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ADAPTIVE OUTPUT-FEEDBACK CONTROL OF A CLASS OF NONLINEAR SYSTEMS

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Adaptive Output-Feedback Control of a Class of Nonlinear Systems*

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Abstract

For a class of single-input single-output nonlinear systems with unknown constant parameters, we present a new adaptive design procedure which requires only output, rather than full-state, measurement. Even though the nonlinearities are not required to satisfy any growth conditions, the stability and tracking (or regulation) properties of the resulting closed-loop adaptive system are global.

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1 Introduction

The problem of designing adaptive output-feedback controllers for nonlinear systems with unknown constant parameters was recently addressed in [1,2]. These papers considered the class of n-dimensional nonlinear systems which have an input-output description expressed globally by the n-th order scalar differential equation

$$D^n y = B(D)\sigma(y)u + \sum_{i=0}^{n-1} D^i \left[\varphi_{0i}(y) + \sum_{j=1}^{p} \theta_j \varphi_{ji}(y)\right], \tag{1.1}$$

where D denotes the differentiation operator, and

- the coefficients $b_0, \ldots, b_m (m \leq n-1)$ of the polynomial $B(D) = b_m D^m + \cdots + b_1 D + b_0$ are unknown, but $B(D)$ is known to be Hurwitz and the sign of b_m is known,
- $\theta_1, \ldots, \theta_p$ are unknown constant parameters, and
- $\sigma(y)$, $\varphi_{ji}(y)$, $0 \leq j \leq n-1$, $0 \leq i \leq p$, are smooth nonlinearities with $\sigma(y) \neq 0 \; \forall y \in \mathbb{R}$, $\varphi_{ji}(0) = 0$, $0 \leq j \leq n-1$, $0 \leq i \leq p$.

In the case of known parameters, systems in this class are linearizable by output (and input) injection, and input-output linearizable (but not necessarily full-state linearizable) by full-state feedback.

In [1], Kanellakopoulos, Kokotovic and Morse extended the direct model-reference adaptive design developed for linear systems by Feuer and Morse [3] to nonlinear systems of the form (1.1), under the additional restriction that for $i = m+1, \ldots, n-1$, the functions $\varphi_{ji}(y)$, $0 \leq j \leq p$, are linear, i.e., that the nonlinearities do not enter the system before the control does.

This restriction was not present in the design developed by Marino and Tomei in [2], which combined their "filtered transformations", introduced in [4], with the adaptive design procedure introduced by Kanellakopoulos, Kokotovic and Morse [5] in the full-state-feedback case.

In this paper we consider the same class of nonlinear systems (1.1), and develop a direct extension of the adaptive design procedure of [5] to the output-feedback case. This new procedure allows us to remove the additional restriction of [1] without using the filtered transformations of [2]. As in [1] and [2], the obtained stability and tracking results are global, despite the fact that the nonlinearities are not restricted by any growth constraints.

2 The Systematic Design Procedure

Consider the class of n-dimensional nonlinear systems with an input-output description given by the differential equation (1.1). An equivalent minimal state representation for such systems is

$$\begin{aligned} \dot{\zeta} &= A\zeta + b\sigma(y)u + \varphi_0(y) + \sum_{i=1}^{p}\theta_i\varphi_i(y) \\ y &= c^{\mathrm{T}}\zeta = \zeta_1 , \end{aligned} \tag{2.1}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & & I \\ 0 & \dots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \varphi_i(y) = \begin{bmatrix} \varphi_{i1}(y) \\ \vdots \\ \varphi_{in}(y) \end{bmatrix}, \quad 0 \le i \le p . \tag{2.2}$$

Suppose now that the control objective is to track a given reference signal $y_{\mathrm{r}}(t)$ with the output y of the system (2.2), and assume that the first ρ derivatives of y_{r} are also given, where $\rho = n - m$. Then, our step-by-step design procedure is as follows:

