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NEAR-OPTIMAL FEEDBACK STABILIZATION OF A CLASS OF NONLINEAR SINGULARLY PERTURBED SYSTEMS

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20. ABSTRACT (continued)

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NEAR-OPTIMAL FEEDBACK STABILIZATION OF A CLASS OF NONLINEAR SINGULARLY PERTURBED SYSTEMS

by

Joe H. Chow and Petar V. Kokotovic

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Near-Optimal Feedback Stabilization

of a Class of Nonlinear Singularly Perturbed Systems[†]

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ABSTRACT

A new series expansion method is developed for a class of nonlinear singularly perturbed optimal regulator problems. The resulting feedback control is near-optimal and can stabilize essentially nonlinear systems when linearized models provide no stability information. The stability domain is shown to include large initial conditions of the fast variables. The control law is implemented in two-time-scales, with the feedback from the fast state variables depending on slow state variables as parameters. The coefficients of the formal expansions of the optimal value function are obtained from equations involving only the slow variables.

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I. Introduction

Compared with the rich literature on linear regulator theory, publications dealing with feedback design of nonlinear systems are a small minority. Realistic approaches to the difficult nonlinear feedback control problem usually exploit properties of special classes of systems to develop approximate methods [1,2]. The approach in this paper exploits multiple time scale properties of a class of nonlinear singularly perturbed systems [3,4] to achieve stabilization and near-optimality. The stabilization results obtained are essentially nonlinear in the sense that they also apply to the critical case when linearized models provide no stability information. Due to a separation of time scales, the proposed design procedure is applicable to higher order systems.

The problem considered is to optimally control the nonlinear system

$$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u , \qquad x(0) = x_o \tag{1a}$$

$$\mu\dot{z} = a_2(x) + A_2(x)z + B_2(x)u , \qquad z(0) = z_o \tag{1b}$$

with respect to the performance index

$$J = \int_0^\infty [p(x) + s'(x)z + z'Q(x)z + u'R(x)u]\,dt \tag{2}$$

where $\mu > 0$ is the small singular perturbation parameter, x, z are n-,m-dimensional states, respectively, u is an r-dimensional control and the prime denotes a transpose. It is assumed that there exists a domain $D \subset R^n$ containing the origin such that for all $x \in D$ and $z \in R^m$ the problem satisfies the following assumptions:

I. The functions a_1, a_2, A_1, A_2, B_1, B_2, p, s, q and R are differentiable with respect to x a sufficient number of times and a_1, a_2, p and s are all zero only at $x = 0$.

II. The matrices Q(x) and R(x) are positive definite, that is, $Q(x) > 0$, $R(x) > 0$. Furthermore, the scalar function $p + s'z + z'Qz$ of x and z is positive definite in both x and z.

III. For every fixed $x \in D$

$$\mathrm{rank}[B_2, A_2B_2, \ldots, A_2^{m-1}B_2] = m \tag{3}$$

and hence $A_2(x)$ is assumed to be nonsingular. (If not, then using $u = \hat{u} + K(x)z$ such that $A_2 + B_2K$ is nonsingular we redefine the problem.)

Assumptions I and II establish that the origin is the desired equilibrium of (1). Assumption III and $Q(x) > 0$ simplify the derivations. Alternatively a less restrictive stabilizability-detectability condition can be used.

Finite time trajectory optimization problems for the same class of systems have been treated in [3,4] via singularly perturbed two point boundary value problems originating from necessary optimality conditions. The resulting controls are open-loop and require boundary layer correction terms at both ends of the interval. For the infinite time regulator problem considered here the Hamilton-Jacobi-Bellman sufficiency condition is more suitable since it readily incorporates stability requirements and leads to feedback solutions. Using this condition we obtain near-optimal stabilizing controls in feedback form and avoid explicit treatment of boundary layer phenomena.

Our procedure is based on a nested power series expansion of the optimal value function in z and μ. An advantage of this procedure is that it uses lower order equations involving only the slow variable x. In applications truncated series are of interest. Stabilizing properties of various truncated designs are discussed and an explicit estimate of the stability domain is given. It is of practical importance that this domain encompasses large initial disturbances of z(0). Furthermore, near-optimality of these truncated designs is established in terms of $O(\mu)$, $O(\mu^2)$, etc. A particularly useful result is that an $O(\mu)$ near-optimal feedback control can be implemented without knowing the value of the small parameter μ.

The paper is organized as follows. In Section II a reduced order problem is formulated for the slow variable x. The crucial assumption is that the properties of its solution are known. Using a truncated expansion of the optimal value function the so called composite control is introduced in Section III. Since the leading term in the series is the optimal value function of the reduced problem, the original problem is well posed. In Section IV it is shown that the composite control guarantees a finite domain of stability for the resulting feedback system. In Section V, a formal expansion of the optimal value function is proposed and near-optimality results are discussed. An example is discussed in Section VI.

