

Decision and Control Laboratory

**ADAPTIVE
OUTPUT-FEEDBACK
CONTROL OF
SYSTEMS WITH
OUTPUT NONLINEARITIES**

**I. Kanellakopoulos
P. V. Kokotovic
A. S. Morse**

*Coordinated Science Laboratory
College of Engineering*
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

REPORT DOCUMENTATION PAGE

1. REPORT SECURITY CLASSIFICATION Unclassified		1b. RESTRICTIVE MARKINGS None	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited	
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE		5. MONITORING ORGANIZATION REPORT NUMBER(S)	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) DC-126 UIIU-ENG-90-2258		7a. NAME OF MONITORING ORGANIZATION National Science Foundation Air Force Office of Scientific Research	
6a. NAME OF PERFORMING ORGANIZATION Coordinated Science Lab University of Illinois	6b. OFFICE SYMBOL (if applicable) N/A	7b. ADDRESS (City, State, and ZIP Code) Washington, DC 20050 Bolling Air Force Base, DC 20332-6448	
6c. ADDRESS (City, State, and ZIP Code) 1101 W. Springfield Ave. Urbana, IL 61801		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER NSF ECS-87-15811 AFOSR 90-0011	
8a. NAME OF FUNDING / SPONSORING ORGANIZATION AFOSR and National Science Foundation	8b. OFFICE SYMBOL (if applicable)	10. SOURCE OF FUNDING NUMBERS	
8c. ADDRESS (City, State, and ZIP Code) Washington, DC 20050 Bolling Air Force Base, DC 20332-6448		PROGRAM ELEMENT NO.	PROJECT NO.
		TASK NO.	WORK UNIT ACCESSION NO.
11. TITLE (Include Security Classification) ADAPTIVE OUTPUT-FEEDBACK CONTROL OF SYSTEMS WITH OUTPUT NONLINEARITIES			
12. PERSONAL AUTHOR(S) KANELLAKOPOULOS, I., P. V. KOKOTOVIC, AND A. S. MORSE			
13a. TYPE OF REPORT Technical	13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Year, Month, Day) 1990 December	15. PAGE COUNT
16. SUPPLEMENTARY NOTATION			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB-GROUP	
19. ABSTRACT (Continue on reverse if necessary and identify by block number) For a class of single-input single-output nonlinear systems with unknown constant parameters, we present a direct model-reference adaptive control scheme, which requires only output, rather than full-state, measurement. The nonlinearities are not required to satisfy any growth conditions. The assumptions on the linear part of the nonlinear system are the same as in the standard adaptive control problem for linear systems, which now appears as a special case of the nonlinear problem solved in this paper.			
20. DISTRIBUTION / AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL		22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL

Adaptive Output-Feedback Control of Systems with Output Nonlinearities*

I. Kanellakopoulos P. V. Kokotovic
Coordinated Science Laboratory
University of Illinois
1101 W. Springfield Ave.
Urbana, IL 61801

A. S. Morse
Dept. of Electrical Engineering
Yale University
New Haven, CT 06520-1968

Technical Report DC-126
Presented at the *1990 Grainger Lectures* on
FOUNDATIONS OF ADAPTIVE CONTROL
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
September 28 – October 1, 1990

Abstract

For a class of single-input single-output nonlinear systems with unknown constant parameters, we present a direct model-reference adaptive control scheme, which requires only output, rather than full-state, measurement. The nonlinearities are not required to satisfy any growth conditions. The assumptions on the linear part of the nonlinear system are the same as in the standard adaptive control problem for linear systems, which now appears as a special case of the nonlinear problem solved in this paper.

*The work of the first two authors was supported in part by the National Science Foundation under Grant ECS-87-15811 and in part by the Air Force Office of Scientific Research under Grant AFOSR 90-0011. The work of the third author was supported in part by Ministero della Università e della Ricerca Scientifica e Tecnologica. The work of the fourth author was supported by the National Science Foundation under Grants ECS-88-05611 and ECS-90-12551.

1 Introduction

Until a few years ago, adaptive linear [1,2] and geometric nonlinear [3,4] methods belonged to two separate areas of control theory. They were helpful in the design of controllers for plants containing either unknown parameters or known nonlinearities, but not both. In the last few years the problem of *adaptive nonlinear control* was formulated to deal with the control of plants containing both unknown parameters and known nonlinearities. A realistic plan of attack to this challenging new problem led through a series of simpler problems, each formulated under certain restrictive assumptions. The two most common assumptions are those of linear parametrization [5-17] and full-state feedback [5-15].

The purpose of this paper is to avoid the full-state feedback assumption and to remove the specific restrictions of previous output-feedback results [16,17].

In the linear case, the adaptive output-feedback designs follow either a direct model-reference path or an indirect path via adaptive observers. Current research on adaptive observers for nonlinear systems [18-20] indicates that the indirect path may become promising for adaptive nonlinear control. However, the major stumbling block along this path continues to be its linear-like proof of stability which imposes restrictive conic conditions on the nonlinearities [16,17]. Under such linear growth constraints the actual nonlinear problem is, in fact, not addressed.

In this paper we formulate and solve a truly nonlinear output-feedback problem by following the direct model-reference path of Feuer and Morse [21]. In contrast to other more popular adaptive linear control methods [1,2], the method of Feuer and Morse offers a possibility to prove stability without any growth constraints. In a companion paper [22] we have exploited this possibility to solve a full-state-feedback adaptive nonlinear control problem. In this paper we present an adaptive output-feedback result without nonlinearity growth constraints.

The results of this paper apply to nonlinear input-output models consisting of a linear transfer function and output-dependent nonlinearities. The coefficients of the transfer function and the parameters multiplying the nonlinearities are unknown. For the linear part, the

assumptions of minimum phase and known sign of the high-frequency gain are the same as in the adaptive linear control theory, which now appears as a special case of the nonlinear theory presented in this paper.

For easier understanding, the new adaptive scheme is first designed for a particular system of sufficient complexity to be illustrative of both the design procedure and the stability properties of the resulting closed-loop adaptive system. In Section 2 we design the adaptive scheme for this system and then prove the stability and tracking properties of the resulting adaptive system in full detail. The design procedure for the general case is presented in Section 3, and the proof of stability and tracking is given in Section 4.

Nonlinear input-output models are intimately tied to state-space equations which originate from nonlinear physical laws expressed in specific state coordinates. In Section 5 we give a state-space form of the class of nonlinear plants which have the desired input-output representation, and characterize this class of plants via a set of geometric conditions.

2 Adaptive Scheme Design: An Example

The purpose of this section is to make both the proposed adaptive scheme and the main features of the Feuer-Morse method more easily accessible to the reader with the usual background in control theory and limited familiarity with adaptive linear control.

2.1. Nonlinear system properties. The nonlinear system is assumed to be minimum-phase [3, Chap. 4] and its nonlinearities depend only on the output variable. This implies that the nonlinear system is linearizable by output injection [23]. The input-output description of a typical nonlinear system of this kind is given by

$$D^5 y = (D^2 + 2D + 1)u + \theta [D^2 p_2(y) + D p_1(y) + p_0(y)] , \quad (2.1)$$

where u and y are the scalar control and output, respectively, $D = \frac{d}{dt}$, and θ is an unknown constant parameter. To address a truly nonlinear problem, we choose the nonlinearities which do not satisfy linear growth constraints:

$$p_0(y) = y^3 , \quad p_1(y) = y^2 + 2y^3 , \quad p_2(y) = ye^y + 2y^2 + y^3 . \quad (2.2)$$

It is important to notice that these nonlinearities are not in the span of u , and, hence, the system (2.1) is not full-state linearizable by static output feedback, or even by static full-state feedback, as shown in Sect. 5. However, it is input-output linearizable by full-state feedback [3, Chap. 4].

The above structural and growth properties of (2.1) and its relative degree [3, Chap. 4] show that (2.1) is a nonlinear system of considerable complexity. However, this system also satisfies a structural constraint under which the results of this paper are applicable: *the nonlinearities do not enter the system before the control input u .*

2.2. Augmenting the CE control. As in most adaptive designs, our first step is to find a dynamic output-feedback control that guarantees the specified stability and tracking properties *when the parameter θ is known*. Most adaptive schemes then replace the unknown θ with its estimate $\hat{\theta}$ and implement the so formed "certainty-equivalence" control. Such certainty-equivalence designs have been satisfactory in adaptive linear control, but have failed to produce truly nonlinear results because of their inherent linear growth constraints [16,17]. To avoid this difficulty we must go beyond the certainty-equivalence approach. Following Feuer and Morse [21], we will *augment the certainty-equivalence control* by an additive term \bar{u} which will counteract the effects of rapidly growing nonlinearities. It will also provide us with additional flexibility in the proof of stability.

The certainty-equivalence part of our control will be designed to match a reference model of the same relative degree as that of the nonlinear plant (2.1). As this plant is input-output linearizable by full-state feedback, we will choose the simplest linear reference model of relative degree three:

$$(D + 1)^3 y_r = r. \quad (2.3)$$

The first step in matching this reference model is to filter the plant equation (2.1) by the strictly proper stable filter F/E_2 , where F is a monic polynomial of degree 2, and E_2 is a monic Hurwitz polynomial of degree 4. This results in

$$\frac{FA}{E_2} y = \frac{FB}{E_2} u + \theta \left[\frac{FD^2}{E_2} p_2(y) + \frac{FD}{E_2} p_1(y) + \frac{F}{E_2} p_0(y) \right], \quad (2.4)$$

where $A = D^5$, $B = D^2 + 2D + 1$ as in (2.1). It is now straightforward to verify that the desired matching is achieved by the control

$$u^* = -\frac{G}{E_2}y + r - \theta\nu(y) - \frac{FB - E_2}{E_2}u^*, \quad (2.5)$$

provided that

$$\nu(y) = \frac{FD^2}{E_2}p_2(y) + \frac{FD}{E_2}p_1(y) + \frac{F}{E_2}p_0(y), \quad (2.6)$$

and that G , a polynomial of degree 4, and F satisfy the polynomial equation

$$FD^5 + G = (D + 1)^3 E_2. \quad (2.7)$$

Note that the polynomial $FB - E_2$ in (2.5) is of degree 3, since FB and E_2 are both monic polynomials of degree 4. As an illustration, the choice $E_2 = (D + 2)^4$ yields the following solution of (2.7):

$$F = D^2 + 11D + 51 \quad (2.8)$$

$$G = 129D^4 + 192D^3 + 168D^2 + 80D + 16. \quad (2.9)$$

When the control (2.5) is applied to the system (2.1) and the initial conditions of the filters used in (2.3), (2.5) and (2.6) are exactly matched with those of the system (2.1), then (2.5) achieves the exact tracking $y(t) = y_r(t)$ for all $t \geq 0$. However, the initial conditions of (2.1) are unknown and the tracking can be achieved only asymptotically, that is,

$$y(t) = y_r(t) + \epsilon(t) \rightarrow y_r(t) \quad \text{as } t \rightarrow \infty, \quad (2.10)$$

where $\epsilon(t)$ is the exponentially decaying tracking error caused by the mismatch of the initial conditions.

When the parameter θ is unknown, we replace it in (2.5) by its estimate $\hat{\theta}$, to be obtained from a parameter update law. To this "certainty-equivalence" part of our control we add a term \bar{u} which will be a handy tool later. So, our adaptive control will be of the following form:

$$u = -\frac{G}{E_2}y + r - \hat{\theta}\nu(y) - \frac{FB - E_2}{E_2}u + \bar{u}. \quad (2.11)$$

When applied to the nonlinear plant (2.1), this control yields the following input-output description of the resulting feedback system:

$$y = \frac{1}{(D+1)^3} [r + (\theta - \hat{\theta}) \nu(y) + \bar{u}] + \epsilon(t), \quad (2.12)$$

where, as in the case when θ was known, $\epsilon(t)$ contains all the exponentially decaying terms caused by the mismatch of the initial conditions. It should be observed that with an exact estimate $\hat{\theta} = \theta$ the linearization of (2.12) is achieved.