<u>Step 0:</u> Choose K_0 such that $A_0 = A - K_0 c^{\mathrm{T}}$ is a Hurwitz matrix, and define the filters

$$\begin{aligned} \dot{\xi}_0 &= A_0\xi_0 + K_0 y + \varphi_0(y) \\ \dot{\xi}_i &= A_0\xi_i + \varphi_i(y) , \quad 1 \le i \le p \\ \dot{v}_j &= A_0 v_j + e_{n-j}\sigma(y)u , \quad 0 \le j \le m , \end{aligned} \tag{2.3}$$

where e_i is the i-th coordinate vector in $\mathbb{R}^n$. It is now easy to see that

$$\frac{d}{dt}\left[\zeta - \left(\xi_0 + \sum_{i=1}^{p}\theta_i\xi_i + \sum_{j=0}^{m} b_j v_j\right)\right] = A_0\left[\zeta - \left(\xi_0 + \sum_{i=1}^{p}\theta_i\xi_i + \sum_{j=0}^{m} b_j v_j\right)\right], \tag{2.4}$$

which implies, in particular, that the derivative of the output y is given by

$$\begin{aligned} \dot{y} &= \zeta_2 + \varphi_{01}(y) + \sum_{i=1}^{p} \theta_i \varphi_{i1}(y) \\ &= \xi_{02} + \varphi_{01}(y) + \sum_{i=1}^{p} \theta_i \left(\xi_{i2} + \varphi_{i1}(y)\right) + \sum_{j=0}^{m} b_j v_{j2} + \epsilon\,, \end{aligned} \tag{2.5}$$

where ϵ is an exponentially decaying term. Next, define

$$x_1 = \zeta_1 - y_r = y - y_r \stackrel{\triangle}{=} \alpha_1(y, y_r)\,, \tag{2.6}$$

and denote by $c_1, c_2, \ldots, c_\rho$ positive coefficients and by $\Gamma_1, \ldots, \Gamma_\rho$ positive definite symmetric matrices to be chosen later.

For convenience of notation, we also define for $i = 0, \ldots, \rho$

$$C_i\xi = [\xi_{0,1}, \ldots, \xi_{0,i+1}, \xi_{1,1}, \ldots, \xi_{1,i+1}, \ldots, \xi_{\rho,1}, \ldots, \xi_{\rho,i+1}] \tag{2.7}$$

$$C_i v = [v_{0,1}, \ldots, v_{0,i+1}, \ldots, v_{m-1,1}, \ldots, v_{m-1,i+1}, v_{m,1}, \ldots, v_{m,i}]\,, \tag{2.8}$$

with the understanding that where $C_i\xi$ or $C_i v$ appear as arguments of a function, that function may depend on any of their elements.

Step 1: Using (2.5), write $\dot{x}_1$ as

$$\begin{aligned} \dot{x}_1 &= \xi_{02} + \varphi_{01}(y) - \dot{y}_r + \sum_{i=1}^{p} \theta_i(\xi_{i2} + \varphi_{i1}(y)) + \sum_{j=0}^{m-1} b_j v_{j2} + b_m v_{m2} + \epsilon \\ &= b_m \left\{ \frac{1}{b_m}\left[\xi_{02} + \varphi_{01}(y) - \dot{y}_r\right] + \sum_{i=1}^{p} \frac{\theta_i}{b_m}\left[\xi_{i2} + \varphi_{i1}(y)\right] + \sum_{j=0}^{m-1} \frac{b_j}{b_m} v_{j2} + v_{m2} \right\} + \epsilon \\ &= -\left(c_1 + \frac{1}{2}\right) x_1 + b_m \left\{ v_{m2} + \kappa_1^{\mathrm{T}} w_1(y, y_r, \dot{y}_r, C_1\xi, C_1 v) \right\} + \epsilon\,, \end{aligned} \tag{2.9}$$

where

$$\kappa_1^{\mathrm{T}} = \left[\frac{1}{b_m}, \frac{\theta_1}{b_m}, \ldots, \frac{\theta_p}{b_m}, \frac{b_0}{b_m}, \ldots, \frac{b_{m-1}}{b_m}\right] \tag{2.10}$$

$$\begin{aligned} w_1^{\mathrm{T}} &= \left[\left(c_1 + \frac{1}{2}\right) x_1 + \xi_{02} + \varphi_{01}(y) - \dot{y}_r, \xi_{12} + \varphi_{11}(y), \right. \\ &\qquad \left. \ldots, \xi_{p2} + \varphi_{p1}(y), v_{02}, \ldots, v_{m-1,2}\right]. \end{aligned} \tag{2.11}$$

