hypothesis. Hence, $\alpha_2 (= T_{k-1}(\beta_2, \beta_3))$ is $\lambda_h^{(k')}$ or an r symbol from Table 1.

(c) From Table 1, $k > k'$. Note that $\lambda_h^{(k')} = T_{k-1}(\beta_1, \beta_2)$. If $k' = k-1$, β_1 must be r_h and β_2 is not an r symbol according to Table 1. Then, it follows from (a) that β_2 is $\lambda_h^{(k-2)}$. Therefore, $\alpha_2 (= T_{k-1}(\lambda_h^{(k-2)}, \beta_3))$ is $\lambda_h^{(k-2)}$ or an r symbol from Table 1. Assume that (c) holds if $k-k' < i$. Consider the case where $k' = k-i \geq 2$ and $i > 1$. Then, $\beta_1 = \lambda_h^{(k')}$ and β_2 is not an r symbol from Table 1. Since $(\beta_1, \beta_2) \in F_{j-1}^{(k-1)}$, β_2 is $\lambda_h^{(k'-1)}$ by the induction hypothesis. Therefore, $\alpha_2 (= T_{k-1}(\lambda_h^{(k'-1)}, \beta_3))$ is $\lambda_h^{(k'-1)}$ or an r symbol from Table 1.

(d) Since $\lambda_h^{(k-1)} = T_{k-1}(\beta_2, \beta_3)$, β_2 must be r_h from Table 1. Then $\alpha_1 = T_{k-1}(\beta_1, r_h)$. Therefore, α_1 is r_h or $\mathcal{E}_h^{(k-1)}$ according to Table 1.

(e) Since $\lambda_h^{(k')} = T_{k-1}(\beta_2, \beta_3)$, β_2 must be $\lambda_h^{(k')}$ from Table 1. By Lemma 1 (f), $k' \geq 1$. Hence, $k \geq 3$. If $k' = k-2$, β_1 is r_h or $e_h^{(k')}$ by (d). Since $\alpha_1 = T_{k-1}(\beta_1, \lambda_h^{(k')})$, α_1 is $\lambda_h^{(k-1)}$ ($= \lambda_h^{(k'+1)}$) or $e_h^{(k')}$. Assume that (e) holds if $k-k' < i$. Consider the case in which $k' = k-i$ and $i > 2$. Since $(\beta_1, \lambda_h^{(k')}) \in F_{j-1}^{(k-1)}$, β_1 is $\lambda_h^{(k'+1)}$ or $e_h^{(k')}$ by the induction hypothesis. Consequently, α_1 is $\lambda_h^{(k'+1)}$ or $e_h^{(k')}$.

(f) Note that $e_h^{(k')} = T_{k-1}(\beta_2, \beta_3)$. If $k' = k-1$, $\beta_2 = l_h$ according to Table 1. Since $(\beta_1, l_h) \in F_{j-1}^{(k-1)}$, it follows from (a), (b), and (c) that β_1 is neither $r_{h'}$, $e_h^{(k'')}$ nor $\lambda_{h'}^{(k''')}$ ($k'' > 1$) if $k > 2$. If $k = 2$, β_1 cannot be an e symbol or a λ symbol by Lemma 1 (f). Since $\alpha_1 = T_1(\beta_1, l_h)$, α_1 is neither $r_{h'}$, $e_h^{(k'')}$ nor $\lambda_{h'}^{(k''')}$ ($k'' > 1$) from Table 1. Assume that (f) holds if $k-k' < i$. Consider the case in which $k' = k-i$ and $i \geq 2$. From Table 1, $\beta_2 = e_h^{(k')}$. Since $(\beta_1, e_h^{(k')}) \in F_{j-1}^{(k-1)}$, it follows from the induction hypothesis that β_1 is neither $r_{h'}$, $e_h^{(k'')}$ nor $\lambda_{h'}^{(k''')}$ ($k'' > 1$). Since $\alpha_1 = T_{k-1}(\beta_1, e_h^{(k')})$, α_1 is neither $r_{h'}$, $e_h^{(k'')}$ nor $\lambda_{h'}^{(k''')}$ ($k'' > 1$).

By a k -chain ($0 \leq k < n$), we shall mean a sequence $\alpha_k, \dots, \alpha_n$ such that $(\alpha_{j-1}, \alpha_j) \vee_j \in F_j^{(k)}$ ($k < j \leq n$) and $\vee_n = 1$.

Lemma 3: The set of quasi-valid \mathcal{S} 's. is identical with that of 0-chains from which the first symbol q_s is deleted.

This lemma is obvious from the definitions of $F_j^{(0)}$ and 0-chain.

Lemma 4: Let $\bar{\alpha}$ ($= \alpha_k, \dots, \alpha_n$) be a k -chain. Then $T_k \bar{\alpha}$ is a $(k+1)$ -chain if and only if α_n is not an l symbol.

This lemma follows directly from the definitions of $T_k \bar{\alpha}$, k -chain and $\pi((\alpha, \beta) \vee)$.

Now we shall consider the inverse of T_k . Let $\alpha_u, \alpha_{u+1}, \dots, \alpha_{u+v}$ be a subsequence of a k -chain $\alpha_k, \dots, \alpha_n$ and let $U_{k,u}(\alpha_u, \dots, \alpha_{u+v}) = \{(\beta_{u-1}, \dots, \beta_{u+v-1}) \mid (\beta_{u+i-1}, \beta_{u+i}) \in F_{u+i}^{(k-1)} \ (0 \leq i < v); \alpha_{u+i} = T_{k-1}(\beta_{u+i-1}, \beta_{u+i}); \exists \beta_{u+v} [(\beta_{u+v-1}, \beta_{u+v}) \in F_{u+v}^{(k-1)}; \alpha_{u+v} = T_{k-1}(\beta_{u+v-1}, \beta_{u+v}); \text{if } u+v = n, \text{ then } (\beta_{n-1}, \beta_n)_1 \in F_n^{(k-1)} \text{ and } \beta_n \text{ is not an } \ell\text{-symbol}]\}$.

Lemma 5: (a) $U_{k,u}(\alpha_u, \alpha_{u+1})$ is not empty.

(b) If α_u is not an r symbol, $\alpha_{u+1}, \dots, \alpha_{u+v-1}$ are r symbols and either α_{u+v} is not an r symbol or $u+v = n$, then $U_{k,u}(\alpha_u, \dots, \alpha_{u+v})$ is not empty.

Proof: (a) The definition of $F_{u+1}^{(k)}$ implies (a).

(b) We shall assume that $k \geq 2$. The case $k = 1$ will be covered by the proof of Lemma 5''(b).

Since $(\alpha_u, \alpha_{u+1}) \in F_{u+1}^{(k)}$ and α_{u+1} is an r symbol, there exist β_{u-1} and β_u such that $(\beta_{u-1}, \beta_u) \in F_u^{(k-1)}$, $(\beta_u, \alpha_{u+1}) \in F_{u+1}^{(k-1)}$ and $T_{k-1}(\beta_{u-1}, \beta_u, \alpha_{u+1}) = (\alpha_u, \alpha_{u+1})$. Since $(\alpha_{u+i-1}, \alpha_{u+i}) \in F_{u+i}^{(k)}$ ($2 \leq i \leq v-1$) and both α_{u+i-1} and α_{u+i} are r symbols, $(\alpha_{u+i-1}, \alpha_{u+i}) \in F_{u+i}^{(k-1)}$. If $\alpha_{u+v-1} = r_h$ and α_{u+v} is not an r symbol, it follows from Lemma 2(a) that $\alpha_{u+v} = \lambda_h^{(k-1)}$. Since $(r_h, \lambda_h^{(k-1)}) \in F_{u+v}^{(k)}$, there exists a non r symbol β_{u+v} such that $(r_h, \beta_{u+v}) = (\alpha_{u+v-1}, \beta_{u+v}) \in F_{u+v}^{(k-1)}$. If $u+v = n$, then there exists β_n such that $(\alpha_{n-1}, \beta_n)_1 \in F_n^{(k-1)}$, $\alpha_n = T_{k-1}(\alpha_{n-1}, \beta_n)$ and β_n is not an ℓ symbol, because $(\alpha_{n-1}, \alpha_n)_1 \in F_n^{(k)}$. It is obvious that $(\beta_{u-1}, \beta_u, \alpha_{u+1}, \dots, \alpha_{u+v-1}) \in U_{k,u}(\alpha_u, \alpha_{u+1}, \dots, \alpha_{u+v})$.

Lemma 5': Let $k \geq 2$ and let $(\beta_{u-1}, \dots, \beta_{u+v-1}) \in U_{k,u}(\alpha_u, \dots, \alpha_{u+v})$.

(a) If $\alpha_u = \epsilon_h^{(k-1)}$, $\beta_{u-1} = \ell_h$.

(b) If $\alpha_u = \lambda_h^{(k-1)}$, $\beta_{u-1} = r_h$.

(c) If α_u is neither r_h , $\epsilon_h^{(k-1)}$ nor $\lambda_h^{(k-1)}$ ($1 \leq h \leq m_1$), $\beta_{u-1} = \alpha_u$.

- (a') If $\alpha_{u+v} = \epsilon_h^{(k-1)}$, $\beta_{u+v-1} = \ell_h$;
 (b') If $\alpha_{u+v} = \lambda_h^{(k-1)}$, $\beta_{u+v-1} = r_h$;
 (c') If α_{u+v} is neither r_h , $\epsilon_h^{(k-1)}$ nor $\lambda_h^{(k-1)}$ ($1 \leq h \leq m_1$),
 $\beta_{u+v-1} = \alpha_{u+v}$.

The proof is obvious from Table 1.

Lemma 5'': Assume that α_u is not an r symbol.