II. The Reduced Control

In singular perturbation techniques [5], a problem for the full order system (1) where $\mu > 0$ is interpreted as a perturbation of a reduced problem

$$\dot{x} = a_1(x) + A_1(x)z + B_1(x)u, \quad x(0) = x_o \tag{4a}$$

$$0 = a_2(x) + A_2(x)z + B_2(x)u \tag{4b}$$

in which $\mu = 0$. Due to Assumption III, z can be solved from (4b) and eliminated from (4a) and (2). Then the reduced problem is to optimally control the system

$$\dot{x} = a_o(x) + B_o(x)u \ , \quad x(0) = x_o \tag{5}$$

with respect to

$$J_o = \int_0^\infty [p_o(x) + 2s_o'(x)u + u'R_o(x)u]\,dt \tag{6}$$

where

$$a_o = a_1 - A_1A_2^{-1}a_2$$

$$B_o = B_1 - A_1A_2^{-1}B_2$$

$$p_o = p - s'A_2^{-1}a_2 + a_2'A_2'^{-1}QA_2^{-1}a_2$$

$$s_o = B_2'A_2'^{-1}(QA_2^{-1}a_2 - \frac{1}{2}s)$$

$$R_o = R + B_2'A_2'^{-1}QA_2^{-1}B_2 \ . \tag{7}$$

The origin $x = 0$ is the desired equilibrium of the optimally controlled reduced system (5) for all $x \in D$, since, in view of Assumption II, $a_o(0) = 0$ and

$$p_o(x) + 2s_o'(x)u + u'R_o(x)u \tag{8}$$

is positive definite in x and u.

The reduced problem (5), (6) is considerably simpler than the original problem (1), (2) because of the elimination of the fast variables and the reduction of the system order. One of the tasks of the singular perturbation analysis is to establish whether the full problem is well posed in the sense that its solution tends to the solution of the reduced problem as $\mu \to 0$. If so, then the next task is to deduce the properties of the original problem from the properties of the reduced problem.

Finally these properties are to serve as a basis for a simplified design procedure.

To formulate our basic assumption about the properties of the solution of the reduced problem we use the optimality principle

$$0 = \min_u [p_o(x) + 2s_o'(x)u + u'R_o(x)u + L_x(a_o(x) + B_o(x)u)] \tag{9}$$

where L is the optimal value function and L_x is its partial derivative with respect to x. This yields the minimizing control

$$u_o = -R_o^{-1}(s_o + \frac{1}{2} B_o'L_x') \tag{10}$$

whose elimination from (9) results in the Hamilton-Jacobi equation

$$0 = (p_o - s_o'R_o^{-1}s_o) + L_x(a_o - B_oR_o^{-1}s_o) - \frac{1}{4} L_xB_oR_o^{-1}B_o'L_x', \quad L(0) = 0. \tag{11}$$

Note that, due to (8), $p_o - s_o'R_o^{-1}s_o$ is positive definite in D. Our crucial assumption is then stated as follows.

IV. The unique positive definite solution L(x) of (11) exists in D and is differentiable with respect to x a sufficient number of times. Furthermore the level surface $L = c_o$ = constant is taken to be the boundary of the set D.

In the special case considered in [1], where the linearization of (5) at $x = 0$ is stabilizable and its states are observable in the quadratic approximation of J_o, our Assumption IV is automatically satisfied for all x near the origin. It follows from Assumption IV that u_o is the unique optimal feedback control for the reduced problem and L is a Lyapunov function of the optimally controlled reduced system

$$\dot{x} = a_o - B_o R_o^{-1}(s_o + \frac{1}{2} B_o' L_x') = \bar{a}_o(x) \tag{12}$$

establishing that the origin is asymptotically stable and the set D belongs to its domain of attraction.

III. The Composite Control

The optimal value function $V(x,z,\mu)$ of the full problem (1), (2) satisfies the equation

$$0 = \min_u [p + s'z + z'Qz + u'Ru + V_x(a_1 + A_1 z + B_1 u) + \frac{1}{\mu} V_z(a_2 + A_2 z + B_2 u)] \tag{13}$$

where V_x, V_z denote the partial derivatives of V with respect to the variables x, z, respectively. The minimizing control of (13) is

$$u = -\frac{1}{2} R^{-1}(B_1' V_x' + \frac{1}{\mu} B_2' V_z') \tag{14}$$

and its substitution into (13) yields the Hamilton-Jacobi equation

$$0 = p + s'z + z'Qz + V_x(a_1 + A_1 z) + \frac{1}{\mu} V_z(a_2 + A_2 z) - \frac{1}{4}(V_x B_1 + \frac{1}{\mu} V_z B_2) R^{-1}(B_1' V_x' + \frac{1}{\mu} B_2' V_z'), \quad V(0,0,\mu) = 0 . \tag{15}$$

Since system (1) is linear in z and J in (2) is quadratic in z, and since $\dot{z}$ is multiplied by μ, we seek a solution of (15) in the form

$$V(x,z,\mu) = \bar{V}_0(x) + \mu \bar{V}_1'(x)z + \mu z'\bar{V}_2(x)z + \mu q(x,z,\mu) \equiv \bar{V}(x,z,\mu) + \mu q(x,z,\mu), \quad \bar{V}_0(0) = 0 \tag{16}$$

where

$$\partial q/\partial x = O(1), \quad \partial q/\partial z = O(\mu). \tag{17}$$

We shall investigate the expansion of q in a later section. The partial derivatives of V with respect to x,z are

$$V_x = \overline{V}_{0x} + 0(\mu)$$
$$V_z = \mu\overline{V}_1' + 2\mu z'\overline{V}_2 + 0(\mu^2). \tag{18}$$

