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by

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I. INTRODUCTION

The invention of the electronic digital computer stimulated the intensive development of numerical methods for the solution of fluid flow problems. The majority of the methods first developed were explicit and of low order because their utility on the small and slow early machines derived from their simplicity. This trend has continued till the present. Such schemes, however, suffer from stringent restrictions on the allowable time step particularly in viscous flows. Implicit schemes are free from this affliction, but this freedom is bought at a price, the solution of large systems of nonlinear equations. The new parallel and pipeline computers have made feasible the large computations required by implicit and high order schemes, and thereby have spurred interest in complex difference methods. Recently Rusanov [4] and Burstein and Mirin [2] have studied explicit third order difference schemes for hyperbolic systems. We present in this report a new unconditionally stable implicit scheme for viscous flows which is uniformly fourth order accurate in time and space.

We first describe the method and examine the local truncation error. We then present a linearized stability analysis of the scheme. Finally we present several examples to illustrate the accuracy and stability of the method.

II. DIFFERENCE SCHEME

Consider the initial-value problem

$$u_t = f(u, u_x, u_{xx}), \qquad (1)$$

$$u(x,0) = g(x), \quad -\infty < x < \infty.$$

Here u is an unknown vector function of x and t, f is a given nonlinear vector function of u, u_x, and u_{xx}, and g is a given vector function of x. Rather than make specific differentiability assumptions, we assume u has derivatives of any order required. Integrating equation (1) from t to t + Δt, we obtain

$$u(x,t+\Delta t) = u(x,t) + \int_t^{t+\Delta t} f(u,u_x,u_{xx})dt.$$

We can discretize this equation by approximating the integral by a quadrature formula employing nodes in the interval $(t,t+\Delta t)$. Approximations to u and hence f at the interior nodes could then be obtained from u and u_t at times t and t + Δt through Hermite interpolation. Since the error in Hermite interpolation is of $0(\Delta t^4)$, we should employ a quadrature formula with error of $0(\Delta t^5)$. Simpson's rule is such a formula and, moreover, employs only one interior node, thus minimizing the number of interpolations. Of course, f cannot be computed exactly because it involves derivatives of u with respect to x. However, we can approximate f by replacing u_x and u_{xx} by appropriate difference quotients. If these quotients are chosen as the centered five-point formulae, the overall accuracy of the scheme is maintained.

We seek a mesh function U which approximates the exact solution u on the mesh

$$(x_{\pm i} = \pm\, i\Delta x,\ t_n = n\Delta t;\ i,\ n = 0,\ 1,\ 2,\ \ldots).$$

We restrict our attention to meshes where $\Delta t = r\Delta x$ for some positive constant r. Let u_i^n and U_i^n denote the values of u and U at the mesh point (x_i, t_n), and let

$$L_d(v) = f(v, d_x v, d_{xx} v),$$
$$L_D(v) = f(v, D_x v, D_{xx} v),$$

where v is a smooth function, $d_x = \partial/\partial x$, $d_{xx} = \partial^2/\partial x^2$, and D_x and D_{xx} are the corresponding five-point centered difference quotients. The method is defined by

$$\begin{aligned} U_i^{n+1} &= U_i^n + (\Delta t/6)[L_D(U_i^n) + 4L_D(\hat{U}_i^{n+1/2}) + L_D(U_i^{n+1})], \\ \hat{U}_i^{n+1/2} &= (U_i^n + U_i^{n+1})/2 + (\Delta t/8)[L_D(U_i^n) - L_D(U_i^{n+1})]. \end{aligned} \tag{2}$$

These implicit equations generate a system of nonlinear algebraic equations for the U_i^{n+1}, which must be solved through some iterative procedure.

