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STABILITY ANALYSIS OF ADAPTIVELY CONTROLLED FLEXIBLE JOINT MANIPULATORS

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Abstract

This report presents a detailed stability analysis of an adaptive composite control strategy for flexible joint robot manipulators. Our so–called slow/fast control strategy, consisting of a slow adaptive controller designed for a rigid robot together with a fast control to damp the elastic oscillations of the joints, was first derived in previous work of the authors and its performance was detailed by both simulations and experimental results. We now present the mathematical details and rigorous stability proofs of our algorithm. Using the composite Lyapunov theory for singularly perturbed systems we present sufficient conditions for adaptive trajectory tracking. For point–to–point motion we show that there is always a range of joint stiffness for which convergence is achieved and we quantify the region of convergence. For tracking of (smooth and bounded) reference trajectories we give sufficient conditions for closed loop stability and uniform boundedness of the tracking error. A residual set to which the tracking error converges is quantified. We also show that for special classes of trajectories, which include step responses generated from reference models and certain joint interpolated trajectories we can achieve asymptotic tracking. We argue that these results are the best that one can expect without additional compensation of the slow subsystem such as with integral manifold based corrective control.

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1 Introduction

The dynamics and control of robot manipulators taking into account the joint flexibility is an interesting and challenging problem, which is attracting attention from an increasing number of researchers. A recent survey, in fact, [22] lists nearly one hundred references dealing with various aspects of this problem, such as feedback linearization, robust control, observer design, and adaptive control.

In the present report we consider the adaptive control problem. Our algorithm, which has appeared previously in [21], [5], and [6], is a composite control law consisting of a slow adaptive control designed on the basis of a rigid robot model and a fast control designed to damp the elastic oscillations at the joints. Such a control strategy is intuitively appealing since it is simple to implement and it exploits the considerable body of knowledge that exists for the adaptive control of rigid robots. This control strategy has been investigated via computer simulation and by experiments performed on a single–link, flexible joint mechanism with excellent results [7], [8]. However, the stability properties of this approach have not been rigorously investigated before now.

2 Synopsis

In this section we will summarize the main results to follow. Since the actual mathematical details are quite involved, the casual reader may gain an understanding of our main results by reading only this section, while the more interested reader can press on.

In the last several years several globally convergent adaptive control algorithms have appeared for rigid robots (See the survey [14]). These algorithms are either adaptive versions of the computed torque approach [3],[13], or they exploit the passivity structure of rigid robot dynamics [19]. In the case of flexible joint robots both of the above approaches fail [22]. As a result, finding a globally convergent adaptive control law which is independent of the joint stiffness is a formidable and still unsolved problem. However, in most practical situations, the joint stiffness is large relative to other parameters in the system. Singular perturbation techniques can thus be used to separate the slow dynamics from the fast dynamics and control each separately using so–called composite control strategies [12]. The fast dynamics involve the joint forces and the slow dynamics involve the link variables.

It usually turns out in practice that the joint resonant modes are poorly damped and this, in fact, is largely the source of the problem associated with joint flexibility in robot control. Our approach can be explained intuitively then as follows: a fast feedback control law is first designed to damp the oscillations of the fast variables. Once the fast transients have decayed, the slow part of the system should appear nearly like the dynamics of a rigid robot, which can then be controlled using any number of techniques. Our strategy is then summarized as

$$control_{composite} = control_{slow} + control_{fast} \tag{1}$$

where $control_{slow}$ is designed using a rigid robot model and $control_{fast}$ is designed solely to provide sufficient damping of the fast dynamics. Any number of techniques for the control of rigid robots can be used to design $control_{slow}$ in the above equation. In this report, we base our design of the slow control on the algorithm of Slotine and Li [19] because it is globally convergent in the absence of joint flexibility, and because its implementation requires only position and velocity measurements. It is significant that our fast control involves only joint velocity measurements. In this way we achieve robustness to parameter uncertainty without the need for acceleration and jerk as would be required by nonlinear feedback linearization results.

Once we have stabilized the fast dynamics with the fast control term, our system can be thought of as the rigid robot model (and rigid adaptive control) with high frequency, stable, unmodeled dynamics represented by the joint flexibility. Once our algorithm is understood in this way, comparisons can be made to some well-known phenomena in adaptive control. For example, the results of Rohrs, et. al. [16] and Ioannou and Kokotović [9] suggest several ways in which such a system might become unstable, even though the slow system is globally convergent and the fast dynamics are well damped! These so-called "instability mechanisms" include:

- 1) Reference trajectories which are "too fast." In other words, if the bandwidth of the reference trajectory is in the same frequency range as the joint resonance, this resonance can be excited and drive the system unstable.
- 2) Parameter drift. The estimated parameters do not necessarily converge to their true values even in the rigid robot case without persistency of excitation conditions on the reference signal. However, it can be shown for rigid robots that the parameter errors are bounded [19]. In the presence of unmodeled dynamics, or in the presence of external disturbances, the parameters can drift along an equilibrium manifold until an instability results [15].
- 3) High Gain instability. This type of instability, when the controller gains are too high, is actually due to the loss of passivity of the flexible joint robot dynamics and can occur even for non-adaptive algorithms [1].

For these reasons, any composite control strategy for flexible joint robots is not likely to be globally convergent independent of the joint stiffness and/or the reference trajectory. In this report we show the following:

- For Point-to-Point motion, i.e., for tracking constant reference inputs, there exists a range of joint stiffness for which the parameter error is bounded and the equilibrium solution is locally asymptotically stable with respect to the tracking error. The stability region is precisely quantified.
- For arbitrary trajectories we give sufficient conditions guaranteeing stability and show that the tracking error converges to a residual set, which we quantify. For a special class of trajectories, including step responses generated from a reference model and joint interpolated trajectories we show that the tracking error converges to zero. This result is slightly stronger than existing results in the literature on adaptive tracking of nonlinear systems and comes about by exploiting the particular structure of robot dynamics. To achieve this, however, the parameter update law of [19] must be modified by the σ-modification scheme of [9] and [10].

Our method of proof is based on the composite Lyapunov theory presented in [17] and our results are similar to the adaptive feedback linearization results in [25]. Our tracking results, in fact, can be thought of as extending the results of [25] from the regulation problem to the tracking problem. The extension is non-trivial and exploits particular nature of robot dynamics and the robot tracking problem.

The report is organized as follows. After defining the notation and terminology in Section 3, we detail the modeling of our system in Section 4. Section 5 gives the detailed derivation of the composite Lyapunov theory applied to our class of systems. Section 6 uses these Lyapunov calculations to derive regulation results, i.e. Point–to–Point motion, while Section 7 presents our results on tracking. Finally, some conclusions are drawn in Section 8.

3 Notation and Terminology

In what follows, we use the following standard notation and terminology [4]: $\mathbf{R}_+$ will denote the set of nonnegative real numbers, and $\mathbf{R}^n$ will denote the usual n-dimensional vector space over $\mathbf{R}$ endowed with the Euclidean norm

$$\|\mathbf{x}\|_2 = \left\{ \sum_{i=1}^{n} |x_i|^2 \right\}^2 . \tag{2}$$

$\mathbf{R}^{n\times n}$ denotes the set of all $n \times n$ matrices with real elements. For each matrix $A \in \mathbf{R}^{n\times n}$, we define the induced matrix norm of A corresponding to the Euclidean vector norm

$$\|A\|_{i2} = \left\{ \lambda_{\max}\left(A^T A\right) \right\}^{\frac{1}{2}}, \tag{3}$$

where $\lambda_{\max}\left(A^T A\right)$ is the maximum eigenvalue of $A^T A$. We define the standard Lebesgue spaces $\mathbf{L}_\infty$ and $\mathbf{L}_2$ as

$$\mathbf{L}_\infty^n(\mathbf{R}_+) = \{ f : \mathbf{R}_+ \longrightarrow \mathbf{R}^n \ \textit{such that } f \textit{ is Lebesgue measurable and } \|f\|_\infty < \infty \} \tag{4}$$

where the $\mathbf{L}_\infty^n$-norm, $\|f\|_\infty$, is defined by

$$\|f\|_\infty = ess \sup_{t\in[0,\infty)} \|f(t)\|, \tag{5}$$

$$\mathbf{L}_2^n(\mathbf{R}_+) = \{ f : \mathbf{R}_+ \longrightarrow \mathbf{R}^n \ \textit{such that } f \textit{ is Lebesgue measurable and } \|f\|_2 < \infty \} \tag{6}$$

where the $\mathbf{L}_2^n$-norm, $\|f\|_2$, is defined by

$$\|f\|_2 = \int_0^\infty \|f(t)\|^2 \, dt. \tag{7}$$

Denote $\mathbf{B_x} \subset \mathbf{R}^{2n}$, $\mathbf{B}_{\boldsymbol{\theta}} \subset \mathbf{R}^r$, $\mathbf{B_y} \subset \mathbf{R}^{2n}$ the closed balls centered at $\mathbf{x} = 0$, $\tilde{\boldsymbol{\theta}} = 0$, and $\mathbf{y} = 0$ respectively, and let

$$\mathbf{B} = \mathbf{B_x} \times \mathbf{B}_{\boldsymbol{\theta}} \times \mathbf{B_y} \subset \mathbf{R}^{2n} \times \mathbf{R}^r \times \mathbf{R}^{2n}. \tag{8}$$

4 Singular Perturbation Model

The dynamic equations of a flexible joint manipulator are given by [20]

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) + K(\mathbf{q}_1 - \mathbf{q}_2) = 0 \tag{9}$$

$$J\ddot{\mathbf{q}}_2 - K(\mathbf{q}_1 - \mathbf{q}_2) = \mathbf{u}_c, \tag{10}$$

where the vectors $\mathbf{q}_1 \in \mathbf{R}^n$ and $\mathbf{q}_2 \in \mathbf{R}^n$ represent the link angles and motor angles, respectively, $D(\mathbf{q}_1)$ is the $n \times n$ inertia matrix for the rigid links, J is a diagonal matrix of actuator inertias reflected to the link side of the gears, $C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1$ represents the Coriolis and Centrifugal terms, $\mathbf{g}(\mathbf{q}_1)$ represents the gravitational terms, and K is a diagonal matrix representing the joint stiffness. For notational simplicity we will assume that all joint stiffness constants are the same in which case K may be taken as a scalar. The composite control law $\mathbf{u}_c$ that we consider is given by [21]

$$\mathbf{u}_c = \mathbf{u}_s(\mathbf{q}_1, \dot{\mathbf{q}}_1, t) + \mathbf{u}_f(\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2), \tag{11}$$

where,

$$\mathbf{u}_f = K_v(\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2). \tag{12}$$

K_v is a constant diagonal matrix, and $\mathbf{u}_s$ is designed using the following rigid model, obtained by letting the joint stiffness K tend to infinity, [20]

$$(D(\mathbf{q}_1) + J)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s. \tag{13}$$

We define the variable

$$\mathbf{z} := K(\mathbf{q}_2 - \mathbf{q}_1), \tag{14}$$

and we assume that K is $O(1/\epsilon^2)$, and K_v is $O(1/\epsilon)$, so that we may write

$$K = \frac{1}{\epsilon^2}K_1 \quad ; \quad K_v = \frac{1}{\epsilon}K_2, \tag{15}$$

where K_1, K_2 are $O(1)$. By substituting the control law (11) into (9)-(10), and using (14)-(15), we obtain the singularly perturbed system [21]

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{z} \tag{16}$$

$$\epsilon^2 J\ddot{\mathbf{z}} + \epsilon K_2\dot{\mathbf{z}} + K_1\mathbf{z} = K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). \tag{17}$$

Let us now choose $\mathbf{u}_s$ as the adaptive control law of Slotine and Li [19] designed for the rigid system (13). We should point out at this point that any control law for rigid robots that provides global tracking can be used as part of our analysis. For example the adaptive inverse dynamics schemes of Craig, et. al. [3] and Middleton and Goodwin [13] could just as well have been used instead. We have chosen to illustrate our analysis using the algorithm of Slotine and Li because of its elegance and simplicity and because it does not require joint acceleration for its implementation. The whole adaptive system can therefore be written as

i) Plant:

$$\begin{aligned} D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) &= \mathbf{z} & (18)\\ \epsilon^2 J\ddot{\mathbf{z}} + \epsilon K_2 \dot{\mathbf{z}} + K_1 \mathbf{z} &= K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). & (19)\end{aligned}$$

ii) Controller (designed for the rigid plant (13)):

$$\mathbf{u}_s = (\hat{D}(\mathbf{q}_1) + \hat{J})\mathbf{a} + \hat{C}(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D \mathbf{r}, \tag{20}$$

where $\hat{D}$, $\hat{J}$, $\hat{C}$ and $\hat{\mathbf{g}}$ represent the terms in (13) with estimated values of the parameters, K_D is a diagonal matrix of positive gains,

$$\tilde{\mathbf{q}}_1 = \mathbf{q}_1 - \mathbf{q}^d, \ \mathbf{v} = \dot{\mathbf{q}}^d - \Lambda\tilde{\mathbf{q}}_1, \mathbf{r} = \dot{\mathbf{q}}_1 - \mathbf{v} = \dot{\tilde{\mathbf{q}}}_1 + \Lambda\tilde{\mathbf{q}}_1, \ \mathbf{a} = \dot{\mathbf{v}}. \tag{21}$$

Λ is a constant diagonal matrix, and $\mathbf{q}^d(t)$ is the reference trajectory which is at least three times continuously differentiable.

iii) Parameter Update Law:

$$\dot{\hat{\boldsymbol{\theta}}} = -\Gamma^{-1} Y^T(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v})\mathbf{r}, \tag{22}$$

where Γ is some symmetric, positive definite matrix, $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is the parameter error, and

$$(D(\mathbf{q}_1) + J)\mathbf{a} + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \mathbf{g}(\mathbf{q}_1) = Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v})\boldsymbol{\theta}. \tag{23}$$

$Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v})$ is an $n \times r$ matrix of known functions (regressor), and $\boldsymbol{\theta}$ is a r-dimensional vector of parameters.

The plant (18)-(19), the controller (20), and the parameter update law (22) are now transformed into a more suitable singularly perturbed form, namely, (see Appendix A for details)

$$\mathcal{S} \ : \ \begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{24}$$

or equivalently,

$$\mathcal{S} \ : \ \begin{cases} \dot{\mathbf{p}} = f(t,\mathbf{p},\mathbf{y}) = \begin{bmatrix} A_1 & \Phi \\ -\Gamma\varphi & 0_{n\times n} \end{bmatrix} \mathbf{p} + \begin{bmatrix} A_3 \\ 0_{n\times n} \end{bmatrix} \mathbf{y} \\ \epsilon\dot{\mathbf{y}} = g(t,\mathbf{p},\mathbf{y},\epsilon) = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{25}$$

where

- $\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix} = \mathcal{T} \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix} \in \mathbf{R}^{2n}$, with the nonsingular linear transformation $\mathcal{T}$ (26)

$$\mathcal{T} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} \\ \Lambda & I_{n\times n} \end{bmatrix}, \quad (27)$$

- $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \in \mathbf{R}^r$ and $(D(\mathbf{q}_1) + J)\mathbf{a} + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \mathbf{g}(\mathbf{q}_1) = Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v})\boldsymbol{\theta}$, (28)

- $\mathbf{p} = \begin{bmatrix} \mathbf{x} \\ \tilde{\boldsymbol{\theta}} \end{bmatrix} \in \mathbf{R}^{2n+r}$, (29)

- $A_1 = A_1(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d) = \begin{bmatrix} -\Lambda & I_{n\times n} \\ -M(\mathbf{q}_1)^{-1}[C(\mathbf{q}_1, \dot{\mathbf{q}}_1) + K_D] & 0_{n\times n} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}$, (30)

- $M(\mathbf{q}_1) = D(\mathbf{q}_1) + J$, (31)

- $\Phi = \Phi(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d) = \begin{bmatrix} 0_{n\times r} \\ M(\mathbf{q}_1)^{-1} Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a}) \end{bmatrix} \in \mathbf{R}^{2n\times r}$, (32)

- $A_3 = A_3(\mathbf{x}, \mathbf{q}_d) = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ M(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}$, (33)

- $\Gamma \in \mathbf{R}^{r\times r}$ is some symmetric positive definite matrix, (34)

- $\varphi = \varphi(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d) = \begin{bmatrix} 0_{r\times n} & Y^T(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v}) \end{bmatrix} \in \mathbf{R}^{r\times 2n}$, (35)

- $A_2 = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -J^{-1}K_1 & -J^{-1}K_2 \end{bmatrix} \in \mathbf{R}^{2n\times 2n}$, (36)

- $B_2 = \begin{bmatrix} 0_{n\times n} \\ J^{-1}K_1 \end{bmatrix} \in \mathbf{R}^{2n\times n}$, (37)

- $\mathbf{u} = \mathbf{u}_s - J\ddot{\mathbf{q}}_1$, (38)

- $\mathbf{y} = \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + A_2^{-1}B_2\mathbf{u} \quad \in \mathbf{R}^{2n}$ where $\mathbf{z} = K(\mathbf{q}_2 - \mathbf{q}_1) = \frac{1}{\epsilon^2}K_1(\mathbf{q}_2 - \mathbf{q}_1)$. (39)

5 Analysis of the Singularly Perturbed System $\mathcal{S}$

System $\mathcal{S}$ is a nonautonomous nonlinear singularly perturbed system in the standard form [12]. $\mathbf{p}$ is the slow variable, and $\mathbf{y}$ is the fast variable. The analysis of system $\mathcal{S}$ follows the techniques of Composite Lyapunov Functions for nonlinear singularly perturbed systems developed in [17]; see also [12].

5.1 Boundary Layer System

The boundary layer system, denoted $\mathcal{S}_{\mathrm{b}}$, is defined as

$$\mathcal{S}_{\mathrm{b}} \; : \; \frac{d\mathbf{y}}{d\tau} = g(t, \mathbf{p}, \mathbf{y}(\tau), \epsilon = 0) = A_2 \mathbf{y}, \tag{40}$$

where $\tau = t/\epsilon$ is a stretching time scale. A_2, given by (36), is a constant Hurwitz matrix. Let P be the symmetric positive definite matrix that satisfies the Lyapunov Equation

$$A_2^T P \; + \; P A_2 \; = \; -Q, \tag{41}$$

where Q is a positive definite matrix. We choose the Lyapunov Function Candidate

$$W(\mathbf{y}) = \mathbf{y}^T P \mathbf{y}. \tag{42}$$

Then the time derivative of W along the solution trajectories of $\mathcal{S}_{\mathrm{b}}$ is obviously

$$\begin{aligned} \dot{W} = \frac{dW(\mathbf{y}(\tau))}{d\tau} \; &= \; [\nabla_{\mathbf{y}} W(\mathbf{y})]^T g(t, \mathbf{p}, \mathbf{y}, \epsilon = 0) \\ &= \; -\mathbf{y}^T Q \mathbf{y} \\ &\leq \; -\lambda_{\min}[Q] \; \|\mathbf{y}\|_2^2 . \end{aligned} \tag{43}$$

Hence, the time derivative of the Lyapunov Function W along the solution trajectories of the boundary layer system $\mathcal{S}_{\mathrm{b}}$ satisfies

$$\left[\begin{array}{l} \forall \mathbf{p} \in \mathbf{R}^{2n+r}, \;\; \forall \mathbf{y} \in \mathbf{R}^{2n}, \;\; \forall t \in \mathbf{R}_+ \\ \dot{W} = \left[\nabla_{\mathbf{y}} W(\mathbf{y}) \right]^T g(t, \mathbf{p}, \mathbf{y}, \epsilon = 0) \;\; \leq -\alpha_2 \, \|\mathbf{y}\|_2^2 \\ \alpha_2 = \lambda_{\min}[Q] \;\;\; \text{with Q satisfying the Lyapunov equation (41).} \end{array} \right. \tag{44}$$

5.2 Reduced System

The reduced system is defined by setting $\epsilon = 0$ in $\mathcal{S}$, that is,

$$\dot{\mathbf{p}} \;=\; f(t, \mathbf{p}, \mathbf{y}) = \begin{bmatrix} A_1 & \Phi \\ -\Gamma\varphi & 0_{n\times n} \end{bmatrix} \mathbf{p} + \begin{bmatrix} A_3 \\ 0_{n\times n} \end{bmatrix} \mathbf{y} \tag{45}$$

$$0 \;=\; g(t, \mathbf{p}, \mathbf{y}, \epsilon = 0) = A_2 \mathbf{y}. \tag{46}$$

Since A_2 is invertible, the algebraic equation (46) has the unique root

$$\mathbf{y} = 0. \tag{47}$$

The reduced system, denoted $\mathcal{S}_{\mathrm{r}}$, is obtained by replacing (47) into (45)

$$\mathcal{S}_{\mathrm{r}} \; : \; \dot{\mathbf{p}} = f(t, \mathbf{p}, \mathbf{y} = 0) = \begin{bmatrix} A_1 & \Phi \\ -\Gamma\varphi & 0_{n\times n} \end{bmatrix} \mathbf{p}, \tag{48}$$

or equivalently,

$$\mathcal{S}_{\mathrm{r}}:\begin{cases}\dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}}\\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x}.\end{cases} \tag{49}$$

Fact 5.1 : The reduced system $\mathcal{S}_{\mathrm{r}}$ is equivalent to the adaptive rigid-joint system, that is,

$$\mathcal{S}_{\mathrm{r}} \;:\; \dot{\mathbf{p}} = \begin{bmatrix} A_1 & \Phi \\ -\Gamma\varphi & 0_{n\times n}\end{bmatrix}\mathbf{p} \quad\Longleftrightarrow$$

$$\begin{cases}(D(\mathbf{q}_1)+J)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s\\ \mathbf{u}_s = (\hat{D}(\mathbf{q}_1)+\hat{J})\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r}\\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\mathbf{r},\end{cases} \tag{50}$$

(see Appendix B.)

□

A consequence of Fact 5.1 is that we can use the same Lyapunov Function Candidate as that of the adaptive rigid-joint system [14], [23], namely,

$$\begin{aligned} V &= \frac{1}{2}\mathbf{r}^T M(\mathbf{q}_1)\mathbf{r} + \tilde{\mathbf{q}}_1^T\Lambda^T K_D\tilde{\mathbf{q}}_1 + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\tilde{\boldsymbol{\theta}}\\ &= V(\tilde{\mathbf{q}}_1,\mathbf{r},\tilde{\boldsymbol{\theta}}) = V(\mathbf{x},\tilde{\boldsymbol{\theta}}) = V(\mathbf{p})\\ &= \frac{1}{2}\mathbf{p}^T P_V\mathbf{p}, \end{aligned} \tag{51}$$

where

$$P_V = \begin{bmatrix} 2\Lambda^T K_D & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & M(\mathbf{q}_1) & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & \Gamma^{-1}\end{bmatrix}. \tag{52}$$

Fact 5.2 : The time derivative of V along the solution trajectories of $\mathcal{S}_{\mathrm{r}}$ is

$$\begin{aligned}\dot{V} = \frac{dV(\mathbf{p}(t))}{dt} &= [\nabla_{\mathbf{p}}V(\mathbf{p})]^T f(t,\mathbf{p},\mathbf{y}=0)\\ &= -\mathbf{x}^T R\mathbf{x}\\ &\leq -\lambda_{\min}[R]\,\|\mathbf{x}\|_2^2\end{aligned} \tag{53}$$

where

$$R = \begin{bmatrix} 2\Lambda^T K_D\Lambda & -\Lambda^T K_D\\ -K_D^T\Lambda & K_D\end{bmatrix} \quad \text{is positive definite.} \tag{54}$$

(see Appendix B.)