Substituting (18) into (15) and neglecting the μ dependent terms, we obtain the equation

$$\begin{aligned}0 = {} & p + \overline{V}_{0x}a_1 + \overline{V}_1'a_2 - \frac{1}{4}(\overline{V}_{0x}B_1 + \overline{V}_1'B_2)R^{-1}(B_1'\overline{V}_{0x}' + B_2'\overline{V}_1) \\ & + [s' + 2a_2'\overline{V}_2 + \overline{V}_{0x}(A_1 - B_1R^{-1}B_2'\overline{V}_2) + \overline{V}_1'(A_2 - B_2R^{-1}B_2'\overline{V}_2)]z \\ & + z'(Q + \overline{V}_2A_2 + A_2'\overline{V}_2 - \overline{V}_2B_2R^{-1}B_2'\overline{V}_2)z. \end{aligned} \tag{19}$$

In order to satisfy (19) identically for all z, we require that

$$0 = p + \overline{V}_{0x}a_1 + \overline{V}_1'a_2 - \frac{1}{4}(\overline{V}_{0x}B_1 + \overline{V}_1'B_2)R^{-1}(B_1'\overline{V}_{0x}' + B_2'\overline{V}_1), \quad \overline{V}_0(0) = 0 \tag{20}$$

$$0 = s' + 2a_2'\overline{V}_2 + \overline{V}_{0x}(A_1 - B_1R^{-1}B_2'\overline{V}_2) + \overline{V}_1'(A_2 - B_2R^{-1}B_2'\overline{V}_2) \tag{21}$$

$$0 = Q + \overline{V}_2A_2 + A_2'\overline{V}_2 - \overline{V}_2B_2R^{-1}B_2'\overline{V}_2. \tag{22}$$

At each fixed value of x, (22) is an algebraic Riccati equation for $\overline{V}_2$. In view of (3) and $Q(x) > 0$, the unique positive definite solution $\overline{V}_2$ exists such that for all $x \in D$, the real parts of the eigenvalues of $\overline{A}_2 = A_2 - B_2R^{-1}B_2'\overline{V}_2$, denoted by $Re\{\lambda(\overline{A}_2)\}$, are less than a negative constant. Thus $\overline{A}_2$ is nonsingular and $\overline{V}_1$ can be expressed in terms of $\overline{V}_{0x}$ and $\overline{V}_2$ as

$$\overline{V}_1' = -[s' + 2a_2'\overline{V}_2 + \overline{V}_{0x}(A_1 - B_1R^{-1}B_2'\overline{V}_2)]\overline{A}_2^{-1}. \tag{23}$$

It is of crucial importance that the elimination of $\overline{V}_1$ from (21) results in an equation involving only $\overline{V}_{0x}$. For the well posedness of the full problem it is necessary that the leading term $\overline{V}_0$ of (16) be identical to the solution L of the reduced problem.

Lemma 1

If Assumptions III and IV are satisfied, then the unique positive definite solution $\overline{V}_0(x)$ of (20)-(22) exists in D and is identical to the solution L(x) of the reduced problem (5),(6).

Proof: It is shown in the Appendix that eliminating $\overline{V}_1$ from (20), we obtain the Hamilton-Jacobi equation (11) with $\overline{V}_{0x}$ in place of L_x, and hence $\overline{V}_0(x) \equiv L(x)$ with properties as in Assumption IV.

By virtue of Lemma 1, $\overline{V}_0$ and $\overline{V}_2$ are solved independently from (11) and (22). This is the separation of time scales in the design of nonlinear regulators, analogous to the linear time-invariant design in [7].

Using $\overline{V}$, we derive the control

$$\begin{aligned}\overline{u} &= -\frac{1}{2}R^{-1}(B_1'\overline{V}_x' + \frac{1}{\mu}B_2'\overline{V}_z') \\ &= -\frac{1}{2}R^{-1}[B_1'\overline{V}_{0x}' + B_2'(\overline{V}_1 + 2\overline{V}_2 z)] + 0(\mu) \qquad (24) \\ &\equiv u_c + 0(\mu)\end{aligned}$$

whose main part u_c is defined as the composite control. Eliminating $\overline{V}_1$ from (24) using (23) and following the derivation in [7], u_c can be written as

$$\begin{aligned}u_c &= -R_o^{-1}(s_o + \frac{1}{2}B_o'\overline{V}_{0x}') - R^{-1}B_2'\overline{V}_2[z + A_2^{-1}(a_2 - B_oR_o^{-1}(s_o + \frac{1}{2}B_o'\overline{V}_{0x}'))] \\ &= u_o - R^{-1}B_2'\overline{V}_2(z + \overline{A}_2^{-1}\overline{a}_2) \qquad (25)\end{aligned}$$

where

$$\overline{A}_2(x) = A_2 - B_2R^{-1}B_2'\overline{V}_2 \qquad (26a)$$

$$\overline{a}_2(x) = a_2 - \frac{1}{2}B_2R^{-1}(B_1'\overline{V}_{0x}' + B_2'\overline{V}_1) \quad , \ \overline{a}_2(0) = 0. \qquad (26b)$$

Hence the composite control u_c consists of a slow control u_o which optimizes the reduced system (5) and a fast control $-R^{-1}B_2'\bar{V}_2(z+\bar{A}^{-1}\bar{a}_2)$ which optimizes the fast part $(z+\bar{A}^{-1}\bar{a}_2)$ of z in the sense that $\bar{V}_2$ satisfies (22). Note that when z is not penalized in (2), that is when $Q(x)=0$, but $\mathrm{Re}\{\lambda(A_2)\}<0$, then $\bar{V}_2$ is identically zero and u_c reduces to u_o of (10). Stabilizing properties of the composite control u_c are established in the next section.