The local truncation error is defined by

$$e(u) = \int_{t_n}^{t_{n+1}} L_d(u)dt - (\Delta t/6)[L_D(u^n) + 4L_D(\hat{u}^{n+1/2}) + L_D(u^{n+1})], \tag{3}$$

where we have suppressed the spatial indices for convenience. Our choice of D_x and D_{xx} insures that

$$L_D(v) = L_d(v) + O(\Delta^4)$$

for $v = u^n$ and u^{n+1}. It is simple to show that

$$\hat{u}^{n+1/2} = u^{n+1/2} + O(\Delta^4),$$

$$D_x \hat{u}^{n+1/2} = d_x u^{n+1/2} + O(\Delta^4),$$

$$D_{xx} \hat{u}^{n+1/2} = d_{xx} u^{n+1/2} + O(\Delta^4),$$

and hence that

$$L_D(\hat{u}^{n+1/2}) = L_d(u^{n+1/2}) + O(\Delta^4).$$

Since the error in Simpson's rule is $O(\Delta^5)$, it follows that

$$e(u) = O(\Delta^5). \tag{4}$$

Hence the scheme is uniformly fourth order accurate in time and space.

III. STABILITY

Consider the vector differential equation

$$u_t = Au_x + Bu_{xx}, \tag{5}$$

where the constant matrices A and B are such that the matrix C, defined by

$$C = B(-30+32\cos\varphi-2\cos2\varphi)/12\Delta x^2 + iA(16\sin\varphi-2\sin2\varphi)/12\Delta x,$$

has eigenvalues $\alpha + i\beta$, satisfying $\alpha \leq 0$ for $0 \leq \varphi \leq 2\pi$. This assumption is satisfied, for example, if A and B commute with the commutator AB - BA, and the eigenvalues of B are nonnegative.

We follow von Neumann and set $L_D(U) = AD_xU + BD_{xx}U$ and $U_j^n = \xi^n \exp(i\omega x_j)$ in equations (2) to obtain

$$U_j^{n+1} = GU_j^n,$$

where the amplification matrix

$$G = (I - \Delta tC/2 + \Delta t^2C^2/12)^{-1}(I + \Delta tC/2 + \Delta t^2C^2/12) \qquad (6)$$

with $\varphi = \omega\Delta x$. A simple calculation shows that the eigenvalues g of G satisfy

$$|g|^2 = [\gamma + \alpha\Delta t + \alpha(\alpha^2+\beta^2)\Delta t^3/12]/[\gamma - \alpha\Delta t - \alpha(\alpha^2+\beta^2)\Delta t^3/12],$$

where $\gamma > 0$. But $\alpha \leq 0$. It follows that $|g| \leq 1$ for any Δt and Δx. Hence the method is unconditionally stable.

Our discussion of the order and stability of the method is valid only for pure initial-value problems and mixed problems with periodic boundary conditions. In a general mixed problem, the difference equations (2) cannot be applied at the nodes adjacent to the boundaries. There are several ways to handle these critical nodes. For example, we might use appropriate uncentered difference schemes, or employ extrapolation to generate any values required by the centered equations (2) at fictitious nodes outside the boundaries. It is easy to invent methods based on these ideas, but they often place undesirable restrictions on the time step. We have developed through numerical experimentation a relatively stable method which employs equations (2) at all nodes and generates values at fictitious nodes through six-point extrapolation. The method introduces errors of $O(\Delta^3)$ and $O(\Delta^4)$ at the nodes near the boundaries. Fortunately these errors, which originate in the interaction between the difference operator D_{xx} and the Hermite interpolation and six-point extrapolation errors, seem to have little effect on the global accuracy. This may be due to the fact that the local error at the majority of nodes is of $O(\Delta^5)$.

The system of nonlinear equations generated by equations (2) must be solved carefully to maintain the stability of the method. We have solved this system using successive substitution, nonlinear overrelaxation, and Broyden's quasi-Newton method [1]. Unfortunately, the first two methods often converge only for $\Delta t = O(\Delta x^2)$, nullifying the main advantage of the scheme. Broyden's method, however, worked extremely well, always allowing us to take $\Delta t = O(\Delta x)$.

IV. NUMERICAL EXAMPLES

We have tested our scheme on a variety of problems for the Burger's, Euler, and Navier-Stokes equations. We present a selection of our results to illustrate the accuracy and stability of the method.