□

Hence, the time derivative of the Lyapunov Function V along the solution trajectories of the reduced system $\mathcal{S}_{\mathrm{r}}$ satisfies

$$\begin{bmatrix} \forall \mathbf{p} \in \mathbf{R}^{2n+r}, \;\; \forall \mathbf{y} \in \mathbf{R}^{2n}, \;\; \forall t \in \mathbf{R}_+ \\ \dot{V} = \Big[\nabla_{\mathbf{p}} V(\mathbf{p})\Big]^T f(t,\mathbf{p},\mathbf{y}=0) \;\; \leq -\alpha_1 \|\mathbf{x}\|_2^2 \\ \alpha_1 = \lambda_{\min}[R] > 0 \quad ; \quad R = \begin{bmatrix} 2\Lambda^T K_D \Lambda & -\Lambda^T K_D \\ -K_D^T \Lambda & K_D \end{bmatrix}. \end{bmatrix} \tag{55}$$

5.3 Interconnection Conditions

The first interconnection condition of interest involves the relationship between the slow part of the original system, $\dot{\mathbf{p}} = f(t,\mathbf{p},\mathbf{y})$, and the reduced system $\mathcal{S}_{\mathbf{r}}$. Precisely, we want to evaluate $[\nabla_{\mathbf{p}} V(\mathbf{p})]^T [f(t,\mathbf{p},\mathbf{y}) - f(t,\mathbf{p},\mathbf{y}=0)]$. Recall that

$$V(\mathbf{p}) = \frac{1}{2}\mathbf{p}^T P_V \mathbf{p}. \tag{56}$$

Consequently,

$$\begin{aligned} [\nabla_{\mathbf{p}} V(\mathbf{p})] &= \frac{1}{2}(P_V^T + P_V)\mathbf{p} \\ &= \frac{1}{2}\begin{bmatrix} \Lambda^T K_D + K_D^T \Lambda & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 2M(\mathbf{q}_1) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \Gamma^{-1} + (\Gamma^{-1})^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \tilde{\boldsymbol{\theta}} \end{bmatrix}. \end{aligned} \tag{57}$$

From the expressions of $f(t,\mathbf{p},\mathbf{y})$ (see system $\mathcal{S}$), and $f(t,\mathbf{p},\mathbf{y}=0)$ (see system $\mathcal{S}_{\mathbf{r}}$), we get

$$\begin{aligned} f(t,\mathbf{p},\mathbf{y}) - f(t,\mathbf{p},\mathbf{y}=0) &= \begin{bmatrix} A_1 & \Phi \\ \Gamma\varphi & 0_{n\times n} \end{bmatrix}\mathbf{p} + \begin{bmatrix} A_3 \\ 0_{n\times n} \end{bmatrix}\mathbf{y} - \begin{bmatrix} A_1 & \Phi \\ \Gamma\varphi & 0_{n\times n} \end{bmatrix}\mathbf{p} \\ &= \begin{bmatrix} A_3 \\ 0_{n\times n} \end{bmatrix}\mathbf{y} \\ &= \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ M(\mathbf{q}_1)^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}\mathbf{y}. \end{aligned} \tag{58}$$

Combining (57) and (58), we obtain

$$\begin{aligned} &[\nabla_{\mathbf{p}} V(\mathbf{p})]^T [f(t,\mathbf{p},\mathbf{y}) - f(t,\mathbf{p},\mathbf{y}=0)] \\ &= \frac{1}{2}\begin{bmatrix} \mathbf{x}^T & \tilde{\boldsymbol{\theta}}^T \end{bmatrix} \begin{bmatrix} \Lambda^T K_D + K_D^T \Lambda & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 2M(\mathbf{q}_1) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \Gamma^{-1} + (\Gamma^{-1})^T \end{bmatrix} \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ M(\mathbf{q}_1)^{-1} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}\mathbf{y} \\ &= \begin{bmatrix} \mathbf{x}^T & \tilde{\boldsymbol{\theta}}^T \end{bmatrix} \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}\mathbf{y} \\ &= \mathbf{x}^T \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix}\mathbf{y} \\ &= \mathbf{x}^T S \mathbf{y} \\ &\leq \|\mathbf{x}\|_2 \, \|S\|_{2i} \, \|\mathbf{y}\|_2 \, . \end{aligned} \tag{59}$$

Hence,

$$\left[\begin{array}{l} \forall \mathbf{p} \in \mathbf{R}^{2n+r}, \quad \forall \mathbf{y} \in \mathbf{R}^{2n}, \quad \forall t \in \mathbf{R}_+ \\ \left[\nabla_{\mathbf{p}} V(\mathbf{p})\right]^T [f(t,\mathbf{p},\mathbf{y}) - f(t,\mathbf{p},\mathbf{y}=0)] \;\; \leq \beta_1 \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \\ \beta_1 = \|S\|_{2i} = 1 \quad ; \quad S = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix}. \end{array}\right. \tag{60}$$

The second interconnection condition of interest involves the relationship between the fast part of the original system, $\epsilon\dot{\mathbf{y}} = g(t,\mathbf{p},\mathbf{y},\epsilon)$, and the boundary layer system $\mathcal{S}_\mathbf{b}$. Precisely, we want to evaluate $[\nabla_\mathbf{y} W(\mathbf{y})]^T[g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)]$. Direct substitution gives

$$\begin{aligned} &[\nabla_\mathbf{y} W(\mathbf{y})]^T[g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)] \\ &\qquad = [2P\mathbf{y}]^T[A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}} - A_2\mathbf{y}] \\ &\qquad = 2\epsilon\mathbf{y}^T P A_2^{-1} B_2 \dot{\mathbf{u}}. \end{aligned} \tag{61}$$

Recall from Appendix A, equation (281), that $B_2 = A_2 \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix}$. Hence, (61) becomes

$$\begin{aligned} &[\nabla_\mathbf{y} W(\mathbf{y})]^T[g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)] \\ &\qquad = 2\epsilon\mathbf{y}^T P A_2^{-1} B_2 \dot{\mathbf{u}} \\ &\qquad = 2\epsilon\mathbf{y}^T P A_2^{-1} A_2 \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \dot{\mathbf{u}} \\ &\qquad = 2\epsilon\mathbf{y}^T P \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \dot{\mathbf{u}} \\ &\qquad \leq 2\epsilon \|\mathbf{y}\|_2 \|P\|_{2i} \left\| \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \right\|_{2i} \|\dot{\mathbf{u}}\|_2, \end{aligned} \tag{62}$$

and therefore,

$$[\nabla_\mathbf{y} W(\mathbf{y})]^T[g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)] \;\leq\; 2\epsilon \|\mathbf{y}\|_2 \|P\|_{2i} \|\dot{\mathbf{u}}\|_2 \,. \tag{63}$$

Fact 5.3 : let

$$F \;:=\; I_{n\times n} + JD(\mathbf{q}_1)^{-1}, \tag{64}$$

$$\rho(t) \;:=\; \frac{\partial \mathbf{u}}{\partial \mathbf{q}_d}\dot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \dot{\mathbf{q}}_d}\ddot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \ddot{\mathbf{q}}_d}\mathbf{q}_d^{(3)}, \tag{65}$$

then,

$$\begin{aligned} \mathbf{u} &= -F^{-1}\begin{bmatrix} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix}\mathbf{y} + F^{-1}\mathbf{u}_s + F^{-1}JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad + F^{-1}JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1) \\ &= \mathbf{u}(\mathbf{x},\mathbf{y},\tilde{\boldsymbol{\theta}},\mathbf{q}_d,\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d), \end{aligned} \tag{66}$$

and,

$$\dot{\mathbf{u}} = F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} - F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi\mathbf{x} + F\rho(t). \tag{67}$$

(see Appendix B.)

□

For $\forall(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$, assume the following

- (a1)

$$\left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}A_3\mathbf{y} + \frac{1}{\epsilon}F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} \right\|_2 \leq (k_3 + \frac{1}{\epsilon}k_2)\left\|\mathbf{y}\right\|_2, \tag{68}$$

where

$$k_3 = \sup_{\mathbf{B}} \left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}A_3 \right\|_{i2}, \tag{69}$$

$$\begin{aligned} k_2 &= \sup_{\mathbf{B}} \left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2 \right\|_{i2} = \sup_{\mathbf{B}} \left\| F\left\{ -F^{-1} \left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{array} \right] \right\} A_2 \right\|_{i2} \\ &= \sup_{\mathbf{B}} \left\| \left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{array} \right] A_2 \right\|_{i2}. \end{aligned} \tag{70}$$

- (a2)

$$\left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\Phi\tilde{\boldsymbol{\theta}} \right\|_2 \leq k_{01}\left\|\mathbf{x}\right\|_2, \tag{71}$$

$$\left\| F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi\mathbf{x} \right\|_2 \leq k_{02}\left\|\mathbf{x}\right\|_2, \tag{72}$$

$$\left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}A_1\mathbf{x} \right\|_2 \leq k_{03}\left\|\mathbf{x}\right\|_2, \tag{73}$$

and hence,

$$\left\| \left\{ F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}A_1 - F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi \right\}\mathbf{x} + F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\Phi\tilde{\boldsymbol{\theta}} \right\|_2 \leq k_1\left\|\mathbf{x}\right\|_2, \tag{74}$$

where

$$k_1 = k_{01} + k_{02} + k_{03}. \tag{75}$$

- (a3)

$$\left\| F\rho(t) \right\| \leq k_4(t). \tag{76}$$

Note that the existence of the various constants k_i in the above estimates requires only continuity of the functions involved since the set $\mathbf{B}$ is compact. Under the above assumptions (a1)-(a3), we conclude that $\forall\ (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$

$$\left\|\dot{\mathbf{u}}\right\| \leq k_1\left\|\mathbf{x}\right\|_2 + (k_3 + \frac{1}{\epsilon}k_2)\left\|\mathbf{y}\right\|_2 + k_4(t). \tag{77}$$

Combining (63) and (77), we therefore conclude that

$$
\begin{aligned}
[\nabla_{\mathbf{y}} W(\mathbf{y})]^T[g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)] &\le 2\epsilon \|\mathbf{y}\|_2 \|P\|_{i2} \|\dot{\mathbf{u}}\|_2 \\
&\le 2\epsilon \|\mathbf{y}\|_2 \|P\|_{i2} \left\{ k_1 \|\mathbf{x}\|_2 + (k_3 + \frac{1}{\epsilon}k_2) \|\mathbf{y}\|_2 + k_4(t) \right\} \\
&\le \epsilon \left\{ 2 \|P\|_{i2} (k_3 + \frac{1}{\epsilon}k_2) \right\} \|\mathbf{y}\|_2^2 + 2k_1\epsilon \|P\|_{i2} \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 + 2\epsilon k_4(t) \|\mathbf{y}\|_2 \|P\|_{i2} \,. \qquad (78)
\end{aligned}
$$

Hence,

$$
\left[
\begin{array}{l}
\forall \mathbf{p} \in \mathbf{B_x} \times \mathbf{B}_{\boldsymbol{\theta}}, \ \forall \mathbf{y} \in \mathbf{B_y}, \ \forall t \in \mathbf{R}_+ \\
\quad \left[\nabla_{\mathbf{y}} W(\mathbf{y})\right]^T [g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)] \\
\qquad\qquad\qquad \le \epsilon(\gamma_1' + \tfrac{1}{\epsilon}\gamma_2) \|\mathbf{y}\|_2^2 + \epsilon\beta_2 \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 + \epsilon\mu(t) \|\mathbf{y}\|_2 \\
\gamma_1' = 2\|P\|_{i2} k_3 \\
\gamma_2 = 2\|P\|_{i2} k_2 \\
\beta_2 = 2\|P\|_{i2} k_1 \\
\mu(t) = 2\|P\|_{i2} k_4(t) \\
\text{P satisfies the Lyapunov equation (41)} \\
k_1, k_2, k_3, \text{ and } k_4(t) \text{ are given by (a1) - (a3).}
\end{array}
\right. \qquad (79)
$$

Remark 5.4: From the analysis of the reduced and boundary layer systems, it is clear that their domains of attraction are $\mathbf{R}^{2n+r}$ and $\mathbf{R}^{2n}$ respectively. Let

$$\Omega_r = \{\mathbf{p} \in \mathbf{B_x} \times \mathbf{B}_{\boldsymbol{\theta}} : V(\mathbf{p}) \le c_r\} \qquad (80)$$

be in the domain of attraction of the reduced subsystem $\mathcal{S}_\mathrm{r}$ and c_r is the largest constant such that Ω_r is contained in $\mathbf{B_x} \times \mathbf{B}_{\boldsymbol{\theta}}$. Similarly, let

$$\Omega_b = \{\mathbf{y} \in \mathbf{B_y} : W(\mathbf{y}) \le c_b\} \qquad (81)$$

be in the domain of attraction of the boundary layer subsystem $\mathcal{S}_\mathrm{b}$ and c_b be the largest constant such that Ω_b is contained in $\mathbf{B_y}$.

5.4 Composite Lyapunov Function for the Singularly Perturbed System S

Consider the following Composite Lyapunov Function Candidate for the singularly perturbed system $\mathcal{S}$

$$\mathcal{V}(\mathbf{p},\mathbf{y}) = (1-d)V(\mathbf{p}) + dW(\mathbf{y}) \quad , \quad 0 < d < 1, \qquad (82)$$

which represents a weighted sum of $V(\mathbf{p})$, the Lyapunov Function of the reduced system $\mathcal{S}_\mathrm{r}$, and $W(\mathbf{y})$, the Lyapunov Function of the boundary layer system $\mathcal{S}_\mathrm{b}$. The derivative of $\mathcal{V}$ along the solution trajectories of $\mathcal{S}$ is

$$\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) = (1-d)\left\{[\nabla_{\mathbf{p}} V(\mathbf{p})]^T f(t,\mathbf{p},\mathbf{y})\right\} + \frac{d}{\epsilon}\left\{[\nabla_{\mathbf{y}} W(\mathbf{y})]^T g(t,\mathbf{p},\mathbf{y},\epsilon)\right\}$$

$$
\begin{aligned}
&= (1-d)\left\{[\nabla_{\mathbf{p}} V(\mathbf{p})]^T f(t,\mathbf{p},\mathbf{y}=0)\right\} \\
&\quad +(1-d)\left\{[\nabla_{\mathbf{p}} V(\mathbf{p})]^T [f(t,\mathbf{p},\mathbf{y}) - f(t,\mathbf{p},\mathbf{y}=0)]\right\} \\
&\quad +\frac{d}{\epsilon}\left\{[\nabla_{\mathbf{y}} W(\mathbf{y})]^T g(t,\mathbf{p},\mathbf{y},\epsilon=0)\right\} \\
&\quad +\frac{d}{\epsilon}\left\{[\nabla_{\mathbf{y}} W(\mathbf{y})]^T [g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)]\right\}.
\end{aligned} \tag{83}
$$

We now substitute equations (44),(55), (60), and (79)

$$
\begin{aligned}
\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\leq -(1-d)\alpha_1 \|\mathbf{x}\|_2^2 + (1-d)\beta_1 \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 - \frac{d}{\epsilon}\alpha_2 \|\mathbf{y}\|_2^2 \\
&\quad +\frac{d}{\epsilon}\left\{\epsilon(\gamma_1' + \frac{1}{\epsilon}\gamma_2)\|\mathbf{y}\|_2^2 + \epsilon\beta_2 \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 + \epsilon\mu(t)\|\mathbf{y}\|_2\right\} \\
&= -(1-d)\alpha_1 \|\mathbf{x}\|_2^2 + \{(1-d)\beta_1 + d\beta_2\}\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \\
&\quad + \left\{-\frac{d}{\epsilon}\alpha_2 + d(\gamma_1' + \frac{1}{\epsilon}\gamma_2)\right\}\|\mathbf{y}\|_2^2 + d\mu(t)\|\mathbf{y}\|_2 \\
&= -(1-d)\alpha_1 \|\mathbf{x}\|_2^2 + \{(1-d)\beta_1 + d\beta_2\}\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \\
&\quad - \left\{\frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1'\right\}\|\mathbf{y}\|_2^2 + d\mu(t)\|\mathbf{y}\|_2 .
\end{aligned} \tag{84}
$$

Let

$$
\begin{aligned}
\Pi^2 &= \left(\sqrt{d}\mu(t) - \frac{1}{2}\sqrt{d}\|\mathbf{y}\|_2\right)^2 \\
&= d\mu^2(t) + \frac{1}{4}d\|\mathbf{y}\|_2^2 - d\mu(t)\|\mathbf{y}\|_2 .
\end{aligned} \tag{85}
$$

Hence,

$$
\begin{aligned}
d\mu(t)\|\mathbf{y}\|_2 &= -\Pi^2 + d\mu^2(t) + \frac{1}{4}d\|\mathbf{y}\|_2^2 \\
&\leq d\mu^2(t) + \frac{1}{4}d\|\mathbf{y}\|_2^2 ,
\end{aligned} \tag{86}
$$

and therefore,

$$
\begin{aligned}
\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\leq -(1-d)\alpha_1 \|\mathbf{x}\|_2^2 + \{(1-d)\beta_1 + d\beta_2\}\|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \\
&\quad - \left\{\frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d(\gamma_1' + \frac{1}{4})\right\}\|\mathbf{y}\|_2^2 + d\mu^2(t).
\end{aligned} \tag{87}
$$

Define

$$
\gamma_1 := \gamma_1' + \frac{1}{4}. \tag{88}
$$

Equation (87) is written

$$
\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) \leq -\begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} + d\mu^2(t), \tag{89}
$$

where,

$$P_d = \begin{bmatrix} (1-d)\alpha_1 & -\frac{(1-d)\beta_1 + d\beta_2}{2} \\ -\frac{(1-d)\beta_1 + d\beta_2}{2} & \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1 \end{bmatrix}. \tag{90}$$

From the expression of $\dot{\mathcal{V}}$, we observe that the right hand side of (89) consists of a quadratic expression and the term $d\mu^2(t)$. First of all note that the quadratic term does not include the state $\tilde{\boldsymbol{\theta}}$. Also, P_d can be made positive definite for some range of ϵ. It will be shown next that $\mu(t)$ is zero if the desired trajectory is a constant vector. Consequently, we can obtain regulation results. In the section that follows these regulation results, we consider arbitrary trajectories and give sufficient conditions guaranteeing stability and show that the tracking error converges to a residual set, which we quantify. To achieve this, however, the parameter update law of [19] must be modified by the σ-modification scheme of [9] and [10].

Recall from assumption (a3) that

$$\|F\rho(t)\|_2 \leq k_4(t), \tag{91}$$

where

$$F = I_{n\times n} + JD(\mathbf{q}_1)^{-1}, \tag{92}$$

$$\rho(t) = \frac{\partial \mathbf{u}}{\partial \mathbf{q}_d}\dot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \dot{\mathbf{q}}_d}\ddot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \ddot{\mathbf{q}}_d}\mathbf{q}_d^{(3)}, \tag{93}$$

and

$$\mu(t) = 2\,\|P\|_{i2}\, k_4(t). \tag{94}$$

Note that F is a bounded function since $D(\mathbf{q}_1)$ and $D(\mathbf{q}_1)^{-1}$ are bounded matrices for all $\mathbf{q}_1$. Three important cases on the nature of $\mu(t)$ are of special interest:

1. **Case 1:** In the regulation problem, the desired trajectory $\mathbf{q}_d$ is a constant vector and hence all higher derivatives of $\mathbf{q}_d$ are zero and $\mathcal{S}$ becomes a time invariant system. All the equations derived earlier hold except for two differences. First, none of the terms is an explicit function of time any more. Second, $\mu^2(t)$ in (89) is zero as is clear from (91)-(94).

2. **Case 2:** If the desired trajectory is three times continuously differentiable with bounded derivatives, so that $\dot{\mathbf{q}}_d$, $\ddot{\mathbf{q}}_d$, $\mathbf{q}_d^{(3)} \in \mathbf{L}_\infty^n$, then $\forall(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$, equation (93) implies that $\rho(t)$, and hence $\mu(t)$ is a bounded function of time ($\mu(t) \in \mathbf{L}_\infty$). So $\exists\ \bar{\mu}$ a positive real constant such that $\mu(t) \leq \bar{\mu}\quad \forall t \in \mathbf{R}_+$.

3. **Case 3:** If $\dot{\mathbf{q}}_d$, $\ddot{\mathbf{q}}_d$, $\mathbf{q}_d^{(3)} \in \mathbf{L}_2^n \cap \mathbf{L}_\infty^n$, then $\forall\ (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$, equation (93) implies that $\rho(t)$, and hence $\mu(t) \in \mathbf{L}_2^n \cap \mathbf{L}_\infty^n$, and furthermore $\lim_{t\longrightarrow\infty} \dot{\mathbf{q}}_d = 0$, and $\lim_{t\longrightarrow\infty} \ddot{\mathbf{q}}_d = 0$. For example, the class of bounded desired trajectories which are eventually constant fits into this category.

6 Regulation Analysis

Since $\mu(t) = 0$ as discussed in Case 1 above, the time derivative of the Lyapunov Function Candidate $\mathcal{V}$ along the solution trajectories of $\mathcal{S}$ is simply given by

$$\dot{\mathcal{V}}(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \leq -\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right], \tag{95}$$

where

$$P_d = \left[\begin{array}{cc} (1-d)\alpha_1 & -\frac{(1-d)\beta_1+d\beta_2}{2} \\ -\frac{(1-d)\beta_1+d\beta_2}{2} & \frac{d}{\epsilon}(\alpha_2-\gamma_2)-d\gamma_1 \end{array}\right]. \tag{96}$$

It is emphasized that the constants α_1, α_2, β_1, β_2, γ_1, and γ_2 are the same as those in the previous section. It should be just kept in mind that all quantities used to derive them are time invariant. We have the following result

Theorem 1 (Regulation) *Assume*

1. *assumptions* (a1)-(a3) *are satisfied* $\forall(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \in \mathbf{B}$.
2. $\alpha_2 - \gamma_2 > 0$.

Then, the equilibrium $\mathbf{x}=0$, $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} = 0$, *and* $\mathbf{y}=0$ *of system* $\mathcal{S}$ *is stable for all* $\epsilon \in (0,\epsilon_d)$ *such that*

$$\epsilon_d = \frac{\alpha_1(\alpha_2-\gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1+d\beta_2]^2} \tag{97}$$

and an estimate of the domain of attraction is given by

$$\Omega_d = \left\{(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \in \mathbf{B} : \mathcal{V}(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \leq min\left[(1-d)c_r, dc_b\right]\right\}. \tag{98}$$

$\Omega_d \subset \mathbf{B}$ *and* c_r *and* c_b *are given by* (80) *and* (81) *respectively.*

Moreover, $\forall\left(\mathbf{x}(0),\tilde{\boldsymbol{\theta}}(0),\mathbf{y}(0)\right) \in \Omega_d$ *we get*

$$\lim_{t\to\infty} \mathbf{x}(t) = 0 \tag{99}$$

$$\lim_{t\to\infty} \mathbf{y}(t) = 0 \tag{100}$$

$$\lim_{t\to\infty} \dot{\tilde{\boldsymbol{\theta}}}(t) = 0. \tag{101}$$

In addition we have

- *the maximum value of* ϵ_d *occurs at* $d^* = \frac{\beta_1}{\beta_1+\beta_2}$ *and is given by*

$$\epsilon^* = \epsilon_{d=d^*} = \frac{\alpha_1(\alpha_2-\gamma_2)}{\alpha_1\gamma_1+\beta_1\beta_2}. \tag{102}$$

The corresponding estimate of the domain of attraction is given by

$$\Omega_{d^*} = \left\{(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \in \mathbf{B} : \mathcal{V}(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \leq min\left[(1-d^*)c_r, d^*c_b\right]\right\}, \tag{103}$$

- *the largest estimate of the domain of attraction occurs at $\bar{d} = \frac{c_r}{c_r + c_b}$ and is given by*

$$\Omega^* = \Omega_{\bar{d}} = \left\{(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B} : \frac{V(\mathbf{x}, \boldsymbol{\theta})}{c_r} + \frac{W(\mathbf{y})}{c_b} \leq 1\right\}. \tag{104}$$

The corresponding upper bound of ϵ is

$$\epsilon_{\bar{d}} = \epsilon_{d=\bar{d}} = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4\bar{d}(1-\bar{d})}[(1-\bar{d})\beta_1 + \bar{d}\beta_2]^2} \tag{105}$$

Proof of Theorem 1: The quadratic term in (95) is negative when P_d is positive definite, i.e. when

$$[(1-d)\alpha_1]\left[\frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1\right] > \frac{1}{4}\left[(1-d)\beta_1 + d\beta_2\right]^2 \tag{106}$$

$$\Longleftrightarrow$$

$$\frac{1}{\epsilon}d(1-d)\alpha_1(\alpha_2 - \gamma_2) - d(1-d)\alpha_1\gamma_1 > \frac{1}{4}\left[(1-d)\beta_1 + d\beta_2\right]^2 \tag{107}$$

$$\Longleftrightarrow$$

$$\frac{1}{\epsilon}d(1-d)\alpha_1(\alpha_2 - \gamma_2) > \frac{1}{4}\left[(1-d)\beta_1 + d\beta_2\right]^2 + d(1-d)\alpha_1\gamma_1 \tag{108}$$

$$\Longleftrightarrow$$

$$\frac{1}{\epsilon}\alpha_1(\alpha_2 - \gamma_2) > \frac{1}{4d(1-d)}\left[(1-d)\beta_1 + d\beta_2\right]^2 + \alpha_1\gamma_1 \tag{109}$$

$$\Longleftrightarrow$$

$$\epsilon < \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2]^2} =: \epsilon_d \tag{110}$$

Given c_r and c_b from (74), (75), respectively, an estimate of the domain of attraction of the singularly perturbed system $\mathcal{S}$ is given by

$$\Omega_d = \left\{(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B} : \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) = (1-d)V(\mathbf{x}, \tilde{\boldsymbol{\theta}}) + dW(\mathbf{y}) \leq min\left[(1-d)c_r, dc_b\right]\right\}. \tag{111}$$

Up to this point, we have $\mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y})$ is a locally positive definite function and $\dot{\mathcal{V}}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \leq 0$ $\forall \epsilon \in (0, \epsilon_d)$, $\forall (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \Omega_d$, and $\forall t \geq 0$. We conclude therefore that the equilibrium $\mathbf{x} = 0$, $\tilde{\boldsymbol{\theta}} = 0$, and $\mathbf{y} = 0$ of $\mathcal{S}$ is stable in the sense of Lyapunov.