IV. Stabilizing Properties

System (1) controlled by u_c is

$$\begin{aligned} \dot{x} &= a_1+A_1z+B_1u_c \equiv \bar{a}_1(x)+\bar{A}_1(x)z, \quad x(0)=x_o \\ \mu\dot{z} &= a_2+A_2z+B_2u_c \equiv \bar{a}_2(x)+\bar{A}_2(x)z, \quad z(0)=z_o \end{aligned} \tag{27}$$

where

$$\begin{aligned} \bar{a}_1 &= a_1-\frac{1}{2}B_1R^{-1}(B_1'\bar{V}_{0x}'+B_2'\bar{V}_1), \quad \bar{a}_1(0)=0 \\ \bar{A}_1 &= A_1-B_1R^{-1}B_2'\bar{V}_2. \end{aligned} \tag{28}$$

With the change of variables

$$\eta = z+\bar{A}_2^{-1}\bar{a}_2 \tag{29}$$

exhibiting η as the fast part of z, system (27) becomes

$$\dot{x} = \bar{a}_o+\bar{A}_1\eta \quad , \; x(0)=x_o \tag{30a}$$

$$\begin{aligned} \mu\dot{\eta} &= \mu(\bar{A}_2^{-1}\bar{a}_2)_x\bar{a}_o+[\bar{A}_2+\mu(\bar{A}_2^{-1}\bar{a}_2)_x\bar{A}_1]\eta \\ &\equiv \mu f(x)+[\bar{A}_2(x)+\mu F(x)]\eta \quad , \; \eta(0) = z_o+\bar{A}_2^{-1}(x_o)\bar{a}_2(x_o). \end{aligned} \tag{30b}$$

Since the right-hand side of (30b) is an $O(\mu)$ perturbation of $\bar{A}_2(x)\eta$ and $\mathrm{Re}\{\lambda(\bar{A}_2)\}<0$ in D we expect that η will rapidly decay to an $O(\mu)$ quantity. This motivates the introduction of

$$U(x,\eta,\varepsilon) = \bar{V}_0(x) + \varepsilon\eta'\bar{V}_2(x)\eta \tag{31}$$

as a tentative Lyapunov function for (30). Here ε is a small positive scalar to be determined. From Assumptions III and IV, $\bar{V}_0(x)$ is positive definite and $\bar{V}_2(x) > 0$ in D. Hence U is positive definite for all $x \epsilon D$ and $\eta \epsilon R^m$. Furthermore, since $\bar{V}_0(x) = c_o > 0$ for all x on the boundary of D, the surface

$$S(x,\eta,\varepsilon) = \{x,\eta : U(x,\eta,\varepsilon) = c_o\} \tag{32}$$

is closed in the (n+m)-dimensional domain $x \epsilon D$, $\eta \epsilon R^m$. We define S_{in} to be the domain in the interior of S.

Let D_1 be a set strictly in the interior of D, that is, the boundary of D_1 does not intersect the boundary of D, and let E be a bounded set in R^m. The presence of ε in U extends S to encompass (x,η) for all $x \epsilon D_1$ and for η in any prescribed set E. This crucial result is stated as follows.

Lemma 2

If Assumptions III and IV are satisfied, then there exists an $\varepsilon > 0$ such that the domain S_{in} contains all $x \epsilon D_1$, $\eta \epsilon E$.

Proof: At each point $\hat{x} \epsilon D_1$, the projection S onto the η subspace is the ellipsoid

$$\eta'\bar{V}_2(\hat{x})\eta = (c_o - \bar{V}_0(\hat{x}))/\varepsilon \tag{33}$$

implying that η extends to $O(1/\sqrt{\varepsilon})$. Hence for every $\hat{x}$, there exists an $\varepsilon(\hat{x})$ sufficiently small such that the ellipsoid (33) includes all $\eta \epsilon E$. (Note that we must exclude the boundary of D because from (33) the projection of S at any point on the boundary of D is a single point $\eta = 0$.) Hence choosing ε^* to be the smallest of such $\varepsilon(\hat{x})$, the domain S_{in} contains all $x \epsilon D_1$, $\eta \epsilon E$ for any $\varepsilon \epsilon (0,\varepsilon^*]$.

By virtue of Lemma 2, the initial condition $\eta(0)$ of (30b), and hence z(0) of (27), can be as far away from zero as $0(1/\sqrt{\varepsilon})$ and still be enclosed by S. We now examine the relationship between ε and μ.

