Coordinated Science Laboratory

UNIVERSITY OF ILLINOIS – URBANA, ILLINOIS

OSCILLATIONS IN A ONE-DIMENSIONAL, INHOMOGENOUS PLASMA

E. A. Jackson and M. Raether

REPORT R-239 JANUARY, 1965

COORDINATED SCIENCE LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Contract DA-28-043-AMC-00073(E)

The research reported in this document was made possible by support extended to the University of Illinois, Coordinated Science Laboratory, jointly by the Department of the Army, Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research) under Department of Army Contract DA-28-043-AMC-00073(E).

OSCILLATIONS IN A ONE-DIMENSIONAL, INHOMOGENEOUS PLASMA

by

E. A. Jackson and M. Raether

Oscillations in finite, inhomogeneous plasmas have received considerable attention recently in connection with the interpretation of the so-called Tonks-Dattner resonances. Although some progress has been made in calculating the resonance frequencies from the moment equations for a cold and warm plasma, attempts to calculate the Landau damping of these modes have not led to tangible results.[1-6]

In this report we give a rigorous formulation of the problem and present detailed calculations of the eigen frequencies and their damping rates for a one-dimensional, inhomogeneous plasma in the long wavelength regime.

1. Eigenfrequencies and Eigenvectors

We start with the one-dimensional Vlassov and Poisson equation.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}E\frac{\partial f}{\partial v} = 0 \tag{1}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - \int f dv) \tag{2}$$

Linearization leads to the following set of equations:

$$v\frac{\partial f_o}{\partial x} - \frac{e}{m}E_o\frac{\partial f_o}{\partial v} = 0 \tag{3}$$

$$\frac{\partial E_o}{\partial x} = 4\pi e(n_i - n_e) \tag{4}$$

and

(5) $$\frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} - \frac{e}{m} E_1 \frac{\partial f_o}{\partial v} - \frac{e}{m} E_o \frac{\partial f_1}{\partial v} = 0$$

(6) $$\frac{\partial E_1}{\partial x} = - 4\pi e \int f_1 \, dv$$

We estimate the ratio of the last two terms in Eq. (5):

$$\frac{E_o \frac{\partial f_1}{\partial v}}{E_1 \frac{\partial f_o}{\partial v}} \approx \frac{E_o n_1}{E_1 n_o} \quad ;$$

From (6) we estimate $\frac{E_1}{\lambda} \sim 4\pi e n_1$ where λ is the wavelength of the oscillation. From (3) we obtain

$$E_o \approx \frac{kT}{eL}$$ where L is the dimension of the plasma.

Hence

$$\frac{E_o n_1}{E_1 n_o} \approx \frac{\lambda_D^2}{\lambda \cdot L}$$ where λ_D is the Debye length.

The fourth term in Eq. (5) is therefore small compared to the third term for most cases of practical interest and will henceforth be neglected.

We therefore consider the equations

(7) $$\frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} - \frac{e}{m} E_1 \frac{\partial f_o}{\partial v} = 0$$

(8) $$\frac{\partial E_1}{\partial x} = - 4\pi e \int f_1 \, dv$$

We enclose the plasma between two specularly reflecting walls at x=0 and x=L. f_1 must satisfy the following boundary conditions:

$$f_1(0,v) = f_1(0,-v)$$

$$f_1(L,v) = f_1(L,-v)$$

Moreover $E_1(0) = E_1(L) = 0$.

Following Weissglas [4] we introduce

$$f_1 = f^+ \text{ for } v > 0$$

$$f_1 = f^- \text{ for } v < 0$$

(7) then can be written

(9)
$$\frac{\partial f_1^+}{\partial t} + v \frac{\partial f^+}{\partial x} - \frac{e}{m} E_1 \frac{\partial f_o}{\partial v} = 0$$

$$\frac{\partial f^-}{\partial t} - v \frac{\partial f^-}{\partial x} + \frac{e}{m} E_1 \frac{\partial f_o}{\partial v} = 0$$

Adding and subtracting these two equations results in

(10)
$$\frac{\partial F^+}{\partial t} + v \frac{\partial F^-}{\partial x} = 0$$

(11)
$$\frac{\partial F^-}{\partial t} + v \frac{\partial F^+}{\partial x} - 2 \frac{e}{m} E_1 \frac{\partial f_o}{\partial v} = 0$$

with $F^+ = f^+ + f^-$ and $F^- = f^+ - f^-$.

The boundary condition on f_1 now simply requires that F^- vanishes at the boundary.

Eq. (8) can be written

(12)
$$\frac{\partial E_1}{\partial x} = - 4\pi e \int_{-\infty}^{+\infty} f_1 dv = - 4\pi e \left\{ \int_o^\infty f^- dv + \int_o^\infty f^+ dv \right\} = - 4\pi e \int_o^\infty F^+ dv \ .$$

We now assume f_1 and E_1 to be proportional to $e^{i\omega t}$.

(10) and (11) thus become

$$i\omega F^{+} + v\frac{\partial F^{-}}{\partial x} = 0 \ ; \quad i\omega F^{-} + v\frac{\partial F^{+}}{\partial x} - 2\frac{e}{m}E_1\frac{\partial f_o}{\partial v} = 0$$

$$F^{+} = i\frac{v}{\omega}\frac{\partial F^{-}}{\partial x} \ .$$

Inserting F^{+} into (12) we obtain

$$\frac{\partial}{\partial x}\left[E_1 + \frac{4\pi ie}{\omega}\int_o^\infty F^{-}v\,dv\right] = 0$$

or

$$E_1 + \frac{4\pi ie}{\omega}\int_o^\infty F^{-}v\,dv = \text{const.} \tag{13}$$

If no external field is present the constant is 0. Expressing everything in terms of F^{-} and E_1 we have

$$F^{-} + \frac{v^2}{\omega^2}\frac{\partial F^{-}}{\partial x^2} + 2i\frac{e}{m\omega}E_1\frac{\partial f_o}{\partial v} = 0 \tag{14}$$

$$E_1 = -\frac{4\pi ie}{\omega}\int_o^\infty F^{-}v\,dv \ . \tag{15}$$

We expand F^{-} and E_1 in sin-series, which automatically satisfy the boundary conditions.

$$F^{-} = \Sigma F_k \sin\frac{\pi kx}{L} \ ; \ E_1 = \Sigma E_k \sin\frac{\pi kx}{L} \tag{16}$$

We put $f_o = n(x) \cdot g(v)$ where $g(v) = \left(\frac{m}{2\pi KT}\right)^{1/2} \exp(-mv^2/2KT)$.

For n(x) we choose the special form (Fig. 1)

(17) $$n(x) = n_o(1 + \nu \cos \frac{2\pi x}{L})$$

A suitable choice of n_i ensures that (17) also satisfies the zero-order equations (3) and (4).