To show (99)-(101), we now exploit the invariance theory for autonomous systems [27]. Let the invariant set $\mathcal{M}$ denote the subset of Ω_d defined by

$$\mathcal{M} = \left\{(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \Omega_d \; : \; \dot{\mathcal{V}}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) = 0\right\}. \tag{112}$$

It is clear from (95) that $\dot{\mathcal{V}}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) = 0$ at $\mathbf{x} = 0$, $\mathbf{y} = 0$, and $\forall \tilde{\boldsymbol{\theta}}$. To find the nature of $\tilde{\boldsymbol{\theta}}$ in this set, we replace $\mathbf{x} = 0$ and $\mathbf{y} = 0$ in the original full system $\mathcal{S}$ (24)

$$\left\{\begin{array}{l} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma^{-1}\varphi\mathbf{x} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}} \end{array}\right. \Bigg|_{\mathbf{x}=0, \mathbf{y}=0} = \left\{\begin{array}{l} 0 = \Phi\tilde{\boldsymbol{\theta}} \\ \dot{\tilde{\boldsymbol{\theta}}} = 0 \\ 0 = \epsilon A_2^{-1}B_2\dot{\mathbf{u}}. \end{array}\right. \tag{113}$$

After replacing the expression of $\dot{\mathbf{u}}$ from (67) we obtain

$$\left\{\begin{array}{l} 0 = \Phi\tilde{\boldsymbol{\theta}} \\ \dot{\tilde{\boldsymbol{\theta}}} = 0 \\ 0 = \epsilon A_2^{-1} B_2 F \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \Phi\tilde{\boldsymbol{\theta}} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \dot{\tilde{\boldsymbol{\theta}}} = 0 \\ \Phi\tilde{\boldsymbol{\theta}} = 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \dot{\tilde{\boldsymbol{\theta}}} = 0 \\ Y\tilde{\boldsymbol{\theta}} = 0. \end{array}\right. \tag{114}$$

Consequently,

$$\mathcal{M} = \left\{(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \Omega_d \,:\, \mathbf{x} = 0\,,\, \mathbf{y} = 0\,,\, \dot{\tilde{\boldsymbol{\theta}}} = 0, Y\tilde{\boldsymbol{\theta}} = 0\right\}, \tag{115}$$

and $\mathcal{M}$ is the largest invariant set of $\mathcal{S}$.

Note that Ω_d is bounded. In addition, $\mathcal{V}$ is bounded from below by zero over the set Ω_d, and $\dot{\mathcal{V}} \leq 0\ \forall(\mathbf{x}, \tilde{\boldsymbol{\theta}}\mathbf{y}) \in \Omega_d$. Hence any solution $(\mathbf{x}(t), \tilde{\boldsymbol{\theta}}(t), \mathbf{y}(t))$ of $\mathcal{S}$ starting from Ω_d remains in it. Furthermore, $\dot{\mathcal{V}} \longrightarrow 0$ as $t \longrightarrow \infty$ and $\mathcal{V} \longrightarrow v_0$, v_0 is a constant. From the invariance theory of autonomous systems, we conclude that solutions $(\mathbf{x}(t), \tilde{\boldsymbol{\theta}}(t), \mathbf{y}(t))$ of $\mathcal{S}$ starting from Ω_d converge to the invariant set $\mathcal{M}$, and hence, $\forall(\mathbf{x}(0), \tilde{\boldsymbol{\theta}}(0), \mathbf{y}(0)) \in \Omega_d$, we achieve $\lim_{t\longrightarrow\infty} \mathbf{x}(t) = 0$, $\lim_{t\longrightarrow\infty} \mathbf{y}(t) = 0$, $\lim_{t\longrightarrow\infty} \dot{\tilde{\boldsymbol{\theta}}}(t) = 0$, and $\lim_{t\longrightarrow\infty} Y\tilde{\boldsymbol{\theta}} = 0$.

Next, referring to Figure 1 we have that the value of d for which ϵ_d achieves its maximum can be determined by minimizing the numerator of (110) and can be easily verified (see [12]) to be

$$d^* = \frac{\beta_1}{\beta_1 + \beta_2}. \tag{116}$$

Therefore

$$\epsilon^* = \epsilon_{d=d^*} = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \beta_1\beta_2}. \tag{117}$$

The corresponding estimate of the domain of attraction is obtained by replacing d by d^* and is given by

$$\Omega_{d^*} = \Omega_{d=d^*} = \left\{(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B} : \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \leq min\left[(1 - d^*)c_r, d^* c_b\right]\right\}. \tag{118}$$

The largest estimate of the domain of attraction is obtained from (111) by maximizing $min\left[(1-d)c_r, dc_b\right]$. Clearly (see also[12]), this minimum is maximized if d is chosen as

$$d = \bar{d} = \frac{c_r}{c_r + c_b}, \tag{119}$$

for which

$$min\left[(1-\bar{d})c_r, \bar{d}c_b\right] = (1-\bar{d})c_r = \bar{d}c_b = \frac{c_r c_b}{c_r + c_b}. \tag{120}$$

Hence,

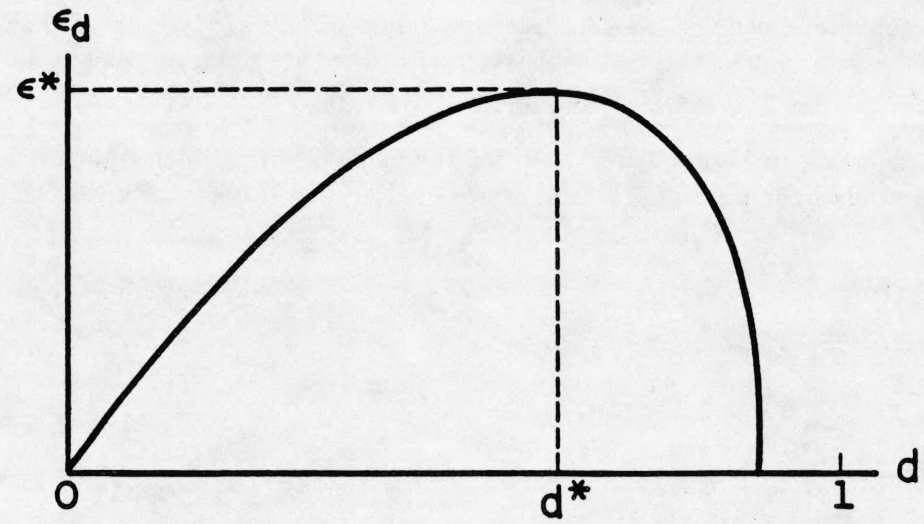

Figure 1: **Upper Bounds of ϵ**

$$\Omega^* = \Omega_{d=\bar{d}} = \left\{ (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B} : \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \le \frac{c_r c_b}{c_r + c_b} \right\}. \tag{121}$$

Note that for $d = \bar{d}$

$$\begin{aligned} \mathcal{V} &= (1-\bar{d})V + \bar{d}W \\ &= \frac{c_b}{c_r + c_b}V + \frac{c_r}{c_r + c_b}W. \end{aligned} \tag{122}$$

Therefore,

$$\mathcal{V} \le \frac{c_r c_b}{c_r + c_b} \iff \frac{c_b}{c_r + c_b}V + \frac{c_r}{c_r + c_b}W \le \frac{c_r c_b}{c_r + c_b} \iff \frac{V}{c_r} + \frac{W}{c_b} \le 1, \tag{123}$$

and (121) becomes

$$\Omega^* = \left\{ (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B} : \frac{V(\mathbf{x}, \tilde{\boldsymbol{\theta}})}{c_r} + \frac{W(\mathbf{y})}{c_b} \le 1 \right\}. \tag{124}$$

The corresponding upper bound of ϵ is

$$\epsilon_{\bar{d}} = \epsilon_{d=\bar{d}} = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4\bar{d}(1-\bar{d})}[(1-\bar{d})\beta_1 + \bar{d}\beta_2]^2}. \tag{125}$$

□

Remark 6.1 Note that since $\lim_{t\longrightarrow\infty} \mathbf{x} = 0$ and $\mathbf{x} = \mathcal{T}\begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix}$ with $\mathcal{T}$ being the nonsingular linear transformation given by (27), we conclude that $\lim_{t\longrightarrow\infty} \tilde{\mathbf{q}}_1 = 0$ and $\lim_{t\longrightarrow\infty} \dot{\tilde{\mathbf{q}}}_1 = 0$.

Remark 6.2 The results of Theorem 1 can be viewed from two converse directions. First, for a desired region of attraction Ω_d, the constants α_1, α_2, β_1, β_2, γ_1, and γ_2 limit the range of ϵ for which regulation is guaranteed. Therefore, the larger the desired region Ω_d is, the smaller the range of ϵ is and hence the larger the stiffness K of the joint must be to guarantee regulation. Second, for a given stiffness value K, that is ϵ, the region of attraction Ω_d for which all inequalities leading to α_1, α_2, β_1, β_2, γ_1, and γ_2 are satisfied is given by Ω_d satisfying (98).

Remark 6.3 One major assumption in Theorem 1 is that $\alpha_2 - \gamma_2 > 0$ so that ϵ_d remains positive. To determine under which conditions this assumption is satisfied, recall from (44) and (79) that

$$\alpha_2 = \lambda_{\min}[Q] \tag{126}$$

$$\begin{aligned} \gamma_2 &= 2\|P\|_{i2} k_2 \\ &= 2\|P\|_{i2} \sup_{\mathbf{B}} \left\| \begin{bmatrix} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} A_2 \right\|_{i2}, \end{aligned} \tag{127}$$

where P satisfies the Lyapunov Equation (41), namely,

$$A_2^T P + PA_2 = -Q. \tag{128}$$

From (128), it follows

$$\|Q\|_{i2} \le 2\|A\|_{i2}\|P\|_{i2}, \tag{129}$$

or equivalently,

$$\|Q\|_{i2} - 2\,\|A\|_{i2}\,\|P\|_{i2} \le 0. \tag{130}$$

For a special choice of Q, we can make $\alpha_2 = \lambda_{\min}[Q] = \|Q\|_{i2}$. Therefore, using (127),

$$\alpha_2 - \gamma_2 = \|Q\|_{i2} - 2\,\|P\|_{i2} \sup_{\mathbf{B}} \left\| \begin{bmatrix} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} A_2 \right\|_{i2}, \tag{131}$$

$$\alpha_2 - \gamma_2 \ge \|Q\|_{i2} - 2\,\|P\|_{i2}\,\|A_2\|_{i2} \sup_{\mathbf{B}} \left\| JD(\mathbf{q}_1)^{-1} \right\|_{i2}. \tag{132}$$

Consequently, we have the following conservative sufficient condition

$$\|Q\|_{i2} - 2\,\|P\|_{i2}\,\|A_2\|_{i2} \sup_{\mathbf{B}} \left\| JD(\mathbf{q}_1)^{-1} \right\|_{i2} > 0 \implies \alpha_2 - \gamma_2 > 0, \tag{133}$$

that is, even though (see (130)) $\|Q\|_{i2} - 2\,\|A\|_{i2}\,\|P\|_{i2} \le 0$, the term $\sup_{\mathbf{B}} \|JD(\mathbf{q}_1)^{-1}\|_{i2}$ has to be small enough to make $\|Q\|_{i2} - 2\,\|P\|_{i2}\,\|A_2\|_{i2} \sup_{\mathbf{B}} \|JD(\mathbf{q}_1)^{-1}\|_{i2} > 0$. This sufficient condition is implying that the relative "size" of J and $D(\mathbf{q}_1)$ is an important factor in the stability of the system. Simulation results for an experimental flexible joint system [1] ($J = 0.004, D = 0.031$) have shown that the system becomes unstable when $J > 1$. Nevertheless the above sufficient condition predicts that stability is insured only for values of $J < D = 0.031$. This shows that this sufficient condition is quite conservative.

Remark 6.4 The condition $\alpha_2 - \gamma_2 > 0$ occurs even if another adaptive control strategy $\mathbf{u}_{r*}$ is used instead of $\mathbf{u}_r$. The reason is that γ_2 originates from the expression $\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\dot{\mathbf{y}}$ which is independent from the choice of $\mathbf{u}_r$ (see Fact 5.3 and the derivations that follows.)

Remark 6.5 In terms of $\mathbf{q}_1$ and $\mathbf{q}_2$, the fact from Theorem 1 that $\lim_{t\to\infty} \mathbf{y}(t) = 0$ is interpreted as follows. Recall that

$$\begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\epsilon^2} K_1(\mathbf{q}_2 - \mathbf{q}_1) \\ \frac{1}{\epsilon} K_1(\dot{\mathbf{q}}_2 - \dot{\mathbf{q}}_1) \end{bmatrix}, \tag{134}$$

and

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + A_2^{-1} B_2 \mathbf{u} \\ &= \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + A_2^{-1} A_2 \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \mathbf{u} \\ &= \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} -\mathbf{u} \\ 0 \end{bmatrix}. \end{aligned} \tag{135}$$

So

$$\mathbf{y} = 0 \Longrightarrow \left\{ \begin{array}{l} \mathbf{z} = \mathbf{u} \\ \epsilon\dot{\mathbf{z}} = 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} \mathbf{q}_2 - \mathbf{q}_1 = \epsilon^2 K_1^{-1}\mathbf{u} = K^{-1}\mathbf{u} \\ \dot{\mathbf{q}}_2 - \dot{\mathbf{q}}_1 = 0, \end{array} \right. \tag{136}$$

[1] For a description of the system, see the section entitled "A Simulation Example" later in the report.

Since $\lim_{t\longrightarrow\infty} \mathbf{q}_1 = \mathbf{q}_d$, and $\lim_{t\longrightarrow\infty} \dot{\mathbf{q}}_1 = 0$,

$$\mathbf{y} = 0 \Longrightarrow \begin{cases} \mathbf{q}_2 = \mathbf{q}_d + K^{-1}\mathbf{u} \\ \dot{\mathbf{q}}_2 = 0. \end{cases} \tag{137}$$

Hence, in regulation, the equilibrium points $\mathbf{x}$ and $\mathbf{y}$ are independent of ϵ (i.e. $\lim_{t\longrightarrow\infty} \mathbf{q}_1 = \mathbf{q}_d$ for all $\epsilon < \epsilon_d$, and $\lim_{t\longrightarrow\infty} \mathbf{z} = \mathbf{u}$ for all $\epsilon < \epsilon_d$.) On the other hand, $\lim_{t\longrightarrow\infty} \mathbf{q}_2$ depends on ϵ.

7 Tracking Analysis

The regulation (set point tracking) results presented above are similar to the adaptive feedback linearization results in [25]. The tracking (of time varying signals) results presented below can be thought of as extending the results of [25] from the regulation problem to the tracking problem. The extension is nontrivial and exploits the particular nature of robot dynamics and robot tracking problem.

7.1 Non-robustness of Tracking

The purpose of this section is to show that based on the Lyapunov analysis presented so far, the tracking of time varying desired trajectories is not robust in the sense that signals are not guaranteed to remain in the domain $\mathbf{B}$. Specifically, it will be shown that parameter drift instability mechanism is not guaranteed to be stopped by the slow adaptive control law used so far. Recall from (89) that the Lyapunov function candidate for the singularly perturbed system $\mathcal{S}$ satisfies

$$\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) \leq -\begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} + d\mu^2(t), \tag{138}$$

where

$$P_d = \begin{bmatrix} (1-d)\alpha_1 & -\frac{(1-d)\beta_1 + d\beta_2}{2} \\ -\frac{(1-d)\beta_1 + d\beta_2}{2} & \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1 \end{bmatrix}. \tag{139}$$

As shown in the regulation case, P_d is positive definite for $\epsilon \in (0, \epsilon_d)$ where

$$\epsilon_d = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2]^2}. \tag{140}$$

In the case where $\mu(t)$ is bounded, (Case 2 and Case 3 above), (138) is written as

$$\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) \leq -\begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} + d\bar{\mu}^2. \tag{141}$$

Define the set

$$\mathcal{B} := \left\{ (\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2) : \ (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \ \in \ \mathbf{B} \right\} \subset \mathbf{R}^3_+. \tag{142}$$

$\mathcal{B}$ is a prism containing the origin and extending along the positive $\|\mathbf{x}\|_2$, $\left\|\tilde{\boldsymbol{\theta}}\right\|_2$, $\|\mathbf{y}\|_2$ axes. Figure 2 shows a three dimensional view and two side views of the prism $\mathcal{B}$. Now define the sets

$$\mathcal{D}_{\bar{\mu}} := \left\{ (\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2) \in \mathcal{B} : \ \left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array} \right] P_d \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array} \right] \leq d\bar{\mu}^2 \right\}, \tag{143}$$

and,

$$\begin{aligned} \mathcal{D}^c_{\bar{\mu}} \ &:= \ \left\{ (\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2) \in \mathcal{B} : \left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array} \right] P_d \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array} \right] > d\bar{\mu}^2 \right\} \\ &= \ \mathcal{B} \setminus \mathcal{D}_{\bar{\mu}}. \end{aligned} \tag{144}$$

The set $\mathcal{D}_{\bar{\mu}}$ is a subset of an elliptic cylinder enclosed in $\mathcal{B}$ extending along the $\left\|\tilde{\boldsymbol{\theta}}\right\|_2$ axis, and with an elliptic cross section in the $\|\mathbf{x}\|_2 - \|\mathbf{y}\|_2$ plane defined by

$$\bar{a}\,\|\mathbf{x}\|_2^2 + \bar{b}\,\|\mathbf{x}\|_2\,\|\mathbf{y}\|_2 + \bar{c}\,\|\mathbf{y}\|_2^2 \leq d\bar{\mu}^2. \tag{145}$$

where

$$\bar{a} \ = \ (1-d)\alpha_1 \tag{146}$$

$$\bar{b} \ = \ -\left\{(1-d)\beta_1 + d\beta_2\right\} \tag{147}$$

$$\bar{c} \ = \ \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1. \tag{148}$$

The axes of the ellipse, denoted $x' - y'$, are obtained by rotating the $\|\mathbf{x}\|_2$-$\|\mathbf{y}\|_2$ axes by an angle ϑ given by [2]

$$\cot(2\vartheta) = \frac{\bar{a} - \bar{c}}{\bar{b}}. \tag{149}$$

Equation (145) is written in the $x' - y'$ axes as

$$\bar{a}'(x')^2 + \bar{c}'(y')^2 \leq d\bar{\mu}^2, \tag{150}$$

or simply,

$$\frac{(x')^2}{\left(\frac{d\bar{\mu}^2}{\bar{a}'}\right)} + \frac{(y')^2}{\left(\frac{d\bar{\mu}^2}{\bar{c}'}\right)} \leq 1, \tag{151}$$

where

$$\bar{a}' \ = \ \bar{a}\cos^2(\vartheta) + \bar{b}\cos(\vartheta)\sin(\vartheta) + \bar{c}\sin^2(\vartheta) \tag{152}$$

$$\bar{c}' \ = \ \bar{a}\sin^2(\vartheta) - \bar{b}\sin(\vartheta)\cos(\vartheta) + \bar{c}\cos^2(\vartheta) \tag{153}$$

(see Figure 3 for a typical situation.) Hence, $\forall \ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \ \in \mathcal{D}_{\bar{\mu}}$, $\dot{\mathcal{V}}$ can be positive or negative, and $\forall \ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \ \in \mathcal{D}^c_{\bar{\mu}}$, $\dot{\mathcal{V}} \leq 0$.

We now define the set

[2] Consult any Calculus book, for example [26].

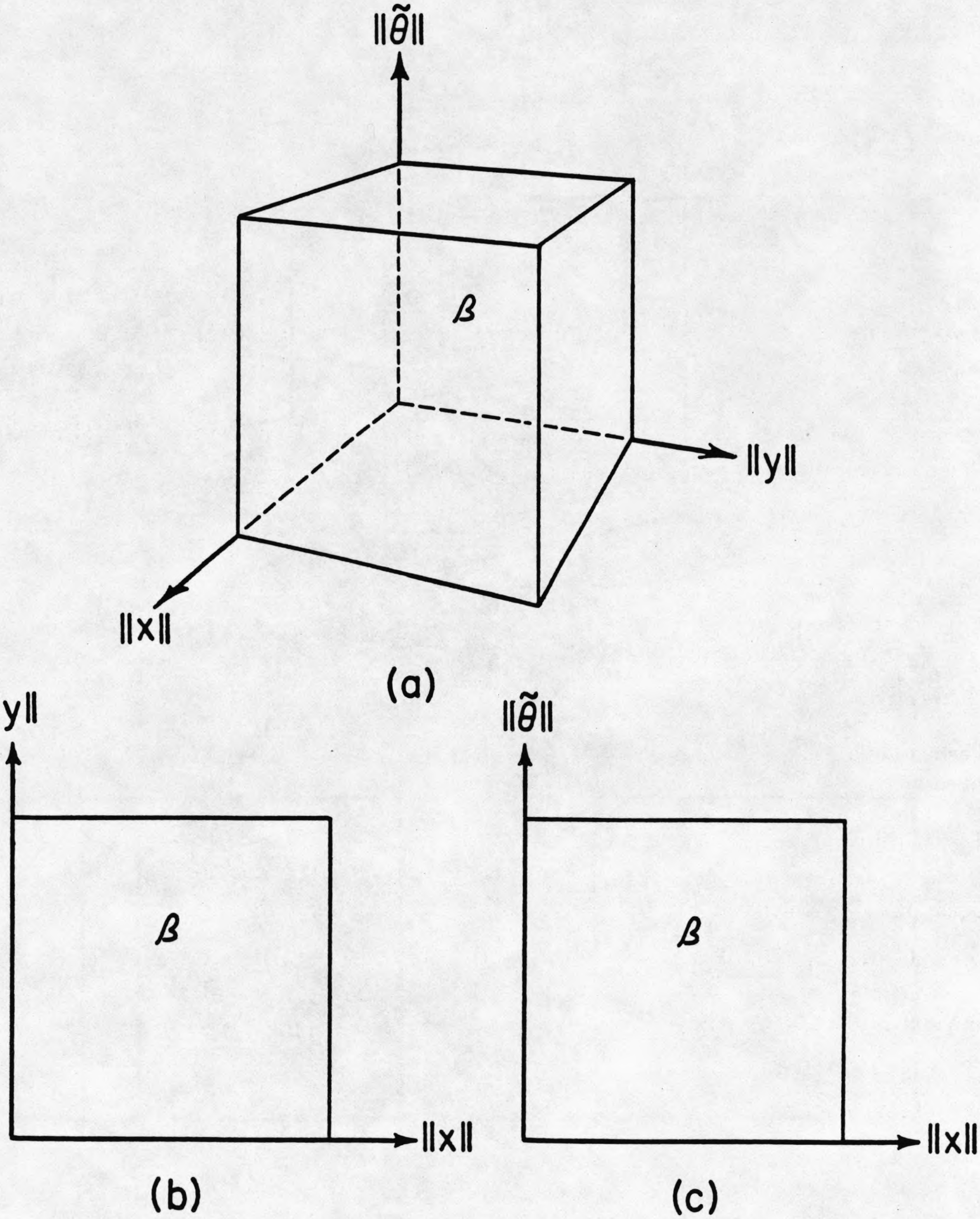

Figure 2: Set $\mathcal{B}$

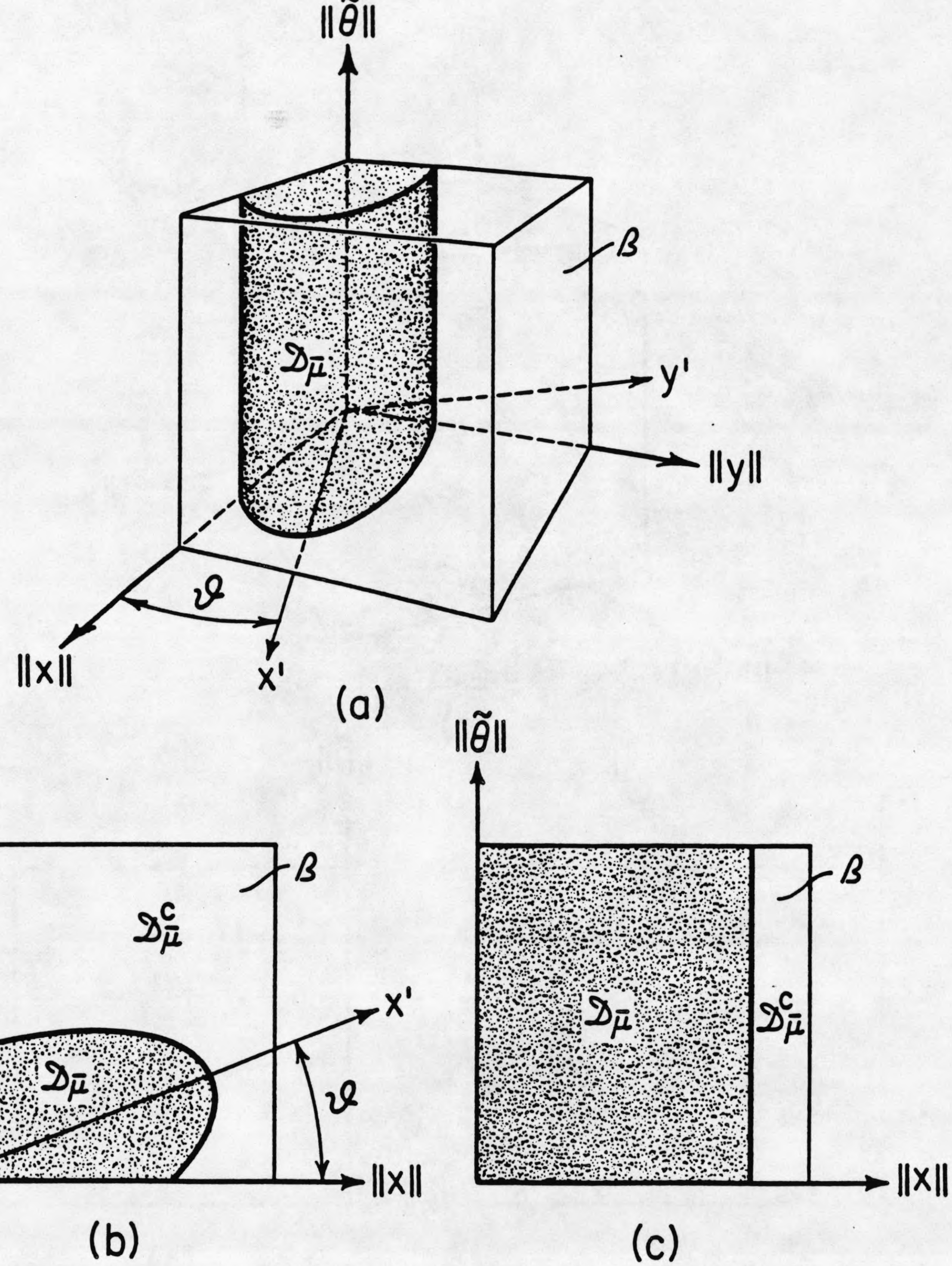

Figure 3: Sets $\mathcal{B}$, $\mathcal{D}_{\bar{\mu}}$ and $\mathcal{D}^c_{\bar{\mu}}$

$$\mathcal{I} := \left\{ (\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2) \in \mathcal{B} : \ \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \leq c \right\}, \tag{154}$$

where c is the largest positive real number such that $\mathcal{I} \subset \mathcal{B}$. To find the nature of $\mathcal{I}$, recall that

$$\begin{aligned} \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) &= (1-d)V(\mathbf{x}, \tilde{\boldsymbol{\theta}}) + dW(\mathbf{y}) \\ &= \frac{1}{2}(1-d)\mathbf{p}^T P_V \mathbf{p} + d\mathbf{y}^T P \mathbf{y} \\ &= \frac{1}{2}(1-d)\mathbf{x}^T \begin{bmatrix} 2\Lambda^T K_D & 0_{n\times n} \\ 0_{n\times n} & M(\mathbf{q}_1) \end{bmatrix} \mathbf{x} + \frac{1}{2}(1-d)\tilde{\boldsymbol{\theta}}^T \Gamma^{-1} \tilde{\boldsymbol{\theta}} + d\mathbf{y}^T P \mathbf{y} \\ &\leq \lambda_x \|\mathbf{x}\|_2^2 + \lambda_\theta \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \lambda_y \|\mathbf{y}\|_2^2, \end{aligned} \tag{155}$$

where

$$\begin{aligned} \lambda_x &= \frac{1}{2} \sup_{\mathbf{B}} \lambda_{\max} \left(\begin{bmatrix} 2\Lambda^T K_D & 0_{n\times n} \\ 0_{n\times n} & M(\mathbf{q}_1) \end{bmatrix} \right) \\ \lambda_\theta &= \frac{1}{2}(1-d)\lambda_{\max}\left(\Gamma^{-1}\right) \\ \lambda_y &= d\lambda_{\max}(P). \end{aligned} \tag{156}$$

Now let

$$\lambda_x \|\mathbf{x}\|_2^2 + \lambda_\theta \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \lambda_y \|\mathbf{y}\|_2^2 \leq c, \tag{157}$$

then, the set $\mathcal{I}$ defined above is a subset of an ellipsoid defined by

$$\frac{\|\mathbf{x}\|_2^2}{\left(\frac{c}{\lambda_x}\right)} + \frac{\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2}{\left(\frac{c}{\lambda_\theta}\right)} + \frac{\|\mathbf{y}\|_2^2}{\left(\frac{c}{\lambda_y}\right)} \leq 1. \tag{158}$$

The case where $\mathcal{I} \cap \mathcal{D}_{\bar{\mu}}^c \neq \emptyset$, is shown in Figure 4.