Using (11),(22) and rearranging, we obtain the time derivative of U with respect to (30) as

$$\dot{U} = -g(x,\varepsilon,\mu) - \frac{\varepsilon}{2\mu}\xi' Q(x)\xi - \frac{\varepsilon}{\mu}\eta' M(x,\eta,\varepsilon,\mu)\eta \tag{34}$$

where

$$\begin{aligned} g &= g_1 - \frac{\mu}{2\varepsilon} y' Q^{-1} y \\ g_1 &= p_o - s_o' R_o^{-1} s_o + \frac{1}{4}\bar{V}_{0x} B_o R_o^{-1} B_o' \bar{V}_{0x}' \\ y &= \bar{A}_1' \bar{V}_{0x}' + 2\varepsilon \bar{V}_2 f \\ \xi &= \eta - \frac{\mu}{\varepsilon} Q^{-1} y \\ M &= \frac{Q}{2} + \bar{V}_2 B_2 R^{-1} B_2' \bar{V}_2 - \mu(\bar{V}_2 F + F'\bar{V}_2) - \mu\dot{\bar{V}}_2. \end{aligned} \tag{35}$$

Since $\bar{V}_2 F + F'\bar{V}_2$ and $\dot{\bar{V}}_2$ are bounded for all x,η in S_{in}, and since $Q(x) > 0$ in D, it follows that there exists a $\mu_1^* > 0$ such that $M > 0$ for all x,η in S_{in} and for $\mu \in (0,\mu_1^*]$. Thus the last two terms in $\dot{U}$ are positive definite. To ensure that $g(x,\varepsilon,\mu)$ is positive definite, we assume that the reduced problem also satisfies

V. The limit

$$\lim_{|x|\to 0} \frac{y'Q^{-1}y}{g_1} = k(\varepsilon) < \infty \tag{36}$$

exists for all fixed $\varepsilon > 0$.

Note that $k \geq 0$ because $y'Q^{-1}y$ is positive semidefinite and g_1 is positive definite. The limit (36) implies that there exists a domain $\tilde{D}$ about $x = 0$ such that

$$y'Q^{-1}y \leq (1+k)g_1 \tag{37}$$

that is such that for $\mu < 2\mathcal{E}/(1+k)$, g is positive definite in $\tilde{D}$, see (35). Let $\bar{k}(\mathcal{E}) > 0$ be the minimum value of g_1 on the boundary of $\tilde{D}$. Hence in the domain

$$\tilde{D}_1(x) = \{x : g_1(x) < \bar{k}\} \tag{38}$$

g is positive definite. On the other hand, since D is bounded, there exists a $k_1(\mathcal{E}) > 0$ such that $y'Q^{-1}y < k_1$ for all $x \in D$, that is such that g is positive definite when x is not in the domain

$$\bar{D}(x) = \{x : g_1(x) < \mu k_1/2\mathcal{E}\} \tag{39}$$

about the origin. But for $\mu < 2\mathcal{E}\bar{k}/k_1$, $\bar{D} \subset \tilde{D}_1$, implying that g is positive definite in D. Thus $\dot{U}$ is negative definite for all x, η contained in S_{in}. We now conclude that U is a Lyapunov function for (30) guaranteeing that $x = 0$, $\eta = 0$ is asymptotically stable for all $x \in D_1$, $\eta \in E$ and for $\mu \in (0, \mu^*]$, where

$$\mu^* = \min\left(\frac{2\mathcal{E}}{1+k}, \frac{2\mathcal{E}\bar{k}}{k_1}, \mu_1^*\right). \tag{40}$$

Returning from the η variable to the z variable via $z = \eta - \bar{A}_2^{-1}\bar{a}_2$, we obtain for all $x \in D_1$, $\eta \in E$ a corresponding bounded domain E_1 for z. We summarize the above discussions on the asymptotic stabilizing property of u_c in (24) as follows.

<u>Theorem 1</u>

If Assumptions I-V are satisfied, then there exists a $\mu^* > 0$ such that for all $\mu \in (0, \mu^*]$ and for all $x \in D_1$ and z in any prescribed bounded set E_1, the origin $x = 0$, $z = 0$ of the feedback system (1) controlled by the composite control u_c is asymptotically stable.

Theorem 1 can be applied in two different directions. As outlined above, for any given D_1 and E_1, we first find $\mathcal{E}^*$ such that S_{in} of (32) contains all $x \in D_1$, $z \in E_1$. Then we find μ^* from (40). This direction is suitable when μ is a parameter at the designer's disposal, such as a gain factor [9]. In the other direction, if μ represents some given physical parameters, such as time constants, we use its value to determine the smallest $\mathcal{E}$ such that $\dot{U}$ of (34) is negative definite, that is we find the largest D_1 and E_1.

As a special case of Assumption V, consider that the origin $x = 0$ of the reduced system (12) is exponentially stable. Then near the origin, $p_o - s_o' R_o^{-1} s_o$, $\overline{V}_0$ grow as $|x|^2$, and $|\overline{V}_{0x}|$, $|a_o|$ grow as $|x|$, and we can find positive constants $k_2, \ldots, k_9$ and δ such that

$$\begin{aligned} k_2|x|^2 &\leq p_o - s_o' R_o^{-1} s_o \leq k_3|x|^2 \\ k_4|x|^2 &\leq \overline{V}_0 \leq k_5|x|^2 \\ k_6|x| &\leq |\overline{V}_{0x}| \leq k_7|x| \\ k_8|x| &\leq |\overline{a}_o| \leq k_9|x| \end{aligned} \tag{41}$$

for all $|x| < \delta$. It follows from (41) that there exists a fixed $k_{10}(\mathcal{E}) > 0$ such that

$$y' Q^{-1} y \leq k_{10}|x|^2 \tag{42}$$

and the limit (36) is bounded by

$$\lim_{|x|\to 0} \frac{y' Q^{-1} y}{g_1} \leq \lim_{|x|\to 0} \frac{k_{10}|x|^2}{k_2|x|^2} = \frac{k_{10}}{k_2} \tag{43}$$

satisfying Assumption V.

In this case a claim stronger than Theorem 1 can be made.