- (a) If α_{u+1} is not an r symbol, $U_{1,u}(\alpha_u, \alpha_{u+1}) = \{(\alpha, \alpha_{u+1}) \mid (\alpha, \alpha_{u+1}) \in F_u^{(0)};$
 $\alpha = \alpha_u \text{ or } r_h \alpha_u \text{ (} 1 \leq h \leq m_1 \text{)}\}$.
 (b) $U_{1,u}(\alpha_u, r_{h_1}, \dots, r_{h_{v-1}}, \alpha_{u+v}) = \{(\alpha, q_i, r_{h_1} q_{i_1}, \dots, r_{h_{v-1}} \gamma) \mid (\alpha, q_i) \in F_u^{(0)};$
 $(q_i, r_{h_1} q_{i_1}) \in F_{u+1}^{(0)}; \alpha = \alpha_u \text{ or } r_h \alpha_u \text{ (} 1 \leq h \leq m_1 \text{)}; q_{i_j} \in R_3(N_2(\ell_{h_j}), \alpha_{u+j-1}),$
 $(1 \leq j \leq v-1); \gamma = \alpha_{u+v} \text{ if } \alpha_{u+v} \text{ is not an r symbol, and}$
 $\gamma \in R_3(N_2(\ell_{h_{v-1}}), \alpha_{u+v}) \text{ otherwise}\}$.
 (c) $U_{1,n-1}(\alpha_{n-1}, r_h) = \{(\alpha, q_i) \mid (\alpha, q_i) \in F_{n-1}^{(0)}, \exists x[(q_i, r_h x)_1 \in F_n^{(0)}]; \alpha = \alpha_{n-1}$
 or $r_{h''} \alpha_{n-1} \text{ (} 1 \leq h'' \leq m_1 \text{)}\}$.

Proof: (a) Since $(\alpha_u, \alpha_{u+1}) \in F_{u+1}^{(1)}$, there exist β_{u-1}, β_u and β_{u+1} such that $(\beta_{u-1}, \beta_u) \in F_u^{(0)}$, $(\beta_u, \beta_{u+1}) \in F_{u+1}^{(0)}$, $T_o(\beta_{u-1}, \beta_u, \beta_{u+1}) = (\alpha_u, \alpha_{u+1})$, and if $u+1 = n$, $(\beta_u, \beta_{u+1})_1 \in F_n^{(0)}$. It follows from Table 1 that $\beta_{u+1} = \alpha_u$ or $r_h \alpha_u$ and $\beta_u = \alpha_{u+1}$ or $r_i \alpha_{u+1}$. Since $\alpha_u (= T_o(\beta_{u-1}, \beta_u))$ is not an r symbol, $\beta_u \neq r_i \alpha_{u+1}$. Conversely, any (α, α_{u+1}) satisfying the condition in Lemma belongs to $U_{1,u}(\alpha_u, \alpha_{u+1})$.

(c) Similarly we can prove (c).

(b) Since sequence $\alpha_u, r_{h_1}, \dots, r_{h_{v-1}}, \alpha_{u+v}$ is a subsequence of a l-chain, it follows from Table 1 that there exist $\beta_{u-1}, \beta_u, x_1, y_1, x_2, y_2, \dots, x_{v-1}, y_{v-1}, \beta_{u+v}$ such that $(\beta_{u-1}, \beta_u) \in F_u^{(0)}$, $(\beta_u, r_{h_1} y_1) \in F_{u+1}^{(0)}$, $(r_{h_{j-1}} x_{j-1}, r_{h_j} y_j) \in F_{u+j}^{(0)}$

$(1 < j \leq v-1)$, $(r_{h_{v-1}} x_{v-1}, \beta_{u+v}) \in F_{u+v}^{(0)}$, $T_0(\beta_{u-1}, \beta_u, r_{h_1} y_1) = (\alpha_u, r_{h_1})$,
 $T_0(r_{h_{v-1}} x_{v-1}, \beta_{u+v}) = \alpha_{u+v}$ and if $u+v = n$, $(r_{h_{v-1}} x_{v-1}, \beta_{u+v})_1 \in F_n^{(0)}$. Since
 α_u is not an r symbol and $\alpha_u = T_0(\beta_{u-1}, \beta_u)$, β_{u-1} is α_u or $r_{h_u} \alpha_u$ from Table 1
 and β_u is a q symbol from Lemma 1(c). If α_{u+v} is not an r symbol, $x_{v-1} = \alpha_{u+v}$.
 By Lemma 1(e), $x_j \in R_3(N_2(\ell_{h_i}), a_{u+i-1})$ ($1 \leq i < v-1$) and x_i is determined
 uniquely. That is, $x_j = q_{i_j}$. If $\alpha_{u+v} = r_{h_v}$, then $\beta_{u+v} = r_{h_v} y$. By Lemma 1(e),
 $x_{v-1} \in R_3(N_2(\ell_{h_v}), a_{u+v-1})$. On the other hand, Lemma 1(d) indicates that we
 can choose y_j ($1 \leq j \leq v-1$) so that $y_j = x_j$. It is obvious that $(\beta_{u-1},$
 $\beta_u, r_{h_1} q_{i_1}, \dots, r_{h_{v-2}} q_{i_{v-2}}, r_{h_{v-1}} x_{v-1}) \in U_{1,u}(\alpha_u, r_{h_1}, \dots, r_{h_{v-1}}, \alpha_{u+v})$.

Remark 3: The first symbol of k -chain is q_s . This is true for $k = 0$.

Note that $T_k(q_s, \alpha)$ is defined only for non- r symbol α and $T_k(q_s, \alpha) = q_s$.

By induction we have this remark.

Lemma 6: Let $\bar{\alpha} = (\alpha_k, \dots, \alpha_n)$ be a k -chain ($k \geq 1$). Then there exists
 a $(k-1)$ -chain $\bar{\beta} = (\beta_{k-1}, \dots, \beta_n)$ such that $T_{k-1} \bar{\beta} = \bar{\alpha}$.

Proof: Let α_{j_n} be the last non- r symbol except for α_n and let $\alpha_{j_{n-1}}$
 be the last non- r symbol preceding α_{j_n} unless $\alpha_{j_n} = q_s$. By Lemma 5, there
 exist a sequence $\beta_{j_{n-1}}, \dots, \beta_n$ and a sequence $\beta_{j_{n-1}-1}, \dots, \beta_{j_n-2}, \beta'_{j_n-1}$ such
 that $(\beta_{j_n-1}, \dots, \beta_{n-1}) \in U_{k, j_n}(\alpha_{j_n}, \dots, \alpha_n)$, $(\beta_{n-1}, \beta_n)_1 \in F_n^{(k-1)}$, β_n is not an
 ℓ -symbol, and $(\beta_{j_{n-1}-1}, \dots, \beta_{j_n-2}, \beta'_{j_n-1}) \in U_{k, j_{n-1}}(\alpha_{j_{n-1}}, \dots, \alpha_{j_n})$. At
 first, assume that $k \geq 2$. According to Lemma 5', we see that if $\alpha_{j_n} =$
 $e_h^{(k-1)}$ (or $\lambda_h^{(k-1)}$), $\beta_{j_n-1} = \beta'_{j_n-1} = \ell_h$ (or r_h) and otherwise, $\beta_{j_n-1} =$
 $\beta'_{j_n-1} = \alpha_{j_n}$. We now assume that $k = 1$. β_{j_n-1} is α_{j_n} or $r_{h_{j_n}} \alpha_{j_n}$ from Lemma 5''.
 Suppose that $\alpha_{j_{n-1}} = r_{h_{j_{n-1}}}$. (Or suppose that $\alpha_{j_{n-1}}$ is not an r -symbol). Then,
 by Lemma 5'''', $\beta'_{j_n-1} = r_{h_{j_{n-1}}} \alpha_{j_n}$ (or $\beta'_{j_n-1} = \alpha_{j_n}$). Since $(\alpha_{j_{n-1}}, \alpha_{j_n}) \in F_{j_n}^{(1)}$,

there exists β such that $(r_h, \alpha_{j_n}, \beta) \in F_{j_n}^{(0)}$ (or $(\alpha_{j_n}, \beta) \in F_{j_n}^{(0)}$). Since $(\beta_{j_n-1}, \beta_{j_n}) \in F_{j_n}^{(0)}$, we have that $(\alpha_{j_n}, \beta_{j_n})$ or $(r_h, \alpha_{j_n}, \beta_{j_n}) \in F_{j_n}^{(0)}$. Using (a) (or (b)) in Lemma 1, we have $(r_h, \alpha_{j_n}, \beta_{j_n}) \in F_{j_n}^{(0)}$ (or $(\alpha_{j_n}, \beta_{j_n}) \in F_{j_n}^{(0)}$). Therefore, according to Lemma 5'' we can choose β_{j_n-1} so that

$$\beta_{j_n-1} = \beta'_{j_n-1} = r_h, \alpha_{j_n}$$

$$(\text{or } \beta_{j_n-1} = \beta'_{j_n-1} = \alpha_{j_n})$$

Consequently, we see that there exists $\beta_{j_n-1-1}, \dots, \beta_n$ such that $(\beta_{j_n-1-1}, \dots, \beta_{n-1}) \in U_{k, j_n-1}(\alpha_{j_n-1}, \dots, \alpha_n)$, $(\beta_{n-1}, \beta_n)_1 \in F_n^{(k-1)}$ and β_n is not an ℓ symbol. By repeating the same arguments, we can prove this Lemma.

Lemma 7: Let $\bar{\alpha}$ be a k -chain ($0 \leq k < n-1$). $T_k \bar{\alpha}$ is a $(k+1)$ -chain satisfying the D-condition if and only if $\bar{\alpha}$ satisfies the D-condition.

Proof: If $\bar{\alpha}$ satisfies the D-condition, α_n , the last symbol of $\bar{\alpha}$, is not an ℓ symbol. Therefore, $T_k \bar{\alpha}$ is a $(k+1)$ -chain by Lemma 4. T_k preserves ℓ symbols and r symbols in order except for cancelling adjacent ℓ_h and r_h pairs and deleting the last symbol unless this symbol is an r symbol. Therefore, $T_k \bar{\alpha}$ satisfies the D-condition. Assume that $\bar{\alpha}$ does not satisfy the D-condition. If α_n is an ℓ symbol, $T_k \bar{\alpha}$ can not be a $(k+1)$ -chain. Only if $\bar{\alpha}$ contains no incompatible adjacent ℓ - r -pairs, $T_k \bar{\alpha}$ is defined, but it can not be a $(k+1)$ -chain satisfying the D-condition because of the property of T_k stated above.

Theorem 2: $\bar{a} (= a_1, \dots, a_n) \in L$ if and only if $F_n^{(n-1)}$ contains an element of the form $(q_s, \alpha)_1$, where α is either p symbol, q symbol or $\mathcal{E}_h^{(k)}$ ($1 \leq h \leq m_1$, $1 \leq k < n-1$).