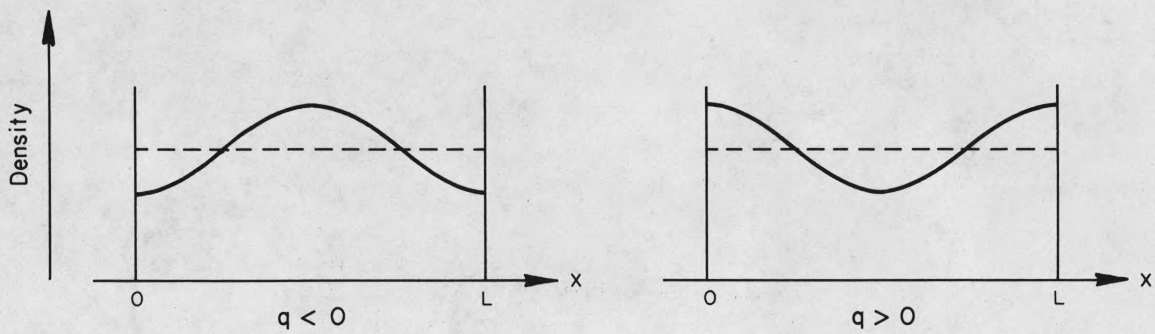

Fig. 1 Schematic Density Profile for $\nu \gtrless 0$.

Although this choice for the density profile appears rather special and arbitrary, it will be shown later that the eigenmodes for more general density profiles can be obtained by perturbation methods starting with $n(x) = n_o(1 + \nu \cos \frac{2\pi x}{L})$ as a zero order approximation.

Inserting the expression (16) into (14) and (15), we obtain

(18) $$E_k = - \frac{4\pi i e}{\omega} \int_o^\infty F_k v dv$$

(19) $$F_k = - \frac{2i \frac{e}{m} n_o \frac{\partial g}{\partial v}}{\omega(1 - \frac{v^2}{\omega^2} \frac{\pi^2 k^2}{L^2})} \quad (E_k + \frac{\nu}{2} E_{k-2} + \frac{\nu}{2} E_{k+2}) \ .$$

Inserting F_k into (18) we have

(20) $$E_k + 2\omega_p^2 (E_k + \frac{\nu}{2} E_{k-2} + \frac{\nu}{2} E_{k+2}) \int_0^\infty \frac{\frac{\partial g}{\partial v} v dv}{\omega^2 - v^2 \frac{\pi^2 k^2}{L^2}} = 0$$

$$2\omega_p^2 \int_0^\infty \frac{\frac{\partial g}{\partial v} v dv}{\omega^2 - v^2 \frac{\pi^2 k^2}{L^2}} = \frac{\omega_p^2}{\pi k} L \int_{-\infty}^{+\infty} \frac{\frac{\partial g}{\partial v} dv}{\omega - \frac{\pi k v}{L}} = \epsilon_k - 1$$

where

(21) $$\epsilon_k(\omega) = 1 + \frac{\omega_p^2}{\varkappa_k} \int_{-\infty}^{+\infty} \frac{\frac{\partial g}{\partial v} dv}{\omega - \varkappa_k v}$$

is the dielectric constant and $\varkappa_k = \frac{\pi k}{L}$.

(20) can now be written

(22) $$\epsilon_k E_k + \frac{\nu}{2} (\epsilon_k - 1) E_{k-2} + \frac{\nu}{2} (\epsilon_k - 1) E_{k+2} = 0.$$

In the first equation for k = 1, we have to observe that we must put $E_{-1} = -E_1$.

The infinite set of equations (22) separates into two systems for odd and even values of k. These sets of equations have a solution only if their determinants vanish. This requires

(23) $$0 = \begin{vmatrix} \epsilon_1 - \frac{\nu}{2}(\epsilon_1 - 1) & \frac{\nu}{2}(\epsilon_1 - 1) & 0 & 0 & \cdots \\ \frac{\nu}{2}(\epsilon_3 - 1) & \epsilon_3 & \frac{\nu}{2}(\epsilon_3 - 1) & 0 & \cdots \\ 0 & \frac{\nu}{2}(\epsilon_5 - 1) & \epsilon_5 & \frac{\nu}{2}(\epsilon_5 - 1) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

for odd values of k. And

$$(24)\quad 0 = \begin{vmatrix} \epsilon_2 & \frac{\nu}{2}(\epsilon_2-1) & 0 & \cdots & \cdots \\ \frac{\nu}{2}(\epsilon_4-1) & \epsilon_4 & \frac{\nu}{2}(\epsilon_4-1) & 0 & \cdots \\ 0 & \frac{\nu}{2}(\epsilon_6-1) & \epsilon_6 & \frac{\nu}{2}(\epsilon_6-1) & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

for even values of k.

In these determinants we divide each row by the off-diagonal elements and obtain

$$(25)\qquad D_o = \begin{vmatrix} a_1 & 1 & \cdots & \cdots \\ 1 & a_3 & 1 & \cdots \\ \cdots & 1 & a_5 & 1 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

and

$$(26)\qquad D_e = \begin{vmatrix} a_2 & 1 & \cdots & \cdots \\ 1 & a_4 & 1 & \cdots \\ \cdots & 1 & a_6 & 1 \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

with $a_k = \dfrac{2\epsilon_k}{\nu(\epsilon_k-1)} - \delta_{ik}$.

To evaluate the determinant D_o we divide the first row by a_1 and subtract it from the second row.

$$D_0 = a_1 \begin{vmatrix} 1 & 1/a_1 & \cdots & \cdots & \cdots \\ 0 & a_3 - 1/a_1 & 1 & \cdots & \cdots \\ \cdots & 1 & a_5 & 1 & \cdots \\ \cdots & \cdots & 1 & a_7 & 1 \end{vmatrix}$$

Proceeding in this fashion we obtain:

$$D_o = a_1\left(a_3 - \frac{1}{a_1}\right)\left(a_5 - \cfrac{1}{a_3 - \cfrac{1}{a_1}}\right) \cdots \left(a_{2n+1} - \cfrac{1}{a_{2n-1} - \cfrac{1}{a_{2n-3} - \ddots}}\right)$$

By calculating the partial numerators and denominators of these continued fractions we can show that all terms cancel, except for the numerator of the last continued fraction. Hence the determinant can be written:

$$(27) \qquad D_o = \lim_{n\to\infty} \text{Num} \quad a_{2n+1} - \cfrac{1}{a_{2n-1} - \cfrac{1}{a_{2n-3} - \cfrac{1}{a_{2n-5} - \ddots}}}$$

If $D_o = 0$ the continued fraction can be inverted and we can write the dispersion relation in the form

(28) $$0 = a_1 - \cfrac{1}{a_3 - \cfrac{1}{a_5 - \cfrac{1}{a_7 - \cfrac{1}{\ddots}}}}$$

An analogous expression is obtained for the determinant D_e. In this form the dispension relation is well adapted for numerical calculation. Considerable simplification results if we use the long-wavelength expression for the dielectric constant.