Let $\hat{\kappa}_1$ be an estimate of κ_1 and define the new state x_2 as

$$\begin{aligned} x_2 &= v_{m2} + \hat{\kappa}_1^{\mathrm{T}} w_1(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, C_1\xi, C_1 v) \\ &\triangleq v_{m2} + \alpha_2(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, C_1\xi, C_1 v, \hat{\kappa}_1)\,. \end{aligned} \tag{2.12}$$

Substitute (2.12) into (2.9) to obtain

$$\dot{x}_1 = -\left(c_1 + \frac{1}{2}\right) x_1 + b_m x_2 + b_m \tilde{\kappa}_1^{\mathrm{T}} w_1 + \epsilon\,, \tag{2.13}$$

where $\tilde{\kappa}_1 = \kappa_1 - \hat{\kappa}_1$. Then, let the update law for $\hat{\kappa}_1$ be

$$\dot{\hat{\kappa}}_1 = \mathrm{sgn}(b_m)\Gamma_1 w_1 x_1 \triangleq \omega_1(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, C_1\xi, C_1 v)\,. \tag{2.14}$$

The time derivative of the nonnegative function

$$V_1 = \frac{1}{2}\left(x_1^2 + \int_t^\infty \epsilon^2(\tau)d\tau\right) + |b_m|\tilde{\kappa}_1^{\mathrm{T}}\Gamma_1^{-1}\tilde{\kappa}_1 \tag{2.15}$$

along the solutions of (2.13)–(2.14) is

$$\begin{aligned} \dot{V}_1 &= -c_1 x_1^2 + b_m x_1 x_2 - \frac{1}{2}x_1^2 + x_1\epsilon - \frac{1}{2}\epsilon^2 \\ &= -c_1 x_1^2 - \frac{1}{2}(x_1 - \epsilon)^2 + b_m x_1 x_2\,. \end{aligned} \tag{2.16}$$

Step 2: Using (2.3), (2.5) and the definitions of x_1, x_2, $\dot{\hat{\kappa}}_1$, write $\dot{x}_2$ as

$$\begin{aligned} \dot{x}_2 &= v_{m3} - K_{02}v_{m1} + \frac{\partial\alpha_2}{\partial y}\left[\xi_{02} + \varphi_{01}(y) + \sum_{i=1}^{p}\theta_i\varphi_{i1}(y) + \sum_{j=0}^{m} b_j v_{j2} + \epsilon\right] \\ &\quad + \frac{\partial\alpha_2}{\partial y_{\mathrm{r}}}\dot{y}_{\mathrm{r}} + \frac{\partial\alpha_2}{\partial \dot{y}_{\mathrm{r}}}\ddot{y}_{\mathrm{r}} + \frac{\partial\alpha_2}{\partial(C_1\xi)}C_1\dot{\xi} + \frac{\partial\alpha_2}{\partial(C_1 v)}C_1\dot{v} + \frac{\partial\alpha_2}{\partial\hat{\kappa}_1}\omega_1(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, C_1\xi, C_1 v) \\ &\triangleq v_{m3} + \beta_2(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ddot{y}_{\mathrm{r}}, C_2\xi, C_2 v, \hat{\kappa}_1) + \kappa^{\mathrm{T}}\bar{w}_2(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, C_1\xi, C_1 v, v_{m2}, \hat{\kappa}_1) + \frac{\partial\alpha_2}{\partial y}\epsilon\,, \end{aligned} \tag{2.17}$$

where

$$\kappa^{\mathrm{T}} = [\theta_1, \ldots, \theta_p, b_0, \ldots, b_m] \tag{2.18}$$

and β_2, $\bar{w}_2$ are defined appropriately, using the fact that the partial derivatives of α_2 with respect to its arguments are known smooth functions of measured variables.