Based on the above discussion, we conclude that the Lyapunov analysis does not guarantee boundedness of signals. Figure 5 illustrates a possible scenario of how signals might leave the domain $\mathcal{B}$. The initial conditions $\left(\|\mathbf{x}(t=0)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t=0)\right\|_2, \|\mathbf{y}(t=0)\|_2\right) \in \mathcal{I} \cap \mathcal{D}_{\bar{\mu}}^c$ where $\dot{\mathcal{V}} \leq 0$. Hence, we know that for subsequent times, as long as $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ is still in $\mathcal{I} \cap \mathcal{D}_{\bar{\mu}}^c$, it either moves to a lower level curve determined by $\mathcal{V} \leq c_1 \leq c$ for some real c_1, or remains in the same level curve. In addition, $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ can either remain in $\mathcal{I} \cap \mathcal{D}_{\bar{\mu}}^c$, or it may converge to $\mathcal{D}_{\bar{\mu}}$ where $\dot{\mathcal{V}}$ has unknown sign. Hence, inside $\mathcal{D}_{\bar{\mu}}$, the sign of $\dot{\mathcal{V}}$ is not necessarily negative, and we can not conclude where $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ might converge. In fact, as shown in Figure 5, it is conceivable that $\left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2$ grows while $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ is still is $\mathcal{D}_{\bar{\mu}}$ until it leaves the domain $\mathcal{B}$. Figure 6 shows other possible scenarios in which $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ leaves the domian $\mathcal{B}$ due to the growth of $\left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2$ inside $\mathcal{D}_{\bar{\mu}}$. We therefore conclude that a parameter drift instability mechanism is conceivable in which $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ leaves the domain $\mathcal{B}$ because $\left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2$ grows while $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ is in $\mathcal{D}_{\bar{\mu}}$. Such a mechanism is possible

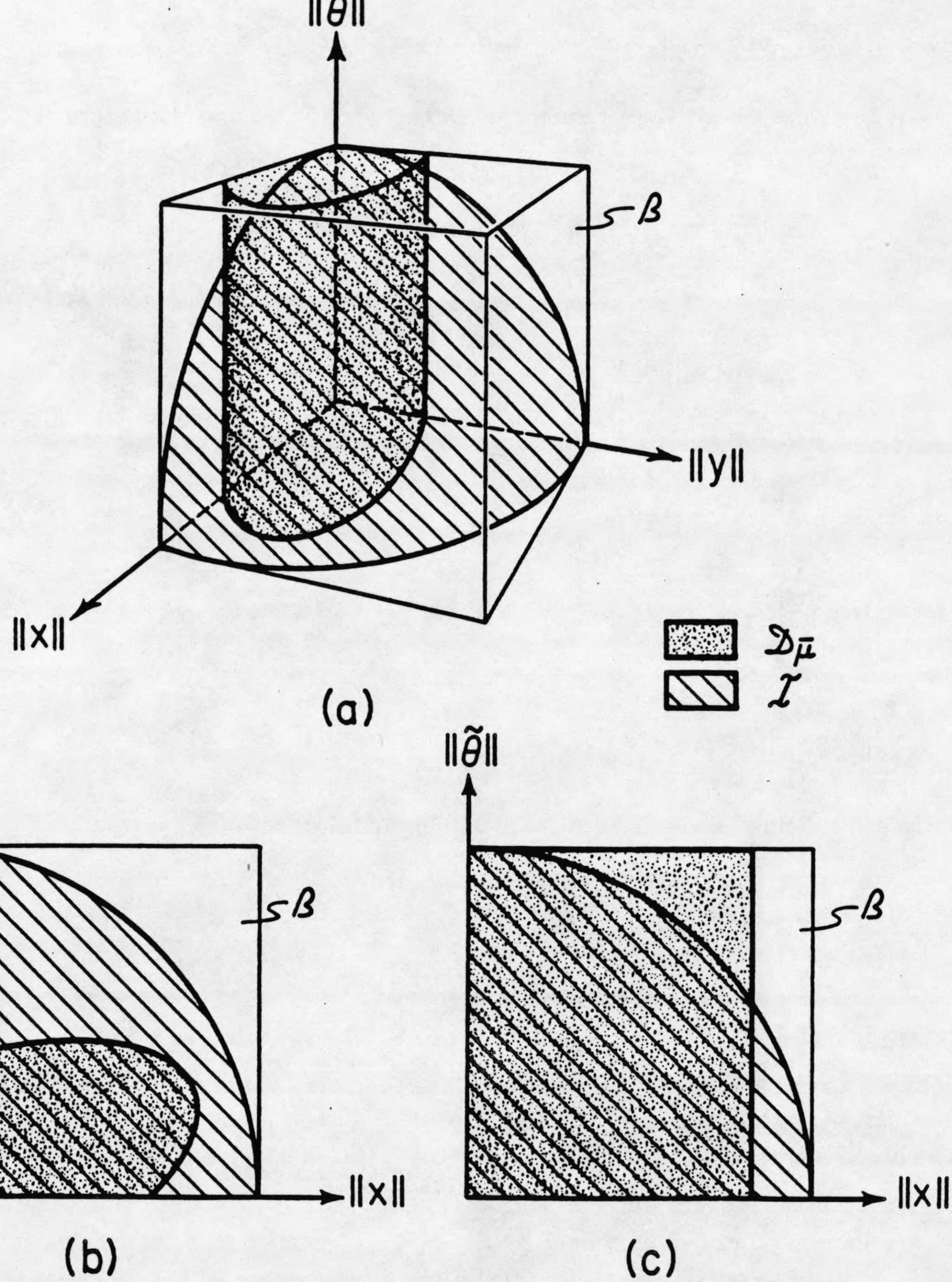

Figure 4: Case $\mathcal{I} \cap \mathcal{D}_{\bar{\mu}}^c \neq \emptyset$

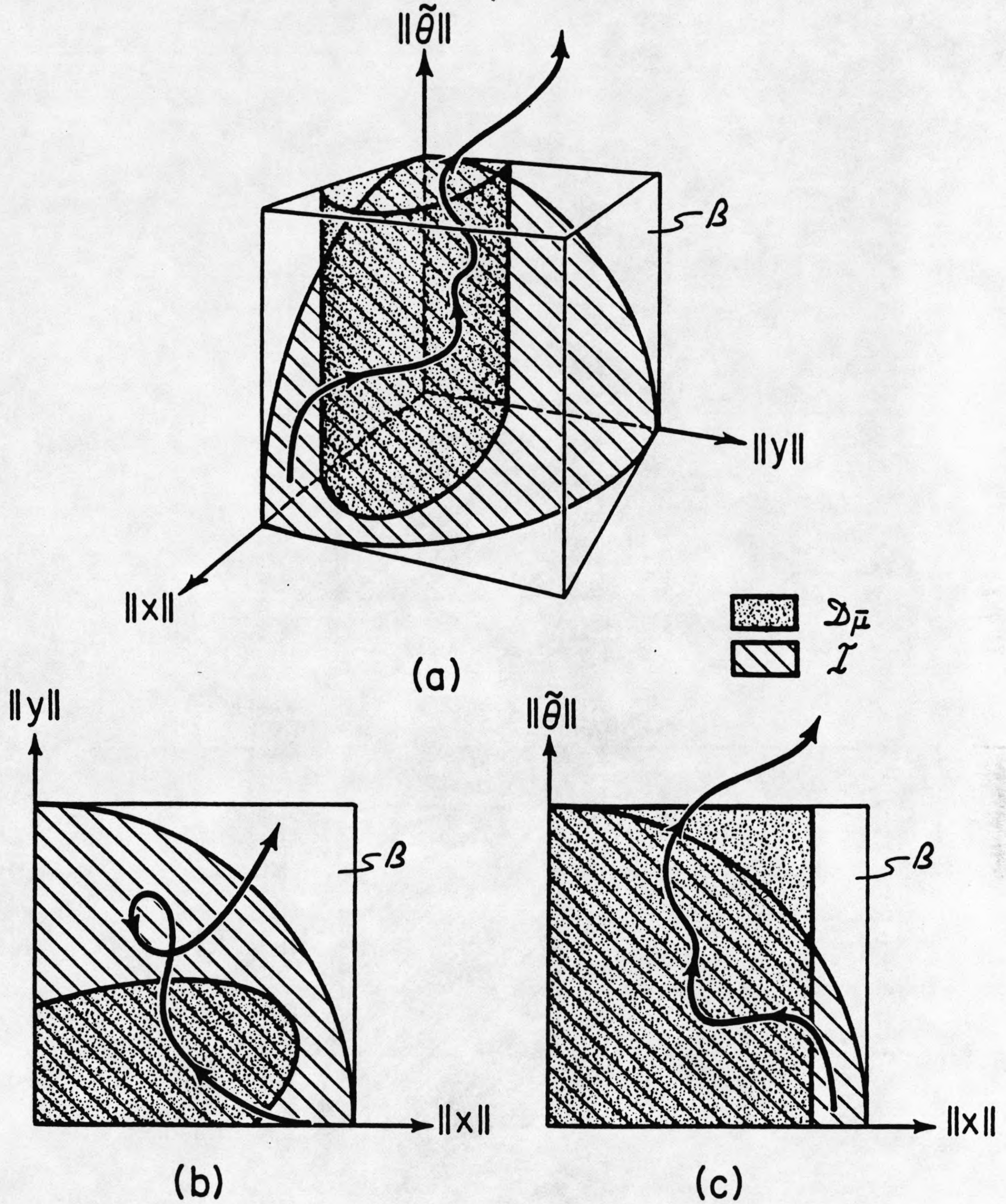

Figure 5: Possible Parameter Drift Instability Mechanism

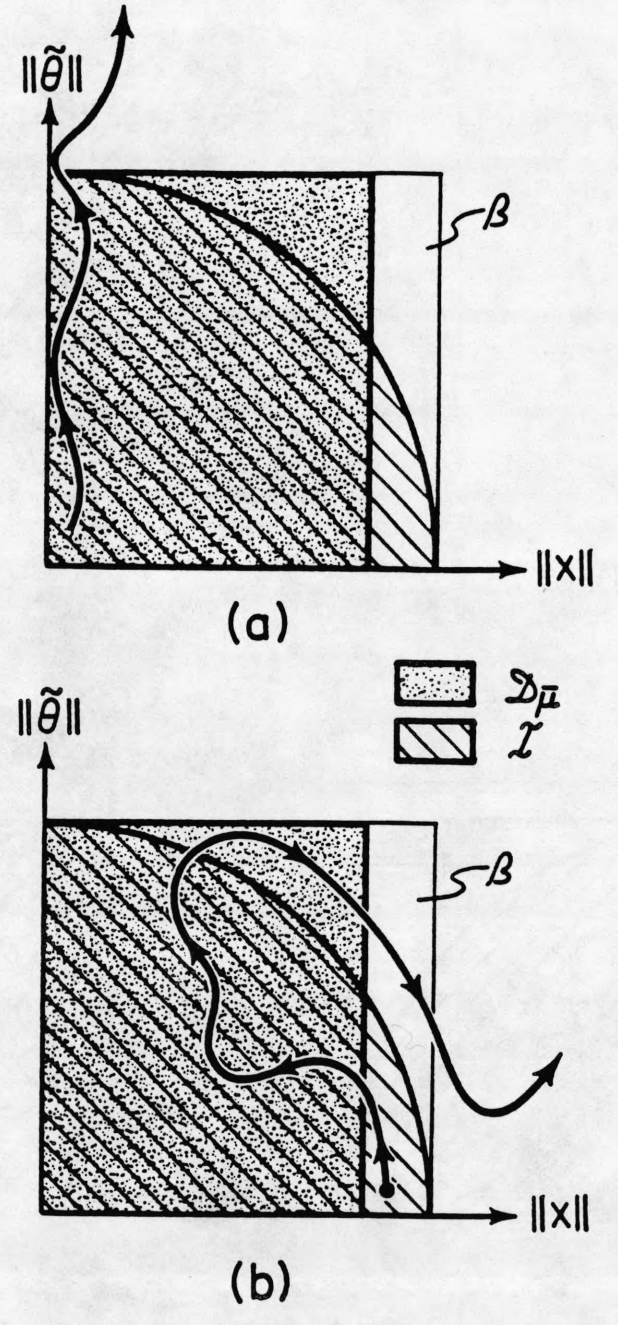

Figure 6: Other Possible Parameter Drift Instability Mechanisms

because the set $\mathcal{D}_{\bar{\mu}}$ (in which the sign of $\dot{\mathcal{V}}$ is unknown) has a cylindrical shape that extends along the whole $\left\|\tilde{\boldsymbol{\theta}}\right\|_2$ axis. A robust adaptive control law in which such an instability mechanism would not happen would result in a set $\mathcal{D}_{\bar{\mu}}$ that does not extend along the whole $\left\|\tilde{\boldsymbol{\theta}}\right\|_2$ axis as will be shown in the next section.

7.2 Robustness Via the fixed σ-modification

In the tracking analysis that follows, we modify the parameter update law in (24), using the fixed σ-modification scheme [9]. The singularly perturbed system $\mathcal{S}$ becomes

$$\mathcal{S}_\sigma \quad : \quad \begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x} - \sigma\Gamma\hat{\boldsymbol{\theta}} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{159}$$

where $\sigma > 0$ is a scalar. The reduced system now becomes

$$\mathcal{S}_\mathrm{r}^\sigma : \begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x} - \sigma\Gamma\hat{\boldsymbol{\theta}}. \end{cases} \tag{160}$$

The boundary layer system $\mathcal{S}_\mathrm{b}$ is still defined by equation (40).

The analysis of system $\mathcal{S}_\sigma$ is very similar to that of the original singularly perturbed system $\mathcal{S}$. In fact we use the same Lyapunov functions candidates V (for the reduced system $\mathcal{S}_\mathrm{r}^\sigma$) and W (for the boundary layer system $\mathcal{S}_\mathrm{b}$.) Consequently, the Composite Lyapunov Function Candidate $\mathcal{V}(\mathbf{p},\mathbf{y})$ given in (82) is also used for the singularly perturbed system $\mathcal{S}_\sigma$. Recall that the time derivative of W along the solution trajectories of $\mathcal{S}_\mathrm{b}$ satisfies (44). We summarize the changes that result from using the σ-modification scheme. The details are given in Appendix B.

Fact 7.1 : The time derivative of the Lyapunov Function V along the solution trajectories of the reduced system $\mathcal{S}_\mathrm{r}^\sigma$ satisfies

$$\left[\begin{array}{l} \forall \mathbf{x} \in \mathbf{R}^{2n}, \ \ \forall\tilde{\boldsymbol{\theta}} \in \mathbf{R}^r, \ \ \forall \mathbf{y} \in \mathbf{R}^{2n}, \ \ \forall t \in \mathbf{R}_+ \\ \dot{V} = \left[\nabla_\mathbf{p} V(\mathbf{p})\right]^T f(t,\mathbf{p},\mathbf{y}=0) \ \ \leq -\alpha_1 \left\|\mathbf{x}\right\|_2^2 - \frac{1}{2}\sigma\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \frac{1}{2}\sigma\left\|\boldsymbol{\theta}\right\|_2^2 \\ \alpha_1 = \lambda_{\min}[R] > 0 \quad ; \quad R = \left[\begin{array}{cc} 2\Lambda^T K_D \Lambda & -\Lambda^T K_D \\ -K_D^T\Lambda & K_D \end{array}\right]. \end{array}\right. \tag{161}$$

Recall that $\boldsymbol{\theta}$ is the constant true parameter vector.

□

Fact 7.2 : The time derivative of $\mathbf{u}$ is given by

$$\dot{\mathbf{u}} = F\frac{\partial\mathbf{u}}{\partial\mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}F\frac{\partial\mathbf{u}}{\partial\mathbf{y}}A_2\mathbf{y} - F\frac{\partial\mathbf{u}}{\partial\tilde{\boldsymbol{\theta}}}\left(\Gamma\varphi\mathbf{x} + \sigma\Gamma\hat{\boldsymbol{\theta}}\right) + F\rho(t). \tag{162}$$

□

Fact 7.3 : Assume

- $(\mathrm{a2})'$

$$\left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\Phi\tilde{\boldsymbol{\theta}}\right\|_2 \le k_{01}\left\|\mathbf{x}\right\|_2, \qquad (\text{ same as in (a2)}) \tag{163}$$

$$\left\| F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\left(\Gamma\varphi\mathbf{x} + \sigma\Gamma\hat{\boldsymbol{\theta}}\right)\right\|_2 \le k'_{02}\left\|\mathbf{x}\right\|_2, \tag{164}$$

$$\left\| F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}A_1\mathbf{x}\right\|_2 \le k_{03}\left\|\mathbf{x}\right\|_2, \qquad (\text{ same as in (a2)}) \tag{165}$$

and hence,

$$\left\|\left\{F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}A_1 - F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi\right\}\mathbf{x} - \sigma F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\hat{\boldsymbol{\theta}} + F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\Phi\tilde{\boldsymbol{\theta}}\right\|_2 \le k'_1\left\|\mathbf{x}\right\|_2, \tag{166}$$

where

$$k'_1 = k_{01} + k'_{02} + k_{03}. \tag{167}$$

Under (a1), $(\mathrm{a2})'$, and (a3), we have $\forall\ (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$

$$\|\dot{\mathbf{u}}\| \le k'_1\left\|\mathbf{x}\right\|_2 + (k_3 + \frac{1}{\epsilon}k_2)\left\|\mathbf{y}\right\|_2 + k_4(t). \tag{168}$$

The second interconnection conditions is therefore given by

$$\left[\begin{array}{l} \forall \mathbf{p} \in \mathbf{B_x} \times \mathbf{B}_{\boldsymbol{\theta}},\ \ \forall \mathbf{y} \in \mathbf{B_y},\ \ \forall t \in \mathbf{R}_+ \\ \quad \left[\nabla_{\mathbf{y}} W(\mathbf{y})\right]^T \left[g(t, \mathbf{p}, \mathbf{y}, \epsilon) - g(t, \mathbf{p}, \mathbf{y}, \epsilon = 0)\right] \\ \qquad\qquad\qquad\qquad \le \epsilon(\gamma'_1 + \frac{1}{\epsilon}\gamma_2)\left\|\mathbf{y}\right\|_2^2 + \epsilon\beta'_2\left\|\mathbf{x}\right\|_2\left\|\mathbf{y}\right\|_2 + \epsilon\mu\left\|\mathbf{y}\right\|_2 \\ \gamma'_1 = 2\left\|P\right\|_{i2} k_3 \\ \gamma_2 = 2\left\|P\right\|_{i2} k_2 \\ \beta'_2 = 2\left\|P\right\|_{i2} k'_1 \\ \mu(t) = 2\left\|P\right\|_{i2} k_4(t) \\ \text{P satisfies the Lyapunov equation (41)} \\ k'_1, k_2, k_3, \text{ and } k_4(t) \text{ are given by (a1)}, (\mathrm{a2})', \text{and(a3)}. \end{array}\right. \tag{169}$$

□

Fact 7.4 : The time derivative of the composite Lyapunov Function $\mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y})$ (82) along the solution trajectories of $\mathcal{S}_\sigma$ becomes

$$\dot{\mathcal{V}}(\mathbf{p}, \mathbf{y}) \le -\left[\begin{array}{cc} \left\|\mathbf{x}\right\|_2 & \left\|\mathbf{y}\right\|_2 \end{array}\right] P'_d \left[\begin{array}{c} \left\|\mathbf{x}\right\|_2 \\ \left\|\mathbf{y}\right\|_2 \end{array}\right] - \frac{1}{2}(1-d)\sigma\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \frac{1}{2}(1-d)\sigma\left\|\boldsymbol{\theta}\right\|_2^2 + d\mu^2(t), \tag{170}$$

where

$$P_d' = \begin{bmatrix} (1-d)\alpha_1 & -\frac{(1-d)\beta_1 + d\beta_2'}{2} \\ -\frac{(1-d)\beta_1 + d\beta_2'}{2} & \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1 \end{bmatrix}. \tag{171}$$

Note that the difference between P_d' and P_d given by (90) is that β_2 in P_d is replaced by β_2' in P_d'.

□

We have the following result.

Theorem 2 (Fixed σ-modification, Boundedness of Tracking Errors) *Assume*

1. $\dot{\mathbf{q}}_d$, $\ddot{\mathbf{q}}_d$, $\mathbf{q}_d^{(3)} \in \mathbf{L}_\infty^n$, *so that* $\exists\ \bar{\mu}$ *a positive real constant such that* $\mu(t) \leq \bar{\mu} \quad \forall t \in \mathbf{R}_+$ *(Case 2 above).*
2. (a1), (a2)$'$, and (a3) *are satisfied* $\forall(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$.
3. $\alpha_2 - \gamma_2 > 0$.

Define the sets $\mathcal{D}_{\bar{\mu},\sigma}$ *and* $\mathcal{R}_{\bar{\mu},\sigma}$ *as follows:*

$$\mathcal{D}_{\bar{\mu},\sigma} := \left\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2 \right) \in \mathcal{B} : \begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d' \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} + \frac{1}{2}(1-d)\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \leq \frac{1}{2}(1-d)\sigma \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \right\}, \tag{172}$$

and

$$\mathcal{R}_{\bar{\mu},\sigma} := \left\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2 \right) \in \mathcal{B} : \ \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \leq c_{\bar{\mu},\sigma} \right\}, \tag{173}$$

where $c_{\bar{\mu},\sigma}$ *is the smallest positive real number such that* $\mathcal{D}_{\bar{\mu},\sigma} \subseteq \mathcal{R}_{\bar{\mu},\sigma}$.