Introducing the error variables

$$e = y - y_r, \quad \tilde{\theta} = \theta - \hat{\theta}, \quad (2.13)$$

and taking the difference between (2.12) and the reference model (2.3), we obtain the tracking error equation:

$$e = \frac{1}{(D+1)^3} [\tilde{\theta} \nu(y) + \bar{u}] + \epsilon(t). \quad (2.14)$$

2.3. Error augmentation and swapping. Following the standard practice in adaptive control, we now set out to construct an error equation in which the parameter error is filtered only by a strictly positive real (SPR) transfer function. As a first step, we rewrite (2.14) in the form

$$\begin{aligned} e = & \frac{1}{(D+1)} \left[\tilde{\theta} \frac{1}{(D+1)^2} \nu(y) \right] + \frac{1}{(D+1)^3} \bar{u} \\ & + \frac{1}{(D+1)^3} [\tilde{\theta} \nu(y)] - \frac{1}{(D+1)} \left[\tilde{\theta} \frac{1}{(D+1)^2} \nu(y) \right] + \epsilon(t). \end{aligned} \quad (2.15)$$

The first term in (2.15) is in the desired SPR form, while the second term is due to the additional control term \bar{u} . As for the third and fourth terms, these are the familiar *swapping terms*, whose presence is caused by the time-varying nature of $\tilde{\theta}$: if $\tilde{\theta}$ were constant, these two terms would cancel out. Let us therefore define the *augmented error* \bar{e} as

$$\bar{e} = e + \eta_0, \quad (2.16)$$

where the term η_0 represents all the undesirable terms in (2.14):

$$\eta_0 = - \left[\frac{1}{(D+1)^3} \bar{u} + \frac{1}{(D+1)^3} [\tilde{\theta} \nu(y)] - \frac{1}{(D+1)} \left[\tilde{\theta} \frac{1}{(D+1)^2} \nu(y) \right] \right]. \quad (2.17)$$

The signal multiplying $\tilde{\theta}$ in the first brackets is of particular importance and is denoted by

$$h_1 = \frac{1}{(D+1)^2} \nu(y). \quad (2.18)$$

Considering $\nu(y)$ as the input and h_1 as the output, we represent (2.18) in the state-space form

$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \dot{h} = A_0 h + b \nu(y), \quad (2.19)$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (2.20)$$

It can now be verified that η_0 is the output of the third order system

$$\dot{\eta}_0 = -\eta_0 + \eta_1 \quad (2.21)$$

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \dot{\eta} = A_0 \eta - b \bar{u} - h \dot{\hat{\theta}}. \quad (2.22)$$

The variables h_1 and η_1 from (2.19) and (2.22) allow us to express the tracking error as

$$e = \frac{1}{D+1} [\tilde{\theta} h_1 - \eta_1] + \epsilon(t). \quad (2.23)$$

The analogous expression for the augmented error is

$$\bar{e} = \frac{1}{D+1} [\tilde{\theta} h_1] + \epsilon(t), \quad (2.24)$$

and it has the desired SPR form: the parameter error $\tilde{\theta}$ multiplied by the "regressor" h_1 is the input into the SPR filter $1/(D+1)$.

2.4. Update law. From this point on, the route prescribed by most of the adaptive linear control literature is to choose a normalized gradient update law and to set $\bar{u} \equiv 0$ (thus returning to a pure certainty equivalence control). In the case of adaptive linear systems, boundedness of the closed-loop signals can then be established using the Gronwall lemma or some type of small-gain argument. Attempts to apply this type of stability proof to nonlinear systems have so far been successful only when conic constraints are imposed on the nonlinearities. Without such linear growth constraints, the term $(\theta - \hat{\theta}) \nu(y)$ can cause

some signals to escape to infinity in finite time if the parameter error $\theta - \hat{\theta}$ is not rapidly decreased. The difficulty with normalizations of update laws is that they don't allow a rapid enough decrease of the parameter error when this error is most harmful. A simulation example of instability of a full-state-feedback scheme with normalization [14] is given in our companion paper [22], where it is also shown that an unnormalized update law preserves global stability. We are, therefore, motivated to look for an unnormalized update law.

The SPR form of (2.24) suggests the unnormalized gradient update law

$$\dot{\hat{\theta}} = h_1 \bar{e}. \quad (2.25)$$

Given the complexity of the nonlinear system (2.1), it is likely that such a simple update law will shift the difficulties in the adaptive design to the proof of stability. Indeed, this is the case. A simple Lyapunov-like function involving \bar{e}^2 and $\tilde{\theta}^2$ has a nonpositive derivative, but fails to prove boundedness of y . It clearly shows, however, that η_0 must be taken into account. Our next attempt is with the function

$$V_\eta = \frac{1}{2} \left(\bar{e}^2 + \tilde{\theta}^2 + \int_t^\infty \epsilon^2(\tau) d\tau \right) + \frac{1}{2} \eta^T P \eta, \quad (2.26)$$

where P is the positive definite solution of the Lyapunov equation

$$PA_0 + A_0^T P = -2I \Rightarrow P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}. \quad (2.27)$$

Using (2.22), (2.24) and (2.25), we compute

$$\dot{V}_\eta = -\frac{1}{2} (\bar{e} - \epsilon(t))^2 - \frac{1}{2} [\bar{e}^2 + 2\eta^T P h h_1 \bar{e} + 2\eta^T P b \bar{u}], \quad (2.28)$$

and try to render it nonpositive. The tool we have prepared for this task is \bar{u} . However, it turns out to be impossible to counteract the effects of the two-dimensional vector $h h_1 \bar{e}$ by \bar{u} alone. Hence, we need an additional degree of freedom involving h_1 . This prompts us to replace η in (2.26) by the new vector

$$\begin{aligned} \zeta_1 &= \eta_1 \\ \zeta_2 &= \xi_1(h_1)\eta_1 + \eta_2, \end{aligned} \quad (2.29)$$

where the nonlinear function $\xi_1(h_1)$ is at our disposal. With the new variables ζ , the non-negative function to be used in our proof is

$$V = \frac{1}{2} \left(\bar{e}^2 + \tilde{\theta}^2 + \int_t^\infty \epsilon^2(\tau) d\tau \right) + \frac{1}{2} \zeta^T P \zeta. \quad (2.30)$$

To evaluate \dot{V} we need $\dot{\zeta}$, which is obtained by differentiating (2.29) and using (2.19)–(2.22):

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 - h_1 \dot{\theta} - \xi(h_1) \zeta_1 \\ \dot{\zeta}_2 &= -\zeta_1 - 2\zeta_2 + 2\xi_1(h_1) \zeta_1 - \bar{u} - h_2 \dot{\theta} \\ &\quad + \xi_1(h_1) \left[\zeta_2 - h_1 \dot{\theta} - \xi_1(h_1) \zeta_1 \right] + \frac{\partial \xi_1}{\partial h_1} h_2 \zeta_1. \end{aligned} \quad (2.31)$$

Introducing the notation

$$w = \begin{bmatrix} -h_1^2 \\ -h_1 h_2 - h_1^2 \xi_1 \end{bmatrix}, \quad (2.32)$$

we compute the time derivative of V as:

$$\begin{aligned} \dot{V} &= -\frac{1}{2} (\bar{e} - \epsilon(t))^2 - \zeta^T \zeta - \left\{ \frac{1}{2} \bar{e}^2 - \zeta^T P w \bar{e} \right. \\ &\quad \left. + \zeta^T P \begin{bmatrix} \xi_1 \zeta_1 \\ -2\xi_1 \zeta_1 + \bar{u} - \xi_1 \zeta_2 + \xi_1^2 \zeta_1 - \frac{\partial \xi_1}{\partial h_1} h_2 \zeta_1 \end{bmatrix} \right\}. \end{aligned} \quad (2.33)$$

2.5. Design equation. We now have two tools to make \dot{V} nonpositive: the function $\xi_1(h_1)$ and the control term \bar{u} . With these tools we will attempt to represent the quantity enclosed in braces in (2.33) as the sum of two squares. It turns out that this is possible to achieve by decomposing P as $P = P_1 + P_2$ such that the following *design equation* holds:

$$P_1 w w^T P_1 \zeta + P_2 w w^T P_2 \zeta = P \begin{bmatrix} \xi_1 \zeta_1 \\ -2\xi_1 \zeta_1 + \bar{u} - \xi_1 \zeta_2 + \xi_1^2 \zeta_1 - \frac{\partial \xi_1}{\partial h_1} h_2 \zeta_1 \end{bmatrix}. \quad (2.34)$$

The substitution of (2.34) into (2.33) yields the desired form for \dot{V} :

$$\dot{V} = -\frac{1}{2} (\bar{e} - \epsilon(t))^2 - \zeta^T \zeta - \left(\frac{\bar{e}}{2} - \zeta^T P_1 w \right)^2 - \left(\frac{\bar{e}}{2} - \zeta^T P_2 w \right)^2 \leq 0. \quad (2.35)$$

Our task is now to find P_1 , P_2 , $\xi_1(h_1)$ and \bar{u} which satisfy the design equation (2.34). For the example considered here, the systematic procedure of Sect. 3 gives the following solution for P_1 and P_2 :

$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (2.36)$$

Substitution of (2.36) into (2.34) results in

$$\begin{aligned} & \left[\begin{array}{c} 2h_1^4 \zeta_1 \\ [(h_1 h_2 + h_1^2 + \xi_1 h_1^2)^2 - 2h_1^4] \zeta_1 + [(h_1 h_2 + h_1^2 + \xi_1 h_1^2)^2] \zeta_2 \end{array} \right] = \\ & = \left[\begin{array}{c} \xi_1 \zeta_1 \\ \left(-2\xi_1 + \xi_1^2 - \frac{\partial \xi_1}{\partial h_1} h_2 \right) \zeta_1 - \xi_1 \zeta_2 + \bar{u} \end{array} \right], \end{aligned} \quad (2.37)$$

which directly yields the following solutions for ξ_1 and \bar{u} :

$$\xi_1 = 2h_1^4 \quad (2.38)$$

$$\begin{aligned} \bar{u} = & \left[8h_1^3 h_2 + 2h_1^4 - 4h_1^8 + (h_1 h_2 + h_1^2 + 2h_1^6)^2 \right] \zeta_1 \\ & + \left[2h_1^4 + (h_1 h_2 + h_1^2 + 2h_1^6)^2 \right] \zeta_2. \end{aligned} \quad (2.39)$$

Using (2.29) and the notation

$$\varphi_1 = 8h_1^3 h_2 + 2h_1^4 + (1 + 2h_1^4)(h_1 h_2 + h_1^2 + 2h_1^6)^2 \quad (2.40)$$

$$\varphi_2 = 2h_1^4 + (h_1 h_2 + h_1^2 + 2h_1^6)^2, \quad (2.41)$$

\bar{u} can be defined in terms of available signals as

$$\bar{u} = \varphi_1 \eta_1 + \varphi_2 \eta_2. \quad (2.42)$$

To summarize, the complete closed-loop adaptive system is

Plant:

$$D^5 y = (D^2 + 2D + 1)u + \theta \left[D^2 p_2(y) + D p_1(y) + p_0(y) \right]$$

Control:

$$u = -\frac{G}{E_2} y + r - \hat{\theta} \nu(y) - \frac{FB - E_2}{E_2} u + \varphi_1 \eta_1 + \varphi_2 \eta_2$$

Update law:

$$\dot{\hat{\theta}} = h_1 \bar{e}, \quad \bar{e} = e + \eta_0 = y - y_r + \eta_0$$

Filters:

$$\dot{h} = A_0 h + b \nu(y)$$

$$\dot{\eta} = A_0 \eta - b(\varphi_1 \eta_1 + \varphi_2 \eta_2) - h h_1 \bar{e}$$

$$\dot{\eta}_0 = -\eta_0 + \eta_1,$$

where y_r is the output of the reference model (2.3) and $\nu(y)$ is defined in (2.6).