Let $\hat{\kappa}_2$ be an estimate of κ and define the new state x_3 as

$$\begin{aligned} x_3 &= v_{m3} + \left[c_2 + \frac{1}{2} \left(\frac{\partial \alpha_2}{\partial y} \right)^2 \right] x_2 + \beta_2 + \hat{\kappa}_2^{\mathrm{T}} \bar{w}_2 + \hat{b}_{m2} x_1 \\ &\triangleq v_{m3} + \alpha_3(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ddot{y}_{\mathrm{r}}, C_2\xi, C_2 v, \hat{\kappa}_1, \hat{\kappa}_2) \,. \end{aligned} \tag{2.19}$$

Substitute (2.19) into (2.17) to obtain

$$\dot{x}_2 = - \left[c_2 + \frac{1}{2} \left(\frac{\partial \alpha_2}{\partial y} \right)^2 \right] x_2 + x_3 + \tilde{\kappa}_2^{\mathrm{T}} w_2 - b_m x_1 + \frac{\partial \alpha_2}{\partial y} \epsilon \,, \tag{2.20}$$

where $\tilde{\kappa}_2 = \kappa - \hat{\kappa}_2$, and

$$w_2^{\mathrm{T}} = \bar{w}_2^{\mathrm{T}} + [0, \ldots, 0, x_1] \,. \tag{2.21}$$

Let the update law for $\hat{\kappa}_2$ be

$$\dot{\hat{\kappa}}_2 = \Gamma_2 w_2 x_2 \triangleq \omega_2(y, y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, C_1\xi, C_1 v, v_{m2}, \hat{\kappa}_1) \,. \tag{2.22}$$

The time derivative of the nonnegative function

$$V_2 = V_1 + \frac{1}{2} \left(x_2^2 + \int_t^\infty \epsilon^2(\tau) d\tau \right) + \tilde{\kappa}_2^{\mathrm{T}} \Gamma_2^{-1} \tilde{\kappa}_2 \tag{2.23}$$

is then given by

$$\dot{V}_2 = -c_1 x_1^2 - \frac{1}{2}(x_1 - \epsilon)^2 - c_2 x_2^2 - \frac{1}{2} \left(\frac{\partial \alpha_2}{\partial y} x_2 - \epsilon \right)^2 + x_2 x_3 \,. \tag{2.24}$$

Step i $(2 < i < \rho)$: Using (2.3), (2.5) and the definitions of $x_1, \ldots, x_i, \dot{\hat{\kappa}}_1, \ldots, \dot{\hat{\kappa}}_{i-1}$, write $\dot{x}_i$ as

$$\begin{aligned} \dot{x}_i &= v_{m,i+1} + \beta_i(y, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i)}, C_i\xi, C_i v, \hat{\kappa}_1, \ldots, \hat{\kappa}_{i-1}) \\ &\quad + \kappa^{\mathrm{T}} w_i(y, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}, C_{i-1}\xi, C_{i-1}v, v_{m,i}, \hat{\kappa}_1, \ldots, \hat{\kappa}_{i-1}) + \frac{\partial \alpha_i}{\partial y} \epsilon \,. \end{aligned} \tag{2.25}$$

Let $\hat{\kappa}_i$ be a *new* estimate of κ and define the new state x_{i+1} as

$$x_{i+1} = v_{m,i+1} + \left[c_i + \frac{1}{2} \left(\frac{\partial \alpha_i}{\partial y} \right)^2 \right] x_i + \beta_i + \hat{\kappa}_i^{\mathrm{T}} w_i + x_{i-1} \,. \tag{2.26}$$