If $\bar{\mu}$ *and* σ *are such that* $\mathcal{R}_{\bar{\mu},\sigma} \subset \mathcal{I}$, *then* $\exists$ *an upper bound of* ϵ, *namely,*

$$\epsilon_d' = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2']^2}, \tag{174}$$

such that all the solution trajectories of the singularly perturbed system $\mathcal{S}_\sigma$ *starting in* $\mathcal{I}$ *converge to the residual set* $\mathcal{R}_{\bar{\mu},\sigma}$ $\forall\ \epsilon\ \in\ (0, \epsilon_d')$ (Refer to Figure 7.)

Proof of Theorem 2: Recall that the Composite Lyapunov Function of the singularly perturbed system $\mathcal{S}_\sigma$ is (see (82))

$$\begin{aligned} \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) &= (1-d)V(\mathbf{p}) + dW(\mathbf{y}) \\ &= \frac{1}{2}(1-d)\mathbf{p}^T P_V \mathbf{p} + \frac{1}{2} d\mathbf{y}^T P \mathbf{y}, \end{aligned} \tag{175}$$

where P_V and P are given by (52) and (41) respectively. Using Assumption 2, the time derivative of $\mathcal{V}$ along the solution trajectories of $\mathcal{S}_\sigma$ is given by (170)

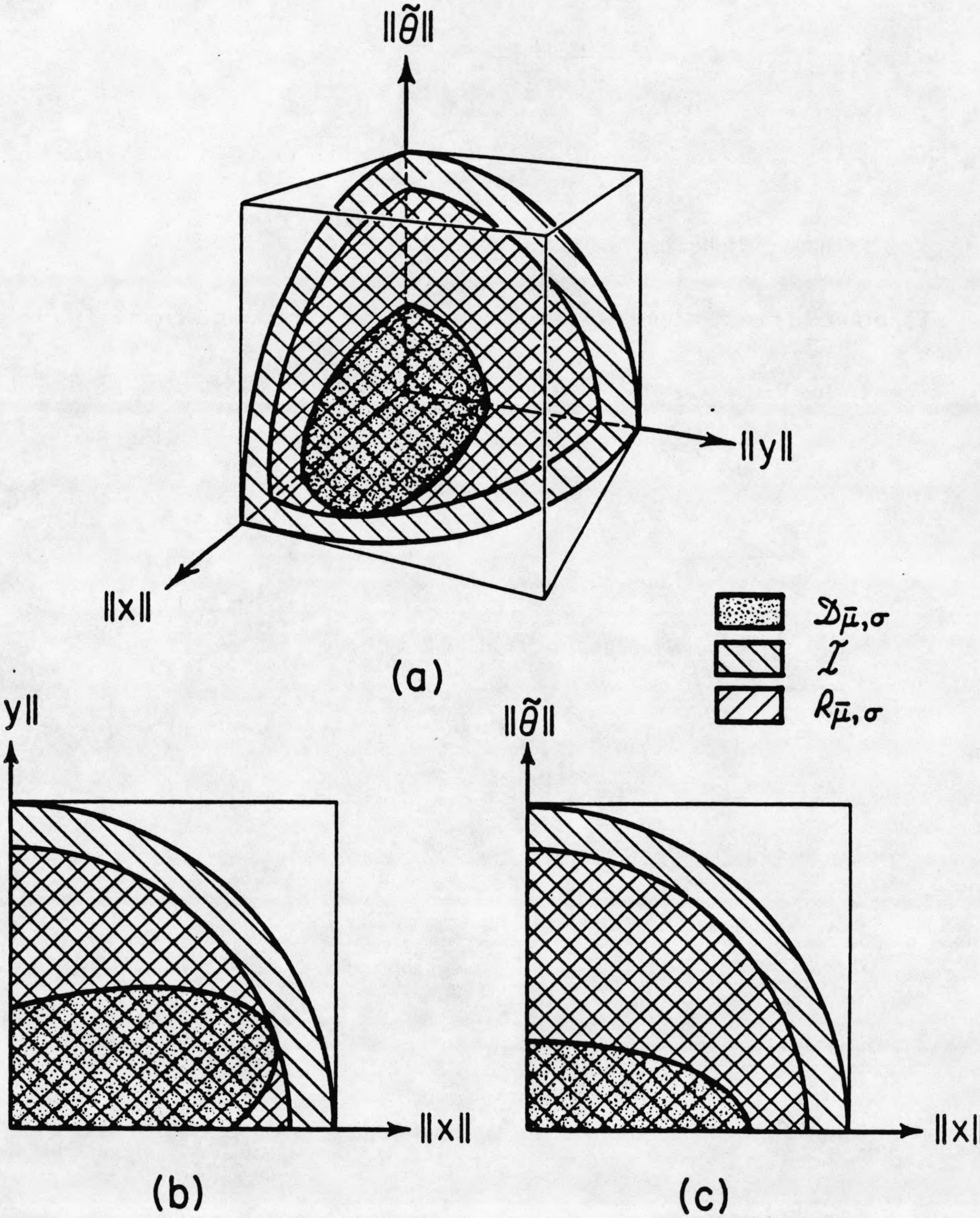

Figure 7: The σ-modification Case

$$\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) \leq -\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] - \frac{1}{2}(1-d)\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \frac{1}{2}(1-d)\sigma \|\boldsymbol{\theta}\|_2^2 + d\mu^2(t), \quad (176)$$

where P_d' is given by (171). Similar to the case in the proof of Theorem 1, and using Assumption 3, the matrix P_d' is positive definite when

$$\epsilon < \epsilon_d' := \frac{\alpha_1(\alpha_2-\gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2']^2}. \quad (177)$$

Using Assumption 1, (176) becomes

$$\begin{aligned} \dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\leq -\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] - \frac{1}{2}(1-d)\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \\ &+ \frac{1}{2}(1-d)\sigma \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2. \end{aligned} \quad (178)$$

Define the sets

$$\begin{aligned} \mathcal{D}_{\bar{\mu},\sigma} := \Bigg\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} : & \left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] \\ & + \frac{1}{2}(1-d)\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \leq \frac{1}{2}(1-d)\sigma \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \Bigg\}, \end{aligned} \quad (179)$$

and

$$\begin{aligned} \mathcal{D}^c_{\bar{\mu},\sigma} := \Bigg\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} : & \left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] \\ & + \frac{1}{2}(1-d)\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 > \frac{1}{2}(1-d)\sigma \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \Bigg\} \\ = \mathcal{B} \setminus \mathcal{D}_{\bar{\mu},\sigma}. \end{aligned} \quad (180)$$

The set $\mathcal{D}_{\bar{\mu},\sigma}$ is a subset of an ellipsoid. To see this, note that

$$\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] + \frac{1}{2}(1-d)\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \leq \frac{1}{2}(1-d)\sigma \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \quad (181)$$

can be written as

$$\underline{a}\,\|\mathbf{x}\|_2^2 + \underline{b}\,\|\mathbf{x}\|_2\,\|\mathbf{y}\|_2 + \underline{c}\,\|\mathbf{y}\|_2^2 + \underline{s}\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \leq \underline{s}\,\|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2, \quad (182)$$

where

$$\begin{aligned} \underline{a} &= \bar{a} = (1-d)\alpha_1 && (183) \\ \underline{b} &= -\{(1-d)\beta_1 + d\beta_2'\} && (184) \\ \underline{c} &= \bar{c} = \frac{d}{\epsilon}(\alpha_2-\gamma_2) - d\gamma_1 && (185) \\ \underline{s} &= \frac{1}{2}(1-d)\sigma. && (186) \end{aligned}$$

This is the equation of an ellipsoid with axes x'-y'- $\left\|\tilde{\boldsymbol{\theta}}\right\|_2$, where the x'-y' axes are obtained by rotating the $\|\mathbf{x}\|_2$-$\|\mathbf{y}\|_2$ axes by an angle $\bar{\vartheta}$ given by

$$\cot(2\bar{\vartheta}) = \frac{\underline{a} - \underline{c}}{\underline{b}}. \tag{187}$$

Equation (182) is written in the $x' - y' - \left\|\tilde{\boldsymbol{\theta}}\right\|_2$ axes as

$$\underline{a}'(x')^2 + \underline{c}'(y')^2 + \underline{s}\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \le \underline{s}\,\|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2, \tag{188}$$

or simply,

$$\frac{(x')^2}{\left(\frac{\underline{s}\|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2}{\underline{a}'}\right)} + \frac{(y')^2}{\left(\frac{\underline{s}\|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2}{\underline{c}'}\right)} + \frac{\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2}{\left(\frac{\underline{s}\|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2}{\underline{s}}\right)} \le 1, \tag{189}$$

where

$$\underline{a}' = \underline{a}\cos^2(\bar{\vartheta}) + \underline{b}\cos(\bar{\vartheta})\sin(\bar{\vartheta}) + \underline{c}\sin^2(\bar{\vartheta}) \tag{190}$$

$$\underline{c}' = \underline{a}\sin^2(\bar{\vartheta}) - \underline{b}\sin(\bar{\vartheta})\cos(\bar{\vartheta}) + \underline{c}\cos^2(\bar{\vartheta}). \tag{191}$$

Hence, $\forall\ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{D}_{\bar{\mu},\sigma}$, $\dot{\mathcal{V}}$ can be positive or negative, and $\forall\ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{D}^c_{\bar{\mu},\sigma}$, $\dot{\mathcal{V}} < 0$. (See Figure 7.) Also, define

$$\mathcal{R}_{\bar{\mu},\sigma} := \left\{\left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} :\ \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \le c_{\bar{\mu},\sigma}\right\}, \tag{192}$$

where $c_{\bar{\mu},\sigma}$ is the smallest positive real number such that $\mathcal{D}_{\bar{\mu},\sigma} \subseteq \mathcal{R}_{\bar{\mu},\sigma}$. If $\bar{\mu}$ and σ are such that $\mathcal{R}_{\bar{\mu},\sigma} \subset \mathcal{I}$ (see Figure 7), choose initial conditions such that $\left(\|\mathbf{x}(t=0)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t=0)\right\|_2, \|\mathbf{y}(t=0)\|_2\right)$ $\in\ (\mathcal{I} \setminus \mathcal{R}_{\bar{\mu},\sigma}) \subset \mathcal{D}^c_{\bar{\mu},\sigma}$. We conclude that $\dot{\mathcal{V}}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) < 0$ as long as $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ is outside $\mathcal{R}_{\bar{\mu},\sigma}$. If $\left(\|\mathbf{x}(t=0)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t=0)\right\|_2, \|\mathbf{y}(t=0)\|_2\right)$ starts inside $\mathcal{R}_{\bar{\mu},\sigma}$, or if it reaches $\mathcal{R}_{\bar{\mu},\sigma}$ for some time $t > 0$, then $\left(\|\mathbf{x}(t)\|_2, \left\|\tilde{\boldsymbol{\theta}}(t)\right\|_2, \|\mathbf{y}(t)\|_2\right)$ remain in $\mathcal{R}_{\bar{\mu},\sigma}$ for all subsequent t. Hence we conclude that all solution trajectories starting in $\mathcal{I}$ converge to the residual set $\mathcal{R}_{\bar{\mu},\sigma}$. □

Remark 7.5 Similar to the regulation case, the maximum value of ϵ_d occurs at

$$d^* = \frac{\beta_1}{\beta_1 + \beta_2} \tag{193}$$

and is given by

$$\epsilon^* = \epsilon_{d=d^*} = \frac{\alpha_1(\alpha_1 - \gamma_2)}{\alpha_1\gamma_1 + \beta_1\beta_2}. \tag{194}$$

Choosing $d = d^*$ fixes the size of $\mathcal{D}_{\bar{\mu},\sigma}$, and hence that of the residual set $\mathcal{R}_{\bar{\mu},\sigma}$. If the size of $\mathcal{R}_{\bar{\mu},\sigma}$ is changed by choosing another d, then a smaller upper bound ϵ_d results as it is clear from Figure 1.

□

Remark 7.6 The advantage of introducing the fixed σ-modification is that the tracking errors and the parameter errors are ensured to converge to a residual set under the conditions of Theorem 2. As far as the desired trajectory is concerned, it is only required that the latter is bounded and is three times continuously differentiable with bounded derivatives. The price paid by introducing the fixed σ-modification is that no conclusion about the convergence to zero of the tracking errors can be made even under further restrictive conditions on the desired trajectory such as those of Case 3 above. Moreover, using the switching σ-modification of [10] for the class of desired trajectories of Case 2, Theorem 2 still applies, and no conclusion about the convergence to zero of the tracking errors can be made. In the next section we show that for the class of desired trajectories of Case 3, the tracking errors converge to zero if the switching σ-modification of [10] is used.

7.3 Asymptotic Tracking with the switching σ-modification

In this section we use the switching σ-modification of [10], and show that for the class of desired trajectories described in Case 3 above, the tracking errors converge to zero, and all other signals are bounded. The singularly perturbed system with this modification becomes

$$\mathcal{S}_{\sigma_s} \quad : \quad \begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\hat{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x} - \sigma_s\Gamma\hat{\boldsymbol{\theta}} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{195}$$

where σ_s is now given by

$$\sigma_s(t) = \begin{cases} 0 & \text{if} \quad \left\|\hat{\theta}(t)\right\|_2 < \theta_0 \\ \sigma_0\left(\frac{\left\|\hat{\theta}(t)\right\|_2}{\theta_0} - 1\right) & \text{if} \quad \theta_0 \le \left\|\hat{\theta}(t)\right\|_2 \le 2\theta_0 \\ \sigma_0 & \text{if} \quad \left\|\hat{\theta}(t)\right\|_2 > 2\theta_0. \end{cases} \tag{196}$$

σ_0 is a positive scalar design parameter. θ_0 is chosen such that

$$\|\boldsymbol{\theta}\|_2 < \theta_0, \tag{197}$$

and hence, it reflects our knowledge of the true parameter vector $\boldsymbol{\theta}$. Determining θ_0 is possible since in general the true parameters have known upper and lower bounds.

Assume

- (a2)$''$: which consists of assumption (a2)$'$ with σ, k_1', and β_2' replaced by σ_s, k_1'', and β_2'' respectively.

Consequently, P_d' given by (171) when (a2)$'$ was assumed, is now denoted P_d'' and is given by

$$P_d'' = \begin{bmatrix} (1-d)\alpha_1 & -\frac{(1-d)\beta_1 + d\beta_2''}{2} \\ -\frac{(1-d)\beta_1 + d\beta_2''}{2} & \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1 \end{bmatrix}. \tag{198}$$

We have the following result

Theorem 3 (Switching σ-modification, Convergence of Tracking Errors) *Assume*

1. $\dot{\mathbf{q}}_d$, $\ddot{\mathbf{q}}_d$, $\mathbf{q}_d^{(3)} \in \mathbf{L}_2^n \cap \mathbf{L}_\infty^n$, *so that* $\exists\ \bar{\mu}$ *a positive real constant such that* $\mu(t) \leq \bar{\mu} \quad \forall t \in \mathbf{R}_+$, *and* $\mu(t) \in \mathbf{L}_2$ *(Case 3 above.)*
2. (a1), (a2)$''$, *and* (a3) *are satisfied* $\forall(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$.
3. $\alpha_2 - \gamma_2 > 0$.

Define the sets $\mathcal{D}_{\bar{\mu},\sigma_0}$ *and* $\mathcal{R}_{\bar{\mu},\sigma_0}$ *as follows:*

$$\mathcal{D}_{\bar{\mu},\sigma_0} := \left\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2 \right) \in \mathcal{B} : \ \begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d'' \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} \right.$$
$$\left. \quad + \frac{1}{2}(1-d)\sigma_0 \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \leq \frac{1}{2}(1-d)\sigma_0 \left\|\boldsymbol{\theta}\right\|_2^2 + d\bar{\mu}^2 \right\}, \tag{199}$$

$$\mathcal{D}^c_{\bar{\mu},\sigma_0} = \mathcal{B} \setminus \mathcal{D}_{\bar{\mu},\sigma_0}, \tag{200}$$

and

$$\mathcal{R}_{\bar{\mu},\sigma_0} := \left\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2 \right) \in \mathcal{B} : \ \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \leq c_{\bar{\mu},\sigma_0} \right\}, \tag{201}$$

where $c_{\bar{\mu},\sigma_0}$ *is the smallest positive real number such that*

- $\mathcal{D}_{\bar{\mu},\sigma_0} \subseteq \mathcal{R}_{\bar{\mu},\sigma_0}$
- $\forall \ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2 \right) \in \left\{ \mathcal{D}^c_{\bar{\mu},\sigma_0} \setminus (\mathcal{R}_{\bar{\mu},\sigma_0} \cap \mathcal{D}_{\bar{\mu},\sigma_0}) \right\}$, $\left\|\hat{\boldsymbol{\theta}}\right\|_2 > 2\theta_0$.

If $\bar{\mu}$, σ_0, *and* θ_0 *are such that* $\mathcal{R}_{\bar{\mu},\sigma_0} \subset \mathcal{I}$, *then* $\exists$ *an upper bound of* ϵ, *namely,*

$$\epsilon_d'' = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2'']^2}, \tag{202}$$

such that all the solution trajectories starting in $\mathcal{I}$ *converge to the residual set* $\mathcal{R}_{\bar{\mu},\sigma_0}$ $\forall\ \epsilon \in (0, \epsilon_d'')$.

Furthermore,

$$\lim_{t \longrightarrow \infty} \mathbf{x}(t) = 0 \tag{203}$$
$$\lim_{t \longrightarrow \infty} \mathbf{y}(t) = 0. \tag{204}$$

Proof of Theorem 3: Using Assumption 2, the time derivative of $\mathcal{V}$ along the solution trajectories of $\mathcal{S}_{\sigma_s}$ is given by (see the proof of Theorem 2)

$$\begin{aligned} \dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\leq -\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d'' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] - \frac{1}{2}(1-d)\sigma_s \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \\ &+ \frac{1}{2}(1-d)\sigma_s \|\boldsymbol{\theta}\|_2^2 + d\mu^2(t), \end{aligned} \tag{205}$$

where P_d'' is given by (198). Using Assumption 3, it is clear that the matrix P_d'' is positive definite when

$$\epsilon < \epsilon_d'' := \frac{\alpha_1(\alpha_2-\gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2'']^2}. \tag{206}$$

Using Assumption 1, (205) becomes

$$\begin{aligned} \dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\leq -\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d'' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] - \frac{1}{2}(1-d)\sigma_s \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \\ &\qquad + \frac{1}{2}(1-d)\sigma_s \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2. \end{aligned} \tag{207}$$

Define the sets

$$\begin{aligned} \mathcal{D}_{\bar{\mu},\sigma_0} := \Bigg\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} : \; &\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d'' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] \\ &+ \frac{1}{2}(1-d)\sigma_0 \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \leq \frac{1}{2}(1-d)\sigma_0 \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \Bigg\}, \end{aligned} \tag{208}$$

and

$$\begin{aligned} \mathcal{D}^c_{\bar{\mu},\sigma_0} := \Bigg\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} : \; &\left[\begin{array}{cc} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{array}\right] P_d'' \left[\begin{array}{c} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{array}\right] \\ &+ \frac{1}{2}(1-d)\sigma_0 \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 > \frac{1}{2}(1-d)\sigma_0 \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \Bigg\} \\ = \; \mathcal{B} \setminus \mathcal{D}_{\bar{\mu},\sigma_0}. \end{aligned} \tag{209}$$

The set $\mathcal{D}_{\bar{\mu},\sigma_0}$ is a subset of an ellipsoid as was shown for the case of $\mathcal{D}_{\bar{\mu},\sigma}$ in the proof of Theorem 2. Now define

$$\mathcal{R}_{\bar{\mu},\sigma_0} := \left\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} : \; \mathcal{V}(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) \leq c_{\bar{\mu},\sigma_0} \right\}, \tag{210}$$

where $c_{\bar{\mu},\sigma_0}$ is the smallest positive real number such that

- $\mathcal{D}_{\bar{\mu},\sigma_0} \subseteq \mathcal{R}_{\bar{\mu},\sigma_0}$
- $\forall \; \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \left\{\mathcal{D}^c_{\bar{\mu},\sigma_0} \setminus (\mathcal{R}_{\bar{\mu},\sigma_0} \cap \mathcal{D}_{\bar{\mu},\sigma_0})\right\}$, $\left\|\hat{\boldsymbol{\theta}}\right\|_2 > 2\theta_0$.

If $\bar{\mu}$, σ_0, and θ_0 are such that $\mathcal{R}_{\bar{\mu},\sigma_0} \subset \mathcal{I}$, then as shown in the proof of Theorem 2 for the fixed σ-modification, we conclude that all solution trajectories starting in $\mathcal{I}$ converge to the residual set $\mathcal{R}_{\bar{\mu},\sigma_0}$, and hence $\mathbf{x}(t) \in \mathbf{L}_\infty^{2n}$, $\tilde{\boldsymbol{\theta}}(t) \in \mathbf{L}_\infty^r$, and $\mathbf{y}(t) \in \mathbf{L}_\infty^{2n}$. Furthermore, the Lyapunov function candidate $\mathcal{V}$ is uniformly bounded. To show the convergence of the tracking error to zero, we first note that it is easy to check (see the proof of Fact 7.1 and Fact 7.4) that the time derivative of the Lyapunov function given by (205) is equivalently written as

$$\begin{aligned}\dot{\mathcal{V}}(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y}) &\le -\left[\ \|\mathbf{x}\|_2 \quad \|\mathbf{y}\|_2\ \right] P_d''\left[\begin{array}{c}\|\mathbf{x}\|_2\\ \|\mathbf{y}\|_2\end{array}\right] - \frac{1}{2}(1-d)\sigma_s\tilde{\boldsymbol{\theta}}^T(t)\hat{\boldsymbol{\theta}}(t) + d\mu^2(t)\\ &\le -\frac{1}{c_2}\left(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2\right) - \frac{1}{2}(1-d)\sigma_s\tilde{\boldsymbol{\theta}}^T(t)\hat{\boldsymbol{\theta}}(t) + d\mu^2(t). \end{aligned} \tag{211}$$

where for $\epsilon < \epsilon_d''$

$$\frac{1}{c_2} = \lambda_{\min}\left(P_d''\right). \tag{212}$$

We have the following Fact (see Appendix B for proof)

Fact 7.7 :

$$-\sigma_s\tilde{\boldsymbol{\theta}}^T(t)\hat{\boldsymbol{\theta}}(t) \le 0 \quad \forall t \in \mathbf{R}_+. \tag{213}$$

□

We now integrate both sides of (211)

$$\begin{aligned}\int_0^T \dot{\mathcal{V}}(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{y})dt &\le -\int_0^T \frac{1}{c_2}\left(\|\mathbf{x}(t)\|_2^2 + \|\mathbf{y}(t)\|_2^2\right)dt - \int_0^T \frac{1}{2}(1-d)\sigma_s\tilde{\boldsymbol{\theta}}(t)\hat{\boldsymbol{\theta}}(t)dt\\ &+ \int_0^T d\mu^2(t)dt. \end{aligned} \tag{214}$$

Taking the limit, rearranging, and recalling that $\mathcal{V}$ is uniformly bounded, we obtain

$$\begin{aligned}\int_0^\infty \frac{1}{c_2}\left(\|\mathbf{x}(t)\|_2^2 + \|\mathbf{y}(t)\|_2^2\right)dt &+ \int_0^\infty \frac{1}{2}(1-d)\sigma_s\tilde{\boldsymbol{\theta}}(t)\hat{\boldsymbol{\theta}}(t)dt - \int_0^\infty d\mu^2(t)dt\\ &\le \mathcal{V}(\mathbf{x}(0),\tilde{\boldsymbol{\theta}}(0),\mathbf{y}(0)) - \lim_{T\longrightarrow\infty}\mathcal{V}(\mathbf{x}(T),\tilde{\boldsymbol{\theta}}(T),\mathbf{y}(T))\\ &< \infty. \end{aligned} \tag{215}$$

Since $\mu(t) \in \mathbf{L}_2$, we conclude that

$$\int_0^\infty \frac{1}{c_2}\left(\|\mathbf{x}(t)\|_2^2 + \|\mathbf{y}(t)\|_2^2\right)dt + \int_0^\infty \frac{1}{2}(1-d)\sigma_s\tilde{\boldsymbol{\theta}}(t)\hat{\boldsymbol{\theta}}(t)dt < \infty. \tag{216}$$

From Fact 7.7, we have $\sigma_s\tilde{\boldsymbol{\theta}}(t)\hat{\boldsymbol{\theta}}(t) \ge 0$. Note therefore that both integrals in (216) are positive, and consequently both integrals are bounded. Hence, $\mathbf{x}(t) \in \mathbf{L}_2^{2n}$, $\mathbf{y}(t) \in \mathbf{L}_2^{2n}$, and combined with the previous boundedness result we conclude that $\mathbf{x}(t) \in \mathbf{L}_2^{2n} \cap \mathbf{L}_\infty^{2n}$ and $\mathbf{y}(t) \in \mathbf{L}_2^{2n} \cap \mathbf{L}_\infty^{2n}$. From the boundedness of $\mathbf{x}(t)$, $\tilde{\boldsymbol{\theta}}(t)$, and $\mathbf{y}(t)$, we conclude that $\dot{\mathbf{x}}(t) \in \mathbf{L}_\infty^{2n}$ (see (24).) Furthermore, using (77), we have from (24) that $\dot{\mathbf{y}}(t) \in \mathbf{L}_{\infty}^{2n}$. Consequently $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are both uniformly continuous. Since in addition they are elements of $\mathbf{L}_2^{2n}$, we conclude that

$$\lim_{t \longrightarrow \infty} \mathbf{x}(t) = 0 \tag{217}$$

$$\lim_{t \longrightarrow \infty} \mathbf{y}(t) = 0. \tag{218}$$

□

Remark 7.8 In the switching σ-modification scheme presented above, an upper bound θ_0 on the norm of the true parameter vector $\boldsymbol{\theta}$ is assumed to be known as described by equation (197). The size of the residual set $\mathcal{R}_{\bar{\mu},\sigma_0}$ to which all states $\mathbf{x}(t)$, $\mathbf{y}(t)$, and $\tilde{\boldsymbol{\theta}}(t)$ are insured to converge depends on θ_0. In general, the larger θ_0 is, the larger the residual set $\mathcal{R}_{\bar{\mu},\sigma_0}$ becomes. Since the tracking results are local, a conservative choice of θ_0 may not insure the existence of a residual set $\mathcal{R}_{\bar{\mu},\sigma_0}$ inside the domain $\mathcal{B}$. Note that a conservative choice of θ_0 can result from the fact that in (197) all elements of $\boldsymbol{\theta}$ are equally weighted even though they may not be of the same order of magnitude. To make the use of the switching σ-modification scheme more efficient in a local context, note that in general we know an upper bound on the magnitude of each individual element θ_i of the true parameter vector $\boldsymbol{\theta}$. Hence, we can modify the use of the switching σ-modification scheme as follows

$$\mathcal{S}_{\Sigma} \quad : \quad \begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x} - \Gamma\Sigma\hat{\boldsymbol{\theta}} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{219}$$

where

$$\Sigma = \operatorname{diag}\left[\sigma_{s1}, \sigma_{s2}, \cdots, \sigma_{sr}\right] \in \mathbf{R}^{r\times r}, \tag{220}$$

and each σ_{si} is now given by

$$\sigma_{si}(t) = \begin{cases} 0 & \text{if} \quad |\, \hat{\theta}_i(t) \,| < \theta_{0i} \\ \sigma_{0i}\left(\frac{|\hat{\theta}_i(t)|}{\theta_{0i}} - 1\right) & \text{if} \quad \theta_{0i} \leq |\, \hat{\theta}_i(t) \,| \leq 2\theta_{0i} \\ \sigma_{0i} & \text{if} \quad |\, \hat{\theta}_i(t) \,| > 2\theta_{0i} \end{cases} \qquad i = 1, 2, \cdots, r. \tag{221}$$

σ_{0i} is a positive scalar design parameter. θ_{0i} is chosen such that

$$|\, \theta_i \,| < \theta_{0i} \qquad\qquad i = 1, 2, \cdots, r. \tag{222}$$

Fact 7.9 : Using the modified switching σ-modification scheme as defined by (219)-(222), the results of Theorem 3 hold (see Appendix B for details.)