Proof: It follows from Corollary 1 and Lemma 3 that $\bar{a} \in L$, if and only if there exists a 0-chain satisfying the D-condition. According to Lemmas 6 and 7, there exists a 0-chain satisfying the D-condition if and only if there exists an $(n-1)$ -chain $(q_s, \alpha)_1$ satisfying the D-condition. An $(n-1)$ -chain $(q_s, \alpha)_1$ satisfies the D-conditions, if and only if α is neither l symbol nor r symbol. By Lemma 2(d) or (e), α can not be $\lambda_h^{(k)}$ ($1 \leq h \leq m_1$, $1 \leq k < n-1$). Thus we have Theorem 2.

Example 6: In Example 5, $F_9^{(8)} = \{(q_s, e_1^{(1)})_1\}$. By Theorem 2, $\bar{a} = (v + (v+v)) \in L_{ex}$.

This theorem gives an efficient recognition and syntax-analysis algorithm for CFL's described in Sections 4 and 5.

4. Recognition Algorithm

Let $\Sigma_1 = \{l_i \ (1 \leq i \leq m_1), p_i \ (1 \leq i \leq m_2), q_i \ (1 \leq i \leq m_3), r_i \ (1 \leq i \leq m_1), \lambda_i^{(1)} \ (1 \leq i \leq m_1), e_i^{(1)} \ (1 \leq i \leq m_1)\}$ and $\Sigma_2 = \{\lambda_h^{(k)}, e_h^{(k)} \ (1 \leq h \leq m_1, 2 \leq k \leq n-1)\}$. Lemma 2 shows that the pairs of $F_j^{(k)}$ containing Σ_2 symbols are:

for $2 \leq l < k$

$$(\lambda_h^{(l)}, \lambda_h^{(l-1)}), (e_h^{(l)}, \lambda_h^{(l)}), (\lambda_h^{(l)}, r_{h'}), (e_h^{(l)}, r_{h'}),$$

$$(\alpha, e_h^{(l)}); \alpha \in \Sigma_1, \alpha \neq r_i \ (1 \leq i \leq m_1)$$

and for $l = k-1$, $(r_h, \lambda_h^{(l)})$.

Let $F_{jl}^{(k)}$ ($2 \leq l < k$) be the subset of $F_j^{(k)}$ consisting of the pairs of the forms shown above and let

$$F_{j1}^{(k)} = F_j^{(k)} - \bigcup_{l=2}^{k-1} F_{jl}^{(k)} \quad (2 < k < n-1),$$

$$F_{j1}^{(k)} = F_j^{(k)} \quad (0 \leq k \leq 2).$$

It is clear that for any k , j , and l

$$|F_{jl}^{(k)}| \leq C_1,$$

where C_1 is a constant independent of input sequence* and $|F_{jl}^{(k)}|$ means the number of the elements of set $F_{jl}^{(k)}$.

Table 3 shows all combinations of $(\alpha_{j-1}, \alpha_j) \in F_{jl}^{(k)}$ ($2 \leq l < k$), $(\alpha_j, \alpha_{j+1}) \in F_{jl'}^{(k)}$ ($2 \leq l' < k$) and $T_k(\alpha_{j-1}, \alpha_j, \alpha_{j+1})$. If there exists $\alpha \in \Sigma_2$ such that $(\alpha, r_h) \in F_j^{(k)}$, we shall add (X, r_h) to $F_{j1}^{(k)}$, where X is a special symbol. Then we can replace two rows indicated by an asterisk in Table 3 by the row

$$(X, r_h), (r_h, \lambda_h^{(k-1)}), (r_h, \lambda_h^{(k)}).$$

This modified table indicates that we need only $F_{j-1i}^{(k)}$ and $F_{ji}^{(k)}$ ($i = 1, l-1, l$) to get $F_{jl}^{(k+1)}$ ($2 \leq l < k$). Moreover, in Table 3 there is only one row indicated by a double asterisk whose third column entry belongs to $F_{j1}^{(k+1)}$. If we use a random access memory and store $F_{j1}^{(k)}, F_{j2}^{(k)}, \dots, F_{jk-1}^{(k)}$ in a block with successive addresses, the number of elementary operations for finding $F_j^{(k+1)}$ from $F_{j-1}^{(k)}$ and $F_j^{(k)}$ can be bounded by $C_2 k$. Let n be the length of the input sequence $\bar{a} (= a_1, \dots, a_n)$. We can find $\{F_j^{(0)}, \dots, F_j^{(j-1)}\}$ from $\{F_{j-1}^{(0)}, \dots, F_{j-1}^{(j-2)}\}$ and a_j sequentially. Therefore, the computing time and the size of memory required to decide whether $\bar{a} \in L$ are bounded by $C_3 n^3$ and $C_4 n^2$ respectively. As shown below, we can get an upperbound of the same order by using suitably organized serial memories.

We shall construct a Turing machine in the sense of Hartmanis and Stearns which can recognize L and has a working tape with one head for reading and

* Hereafter, C_i is a constant independent of input sequence even though it is not stated.

Table 3

(α_{j-1}, α_j)	(α_j, α_{j+1})	$T_k(\alpha_{j-1}, \alpha_j, \alpha_{j+1})$
$\lambda_h^{(l)}, \lambda_h^{(l-1)}$	$\lambda_h^{(l-1)}, \lambda_h^{(l-2)}$	$\lambda_h^{(l)}, \lambda_h^{(l-1)}$
$\lambda_h^{(l)}, \lambda_h^{(l-1)}$	$\lambda_h^{(l-1)}, r_{h'}$	$\lambda_h^{(l)}, r_{h'}$
$\epsilon_h^{(l)}, \lambda_h^{(l)}$	$\lambda_h^{(l)}, \lambda_h^{(l-1)}$	$\epsilon_h^{(l)}, \lambda_h^{(l)}$
$\epsilon_h^{(l)}, \lambda_h^{(l)}$	$\lambda_h^{(l)}, r_{h'}$	$\epsilon_h^{(l)}, r_{h'}$
$\alpha, \epsilon_h^{(l)}$	$\epsilon_h^{(l)}, \lambda_h^{(l)}$	$\alpha, \epsilon_h^{(l)}$
** $\alpha, \epsilon_h^{(l)}$	$\epsilon_h^{(l)}, r_{h'}$	$\alpha, r_{h'}$
$r_{h'}, \lambda_h^{(k-1)}$	$\lambda_h^{(k-1)}, r_{h'}$	$\lambda_h^{(k)}, r_{h'}$
$r_{h'}, \lambda_h^{(k-1)}$	$\lambda_h^{(k-1)}, \lambda_h^{(k-2)}$	$\lambda_h^{(k)}, \lambda_h^{(k-1)}$
* $\lambda_h^{(l)}, r_{h'}$	$r_{h'}, \lambda_h^{(k-1)}$	$r_{h'}, \lambda_h^{(k)}$
* $\epsilon_h^{(l)}, r_{h'}$	$r_{h'}, \lambda_h^{(k-1)}$	$r_{h'}, \lambda_h^{(k)}$

$$\alpha \in \Sigma_1 \text{ and } \alpha \neq r_i \text{ (} 1 \leq i \leq m_1 \text{)}$$

another independent head for writing besides one-way input and output tapes. Let us divide the working tape into sections of the same length, $T_{00}, T'_{00}, T_{11}, T'_{11}, T_{22}, T'_{22}, T_{32}, T'_{32}, T_{33}, T'_{33}, \dots, T_{k2}, T'_{k2}, \dots, T_{kk}, T'_{kk}, \dots$, where T_{kl} (or T'_{kl}) ($2 \leq l < k$) is used to store $F_{jl}^{(k)}$ for odd (or even) j and some control marks, and T_{kk} (or T'_{kk}) ($k = 0, 1, \dots$) is used to store $F_{j1}^{(k)}$ for odd (or even) j and some control marks.

Suppose that the first $j-1$ symbols on the input tape have been read and $j-1$ is odd (or even) and that T_{kl} (or T'_{kl}) ($0 \leq k < j-1, 2 \leq l < j-1$) contains the information on $F_{j-1l}^{(k)}$ and T_{kk} (or T'_{kk}) contains that on $F_{j-11}^{(k)}$. Then, the operation of this machine proceeds as follows:

1) Read the j -th input symbol a_j . From a_j and $F_{j-11}^{(0)}$ on T_{00} (or T'_{00}) calculate $F_{j1}^{(0)}$ and store it into T'_{00} (or T_{00}).

2) Assume that $F_{jl}^{(h)}$'s ($0 \leq h \leq k, 1 \leq l < h$) have been obtained and stored in T'_{hl} (or T_{hl}) and T'_{kk} (or T_{kk}). Copy the information on $F_{j-11}^{(k)}$ and $F_{j1}^{(k)}$ in T_{kk} and T'_{kk} into a finite working memory W_1 of the control unit. Obtain $F_{j2}^{(k+1)}$ from $F_{j-12}^{(k)}$ in T_{k2} (or T'_{k2}), $F_{j2}^{(k)}$ in T'_{k2} (or T_{k2}), $F_{j-11}^{(k)}$ and $F_{j1}^{(k)}$ in W_1 and store it into T'_{k+12} (or T_{k+12}). Simultaneously store the obtained partial results for $F_{j1}^{(k+1)}$ in a working finite memory W_2 of the control unit. Suppose that $F_{j2}^{(k+1)}, \dots, F_{ji-1}^{(k+1)}$ ($i \leq k$) have been obtained and stored in $T'_{k+12}, \dots, T'_{k+1 i-1}$ (or $T_{k+12}, \dots, T_{k+1 i-1}$). Then, calculate $F_{ji}^{(k+1)}$ from $F_{j-1i}^{(k)}$ ($i' = i-1, i$) in $T_{ki'}$ (or $T'_{ki'}$), $F_{ji'}^{(k)}$ in $T'_{ki'}$ (or $T_{ki'}$) and $F_{j-11}^{(k)}$ and $F_{j1}^{(k)}$ in W_1 and store it in T'_{k+1i} (or T_{k+1i}) (Fig. 1). Simultaneously store the obtained partial results for $F_{j1}^{(k+1)}$ in W_2 . For each i , the number of these operations is bounded by C_3 , because this machine has one head for reading and another independent head for writing. Repeat this cycle on i

up to $i = k$. Copy $F_{j1}^{(k+1)}$ which has been obtained and stored in W_2 into $T_{k+1, k+1}^{(k+1)}$ (or $T_{k+1, k+1}$). For each k , the total number of these operations is bounded by $C_3 k$.