For real values of ω, ϵ_k can be written

(29) $$\epsilon_k = 1 + 2\,\frac{\alpha^2}{k^2}\left(1 - 2\Omega^2\,\frac{\alpha^2}{k^2}\,Y\left(\frac{\alpha\Omega}{k}\right) + i\sqrt{\pi}\,\frac{\alpha\Omega}{k}\exp\frac{-\alpha^2\Omega^2}{k^2}\right)$$

with

$$\frac{\omega}{\omega_p} = \Omega \; ; \quad \frac{\beta\omega_p L}{\pi} = \alpha \; ; \quad \beta^2 = \frac{m}{2KT} \; .$$

Neglecting the imaginary part for the moment and using the high frequency expansion for $Y(z)$

(30) $$Y(z) = \frac{1}{2z^2}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \dots\right)$$

We obtain for a_k (for $k > 1$)

$$a_k = \frac{2\epsilon_k}{\upsilon(\epsilon_k - 1)} = \frac{2}{\upsilon}\left(1 - \frac{\Omega^2}{1 + \frac{3}{2}\,\frac{k^2}{\alpha^2\Omega^2}}\right)$$

For $\frac{3}{2} \frac{k^2}{\alpha^2\Omega^2} \ll 1$ we may expand the denominator and obtain

$$(31) \qquad a_k = \frac{2}{\nu} \left(1 - \Omega^2 + \frac{3}{2} \frac{k^2}{\alpha^2} \right)$$

If we introduce this value into the recurrence relation for the E_k

$$(32) \qquad E_{k-2} + a_k E_k + E_{k+2} = 0 \qquad (k > 1)$$

We find

$$E_{k-2} + \frac{2}{\nu} (1 - \Omega^2 + \frac{3}{2} \frac{k^2}{\alpha^2}) E_k + E_{k+2} = 0 .$$

This can be written

$$-\frac{\nu}{3} \alpha^2 (E_{k-2} + E_{k+2}) + \left(\frac{2}{3} \alpha^2 (\Omega^2 - 1) - k^2 \right) E_k = 0.$$

With the notation $q = \frac{\nu}{3} \alpha^2$ and $a = \frac{2}{3} \alpha^2 (\Omega^2 - 1)$

$$(33) \qquad q = \frac{\nu}{2} \frac{1}{3\pi^2} \frac{L^2}{\lambda_D^2} \quad ; \quad a = \frac{1}{3\pi^2} \frac{L^2}{\lambda_D^2} (\Omega^2 - 1)$$

We see that this recurrence relation coincides with that for the Mathieu equation [8]

$$(34) \qquad \frac{d^2E}{d\xi^2} + (a - 2q \cos 2\xi) E = 0$$

where $\xi = \pi \frac{x}{L}$.

We can therefore conclude that the eigenfunctions of the electric field are the Mathieu functions $se_r(\xi,q)$. The eigenmodes of oscillation are determined by corresponding values of a and q belonging to the eigenfunctions $se_r(\xi,q)$.

The function $a_r = a_r(q)$ corresponding to $se_r(\xi,q)$ are plotted in Fig. 2.

For large values of q the following asymptotic expressions hold.

For $q < 0$

$$a_{2r+1} = -2q + (8r+6)\sqrt{q}$$

(35)

$$a_{2r+2} = -2q + (8r+6)\sqrt{q}$$

For $q > 0$

$$a_{2r+1} = -2q + (8r+2)\sqrt{q}$$

(36)

$$a_{2r+2} = -2q + (8r+6)\sqrt{q}\ .$$

Similar results have been obtained by Weissglas for the same density profile using the moment equations.[3].

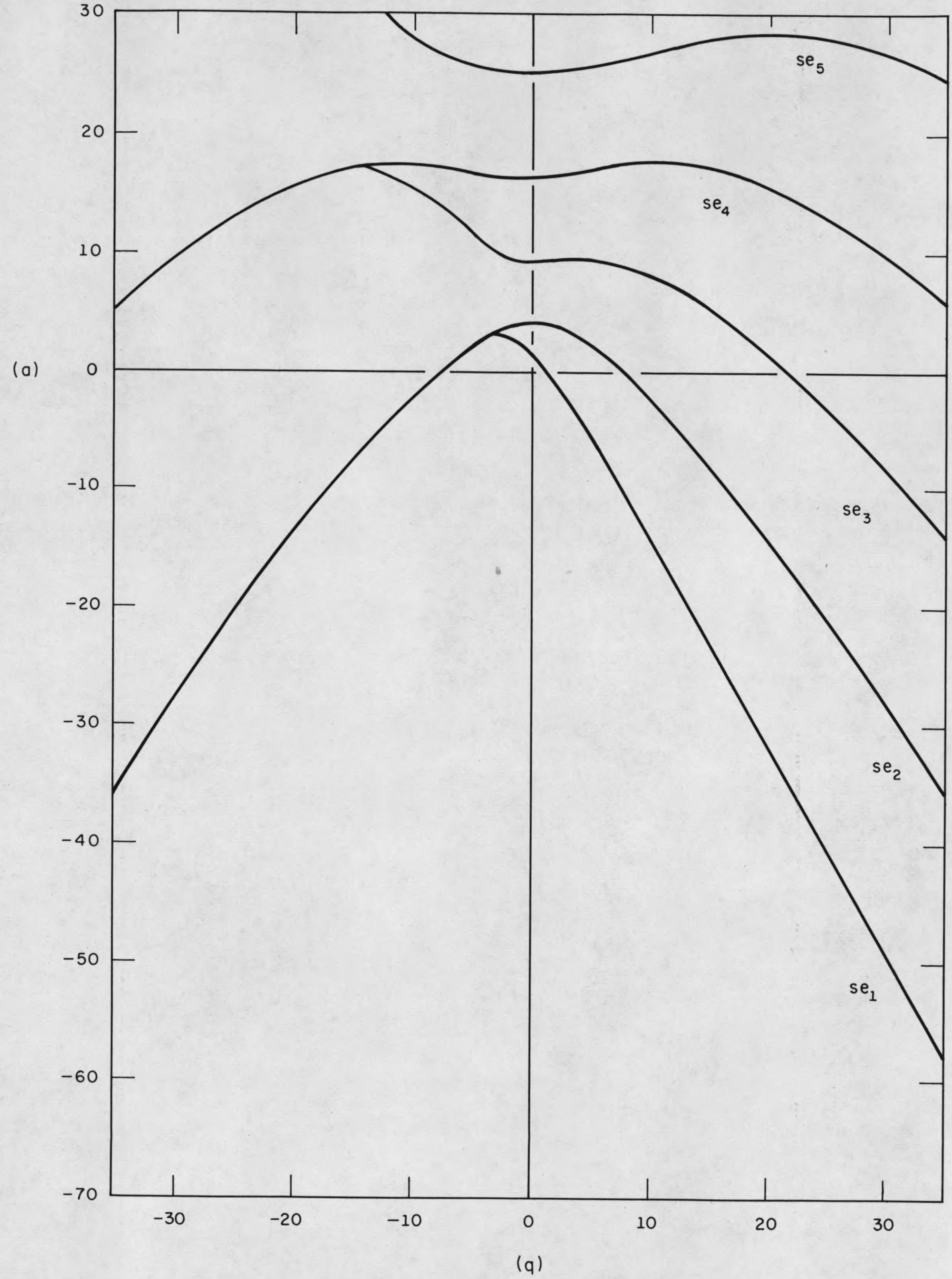

Fig. 2 a-q-diagram for the Mathieu functions $se_r(\xi,q)$.