Burger's equation is often used as a model for the one-dimensional time-dependent Navier-Stokes equation. It approximates, to first order, the motion of a plane wave of small but finite amplitude. The equation takes account of convection and diffusion, and has the form

$$u_t + (u^2/2)_x = \lambda u_{xx}, \tag{7}$$

where u is the excess wavelet velocity, and λ is the diffusivity of sound.

We studied Burger's equation subject to the initial condition

$$u(x,0) = \sin \pi x, \quad -\infty < x < \infty.$$

This problem has the exact solution

$$u(x,t) = -2\lambda\varphi_x(x,t)/\varphi(x,t),$$

where

$$e^{\mu}\varphi(x,t) = I_0(\mu) + 2\sum_{n=1}^{\infty} I_n(\mu)\exp(-\lambda n^2\pi^2 t)\cos n\pi x,$$

$\mu = 1/2\pi\lambda$, and I_n is the modified Bessel function of order n. The solution has a period of 2 and is antisymmetric with respect to the lines $x = 0, \pm 1, \pm 2, \ldots$. Hence we need only consider the interval $0 \le x \le 1$. Moreover, we need not extrapolate because of the antisymmetry in the solution. We computed the solution of this problem on the region $0 \le x,t \le 1$ for $\lambda = .08$, and $\Delta t = \Delta x = .1, .05$, and $.025$. We also tested the stability of the scheme by solving this problem with $\Delta t \gg \Delta x$. As expected, it worked extremely well. For example, we used $\Delta t/\Delta x = .5/.0125 = 40$ without any difficulty.

We also studied Burger's equation with $\lambda = 1$ subject to the initial and boundary conditions

$$u(x,0) = [\sin(x) + \cos(x)]/[e^x + \cos(x)/2],$$

$$u(0,t) = [\cos(2t) - \sin(2t)]/[1 + \cos(2t)/2],$$

$$\lim_{x\to\infty} u(x,t) = 0.$$

This problem has the exact solution

$$u(x,t) = [\sin(x-2t) + \cos(x-2t)]/[e^x + \cos(x-2t)/2].$$

We computed the solution of this problem on the region $0 \le x,t \le 5$ for $\Delta t = \Delta x = .5, .25$, and $.125$. The boundary condition at $x = 5$ was obtained from the exact solution.

We also studied Burger's equation subject to the initial condition

$$u(x,0) = 2ch(-x), \quad -\infty < x < \infty,$$

where h is Heaviside's step function. The solution is a shock-like wave moving with velocity c. For computational convenience, we transformed the space coordinate into a coordinate moving with velocity c. In this new coordinate, the solution is

$$u(x,t) = 2c/[1 + \exp(cx/\lambda)\mathrm{erfc}[-(x+ct)/2\sqrt{(\lambda t)}]/\mathrm{erfc}[(x-ct)/2\sqrt{(\lambda t)}]].$$

We computed the solution of the transformed problem on the region $-1 \leq x \leq 1$, $.05 \leq t \leq 3.05$ for $\lambda = .1$, $c = .5$, and $\Delta t = \Delta x = .2$, $.1$, and $.05$. The initial and boundary conditions were obtained from the exact solution. The calculation was started at $t = .05$, rather than $t = 0$, to eliminate the systematic phase error introduced by the ambiguity in the origin of the discretized step function. The scheme produces smooth profiles when started with the step function, but the phase error masks the truncation error, making the comparison of errors for different Δt and Δx difficult.

Figures 1 - 3 show selected profiles for these three problems. In each case the errors were graphically insignificant so only the computed results are plotted. Table 1 lists the error norms and computed orders

$$||e||_q = (\sum_i |e_i|^q/m)^{1/q}, \quad r_q(\Delta) = \ln(||e(\Delta)||_q/||e(\Delta/2)||_q)/\ln(2),$$

where $e_i = u_i - U_i$, and the sum is taken over the m interior nodes. The computed orders should be 4 in the first problem because the assumptions made in our analysis of the truncation error are satisfied. However, the orders may differ from 4 in the second and third problems because of the errors introduced by the extrapolation procedure. The scheme clearly fulfills our expectations. The extrapolation errors have a remarkably small effect on the global accuracy for sufficiently small Δt.