Substitute (2.25) into (2.25) to obtain

$$\dot{x}_i = -\left[c_i + \frac{1}{2}\left(\frac{\partial \alpha_i}{\partial y}\right)^2\right] x_i + x_{i+1} + \tilde{\kappa}_i^{\mathrm{T}} w_i - x_{i-1} + \frac{\partial \alpha_i}{\partial y}\epsilon\,, \tag{2.27}$$

where $\tilde{\kappa}_i = \kappa - \hat{\kappa}_i$. Let the update law for $\hat{\kappa}_i$ be

$$\dot{\hat{\kappa}}_i = \Gamma_i w_i x_i \triangleq \omega_i(y, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i-1)}, C_{i-1}\xi, C_{i-1}v, v_{m,i}, \hat{\kappa}_1, \ldots, \hat{\kappa}_{i-1})\,. \tag{2.28}$$

The time derivative of the nonnegative function

$$V_i = V_{i-1} + \frac{1}{2}\left(x_i^2 + \int_t^\infty \epsilon^2(\tau)d\tau\right) + \tilde{\kappa}_i^{\mathrm{T}}\Gamma_i^{-1}\tilde{\kappa}_i \tag{2.29}$$

is then given by

$$\dot{V}_i = -\sum_{j=1}^{i}\left[x_j^2 + \frac{1}{2}\left(\frac{\partial \alpha_j}{\partial y}x_j - \epsilon\right)^2\right] + x_i x_{i+1}\,. \tag{2.30}$$

<u>Step ρ:</u> Using (2.3), (2.5) and the definitions of $x_1, \ldots, x_\rho$, $\dot{\hat{\kappa}}_1, \ldots, \dot{\hat{\kappa}}_{\rho-1}$, write $\dot{x}_\rho$ as

$$\begin{aligned} \dot{x}_\rho \;&=\; \sigma(y)u + v_{m,\rho+1} + \beta_\rho(y, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho)}, C_\rho\xi, C_\rho v, \hat{\kappa}_1, \ldots, \hat{\kappa}_{\rho-1}) \\ &\quad + \kappa^{\mathrm{T}} w_\rho\left(y, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-1)}, C_{\rho-1}\xi, C_{\rho-1}v, v_{m,\rho}, \hat{\kappa}_1, \ldots, \hat{\kappa}_{\rho-1}\right) + \frac{\partial \alpha_\rho}{\partial y}\epsilon\,. \end{aligned} \tag{2.31}$$

Let $\hat{\kappa}_\rho$ be a *new* estimate of κ and define the control u as

$$u = -\frac{1}{\sigma(y)}\left\{v_{m,\rho+1} + \left[c_\rho + \frac{1}{2}\left(\frac{\partial \alpha_\rho}{\partial y}\right)^2\right] x_\rho + \beta_\rho + \hat{\kappa}_\rho^{\mathrm{T}} w_\rho + x_{\rho-1}\right\}. \tag{2.32}$$

Substitute (2.32) into (2.31) to obtain

$$\dot{x}_\rho = -\left[c_\rho + \frac{1}{2}\left(\frac{\partial \alpha_\rho}{\partial y}\right)^2\right] x_\rho + \tilde{\kappa}_\rho^{\mathrm{T}} w_\rho - x_{\rho-1} + \frac{\partial \alpha_\rho}{\partial y}\epsilon\,, \tag{2.33}$$

where $\tilde{\kappa}_\rho = \kappa - \hat{\kappa}_\rho$. Let the update law for $\hat{\kappa}_\rho$ be

$$\dot{\hat{\kappa}}_\rho = \Gamma_\rho w_\rho x_\rho \triangleq \omega_\rho\left(y, y_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(\rho-1)}, C_{\rho-1}\xi, C_{\rho-1}v, v_{m,\rho}, \hat{\kappa}_1, \ldots, \hat{\kappa}_{\rho-1}\right). \tag{2.34}$$