Repeat this cycle on k until $F_{ji}^{(j-1)}$'s ($1 \leq i < j-1$) are found. In the last cycle in which $k = j-1$, test whether $F_{ji}^{(j-1)}$'s ($1 \leq i < j-1$) contain an element of the form $(q_s, \alpha)_1$, where α is either a p symbol, a q symbol, or an \mathcal{E} symbol. If so, print a "1" as the j -th output digit on the output tape, and otherwise print a "0".

For each j , the total number of the operations mentioned above is bounded by $C_4 j^2$. Therefore, this machine prints the j -th output digit in $C_5 j^3$ of fewer operations.

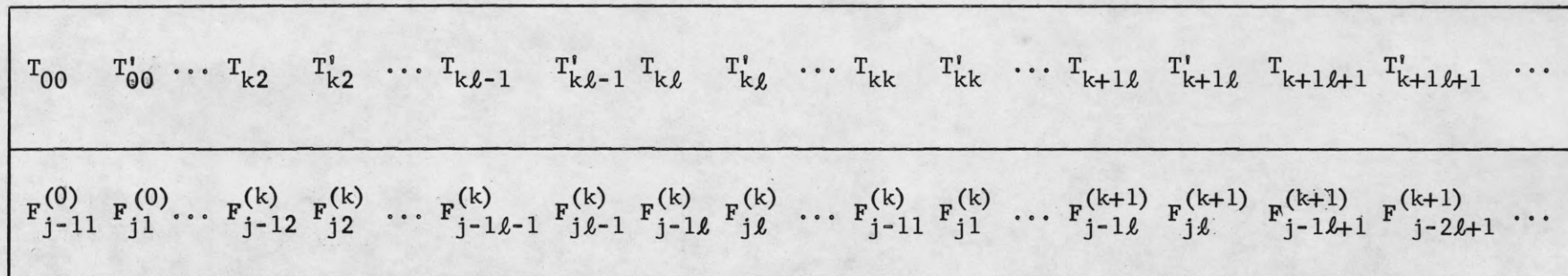
If the machine has only one head, then in each cycle on i the head-shifts from T_{ki-1} , T_{ki-1}^i , T_{ki} or T_{ki}^i to T_{k+li}^i or T_{k+li} must be taken into account, and the number of the operations in this i cycle can be bounded by $C'_3 k$ instead of C_3 . Consequently, it is easily shown that the machine prints the j -th output digit in $C''_5 j^4$ or fewer operations. Similarly we can form a double tape Turing machine which recognizes L and prints the j -th output digit in $C'''_5 j^3$ or fewer operations. In this machine, the additional tape is used as a temporary memory for $F_{j2}^{(k)}, F_{j3}^{(k)}, \dots, F_{jk-1}^{(k)}, F_{j1}^{(k)}$. After all the $F_{ji}^{(k)}$ ($0 < i < k$) have been found, the contents of the second tape are transferred to the first main tape. We can form a Turing machine of the same type for which the constant C_5 (or C'_5 , or C'''_5) is equal to one (11). We summarize the results above in Theorem 3.

Theorem 3: Any context free language is n^3 -recognizable (or n^4 -recognizable) by a double tape or a double-head single-tape (or a single-

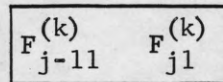
Working Tape

Reading Head

Writing Head



W_1



W_2

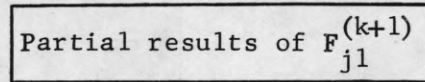


Figure 1

head single-tape) Turing machine in the sense of Hartmanis and Stearns.

5. Syntax-Analysis Algorithm

We shall show a syntax-analysis algorithm for a CFL. Let $\bar{a} (= a_1, a_2, \dots, a_n)$ be in L. The problem is to find all valid S-d.s.'s for \bar{a} . Hereafter we shall fix the input sequence \bar{a} . If a sequence $\bar{\alpha} = \alpha_{w+1}, \dots, \alpha_{w+k}$ satisfies the D-condition and the following conditions:

- 1) $(l_h, \alpha_{w+1}) \in F_{w+1}^{(0)}$ or $(r_h, l_h, \alpha_{w+1}) \in F_{w+1}^{(0)}$ for some h' ,
- 2) $(\alpha_{w+i-1}, \alpha_{w+i}) \in F_{w+i}^{(0)}$ ($1 < i < k+1$),
- 3) $(\alpha_{w+k}, r_h x) \in F_{w+k+1}^{(0)}$ for some x ,

then we shall call this sequence $\bar{\alpha}a(w, k, h)$ - s.d.s. or a s.d.s.

Let $\bar{\alpha} = \alpha_u, \alpha_{u+1}, \dots, \alpha_w, \dots, \alpha_{w+k}, \dots, \alpha_v$ be a valid S-d.s. or a s.d.s. If α_w is l_h or $r_i l_h$ for some i , $\bar{\alpha}' = \alpha_{w+1}, \dots, \alpha_{w+k}$ is a (w, k, h) -s.d.s. and α_{w+k+1} is $r_h x$ for some x , then the (w, k, h) - s.d.s. $\bar{\alpha}'$ is said to be in $\bar{\alpha}$ and this relation is denoted by $\bar{\alpha}' \subset \bar{\alpha}$. In the transformation process of $\bar{\alpha}$ into \wedge by applying the rules $l_h r_h \rightarrow \wedge$ ($1 \leq h \leq m_1$), an l_h symbol* and an r_h symbol to which the rule " $l_h r_h \rightarrow \wedge$ " is applied will be called paired symbols. It is obvious that this pairing is unique. If $\bar{\alpha}' = \alpha_{w+1}, \dots, \alpha_{w+k}$ is a (w, k, h) - s.d.s. in $\bar{\alpha}$, then the l_h symbol in α_w and the r_h symbol in α_{w+k+1} are paired symbols. For $\bar{\alpha}'$ satisfies the D-condition. Consequently, if two s.d.s.'s $\bar{\alpha}'$ and $\bar{\alpha}''$ in $\bar{\alpha}$ overlap, then $\bar{\alpha}' \subset \bar{\alpha}''$ or $\bar{\alpha}'' \subset \bar{\alpha}'$, because the r symbol paired with an l symbol in α_{w+i} ($0 < i \leq k$) must be in α_{w+j} ($1 < j \leq k$). Therefore, for any s.d.s. $\bar{\alpha}'$ in $\bar{\alpha}$, there is a unique sequence of $\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_t$ of s.d.s.'s in $\bar{\alpha}$ such that

*We discern an l_n symbol in a position from an l_h symbol in another position.

1) $\bar{\beta}_0 = \bar{\alpha}$ 2) for $i = 1, 2, \dots, t$, $\bar{\beta}_{i-1} \supset \bar{\beta}_i$ and there exists no s.d.s. $\bar{\beta}$ such that $\bar{\beta}_{i-1} \supset \bar{\beta} \supset \bar{\beta}_i$, and 3) $\bar{\beta}_t = \bar{\alpha}'$. Then, $\bar{\alpha}'$ will be said to be of order t in $\bar{\alpha}$. If $\bar{\alpha}'$ is of order t_1 in $\bar{\alpha}$ and $\bar{\alpha}''$ is of order t_2 in $\bar{\alpha}'$, then $\bar{\alpha}''$ is of order $t_1 + t_2$ in $\bar{\alpha}$ by definition. If $\bar{\alpha}'$ and $\bar{\alpha}''$ are of the same order in $\bar{\alpha}$, $\bar{\alpha}'$ and $\bar{\alpha}''$ do not overlap.

Let $\bar{\alpha}$ be a valid S-d.s. or a s.d.s. By $P_t \bar{\alpha}$, we mean the sequence derived from $\bar{\alpha}$ by the following steps: 1) Replace each (w, k, h) - s.d.s. $\bar{\alpha}'$ of order t in $\bar{\alpha}$ by sequence $\lambda_h, \dots, \lambda_h$ of length k , and 2) delete the remaining r symbols. Here, λ_h is a special symbol. We shall call the sub-sequence $\lambda_h, \dots, \lambda_h$ from the $(w+1)$ -st place to the $(w+k)$ -th place a (w, k, h) - λ - s.

Remark 4: If in $\bar{\alpha}$ we replace each (w, k, h) - s.d.s. $\bar{\alpha}_{w, k, h}$ of order $(t-1)$ by $P_1 \bar{\alpha}_{w, k, h}$ and delete the remaining r symbols, we can get $P_t \bar{\alpha}$. In other words, if in $P_{t-1} \bar{\alpha}$ we replace each (w, k, h) - λ - s. by a $P_1 \bar{\alpha}_{w, k, h}$, we can get $P_t \bar{\alpha}$. This follows directly from the definition.

Example 7: Let $\bar{a} = (v + (v+v))$ and

$$\bar{\alpha} = \underbrace{l_1, q_1, r_1}_{\text{Example 2)}, \quad \underbrace{l_3, l_1, q_1, r_1, l_3, q_1, r_3 q_2, r_3 q_2}_{\text{Example 2)}}.$$

$$\begin{aligned} P_1 \bar{\alpha} &= l_1, \quad \lambda_1, \quad l_3, \quad \lambda_3, \quad \lambda_3, \quad \lambda_3, \quad \lambda_3, \quad \lambda_3, \quad q_2, \\ P_2 \bar{\alpha} &= l_1, \quad q_1, \quad l_3, \quad l_1, \quad \lambda_1, \quad l_3, \quad \lambda_3, \quad q_2, \quad q_2, \\ P_3 \bar{\alpha} &= l_1, \quad q_1, \quad l_3, \quad l_1, \quad q_1, \quad l_3, \quad q_1, \quad q_2, \quad q_2, \end{aligned}$$

Remark 5: If there is a s.d.s. of order d but no s.d.s. of order $d+1$ in $\bar{\alpha}$, then $P_d \bar{\alpha} = P_{d+1} \bar{\alpha}$ is obtained from $\bar{\alpha}$ by deleting all r symbols. Therefore, if $\bar{\alpha}$ is a valid S-d.s. for \bar{a} , then $P_d \bar{\alpha}$ is a valid d.s. for \bar{a} . Since a symbol preceding an r symbol is a q symbol, $d \leq n/2$.

Lemma 8: Let $\bar{\alpha}$ be a valid S-d.s. for \bar{a} and $\bar{\alpha}'$ be a (w, k, h) - s.d.s. in $\bar{\alpha}$. Let $\bar{\beta}$ be a sequence obtained from $\bar{\alpha}$ by replacing $\bar{\alpha}'$ by $\bar{\alpha}''$, another (w, k, h) - s.d.s. Then, $\bar{\beta}$ is also a valid S-d.s. for \bar{a} .