2.6. Stability and tracking. The stability and tracking properties of (2.43) are now established using the nonnegative function V from (2.30), whose derivative, given in (2.35), is nonpositive. Because of the piecewise continuity of $r(t)$ and the smoothness of the nonlinear functions p_0, p_1, p_2 , the solution of (2.43) has a maximum interval of definition, which we denote by $[0, t_f]$. We will now show that $t_f = \infty$.

From (2.30) and (2.35) we conclude that \bar{e} , ζ , and $\tilde{\theta}$ are bounded on $[0, t_f]$. Solving (2.29) for η and using (2.38) we obtain

$$\begin{aligned}\eta_1 &= \zeta_1 \\ \eta_2 &= \zeta_2 - 2h_1^4 \zeta_1.\end{aligned}\quad (2.44)$$

Thus, the boundedness of ζ implies that η_1 is bounded, which, in turn, implies that η_0 is bounded: $\dot{\eta}_0 = -\eta_0 + \eta_1$. Since \bar{e} and η_0 are bounded, e is bounded: $e = \bar{e} - \eta_0$. By the boundedness of y_r , this implies that y is bounded: $y = e + y_r$. Hence, $\nu(y)$ is bounded, which means that h is bounded: $\dot{h} = A_0 h + b\nu(y)$. (2.44), The boundedness of h and ζ implies that η and \bar{u} are bounded (cf. (2.44) and (2.39)).

This does not yet prove that u is bounded. From (2.11), to prove the boundedness of u we only need to show that $\frac{FB - E_2}{E_2}u$ is bounded. Since $\frac{FB - E_2}{E_2}$ is of relative degree 1, we can express $\frac{FB - E_2}{E_2}u$ in the form

$$\frac{FB - E_2}{E_2}u = \frac{E_2}{(D+1)^3(D+\lambda_1)} \left[\frac{E_3}{(D+1)^3}u + \frac{1}{D+\lambda_1} \frac{E_4}{(D+1)^3}u \right] + \epsilon(t), \quad (2.45)$$

where λ_1 is a positive constant and E_3, E_4 are polynomials of degree 2. Now (2.45) clearly shows that u is bounded if $\frac{D^i}{(D+1)^3}u$ is bounded for $i = 0, 1, 2$. Since y is bounded and the plant is minimum phase and of relative degree 3, it follows that $\frac{1}{(D+1)^3}u$ is bounded. Differentiating (2.14) and substituting $-\hat{\theta}\nu(y) + \bar{u}$ from (2.11), we obtain for $i = 1, 2$:

$$\begin{aligned}e^{(i)} &= \frac{D^i}{(D+1)^3} [\tilde{\theta}\nu(y) + \bar{u}] + \epsilon(t) \\ &= \frac{D^i}{(D+1)^3} \left[\theta\nu(y) + \frac{FB}{E_2}u + \frac{G}{E_2}y - r \right] + \epsilon(t).\end{aligned}\quad (2.46)$$

Using $h_i = \frac{D^{i-1}}{(D+1)^2} \nu(y)$, $i = 1, 2$, from (2.19) and rearranging terms in (2.46) we get

$$\frac{D^i}{(D+1)^3} \left[\frac{FB}{E_2} u \right] = e^{(i)} - \frac{D^i}{(D+1)^3} \left[\frac{G}{E_2} y - r \right] - \frac{D}{D+1} h_i \theta + \epsilon(t). \quad (2.47)$$

From (2.24) we have

$$\dot{e} = \dot{\bar{e}} - \dot{\eta}_0 = -\bar{e} + \tilde{\theta} h_1 + \eta_0 - \eta_1 + \epsilon(t) \quad (2.48)$$

$$\begin{aligned} \ddot{e} &= -\dot{\bar{e}} + \dot{\tilde{\theta}} h_1 + \tilde{\theta} \dot{h}_1 + \dot{\eta}_0 - \dot{\eta}_1 + \dot{\epsilon}(t) \\ &= \bar{e} - \tilde{\theta} h_1 - h_1^2 \bar{e} + \tilde{\theta} h_2 - \eta_0 + \eta_1 - \eta_2 + h_1^2 \bar{e} + \dot{\epsilon}(t). \end{aligned} \quad (2.49)$$

Since \bar{e} , h , η , $\tilde{\theta}$ are bounded, (2.48) and (2.49) imply that \dot{e} , \ddot{e} are bounded. Hence, by (2.47), $\frac{D^i}{(D+1)^3} \left[\frac{FB}{E_2} u \right]$ is bounded for $i = 1, 2$. The boundedness of $\frac{D^i}{(D+1)^3} u$ for $i = 1, 2$ then follows from the boundedness of $\frac{1}{(D+1)^3} u$ and the recursive expression

$$\frac{D^i}{(D+1)^3} u = \frac{D^i}{(D+1)^3} \frac{FB}{E_2} u - \frac{D(FB - E_2)}{E_2} \frac{D^{i-1}}{(D+1)^3} u + \epsilon(t). \quad (2.50)$$

Next, we prove that the state of the plant is bounded. From (2.12) it follows that $D^i y$, $0 \leq i \leq 3$ are bounded. Combining this with the fact that the plant is minimum phase, we conclude that the state of any minimal realization of (2.1) is bounded on $[0, t_f]$.

Thus, we have shown that the state of the closed-loop adaptive system (2.43) is bounded on its maximum interval of existence $[0, t_f]$. Therefore, $t_f = \infty$.

Finally, we prove convergence of the tracking error to zero. From $t_f = \infty$ and (2.35) we conclude that \dot{V} is bounded and integrable on $[0, \infty)$. Furthermore, the boundedness of \dot{e} (cf. (2.24)), $\dot{\zeta}$ (cf. (2.31)), and \dot{h} (cf. (2.19)), implies that \ddot{V} is bounded. Hence, $\dot{V} \rightarrow 0$ as $t \rightarrow \infty$, which implies (cf. (2.35)) that $\bar{e} \rightarrow 0$, $\zeta \rightarrow 0$ (since h is bounded). This, in turn, implies that $\eta_0 \rightarrow 0$ as $t \rightarrow \infty$ by (2.21). We conclude that

$$y - y_r = e = \bar{e} - \eta_0 \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (2.51)$$

□

3 The Systematic Design Procedure

Even though the expressions in the general case become more complicated than in the preceding section, the main steps of the design procedure remain essentially the same.

3.1. Nonlinear system properties. We consider the class of n -dimensional nonlinear systems which have an input-output description expressed globally by the n -th order scalar differential equation

$$A(D)y = B(D)[q(y)u] + \sum_{i=0}^m D^i [p_{i0}(y) + p_i^T(y)\theta_1], \quad (3.1)$$

where

- the coefficients a_0, \dots, a_{n-1} of the denominator polynomial $A(D) = D^n + a_{n-1}D^{n-1} + \dots + a_0$ are unknown,
- the coefficients b_0, \dots, b_m ($m \leq n - 1$) of the numerator polynomial $B(D) = b_m D^m + \dots + b_0$ are unknown, but $B(D)$ is known to be Hurwitz, and the sign of b_m is known,
- θ_1 is an ℓ -dimensional vector of unknown parameters,
- $q(y), p_{ij}(y), 0 \leq i \leq m, 0 \leq j \leq \ell$, are smooth nonlinearities with $q(y) \neq 0 \forall y \in \mathbb{R}$, $p_{ij}(0) = 0, 0 \leq i \leq m, 0 \leq j \leq \ell$.

Systems in this class are linearizable by output injection, and input-output linearizable by full-state feedback, but not necessarily full-state linearizable, even by full-state feedback, as will be shown in Sect. 5.

3.2. Augmenting the CE control. The design objective of the certainty-equivalence part of our control is to match a reference model of the same relative degree as that of the nonlinear plant (3.1). As this plant is input-output linearizable by output feedback, we choose the linear reference model:

$$E_1(D)E_2(D)y_r = Rr, \quad E_2(D) = E_{21}(D)E_{22}(D), \quad (3.2)$$

where $E_1(D)$, $E_2(D)$, $E_{21}(D)$, $E_{22}(D)$ are monic Hurwitz polynomials of degree $n - m$, $n - 1$, $n - m - 1$, and m , respectively, and $R(D)$ is a polynomial of degree $h \leq n - 1$. Filtering (3.1) by the strictly proper stable filter F/E_2 , where F is a monic polynomial of degree $n - m - 1$, we obtain

$$\frac{FA}{E_2}y = \frac{FB}{E_2}[q(y)u] + \sum_{i=0}^m \frac{FD^i}{E_2} [p_{i0}(y) + p_i^T(y)\theta_1]. \quad (3.3)$$

It is now straightforward to verify that in the case when the coefficients of $A(D)$ and $B(D)$ and the parameters θ_1 are known, the desired matching is achieved by the control

$$u^* = \frac{1}{q(y)} \left\{ -\frac{G}{E_2}y + \frac{M}{E_2}r - \sum_{j=0}^{\ell} \left[\frac{T_j}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{ij}(y) \right] - \frac{L_0}{E_2} [q(y)u^*] \right\}, \quad (3.4)$$

provided that G , a polynomial of degree $n - 1$, F , M , T_j and L_0 satisfy the polynomial equations

$$FA + b_m G = E_1 E_2 \quad (3.5)$$

$$M = \frac{1}{b_m} R \quad (3.6)$$

$$T_j = \frac{1}{b_m} \theta_{1j} F, \quad 0 \leq j \leq \ell, \quad \theta_{10} \equiv 1 \quad (3.7)$$

$$L_0 = \frac{1}{b_m} FB - E_2. \quad (3.8)$$

Note that L_0 is a polynomial of degree $n - 2$, since both $\frac{1}{b_m} FB$ and E_2 are monic polynomials of degree $n - 1$. When the control (3.4) is applied to the system (3.1), asymptotic tracking is achieved:

$$y(t) = y_r(t) + \epsilon(t) \rightarrow y_r(t) \quad \text{as } t \rightarrow \infty, \quad (3.9)$$

where $\epsilon(t)$ is the exponentially decaying tracking error caused by the mismatch of the initial conditions.