Corollary 1

If Assumptions I-IV are satisfied and the origin $x=0$ of the reduced system is exponentially stable, then the conclusion of Theorem 1 holds and moreover the origin $x=0$, $z=0$ of (27) is exponentially stable.

Proof: The first part of the corollary follows from Theorem 1. The second part follows from the linearization of (27) at the origin

$$\begin{bmatrix} \dot{\delta x} \\ \dot{\delta z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{a}_1(0)}{\partial x} & \bar{A}_1(0) \\ \frac{1}{\mu}\frac{\partial \bar{a}_2(0)}{\partial x} & \frac{1}{\mu}\bar{A}_2(0) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta z \end{bmatrix} . \tag{44}$$

The system matrix of (44) has one group of n small eigenvalues $0(\mu)$ close to those of $\frac{\partial \bar{a}_1}{\partial x} - \bar{A}_1\bar{A}_2^{-1}\frac{\partial \bar{a}_2}{\partial x}\Big|_{x=0}$ and another group of m large eigenvalues $0(1)$ close to those of $\frac{1}{\mu}\bar{A}_2(0)$ [8]. But $\bar{a}_1 - \bar{A}_1\bar{A}_2^{-1}\bar{a}_2 = \bar{a}_o$ and $\frac{\partial \bar{a}_o}{\partial x}\Big|_{x=0} = \frac{\partial \bar{a}_1}{\partial x} - \bar{A}_1\bar{A}_2^{-1}\frac{\partial \bar{a}_2}{\partial x}\Big|_{x=0}$ as $\bar{a}_2(0) = 0$. Thus the real parts of the eigenvalues of the system matrix of (44) are all negative and $x=0$, $z=0$ is exponentially stable.

If the origin $x=0$ of the reduced system is only asymptotically stable but not exponentially stable, then in general g need not be positive definite for all $x \in D$. This situation includes the critical case when the linearized model does not provide any stability information as clarified by the example in Section VI. For this situation the system is now shown to possess a weaker stability property, that is, its trajectories tend to a small sphere around the origin. Define the domain in R^n

$$\rho(x) = \{x : g(x,\mathcal{E},\mu) \leq 0\} \tag{45}$$

which is contained in the domain $\bar{D}$ of (39). Due to the presence of μ in

(34), $\dot{U}$ may be positive only if $x \epsilon \rho(x)$ and $\eta = O(\mu)$. Otherwise, $\dot{U}$ is negative. Defining the surface

$$\pi(x,z) = \{x,z : x\epsilon\rho(x;\mu),\ z = -\bar{A}_2^{-1}(x)\bar{a}_2(x)\} \tag{46}$$

about the origin in R^{m+n}, u_c defined by (24) is a stabilizing control in the following sense.

Theorem 2

If Assumptions I-IV are satisfied, then there exists a $\mu^* > 0$ such that for all $\mu \epsilon (0,\mu^*]$, the feedback control (24) steers all $x \epsilon D_1$, $z \epsilon E_1$ of the full system $O(\mu)$ close to the surface $\pi(x,z)$.

Proof: Since $U > 0$ and $\dot{U} < 0$ except for $x\epsilon\rho(x)$ and $\eta = O(\mu)$, x converges to $\rho(x)$ and η decays to an $O(\mu)$ quantity. Thus in the x,z variables, (x,z) converges to an $O(\mu)$ neighborhood of the surface $\pi(x,z)$.

In the case where the fast transients of z in (1) are exponentially stable, that is, $A_2(x)$ is stable for all $x \epsilon D$, and we are only concerned with the optimality of the reduced system (5), then the z-independent reduced control u_o of (10) stabilizes the full system (1) with essentially the same stabilizing properties as u_c of (24). We shall not repeat the argument.

An attractive feature of the controls u_c and u_o is that they do not require the knowledge of the actual value of μ provided that it is sufficiently small. When appropriately implemented, these controls stabilize the full system (1) and achieve optimality of the reduced system, and in the case of u_c, also optimality of the fast part of z. The above results also answer the question of well posedness by giving the conditions under which the same optimal reduced order system is obtained when μ is set equal to zero either when system (1) is uncontrolled or when it is controlled by

u_c or u_o. In contrast to many other singular perturbation results which require μ to be sufficiently small, this section provides a method to compute an estimate of allowable values of μ given a stability domain or vice versa.

V. A Formal Expansion and Near-Optimality

The expansion (16) only satisfies the Hamilton-Jacobi equation (15) to $O(\mu)$ order. We now propose to solve (15) by expanding V formally as a nested infinite power series. If this power series is convergent, then the optimal solution V of (15) exists. For x,z near the origin, it has been shown in [1] that the optimal solution exists and possesses a power series expansion when system (1) after linearization at the origin is stabilizable and the state in the quadratic approximation of J is observable. Here we are interested in a power series of V which satisfies (15) to any order of μ.