Then, the time derivative of the nonnegative function

$$\begin{aligned} V_\rho \;&=\; V_{\rho-1} + \frac{1}{2}\left(x_\rho^2 + \int_t^\infty \epsilon^2(\tau)d\tau\right) + \tilde{\kappa}_\rho^{\mathrm{T}}\Gamma_\rho^{-1}\tilde{\kappa}_\rho \\ &=\; \frac{1}{2}\sum_{j=1}^{\rho} x_j^2 + |b_m|\tilde{\kappa}_1^{\mathrm{T}}\Gamma_1^{-1}\tilde{\kappa}_1 + \sum_{j=2}^{\rho}\tilde{\kappa}_j^{\mathrm{T}}\Gamma_j^{-1}\tilde{\kappa}_j + \frac{\rho}{2}\int_t^\infty \epsilon^2(\tau)d\tau \end{aligned} \tag{2.35}$$

is rendered nonpositive (since $c_j > 0$, $j = 1, \ldots, \rho$):

$$\dot{V}_\rho = -\sum_{j=1}^{\rho} \left[c_j x_j^2 + \frac{1}{2} \left(\frac{\partial \alpha_j}{\partial y} x_j - \epsilon \right)^2 \right] \leq 0\,. \tag{2.36}$$

3 Stability and Tracking

We are now ready to state and prove our main result:

Theorem 3.1. *Assume that $y_\mathrm{r}, \dot{y}_\mathrm{r}, \ldots, y_\mathrm{r}^{(\rho)}$ are uniformly bounded, and that $y_\mathrm{r}^{(\rho)}$ is piecewise continuous. Then, if the design procedure of Section 2 is applied to the nonlinear system (1.1), all the signals in the resulting closed-loop adaptive system are well-defined and uniformly bounded on $[0, \infty)$, and, in addition,*

$$\lim_{t \to \infty} [y(t) - y_\mathrm{r}(t)] = 0\,. \tag{3.1}$$

Proof. Due to the piecewise continuity of $y_\mathrm{r}^{(\rho)}$ and the smoothness of the nonlinearities, the solution of the closed-loop system has a maximum interval of definition $[0, t_\mathrm{f})$. On this interval, the time derivative of the nonnegative function V_ρ defined in (2.35) is nonpositive, as shown in (2.36). We conclude that $x_1, \ldots, x_\rho$ and $\hat{\kappa}_1, \ldots, \hat{\kappa}_\rho$ are bounded on $[0, t_\mathrm{f})$ by constants depending only on initial conditions. In particular, since x_1 and y_r are bounded, we have that y is bounded, which, by (2.3), implies that ξ_0, $\xi_1, \ldots, \xi_\rho$ are bounded and $\sigma(y)$ is bounded away from zero. Furthermore, from the differential equation (1.1), the boundedness of y, together with the fact that $B(D)$ is Hurwitz, imply that $H_\rho(D)\sigma(y)u$ is bounded, where $H_i(s)$ denotes any asymptotically stable transfer function of relative degree greater than or equal to i. This in turn implies that $F_j v_{m-j}$, $0 \leq j \leq m$, are bounded, where $F_i v_k = [v_{k,1}, \ldots, v_{k,i+1}]$. In particular, it implies that $C_1 v$ is bounded. By (2.12), the boundedness of y, y_r, $\dot{y}_\mathrm{r}$, $\hat{\kappa}_1$, $C_1 \xi$, $C_1 v$ and x_2 implies that v_{m2} is bounded. Hence, $H_{\rho-1}(D)\sigma(y)u$ is bounded, which means that $F_{j+1} v_{m-j}$, $0 \leq j \leq m$, are bounded. This again implies that $C_2 v$ is bounded, which, together with the boundedness of x_3, implies by (2.19) that v_{m3} is bounded. Continuing in the same fashion, we can prove that $H_1(D)\sigma(y)u$ is bounded, which implies that v is bounded. Since $\sigma(y)$ is bounded away from zero, we conclude now from (2.32) that u is bounded. From (1.1), this implies that y, $\dot{y}_1, \ldots, y^{(n-m)}$ are bounded. Since the m-dimensional zero dynamics of (1.1) are linear and exponentially stable, a standard argument proves that the state of any minimal realization of (1.1) is bounded, and, hence, ζ is bounded.