We now rewrite the control (3.4) as

$$u^* = -\frac{1}{q(y)} [\phi^T(y, u^*, r)\theta], \quad (3.10)$$

where the n_θ -dimensional vectors θ and ϕ (with $n_\theta = 2n + (\ell + 1)(n - m)$) are defined as

$$\begin{aligned} \theta^T = & [g_0, \dots, g_{n-1}, -m_0, t_{00}, \dots, t_{0n-m-1}, t_{10}, \dots, t_{1n-m-1}, \\ & \dots, t_{\ell 0}, \dots, t_{\ell n-m-1}, \ell_{00}, \dots, \ell_{0n-2}] \end{aligned} \quad (3.11)$$

$$\phi^T(y, u^*, r) = \left[\frac{\bar{D}_{n-1}}{E_2} y, \frac{R}{E_2} r, \frac{\bar{D}_{n-m-1}}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{i0}(y), \frac{\bar{D}_{n-m-1}}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{i1}(y), \dots, \frac{\bar{D}_{n-m-1}}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{i\ell}(y), \frac{\bar{D}_{n-2}}{E_2} [q(y)u^*] \right], \quad (3.12)$$

with \bar{D}_k defined as the $(k+1)$ -dimensional row operator $\bar{D}_k = [1, D, \dots, D^k]$. The form (3.10) is particularly useful in the case where the coefficients of $A(D)$ and $B(D)$ and the components of the vector θ_1 are unknown. Since in that case the parameter vector θ defined in (3.11) and used in (3.10) cannot be computed, it is replaced by an estimate $\hat{\theta}$. The so formed "certainty-equivalence" control is then *augmented by an additive term* \bar{u} which is yet to be designed. Hence, the adaptive control will be of the form:

$$u = \frac{1}{q(y)} [-\phi^T(y, u, r)\hat{\theta} + \bar{u}]. \quad (3.13)$$

Filtering the system equation (3.1) by the strictly proper stable filter $\frac{F}{E_1 E_2}$, and using (3.5)-(3.8) and (3.11)-(3.12), we obtain:

$$\frac{FA}{E_1 E_2} y = \frac{BF}{E_1 E_2} [q(y)u] + \sum_{i=0}^m \frac{D^i F}{E_1 E_2} [p_{i0}(y) + p_i^T(y)\theta_i] + \epsilon(t) \quad (3.14)$$

$$y = \frac{b_m}{E_1} \left\{ \frac{L}{E_2} [q(y)u] + \frac{G}{E_2} y + \sum_{j=0}^{\ell} \left[\frac{T_j}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{ij}(y) \right] \right\} + \epsilon(t) \quad (3.15)$$

$$y = \frac{b_m}{E_1} \left\{ q(y)u + \phi^T(y, u, r)\theta + \frac{M}{E_2} r \right\} + \epsilon(t), \quad (3.16)$$

where, as in the case when θ is known, $\epsilon(t)$ denotes a linear combination of exponentially decaying terms caused by the mismatch of the initial conditions. Substitution of (3.13) into (3.16) yields the following description of the resulting feedback system:

$$y = \frac{b_m}{E_1} \left\{ \frac{R}{b_m E_2} r + \phi^T(y, u, r) (\theta - \hat{\theta}) + \bar{u} \right\} + \epsilon(t). \quad (3.17)$$

Introducing the error variables

$$e = y - y_r, \quad \tilde{\theta} = \theta - \hat{\theta}, \quad (3.18)$$

and taking the difference between (3.17) and the reference model (3.2), we obtain the *tracking error equation*:

$$e = \frac{b_m}{E_1} [\phi^T(y, u, r)\tilde{\theta} + \bar{u}] + \epsilon(t). \quad (3.19)$$

In the special case of relative degree one ($n - m = 1$), the design is extremely simple. Since the transfer function $|b_m|/E_1$ is SPR, the parameter update law

$$\dot{\hat{\theta}} = \text{sgn}(b_m)\Gamma\phi(y, u, r)e, \quad (3.20)$$

where $\Gamma = \Gamma^T > 0$ is the *adaptive gain*, guarantees boundedness of all the closed-loop signals and convergence of the tracking error e to zero [1, Chap. 5], and the control augmentation is not needed: $\bar{u} \equiv 0$.

3.3. Error augmentation and swapping. For relative degree higher than one ($n - m > 1$), the design becomes considerably more complicated, since $|b_m|/E_1$ is no longer SPR. We first rewrite (3.19) in the form

$$e = \frac{1}{D + \lambda_0} \left[b_m \tilde{\theta}^T \frac{1}{E_0} \phi(y, u, r) \right] + \frac{b_m}{E_1} \bar{u} + \frac{b_m}{D + \lambda_0} \left[\frac{1}{E_0} [\tilde{\theta}^T \phi] - \tilde{\theta}^T \frac{1}{E_0} \phi \right] + \epsilon(t), \quad (3.21)$$

where

$$E_0(D)(D + \lambda_0) = E_1(D). \quad (3.22)$$

In contrast to the example of Sect. 2, where the high-frequency gain b_m was known, here it is unknown. Therefore, using an estimate \hat{b}_m and denoting $\tilde{b}_m = b_m - \hat{b}_m$, we rewrite (3.21) in the form

$$e = \frac{1}{D + \lambda_0} \left[b_m \tilde{\theta}^T \frac{1}{E_0} \phi + \tilde{b}_m \left(\frac{1}{E_0} \bar{u} - \tilde{\theta}^T \frac{1}{E_0} \phi + \frac{1}{E_0} [\tilde{\theta}^T \phi] \right) \right] + \frac{1}{D + \lambda_0} \left[\hat{b}_m \left(\frac{1}{E_0} \bar{u} - \tilde{\theta}^T \frac{1}{E_0} \phi + \frac{1}{E_0} [\tilde{\theta}^T \phi] \right) \right] + \epsilon(t). \quad (3.23)$$

Since the first summand in (3.23) is in the desired SPR form, we define the augmented error

$$\bar{e} = e + \eta_0, \quad (3.24)$$

where the term η_0 represents all the undesirable terms in (3.23):

$$\eta_0 = -\frac{1}{D + \lambda_0} \left[\hat{b}_m \left(\frac{1}{E_0} \bar{u} - \tilde{\theta}^T \frac{1}{E_0} \phi + \frac{1}{E_0} [\tilde{\theta}^T \phi] \right) \right]. \quad (3.25)$$

The vector multiplying $\tilde{\theta}$ in the first term is denoted by

$$\psi = \frac{1}{E_0} \phi(y, u, r). \quad (3.26)$$

Considering $\phi(y, u, r)$ as the input and ψ as the output, we represent (3.26) in the matrix state-space form:

$$\dot{H} = A_0 H + b \phi^T(y, u, r) \quad (3.27)$$

$$\psi^T = c^T H,$$

where (c, A_0, b) is a minimal realization of $1/E_0$:

$$c^T (sI - A_0)^{-1} b = \frac{1}{E_0(s)}. \quad (3.28)$$

Now η_0 is the output of the $(n - m)$ -dimensional system

$$\dot{\eta}_0 = -\lambda_0 \eta_0 + \hat{b}_m c^T \eta \quad (3.29)$$

$$\dot{\eta} = A_0 \eta - b \bar{u} - H \hat{\theta}. \quad (3.30)$$

The variables ψ and $c^T \eta$ from (3.27) and (3.30) allow us to express the tracking error as (cf. (3.21))

$$e = \frac{1}{s + \lambda_0} [b_m \tilde{\theta}^T \psi - b_m c^T \eta] + \epsilon(t). \quad (3.31)$$

The analogous expression for the augmented error is

$$\bar{e} = \frac{1}{s + \lambda_0} [b_m \tilde{\theta}^T \psi - \tilde{b}_m c^T \eta] + \epsilon(t), \quad (3.32)$$

and it has the desired SPR form: the parameter errors $\tilde{\theta}$ and \tilde{b}_m are filtered only by the SPR filter $1/(s + \lambda_0)$.

3.4. Update law. As in Sect. 2, we choose the unnormalized gradient update laws suggested by the SPR form of (3.32):

$$\dot{\hat{\theta}} = \text{sgn}(b_m) \Gamma \psi \bar{e} \quad (3.33)$$

$$\dot{\hat{b}}_m = -\gamma c^T \eta \bar{e}, \quad (3.34)$$

where $\Gamma = \Gamma^T > 0$ and $\gamma > 0$ are the adaptive gains.

From Sect. 2 we know that in the proof of stability there will be a need to balance the interaction between η and H . Therefore, we introduce the new variables ζ :

$$\zeta = S^{-1}\eta, \quad S \equiv C_{\bar{n}}^{-1}\Xi C_{\bar{n}}, \quad (3.35)$$

where $\bar{n} \equiv n - m - 1$,

$$C_i \equiv \left[c, A_0^T c, \dots, (A_0^T)^{i-1} c \right]^T, \quad i = 1, \dots, \bar{n} \quad (3.36)$$

$$\Xi = I_{\bar{n} \times \bar{n}} + \begin{bmatrix} 0 \\ \xi_1 E_{1, \bar{n}} \\ \xi_2 E_{2, \bar{n}} \\ \vdots \\ \xi_{\bar{n}-1} E_{\bar{n}-1, \bar{n}} \end{bmatrix} \quad (3.37)$$

$$E_{j,i} = [I_{j \times j}, 0]_{j \times i}, \quad 0 < j \leq i, \quad (3.38)$$

and $I_{i \times i}$ is the $i \times i$ identity matrix. The components of the i -dimensional row vectors ξ_i are nonlinear functions of the elements of H which represent the aforementioned additional degrees of design freedom. In order to show that the matrix S defined in (3.35) is invertible, we note that, because of the structure of the matrices $E_{i,j}$ defined by (3.38), the matrix Ξ is lower triangular with ones on its diagonal. From this and the aforementioned functional dependence of ξ_i on H , it follows that Ξ^{-1} always exists and that the elements of both Ξ and Ξ^{-1} are polynomial functions of the elements of H . Furthermore, $C_{\bar{n}}^{-1}$ exists because (c^T, A_0) is assumed to be an observable pair.

The nonnegative function to be used in our stability proof is

$$V = \frac{1}{2} \left(\bar{e}^2 + |b_m| |\bar{\theta}^T \Gamma^{-1} \bar{\theta}| + \frac{1}{\gamma} \bar{b}_m^2 + \frac{1}{\lambda_0} \int_t^\infty \epsilon^2(\tau) d\tau \right) + \frac{\lambda_0}{\bar{n}} \zeta^T P \zeta. \quad (3.39)$$

The form of (3.39) is the same as that of (2.30), where P is the positive definite solution of the Lyapunov equation

$$P A_0 + A_0^T P = -Q_0. \quad (3.40)$$

To evaluate \dot{V} we need $\dot{\zeta}$, which is obtained by differentiating (3.35) and using (3.30), (3.33) and (3.36)–(3.38):

$$\begin{aligned}
\dot{\zeta} &= S^{-1}(A_0\eta - b\bar{u} - \text{sgn}(b_m)H\Gamma\psi\bar{e}) + \frac{d}{dt}(S^{-1})\eta \\
&= S^{-1}A_0S\zeta + S^{-1}(b\bar{u} + \text{sgn}(b_m)H\Gamma\psi\bar{e}) - S^{-1}\dot{S}\zeta \\
&= A_0\zeta - \text{sgn}(b_m)S^{-1}H\Gamma\psi\bar{e} + S^{-1}(A_0S - SA_0 - \dot{S})\zeta - S^{-1}b\bar{u}. \tag{3.41}
\end{aligned}$$

Introducing the notation

$$w = -\text{sgn}(b_m)S^{-1}H\Gamma\psi \tag{3.42}$$

$$W = S^{-1}(SA_0 - A_0S + \dot{S}), \tag{3.43}$$

we rewrite (3.41) as

$$\dot{\zeta} = A_0\zeta + w\bar{e} - W\zeta - S^{-1}b\bar{u}, \tag{3.44}$$

and compute the time derivative of V as

$$\dot{V} = -\frac{1}{2}\left(\sqrt{\lambda_0}\bar{e} - \frac{1}{\sqrt{\lambda_0}}\epsilon\right)^2 - \frac{\lambda_0}{\bar{n}}\zeta^T Q_0\zeta - \frac{2\lambda_0}{\bar{n}}\left\{\frac{\bar{n}}{4}\bar{e}^2 - \zeta^T P(w\bar{e} - W\zeta - S^{-1}b\bar{u})\right\}. \tag{3.45}$$

3.5. Design equation. The tools we have at our disposal to make \dot{V} nonpositive are the functions $\xi_i(H)$ and the control term \bar{u} . With these tools we will attempt to represent the quantity enclosed in braces in (3.45) as the sum of \bar{n} squares. It turns out that this is possible to achieve by decomposing P as $P = \sum_{i=1}^{\bar{n}} P_i$ such that the following *design equation* holds:

$$\sum_{i=1}^{\bar{n}} P_i w w^T P_i \zeta = P(W\zeta + S^{-1}b\bar{u}). \tag{3.46}$$