The Navier-Stokes equations may be written in the form

$$w_t = f_x + s, \tag{8}$$

where

$$w = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \quad f = -\begin{bmatrix} m \\ p+m^2/\rho \\ m(p+E)/\rho \end{bmatrix},$$

$$s = \begin{bmatrix} 0 \\ (4/3Re)(m/\rho)_{xx} \\ [(\gamma/PrRe)(p/(\gamma-1)\rho)_{xx}+(4/3Re)[((m/\rho)_x)^2+(m/\rho)(m/\rho)_{xx}]] \end{bmatrix},$$

$$p = (\gamma-1)(E-m^2/2\rho).$$

Here ρ,m,E, and p are the density, momentum, energy, and pressure per unit volume, and Re and Pr are the Reynolds and Prandtl numbers. The Euler equations result if we set Re = ∞.

Our scheme can be applied to smooth inviscid flows as well as viscous flows. A simple calculation shows that it will remain fourth order at the noncritical interior nodes if the solution w has five continuous derivatives with respect to x. To illustrate this fact, we studied the evolution of an isentropic expansion wave using the Euler equations. The initial velocity

$u(x,0)$ was

$$u(x,0) = \begin{cases} u_0 & -\infty < x \le -1, \\ u(x) & -1 \le x \le 1, \\ u_1 & 1 \le x < \infty, \end{cases}$$

where $u(x)$ is the monotonically increasing polynomial

$$u(x) = (u_1-u_0)(63x^{11}-385x^9+990x^7-1386x^5+1155x^3-693x)/512 + (u_0+u_1)/2,$$

chosen so that $u(x,0) \in C^5$. Given $u(x,0)$, the solution at (x,t) is easily computed. It is

$$\rho = (1 + (\gamma-1)(u-u_0)/2\sqrt{\gamma})^{2/(\gamma-1)},$$

$$u = u(\xi,0), \quad m = \rho u, \quad E = \rho^\gamma/(\gamma-1) + \rho u^2/2,$$

where ξ is the solution of the nonlinear equation

$$\xi + (\gamma+1)u(\xi,0)t/2 - x + ((1-\gamma)u_0/2+\sqrt{\gamma})t = 0.$$

For computational convenience, we transformed the space coordinate into a wave-centered coordinate. We computed the solution of the transformed problem on the region $-1.2 \le x \le 1.2$, $0 \le t \le 1.6$, for $\gamma = 1.4$, $u_0 = 1.75$, $u_1 = 2.16666667$, and $\Delta t = \Delta x/2 = .1$ and $.05$. Table 2 summarizes the error norms and computed orders for the calculation. The orders are initially close to 4, but decrease, as expected, as the most rapidly varying portions of the wave approach the boundaries.

The steady shock profiles for the Navier-Stokes equations can be obtained analytically for $Pr = .75$. The solution in a shock-centered coordinate system for shock Mach number M is

$$\rho = u_0/(u_0-u), \quad m = \rho u,$$

$$T = 1 + (\gamma-1)(2u_0u-u^2)/2\gamma, \quad E = \rho T/(\gamma-1) + \rho u^2/2,$$

where u is the solution of the nonlinear equation

$$u - (\alpha-u)^{u_1/u_0}(\alpha/2)^{\alpha/u_0} \exp(-\alpha\beta x/u_0) = 0,$$

and

$$u_0 = \sqrt{\gamma}M, \quad u_1 = \sqrt{\gamma}(2+(\gamma-1)M^2)/(M(\gamma+1)),$$

$$\alpha = u_0-u_1, \quad \beta = 3u_0\mathrm{Re}(\gamma+1)/8\gamma.$$

We transformed the Navier-Stokes equations to a shock-centered coordinate system, used the steady solution to obtain initial and boundary conditions, and solved the difference equations on the region $-2 \leq x \leq 2.2$, $0 \leq t \leq 6$, for $\gamma = 1.4$, $M = 1.6$, $Pr = .75$, $Re = 4.5$, and $\Delta t = \Delta x = .2$. Figure 4 shows the ρ, m, and E profiles at $t = 6$, and Table 3 summarizes the error norms at $t = 6$. Rubin and Burstein [3] have performed a related computation. Our profiles, however, exhibit no overshoot in contrast to their profiles.