2. Landau-Damping of the Modes

The knowledge of the eigenfunctions of the electric field enables us to calculate the collisionless damping of the corresponding modes by perturbation methods. Using the matrix corresponding to (25) and (26) we can write (32) in the form

(37) $\underline{\underline{M}} \cdot \underline{E} = 0$

where

$$M_{jk} = (a_j \delta_{jk} + \delta_{j,k+2} + \delta_{j,k-2}) \; ; \quad a_j = \frac{2\varepsilon_j}{\nu(\varepsilon_j - 1)} - \delta_{1j}$$

where now the a_j's are complex. Separating the a_j into real and imaginary parts we have

$$\underline{\underline{M}} = \underline{\underline{M}}_0 + i\, \underline{\underline{M}}_1$$

where

(38) $M_{0_{jk}} = (\mathrm{Re}\, a_j \delta_{jk} + \delta_{j,k+2} + \delta_{j,k-2}$

and

(39) $M_{1_{jk}} = \mathrm{Im}\, a_j \delta_{jk}$.

In the last section we neglected the Im a_j and obtained a solution to the equation

(40) $\underline{\underline{M}}_0 \cdot \underline{E}_0 = 0$.

Setting $\underline{E} = \underline{E}_0 + i\, \underline{E}_1$ equation (37) becomes

(41) $\underline{\underline{M}}_0 \cdot \underline{E}_0 + i\, \underline{\underline{M}}_1 \cdot \underline{E}_0 + i\, \underline{\underline{M}}_0 \cdot \underline{E}_1 = 0$

to first order in the perturbed quantities. The first term vanishes because of equation (40). We dot $\underline{E}_0$ into the remainder of (41) and obtain

$$\underline{E}_0 \cdot \underline{\underline{M}}_1 \cdot \underline{E}_0 + \underline{E}_0 \cdot \underline{\underline{M}}_0 \cdot \underline{E}_1 = 0 \ .$$

Since $\underline{\underline{M}}_0$ is a symmetric matrix, $\underline{E}_0 \cdot \underline{\underline{M}}_0 \cdot \underline{E}_1 = \underline{E}_1 \cdot \underline{\underline{M}}_0 \cdot \underline{E}_0 = 0$ in virtue of equation (40). Thus

$$\underline{E}_0 \cdot \underline{\underline{M}}_1 \cdot \underline{E}_0 = 0 \tag{42}$$

where $\underline{E}_0$ is a column matrix whose elements are the Fourier coefficients of the Mathieu functions, D_k, which are defined by

$$se_r(\xi,q) = \sum_{k=\left\{\substack{\text{even}\\ \text{odd}}\right.} D_k^{\,r}(q) \sin (k\xi) \ .$$

Thus, using (39), equation (42) becomes

$$\sum_{k=\left\{\substack{\text{even}\\ \text{odd}}\right.} D_k^{\,2} \operatorname{Im} a_k = 0 \ . \tag{43}$$

Now

$$\operatorname{Im} a_k = - \frac{2 \operatorname{Im} \epsilon_k}{v|\epsilon_k - 1|^2} \ . \tag{44}$$

We approximate $|\epsilon_k - 1|^2$ by $\omega_p^{\,2}/\omega_r^{\,2}$ (where $\omega = \omega_r + i\,\omega_i$) and expand $\operatorname{Im} \epsilon_k$ in a power series in ω_i/ω_r. To first order in ω_i/ω_r we obtain [7]

$$\operatorname{Im} \epsilon_k = - \pi \frac{\omega_p^{\,2}}{\kappa_k^{\,2}} \left.\frac{\partial g}{\partial v}\right|_{v = \frac{\omega_r}{\kappa_k}} - \frac{\omega_i}{\omega_r} \frac{\omega_p^{\,2}}{\kappa_k^{\,2}} \, P \int \frac{\partial^2 g/\partial v^2 dv}{v - \frac{\omega_r}{\kappa_k}} \ .$$

P denotes the principle value part of the integral. In the long wavelength approximation this becomes

$$\operatorname{Im} \epsilon_k \simeq \omega_p^2 \left(2\sqrt{\pi} \frac{\omega_r}{\kappa_k^3} \beta^3 \exp(-\beta^2 \omega_r^2/\kappa_k^2) - \frac{\omega_i}{\omega_r^3} \right) .$$

Substituting these into (44), equation (43) becomes

$$(45) \qquad \Sigma D_k^2 \left(2\sqrt{\pi} \frac{\omega_r}{\kappa_k^3} \beta^3 e^{-\beta^2 \omega_r^2/\kappa_k^2} - \frac{\omega_i}{\omega_r^3} \right) = 0.$$

The D_k's are normalized according to

$$\Sigma D_k^2 = 1.$$

Therefore (45) reduces to

$$\frac{\omega_i}{\omega_r} = \Sigma D_k^2 \, 2\sqrt{\pi} \frac{\omega_r^3}{\kappa_k^3} \beta^3 e^{-\beta^2 \omega_r^2/\kappa_k^2} .$$

In this approximation the damping rate is just the linear superposition of the individual damping rates of the Fourier components contributing to the eigenvectors.

Fig. 3 shows as an example the damping of the first 3 modes for one particular value of $\frac{\lambda_D}{L}$. We notice a tremendous increase in the damping rate for even small inhomogenities. In view of these results one may expect that even in the long wavelength limit Landau-damping can become the dominant damping mechanism for oscillations in inhomogeneous plasmas.