We have thus shown that the state of the closed-loop adaptive system is bounded on $[0, t_f)$. Hence, $t_f = \infty$. To prove the convergence of the tracking error to zero, we first note that (2.35) and (2.36) imply that $\dot{V}_\rho$ is bounded and integrable on $[0, \infty)$. Furthermore, the boundedness of all the closed-loop signals implies that $\ddot{V}_\rho$ is bounded. Hence, $\dot{V}_\rho \to 0$ as $t \to \infty$, which proves that $x_1, \ldots, x_\rho \to 0$ as $t \to \infty$. This, in particular, implies that $y - y_r \to 0$ as $t \to \infty$. □

4 The Class of Nonlinear Systems

Most models of nonlinear systems are expressed in specific state coordinates. From that state-space form it may not always be obvious whether or not the nonlinear system at hand has an input-output description of the form (1.1). Therefore, we now give coordinate-free geometric conditions which are necessary and sufficient for a single-input single-output nonlinear system of the form

$$\begin{aligned} \dot{z} &= f(z; \vartheta) + g(z; \vartheta) u \\ y &= h(z; \vartheta) \end{aligned} \tag{4.1}$$

to have an input-output description of the form (1.1), which is repeated here for convenience:

$$D^n y = B(D)\sigma(y) u + \sum_{i=0}^{n-1} D^i \left[\varphi_{0i}(y) + \sum_{j=1}^{p} \theta_j \varphi_{ji}(y) \right] . \tag{4.2}$$

In (4.1), $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, ϑ is a vector of unknown constant parameters, and f, g, h are smooth vector fields with $f(0; \vartheta) = 0$ and $g(z; \vartheta) \neq 0$ $\forall z \in \mathbb{R}^n$. Accordingly, in (4.2), $\sigma(y)$ and $\varphi_{ji}(y)$, $0 \leq i \leq n-1$, $0 \leq j \leq p$ are smooth nonlinearities with $\varphi_{ji}(0) = 0$, $\sigma(y) \neq 0$ $\forall y \in \mathbb{R}$. The coefficients $b_0, \ldots, b_m (m \leq n-1)$ of the polynomial $B(D)$, as well as $\theta_1, \ldots, \theta_p$, are unknown parameters resulting from a possible reparametrization in which functions of the original unknown parameters ϑ are treated as new parameters.

The following proposition is a corollary of [1, Proposition 5.2].

Proposition 4.1. *The system (4.1) has an input-output description of the form (4.2) if and only if the following conditions are satisfied for all $z \in \mathbb{R}^n$ and for the true value of the parameter vector ϑ:*

(i) the one-forms $dh, dL_f h, \ldots, dL_f^{n-1} h$ are linearly independent,

(ii) $[\mathrm{ad}_f^i \bar{g}, \mathrm{ad}_f^j \bar{g}] = 0$, $i, j = 0, \ldots, n-1$, where $\bar{g}$ is uniquely defined by

$$L_{\bar{g}} L_f^i h = \begin{cases} 0, & i = 0, \ldots, n-2 \\ 1, & i = n-1, \end{cases} \tag{4.3}$$

(iii) $\mathrm{ad}_f^n \bar{g} = \sum_{i=0}^{n-1} \left[\varphi_{0i}'(y) + \sum_{j=1}^{p} \theta_i \varphi_{ji}'(y) \right] (-1)^{n-i} \mathrm{ad}_f^i \bar{g}$,
with $\varphi_{ji}(y) = \int_0^y \varphi_{ji}'(s) ds$, $0 \le i \le n-1$, $0 \le j \le p$,

(iv) $g = \sigma(y) \sum_{i=0}^{m} b_i (-1)^i \mathrm{ad}_f^i \bar{g}$, and

(v) the vector fields f and $\bar{g}$ are complete.

□

Example 4.2 (Single-link flexible-joint manipulator). In order to demonstrate why a reparametrization may be required to write a system of the form (4.1) into the input-output form (4.2), we consider a single-link robotic manipulator whose rotary motion is controlled by means of an elastically coupled actuator. If the effect of elastic coupling is modeled as a linear torsional spring, then the dynamic equations of the system are (cf. [6, p. 231]):

$$\begin{aligned} J_1 \ddot{q}_1 + F_1 \dot{q}_1 + K \left(q_1 - \frac{q_2}{N} \right) mgd \cos q_1 &= 0 \\ J_2 \ddot{q}_2 + F_2 \dot{q}_2 - \frac{K}{N} \left(q_1 - \frac{q_2}{N} \right) &= u, \end{aligned} \tag{4.4}$$

where q_1 and q_2 are the angular positions of the link and the actuator, and u is the torque produced at the actuator axis. The inertias J_1, J_2, the viscous friction constants F_1, F_2, the elasticity constant K, the link mass M, the position of the link's center of gravity d, the transmission gear ratio N and the acceleration of gravity g can all be unknown.