problem	time	Δ	$\|\|e\|\|_1$	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	r_1	r_2	r_∞
1	1.00	.100	2.63-4	3.86-4	7.46-4	3.8	3.8	3.7
		.050	1.84-5	2.68-5	5.68-5	4.0	4.0	3.8
		.025	1.16-6	1.72-6	3.96-6			
2	5.00	.500	2.11-2	2.37-2	3.63-2	3.4	3.3	2.7
		.250	1.95-3	2.42-3	5.49-3	3.9	3.9	4.1
		.125	1.30-4	1.61-4	3.14-4			
3	3.05	.200	8.18-3	8.89-3	1.11-2	7.3	7.0	6.3
		.100	5.23-5	7.16-5	1.43-4	4.0	4.0	3.8
		.050	3.35-6	4.59-6	1.04-5			

Table 1. Error norms and computed orders for Burger's equation.

time	variable	Δx	$\|\|e\|\|_1$	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	r_1	r_2	r_∞
.6	ρ	.2	5.14-4	6.23-4	1.41-3	3.7	3.6	3.9
		.1	4.09-5	4.97-5	9.68-5			
	m	.2	1.66-3	2.03-3	4.60-3	3.7	3.6	3.8
		.1	1.32-4	1.63-4	3.28-4			
	E	.2	4.24-3	5.28-3	1.20-2	3.7	3.7	3.8
		.1	3.37-4	4.21-4	8.62-4			
1.6	ρ	.2	6.73-4	8.53-4	1.85-3	3.4	3.4	3.0
		.1	6.35-5	8.28-5	2.25-4			
	m	.2	2.18-3	2.83-3	6.54-3	3.4	3.4	3.1
		.1	2.06-4	2.72-4	7.80-4			
	E	.2	5.70-3	7.37-3	1.74-2	3.4	3.4	3.1
		.1	5.27-4	7.06-4	2.09-3			

Table 2. Error norms and computed orders for the Euler equations ($\gamma = 1.4$, $u_0 = 1.75$, $u_1 = 2.166667$).

variable	$\|\|e\|\|_1$	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$
ρ	3.53-4	5.79-4	2.05-3
m	5.06-4	9.63-4	3.48-3
E	1.70-3	2.94-3	1.06-2

Table 3. Error norms at t = 6 for the Navier-Stokes equations (γ = 1.4, M = 1.6, Pr = .75, Re = 4.5, $\Delta x = \Delta t$ = .2).

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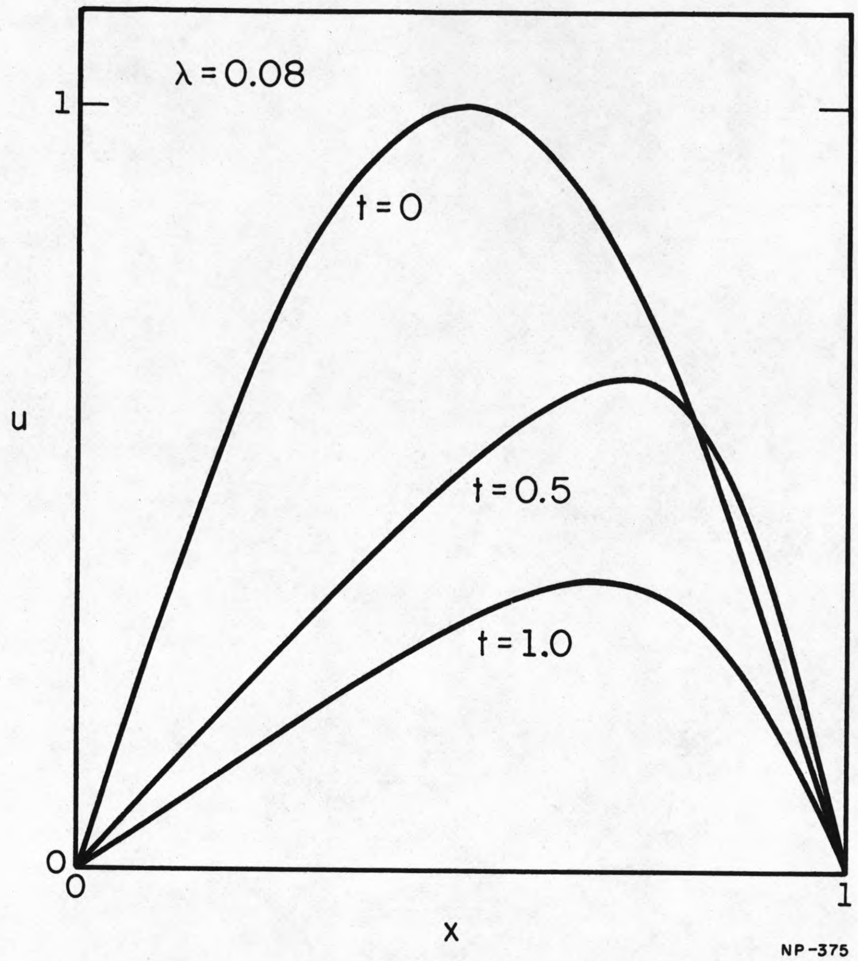