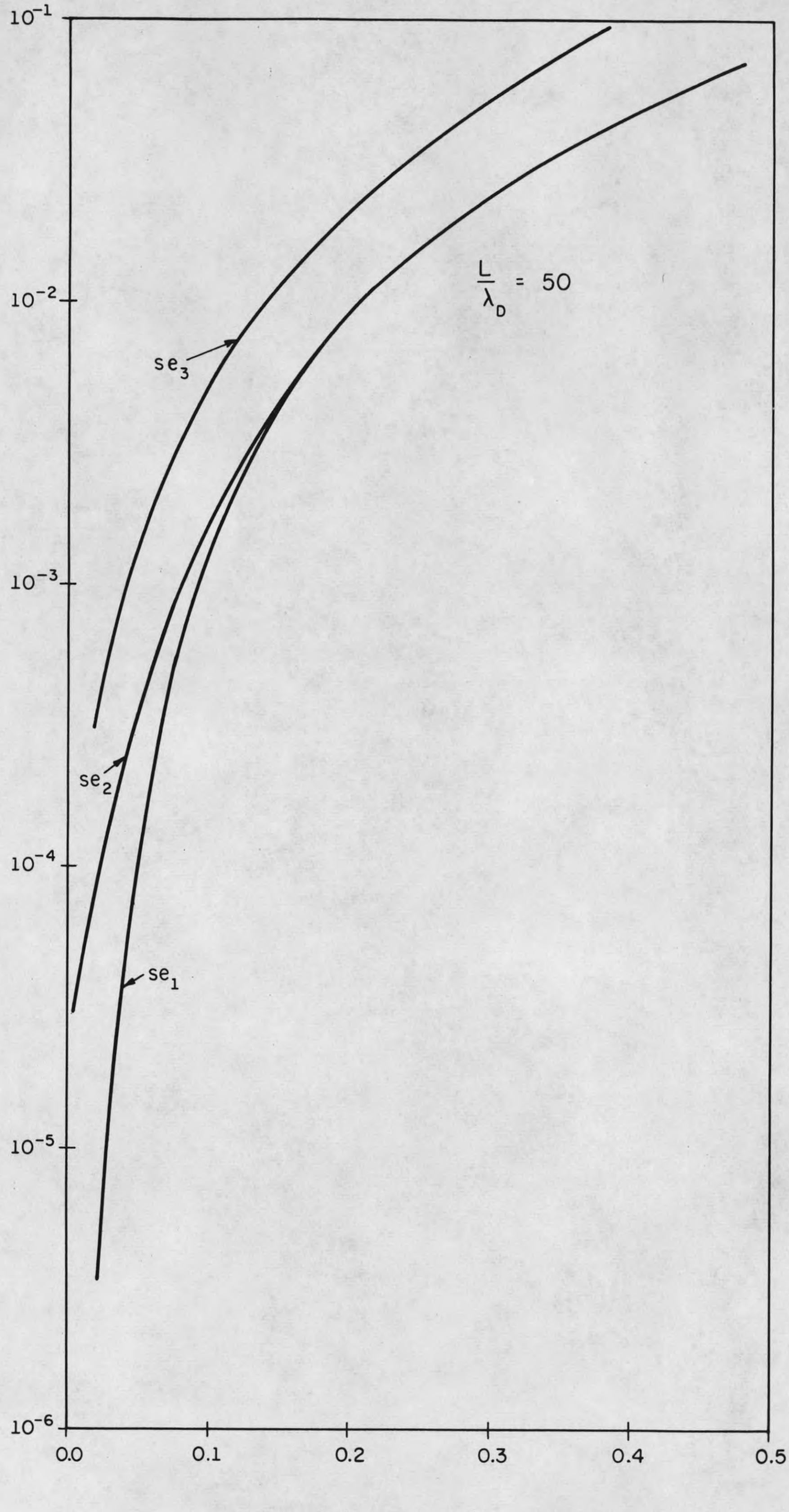

Fig. 3 Damping of modes as a function of inhomogeneity.

3. Example of a Different Density Profile

The solution that has been obtained for the density profile

$$n(x) = n_o \left(1 \pm \nu \cos \frac{2\pi x}{L}\right)$$

provides a zero order approximation for the treatment of other density profiles by perturbation theory.

We demonstrate this for the profile

$$n'(x) = n'_o \left(1 + \xi \sin \frac{\pi x}{L}\right)$$

$\sin \frac{\pi x}{L}$ can be expanded in a series in $\cos \frac{2\pi k x}{L}$.

$$\sin \frac{\pi x}{L} = \frac{4}{\pi} \left(\frac{1}{2} - \frac{1}{3} \cos \frac{2\pi x}{L} - \frac{1}{3.5} \cos \frac{4\pi x}{L} - \frac{1}{5.7} \cos \frac{6\pi x}{L} \ldots \right)$$

$$n'(x) = n'_o \left(1 + \frac{2}{\pi} \xi - \frac{4}{3\pi} \xi \cos \frac{2\pi x}{L} - \frac{4}{3.5\pi} \xi \cos \frac{4\pi x}{L} - \ldots \right)$$

We introduce

$$n_o = n'_o \left(1 + \frac{2}{\pi} \xi\right)$$

$$\mu = - \frac{4}{3\pi} \frac{\xi}{1 + \frac{2}{\pi} \xi}$$

$n'(x)$ then becomes

$$n'(x) = n_o \left(1 + \mu \cos \frac{2\pi x}{L} + \frac{\mu}{5} \cos \frac{4\pi x}{L} + \frac{3\mu}{5.7} \cos \frac{6\pi x}{L} \ldots \right) .$$

We shall consider the solution for $n_o \left(1 + \mu \cos \frac{2\pi x}{L}\right)$ as a zero order approximation and treat the remaining terms as a perturbation. The recurrence relation now becomes

$$\epsilon_k E_k + \frac{\mu}{2} (\epsilon_k - 1) (E_{k-2} + E_{k+2}) + \frac{\mu}{2.5} (\epsilon_k - 1) (E_{k-4} + E_{k+4})$$

$$+ \frac{3\mu}{2.5.7} (\epsilon_k - 1) (E_{k-6} + E_{k+6}) + \ldots = 0$$

with

$$\frac{2\epsilon_k}{\mu(\epsilon_k - 1)} = \frac{2}{\mu\omega_p^2}\left(\omega_p^2 - \omega^2 + \frac{3}{2}\,\frac{k^2}{\beta^2}\,\frac{\pi^2}{L^2}\right) .$$

This may be written

$$\left(\omega_p^2 - \omega^2 + \frac{3}{2}\,\frac{k^2\pi^2}{\beta^2 L^2}\right) E_k + \frac{\mu}{2}\,\omega_p^2\,(E_{k-2} + E_{k+2})$$

$$+ \frac{\mu\omega_p^2}{2.5}\,(E_{k-4} + E_{k+4}) + \frac{3\mu\omega_p^2}{2.5.7}\,(E_{k-6} + E_{k+6}) + \ldots = 0 .$$

This set of equations can be written as an eigenvalue equation for ω^2

$$\underline{\underline{M}}\,\underline{E} = \omega^2\,\underline{E} .$$

We split $\underline{\underline{M}}$ into $\underline{\underline{M}}_0$ and $\underline{\underline{M}}_1$ and attempt a perturbation solution

$$(\underline{\underline{M}}_0 + \underline{\underline{M}}_1)\,(\underline{E}_0 + \underline{E}_1) = (\omega_0^2 + \omega_1^2)\,(\underline{E}_0 + \underline{E}_1) .$$

In zero order we have

$$\underline{\underline{M}}_0\,\underline{E}_0 = \omega_0^2\,\underline{E}_0$$

The solution to this equation is known. In first order we obtain for the perturbed eigenvalue

$$\omega_1^2 = (\underline{E}_0 \cdot \underline{\underline{M}}_1\,\underline{E}_0)$$

$$M_1 = \frac{3}{2}\mu\omega_p^2 \begin{pmatrix} 0 & -\frac{1}{3.5} & \frac{4}{3.5.7} & \frac{4}{5.7.9} & \cdots \\ -\frac{1}{3.5} & -\frac{1}{5.7} & -\frac{1}{7.9} & \frac{4.7}{5.9.11} & \cdots \\ \frac{4}{3.5.7} & -\frac{1}{7.9} & -\frac{1}{9.11} & -\frac{1}{11.13} & \cdots \\ \frac{4}{5.7.9} & \frac{4.7}{5.9.11} & -\frac{1}{11.13} & -\frac{1}{13.15} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Representing $\underline{E}_0$ by the Fourier coefficients of the respective Mathieu functions $E_0 = (D_1, D_3, D_5, \ldots)$ and the matrix elements of M_1 by M_{ik}, we obtain

$$\begin{aligned} \omega_1^2 = \frac{3}{2}\mu\omega_p^2 \Big(& M_{33} D_3^2 + M_{55} D_5^2 + M_{77} D_7^2 + \ldots \\ & + 2M_{13} D_1 D_3 + 2M_{15} D_1 D_5 + 2M_{17} D_1 D_7 + \ldots \\ & + 2M_{35} D_3 D_5 + 2M_{37} D_3 D_7 + \ldots \\ & + 2M_{57} D_5 D_7 + \ldots \Big) \\ = {} & \frac{3}{2}\mu\omega_p^2 \, S \, . \end{aligned}$$

In order to compare results for the two profiles we have to express density and inhomogenity in terms of common variables. We choose as such variables the maximum density $\overline{n}$ and the total inhomogenity η.