The substitution of (3.46) into (3.45) yields the desired form for \dot{V} :

$$\dot{V} = -\frac{1}{2}\left(\sqrt{\lambda_0}\bar{e} - \frac{1}{\sqrt{\lambda_0}}\epsilon\right)^2 - \frac{\lambda_0}{\bar{n}}\zeta^T Q_0\zeta - \frac{2\lambda_0}{\bar{n}}\sum_{i=1}^{\bar{n}}\left(\frac{\bar{e}}{2} - \zeta^T P_i w\right)^2 \leq 0. \tag{3.47}$$

Our task now is to find P_i , $\xi_i(H)$ and \bar{u} which satisfy the design equation (3.46). Following the development in [21], we define

$$P_i = C_i^T M_i C_i, \tag{3.48}$$

where

$$\begin{aligned} M_1 &= (C_1 P^{-1} C_1)^{-1} \\ M_i &= (C_i P^{-1} C_i^T)^{-1} \left(I_{i \times i} - \sum_{j=1}^{i-1} C_i P^{-1} C_j^T M_j E_{j,i} \right), \quad i = 2, \dots, \bar{n}. \end{aligned} \quad (3.49)$$

In [21, Lemma 1] it is proved that (3.48)–(3.49) result in

$$\sum_{i=1}^{\bar{n}} P_i = P \quad (3.50)$$

$$C_j P^{-1} C_i^T M_i C_i = 0, \quad 1 \leq j < i \leq \bar{n}. \quad (3.51)$$

This proof is now given for completeness. From (3.49) we have

$$\begin{aligned} I_{i \times i} &= C_i P^{-1} C_i^T M_i + \sum_{j=1}^{i-1} C_i P^{-1} C_j^T M_j E_{j,i} \\ &= C_i P^{-1} \sum_{k=1}^i C_k^T M_k E_{k,i}. \end{aligned} \quad (3.52)$$

Premultiplying both sides of (3.52) by C_i and using the identity

$$C_k = E_{k,i} C_i, \quad k \leq i, \quad (3.53)$$

we obtain

$$C_i = C_i P^{-1} \sum_{k=1}^i C_k^T M_k C_k. \quad (3.54)$$

Evaluating (3.54) at $i = \bar{n}$ and using the nonsingularity of $C_{\bar{n}}$ we obtain (3.50). Furthermore, premultiplying (3.54) by $E_{j,i}$, where $j < i$, and using (3.53) again, we obtain

$$C_j = C_j P^{-1} \sum_{k=1}^i C_k^T M_k C_k, \quad i > j. \quad (3.55)$$

But from (3.54) we have

$$C_j = C_j P^{-1} \sum_{k=1}^j C_k^T M_k C_k,$$

which, combined with (3.55), results in (3.51).

Having established (3.50) and (3.51), we now set out to find $\xi_i(H)$ and \bar{u} which, along with P_i defined by (3.48)–(3.49), satisfy the design equation (3.46). Substituting (3.48) into (3.46) we obtain

$$W\zeta + S^{-1}b\bar{u} = P^{-1} \sum_{i=1}^{\bar{n}} C_i^T M_i C_i w w^T C_i^T M_i^T C_i. \quad (3.56)$$

Using (3.43) we rewrite the design equation (3.56) as

$$(SA_0 - A_0S + \dot{S})\zeta + b\bar{u} = SP^{-1} \sum_{i=1}^{\bar{n}} C_i^T M_i C_i w w^T C_i^T M_i^T C_i \zeta. \quad (3.57)$$

Premultiplying both sides of (3.57) by $c^T A_0^{i-1}$ for $i = 1, \dots, \bar{n}$, we obtain

$$\begin{aligned} c^T A_0^{i-1} (A_0 S \zeta - b\bar{u}) &= c^T A_0^{i-1} \left[S \left(A_0 - P^{-1} \sum_{j=1}^{\bar{n}} C_j^T M_j C_j w w^T C_j^T M_j C_j \right) + \dot{S} \right] \zeta \\ &\quad + c^T A_0^{i-1} b\bar{u}, \quad 1 \leq i \leq \bar{n}. \end{aligned} \quad (3.58)$$

From (3.35), (3.37) and (3.53) we have

$$C_{\bar{n}} S = \Xi C_{\bar{n}} = C_{\bar{n}} + \begin{bmatrix} 0 \\ \xi_1 C_1 \\ \xi_2 C_2 \\ \vdots \\ \xi_{\bar{n}-1} C_{\bar{n}-1} \end{bmatrix}, \quad (3.59)$$

which gives

$$c^T S = c^T \quad (3.60)$$

$$c^T A_0^{i-1} S = c^T A_0^{i-1} + \xi_{i-1} C_{i-1}, \quad 1 \leq i \leq \bar{n}. \quad (3.61)$$

Furthermore, from (3.35) and (3.37) we have

$$\begin{aligned} c^T A_0^{i-1} \dot{S} &= c^T A_0^{i-1} C_{\bar{n}}^{-1} \dot{\Xi} C_{\bar{n}} = c^T A_0^{i-1} C_{\bar{n}}^{-1} \begin{bmatrix} 0 \\ \dot{\xi}_1 C_1 \\ \vdots \\ \dot{\xi}_{\bar{n}-1} C_{\bar{n}-1} \end{bmatrix} \\ &= \dot{\xi}_{i-1} C_{i-1}, \quad 1 \leq i \leq \bar{n}, \end{aligned} \quad (3.62)$$

where we have used the definition of $C_{\bar{n}}$ to obtain the last equality. Finally, from the definition (3.28) of the triple (c, A_0, b) we have:

$$c^T A_0^{i-1} b = 0, \quad 1 \leq i \leq \bar{n} - 1 \quad (3.63)$$

$$c^T A_0^{\bar{n}-1} b = 1. \quad (3.64)$$

Substituting (3.60)–(3.64) into (3.58) results in

$$c^T A_0^i + \xi_i C_i = \dot{\xi}_{i-1} C_{i-1} + c^T A_0^i + \xi_{i-1} C_{i-1} A_0 - (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) \\ \times P^{-1} \sum_{j=1}^{\bar{n}} C_j^T M_j C_j w w^T C_j^T M_j^T C_j, \quad 1 \leq i \leq \bar{n} - 1 \quad (3.65)$$

$$\bar{u} = \left[c^T A_0^{\bar{n}} S - \dot{\xi}_{\bar{n}-1} C_{\bar{n}-1} - c^T A_0^{\bar{n}} - \xi_{\bar{n}-1} C_{\bar{n}-1} A_0 + (c^T A_0^{\bar{n}-1} + \xi_{\bar{n}-1} C_{\bar{n}-1}) \right. \\ \left. \times P^{-1} \sum_{j=1}^{\bar{n}} C_j^T M_j C_j w w^T C_j^T M_j^T C_j \right] \zeta. \quad (3.66)$$

At this point, we have almost achieved our goal of finding ξ_i and \bar{u} which satisfy the design equation and thus render \dot{V} nonpositive. Still, (3.65)–(3.66) are in a rather complicated form and, moreover, they involve the time derivatives of the functions ξ_i . Therefore, we now set out to simplify (3.65)–(3.66) and to express $\dot{\xi}_{i-1}$ as explicit functions of available signals.

Motivated by the appearance of the terms $C_j w$ in (3.65)–(3.66), we introduce the i -dimensional column vectors $w_1, \dots, w_{\bar{n}}$ which are defined as

$$w_i = C_i w, \quad 1 \leq i \leq \bar{n}. \quad (3.67)$$

Combining (3.67) with (3.53) we see that these vectors satisfy the recursive expressions

$$w_{\bar{n}} = C_{\bar{n}} w \quad (3.68)$$

$$w_i = E_{i,i+1} w_{i+1}, \quad 1 \leq i \leq \bar{n} - 1. \quad (3.69)$$

Using (3.38) we can rewrite (3.69) as

$$w_i^T = [w_{i-1}^T, w_{i,i}], \quad 1 \leq i \leq \bar{n} - 1. \quad (3.70)$$

We now set out to obtain explicit expressions for $w_1, \dots, w_{\bar{n}}$ in terms of ξ_i . Substituting (3.42) into (3.68) results in

$$\Xi w_{\bar{n}} = -\text{sgn}(b_m) C_{\bar{n}} H \Gamma \psi \quad (3.71)$$

We then use (3.27), (3.37) and (3.70) to rewrite (3.71) as

$$\left(I_{\bar{n} \times \bar{n}} + \begin{bmatrix} 0 \\ \xi_1 E_{1,\bar{n}} \\ \vdots \\ \xi_{\bar{n}-1} E_{\bar{n}-1,\bar{n}} \end{bmatrix} \right) \begin{bmatrix} w_{\bar{n}-1} \\ w_{\bar{n},\bar{n}} \end{bmatrix} = -\text{sgn}(b_m) \begin{bmatrix} c^T \\ c^T A_0 \\ \vdots \\ c^T A_0^{\bar{n}-1} \end{bmatrix} H \Gamma H^T c. \quad (3.72)$$

By (3.38), (3.72) is equivalent to

$$\left(I_{(\bar{n}-1) \times (\bar{n}-1)} + \begin{bmatrix} 0 \\ \xi_1 E_{1, \bar{n}-1} \\ \vdots \\ \xi_{\bar{n}-2} E_{\bar{n}-2, \bar{n}-1} \end{bmatrix} \right) w_{\bar{n}-1} = -\text{sgn}(b_m) C_{\bar{n}-1} H \Gamma H^T c \quad (3.73)$$

$$w_{\bar{n}, \bar{n}} + \xi_{\bar{n}-1} w_{\bar{n}-1} = -\text{sgn}(b_m) c^T A_0^{\bar{n}-1} H \Gamma H^T c. \quad (3.74)$$

Starting from (3.73)–(3.74), one can repeat the above procedure to show that (3.72) is equivalent to

$$\begin{bmatrix} w_1 \\ w_{2,2} + \xi_1 w_1 \\ \vdots \\ w_{\bar{n}, \bar{n}} + \xi_{\bar{n}-1} w_{\bar{n}-1} \end{bmatrix} = -\text{sgn}(b_m) \begin{bmatrix} c^T \\ c^T A_0 \\ \vdots \\ c^T A_0^{\bar{n}-1} \end{bmatrix} H \Gamma H^T c. \quad (3.75)$$

Hence, the explicit expressions for the vectors $w_1, \dots, w_{\bar{n}}$ are

$$w_1^T = -\text{sgn}(b_m) (c^T H \Gamma H^T c) \quad (3.76)$$

$$w_i^T = [w_{i-1}^T, -\text{sgn}(b_m) (c^T A_0^{i-1} H \Gamma H^T c) - \xi_{i-1} w_{i-1}], \quad i = 2, \dots, \bar{n}. \quad (3.77)$$

Combining (3.51) and (3.67) we obtain

$$\begin{aligned} C_i P^{-1} \sum_{j=1}^{\bar{n}} C_j^T M_j C_j w w^T C_j^T M_j^T C_j &= C_i P^{-1} \sum_{j=1}^i C_j^T M_j C_j w w^T C_j^T M_j^T C_j \\ &= C_i \left[P^{-1} \sum_{j=1}^i C_j^T M_j w_j w_j^T M_j^T E_{j,i} \right] C_i \\ &= C_i N_i C_i, \end{aligned} \quad (3.78)$$

where

$$N_i = P^{-1} \sum_{j=1}^i C_j^T M_j w_j w_j^T M_j^T E_{j,i}, \quad i = 1, \dots, \bar{n}. \quad (3.79)$$