Proof: Let $\bar{\alpha} = \alpha_1, \alpha_2, \dots, \alpha_n$ and $\bar{\alpha}' = \alpha'_{w+1}, \dots, \alpha'_{w+k}$. Then, by definition, (ℓ_h, α'_{w+1}) or $(r_h, \ell_h, \alpha'_{w+1}) \in F_{w+1}^{(0)}$ for some h' , $(\alpha'_{w+i-1}, \alpha'_{w+i}) \in F_{w+i}^{(0)}$ ($1 < i < k+1$) and $(\alpha'_{w+k}, r_h x) \in F_{w+k+1}^{(0)}$ for some x . On the other hand, $\alpha_w = \ell_h$ or $r_j \ell_h$ and $\alpha_{w+k+1} = r_h y$. Therefore, $(\alpha_w, \alpha'_{w+1}) \in F_{w+1}^{(0)}$ by Lemma 1(a) or (b) and $(\alpha'_{w+k}, \alpha_{w+k+1}) \in F_{w+k+1}^{(0)}$ by Lemma 1(d). Hence, $\bar{\beta}$ is a quasi-valid S-s. Also, $\bar{\beta}$ satisfies the D-condition by definition. Consequently, this lemma follows from Theorem 1.

We shall now show a procedure for finding $\{P_t \bar{\alpha} | \bar{\alpha} \text{ is a valid S-d.s. for } \bar{a}\}$ from $F_j^{(k)}$ ($0 \leq k < j$, $0 < j \leq n$).

Let $\bar{\alpha}^{(1)} = \alpha_{j_1}, \alpha_{j_1+1}, \dots, \alpha_{j_1}$, $\bar{\alpha}^{(2)} = \alpha_{j_2}, \dots, \alpha_{j_2}$ ($j_1^1 < j_2^1$), \dots , $\bar{\alpha}^{(u)} = \alpha_{j_u}, \dots, \alpha_{j_u}$ ($j_{u-1}^1 < j_u^1$) be the s.d.s.'s of the first order in $\bar{\alpha}$ and let $\alpha_{j_i^1+1} = r_{h_i} x_i$ ($1 \leq i \leq u$). Let $\beta_{00} = q_s$, $\beta_{0j} = \alpha_j$ ($1 \leq j \leq n$), $\bar{\gamma}_0 = q_s$, $\bar{\alpha}$ and

$$T_{k-1} \dots T_1 T_0 \bar{\gamma}_0 = \beta_{kk}, \beta_{kk+1}, \beta_{kk+2}, \dots, \beta_{kn} \quad (1 \leq k < n)$$

From the definitions of T_k and $F_j^{(k)}$, we have the following:

(1) For $j_i^1 + 1 < j < j_{i+1}^1 - 1$ ($1 \leq i \leq u$),

$$\beta_{0j} = \beta_{1j+1} = \beta_{2j+2} = \dots = \beta_{n-jn} = \alpha_j.$$

(2) For $j_i^1 + 1 = j < j_{i+1}^1 - 1$,

$$\beta_{1j+1} = \beta_{2j+2} = \dots = \beta_{n-jn} = x_i.$$

(3) For $j = j_i^1 - 1$,

$$\beta_{1j+1} = \beta_{2j+2} = \dots = \beta_{j_i^i - j_i + 1j_i^i} = l_{h_i},$$

$$\beta_{j_i^i - j_i + 2j_i^i + 1} = \dots = \beta_{n - j_i + 1n} = e_{h_i}^{(j_i^i - j_i + 1)}$$

(4) For $j_i \leq j \leq j_i^i$ ($1 \leq i \leq u$),

$$\beta_{j_i^i - j + 2j_i^i + 2} = \beta_{j_i^i - j + 3j_i^i + 3} = \dots = \beta_{n - jn} = \lambda_{h_i}^{(j_i^i - j + 1)}$$

(5) If $j_u^i = n-1$,

$$\beta_{1n} = \beta_{2n} = \dots = \beta_{n - j_u n} = r_{h_u}, \beta_{n - j_u + 1n} = e_{h_u}^{(n - j_u)}$$

(6.1) β_{n-1n} is neither an l symbol nor an r symbol (by Lemma 7).

(6.2) $\beta_{n-1n-1} = q_s$ (Remark 3).

(6.3) $(\beta_{kn-1}, \beta_{kn})_1 \in F_n^{(k)}$ ($0 \leq k < n$).

(6.4) $T_k(\beta_{kn-1}, \beta_{kn}) = \beta_{k+1n}$ ($0 \leq k < n-1$).

From (1) through (5), we have the following:

(7) If in sequence $\beta_{n-1n}, \beta_{n-2n}, \dots, \beta_{1n}, \beta_{on}$, we replace $e_h^{(k)}$ by l_h and $\lambda_h^{(k)}$ or r_h by λ_h and, in case of $\beta_{on} = r_h x$, delete this r_h , then we obtain $P_1 \bar{\alpha}$.

Example 8: Let $\bar{\alpha} = l_1, q_1, r_1 l_3, l_1, q_1, r_1 l_3, q_1, r_3 q_2, r_3 q_2$ (Example 7).

$$\begin{array}{l} \bar{\alpha} = l_1 \quad q_1 \quad r_1 l_3 \quad l_1 \quad q_1 \quad r_1 l_3 \quad q_1 \quad r_3 q_2 \quad r_3 q_2 \\ T_0 \bar{\alpha} = \quad \quad l_1 \quad r_1 \quad l_3 \quad l_1 \quad r_1 \quad l_3 \quad r_3 \quad r_3 \\ T_1 T_0 \bar{\alpha} = \quad \quad \quad e_1^{(1)} \quad \lambda_1^{(1)} l_3 \quad e_1^{(1)} \quad \lambda_1^{(1)} \quad e_3^{(1)} \quad r_3 \\ T_2 T_1 T_0 \bar{\alpha} = \quad \quad \quad \quad e_1^{(1)} \lambda_1^{(1)} \quad l_3 \quad e_1^{(1)} \quad \lambda_1^{(1)} \quad r_3 \\ T_3 T_2 T_1 T_0 \bar{\alpha} = \quad \quad \quad \quad \quad e_1^{(1)} \lambda_1^{(1)} \quad l_3 \quad e_1^{(1)} \quad r_3 \\ T_4 T_3 T_2 T_1 T_0 \bar{\alpha} = \quad \quad \quad \quad \quad \quad e_1^{(1)} \lambda_1^{(1)} \quad l_3 \quad r_3 \\ T_5 \dots T_0 \bar{\alpha} = \quad \quad \quad \quad \quad \quad \quad e_1^{(1)} \lambda_1^{(1)} \quad e_3^{(5)} \\ T_6 \dots T_0 \bar{\alpha} = \quad \quad \quad \quad \quad \quad \quad \quad e_1^{(1)} \quad \lambda_1^{(1)} \\ T_7 \dots T_0 \bar{\alpha} = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad e_1^{(1)} \\ P_1 \bar{\alpha} = l_1 \quad \lambda_1 \quad l_3 \quad \lambda_3 \quad \lambda_3 \quad \lambda_3 \quad \lambda_3 \quad \lambda_3 \quad q_2 \end{array}$$

Conversely, suppose that sequences $\beta_1^i, \beta_2^i, \dots, \beta_n^i$ and $\beta_1^{i'}, \beta_2^{i'}, \dots, \beta_n^{i'}$ satisfy the following conditions:

- 1) β_1^i is neither an ℓ symbol nor an r symbol
- 2) $\beta_1^{i'} = q_s$
- 3) $(\beta_j^{i'}, \beta_j^i)_1 \in F_n^{(n-j)}$
- 4) $T_{n-j}(\beta_j^{i'}, \beta_j^i) = \beta_{j-1}^i$ ($1 < j \leq n$).

Now, if $\beta_j^i = \mathcal{E}_n^{(k)}$, let $\beta_j = \ell_h$. If $\beta_j^i = \lambda_h^{(k)}$ or r_h , let $\beta_j = \lambda_h$. If $\beta_n^i = r_h x$, let $\beta_n = x$. Otherwise, let $\beta_j = \beta_j^i$. The sequences $\bar{\beta} = \beta_1, \beta_2, \dots, \beta_n$ and $\bar{\beta}^i = \beta_1^i, \beta_2^i, \dots, \beta_n^i$ will be said to be a P_1 -chain and the sequence associated with $\bar{\beta}$ respectively.

Now, let $\bar{\gamma}_{n-1} = \beta_1^{i'}, \beta_1^i$. Then, $\bar{\gamma}_{n-1}$ is an $(n-1)$ -chain satisfying the D-condition. We shall prove that there exist k -chain $\bar{\gamma}_k$ ($0 \leq k < n$) such that (a) $\bar{\gamma}_k$ satisfies the D-condition, (b) the last symbol of $\bar{\gamma}_k$ is β_{n-k}^i and (c) $\bar{\gamma}_{k+1} = T_k \bar{\gamma}_k$.

Proof: Assume that for $n-1 \geq k > i \geq 0$ there exists $\bar{\gamma}_k$ satisfying the conditions (a), (b) and (c). From Lemmas 6 and 7, there exists i -chain $\bar{\gamma}_i^i$ satisfying conditions (a) and (c). Let $\bar{\gamma}_i$ be the sequence obtained from $\bar{\gamma}_i^i$ by replacing the last symbol by β_{n-i}^i . Now, consider α such that

$$T_i(\alpha, \beta) = \beta_{n-i-1}^i$$

If β_{n-i-1}^i is not an r symbol and $i \geq 1$, then α is determined uniquely and independently of β from Table 1. Then it follows from 4) that the second last symbol of $\bar{\gamma}_i^i$ is $\beta_{n-i}^{i'}$. Consequently, $\bar{\gamma}_i$ is an i -chain from 3). On the other hand, β_{n-i}^i is neither an ℓ symbol nor an r symbol by 3) and 4). Hence, $\bar{\gamma}_i$ satisfies the conditions (a), (b) and (c).

If $i = 0$ and β'_{n-1} is not an r symbol, then $\alpha = \beta'_{n-1}$ or $r_h \beta'_{n-1}$ for some h from Table 1. Therefore, $\beta'_n = \beta'_{n-1}$ or $r_h \beta'_{n-1}$ by 4). Let γ_{n-1} and γ_n denote the last two symbols of $\bar{\gamma}'_0$. Then, $(\gamma_{n-1}, \gamma_n)_1 \in F_n^{(0)}$ and $\gamma_{n-1} = \beta'_{n-1}$ or $r_h \beta'_{n-1}$. By Lemma 1(a) or (b) and 3), $(\gamma_{n-1}, \beta'_n)_1 \in F_n^{(0)}$. Since β'_n contains neither an ℓ symbol nor an r symbol from 3) and 4), $\bar{\gamma}'_0$ satisfies the conditions (a), (b) and (c).