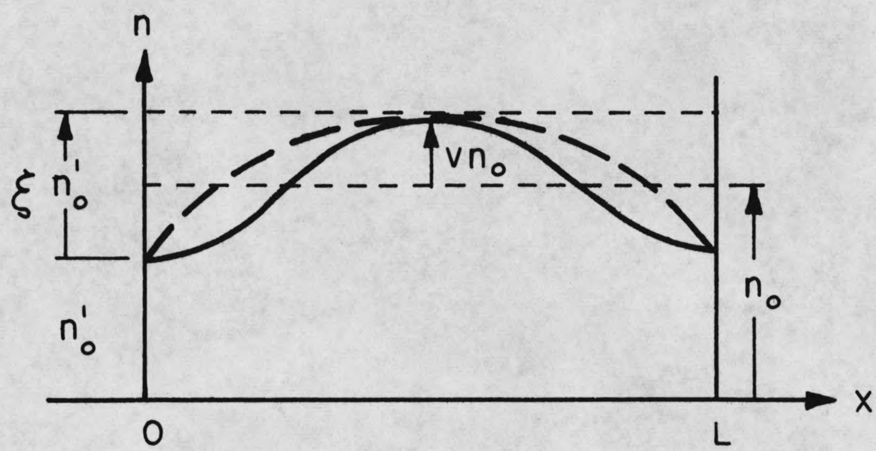

Fig. 4 Schematic sketch of the sin- and cos-density profile.

The following relations hold

$$\xi\, n_o' = 2\nu\, n_o = \eta\, \bar{n}$$

$$n_o(1 + \nu) = n_o' \,(1 + \xi) = \bar{n} \quad .$$

In terms of the new variables we have

$$\xi = \frac{\eta}{1-\eta} \quad ; \quad \nu = \frac{\eta}{2-\eta}$$

$$n_o = \bar{n}\,(1 - \frac{1}{2}\,\eta) \;;\; n_o' = \bar{n}\,(1 - \eta) \;.$$

Let us denote by a_1, q_1 the characteristic values for the cos-profile, and by a_2, q_2 those for the zero-order of the sin-profile. We find

$$q_1 = -\frac{1}{6}\eta\frac{\beta^2L^2}{\pi^2}\bar{\omega}_p^{\ 2}$$

$$a_1 = \frac{2}{3}\frac{\beta^2L^2}{\pi^2}(\omega^2 - \bar{\omega}_p^{\ 2}(1 - \frac{1}{2}\eta)$$

$$q_2 = -\frac{4}{9\pi}\eta\frac{\beta^2L^2}{\pi^2}\bar{\omega}_p^{\ 2}$$

$$a_2 = \frac{2}{3}\frac{\beta^2L^2}{\pi^2}(\omega_o^{\ 2} - \bar{\omega}_p^{\ 2}(1 - (1 - \frac{2}{\pi})\eta))$$

$$\omega_2 = \omega_o^{\ 2} + \omega_1^{\ 2} = \omega_o^{\ 2} - \frac{2}{\pi}\bar{\omega}_p^{\ 2}\eta S$$

or

$$\frac{\omega^2}{\omega_p^{\ 2}} = 3\pi^2\left(\frac{\lambda_D}{L}\right)^2 a_2 + 1 - \eta + \frac{2}{\pi}\eta(1 - S) \qquad \text{(sin-profile)}$$

$$\frac{\omega^2}{\omega_p^{\ 2}} = 3\pi^2\left(\frac{\lambda_D}{L}\right)^2 a_1 + 1 - \frac{1}{2}\eta \qquad \text{(cos-profile)}$$

λ_D is referred to the maximum density.

Fig. 5 shows a comparison of the frequencies of the lowest mode as a function of inhomogeneity for the sin- and the cos-profile.