Since system (1) is linear in z and J is quadratic in z, the optimal value function can be expanded as a power series in the components of z [2]. In addition, since z is the fast variable, the z terms in the optimal value function are multiplied by appropriate powers of μ [5]. In view of these two characteristics, we seek a solution of (15) in the form

$$\begin{aligned} V(x,z,\mu) = & V_0(x,\mu) + \mu \sum_{j=1}^{m} V_{1j}(x,\mu) z_j + \mu \sum_{j=1}^{m} \sum_{k=1}^{m} V_{2jk}(x,\mu) z_j z_k \\ & + \mu^2 \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{q=1}^{m} V_{2jkq}(x,\mu) z_j z_k z_q + \cdots \\ & + \mu^{i-1} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_i=1}^{m} V_{ij_1j_2\cdots j_i}(x,\mu) z_{j_1} z_{j_2} \cdots z_{j_i} + \cdots , \\ & V_0(0,\mu) = 0 \end{aligned} \tag{47}$$

where $V_{ij_1j_2\cdots j_i}$ is the $(j_1,j_2,\ldots,j_i)$ element of the completely symmetric generalized matrix† V_i of dimension m^i and z_j is the jth component of z. The summation signs in (47) and in other equations in the paper will be omitted when there is no confusion as to which indices $j_1,j_2,\ldots,j_i$ are being summed. The partial derivatives $V_x, V_{z_1},\ldots,V_{z_m}$ expressed in terms of the vector x and the scalars $z_1,\ldots,z_m$ are

$$V_x = V_{0x} + \mu V_{1jx}z_j + \mu V_{2jkx}z_jz_k + \cdots \tag{48a}$$

$$V_{z_i} = \mu V_{1i} + 2\mu V_{2ij}z_j + 3\mu^2 V_{3ijk}z_jz_k + \cdots,\quad i=1,2,\ldots,m \tag{48b}$$

where the summation signs over j,k are omitted.

For the series (47) to satisfy (15) as an identity, we first rewrite (15) in terms of the vector x and the scalars $z_1,\ldots,z_m$,

$$0 = p + s_iz_i + Q_{ij}z_iz_j + V_x(a_1 + A_{1i}z_i) + \frac{1}{\mu}V_{z_i}(a_{2i} + A_{2ij}z_j)$$
$$- \frac{1}{4}(V_xB_1 + \frac{1}{\mu}V_{z_i}B_{2i})R^{-1}(B_1'V_x' + \frac{1}{\mu}B_{2i}'V_{z_i}) \tag{49}$$

where s_i, a_{2i} are the ith components of the vectors s, a_2, respectively, A_{1i} is the ith column of the matrix A_1, B_{2i} is the ith row of B_2, Q_{2ij}, A_{2ij} are the (i,j) elements of Q, A_2, respectively, and the summation signs over the indices i,j are omitted. Then, upon substituting (48) into (49) and equating the coefficients of the like powers of z_i, we obtain

---

†The $(j_1,j_2,\ldots,j_i)$ elements of V_i are identical for all permutations of the indices $j_1,j_2,\ldots,j_i$ [6].

$$0 = p + V_{0x}a_1 + V_{1i}a_{2i} - \frac{1}{4}(V_{0x}B_1 + V_{1i}B_{2i})R^{-1}(B_1'V_{0x}' + B_{2i}'V_{1i}),$$

$$V_0(0,\mu) = 0 \qquad (50a)$$

$$0 = s_i + V_{0x}A_{1i} + \mu V_{1ix}a_1 + V_{1j}A_{2ji} + 2V_{2ij}a_{2j} - \frac{1}{2}(V_{0x}B_1$$

$$+ V_{1j}B_j)R^{-1}(\mu B_1'V_{1ix}' + 2B_{2j}'V_{2ji}), \quad i = 1,2,\ldots,m \qquad (50b)$$

$$0 = Q_{ij} + \mu V_{2ijx}a_1 + \mu(V_{1ix}A_{1j})_s + 2(V_{2ik}A_{2kj})_s + 3\mu V_{3ijk}a_{2k}$$

$$- \frac{1}{2}(V_{x}B_1 + V_{1k}B_{2k})R^{-1}(\mu B_1'V_{2ijx}' + 3\mu B_{2k}'V_{3kij})$$

$$- \frac{1}{4}(\mu V_{1ix}B_1 + 2V_{2ik}B_{2k})R^{-1}(\mu B_1'V_{1jx}' + 2B_{2k}'V_{2kj}),$$

$$i,j = 1,2,\ldots,m \qquad (50c)$$

$$0 = \mu^2 V_{3ijkx}a_1 + \mu(V_{2ijx}A_{1k})_s + 4\mu^2 V_{4ijkq}a_{2q} + 3\mu(V_{3ijq}A_{2qk})_s$$

$$- \frac{1}{2}(V_{0x}B_1 + V_{1q}B_{2q})R^{-1}(\mu^2 B_1'V_{3ijkx}' + 4\mu^2 B_{2q}'V_{4ijkq})$$

$$- \frac{1}{2}(\mu V_{1ix}B_1 + 2V_{2iq}B_{2q})R^{-1}(\mu B_1'V_{2jkx}' + 3\mu B_{2q}'V_{3qjk}))_s,$$

$$i,j,k = 1,2,\ldots,m \qquad (50d)^\dagger$$

$$\vdots$$

where the right hand sides of (50a),(50b),(50c),(50d),..., are the coefficients of the z-independent terms and of the z_i, z_iz_j, $z_iz_jz_k$,..., terms, respectively. Because of symmetry, there are m(m+1)/2 equations in (50c), m(m+1)(m+2)/6 equations in (50d) and in general, $(\prod_{k=0}^{i-1}(m+k))/i!$ equations when the coefficients of $z_{j_1}z_{j_2}\ldots z_{j_i}$, $j_1,j_2,\ldots,j_i = 1,2,\ldots,m$, are equated.