Thus, the last term in (3.65) and (3.66) can be rewritten as

$$\begin{aligned} (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) P^{-1} \sum_{j=1}^{\bar{n}} C_j^T M_j C_j w w^T C_j^T M_j^T C_j &= \\ &= ([0 \dots 0 1] + \xi_{i-1} E_{i-1, i}) C_i P^{-1} \sum_{j=1}^{\bar{n}} C_j M_j C_j w w^T C_j^T M_j^T C_j \\ &= ([0 \dots 0 1] + \xi_{i-1} E_{i-1, i}) C_i N_i C_i \\ &= (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) N_i C_i. \end{aligned} \quad (3.80)$$

Introducing the $i \times (i+1)$ matrix $R_{i+1} = [0, I_{i \times i}]$ and Substituting (3.80) into (3.65)–(3.66) we obtain

$$\begin{aligned} \xi_i C_i &= \dot{\xi}_{i-1} C_{i-1} + \xi_{i-1} C_{i-1} A_0 - (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) N_i C_i \\ &= \left[\dot{\xi}_{i-1} E_{i-1,i} + \xi_{i-1} R_i - (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) N_i \right] C_i, \quad i = 1, \dots, \bar{n} - 1 \end{aligned} \quad (3.81)$$

$$\begin{aligned} \bar{u} &= \left\{ c^T A_0^{\bar{n}} S - c^T A_0^{\bar{n}} - \left[\dot{\xi}_{\bar{n}-1} E_{\bar{n}-1,\bar{n}} + \xi_{\bar{n}-1} R_{\bar{n}} \right. \right. \\ &\quad \left. \left. - (c^T A_0^{\bar{n}-1} + \xi_{\bar{n}-1} C_{\bar{n}-1}) N_{\bar{n}} \right] C_{\bar{n}} \right\} \zeta. \end{aligned} \quad (3.82)$$

This form makes it apparent that the design equation (3.56) is satisfied by the recursive expressions

$$\xi_1 = -c^T N_1 \quad (3.83)$$

$$\xi_i = \dot{\xi}_{i-1} E_{i-1,i} + \xi_{i-1} R_i - (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) N_i, \quad i = 2, \dots, \bar{n}, \quad (3.84)$$

$$\bar{u} = \left[c^T A_0^{\bar{n}} - (c^T A_0^{\bar{n}} + \xi_{\bar{n}} C_{\bar{n}}) S^{-1} \right] \eta. \quad (3.85)$$

To finally solve the design equation, we need to express $\dot{\xi}_{i-1}$ in (3.84) as an explicit function of available signals. We first show via an induction argument that (3.76), (3.77), (3.79), (3.83) and (3.84) imply that the elements of w_i , N_i and ξ_i are polynomial functions of the elements of $C_i H$:

- For $i = 1$, this fact is obvious from the definitions:

$$w_1 = -\text{sgn}(b_m) C_1 H \Gamma (C_1 H)^T$$

$$N_1 = P^{-1} C_1^T M_1 w_1 w_1^T M_1^T$$

$$\xi_1 = -c^T N_1.$$

- For $i = k \leq \bar{n} - 1$, suppose the elements of w_k , N_k and ξ_k are polynomial functions of the elements of $C_k H$. Then, the derivative of ξ_k can be expressed as

$$\dot{\xi}_k = \sum_{j=1}^{n_\theta} \dot{h}_{k,j}^T \frac{\partial \xi_k}{\partial h_{k,j}}, \quad (3.86)$$

where $h_{k,j}$ is the j -th column of $C_k H$ and $\partial \xi_k / \partial h_{k,j}$ is the $k \times j$ matrix of partial derivatives of $\xi_k(C_k H)$ with respect to the elements of $h_{k,j}$. But from (3.27), (3.63),

(3.64), and the definition of R_{k+1} , we obtain

$$C_k \dot{H} = C_k A_0 H = R_{k+1} C_{k+1} H, \quad 1 \leq k \leq \bar{n} - 1. \quad (3.87)$$

Combining (3.86) and (3.87) we can express $\dot{\xi}_k$ as

$$\dot{\xi}_k = \sum_{j=1}^{n_\theta} h_{k+1,j}^T R_{k+1}^T \frac{\partial \xi_k}{\partial h_{k,j}}. \quad (3.88)$$

• For $i = k + 1$, we have

$$\begin{aligned} w_{k+1}^T &= [w_k^T, -\text{sgn}(b_m)[0 \dots 0 \ 1] C_{k+1} H \Gamma (C_1 H)^T - \xi_k w_k] \\ N_{k+1} &= N_k E_{k,k+1} + P^{-1} C_{k+1}^T M_{k+1} w_{k+1} w_{k+1}^T M_{k+1}^T \\ \xi_{k+1} &= \sum_{j=1}^{n_\theta} h_{k+1,j}^T R_{k+1}^T \frac{\partial \xi_k}{\partial h_{k,j}} E_{k,k+1} + \xi_k R_{k+1} - (c^T A_0^k + \xi_k C_k) N_{k+1}. \end{aligned}$$

Hence, the elements of w_{k+1} , N_{k+1} and ξ_{k+1} are polynomial functions of the elements of $C_{k+1} H$.

Thus, the term $\dot{\xi}_{i-1}$ in (3.84) can be calculated explicitly from (3.88).

The design procedure is now complete. The expressions for ξ_i and \bar{u} , which guarantee that the nonnegative function V in (3.39) has the nonpositive derivative (3.47), are

$$\begin{aligned} \xi_1 &= -c^T N_1 \\ \xi_i &= \sum_{j=1}^{n_\theta} h_{i,j}^T R_i^T \frac{\partial \xi_{i-1}}{\partial h_{i-1,j}} E_{i-1,i} + \xi_{i-1} R_i - (c^T A_0^{i-1} + \xi_{i-1} C_{i-1}) N_i, \quad i = 2, \dots, \bar{n} \\ \bar{u} &= \varphi^T \eta \\ \varphi^T &= c^T A_0^{\bar{n}} - (c^T A_0^{\bar{n}} + \xi_{\bar{n}} C_{\bar{n}}) (C_{\bar{n}}^{-1} \Xi C_{\bar{n}})^{-1}. \end{aligned} \quad (3.89)$$

The designed closed-loop adaptive system is:

Plant:

$$\begin{aligned}\dot{x} &= A_{\Sigma}x + b_{\Sigma}q(y)u + \sum_{i=0}^m \varepsilon_{n-i} [p_{i0}(y) + p_i^T(y)\theta_1] \\ y &= c_{\Sigma}^T x\end{aligned}$$

Control:

$$u = \frac{1}{q(y)} [-\phi^T(y, u, r)\hat{\theta} + \varphi^T \eta]$$

Update law:

$$\dot{\hat{\theta}} = \text{sgn}(b_m)\Gamma\psi\bar{e}$$

$$\dot{\hat{b}}_m = -\gamma c^T \eta \bar{e}$$

$$\bar{e} = e + \eta_0 = y - y_r + \eta_0$$

Filters:

$$\dot{H} = A_0 H + b\phi^T(y, u, r)$$

$$\psi^T = c^T H$$

$$\dot{\eta}_0 = -\lambda_0 \eta_0 + \hat{b}_m c^T \eta$$

$$\dot{\eta} = A_0 \eta - b\varphi^T \eta - H \text{sgn}(b_m)\Gamma\psi\bar{e},$$

where y_r is the output of the reference model (3.2), $\phi(y, u, r)$ is defined in (3.12), and $(c_{\Sigma}, A_{\Sigma}, b_{\Sigma})$ is a minimal state representation of the plant equation (3.1):

$$A_{\Sigma} = \begin{bmatrix} -a_{n-1} & & & & \\ & \vdots & & & \\ & & I & & \\ -a_0 & & 0 & \dots & 0 \end{bmatrix}, \quad b_{\Sigma} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}, \quad c_{\Sigma} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (3.91)$$

with ε_{n-i} the $(n-i)$ -th coordinate vector in \mathbb{R}^n .

The stability and tracking properties of (3.90) are established in the next section.

4 Stability and Tracking

We are now ready to state and prove our main result:

Theorem 4.1. *For any uniformly bounded and piecewise continuous reference input r , all the signals in the closed-loop adaptive system (3.90) are well-defined and uniformly bounded on $[0, \infty)$, and, in addition,*

$$\lim_{t \rightarrow \infty} e(t) = 0, \quad \lim_{t \rightarrow \infty} \eta(t) \rightarrow 0, \quad \lim_{t \rightarrow \infty} \eta_0(t) = 0. \quad (4.1)$$

Proof. Due to the piecewise continuity of $r(t)$ and the smoothness of the nonlinear functions appearing in the definitions of various terms in the closed-loop system (3.90), the solution of (3.90) has a maximum interval of existence $[0, t_f)$. On this interval, the time derivative of the nonnegative function V defined in (3.39)

$$V = \frac{1}{2} \left(\bar{e}^2 + |b_m| \bar{\theta}^T \Gamma^{-1} \bar{\theta} + \frac{1}{\gamma} \bar{b}_m^2 + \frac{1}{\lambda_0} \int_t^\infty \epsilon^2(\tau) d\tau \right) + \frac{\lambda_0}{\bar{n}} \zeta^T P \zeta,$$

computed along the solutions of (3.90), is given by (3.47):

$$\dot{V} = -\frac{1}{2} \left(\sqrt{\lambda_0} \bar{e} - \frac{1}{\sqrt{\lambda_0}} \epsilon \right)^2 - \frac{\lambda_0}{\bar{n}} \zeta^T Q_0 \zeta - \frac{2\lambda_0}{\bar{n}} \sum_{i=1}^{\bar{n}} \left(\frac{\bar{e}}{2} - \zeta^T P_i w \right)^2 \leq 0.$$

We conclude that V , \bar{e} , $\bar{\theta}$, \bar{b}_m and ζ are bounded on $[0, t_f)$ by constants depending only on the initial conditions of (3.90). This implies that $\hat{\theta}$, \hat{b}_m are bounded on $[0, t_f)$. The boundedness of ζ together with (3.60) implies that $c^T \eta$ is bounded; from the definition of η_0 in (3.90) and the boundedness of \hat{b}_m and $c^T \eta$ it follows that η_0 is bounded. But since $e = \bar{e} - \eta_0$, and \bar{e} , η_0 are bounded, we have that e is bounded. Now from the boundedness of r we have that y_r is bounded, and, hence, y is bounded, since $y = e + y_r$. The boundedness of y implies that all the nonlinearities appearing are bounded, and, furthermore, that $q(y)$ is bounded away from zero. Filtering the system equation (3.1) with the strictly proper stable filter $1/BE_1$ and rearranging terms, we obtain

$$\frac{1}{E_1} [q(y)u] = \frac{A}{BE_1} y + \sum_{i=0}^m \frac{D^i}{BE_1} [p_{i0}(y) + p_i^T(y)\theta_1], \quad (4.2)$$

which, by the boundedness of y , implies that $\frac{1}{E_1}q(y)u$ is bounded. The boundedness of H is now established by proving that the row vectors $c^T A_0^i H$, $0 \leq i \leq \bar{n} - 1$ are bounded. This, in turn, is proved as follows: First, from [21, Lemma 5] we have that the first \bar{n} derivatives of e can be expressed as

$$e^{(i)} = \mu_i \left(\bar{e}, \tilde{\theta}, \tilde{b}_m, \eta_0, C_i H, C_i \zeta, \epsilon_i \right), \quad i = 1, \dots, \bar{n}, \quad (4.3)$$

where the μ_i 's are continuous functions of their arguments and the ϵ_i 's are exponentially decaying terms. This is straightforward to show (cf. (2.48)–(2.49) in the example of Sect. 2), starting from $e = \bar{e} - \eta_0$ and using (3.90) and the facts that the derivative of ζ is given by