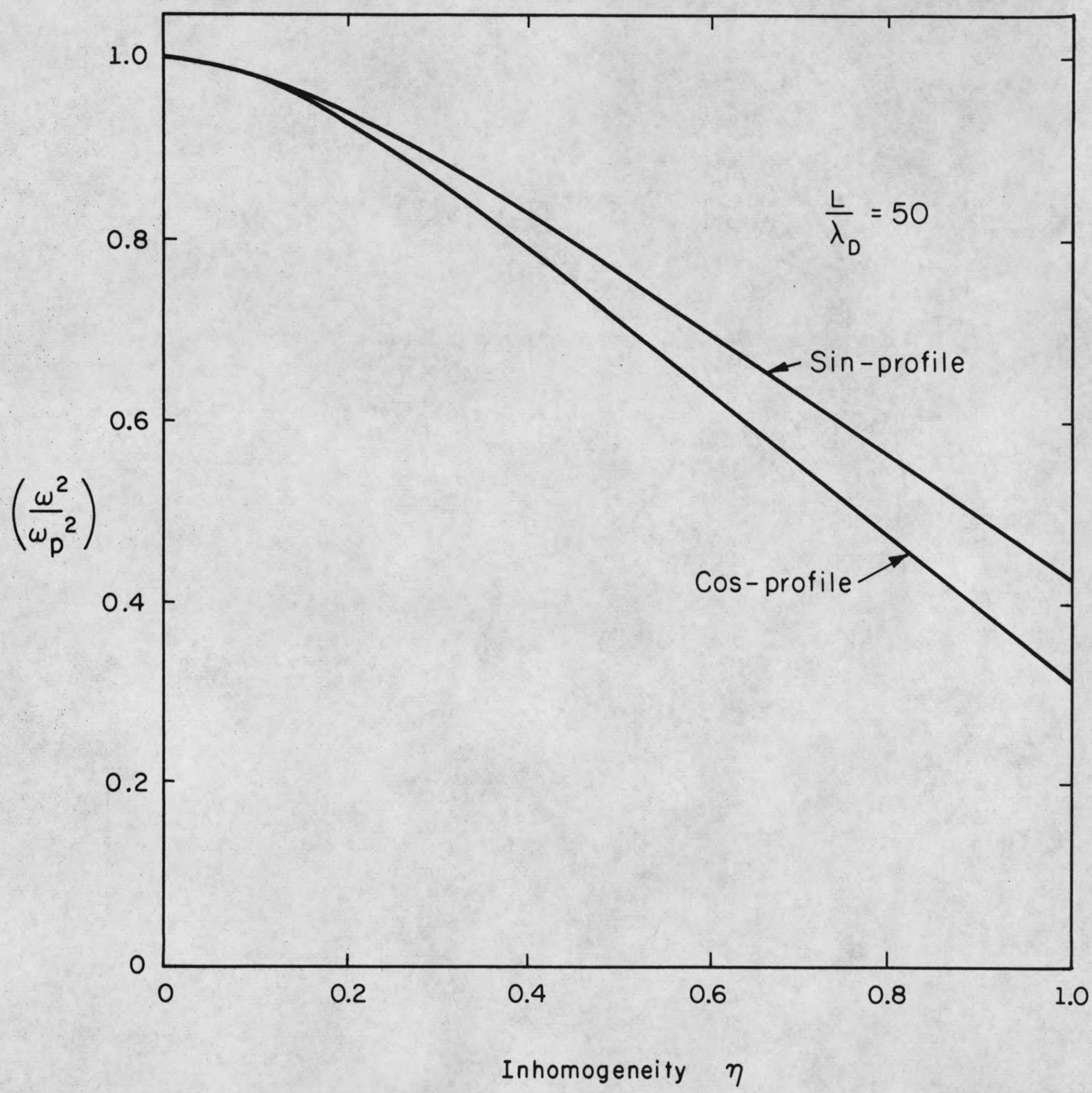

Fig. 5 Eigenfrequencies of the first mode as a function of inhomogeneity for the sin- and cos-profile.

REFERENCES

1. A Dattner, Phys. Rev. Letters 10, 205 (1963).

2. J. C. Nickel, J. V. Parker and R. W. Gould, Phys. Rev. Letters 11, 185 (1963).

3. P. Weissglas, Phys. Rev. Letters 10, 206 (1963).

4. P. Weissglas, J. Nucl. Energy, Part C, 6, 251 (1964). (This paper also contains numerous references to earlier work.)

5. F. C. Hoh, Phys. Rev. 133, A1016 (1964).

6. S. J. Buchsbaum and A. Hasegawa, Phys. Rev. Letters 12, 685 (1964).

7. J. D. Jackson, J. Nucl. Energy, Part C, 1, 171 (1960).

8. N. W. McLachlan, Theory and Application of Mathieu Functions, Oxford, 1951.

Distribution list as of September 1, 1964

1 Director
Air University Library
Maxwell Air Force Base, Alabama
Attn: CR-4803a

1 Redstone Scientific Information Center
U.S. Army Missile Command
Redstone Arsenal, Alabama

1 Electronics Research Laboratory
University of California
Berkeley 4, California

2 Hughes Aircraft Company
Florence and Teale
Culver City, California
Attn: N.E. Devereux
Technical Document Center

3 Autonetics
9150 East Imperial Highway
Downey, California
Attn: Tech. Library, 3041-11

1 Dr. Arnold T. Nordsieck
General Motors Corporation
Defense Research Laboratories
6767 Hollister Avenue
Goleta, California

1 University of California
Lawrence Radiation Laboratory
P.O. Box 808
Livermore, California

1 Mr. Thomas L. Hartwick
Aerospace Corporation
P.O. Box 95085
Los Angeles 45, California

1 Professor Zorab Kaprelian
University of Southern California
University Park
Los Angeles 7, California

1 Sylvania Electronic Systems-West
Electronic Defense Laboratories
P.O. Box 205
Mountain View, California
Attn: Documents Center

1 Varian Associates
611 Hansen Way
Palo Alto, California 94303
Attn: Technical Library

1 Huston Denslow
Library Supervisor
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

1 Professor Nicholas George
California Institute of Technology
Electrical Engineering Department
Pasadena, California

1 Space Technology Labs., Inc.
One Space Park
Redondo Beach, California
Attn: Acquisitions Group
STL Technical Library

2 Commanding Officer and Director
U.S. Navy Electronics Laboratory
San Diego, California 92052
Attn: Code 2800, C.S. Manning

1 Commanding Officer and Director
U.S. Navy Electronics Laboratory
San Diego, California 92052

1 Commanding Officer
Office of Naval Research Branch Office
1000 Geary Street
San Francisco, California 94109

1 The RAND Corporation
1700 Main Street
Santa Monica, California
Attn: Library

1 Stanford Electronics Laboratories
Stanford University
Stanford, California
Attn: SEL Documents Librarian

1 Dr. L.F. Carter
Chief Scientist Air Force
Room 4E-324, Pentagon
Washington 25, D.C.

1 Mr. Robert L. Feik
Associate Director for Research
Research and Technology Division
AFSC
Bolling Air Force Base 25, D.C.

1 Captain Paul Johnson (USN-Ret)
National Aeronautics and Space
Administration
1520 H. Street, N.W.
Washington 25, D.C.

1 Major Edwin M. Myers
Headquarters USAF (AFRDR)
Washington 25, D.C.

1 Dr. James Ward
Office of Deputy Director
(Research and Info)
Department of Defense
Washington 25, D.C.

1 Dr. Alan T. Waterman
Director, National Science Foundation
Washington 25, D.C.

1 Mr. G.D. Watson
Defense Research Member
Canadian Joint Staff
2450 Massachusetts Ave., N.W.
Washington 8, D.C.

1 Mr. Arthur G. Wimer
Chief Scientist
Air Force Systems Command
Andrews Air Force Base
Washington 25, D.C.

1 Director Advanced Research
Projects Agency
Washington 25, D.C.

1 Air Force Office of Scientific Research
Directorate of Engineering Sciences
Washington 25, D.C.
Attn: Electronics Division

1 Director of Science and Technology
Headquarters, USAF
Washington 25, D.C.
Attn: AFRST-EL/GU

1 AFRST - SC
Headquarters, USAF
Washington 25, D.C.

1 Headquarters, R & T Division
Bolling Air Force Base
Washington 25, D.C.
Attn: RTHR

1 Headquarters, U.S. Army Materiel Command
Research Division, R & D Directorate
Washington 24, D.C.
Attn: Physics and Electronics Branch
Electronics Section

1 Commanding General
U.S. Army Materiel Command
Washington 25, D.C.
Attn: R & D Directorate

1 Commanding Officer
Diamond Ordnance Fuze Laboratories
Washington 25, D.C.
Attn: Librarian, Room 211, Bldg. 92

1 Operations Evaluation Group
Chief of Naval Operations (OP-13EG)
Department of Navy
Washington, D.C. 20350

1 Chief of Naval Operations (Code OP-07T)
Department of the Navy
Washington , D.C. 20350

1 Commanding Officer
U.S. Army Personnel Research Office
Washington 25, D.C.

1 Commanding Officer & Director
Code 142 Library
David W. Taylor Model Basin
Washington, D.C. 20360

1 Chief, Bureau of Ships (Code 686)
Department of the Navy
Washington, D.C. 20360

1 Chief, Bureau of Ships (Code 732)
Department of the Navy
Washington, D.C. 20360

1 Chief, Bureau of Naval Weapons
Technical Library, DLI-3
Department of the Navy
Washington, D.C. 20360

1 Director, (Code 5140)
U.S. Naval Research Laboratory
Washington, D.C. 20390

1 Chief of Naval Research (Code 437)
Department of the Navy
Washington, D.C. 20360

1 Dr. H. Wallace Sinaiko
Institute for Defense Analyses
Research & Engineering Support Division
1666 Connecticut Ave., N.W.
Washington 9, D.C.