In order to find the input-output description of the system (4.4), where u is the input and $y = q_1$ is the measured output, we use the following minimal state representation of (4.4), where $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{mgd}{J_1}\cos x_1 - \frac{F_1}{J_1}x_2 - \frac{K}{J_1}\left(x_1 - \frac{x_3}{N}\right) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{K}{J_2 N}\left(x_1 - \frac{x_3}{N}\right) - \frac{F_2}{J_2}x_4 + \frac{1}{J_2}u \\ y &= x_1\,. \end{aligned} \tag{4.5}$$

Differentiating y twice, we obtain $x_2 = Dy$ and

$$D^2 y = -\frac{mgd}{J_1}\cos y - \frac{F_1}{J_1}Dy - \frac{K}{J_1}\left(y - \frac{x_3}{N}\right), \tag{4.6}$$

which implies that

$$x_3 = \frac{J_1 N}{K}\left(D^2 y + \frac{mgd}{J_1}\cos y + \frac{F_1}{J_1}Dy + \frac{K}{J_1}y\right) \tag{4.7}$$

$$x_4 = Dx_3 = \frac{J_1 N}{K}\left(D^3 y + \frac{mgd}{J_1}D\cos y + \frac{F_1}{J_1}D^2 y + \frac{K}{J_1}Dy\right). \tag{4.8}$$

Differentiating (4.8) and substituting x_3 and x_4 from (4.7) and (4.8), we arrive at the input-output description of (4.4):

$$\begin{aligned} D^4 y &= \frac{K}{J_1 J_2 N}u + D^3\left(\frac{F_2}{J_2} - \frac{F_1}{J_1}\right)y - D^2\left[\left(\frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2 N}{J_2 K}\right)y + \frac{mgd}{J_1}\cos y\right] \\ &\quad -D\left[\left(\frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 N}{J_2}\right)y + \frac{mgdF_2 N}{J_2 K}\cos y\right] - \frac{mgdK}{J_1 J_2 N^2}\cos y\,, \end{aligned} \tag{4.9}$$

which is in the form (4.2), if we define

$$\begin{aligned} &b_0 = \frac{K}{J_1 J_2 N} > 0\,,\ \theta_1 = \frac{F_2}{J_2} - \frac{F_1}{J_1}\,,\ \theta_2 = \frac{K}{J_1} + \frac{K}{J_2 N^2} + \frac{F_1 F_2 N}{J_2 K}\,,\ \theta_3 = \frac{mgd}{J_1}\,, \\ &\theta_4 = \frac{F_1 K}{J_1 J_2 N^2} + \frac{F_2 N}{J_2}\,,\ \theta_5 = \frac{mgdF_2 N}{J_2 K}\,,\ \theta_6 = \frac{mgdK}{J_1 J_2 N^2}\,. \end{aligned} \tag{4.10}$$

□

5 Concluding Remarks

For the class of nonlinear systems considered in [2], we have developed a new systematic design procedure for adaptive output-feedback control. The adaptive controller resulting from this new procedure has dimension $n(m+p+2)+\rho(m+p+1)$. Comparing this to the controller of [2], which has dimension $(n-1)\left[\frac{1}{2}n+p+\rho\right]+p\rho+2\rho+n+1$, we see that, depending on the values of n, m, p (recall that $\rho = n-m$), either our new procedure or the procedure of [2] may yield the controller of lower dimension. Finally, we should note that in both cases the controller dimensions can be reduced if, instead of using the design procedure of [5], one employs the improved version of that procedure developed by Jiang and Praly [7].

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