$$\dot{\zeta} = A_0 \zeta + w \bar{e} - P^{-1} \sum_{i=1}^{\bar{n}} C_i^T M_i C_i w w^T C_i^T M_i^T C_i \zeta, \quad (4.4)$$

and that the elements of w_i , N_i and ξ_i are polynomial functions of the elements of $C_i H$. Second, from (3.27) we have

$$c^T A_0^i H = \frac{D^i}{E_0} \phi^T(y, u, r) + \bar{\epsilon}_i(t), \quad 0 \leq i \leq \bar{n} - 1, \quad (4.5)$$

where $\bar{\epsilon}_i(t)$ are n_θ -dimensional row vectors of exponentially decaying terms. Then, we use (3.12) to express ϕ as

$$\phi^T(y, u, r) = \left[\bar{v}(y, r), \frac{\bar{D}_{n-2}}{E_2} [q(y)u] \right], \quad (4.6)$$

with

$$\bar{v}(y, r) = \left[\frac{\bar{D}_{n-1}}{E_2} y, \frac{R}{E_2} r, \frac{\bar{D}_{n-m-1}}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{i0}(y), \dots, \frac{\bar{D}_{n-m-1}}{E_{21}} \sum_{i=0}^m \frac{D^i}{E_{22}} p_{i\ell}(y) \right] \quad (4.7)$$

being bounded, since y and r are bounded. Combining (3.13), (3.19) and (4.5) we obtain the following expressions for the first \bar{n} derivatives of the tracking error e :

$$\begin{aligned} e^{(i)} &= \frac{b_m D^i}{E_1} \left[\phi^T(y, u, r) \tilde{\theta} + \bar{u} \right] + \epsilon(t) \\ &= \frac{b_m D^i}{E_1} \left[q(y)u + \phi^T(y, u, r) \theta \right] + \epsilon(t) \\ &= \frac{b_m D^i}{E_1} [q(y)u] + \frac{b_m E_0 D}{E_1} \frac{D^{i-1}}{E_0} \phi^T(y, u, r) \theta + \epsilon(t) \\ &= \frac{b_m D^i}{E_1} [q(y)u] + \frac{b_m E_0 D}{E_1} c^T A_0^{i-1} H \theta + \epsilon(t), \quad 1 \leq i \leq \bar{n} - 1. \end{aligned} \quad (4.8)$$

It is important to note that $b_m E_0 D / E_1$ is stable and proper. The boundedness of $c^T A_0^i H$ is now established for $i = 0, \dots, \bar{n} - 1$ by the following induction argument:

- For $i = 0$, (4.5)–(4.6) give

$$\begin{aligned} c^T H &= \left[\frac{1}{E_0} \bar{v}(y, r), \frac{\bar{D}_{n-2}}{E_0 E_2} [q(y)u] \right] \\ &= \left[\frac{1}{E_0} \bar{v}(y, r), \frac{E_1 \bar{D}_{n-2}}{E_0 E_2} \frac{1}{E_1} [q(y)u] \right]. \end{aligned} \quad (4.9)$$

Since $\bar{v}(y, r)$, $\frac{1}{E_1} [q(y)u]$ are bounded and $\frac{E_1 \bar{D}_{n-2}}{E_0 E_2}$ is a row of stable proper filters, (4.9) implies that $c^T H$ is bounded. Furthermore, we have already shown that e is bounded.

- For $1 \leq i \leq \bar{n} - 1$, assume that $c^T A_0^k H$ and $e^{(k)}$ are bounded for $0 \leq k \leq i - 1$. Hence, $C_i H$ is bounded, and, by (4.3), $e^{(i)}$ is bounded. Then, rewriting (4.8) as

$$\frac{D^i}{E_1} [q(y)u] = \frac{1}{b_m} e^{(i)} - \frac{E_0 D}{E_1} c^T A_0^{i-1} H \theta + \epsilon(t), \quad (4.10)$$

we conclude that $\frac{D^i}{E_1} [q(y)u]$ is bounded. Finally, using (4.5)–(4.7), we obtain

$$\begin{aligned} c^T A_0^i H &= \left[\frac{D^i}{E_0} \bar{v}(y, r), \frac{D^i \bar{D}_{n-2}}{E_0 E_2} [q(y)u] \right] \\ &= \left[\frac{D^i}{E_0} \bar{v}(y, r), \frac{E_1 \bar{D}_{n-2}}{E_0 E_2} \frac{D^i}{E_1} [q(y)u] + \bar{\epsilon}(t) \right]. \end{aligned} \quad (4.11)$$

Hence, $c^T A_0^i H$ is bounded.

This proves that H is bounded, which, by (4.3), means that $e^{(i)}$, $1 \leq i \leq \bar{n}$, are bounded.

Next, we prove the boundedness of u . From (3.13), the boundedness of $\hat{\theta}$ and the fact that $q(y)$ is bounded away from zero, it follows that u is bounded if \bar{u} and $\phi(y, u, r)$ are bounded. The boundedness of H implies the boundedness of φ , Ξ and Ξ^{-1} . Since $\eta = S\zeta = (C_{\bar{n}})^{-1} \Xi C_{\bar{n}}$, and Ξ and ζ are bounded, η is bounded as well. Hence, \bar{u} is bounded. From (4.6)–(4.7), to prove boundedness of $\phi(y, u, r)$ we only need to show that $\frac{\bar{D}_{n-2}}{E_2} [q(y)u]$. We first rewrite (4.8) as

$$\frac{D^i}{E_1} [q(y)u] = \frac{1}{b_m} e^{(i)} - \frac{E_0 D}{E_1} c^T A_0^{i-1} H \theta + \epsilon(t), \quad 1 \leq i \leq \bar{n} - 1, \quad (4.12)$$

which implies that $\frac{D^i}{E_1} [q(y)u]$ is bounded for $i = 1, \dots, \bar{n} - 1$. Combining this with the fact that $\frac{1}{E_1} [q(y)u]$ it follows that $\frac{D^i}{E_2} [q(y)u]$ is bounded for $i = 1, \dots, n - 3$. Differentiating (4.12) with $i = \bar{n} - 1$ and using (3.27) and (3.64) we obtain

$$\frac{D^{\bar{n}}}{E_1} [q(y)u] = \frac{1}{b_m} e^{(\bar{n})} - \frac{E_0 D}{E_1} [c^T A_0^{\bar{n}} H + \phi^T(y, u, r)] \theta + \epsilon(t). \quad (4.13)$$

Substituting $\phi^T(y, u, r)$ from (4.6) and rewriting (3.11) as $\theta^T = [\bar{\theta}^T, \ell_{0n-2}]$, we express (4.13) as

$$\begin{aligned} \frac{L}{E_2} [q(y)u] &= \frac{D^{\bar{n}} E_2 + \ell_{0n-2} D^{n-1} E_0}{E_1 E_2} [q(y)u] \\ &= \frac{1}{b_m} e^{(\bar{n})} - \frac{E_0 D}{E_1} \left[c^T A_0^{\bar{n}} H \theta + \left[\bar{v}(y, r), \frac{D^{n-3}}{E_2} [q(y)u] \right] \bar{\theta} \right] + \epsilon(t), \end{aligned} \quad (4.14)$$

which implies that $\frac{L}{E_2} [q(y)u]$ is bounded. Since L is of degree $n - 2$ and $\frac{D^i}{E_2} [q(y)u]$ is bounded for $i = 1, \dots, n - 3$, it follows that $\frac{D^{n-2}}{E_2} [q(y)u]$ is bounded. Hence, $\frac{D^{n-2}}{E_2} [q(y)u]$ is bounded, which proves that u is bounded.

In order to show the boundedness of the state of the plant, we note that the boundedness of u and (3.16) imply that $D^i y$, $0 \leq i \leq n - m$, are bounded. From this and the fact that $B(D)$ is Hurwitz, we conclude that the state x in (3.90) is bounded.

We have thus proved that the state of the closed-loop adaptive system (3.90) is bounded on $[0, t_f]$. Hence, $t_f = \infty$.

To prove the convergence of the tracking error e to zero, we first note that (3.39) and (3.47) imply that \dot{V} is bounded and integrable on $[0, \infty)$. Furthermore, the boundedness of \dot{e} (cf. (3.32)), $\dot{\zeta}$ (cf. (4.4)) and \dot{H} (cf. (3.90)) implies that \ddot{V} is bounded. Hence, $\dot{V} \rightarrow 0$ as $t \rightarrow \infty$, which, in view of (3.47), proves that $\bar{e} \rightarrow 0$, $\zeta \rightarrow 0$ as $t \rightarrow \infty$. Since $\eta = S\zeta$ and S is bounded, $\eta \rightarrow 0$ as $t \rightarrow \infty$. Combined with (3.90) and the boundedness of \hat{b}_m , this also proves that $\eta_0 \rightarrow 0$ as $t \rightarrow \infty$. Thus,

$$\lim_{t \rightarrow \infty} [y(t) - y_r(t)] = \lim_{t \rightarrow \infty} [\bar{e}(t) - \eta_0(t)] = 0. \quad (4.15)$$

□

5 The Class of Nonlinear Systems

Most models of nonlinear systems are expressed in specific state coordinates. From that state-space form it may not always be obvious whether or not the nonlinear system at hand has the input-output description assumed in Sect.3. Therefore, we now give coordinate-free geometric conditions which are necessary and sufficient for a single-input single-output nonlinear system of the form

$$\begin{aligned} \dot{z} &= f(z; \alpha) + g(z; \alpha)u \\ y &= h(z; \alpha) \end{aligned} \quad (5.1)$$

to have an input-output description of the form (3.1), which is repeated here for convenience:

$$A(D)y = B(D)[q(y)u] + \sum_{i=0}^m D^i [p_{i0}(y) + p_i^T(y)\theta_1]. \quad (5.2)$$

In (5.1) $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $\alpha = [\alpha_1 \dots \alpha_r]^T \in \mathbb{R}^r$ is a vector of unknown constant parameters, and f, g, h , are smooth vector fields with $f(0; \alpha) = 0, h(0; \alpha) = 0$, for all $\alpha \in \mathbb{R}^r, g(z) \neq 0$ for all $z \in \mathbb{R}^n$. In (5.2)

- the coefficients a_0, \dots, a_{n-1} of the denominator polynomial $A(D) = D^n + a_{n-1}D^{n-1} + \dots + a_0$ are unknown,
- the coefficients $b_0, \dots, b_m (m \leq n-1)$ of the numerator polynomial $B(D) = b_m D^m + \dots + b_0$ are unknown,
- θ_1 is an ℓ -dimensional vector of unknown parameters, resulting from a possible overparameterization in which products and powers of the original unknown parameters α_i are treated as new parameters (so that $\ell \geq r$),
- $q(y), p_{ij}(y), 0 \leq i \leq m, 0 \leq j \leq \ell$ are smooth nonlinearities with $q(y) \neq 0 \forall y \in \mathbb{R}, p_{ij}(0) = 0, 0 \leq i \leq m, 0 \leq j \leq \ell$.