If $\beta'_{n-i-1} = r_h$ and $i \geq 1$, then $\beta'_{n-i} = r_h$ by 4) and the last symbol of $\bar{\gamma}'_i$ is also r_h by (c). Hence, $\bar{\gamma}'_i = \bar{\gamma}'_i$. If $i = 0$ and $\beta'_{n-1} = r_h$, then $\gamma_n = r_h x$ and $\beta'_n = r_h y$. Since $(\gamma_{n-1}, \gamma_n)_1 \in F_n^{(0)}$ and $(\beta'_{n-1}, \beta'_n)_1 \in F_n^{(0)}$, x and y are q symbols and therefore x and $y \in R_3(N_2(\ell_h), a_n)$. Since $R_3(N_2(\ell_h), a_n)$ contains at most one element, $\bar{\gamma}'_0 = \bar{\gamma}'_0$.

Let $\bar{\gamma}$ be the sequence obtained from $\bar{\gamma}'_0$ by deleting the first symbol q_s . Then $\bar{\gamma}$ is a valid S-d.s. By (7) and the definition of $\bar{\gamma}$,

$$\bar{\beta} = P_1 \bar{\gamma}.$$

To summarize, we have Lemma 9.

Lemma 9: If $\bar{\alpha}$ is a valid S-d.s., then $P_1 \bar{\alpha}$ is a P_1 -chain and conversely if $\bar{\beta}$ is a P_1 -chain, then there exists a valid S-d.s. $\bar{\alpha}$ such that $P_1 \bar{\alpha} = \bar{\beta}$.

In order to extend this lemma we need several simple lemmas.

Lemma 10: (1) If and only if there is a $(w, k, h) - \lambda - s.$ in a P_1 -chain $\bar{\beta}$, then the w -th symbol β'_w of the sequence associated with $\bar{\beta}$ is $\mathcal{E}_h^{(k)}$. (2) If there is a $(w, k, h) - \lambda - s.$ in a P_1 -chain, then $(\ell_h, r_h) \in F_{w+k+1}^{(k)}$.

Proof: The same notations as those in the definition of a P_1 -chain will be used.

a) Suppose that $\beta_j^i = \lambda_h^{(k)}$ ($k > 1$). Then, it follows from Table 1 and 4) of the definition of P_1 -chain that if $k = n - j - 1$, $\beta_{j+1}^{i'} = r_h$ and otherwise $\beta_{j+1}^{i'} = \lambda_h^{(k)}$ and that β_{j+1}^i is not an r symbol. Hence, $\beta_{j+1}^i = \lambda_h^{(k-1)}$ by Lemma 2(a) or (c). On the other hand, if $k = n - j - 1$, $\beta_j^{i'}$ is either r_h or $\mathcal{E}_h^{(k)}$ by 3) and Lemma 2(d), and otherwise $\beta_j^{i'}$ is either $\mathcal{E}_h^{(k)}$ or $\lambda_h^{(k+1)}$ by 3) and Lemma 2(e). Therefore, β_{j-1}^i is $\mathcal{E}_h^{(k)}$ or $\lambda_h^{(k+1)}$ from 4).

b) Suppose that $\beta_j^i = r_h$. Then, $\beta_{j-1}^i = r_h$ or $\mathcal{E}_h^{(n-j)}$ by 4) and Table 1 and if $j = n - 1$, $\beta_{j+1}^i = r_h$ and otherwise $\beta_{j+1}^i = r_h$.

c) Suppose that $\beta_j^i = \mathcal{E}_h^{(k)}$. If $k = n - j - 1$, then $\beta_{j+1}^i = r_h$ by 4) and Table 1. Otherwise, β_{j+1}^i is not an r symbol and $\beta_{j+1}^{i'} = \mathcal{E}_h^{(k)}$. Therefore, $\beta_{j+1}^i = \lambda_h^{(k)}$ by Lemma 2(b).

The first part of this lemma follows immediately from a), b), c) and the definition of $\bar{\beta}$.

d) If $(\alpha, \mathcal{E}_h^{(k)}) \in F_j^{(i)}$, then $(\mathcal{L}_h, r_h) \in F_{j-i+k+1}^{(k)}$. If $i = k + 1$, this follows from Table 1. Otherwise, there exists α^i such that $(\alpha^i, \mathcal{E}_h^{(k)}) \in F_{j-1}^{(i-1)}$. Thus, d) can be proved by induction.

From the first part of this lemma it follows that $(\beta_w^{i'}, \mathcal{E}_h^{(k)}) \in F_n^{(n-w)}$. Therefore, the second part of this lemma follows from d).

Lemma 11: Suppose that $(\mathcal{L}_h, r_h) \in F_{w+k+1}^{(k)}$. Then, there exist sequences $\beta_1^i, \beta_2^i, \dots, \beta_k^i$ and $\beta_1^{i'}, \beta_2^{i'}, \dots, \beta_k^{i'}$ such that 1) β_1^i is neither an \mathcal{L} symbol nor an r symbol, 2) $\beta_1^{i'} = \mathcal{L}_h$ or $r_h \mathcal{L}_h$ (only if $k = 1$), 3) $(\beta_j^{i'}, \beta_j^i)_1 \in F_{w+k}^{(k-j)}$ ($1 \leq j \leq k$) and 4) $T_{k-j}(\beta_j^{i'}, \beta_j^i) = \beta_{j-1}^i$ ($1 < j \leq k$).

Proof: From the assumption and the definition of $F_j^{(k)}$, there exist $\beta_1^i, \dots, \beta_k^i, \beta_1^{i'}, \dots, \beta_k^{i'}$ such that

a) $\beta_1^{i'} = \mathcal{L}_h$ or $r_h \mathcal{L}_h$ (only if $k = 1$),

- b) $(\beta_j^v, \beta_j^v) \in F_{w+k}^{(k-j)}$, $(\beta_j^v, r_h) \in F_{w+k+1}^{(k-j)}$ ($1 \leq j \leq k$),
 $(\beta_k^v, r_h x) \in F_{w+k+1}^{(0)}$ for some x , and
- c) $T_{k-j}(\beta_j^v, \beta_j^v, r_h) = (\beta_{j-1}^v, r_h)$ ($1 < j < k$),
 $T_0(\beta_k^v, \beta_k^v, r_h x) = (\beta_{k-1}^v, r_h)$.

Here, β_j^v ($1 < j \leq k$) can not be an ℓ symbol by c) and β_k^v is a q symbol or a combination of r and q symbols by Lemma 1(c). Therefore, $(\beta_j^v, \beta_j^v)_1 \in F_{w+k}^{(k-j)}$ ($1 \leq j \leq k$).

Assume that $(\ell_h, r_h) \in F_{w+k+1}^{(k)}$ and let $\bar{\beta}^v = \beta_1^v, \dots, \beta_k^v$ be the sequence in Lemma 11. If $\beta_j^v = \ell_{h^v}^{(k^v)}$, let $\beta_j = \ell_{h^v}$. If $\beta_j^v = \lambda_{h^v}^{(k^v)}$ or r_{h^v} , let $\beta_j = \lambda_{h^v}$. If $\beta_k^v = r_{h^v} x$, let $\beta_k = x$. Otherwise, let $\beta_j = \beta_j^v$. The sequence $\bar{\beta} = \beta_1, \dots, \beta_k$ and $\bar{\beta}^v = \beta_1^v, \dots, \beta_k^v$ will be said to be a (w, k, h) -chain and the sequence associated with $\bar{\beta}$ respectively. We shall define a (w, k, h) -chain only if $(\ell_h, r_h) \in F_{w+k+1}^{(k)}$. The following lemma can be proved by using almost the same argument as that in the proof of Lemma 9.

Lemma 9': If $\bar{\alpha}$ is a (w, k, h) -s.d.s., then $P_1 \bar{\alpha}$ is a (w, k, h) -chain and conversely if $\bar{\beta}$ is a (w, k, h) -chain, then there exists a (w, k, h) -s.d.s. $\bar{\alpha}$ such that $P_1 \bar{\alpha} = \bar{\beta}$.

The proof of the following lemma is analogous to the proof of Lemma 10.

Lemma 10': (1) If and only if there is a (w, k, h) - λ - s. in a (w', k', h') -chain $\bar{\beta}$, then the w -th symbol of the sequence associated with $\bar{\beta}$ is $\ell_h^{(k)}$. (2) If there is a (w, k, h) - λ - s. in a (w', k', h') -chain, then $(\ell_h, r_h) \in F_{w+k+1}^{(k)}$.

We shall now define P_{t+1} -chain as a sequence derived from a P_t -chain by replacing each (w, k, h) - λ - s. in the P_t -chain by a (w, k, h) -chain. We shall prove that if there is a (w, k, h) - λ - s. in a P_t -chain, there is

always a (w,k,h) -chain. If $t = 1$, this follows from Lemmas 10 and 11. If $t \geq 2$, it follows from the definition of P_t that this $(w,k,h) - \lambda - s.$ is in a (w',k',h') -chain which is substituted for a $(w',k',h') - \lambda - s.$ in a P_{t-1} -chain. Therefore, the proof is obvious from Lemmas 10' and 11.

Now we have Lemma 12.

Lemma 12: (1) If $\bar{\alpha}$ is a valid S-d.s., then $P_t \bar{\alpha}$ is a P_t -chain. (2)

If $\bar{\beta}$ is a P_t -chain, then there exists a valid S-d.s. $\bar{\alpha}$ such that

$$P_t \bar{\alpha} = \bar{\beta}.$$

Proof: We shall prove this lemma by induction. For $t = 1$, this lemma holds from Lemma 9. Assume that for $t - 1$ this lemma holds.

The proof of (1). Let $\bar{\alpha}_{w,k,h}$ denote a (w,k,h) -s.d.s. of order $(t-1)$ in $\bar{\alpha}$. From induction hypothesis, $P_{t-1} \bar{\alpha}$ is a P_{t-1} -chain. By Lemma 9', $P_1 \bar{\alpha}_{w,k,h}$ is a (w,k,h) -chain. Therefore since $P_t \bar{\alpha}$ is derived from $P_{t-1} \bar{\alpha}$ by replacing each $(w,k,h) - \lambda - s.$ by $P_1 \bar{\alpha}_{w,k,h}$ (Remark 4), $P_t \bar{\alpha}$ is a P_t -chain by definition.