□

7.4 A Simulation Example

References [6] and [7] contain experimental results that show the effectiveness of the proposed composite control technique. In these references, experiments showed excellent results for tracking a slow desired trajectory (step response generated from a second order linear system) with no modification in the parameter update law. Of course, good tracking of a time varying desired trajectory is not guaranteed in general as predicted by the stability analysis in this report. In the simulation example that follows, we consider a sinusoidal desired trajectory (Case 2), and illustrate the following points

- Without modification of the parameter update law, parameter drift instability mechanism is possible.
- Adding σ-modification to the parameter update law, we get boundedness of all signals, but nonzero tracking errors (Theorem 2.)

The hardware of a specially constructed single-link flexible-joint arm is shown in Figure 8 (see [6] and [7].) The dynamics of this system (see Figure 9) are modeled as

$$\begin{aligned} I\ddot{q}_1 + Mgl\, sin(q_1) + k(q_1 - q_2) &= 0 & (223)\\ J\ddot{q}_2 + B\dot{q}_2 - k(q_1 - q_2) &= u_c. & (224) \end{aligned}$$

Nominal and true values for the arm parameters are shown in Table 1. The related rigid model, obtained in the limit as $k \rightarrow \infty$, is

$$(I + J)\ddot{q}_1 + B\dot{q}_1 + Mgl\, sin(q_1) = u_s. \qquad (225)$$

The damping coefficient B is assumed to be known. We define the parameter vector θ as

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} I + J \\ Mgl \end{bmatrix}. \qquad (226)$$

The design of the rigid control law u_s is now based on the rigid model (225). Using the algorithm of Slotine and Li [19] for this term, the complete control law with correction is $u_c = u_s + u_f$

$$u_s = \hat{\theta}_1 a + \hat{\theta}_2 sin(q_1) + Bv - K_D r \quad , u_f = K_v(\dot{q}_1 - \dot{q}_2). \qquad (227)$$

Recall that

$$\begin{aligned} \tilde{q}_1 &= q_1 - q^d, & (228)\\ v &= \dot{q}^d - \lambda\tilde{q}_1, & (229)\\ r &= \dot{\tilde{q}}_1 + \lambda\tilde{q}_1, & (230)\\ a &= \ddot{q}_d - \lambda\dot{\tilde{q}}_1, & (231) \end{aligned}$$

where $q^d(t)$ is the desired trajectory. We choose for the desired trajectory $q^d(t)$ a sinusoidal function given in Table 2.

Choosing $\Gamma^{-1} = diag(\gamma_1, \gamma_2)$, the parameter update law with the modified switching σ-modification is given by [3]

$$\begin{aligned} \dot{\hat{\theta}}_1 &= -\gamma_1 ar - \gamma_1\sigma_{s1}\hat{\theta}_1 & (232)\\ \dot{\hat{\theta}}_2 &= -\gamma_2 sin(q_1) r - \gamma_2\sigma_{s2}\hat{\theta}_2, & (233) \end{aligned}$$

where

[3] As discussed in Remark 7.6, either fixed or switching σ modification can be used in the results of Theorem 2.

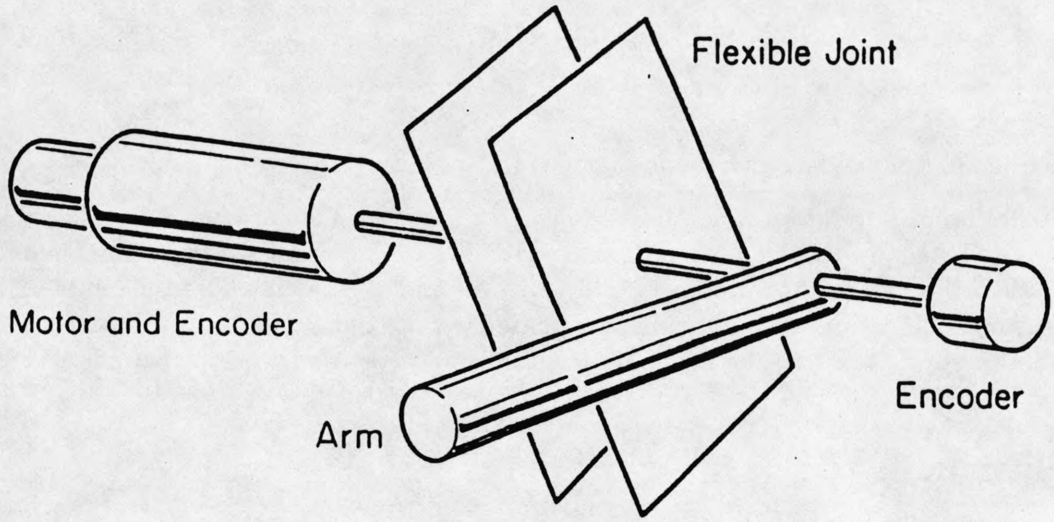

Figure 8: **Sketch of Experimental Hardware**

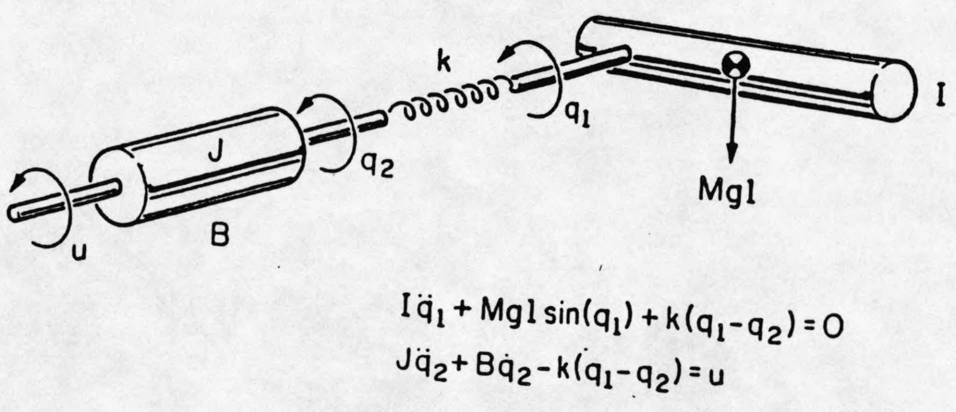

Figure 9: **Model of Single-link Flexible-joint**

$$\sigma_{si}(t) = \begin{cases} 0 & \text{if} \quad |\hat{\theta}_i(t)| < \theta_{0i} \\ \sigma_{0i}\left(\frac{|\hat{\theta}_i(t)|}{\theta_{0i}} - 1\right) & \text{if} \quad \theta_{0i} \leq |\hat{\theta}_i(t)| \leq 2\theta_{0i} \\ \sigma_{0i} & \text{if} \quad |\hat{\theta}_i(t)| > 2\theta_{0i} \end{cases} \qquad i = 1, 2. \tag{234}$$

The gains used in the control law and the parameter update law are shown in Table 3.

The simulation results shown in Figure 10-Figure 12 illustrate the parameter drift instability mechanism as predicted by the analysis (compare with Figure 5.) The states remain bounded for sometime, then the parameter estimates rapidly diverge and all the states become unbounded. Introducing the σ modification in the parameter update law, the signals are bounded as predicted by Theorem 2. Note that the tracking errors do not converge to zero but remain bounded.

8 Conclusions

In this report we have given stability proofs for a composite adaptive control law for flexible joint robot manipulators. The complexity of the analysis points out the difficulty of the control problem for this class of systems. Although our results give only sufficient conditions for local stability it can be argued, based on what is known about the behavior of adaptive control systems, that this is the best one can do without additional compensation. One promising approach to extend these results would be to incorporate the integral manifold based corrective control idea. We are currently investigating this extension. In addition we have already produced some experimental results of this scheme for a single–link, flexible joint mechanism that we have constructed [6]. Further experimental results, including an illustration of the instability mechanisms shown in the simulation example, are under investigation.

Parameter	Nominal Value	True Value
Link Inertia: I $(kg-m^2)$	0.031	(1+0.5)0.031=0.0465
Rotor Inertia: J $(kg-m^2)$	0.004	0.004
Rotor Friction: B $(N-m-sec/rad)$	0.007	0.007
Nominal Load: Mgl $(N-m)$	0.8	(1+0.5)0.8=1.2
Joint Stiffness k $(N-m/rad)$	5	5

Table 1: **Nominal and True Values of the Arm Parameters**

$q^d(t)$	**Amplitude A**	**Frequency ω**
$A\,sin(\omega\,t)$	0.1	17

Table 2: **Desired Trajectory**

Gain	λ	K_D	K_v	γ_1	γ_2	σ_{01}	σ_{02}	θ_{01}	θ_{02}
Value	10	0.2	$0.2\sqrt{2}$	0.001	10	100	0.1	0.051005	1.212

Table 3: **Control Law and Parameter Update Law Gain Values**

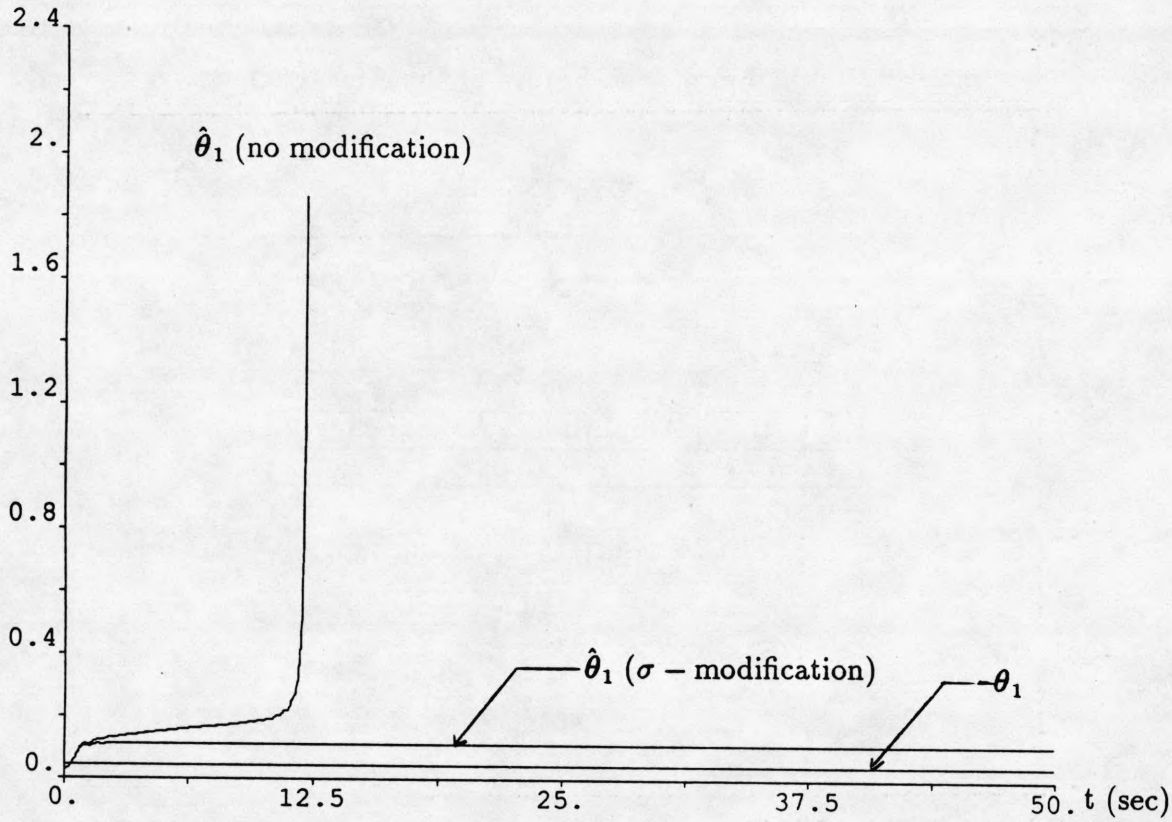

Figure 10: **Parameter Drift Instability Example :** $\hat{\theta}_1$

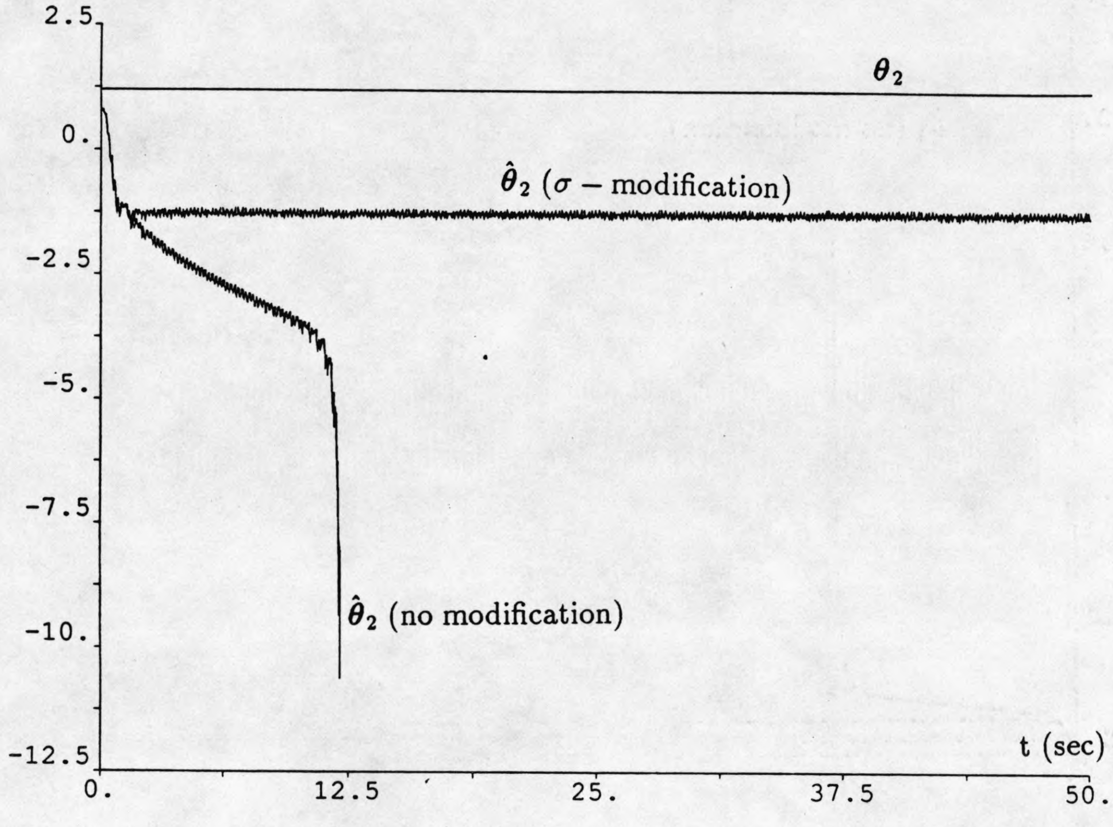

Figure 11: **Parameter Drift Instability Example :** $\hat{\theta}_2$

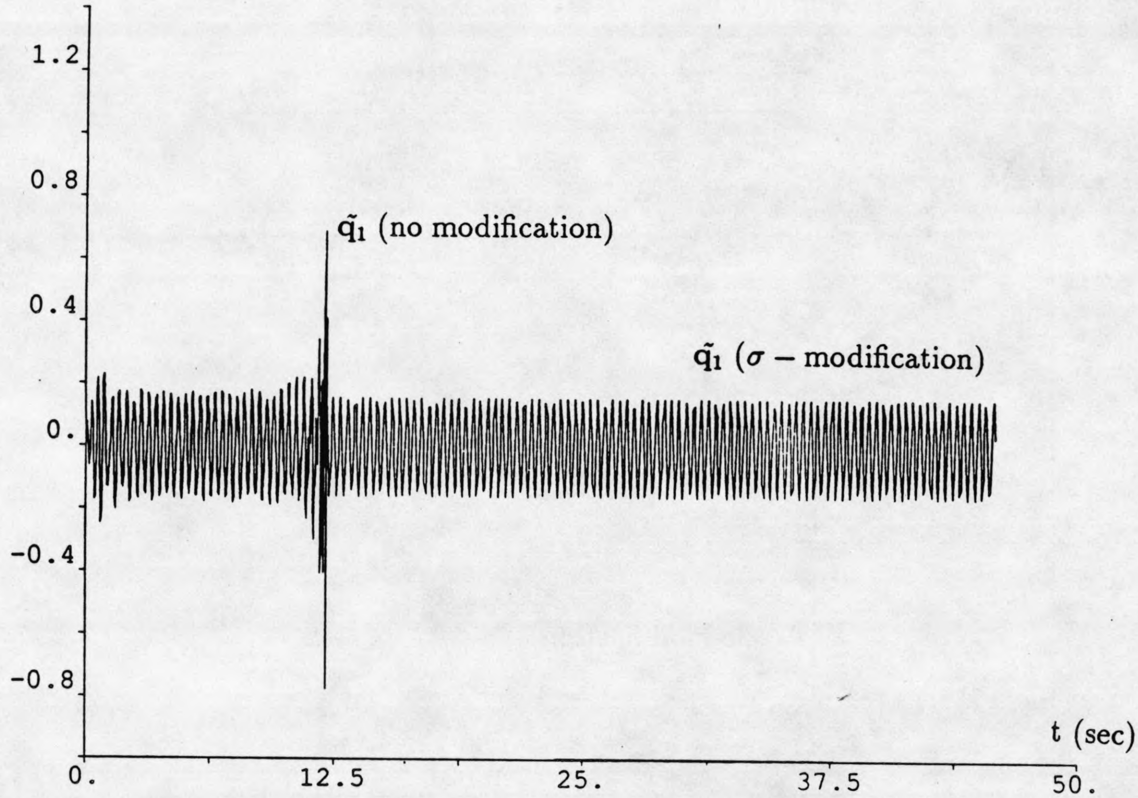

Figure 12: **Parameter Drift Instability Example :** $\tilde{q}_1$

9 Appendices

A Singular Perturbation Model Development

The original singularly perturbed adaptive control system is defined as follows.

i) Plant:

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{z} \tag{235}$$

$$\epsilon^2 J\ddot{\mathbf{z}} + \epsilon K_2\dot{\mathbf{z}} + K_1\mathbf{z} = K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). \tag{236}$$

ii) Controller (designed for the rigid plant (13)):

$$\mathbf{u}_s = (\hat{D}(\mathbf{q}_1) + \hat{J})\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r}, \tag{237}$$

where $\hat{D}$, $\hat{J}$, $\hat{C}$ and $\hat{\mathbf{g}}$ represent the terms in (13) with estimated values of the parameters, K_D is a diagonal matrix of positive gains, and

$$\mathbf{v} = \dot{\mathbf{q}}^d - \Lambda\tilde{\mathbf{q}}_1, \tag{238}$$

$$\mathbf{r} = \dot{\mathbf{q}}_1 - \mathbf{v} = \dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_d + \Lambda\tilde{\mathbf{q}}_1 = \dot{\tilde{\mathbf{q}}}_1 + \Lambda\tilde{\mathbf{q}}_1, \tag{239}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{q}}_d - \Lambda\dot{\tilde{\mathbf{q}}}_1, \tag{240}$$

$$\tilde{\mathbf{q}_1} = \mathbf{q}_1 - \mathbf{q}^d. \tag{241}$$

Note that

$$\ddot{\mathbf{q}}_1 = \dot{\mathbf{r}} + \mathbf{a}, \tag{242}$$

$$\dot{\mathbf{q}}_1 = \mathbf{r} + \mathbf{v}. \tag{243}$$

Λ is a constant diagonal matrix, and $\mathbf{q}^d(t)$ is the reference trajectory which is at least three times continuously differentiable.

iii) Parameter Update Law:

$$\dot{\tilde{\boldsymbol{\theta}}} = -\Gamma^{-1}Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\mathbf{r}, \tag{244}$$

where Γ is some symmetric, positive definite matrix, $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is the parameter error, and

$$(\hat{D}(\mathbf{q}_1) + \hat{J})\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) = Y(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\hat{\boldsymbol{\theta}}. \tag{245}$$

Let's take equation (236) and rewrite it as

$$\epsilon^2 J\ddot{\mathbf{z}} = -\epsilon K_2\dot{\mathbf{z}} - K_1\mathbf{z} + K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). \tag{246}$$

Premultiplying both sides by J^{-1}, we obtain

$$\epsilon^2\ddot{\mathbf{z}} = -\epsilon J^{-1}K_2\dot{\mathbf{z}} - J^{-1}K_1\mathbf{z} + J^{-1}K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). \tag{247}$$

Define

$$\mathbf{w} := \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix}, \tag{248}$$

and

$$\mathbf{u} := \mathbf{u}_s - J\ddot{\mathbf{q}}_1. \tag{249}$$

Using (248)-(249), equation (247) can be written in matrix form as follows

$$\epsilon\dot{\mathbf{w}} = \begin{bmatrix} \epsilon\dot{\mathbf{z}} \\ \epsilon^2\ddot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -J^{-1}K_1 & -J^{-1}K_2 \end{bmatrix}\begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ J^{-1}K_1 \end{bmatrix}\mathbf{u}. \tag{250}$$

Hence, we can write equation (247) as

$$\epsilon\dot{\mathbf{w}} = A_2\mathbf{w} + B_2\mathbf{u}, \tag{251}$$

where

$$A_2 = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -J^{-1}K_1 & -J^{-1}K_2 \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{252}$$

and

$$B_2 = \begin{bmatrix} 0_{n\times n} \\ J^{-1}K_1 \end{bmatrix} \in \mathbf{R}^{2n\times n}. \tag{253}$$

$\mathbf{w}$ and $\mathbf{u}$ are given by (248) and (249) respectively.