We first note that a minimal state representation of (5.2) is given by

$$\begin{aligned} \dot{x} &= A_\Sigma x + b_\Sigma q(y)u + \sum_{i=0}^m \varepsilon_{n-i} [p_{i0}(y) + p_i^T(y)\theta_1] \\ y &= c_\Sigma^T x, \end{aligned} \quad (5.3)$$

with $A_\Sigma, b_\Sigma, c_\Sigma, \varepsilon_{n-i}$ as defined in (3.91). Hence, the following statement becomes obvious:

Fact 5.1. *The nonlinear system (5.1) has an input-output description of the form (5.2) if and only if there exists a global in z , possibly parameter-dependent, diffeomorphism transforming (5.1) into (5.3).*

Using this fact, we now state the following result:

Proposition 5.2. *The system (5.1) has an input-output description of the form (5.2) if and only if the following conditions are satisfied for all $z \in \mathbb{R}^n$ and for the true value of the parameter vector α :*

(C1) *the one-forms $dh, dL_f h, \dots, dL_f^{n-1} h$ are linearly independent*

(C2) $[\text{ad}_f^i \bar{g}, \text{ad}_f^j \bar{g}] = 0, \quad i, j = 0, \dots, n-1$, *where \bar{g} is uniquely defined by*

$$L_{\bar{g}} L_f^i h = \begin{cases} 0, & i = 0, \dots, n-2 \\ 1, & i = n-1 \end{cases}$$

(C3)

$$\begin{aligned} \text{ad}_f^n \bar{g} &= \sum_{j=0}^{n-1} d_j(\alpha) \text{ad}_f^j \bar{g} + \sum_{j=0}^m [p_{j0}'(y) + p_j^T(y) \theta_1] \text{ad}_f^j \bar{g} \\ [g, \text{ad}_f^j \bar{g}] &= 0, \quad j = 0, \dots, n-2 \\ g &= q(y) \sum_{j=0}^m c_j(\alpha) \text{ad}_{f_0}^j \bar{g}, \end{aligned}$$

with $d_j(\alpha), c_j(\alpha)$ polynomial functions of α , θ_1 the new unknown parameter vector, and $p_i(y) = \int_0^y p_i'(v) dv, i = 0, \dots, \ell$

(C4) *the vector fields f and \bar{g} are complete.*

Proof. Using Proposition 3 of [23], it is straightforward to show that conditions (C1)–(C3) are necessary and sufficient for the existence of a local diffeomorphism such that in the new

coordinates the system (5.1) is expressed as

$$\begin{aligned}
 \dot{x} &= \begin{bmatrix} (-1)d_{n-1}(\alpha) & & & \\ & \vdots & & \\ & & I & \\ (-1)^n d_0(\alpha) & 0 & \dots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{n-m} c_m \\ \vdots \\ (-1)^n c_0 \end{bmatrix} q(y)u \\
 &+ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{n-m} [p_{m0}(y) + p_m^T(y)\theta_1] \\ \vdots \\ (-1)^n [p_{00}(y) + p_0^T(y)\theta_1] \end{bmatrix} \\
 y &= x_1,
 \end{aligned} \tag{5.4}$$

which is exactly in the form (5.3), where the coefficients $a_0, \dots, a_{n-1}, b_0, \dots, b_m$ depend on the physical parameters α . From [24], condition (C4) is necessary and sufficient for the above diffeomorphism to be global. \square

Remark 5.3. The above proposition gives a set of geometric conditions characterizing the class of nonlinear systems to which our adaptive scheme can be applied. Whenever the conditions (C1)–(C4) can be verified a priori, the input-output description (5.2) of the nonlinear system at hand is determined directly from (C3), without the need to compute the diffeomorphism of Proposition 5.2. Unfortunately, the verification of (C1)–(C4) may require some a priori information about the unknown parameter vector α .

Remark 5.4. The conditions of Proposition 5.2 are satisfied by nonlinear systems that are linearizable by output injection and input-output linearizable by full-state feedback. However, they need not be full-state feedback linearizable.

We illustrate these two remarks with an example.

Example 5.6. It may not be obvious that (2.1) is the input-output description of the nonlinear system

$$\begin{aligned}
 \dot{z}_1 &= z_2 + \alpha(3ye^y + 2y^2) \\
 \dot{z}_2 &= z_3 - \alpha(2ye^y + y^2) \\
 \dot{z}_3 &= z_4 + \alpha ye^y \\
 \dot{z}_4 &= z_5 + \alpha y^2 \\
 \dot{z}_5 &= u + \alpha y^3 \\
 y &= z_1 + 2z_2 + z_3.
 \end{aligned} \tag{5.5}$$

However, this can be established by checking the conditions (C1)–(C4). Straightforward calculations show that for (5.5) we have

$$\bar{g} = 5\frac{\partial}{\partial z_1} - 4\frac{\partial}{\partial z_2} + 3\frac{\partial}{\partial z_3} - 2\frac{\partial}{\partial z_4} + \frac{\partial}{\partial z_5} \tag{5.6}$$

$$\text{ad}_f^5 \bar{g} = \alpha \left[-3y^2 \bar{g} + (2y + 6y^2) \text{ad}_f \bar{g} - (ye^y + e^y + 4y + 3y^2) \text{ad}_f^2 \bar{g} \right] \tag{5.7}$$

$$g = -\bar{g} + 2\text{ad}_f \bar{g} - \text{ad}_f^2 \bar{g}. \tag{5.8}$$

Hence, the conditions (C1)–(C4) are satisfied for all α and the input-output description of (5.5) is

$$D^5 y = (D^2 + 2D + 1)u + \theta [D^2(ye^y + 2y^2 + y^3) + D(y^2 + 2y^3) + y^3], \tag{5.9}$$

where $\theta = \alpha$. It is important to note that to determine this input-output description no explicit change of coordinates was required. In this simple example, however, one can find the corresponding change of coordinates by inspection:

$$\begin{aligned}
 x_1 &= z_1 + 2z_2 + z_3 \\
 x_2 &= z_2 + 2z_3 + z_4 \\
 x_3 &= z_3 + 2z_4 + z_5 \\
 x_4 &= z_4 + 2z_5 \\
 x_5 &= z_5.
 \end{aligned} \tag{5.10}$$

In these coordinates, (5.5) becomes

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 + u + \alpha (ye^y + 2y^2 + y^3) \\ \dot{x}_4 &= x_5 + \alpha (y^2 + 2y^3) \\ \dot{x}_5 &= u + \alpha y^3 \\ y &= x_1.\end{aligned}\tag{5.11}$$

We immediately see that (5.11) has the input-output description (5.9).

However, it should also be pointed out that (5.5) is *not* full-state feedback linearizable, since the distribution

$$\mathcal{G}^3 = \text{span} \{g, \text{ad}_f g, \text{ad}_f^2 g, \text{ad}_f^3 g\}\tag{5.12}$$

is not involutive[3,4]. □

6 Conclusions

This paper has extended the theory of adaptive control for linear systems to a class of systems which are essentially nonlinear in the sense that their nonlinearities are not restricted by any growth constraints. In spite of this absence of growth constraints, all the stability and tracking results are global.

The assumptions on the linear part of the system are the same as in the standard adaptive theory for linear systems. However, to guarantee the aforementioned global properties, the systematic design procedure has departed from the two main ingredients of most adaptive schemes for linear systems: the certainty-equivalence control and the normalization of the update law. In addition to the certainty-equivalence part, the control contains a term which counteracts the effects of rapidly growing nonlinearities. Thanks to the presence of this term, the normalization of the update law is avoided, which allows the rapid decrease of the parameter error. This proved to be crucial in preventing finite escape times common in systems with rapidly growing nonlinearities.

The class of nonlinear systems has been restricted by coordinate-free geometric conditions which are equivalent to the structural requirements that the nonlinearities depend only on the output and do not enter the system before the control does, and that the zero dynamics are linear and exponentially stable. Relaxing these restrictions, and thus enlarging the class of systems that can be adaptively controlled using only output measurement, is a topic of further research.

Acknowledgement

The authors would like to thank Professor Riccardo Marino for many helpful discussions which led to the development of Section 5.

References

- [1] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1989.
- [2] S. S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1989.
- [3] A. Isidori, *Nonlinear Control Systems*, 2nd ed., Springer-Verlag, Berlin, 1989.
- [4] H. Nijmeijer and A. J. van der Shaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, Berlin, 1990.
- [5] G. Campion and G. Bastin, "Indirect adaptive state feedback control of linearly parametrized nonlinear systems," *Int. J. Adapt. Control Sig. Proc.*, vol. 4, pp. 345-358, Sept. 1990.
- [6] I. Kanellakopoulos, P. V. Kokotovic, and R. Marino, "Robustness of adaptive nonlinear control under an extended matching condition," *Prepr. IFAC Symp. Nonlinear Control Syst. Design*, pp. 192-197, Capri, Italy, June 1989.
- [7] I. Kanellakopoulos, P. V. Kokotovic, and R. Marino, "An extended direct scheme for robust adaptive nonlinear control," *Automatica*, to appear, March 1991.
- [8] R. Marino, I. Kanellakopoulos, and P. V. Kokotovic, "Adaptive tracking for feedback linearizable SISO systems," *Proc. 28th IEEE Conf. Dec. Control*, pp. 1002-1007, Tampa, FL, Dec. 1989.

- [9] J.-J. E. Slotine and J. A. Coetsee, "Adaptive sliding controller synthesis for non-linear systems," *Int. J. Control*, vol. 43, pp. 1631-1651, June 1986.
- [10] D. G. Taylor, P. V. Kokotovic, R. Marino, and I. Kanellakopoulos, "Adaptive regulation of nonlinear systems with unmodeled dynamics," *IEEE Trans. Aut. Control*, vol. 34, pp. 405-412, April 1989.
- [11] K. Nam and A. Arapostathis, "A model-reference adaptive control scheme for pure-feedback nonlinear systems," *IEEE Trans. Aut. Control*, vol. 33, pp. 803-811, Sept. 1988.
- [12] J.-B. Pomet and L. Praly, *Adaptive nonlinear control: an estimation-based algorithm*, in *New Trends in Nonlinear Control Theory*, J. Descusse, M. Fliess, A. Isidori and D. Leborgne Eds., Springer-Verlag, Berlin, 1989.
- [13] J.-B. Pomet and L. Praly, "Adaptive nonlinear regulation: equation error from the Lyapunov equation," *Proc. 28th IEEE Conf. Dec. Control*, pp. 1008-1013, Tampa, FL, Dec. 1989.
- [14] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Aut. Control*, vol. 34, pp. 1123-1131, Nov. 1989.
- [15] A. Teel, R. Kadiyala, P. V. Kokotovic, and S. S. Sastry, "Indirect techniques for adaptive input output linearization of nonlinear systems," *Int. J. Control*, to appear.
- [16] I. Kanellakopoulos, P. V. Kokotovic, and R. H. Middleton, "Observer-based adaptive control of nonlinear systems under matching conditions," *Proc. 1990 Amer. Control Conf.*, pp. 549-555, San Diego, CA, May 1990.
- [17] I. Kanellakopoulos, P. V. Kokotovic, and R. H. Middleton, "Indirect adaptive output-feedback control of a class of nonlinear systems," *Proc. 29th IEEE Conf. Dec. Control*, pp. 2714-2719, Honolulu, HI, Dec. 1990.
- [18] G. Bastin and M. R. Gevers, "Stable adaptive observers for nonlinear time-varying systems," *IEEE Trans. Aut. Control*, vol. 33, pp. 650-658, July 1988.
- [19] R. Marino, "Adaptive observers for single output nonlinear systems," *IEEE Trans. Aut. Control*, vol. 35, pp. 1054-1058, Sept. 1990.
- [20] R. Marino and P. Tomei, "Global adaptive observers for nonlinear systems via filtered transformations," Submitted to *IEEE Trans. Autom. Control*, 1990.
- [21] A. Feuer and A. S. Morse, "Adaptive control of single-input single-output linear systems," *IEEE Trans. Aut. Control*, vol. AC-23, pp. 557-569, Aug. 1978.
- [22] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," Technical Report DC-124, Coordinated Science Laboratory, University of Illinois, Urbana, IL, Oct. 1990.

- [23] A. J. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Syst. Control Lett.*, vol. 3, pp. 47-52, June 1983.
- [24] W. Respondek, *Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems*, in *Algebraic and Geometric Methods in Nonlinear Control Theory*, M. Fliess and M. Hazewinkel (Eds.), pp. 257-284, D. Reidel Publishing Co., Dordrecht, 1986.