The proof of (2). (a) By definition, a P_t -chain $\bar{\beta}$ is derived from a P_{t-1} -chain $\bar{\beta}'$ by replacing each $(w,k,h) - \lambda - s.$ in $\bar{\beta}'$ by a (w,k,h) -chain $\bar{\beta}_{w,k,h}$. (b) By induction hypothesis there exists a valid S-d.s. $\bar{\alpha}'$ such that $P_{t-1} \bar{\alpha}' = \bar{\beta}'$. (c) For each $\bar{\beta}_{w,k,h}$, it follows from Lemma 9' that there exists a (w,k,h) -s.d.s. $\bar{\alpha}_{w,k,h}$ such that $P_1 \bar{\alpha}_{w,k,h} = \bar{\beta}_{w,k,h}$. (d) Let $\bar{\alpha}$ denote the sequence obtained from $\bar{\alpha}'$ by replacing each (w,k,h) -s.d.s. of order $(t-1)$ by $\bar{\alpha}_{w,k,h}$. By Lemma 8, $\bar{\alpha}$ is a valid S-d.s. By construction, $P_{t-1} \bar{\alpha} = P_{t-1} \bar{\alpha}' = \bar{\beta}'$. (e) From Remark 4, $P_t \bar{\alpha}$ is derived from $P_{t-1} \bar{\alpha}$ ($= \bar{\beta}'$) by replacing each $(w,k,h) - \lambda - s.$ by $P_1 \bar{\alpha}_{w,k,h}$ ($= \bar{\beta}_{w,k,h}$). Consequently, by (a), $P_t \bar{\alpha} = \bar{\beta}$.

In forming a P_1 -chain or a (w,k,h) -chain, there may be permissible alternatives. From Lemma 12, if $\bar{\beta}$ and $\bar{\delta}$ are different P_t -chains, there exist at least two different valid S-d.s.'s $\bar{\alpha}$ and $\bar{\gamma}$ such that

$$P_t \bar{\alpha} = \bar{\beta} \text{ and } P_t \bar{\gamma} = \bar{\delta} .$$

Therefore, there exist at least two different valid d.s.'s (Remark 2).

If grammar G is unambiguous, there are no alternatives. To summarize, we have Theorem 4.

Theorem 4: (1) If a P_t -chain $\bar{\beta}$ contains no λ_h -symbols ($1 \leq h \leq m_1$), $\bar{\beta}$ is a valid d.s. (2) Any valid d.s. is a P_t -chain for some t ($t \leq n/2$). (3) Any choice of alternatives in forming P_t -chain leads to a valid d.s. and, furthermore, different choices of alternatives generate different valid d.s.'s.

Example 9: Let $\bar{a} = (v + (v+v))$. Referring to Table 2, we have the following:

P_1 -chain: $l_1, \lambda_1, l_3, \lambda_3, \lambda_3, \lambda_3, \lambda_3, \lambda_3, q_2$;
 (1, 1, 1)-chain: q_1 ; (3, 5, 3)-chain: $l_1, \lambda_1, l_3, \lambda_3, q_2$;
 P_2 -chain: $l_1, q_1, l_3, l_1, \lambda_1, l_3, \lambda_3, q_2, q_2$;
 (4, 1, 1)-chain: q_1 ; (6, 1, 3)-chain: q_1 ;
 P_3 -chain: $l_1, q_1, l_3, l_1, q_1, l_3, q_1, q_2, q_2$.

Lemma 12 is verified for this example by comparing these with $P_t \bar{\alpha}$ ($1 \leq t \leq 3$) in Example 7. Since the P_3 -chain contains no λ_h symbols, this is the valid d.s. for \bar{a} , which is determined uniquely because there is no alternative in forming the P_1 -chain, the (1, 1, 1)-chain, the (3, 5, 3)-chain, the (4, 1, 1)-chain and the (6, 1, 3)-chain.

Let CT denote the computing time required for syntax-analysis and let N denote the number of different nonequivalent valid d.s. for a given input sequence. If we use a random-access memory of size $C_{10}n^3$, we can proceed as follows: Form the table of $F_j^{(k)}$ ($0 \leq k < j$, $1 \leq j \leq n$). Test whether the given input sequence \bar{a} is in L. If $\bar{a} \in L$, form an initial part of P_1 -chain. In order to find a next symbol of a partially formed P_1 -chain or (w,k,h) -chain, look up the table of $F_j^{(k)}$ from the bottom of the table. If there are alternatives, choose the first one and write a special mark on the chosen one which is used for tracing the whole tree of alternatives without repetition.

Whenever we encounter $\mathcal{E}_h^{(k)}$ in the w -th place of the sequence associated with a P_1 -chain or a (w',k',h') -chain, let $\beta_w = \mathcal{L}_h$ and find β_{w+1}, \dots from the conditions of the (w,k,h) -chain and proceed as far as a new $\mathcal{E}_{h'}^{(k')}$ symbol is not chosen. If we reach the end of the current (w,k,h) -chain, then return to the corresponding place in the latest (w',k',h') -chain or P_1 -chain and restart from this point. The linkages of such jumps can be controlled efficiently by using a push down store.

As it will be shown below, the size of required memory can be reduced to the order of n^2 . It follows from the proof of Lemma 10 that the entry in $F_j^{(i)}$ required for obtaining a (w,k,h) -chain are of the form $(\alpha, \beta)_1$, where α is an \mathcal{L} symbol, a p symbol, a q symbol or $\lambda_h^{(1)}$ ($1 \leq h \leq m_1$) and β is a p symbol, a q symbol, an r symbol, a combination of an r symbol and a q symbol or an \mathcal{E} symbol. Except for entries of the form $(\alpha, \mathcal{E}_h^{(k)})_1$, the number of such entries in $F_j^{(i)}$ is bounded above by a constant. Furthermore, the following lemmas show that the all valid d.s.'s can be generated without referring to entries of the form $(\alpha, \mathcal{E}_h^{(k)})_1$.

Lemma 13: Let $i \geq k+2$. Then, $(\alpha, \mathcal{E}_h^{(k)}) \in F_j^{(i)}$, if and only if

$$(\alpha, \mathcal{E}_h^{(k)}) \in F_{j-i+k+1}^{(k+1)} \quad (1)$$

and there exists non r -symbol β such that for $i = k+2$

$$(r_h, \beta) \in F_j^{(i-k-1)} = F_j^{(1)} \quad (2)$$

and for $i > k+2$

$$(\lambda_h^{(1)}, \beta) \in F_j^{(i-k-1)} \quad (3)$$

Proof: (1) The "only if" part follows from a), c) and d) of the proof of Lemma 10. (2) Consider the case of $i-k = 2$. Assume that (1) and (2) hold. Then, from (1)

$$(\ell_h, r_h) \in F_{j-1}^{(k)} \quad (4)$$

Thus, for $1 \leq i' < k$, there exists non ℓ -symbol $\beta_{i'}$ such that

$$(\beta_{i'}, r_h) \in F_{j-1}^{(i')} \quad (5)$$

Hence, it follows from (2), (4) and (5) that

$$(r_h, \lambda_h^{(i')}) \in F_j^{(i'+1)} \quad (1 \leq i' < k)$$

$$(\mathcal{E}_h^{(k)}, \lambda_h^{(k)}) \in F_j^{(k+1)},$$

thus, by (1)

$$(\alpha, \mathcal{E}_h^{(k)}) \in F_j^{(k+2)} = F_j^{(i)}$$

(3) Suppose that the "if" part holds for $i-k < m$. Let $i-k = m > 2$. From (3), there exists non r -symbol γ such that

$$(r_h, \gamma) \in F_{j-1}^{(i-k-2)}, \quad \text{if } m = 3, \quad (6)$$

$$(\lambda_h^{(1)}, \gamma) \in F_{j-1}^{(i-k-2)}, \quad \text{if } m > 3. \quad (7)$$

By (6) or (7) and the induction hypothesis,

$$(\alpha, \mathcal{E}_h^{(k)}) \in F_{j-1}^{(i-1)} \quad (8)$$

The same argument as that of (a) and (c) of the proof of Lemma 10 gives

$$(\mathcal{r}_h, \lambda_h^{(i)}) \text{ or } (\lambda_h^{(i'+1)}, \lambda_h^{(i')}) \in F_{j-1}^{(i-k+i'-2)} \\ (1 \leq i' < k)$$

$$(\mathcal{E}_h^{(k)}, \lambda_h^{(k)}) \in F_{j-1}^{(i-2)}.$$

By ~~combining~~ these relations with (3) and (8),

$$(\lambda_h^{(i'+1)}, \lambda_h^{(i')}) \in F_j^{(i-k+i'-1)}, \quad (1 \leq i' < k)$$

$$(\mathcal{E}_h^{(k)}, \lambda_h^{(k)}) \in F_j^{(i-1)},$$

$$(\alpha, \mathcal{E}_h^{(k)}) \in F_j^{(i)}.$$

The next lemma follows directly from Lemma 13 and the definition of

$\pi(\alpha_{i-1}, \alpha_i)_v$.

Lemma 13': Let $i \geq k+2$. Then, $(\alpha, \mathcal{E}_h^{(k)})_1 \in F_j^{(i)}$, if and only if $(\alpha, \mathcal{E}_h^{(k)}) \in F_{j-i+k+1}^{(k+1)}$ and there exists non $\#$ -symbol β such that for $i = k+2$, $(\mathcal{r}_h, \beta)_1 \in F_j^{(i-k-1)}$ and for $i > k+2$, $(\lambda_h^{(1)}, \beta)_1 \in F_j^{(i-k-1)}$.

Let $\overline{F}_j^{(i)}$ be the set consisting of the following elements:

- 1) (α, β) such that $(\alpha, \beta)_1 \in F_j^{(i)}$ and α and β are neither $\mathcal{E}_h^{(k)}$ -symbols nor $\lambda_h^{(k)}$ ($k > 1$) symbols.
- 2) (α, \mathcal{E}_h) if for some k , $(\alpha, \mathcal{E}_h^{(k)})_1 \in F_j^{(i)}$.
- 3) (α, \mathcal{E}_h') if $(\alpha, \mathcal{E}_h^{(i-1)}) \in F_j^{(i)}$.