Now let's consider equation (235) and rewrite it for convenience

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{z}. \tag{254}$$

We add and subtract $\mathbf{u}$ in the right hand side of (254)

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u} + \mathbf{z} - \mathbf{u}. \tag{255}$$

Now, we replace the expression of $\mathbf{u}$ from (249)

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s - J\ddot{\mathbf{q}}_1 + \mathbf{z} - \mathbf{u}, \tag{256}$$

$\Longleftrightarrow$

$$[D(\mathbf{q}_1) + J]\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s + \mathbf{z} - \mathbf{u}. \tag{257}$$

Next, we replace (242)-(243) and obtain

$$[D(\mathbf{q}_1) + J][\dot{\mathbf{r}} + \mathbf{a}] + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)[\mathbf{r} + \mathbf{v}] + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s + \mathbf{z} - \mathbf{u}, \tag{258}$$

$\Longleftrightarrow$

$$[D(\mathbf{q}_1) + J]\dot{\mathbf{r}} + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{r} = \mathbf{u}_s - \mathbf{g}(\mathbf{q}_1) - [D(\mathbf{q}_1) + J]\mathbf{a} - C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \mathbf{z} - \mathbf{u}. \tag{259}$$

Now replace $\mathbf{u}_s$ from (237)

$$\begin{aligned} [D(\mathbf{q}_1) + J]\dot{\mathbf{r}} + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{r} &= [\hat{D}(\mathbf{q}_1) + \hat{J}]\mathbf{a} + \hat{C}(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r} - \mathbf{g}(\mathbf{q}_1) \\ &- [D(\mathbf{q}_1) + J]\mathbf{a} - C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \mathbf{z} - \mathbf{u}, \end{aligned} \tag{260}$$

$\Longleftrightarrow$

$$[D(\mathbf{q}_1) + J]\dot{\mathbf{r}} + [C(\mathbf{q}_1, \dot{\mathbf{q}}_1) + K_D]\mathbf{r} = [\tilde{D}(\mathbf{q}_1) + \tilde{J}]\mathbf{a} + \tilde{C}(\mathbf{q}_1, \dot{\mathbf{q}}_1)\mathbf{v} + \tilde{\mathbf{g}}(\mathbf{q}_1) + \mathbf{z} - \mathbf{u}, \tag{261}$$

$\Longleftrightarrow$

$$[D(\mathbf{q}_1) + J]\dot{\mathbf{r}} + [C(\mathbf{q}_1, \dot{\mathbf{q}}_1) + K_D]\mathbf{r} = Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a})\tilde{\boldsymbol{\theta}} + \mathbf{z} - \mathbf{u}. \tag{262}$$

Define

$$\mathbf{x} := \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix}. \tag{263}$$

Consequently, using (239)-(240),

$$\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} \\ \Lambda & I_{n\times n} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix}. \tag{264}$$

Also, define

$$M(\mathbf{q}_1) := D(\mathbf{q}_1) + J. \tag{265}$$

Using definitions (263) and (265), equation (262) is written in matrix form as follows

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} \dot{\tilde{\mathbf{q}}}_1 \\ \dot{\mathbf{r}} \end{bmatrix} \\ &= \begin{bmatrix} -\Lambda & I_{n\times n} \\ -M(\mathbf{q}_1)^{-1}[C(\mathbf{q}_1, \dot{\mathbf{q}}_1) + K_D] & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ M(\mathbf{q}_1)^{-1}Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a}) \end{bmatrix} \tilde{\boldsymbol{\theta}} \\ &+ \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ M(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ -M(\mathbf{q}_1)^{-1} \end{bmatrix} \mathbf{u}. \end{aligned} \tag{266}$$

Hence, we can write equation (266) as

$$\dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{w} + B_1\mathbf{u}. \tag{267}$$

where

$$A_1 = A_1(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d) = \begin{bmatrix} -\Lambda & I_{n\times n} \\ -M(\mathbf{q}_1)^{-1}[C(\mathbf{q}_1, \dot{\mathbf{q}}_1) + K_D] & 0_{n\times n} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{268}$$

$$\Phi = \Phi(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d) = \begin{bmatrix} 0_{n\times r} \\ M(\mathbf{q}_1)^{-1}Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a}) \end{bmatrix} \in \mathbf{R}^{2n\times r}, \tag{269}$$

$$A_3 = A_3(\mathbf{x}, \mathbf{q}_d) = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ M(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{270}$$

$$B_1 = B_1(\mathbf{x}, \mathbf{q}_d) = \begin{bmatrix} 0_{n\times n} \\ -M(\mathbf{q}_1)^{-1} \end{bmatrix} \in \mathbf{R}^{2n\times n}. \tag{271}$$

The Parameter Update Law (244) is now rewritten in terms of $\mathbf{x}$ as follows

$$\begin{aligned} \dot{\tilde{\boldsymbol{\theta}}} &= -\Gamma^{-1}Y^T(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v})\mathbf{r} \\ &= -\Gamma^{-1}\begin{bmatrix} 0_{r\times n} & Y^T(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v}) \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix}. \end{aligned} \tag{272}$$

Hence, we can write (272) as

$$\dot{\tilde{\boldsymbol{\theta}}} = -\Gamma^{-1}\varphi\mathbf{x}, \tag{273}$$

where

$$\varphi = \varphi(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d) = \begin{bmatrix} 0_{r\times n} & Y^T(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{a}, \mathbf{v}) \end{bmatrix} \in \mathbf{R}^{r\times 2n}. \tag{274}$$

Remark a1 : Note that using (249) and (235) we obtain

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_s - J\ddot{\mathbf{q}}_1 \\ &= \mathbf{u}_s - JD(\mathbf{q}_1)^{-1}[\mathbf{z} - C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 - \mathbf{g}(\mathbf{q}_1)] \\ &= \mathbf{u}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{w}, \mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d). \end{aligned} \tag{275}$$

Remark a2 : Up to this point we have transformed the original singularly perturbed system given by the Plant (235)-(236), the Controller (237), and the Parameter Update Law (244) into an equivalent set of singularly perturbed equations given by a slow subsystem, equations (267) and (273), and a fast subsystem , equation (251).

To make the stability analysis even more tractable, let's further define [2]

$$\mathbf{y} := \mathbf{w} + A_2^{-1}B_2\mathbf{u}. \tag{276}$$

Using this definition of $\mathbf{y}$, we can write

$$\epsilon\dot{\mathbf{y}} := \epsilon\dot{\mathbf{w}} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}. \tag{277}$$

We now substitute in (277) $\epsilon\dot{\mathbf{w}}$ from the equation of the fast subsystem (251)

$$\begin{aligned} \epsilon\dot{\mathbf{y}} &= A_2\mathbf{w} + B_2\mathbf{u} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}} \\ &= A_2[\mathbf{w} + A_2^{-1}B_2\mathbf{u}] + \epsilon A_2^{-1}B_2\dot{\mathbf{u}} \\ &= A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}. \end{aligned} \tag{278}$$

Hence, the fast subsystem in terms of the fast variable $\mathbf{y}$ is given by

$$\epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}. \tag{279}$$

Let's now express equation (267) in terms of the new fast variable $\mathbf{y}$.

Remark a3 : We should verify that

$$A_3\mathbf{w} + B_1\mathbf{u} = A_3\mathbf{y}. \tag{280}$$

To see this, note first that it is easy to verify that

$$B_2 = A_2 \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix}. \tag{281}$$

Consequently,

$$\begin{aligned} A_3A_2^{-1}B_2 &= A_3A_2^{-1}A_2 \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} = A_3 \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \\ &= \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ -M(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} \\ -M(\mathbf{q}_1)^{-1} \end{bmatrix} = B_1. \end{aligned} \tag{282}$$

Using (282) and the definition of $\mathbf{y}$, equation (276), we conclude that

$$\begin{aligned} A_3\mathbf{w} + B_1\mathbf{u} &= A_3\mathbf{w} + A_3A_2^{-1}B_2\mathbf{u} \\ &= A_3[\mathbf{w} + A_2^{-1}B_2\mathbf{u}] \\ &= A_3\mathbf{y}. \end{aligned} \tag{283}$$

Equations (283) suggest that we write (267) as follows

$$\dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}. \tag{284}$$

In summary, the original singularly perturbed system given by the Plant (235)-(236), the Controller (237), and the Parameter Update Law (244) are transformed into an equivalent set of singularly perturbed equations given by a slow subsystem, equations (284) and (273), and a fast subsystem , equation (279). To conclude, we write the resulting system of equations into two equivalent forms both of which are helpful in the analysis,

$$\begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma^{-1}\varphi\mathbf{x} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{285}$$

or equivalently,

$$\begin{cases} \dot{\mathbf{p}} = f(t,\mathbf{p},\mathbf{y}) = \begin{bmatrix} A_1 & \Phi \\ -\Gamma^{-1}\varphi & 0_{n\times n} \end{bmatrix} \mathbf{p} + \begin{bmatrix} A_3 \\ 0_{n\times n} \end{bmatrix} \mathbf{y} \\ \epsilon\dot{\mathbf{y}} = g(t,\mathbf{p},\mathbf{y},\epsilon) = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \tag{286}$$

where

$$\mathbf{p} = \begin{bmatrix} \mathbf{x} \\ \tilde{\boldsymbol{\theta}} \end{bmatrix}, \tag{287}$$

$$\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix}, \tag{288}$$

$$A_1 = A_1(\mathbf{x},\mathbf{q}_d,\dot{\mathbf{q}}_d) = \begin{bmatrix} -\Lambda & I_{n\times n} \\ -M(\mathbf{q}_1)^{-1}[C(\mathbf{q}_1,\dot{\mathbf{q}}_1) + K_D] & 0_{n\times n} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{289}$$

$$M(\mathbf{q}_1) \;=\; D(\mathbf{q}_1) \;+\; J, \tag{290}$$

$$\Phi = \Phi(\mathbf{x},\mathbf{q}_d,\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d) = \begin{bmatrix} 0_{n\times r} \\ M(\mathbf{q}_1)^{-1}Y(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a}) \end{bmatrix} \in \mathbf{R}^{2n\times r}, \tag{291}$$

$$A_3 = A_3(\mathbf{x},\mathbf{q}_d) = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ M(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{292}$$

$$\Gamma \in \mathbf{R}^{r\times r} \quad \text{is some symmetric positive definite matrix,} \tag{293}$$

$$\varphi = \varphi(\mathbf{x},\mathbf{q}_d,\dot{\mathbf{q}}_d) = \begin{bmatrix} 0_{r\times n} & Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v}) \end{bmatrix} \in \mathbf{R}^{r\times 2n}, \tag{294}$$

$$\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} \quad \text{and} \quad (D(\mathbf{q}_1) + J)\mathbf{a} + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \mathbf{g}(\mathbf{q}_1) = Y(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\boldsymbol{\theta}, \tag{295}$$

$$A_2 = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -J^{-1}K_1 & -J^{-1}K_2 \end{bmatrix} \in \mathbf{R}^{2n\times 2n}, \tag{296}$$

$$B_2 = \begin{bmatrix} 0_{n\times n} \\ J^{-1}K_1 \end{bmatrix} \in \mathbf{R}^{2n\times n}, \tag{297}$$

$$\mathbf{u} = \mathbf{u}_s - J\ddot{\mathbf{q}}_1, \tag{298}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} + A_2^{-1}B_2\mathbf{u} \qquad ; \qquad \mathbf{z} = K(\mathbf{q}_2 - \mathbf{q}_1) = \frac{1}{\epsilon^2}K_1(\mathbf{q}_2 - \mathbf{q}_1) \tag{299}$$

B Detailed Verification of Facts

Fact 5.1 The rigid joint plant, the adaptive control law, and the parameter update law are given by

$$\begin{cases} (D(\mathbf{q}_1)+J)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s \\ \mathbf{u}_s = (\hat{D}(\mathbf{q}_1)+\hat{J})\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma^{-1}Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\mathbf{r}. \end{cases} \tag{300}$$

Combining the plant and the Controller from (300), and recalling from (265) that $M(\mathbf{q}_1) = D(\mathbf{q}_1)+J$, we obtain

$$\begin{aligned} M(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) &= \mathbf{u}_s \\ &= \hat{M}(\mathbf{q}_1)\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r}. \end{aligned} \tag{301}$$

Using the fact that $\ddot{\mathbf{q}}_1 = \dot{\mathbf{r}} + \mathbf{a}$ and $\dot{\mathbf{q}}_1 = \mathbf{r} + \mathbf{v}$ (see (242) and (243)), equation (301) becomes

$$\begin{aligned} M(\mathbf{q}_1)[\dot{\mathbf{r}}+\mathbf{a}] &+ C(\mathbf{q}_1,\dot{\mathbf{q}}_1)[\mathbf{r}+\mathbf{v}] + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}_s \\ &= \hat{M}(\mathbf{q}_1)\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r}, \end{aligned} \tag{302}$$

$\Longleftrightarrow$

$$\begin{aligned} M(\mathbf{q}_1)\dot{\mathbf{r}} &+ C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{r} + [M(\mathbf{q}_1) - \hat{M}(\mathbf{q}_1)]\mathbf{a} \\ &+ [C(\mathbf{q}_1,\dot{\mathbf{q}}_1) - \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)]\mathbf{v} + \mathbf{g}(\mathbf{q}_1) - \hat{\mathbf{g}}(\mathbf{q}_1) + K_D\mathbf{r} = 0, \end{aligned} \tag{303}$$

$\Longleftrightarrow$

$$M(\mathbf{q}_1)\dot{\mathbf{r}} + [C(\mathbf{q}_1,\dot{\mathbf{q}}_1) + K_D]\mathbf{r} + [\tilde{M}(\mathbf{q}_1)\mathbf{a} + \tilde{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \tilde{\mathbf{g}}(\mathbf{q}_1)] = 0, \tag{304}$$

$\Longleftrightarrow$

$$M(\mathbf{q}_1)\dot{\mathbf{r}} + [C(\mathbf{q}_1,\dot{\mathbf{q}}_1) + K_D]\mathbf{r} - Y(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\tilde{\boldsymbol{\theta}} = 0. \tag{305}$$

In matrix form, equation (305) is written as

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} \dot{\tilde{\mathbf{q}}}_1 \\ \dot{\mathbf{r}} \end{bmatrix} \\ &= \begin{bmatrix} -\Lambda & I_{n\times n} \\ -M(\mathbf{q}_1)^{-1}[C(\mathbf{q}_1,\dot{\mathbf{q}}_1)+K_D] & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ M(\mathbf{q}_1)^{-1}Y(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a}) \end{bmatrix} \tilde{\boldsymbol{\theta}} \\ &= A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}}. \end{aligned} \tag{306}$$

From Appendix A, it is obvious that

$$\begin{aligned} \dot{\tilde{\boldsymbol{\theta}}} &= -\Gamma^{-1}Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\mathbf{r} \\ &= -\Gamma^{-1}\varphi\mathbf{x}. \end{aligned} \tag{307}$$

Thus, we conclude that

$$\left\{\begin{array}{l}(D(\mathbf{q}_1)+J)\ddot{\mathbf{q}}_1+C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1+\mathbf{g}(\mathbf{q}_1)=\mathbf{u}_s\\ \mathbf{u}_s=(\hat{D}(\mathbf{q}_1)+\hat{J})\mathbf{a}+\hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v}+\hat{\mathbf{g}}(\mathbf{q}_1)-K_D\mathbf{r}\\ \dot{\tilde{\boldsymbol{\theta}}}=-\Gamma^{-1}Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{a},\mathbf{v})\mathbf{r}\end{array}\right.$$

$$\Longleftrightarrow$$

$$\left\{\begin{array}{l}\dot{\mathbf{x}}=A_1\mathbf{x}+\Phi\tilde{\boldsymbol{\theta}}\\ \dot{\tilde{\boldsymbol{\theta}}}=-\Gamma^{-1}\varphi\mathbf{x}\end{array}\right.$$

$$\Longleftrightarrow$$

$$\mathcal{S}_{\mathbf{r}}\;:\;\dot{\mathbf{p}}=\left[\begin{array}{cc}A_1 & \Phi\\ -\Gamma^{-1}\varphi & 0_{n\times n}\end{array}\right]\mathbf{p}. \tag{308}$$

□

Fact 5.2 The Lyapunov Function Candidate for the reduced system $\mathcal{S}_{\mathbf{r}}$ is

$$V(\mathbf{p})=\frac{1}{2}\mathbf{r}^TM(\mathbf{q}_1)\mathbf{r}+\tilde{\mathbf{q}}_1^T\Lambda^TK_D\tilde{\mathbf{q}}_1+\frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\tilde{\boldsymbol{\theta}}. \tag{309}$$

The time derivative of V along the trajectory solution of $\mathcal{S}_{\mathbf{r}}$ is

$$\begin{aligned}
\dot{V} &= [\nabla_{\mathbf{p}}V(\mathbf{p})]^Tf(t,\mathbf{p},\mathbf{y}=0)\\
&= \mathbf{r}^TM(\mathbf{q}_1)\dot{\mathbf{r}}+\frac{1}{2}\mathbf{r}^T\dot{M}(\mathbf{q}_1)\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1+\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\dot{\tilde{\boldsymbol{\theta}}}\\
&= \mathbf{r}^T[-C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{r}-K_D\mathbf{r}+Y(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a})\tilde{\boldsymbol{\theta}}]+\frac{1}{2}\mathbf{r}^T\dot{M}(\mathbf{q}_1)\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1+\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\dot{\tilde{\boldsymbol{\theta}}}\\
&= \frac{1}{2}\mathbf{r}^T[\dot{M}(\mathbf{q}_1)-2C(\mathbf{q}_1,\dot{\mathbf{q}}_1)]\mathbf{r}-\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1+\mathbf{r}^TY(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a})\tilde{\boldsymbol{\theta}}+\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}\dot{\tilde{\boldsymbol{\theta}}}\\
&= -\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1+\mathbf{r}^TY(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a})\tilde{\boldsymbol{\theta}}+\tilde{\boldsymbol{\theta}}^T\Gamma^{-1}[-\Gamma Y^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a})\mathbf{r}]\\
&= -\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1+\mathbf{r}^TY(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a})\tilde{\boldsymbol{\theta}}-\tilde{\boldsymbol{\theta}}^TY^T(\mathbf{q}_1,\dot{\mathbf{q}}_1,\mathbf{v},\mathbf{a})\mathbf{r}\\
&= -\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1\\
&= -\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D[\mathbf{r}-\Lambda\tilde{\mathbf{q}}_1]\\
&= -\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\mathbf{r}-2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\Lambda\tilde{\mathbf{q}}_1\\
&= -\mathbf{r}^TK_D\mathbf{r}+\tilde{\mathbf{q}}_1^T\Lambda^TK_D\mathbf{r}+\mathbf{r}^TK_D^T\Lambda\tilde{\mathbf{q}}_1-2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\Lambda\tilde{\mathbf{q}}_1\\
&= -\left[\begin{array}{cc}\tilde{\mathbf{q}}_1^T & \mathbf{r}^T\end{array}\right]\left[\begin{array}{cc}2\Lambda^TK_D\Lambda & -\Lambda^TK_D\\ -K_D^T\Lambda & K_D\end{array}\right]\left[\begin{array}{c}\tilde{\mathbf{q}}_1\\ \mathbf{r}\end{array}\right]\\
&= -\mathbf{x}^TR\mathbf{x}.
\end{aligned} \tag{310}$$

where

$$R=\left[\begin{array}{cc}2\Lambda^TK_D\Lambda & -\Lambda^TK_D\\ -K_D^T\Lambda & K_D\end{array}\right]. \tag{311}$$

We should show that R is positive definite. It is clear from above that

$$-\mathbf{x}^TR\mathbf{x} \;=\; -\mathbf{r}^TK_D\mathbf{r}+2\tilde{\mathbf{q}}_1^T\Lambda^TK_D\dot{\tilde{\mathbf{q}}}_1$$

$$
\begin{aligned}
&= -[\dot{\tilde{\mathbf{q}}}_1^T + \tilde{\mathbf{q}}_1^T\Lambda^T]K_D[\dot{\tilde{\mathbf{q}}}_1 + \Lambda\tilde{\mathbf{q}}_1] + 2\tilde{\mathbf{q}}_1^T\Lambda^T K_D\dot{\tilde{\mathbf{q}}}_1 \\
&= -\dot{\tilde{\mathbf{q}}}_1^T K_D\dot{\tilde{\mathbf{q}}}_1 - \dot{\tilde{\mathbf{q}}}_1^T K_D\Lambda\tilde{\mathbf{q}}_1 - \tilde{\mathbf{q}}_1^T\Lambda^T K_D\dot{\tilde{\mathbf{q}}}_1 - \tilde{\mathbf{q}}_1^T\Lambda^T K_D\Lambda\tilde{\mathbf{q}}_1 + 2\tilde{\mathbf{q}}_1^T\Lambda^T K_D\dot{\tilde{\mathbf{q}}}_1 \\
&= -\tilde{\mathbf{q}}_1^T\Lambda^T K_D\Lambda\tilde{\mathbf{q}}_1 - \dot{\tilde{\mathbf{q}}}_1^T K_D\dot{\tilde{\mathbf{q}}}_1 \\
&= -\begin{bmatrix} \tilde{\mathbf{q}}_1^T & \dot{\tilde{\mathbf{q}}}_1^T \end{bmatrix} \begin{bmatrix} \Lambda^T K_D\Lambda & 0_{n\times n} \\ 0_{n\times n} & K_D \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix} \\
&= -\begin{bmatrix} \tilde{\mathbf{q}}_1^T & \dot{\tilde{\mathbf{q}}}_1^T \end{bmatrix} R^* \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix},
\end{aligned} \tag{312}
$$

where

$$
R^* = \begin{bmatrix} \Lambda^T K_D\Lambda & 0_{n\times n} \\ 0_{n\times n} & K_D \end{bmatrix}. \tag{313}
$$

Clearly R^* is positive definite, and R can obtained from R^* using the nonsingular linear transformation $\mathcal{T}$ as follows. Recall that $\mathcal{T}$ was defined as

$$
\mathbf{x} = \mathcal{T}\begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix} = \mathcal{T}^{-1}\mathbf{x} \quad ; \quad \mathcal{T} = \begin{bmatrix} I_{n\times n} & 0_{n\times n} \\ \Lambda & I_{n\times n} \end{bmatrix} \tag{314}
$$

Hence (312) can be written as

$$
-\mathbf{x}^T R\mathbf{x} = -\mathbf{x}^T(\mathcal{T}^{-1})^T R^*\mathcal{T}^{-1}\mathbf{x}. \tag{315}
$$

Since definiteness is invariant under nonsingular linear transformations, we conclude that R, similarly to R^*, is positive definite.