Figure 1.

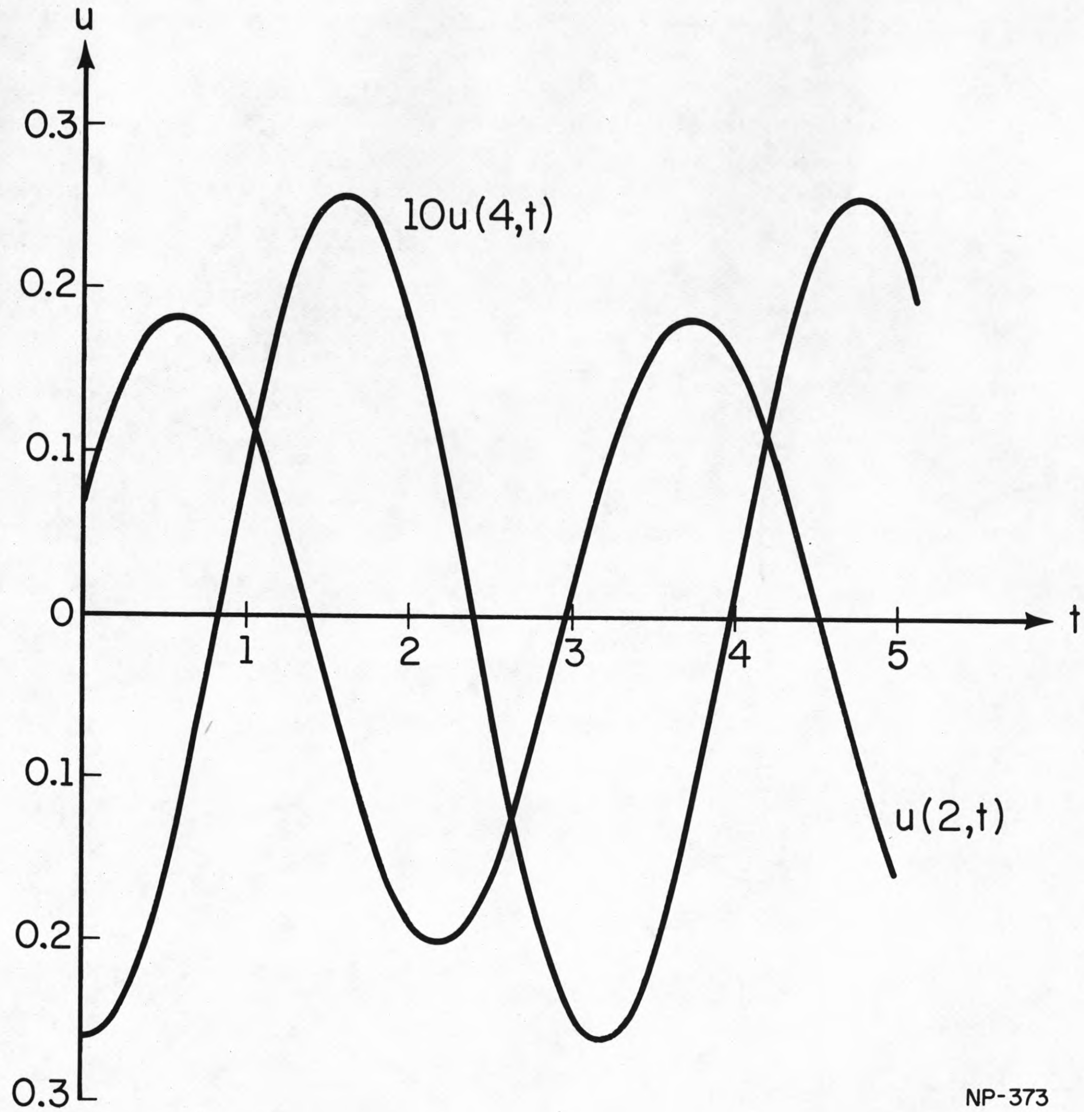

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Figure 2

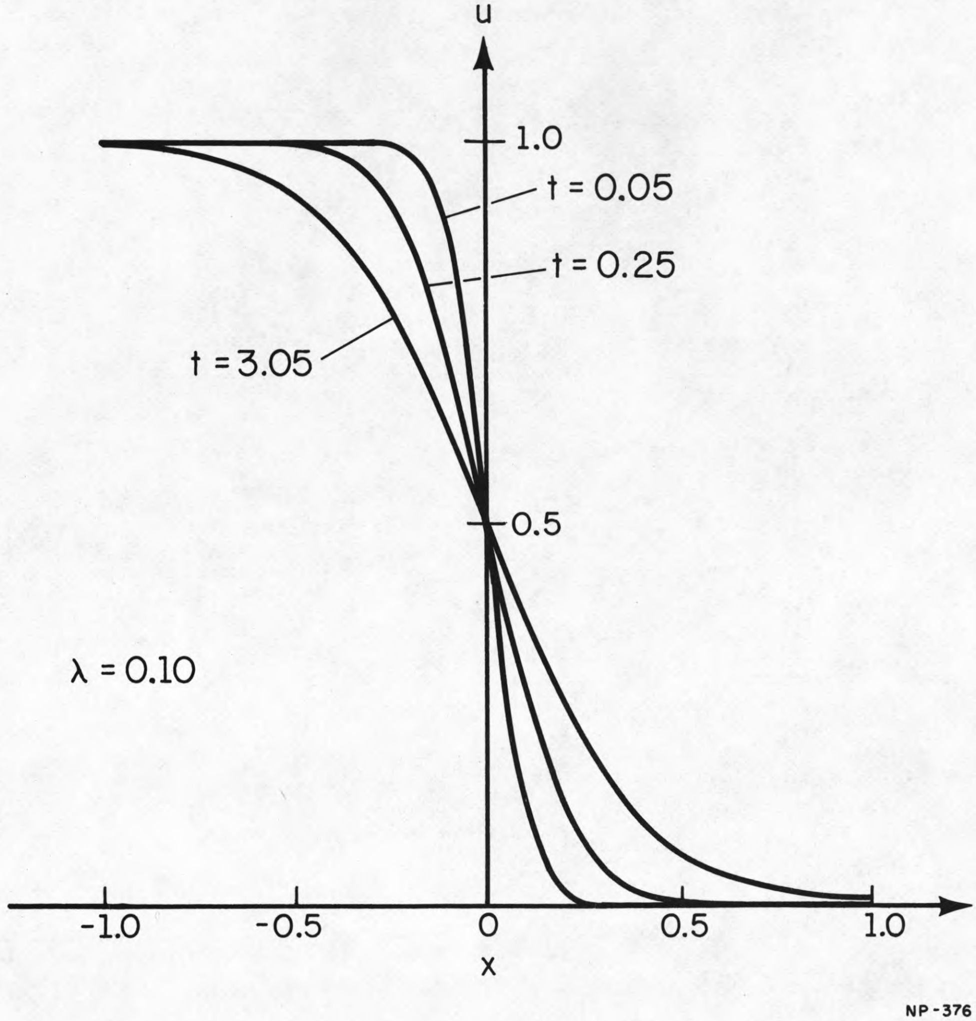

Figure 3

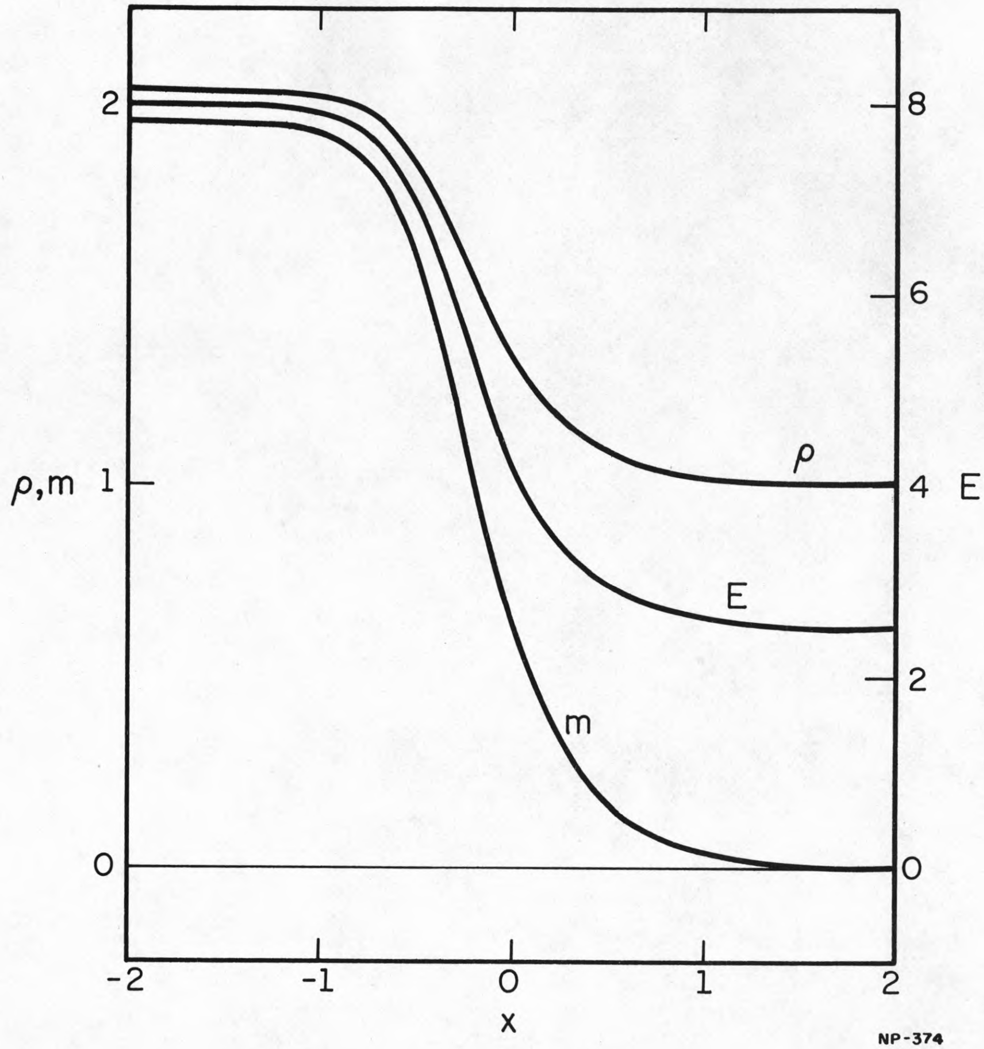

Figure 4.

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13. ABSTRACT

An implicit finite difference scheme for viscous flows is presented. The scheme is based on Simpson's rule and two-point Hermite interpolation, has a truncation error of $O(\Delta t^5)$ for fixed $\Delta t/\Delta x$, and is unconditionally stable according to a Fourier stability analysis. Numerical solutions of the Burger's, Euler, and Navier-Stokes equations are presented to illustrate the order and accuracy of the scheme.

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