Then, the number of elements in $\overline{F}_j^{(i)}$ can be bounded above by a constant independent of input sequences. It follows from Lemma 13' that $\overline{F}_j^{(i)}$'s ($1 \leq j \leq n$; $0 \leq i < j$) have enough information to generate all valid d.s.'s.

The procedure for generating valid d.s. from $\overline{F}_j^{(i)}$ is almost the same as the one for generating them from $\overline{F}_j^{(i)}$ except for the operations on $(\alpha, \mathcal{E}_h^{(k)})$.

The Tables of $\overline{F}_j^{(i)}$ ($1 \leq j \leq n$; $0 \leq i < j$) can be formed as follows:

For each j , obtain $\{F_j^{(0)}, F_j^{(1)}, \dots, F_j^{(j-1)}\}$ and $\{\overline{F}_j^{(0)}, \overline{F}_j^{(1)}, \dots, \overline{F}_j^{(j-1)}\}$ from $\{F_{j-1}^{(0)}, F_{j-1}^{(1)}, \dots, F_{j-1}^{(j-2)}\}$ and a_j . Erase $F_{j-1}^{(0)}, F_{j-1}^{(1)}, \dots, F_{j-1}^{(j-2)}$. Repeat this step up to $j = n$. The size of required memory can be bounded above by $C_{20}n^2$. If we use a random access memory or two tapes of length $C_{30}n^2$ and one tape of length $C_{31}n$, then the computing time is bounded above by $C_{21}n^3$ or $C_{32}n^3$, respectively.

One of the procedures which produce all the valid d.s. serially in some order without repetition will be shown below. A special mark indicates the end of each valid d.s. and the maximal initial subsequence of each valid d.s. which is also an initial subsequence of the immediately preceding valid d.s. will be omitted. Let j , i and w be indices, PL be a push down store and [PL] be the context of the top cell of PL. Index w indicates that the w -th symbol of a valid d.s. is looked for in the current step. Let α and β be working registers. For simplicity of notations, the context of register α or β will be denoted by α or β respectively.

- 1) Initial setting: $1 \rightarrow w$, $n \rightarrow j$, and $q_s \rightarrow \alpha$. Go to 2).
- 2) Selection of the w -th rule name of a current valid d.s. If $w > n$, then go to 4). Find the first symbol-pair in $\overline{F}_j^{(j-w)}$ whose first symbol is α . Store the second symbol of this pair into register β . Mark this pair (α, β) with *.
- 2.0) If $\beta = \mathcal{E}_h$ for some h , then go to 2.3).

- 2.1) Write β as the w -th symbol of the current valid d.s. If $j \neq w$, go to 2.2). $j + [PL] \rightarrow j$. $w+1 \rightarrow w$.
- If $[PL] = 1$, then $\beta \rightarrow \alpha$, pop up PL and go to 3).
- If $[PL] = 2$, then $r_h \rightarrow \alpha$.
- If $[PL] > 2$, then $\lambda_h^{(1)} \rightarrow \alpha$.
- Pop up PL and go to 2).
- 2.2) $\beta \rightarrow \alpha$
- $w+1 \rightarrow w$
- Go to 2).
- 2.3) If $(\alpha, \mathcal{E}_h^i) \in \overline{F}_j^{(j-w)}$, then set $i = 0$ and go to 2.4).
- Otherwise, find the smallest i such that $(\alpha, \mathcal{E}_h^i) \in \overline{F}_{j-i}^{(j-w-i)}$ and there exists a non r -symbol γ such that (r_h, γ) or $(\lambda_h^{(1)}, \gamma) \in \overline{F}_j^{(i)}$.
- 2.4) Mark the pair $(\alpha, \mathcal{E}_h^i)$ with $*$.
- $i+1 \rightarrow PL$
- Write l_h as the w -th symbol of the current valid d.s.
- $j-i-1 \rightarrow j$
- $l_h \rightarrow \alpha$
- $w+1 \rightarrow w$
- Go to 2).
- 3) Find the first symbol-pair of the form $(\alpha, r_h x)$ in $\overline{F}_j^{(0)}$.
- Mark this pair with $*$.
- $x \rightarrow \beta$
- Go to 2.1).
- 4) Write a special mark indicating the end of the current valid d.s. as the output.
- Go to 5).

5) Searching for the starting symbol of the next valid d.s.:

If $w = 0$, then stop.

Otherwise, find the marked pair in $\overline{F}_j^{(j-w)}$ whose second symbol γ is not an \mathcal{E}' -symbol.

Store the first symbol of this pair to register α .

If $\gamma = \mathcal{E}_h$ for some h , then go to 5.2).

Erase the mark on the pair (α, γ) .

5.0) Look for a symbol-pair in $\overline{F}_j^{(j-w)}$ whose first symbol is α and which follows the previously marked entry (α, γ) . If any, store the second symbol of this pair to register β and go to 2.0).

Otherwise, go to 5.1).

5.1) $w-1 \rightarrow w$.

If $\gamma = r_h x$, then $j-1 \rightarrow j$ and $1 \rightarrow PL$.

If $\alpha = r_h$ or $\lambda_h^{(1)}$, then $j-w \rightarrow PL$ and $w \rightarrow j$.

If $\alpha = l_h$, then $j + [PL] \rightarrow j$ and pop up PL .

Go to 5).

5.2) Find a marked pair $(\alpha, \mathcal{E}_h^{i'})$ in $\overline{F}_{j-i'}^{(j-w-i')}$ for some i' ($0 \leq i' < j$).

Erase the mark on this pair.

Look for the smallest i such that $i > i'$, $(\alpha, \mathcal{E}_h^{i'}) \in \overline{F}_{j-i}^{(j-w-i)}$ and there exists a non r -symbol δ such that (r_h, δ) or $(\lambda_h^{(1)}, \delta) \in \overline{F}_j^{(i)}$.

If there exists such an i , then go to 2.4).

Otherwise, erase the mark on the pair (α, γ) and go to 5.0).

If the access time to memory is assumed to be independent of n , then the computing time at each step 2), 3) or 5) can be bounded above by $C_{22}n$. For each valid d.s., step 2), 3) or 5) is repeated at most n times.

Therefore, the computing time for each valid d.s. can be bounded above by $C_{23}n^2$. After the last valid d.s. is typed out, step 5) is repeated n times and the procedure terminates. Thus, we have

$$CT \leq C_{21}n^3 + C_{24}n^2N.$$

Consider the case where there are used two working tapes of length $C_{30}n^2$ and two tapes of length $C_{31}n$ for counting and copying. Form Tables $\overline{F}_1^{(0)}, \overline{F}_2^{(0)}, \overline{F}_2^{(1)}, \overline{F}_3^{(0)}, \dots, \overline{F}_n^{(0)}, \overline{F}_n^{(1)}, \dots, \overline{F}_n^{(n-1)}$ in this order on the maintape of length $C_{30}n^2$. Write a special mark at the end of each $\overline{F}_j^{(i)}$ ($0 \leq i < j-1$) and another mark at the end of each $\overline{F}_j^{(j-1)}$. Use one tape for index w and another for push down store PL which keeps the current and previous values of index i . The position of the head on the main tape or a special mark can indicate the current value of index j . At the beginning of step 2.3) or 5.2) copy Tables $\overline{F}_j^{(0)}, \overline{F}_j^{(1)}, \dots, \overline{F}_j^{(j-w-1)}$ to a working tape from the main tape. Then, it is easily shown that the computing time at each step 2), 3) or 5) can be bounded above by $C_{33}n^2$. Consequently, we have

$$CT \leq C_{32}n^3 + C_{34}n^3N.$$

We have assumed that grammar G is in standard 2-form. If given grammar G is not in standard 2-form, we can effectively construct grammar G_s in standard 2-form strongly equivalent to G as shown by Greibach (6), (16). It can be easily seen that the additional computing time to convert a derivation sequence in G_s into the corresponding one in G is asymptotically dominated by the terms derived above (16). Moreover, our algorithm can be applied directly to grammar in general standard form with some modifications. If a rule is of the form:

$$Y \rightarrow a Y_1 Y_2 \dots Y_u \quad (u \geq 2),$$

we name this rule as l_h and rewrite it as follows:

$$l_h: Y \rightarrow a Y_1^{r_{h1}} Y_2^{r_{h2}} \dots r_{hv_h} Y_u; \quad v_h = u-1.$$

For the D-condition, we use the rules of the form:

$x \rightarrow \wedge$ if x is neither an l symbol nor an r symbol, and

$$l_h^{r_{h1}} \dots r_{hv_h} \rightarrow \wedge.$$

The extension is straightforward. This extended algorithm may be somewhat more efficient and practical refinements are currently under investigation at Osaka University.

The advantage of using a standard form grammar is that we can simplify the procedure for generating all valid derivation sequences which is essentially much more complicated than the procedure for converting each derivation sequence in a standard form grammar to the corresponding one in the original grammar.

Concluding Remarks

Hartmanis and Stearns showed an example of a CFL which is not n -recognizable by multi-head multi-tape Turing machine (11). There is a gap between n and n^3 . It is not known whether there is a CFL which is not n^2 -recognizable. It is also not known whether a general syntax-analyzer is possible which would require a memory space proportional only to n without an exponential growth of computing time.

For a linear grammar (1), the procedures can be so simplified that the upperbound of computing time is reduced by one degree (14). The

framework of the algorithm presented in this paper is relatively suitable to be incorporated with a capability of syntax-error analysis (15). The details are under investigation at Osaka University.

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13. ABSTRACT An efficient algorithm of recognition and syntax-analysis for the full class of context-free languages without the difficulty of exponential growth of computing time with the length n of input sequence is presented. This algorithm makes use of a fundamental algebraic property of a context-free language. It is shown in this paper that a context-free language is n^3 -recognizable in the sense of Hartmanis and Stearns by a double-tape or double-head single-tape Turing machine and it is n^4 -recognizable by a single-head single-tape Turing machine. The size of memory required for recognition is proportional to n^2 . If we use a random-access memory whose size is proportional to n^2 , the computing time required for syntax-analysis is upper-bounded by $C_1 n^3 + C_2 n^2 N$, where N denotes the number of non-equivalent valid derivation sequences for a given input sequence and C_i 's are constants independent of input sequences. If we use two tapes of length $C_3 n^2$ and two tapes of length $C_4 n$ as working memories, the computing time for syntax-analysis is upper-bounded by $n^3(C_5 + C_6 N)$.		

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
recognition syntax-analysis context-free languages algorithm Turing machine n ³ recognizable						

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