□

Fact 5.3 Recall that

$$
\begin{aligned}
\mathbf{u} &= \mathbf{u}_s - J\ddot{\mathbf{q}}_1 \\
&= \mathbf{u}_s - JD(\mathbf{q}_1)^{-1}[\mathbf{z} - C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 - \mathbf{g}(\mathbf{q}_1)] \\
&= \mathbf{u}_s - JD(\mathbf{q}_1)^{-1}\mathbf{z} + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1)
\end{aligned} \tag{316}
$$

$\Longleftrightarrow$

$$
\mathbf{u} + JD(\mathbf{q}_1)^{-1}\mathbf{z} = \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1) \tag{317}
$$

$\Longleftrightarrow$

$$
\mathbf{u} + \begin{bmatrix} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \epsilon\dot{\mathbf{z}} \end{bmatrix} = \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1). \tag{318}
$$

Substituting from (39), we obtain

$$
\begin{aligned}
&\mathbf{u} + \begin{bmatrix} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix} \left\{\mathbf{y} - A_2^{-1}B_2\mathbf{u}\right\} = \\
&\qquad \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1)
\end{aligned} \tag{319}
$$

$\Longleftrightarrow$

$$\left\{I_{n\times n} - \left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right] A_2^{-1}B_2\right\}\mathbf{u} =$$
$$-\left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right]\mathbf{y} + \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1). \quad (320)$$

Recall from Appendix A, equation (281) that $B_2 = A_2\begin{bmatrix} -I_{n\times n} \\ 0_{n\times n}\end{bmatrix}$, then (320) becomes

$$\left\{I_{n\times n} - \left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right]\begin{bmatrix} -I_{n\times n} \\ 0_{n\times n}\end{bmatrix}\right\}\mathbf{u} =$$
$$-\left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right]\mathbf{y} + \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1) \quad (321)$$

$\Longleftrightarrow$

$$\left\{I_{n\times n} + JD(\mathbf{q}_1)^{-1}\right\}\mathbf{u} =$$
$$-\left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right]\mathbf{y} + \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1). \quad (322)$$

Define

$$F := I_{n\times n} + JD(\mathbf{q}_1)^{-1}, \quad (323)$$

then (322) is written

$$F\mathbf{u} = -\left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right]\mathbf{y} + \mathbf{u}_s + JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1). \quad (324)$$

To invert F, we use the Modified Matrices Formula ([11], page 656) which states that for $\mathcal{A}$ and $\mathcal{C}$ nonsingular $m\times m$ and $n\times n$ matrices, respectively, we have

$$[\mathcal{A} + \mathcal{B}\mathcal{C}\mathcal{D}]^{-1} = \mathcal{A}^{-1} - \mathcal{A}^{-1}\mathcal{B}[\mathcal{D}\mathcal{A}^{-1}\mathcal{B} + \mathcal{C}^{-1}]^{-1}\mathcal{D}\mathcal{A}^{-1}. \quad (325)$$

Using the following correspondence $\mathcal{A} \longleftrightarrow I_{n\times n}$, $\mathcal{B} \longleftrightarrow J$, $\mathcal{C} \longleftrightarrow D(\mathbf{q}_1)^{-1}$, and $\mathcal{D} \longleftrightarrow I_{n\times n}$, then

$$F^{-1} = [I_{n\times n} + JD(\mathbf{q}_1)^{-1}]^{-1} = [I_{n\times n} + JD(\mathbf{q}_1)^{-1}I_{n\times n}]^{-1} = I_{n\times n} - J[J + D(\mathbf{q}_1)]^{-1}. \quad (326)$$

Hence, F^{-1} is a well defined matrix, and we can write (324) as

$$\begin{aligned}\mathbf{u} = & - F^{-1}\left[\begin{array}{cc} JD(\mathbf{q}_1)^{-1} & 0_{n\times n}\end{array}\right]\mathbf{y} + F^{-1}\mathbf{u}_s + F^{-1}JD(\mathbf{q}_1)^{-1}C(\mathbf{q}_1,\dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 \\ & + F^{-1}JD(\mathbf{q}_1)^{-1}\mathbf{g}(\mathbf{q}_1). \end{aligned} \quad (327)$$

Recall from (20) that

$$\begin{aligned}\mathbf{u}_s &= (\hat{D}(\mathbf{q}_1) + \hat{J})\mathbf{a} + \hat{C}(\mathbf{q}_1,\dot{\mathbf{q}}_1)\mathbf{v} + \hat{\mathbf{g}}(\mathbf{q}_1) - K_D\mathbf{r} \\ &= \mathbf{u}_s(\mathbf{x},\tilde{\boldsymbol{\theta}},\mathbf{q}_d,\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d). \end{aligned} \quad (328)$$

Equations (327) and (328) imply that

$$\mathbf{u} = \mathbf{u}(\mathbf{x},\mathbf{y},\tilde{\boldsymbol{\theta}},\mathbf{q}_d,\dot{\mathbf{q}}_d,\ddot{\mathbf{q}}_d), \quad (329)$$

and therefore

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\dot{\mathbf{y}} + \frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\dot{\tilde{\boldsymbol{\theta}}} + \frac{\partial \mathbf{u}}{\partial \mathbf{q}_d}\dot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \dot{\mathbf{q}}_d}\ddot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \ddot{\mathbf{q}}_d}\mathbf{q}_d^{(3)}. \tag{330}$$

Define

$$\rho(t) \;:=\; \frac{\partial \mathbf{u}}{\partial \mathbf{q}_d}\dot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \dot{\mathbf{q}}_d}\ddot{\mathbf{q}}_d + \frac{\partial \mathbf{u}}{\partial \ddot{\mathbf{q}}_d}\mathbf{q}_d^{(3)}, \tag{331}$$

then

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\dot{\mathbf{y}} + \frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\dot{\tilde{\boldsymbol{\theta}}} + \rho(t). \tag{332}$$

We now replace the expressions of $\dot{\mathbf{x}}$, $\dot{\mathbf{y}}$, and $\dot{\tilde{\boldsymbol{\theta}}}$ from the equations of system $\mathcal{S}$ in (332)

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}[\frac{1}{\epsilon}A_2\mathbf{y} + A_2^{-1}B_2\dot{\mathbf{u}}] + \frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}[-\Gamma\varphi\mathbf{x}] + \rho(t). \tag{333}$$

Rearranging (333), we obtain

$$[I_{n\times n} - \frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2^{-1}B_2]\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} - \frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi\mathbf{x} + \rho(t). \tag{334}$$

Note that

$$\begin{aligned}
I_{n\times n} - \frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2^{-1}B_2 &= I_{n\times n} - \frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2^{-1}A_2\begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \\
&= I_{n\times n} - \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \\
&= I_{n\times n} - \left\{-F^{-1}\begin{bmatrix} JD(\mathbf{q}_1)^{-1} & 0_{n\times n} \end{bmatrix}\right\}\begin{bmatrix} -I_{n\times n} \\ 0_{n\times n} \end{bmatrix} \\
&= I_{n\times n} - F^{-1}JD(\mathbf{q}_1)^{-1} \\
&= I_{n\times n} - F^{-1}[F - I_{n\times n}] \\
&= F^{-1}.
\end{aligned} \tag{335}$$

Equation (334) becomes

$$F^{-1}\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} - \frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi\mathbf{x} + \rho(t), \tag{336}$$

and

$$\dot{\mathbf{u}} = F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} - F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\Gamma\varphi\mathbf{x} + F\rho(t). \tag{337}$$

□

Fact 7.1 The Lyapunov Function Candidate for the reduced system $\mathcal{S}_{\mathrm{r}}^{\sigma}$ is the same as that of the reduced system $\mathcal{S}_r$, namely,

$$V(\mathbf{p}) = \frac{1}{2}\mathbf{r}^T M(\mathbf{q}_1)\mathbf{r} + \tilde{\mathbf{q}}_1^T \Lambda^T K_D \tilde{\mathbf{q}}_1 + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T \Gamma^{-1}\tilde{\boldsymbol{\theta}}. \tag{338}$$

Using the derivations in the proof of Fact 5.2, we conclude that the time derivative of V along the solution trajectories of $\mathcal{S}_{\mathrm{r}}^{\sigma}$ is

$$\begin{aligned} \dot{V} &= [\nabla_{\mathbf{p}} V(\mathbf{p})]^T f(t, \mathbf{p}, \mathbf{y} = 0) \\ &= -\mathbf{r}^T K_D \mathbf{r} + 2\tilde{\mathbf{q}}_1^T \Lambda^T K_D \dot{\tilde{\mathbf{q}}}_1 + \mathbf{r}^T Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a})\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T \Gamma^{-1}\dot{\tilde{\boldsymbol{\theta}}} \\ &= -\mathbf{r}^T K_D \mathbf{r} + 2\tilde{\mathbf{q}}_1^T \Lambda^T K_D \dot{\tilde{\mathbf{q}}}_1 + \mathbf{r}^T Y(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a})\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T \Gamma^{-1}[-\Gamma Y^T(\mathbf{q}_1, \dot{\mathbf{q}}_1, \mathbf{v}, \mathbf{a})\mathbf{r} - \sigma\Gamma\hat{\boldsymbol{\theta}}] \\ &= -\mathbf{r}^T K_D \mathbf{r} + 2\tilde{\mathbf{q}}_1^T \Lambda^T K_D \dot{\tilde{\mathbf{q}}}_1 - \sigma\tilde{\boldsymbol{\theta}}^T\hat{\boldsymbol{\theta}} \\ &= -\mathbf{x}^T R\mathbf{x} - \sigma\tilde{\boldsymbol{\theta}}^T\hat{\boldsymbol{\theta}}, \end{aligned} \tag{339}$$

where R is given by (311). Note that

$$\begin{aligned} -\sigma\tilde{\boldsymbol{\theta}}^T\hat{\boldsymbol{\theta}} &= -\sigma\tilde{\boldsymbol{\theta}}^T\left(\boldsymbol{\theta} + \tilde{\boldsymbol{\theta}}\right) \\ &= -\sigma\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} - \sigma\tilde{\boldsymbol{\theta}}^T\boldsymbol{\theta}. \end{aligned} \tag{340}$$

Let

$$\begin{aligned} \Theta^2 &= \left(\sqrt{\frac{\sigma}{2}}\tilde{\boldsymbol{\theta}} + \sqrt{\frac{\sigma}{2}}\boldsymbol{\theta}\right)\cdot\left(\sqrt{\frac{\sigma}{2}}\tilde{\boldsymbol{\theta}} + \sqrt{\frac{\sigma}{2}}\boldsymbol{\theta}\right) \\ &= \frac{\sigma}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{\sigma}{2}\boldsymbol{\theta}^T\boldsymbol{\theta} + \sigma\tilde{\boldsymbol{\theta}}^T\boldsymbol{\theta}. \end{aligned} \tag{341}$$

Hence

$$-\sigma\tilde{\boldsymbol{\theta}}^T\boldsymbol{\theta} = -\Theta^2 + \frac{\sigma}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{\sigma}{2}\boldsymbol{\theta}^T\boldsymbol{\theta}. \tag{342}$$

Combining (340) and (342), we get

$$\begin{aligned} -\sigma\tilde{\boldsymbol{\theta}}^T\hat{\boldsymbol{\theta}} &= -\sigma\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} - \Theta^2 + \frac{\sigma}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{\sigma}{2}\boldsymbol{\theta}^T\boldsymbol{\theta} \\ &\leq -\frac{\sigma}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{\sigma}{2}\boldsymbol{\theta}^T\boldsymbol{\theta}. \end{aligned} \tag{343}$$

Using (343) in (339), we get

$$\dot{V} \leq -\mathbf{x}^T R\mathbf{x} - \frac{\sigma}{2}\tilde{\boldsymbol{\theta}}^T\tilde{\boldsymbol{\theta}} + \frac{\sigma}{2}\boldsymbol{\theta}^T\boldsymbol{\theta}. \tag{344}$$

We thus conclude that

$$\left[\begin{array}{l} \forall \mathbf{p} \in \mathbf{R}^{2n+r}, \ \ \forall \mathbf{y} \in \mathbf{R}^{2n}, \ \ \forall t \in \mathbf{R}_+ \\ \dot{V} = \left[\nabla_{\mathbf{p}} V(\mathbf{p})\right]^T f(t, \mathbf{p}, \mathbf{y} = 0) \ \ \leq -\alpha_1 \|\mathbf{x}\|_2^2 - \tfrac{1}{2}\sigma \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \tfrac{1}{2}\sigma\|\boldsymbol{\theta}\|_2^2 \\ \alpha_1 = \lambda_{\min}[R] > 0. \end{array}\right. \tag{345}$$

□

Fact 7.2 Note that $\mathbf{u}$ is the same when the σ-modification scheme is used. Hence, using (332) (see the proof of Fact 5.3), we conclude that

$$\begin{aligned}\dot{\mathbf{u}} &= F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} + F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\dot{\tilde{\boldsymbol{\theta}}} + F\rho(t) \\ &= F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} - F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\left(\Gamma\varphi\mathbf{x} + \sigma\Gamma\hat{\boldsymbol{\theta}}\right) + F\rho(t). \qquad (346)\end{aligned}$$

□

Fact 7.3 (169) is derived exactly like the second interconnection condition given by (79).

□

Fact 7.4 The Composite Lyapunov Function Candidate for the singularly perturbed system $\mathcal{S}_\sigma$ is given by (82), namely,

$$\mathcal{V}(\mathbf{p},\mathbf{y}) = (1-d)V(\mathbf{p}) + dW(\mathbf{y}) \quad , \quad 0 < d < 1. \qquad (347)$$

The derivative of $\mathcal{V}$ along the solution trajectories of $\mathcal{S}$ is given by (83)

$$\begin{aligned}\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &= (1-d)\left\{[\nabla_{\mathbf{p}}V(\mathbf{p})]^T f(t,\mathbf{p},\mathbf{y})\right\} + \frac{d}{\epsilon}\left\{[\nabla_{\mathbf{y}}W(\mathbf{y})]^T g(t,\mathbf{p},\mathbf{y},\epsilon)\right\} \\ &= (1-d)\left\{[\nabla_{\mathbf{p}}V(\mathbf{p})]^T f(t,\mathbf{p},\mathbf{y}=0)\right\} \\ &\quad +(1-d)\left\{[\nabla_{\mathbf{p}}V(\mathbf{p})]^T [f(t,\mathbf{p},\mathbf{y}) - f(t,\mathbf{p},\mathbf{y}=0)]\right\} \\ &\quad +\frac{d}{\epsilon}\left\{[\nabla_{\mathbf{y}}W(\mathbf{y})]^T g(t,\mathbf{p},\mathbf{y},\epsilon=0)\right\} \\ &\quad +\frac{d}{\epsilon}\left\{[\nabla_{\mathbf{y}}W(\mathbf{y})]^T [g(t,\mathbf{p},\mathbf{y},\epsilon) - g(t,\mathbf{p},\mathbf{y},\epsilon=0)]\right\}. \qquad (348)\end{aligned}$$

We now substitute equations (44),(161), (60), and (169)

$$\begin{aligned}\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\leq -(1-d)\alpha_1\|\mathbf{x}\|_2^2 - (1-d)\frac{1}{2}\sigma\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + (1-d)\frac{1}{2}\sigma\|\boldsymbol{\theta}\|_2^2 \\ &\quad +(1-d)\beta_1\|\mathbf{x}\|_2\|\mathbf{y}\|_2 - \frac{d}{\epsilon}\alpha_2\|\mathbf{y}\|_2^2 \\ &\quad +\frac{d}{\epsilon}\left\{\epsilon(\gamma_1' + \frac{1}{\epsilon}\gamma_2)\|\mathbf{y}\|_2^2 + \epsilon\beta_2'\|\mathbf{x}\|_2\|\mathbf{y}\|_2 + \epsilon\mu(t)\|\mathbf{y}\|_2\right\}. \qquad (349)\end{aligned}$$

Using (85) and (88), we can write (349) as

$$\begin{aligned}\dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) \leq &- \begin{bmatrix}\|\mathbf{x}\|_2 & \|\mathbf{y}\|_2\end{bmatrix} P_d' \begin{bmatrix}\|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2\end{bmatrix} - \frac{1}{2}(1-d)\sigma\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \\ &+ \frac{1}{2}(1-d)\sigma\|\boldsymbol{\theta}\|_2^2 + d\mu^2(t), \qquad (350)\end{aligned}$$

where

$$P_d' = \begin{bmatrix}(1-d)\alpha_1 & -\frac{(1-d)\beta_1 + d\beta_2'}{2} \\ -\frac{(1-d)\beta_1 + d\beta_2'}{2} & \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1\end{bmatrix}. \qquad (351)$$

□ (352)

Fact 7.7

$$\begin{aligned}
\sigma_s \tilde{\boldsymbol{\theta}}^T(t)\hat{\boldsymbol{\theta}}(t) &= \sigma_s \left(\hat{\boldsymbol{\theta}}^T(t) - \boldsymbol{\theta}\right)\hat{\boldsymbol{\theta}}(t) \\
&= \sigma_s \left\|\hat{\boldsymbol{\theta}}(t)\right\|_2^2 - \sigma_s \hat{\boldsymbol{\theta}}^T(t)\boldsymbol{\theta} \\
&\geq \sigma_s \left\|\hat{\boldsymbol{\theta}}(t)\right\|_2^2 - \sigma_s \left\|\hat{\boldsymbol{\theta}}(t)\right\|_2 \|\boldsymbol{\theta}\|_2 \\
&= \sigma_s \left\|\hat{\boldsymbol{\theta}}(t)\right\|_2 \left(\left\|\hat{\boldsymbol{\theta}}(t)\right\|_2 - \theta_0 + \theta_0 - \|\boldsymbol{\theta}\|_2\right) \\
&= \sigma_s \left\|\hat{\boldsymbol{\theta}}(t)\right\|_2 \left(\left\|\hat{\boldsymbol{\theta}}(t)\right\|_2 - \theta_0\right) + \sigma_s \left\|\hat{\boldsymbol{\theta}}(t)\right\|_2 (\theta_0 - \|\boldsymbol{\theta}\|_2) \qquad (353) \\
&\geq 0. \qquad (354)
\end{aligned}$$

(354) follows from (353) using (196) and (197).

□

Fact 7.9

Let's rewrite the equations of the singularly perturbed system $\mathcal{S}_\Sigma$ for convenience

$$\mathcal{S}_\Sigma \quad : \quad \begin{cases} \dot{\mathbf{x}} = A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y} \\ \dot{\tilde{\boldsymbol{\theta}}} = -\Gamma\varphi\mathbf{x} - \Gamma\Sigma\hat{\boldsymbol{\theta}} \\ \epsilon\dot{\mathbf{y}} = A_2\mathbf{y} + \epsilon A_2^{-1}B_2\dot{\mathbf{u}}, \end{cases} \qquad (355)$$

where

$$\Sigma = \operatorname{diag}\left[\sigma_{s1}, \sigma_{s2}, \cdots, \sigma_{sr}\right] \in \mathbf{R}^{r\times r}. \qquad (356)$$

and each σ_{si} is given by

$$\sigma_{si}(t) = \begin{cases} 0 & \text{if} \quad |\,\hat{\theta}_i(t)\,| < \theta_{0i} \\ \sigma_{0i}\left(\frac{|\hat{\theta}_i(t)|}{\theta_{0i}} - 1\right) & \text{if} \quad \theta_{0i} \leq |\,\hat{\theta}_i(t)\,| \leq 2\theta_{0i} \\ \sigma_{0i} & \text{if} \quad |\,\hat{\theta}_i(t)\,| > 2\theta_{0i} \end{cases} \qquad i = 1, 2, \cdots, r. \qquad (357)$$

Using the same analysis methodology as in the original switching σ-modification scheme, we first follow the lines of Fact 7.1, and choose the same Lyapunov Function Candidate V for the reduced system. It is easy to verify that the time derivative of V along the solution trajectories of the reduced system is given by

$$\dot{V} = -\mathbf{x}^T R\mathbf{x} - \tilde{\boldsymbol{\theta}}^T\Sigma\hat{\boldsymbol{\theta}}, \qquad (358)$$

where R is given by (311). Note that

$$-\tilde{\boldsymbol{\theta}}^T\Sigma\hat{\boldsymbol{\theta}} = -\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\Sigma\boldsymbol{\theta}. \qquad (359)$$

Define

$$\Sigma^{\frac{1}{2}} := \mathrm{diag}\left[\sqrt{\sigma_{s1}}, \sqrt{\sigma_{s2}}, \cdots, \sqrt{\sigma_{sr}}\right]. \tag{360}$$

Let

$$\begin{aligned}\Xi^2 &= \left(\sqrt{\frac{1}{2}}\Sigma^{\frac{1}{2}}\tilde{\boldsymbol{\theta}} + \sqrt{\frac{1}{2}}\Sigma^{\frac{1}{2}}\boldsymbol{\theta}\right) \cdot \left(\sqrt{\frac{1}{2}}\Sigma^{\frac{1}{2}}\tilde{\boldsymbol{\theta}} + \sqrt{\frac{1}{2}}\Sigma^{\frac{1}{2}}\boldsymbol{\theta}\right) \\ &= \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} + \frac{1}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta} + \tilde{\boldsymbol{\theta}}^T\Sigma\boldsymbol{\theta}. \end{aligned} \tag{361}$$

Hence,

$$-\tilde{\boldsymbol{\theta}}^T\Sigma\boldsymbol{\theta} = -\Xi^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} + \frac{1}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta}. \tag{362}$$

Combining (359) and (362), we get

$$\begin{aligned}-\tilde{\boldsymbol{\theta}}^T\Sigma\hat{\boldsymbol{\theta}} &= -\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} - \Xi^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} + \frac{1}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta} \\ &\leq -\frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} + \frac{1}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta}. \end{aligned} \tag{363}$$

Using (363) in (358), we get

$$\dot{V} \leq -\mathbf{x}^T R\mathbf{x} - \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\Sigma\tilde{\boldsymbol{\theta}} + \frac{1}{2}\boldsymbol{\theta}^T\Sigma\boldsymbol{\theta} \tag{364}$$

$$\leq -\alpha_1\|\mathbf{x}\|_2^2 - \frac{1}{2}\sigma_m\left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 + \frac{1}{2}\sigma_M\|\boldsymbol{\theta}\|_2^2, \tag{365}$$

where

$$\alpha_1 = \lambda_{\min}[R] \quad (\text{see } (55)), \tag{366}$$

$$\sigma_m := \min\{\sigma_{s1}, \sigma_{s2}, \cdots, \sigma_{sr}\}, \tag{367}$$

$$\sigma_M := \max\{\sigma_{s1}, \sigma_{s2}, \cdots, \sigma_{sr}\}. \tag{368}$$

The time derivative of $\mathbf{u}$ is given by

$$\dot{\mathbf{u}} = F\frac{\partial \mathbf{u}}{\partial \mathbf{x}}[A_1\mathbf{x} + \Phi\tilde{\boldsymbol{\theta}} + A_3\mathbf{y}] + \frac{1}{\epsilon}F\frac{\partial \mathbf{u}}{\partial \mathbf{y}}A_2\mathbf{y} - F\frac{\partial \mathbf{u}}{\partial \tilde{\boldsymbol{\theta}}}\left(\Gamma\varphi\mathbf{x} + \Gamma\Sigma\hat{\boldsymbol{\theta}}\right) + F\rho(t). \tag{369}$$

Assume

- (a2)$'''$: which consists of (a2)$''$ with σ_s, k_1'', and β_2'' replaced by Σ, k_1''', and β_2''' respectively.

Consequently, P_d'' given by (198) when (a2)$''$ was assumed, is now denoted P_d''' and is given by

$$P_d''' = \begin{bmatrix} (1-d)\alpha_1 & -\frac{(1-d)\beta_1 + d\beta_2'''}{2} \\ -\frac{(1-d)\beta_1 + d\beta_2'''}{2} & \frac{d}{\epsilon}(\alpha_2 - \gamma_2) - d\gamma_1 \end{bmatrix}. \tag{370}$$

Assume the following

1. $\dot{\mathbf{q}}_d,\ \ddot{\mathbf{q}}_d,\ \mathbf{q}_d^{(3)} \in \mathbf{L}_2^n \cap \mathbf{L}_\infty^n$, so that $\exists\ \bar{\mu}$ a positive real constant such that $\mu(t) \le \bar{\mu} \quad \forall t \in \mathbf{R}_+$, and $\mu(t) \in \mathbf{L}_2$ (Case 3 above.)
2. (a1), (a2)$'''$, and (a3) are satisfied $\forall (\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \in \mathbf{B}$.
3. $\alpha_2 - \gamma_2 > 0$.

Similar to the proof in Theorem 2 and Theorem 3, the time derivative of the Lyapunov Function candidate $\mathcal{V}$ along the solution trajectories of $\mathcal{S}_\Sigma$ can be verified to be (using (365))

$$\begin{aligned} \dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\le -\begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d''' \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} - \frac{1}{2}(1-d)\sigma_m \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \\ &+ \frac{1}{2}(1-d)\sigma_M \|\boldsymbol{\theta}\|_2^2 + d\mu^2(t), \end{aligned} \tag{371}$$

where P_d''' is given by (370). Similar to the case in the proof of Theorem 1, and using Assumption 3, the matrix P_d''' is positive definite when

$$\epsilon < \epsilon_d''' := \frac{\alpha_1(\alpha_2 - \gamma_2)}{\alpha_1\gamma_1 + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2''']^2}. \tag{372}$$

Using Assumption 1, (371) becomes

$$\begin{aligned} \dot{\mathcal{V}}(\mathbf{p},\mathbf{y}) &\le -\begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d''' \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} - \frac{1}{2}(1-d)\sigma_m \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \\ &+ \frac{1}{2}(1-d)\sigma_M \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2. \end{aligned} \tag{373}$$

Define the sets

$$\begin{aligned} \mathcal{D}_{\bar{\mu},\Sigma} := \Bigg\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} :\ & \begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d'' \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} \\ & + \frac{1}{2}(1-d)\sigma_0^m \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 \le \frac{1}{2}(1-d)\sigma_0^M \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \Bigg\}, \end{aligned} \tag{374}$$

and

$$\begin{aligned} \mathcal{D}^c_{\bar{\mu},\Sigma} := \Bigg\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} :\ & \begin{bmatrix} \|\mathbf{x}\|_2 & \|\mathbf{y}\|_2 \end{bmatrix} P_d'' \begin{bmatrix} \|\mathbf{x}\|_2 \\ \|\mathbf{y}\|_2 \end{bmatrix} \\ & + \frac{1}{2}(1-d)\sigma_0^m \left\|\tilde{\boldsymbol{\theta}}\right\|_2^2 > \frac{1}{2}(1-d)\sigma_0^M \|\boldsymbol{\theta}\|_2^2 + d\bar{\mu}^2 \Bigg\} \\ = \ \mathcal{B} \setminus \mathcal{D}_{\bar{\mu},\Sigma}. \end{aligned} \tag{375}$$

where

$$\sigma_0^m := \min\{\sigma_{01}, \sigma_{02}, \cdots, \sigma_{0r}\}, \tag{376}$$
$$\sigma_0^M := \max\{\sigma_{01}, \sigma_{02}, \cdots, \sigma_{0r}\}. \tag{377}$$

Now define

$$\mathcal{R}_{\bar{\mu},\Sigma} := \left\{ \left(\|\mathbf{x}\|_2, \left\|\tilde{\boldsymbol{\theta}}\right\|_2, \|\mathbf{y}\|_2\right) \in \mathcal{B} :\ \mathcal{V}(\mathbf{x}, \tilde{\boldsymbol{\theta}}, \mathbf{y}) \le c_{\bar{\mu},\Sigma} \right\}, \tag{378}$$

where $c_{\bar{\mu},\Sigma}$ is the smallest positive real number such that

- $\mathcal{D}_{\bar{\mu},\Sigma} \subseteq \mathcal{R}_{\bar{\mu},\Sigma}$
- $\forall \ \left(\|\mathbf{x}\|_2, \left\|\tilde{\theta}\right\|_2, \|\mathbf{y}\|_2\right) \in \left\{\mathcal{D}^c_{\bar{\mu},\Sigma} \setminus (\mathcal{R}_{\bar{\mu},\Sigma} \cap \mathcal{D}_{\bar{\mu},\Sigma})\right\}, \left\|\hat{\theta}_i\right\|_2 > 2\theta_{0i} \quad i = 1, 2, \cdots, r.$

If $\bar{\mu}$, σ_{0i}, and θ_{0i}, $\quad i = 1, 2, \cdots, r$ are such that $\mathcal{R}_{\bar{\mu},\Sigma} \subset \mathcal{I}$, then using the same reasoning as in the proof Theorem 3, we conclude that for the range of ϵ defined by (372), all solution trajectories starting in $\mathcal{I}$ converge to the residual set $\mathcal{R}_{\bar{\mu},\Sigma}$, furthermore,

$$\lim_{t \longrightarrow \infty} \mathbf{x}(t) = 0 \tag{379}$$

$$\lim_{t \longrightarrow \infty} \mathbf{y}(t) = 0. \tag{380}$$

□

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