1 Data Processing Systems Division
National Bureau of Standards
Conn. at Van Ness
Room 239, Bldg. 10
Washington 25, D.C.
Attn: A.K. Smilow

1 National Bureau of Standards
Research Information Center &
Advisory Service on Information
Processing
Data Processing Systems Division
Washington 25, D.C.

1 Exchange and Gift Division
The Library of Congress
Washington 25, D.C.

1 NASA Headquarters
Office of Applications
400 Maryland Avenue, s.w.
Washington 25, D.C.
Attn: Mr. A.M. Greg Andrus
Code FC

1 APGC (PGAPI)
Eglin Air Force Base
Florida

1 Martin Company
P.O. Box 5837
Orlando, Florida
Attn: Engineering Library
MP-30

1 Commanding Officer
Office of Naval Research, Branch Office
230 North Michigan
Chicago , Illinois 60601

1 Laboratories for Applied Sciences
University of Chicago
6220 South Drexel
Chicago, Illinois 60637

1 Librarian
School of Electrical Engineering
Purdue University
Lafayette, Indiana

1 Donald L. Epley
Department of Electrical Engineering
State University of Iowa
Iowa City, Iowa

1 Commanding Officer
U.S. Army Medical Research Laboratory
Fort Knox, Kentucky

2 Keats A. Pullen, JR.
Balistic Research Laboratories
Aberdeen Proving Ground, Maryland

1 Director
U.S. Army Human Engineering Laboratories
Aberdeen Proving Ground, Maryland

1 Mr. James Tippett
National Security Agency
Fort Meade, Maryland

1 Commander
Air Force Cambridge Research Laboratories
Laurence G. Hanscom Field
Bedford, Massachusetts

1 Dr. Lloyd Hollingsworth
Director, ERD AFCRL
L.G. Hanscom Field
Bedford, Massachusetts

1 Data Sciences Laboratory
Air Force Cambridge Research Lab.
Office of Aerospace Research, USAF
L.G. Hanscom Field
Bedford, Massachusetts
Attn: Lt. Stephen J. Kahne - CRB

1 Instrumentation Laboratory
Massachusetts Institute of Technology
68 Albany Street
Cambridge 39, Massachusetts
Attn: Library WI-109

1 Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge 39, Massachusetts
Attn: Document Room 26-327

1 Dr. Robert Kingston
Lincoln Laboratories
Lexington, Massachusetts

1 Lincoln Laboratory
Massachusetts Institute of Technology
P.O. Box 73
Lexington 73, Massachusetts
Attn: Library, A-082

1 Sylvania Electric Products, Inc.
Electronic Systems
Waltham Labs. Library
100 First Avenue
Waltham 54, Massachusetts

1 Minneapolis-Honeywell Regulator Co.
Aeronautical Division
2600 Ridgeway Road
Minneapolis 13, Minnesota
Attn: Dr. D.F. Elwell
Main Station : 625

1 Inspector of Naval Material
Bureau of Ships Technical Representative
1902 West Minnehaha Avenue
St. Paul 4, Minnesota

20 Activity Supply Officer, USAELRDL
Building 2504, Charles Wood Area
Fort Monmouth, New Jersey
For: Accountable Property Officer
Marked: For Inst. for Exploratory Research
Inspect at Destination
Order No. 576-PM-63-91

1 Commanding General
U.S. Army Electronic Command
Fort Monmouth, New Jersey
Attn: AMSEL-RE

1 Miss F. Cloak
Radio Corporation of America
RCA Laboratories
David Sarnoff Research Center
Princeton, New Jersey

1 Mr. A.A. Lundstrom
Bell Telephone Laboratories
Room 2E-127
Whippany Road
Whippany, New Jersey

1 AFMDC (MDSGO Maj. P. Wheeler, Jr.)
Holloman Air Force Base
New Mexico 88330

1 Commanding General
White Sands Missile Range
New Mexico

1 Microwave Research Institute
Polytechnic Institute of Brooklyn
55 John Street
Brooklyn 1, New York

1 Cornell Aeronautical Laboratory, Inc.
4455 Genesee Street
Buffalo 21, New York
Attn: J.P. Desmond, Librarian

1 Sperry Gyroscope Company
Marine Division Library
155 Glen Cove Road
Carle Place, L.I., New York
Attn; Mrs. Barbara Judd

1 Major William Harris
RADC (RAWI)
Griffiss Air Force Base
New York

1 Rome Air Development Center
Griffiss Air Force Base
New York
Attn: Documents Library
RAALD

1 Library
Light Military Electronics Department
General Electric Company
Armament & Control Products Section
Johnson City, New York

1 Columbia Radiation Laboratory
Columbia University
538 West 120th Street
New York 57, New York

1 Mr. Alan Barnum
Rome Air Development Center
Griffiss Air Force Base
Rome, New York

1 Dr. E. Howard Holt
Director
Plasma Research Laboratory
Rensselaer Polytechnic Institute
Troy, New York

3 Commanding Officer
U.S. Army Research Office (Durham)
Box CM, Duke Station
Durham, North Carolina
Attn: CRD-AA-1P, Mr. Ulsh

1 Battelle-DEFENDER
Battelle Memorial Institute
505 King Avenue
Columbus 1, Ohio

1 Aeronautical Systems Division
Navigation and Guidance Laboratory
Wright-Patterson Air Force Base
Ohio

1 Aeronautical Systems Division
Directorate of Systems Dynamic Analysis
Wright-Patterson Air Force Base
Ohio

1 Commander
Research & Technology Div.
Wright-Patterson Air Force Base
Ohio 45433
Attn: MAYT (Mr. Evans)

1 Commanding Officer (AD-5)
U.S. Naval Air Development Center
Johnsville, Pennsylvania
Attn: NADC Library

2 Commanding Officer
Frankford Arsenal
Philadelphia 37, Pennsylvania
Attn: SMUFA-1300

1 H.E. Cochran
Oak Ridge National Laboratory
P.O. Box X
Oak Ridge, Tennessee

1 U.S. Atomic Energy Commission
Office of Technical Information Extension
P.O. Box 62
Oak Ridge, Tennessee

1 President
U.S. Army Air Defense Board
Fort Bliss, Texas

1 Director
Human Resources Research Office
The George Washington University
300 North Washington Street
Alexandria, Virginia

20 Defense Documentation Center
for Scientific & Technical Information
Cameron Station
Alexandria, Virginia 22314

1 Commander
U.S. Army Research Office
Highland Building
3045 Columbia Pike
Arlington 4, Virginia

1 U.S. Naval Weapons Laboratory
Computation and Analysis Laboratory
Dahlgren, Virginia
Attn: Mr. Ralph A. Niemann