---

† The subscript s denotes the symmetrization operation of generalized matrices [6]. For example,

$$(V_{2ik}A_{2kj})_s = \frac{1}{2}(V_{2ik}A_{2kj} + V_{2jk}A_{2ki})$$

$$(V_{3ijq}A_{2qk})_s = \frac{1}{6}(V_{3ijq}A_{2qk} + V_{3jiq}A_{2qk} + V_{3ikq}A_{2qj} + V_{3kiq}A_{2qj} + V_{3jkq}A_{2qi} + V_{3kjq}A_{2qi}).$$

For a simplified treatment of these equations we now exploit the presence of the small singular perturbation parameter μ. We expand each coefficient of (47) as a power series in μ

$$V_i(x,\mu) = \sum_{j=0}^{\infty} \mu^j V_i^j(x) \quad , \quad i = 0,1,2,\ldots \tag{51}$$

where the boundary condition of V_0^j is $V_0^j(0) = 0$, $j = 0,1,2,\ldots$ The expressions (51) substituted into equations (50) are to satisfy them as identities in μ. Equating the coefficients of the like powers in μ, we generate sets of equations for V_i^j, $i,j = 0,1,2,\ldots$ The first set of equations obtained by equating the μ-independent parts in (50a), (50b), (50c), are precisely equations (20), (21), (22), respectively. Hence from the uniqueness of solutions to (20), (21), (22). We conclude that

$$V_0^0 = \overline{V}_0 = L, \quad V_1^0 = \overline{V}_1, \quad V_2^0 = \overline{V}_2 \tag{52}$$

and $\overline{V}$ thus consists of the leading terms of V.

The second set of equations in matrix form

$$0 = V_{0x}^1 \overline{a}_1 + V_1^{1'} \overline{a}_2 \ , \quad V_0^1(0) = 0 \tag{53a}$$

$$0 = V_{0x}^1 \overline{A}_1 + \overline{a}_1' V_{1x}^{0'} + V_1^{1'} \overline{A}_2 + 2\overline{a}_2' V_2^1 \tag{53b}$$

$$0 = V_{2x}^0 \overline{a}_1 + \frac{1}{2}(V_{1x}^0 \overline{A}_1 + \overline{A}_1' V_{1x}^{0'}) + V_2^1 \overline{A}_2 + \overline{A}_2' V_2^1 + 3(V_3^0 \overline{a}_2) \tag{53c}$$

$$0 = 3(V_3^0 \overline{A}_2)_s + (V_{2x}^0 \overline{A}_1)_s \tag{53d}$$

obtained by equating the μ terms in (50a), (50b), (50c), (50d), respectively, involve only the unknown terms V^1_{0x}, V^1_1, V^1_2 and V^0_3. In (53) the multiplication of an $n_1 \times n_2 \times n_3$ matrix by an $n_3 \times n_4$ matrix results in an $n_1 \times n_2 \times n_4$ matrix. For convenience we suppress the last dimension of the $m \times m \times 1$ matrices $(V^0_{2x}\bar{a}_1)$ and $(V^0_3\bar{a}_2)$ and regard them as $m \times m$ matrices. Since $\bar{A}_2$ is stable, (53d) and (53c) can be solved sequentially for V^0_3 and V^1_2, respectively. Then V^1_1 can be solved from (53b) and its substitution into (53a) results in the partial differential equation

$$0 = V^1_{0x}\bar{a}_o - (\bar{a}_1' V^{0'}_{1x} + 2\bar{a}_2' V^1_2)\bar{A}_2^{-1}\bar{a}_2 , \qquad V^1_0(0) = 0.$$

In general, in equating the μ^i terms we obtain the (i+1)st set of equations involving the unknown terms V^i_{0x}, V^i_1, V^i_2, $V^{i-1}_3, \ldots, V^0_{i+2}$. The terms $V^0_{i+1}, V^1_i, \ldots, V^{i-1}_2$ are solved for sequentially and then V^{i-1}_0 is to be solved from an equation similar to (41).

The main accomplishment of the nested expansions is that the first set of equations (20)-(22) can be solved independently for the first three zeroth order terms V^0_0, V^0_1, and V^0_2. Similarly, (53) and the subsequent sets of equations can be solved independently for $V^i_0, V^i_1, \ldots, V^0_{i+2}$. These equations are dependent only on x and not on z or μ. A further simplifying property is that at the first stage the equations (11),(22) for V^0_0 and V^0_2 are decoupled.

The approximation obtained by expanding V of (47), (51) to the ith set of equations is stated in the following theorem.

<u>Theorem 3</u>

Suppose that the solutions to the ith set of equations of V exist and let V^i be the truncated series of (47), (51) including all the terms

V_i^j up to the ith set. Then the control

$$u_i = -\frac{1}{2} R^{-1}(B_1' V_x^{i'} + \frac{1}{\mu} B_2' V_z^{i'}) \quad (55)$$

is near optimal in the sense that V^i satisfies the Hamilton-Jacobi equation (15) to an $O(\mu^i)$ error.

Proof: Substituting the V_i^j terms into (15) and using the first i set of equations of V, the coefficients of μ^k terms, $k < i$, in the resulting equation vanish, implying $O(\mu^i)$ near-optimality.

Thus Theorem 3 implies that u_c of (24) is an $O(\mu)$ near-optimal control because it is an $O(\mu)$ approximation of u_1 which achieves $O(\mu)$ near-optimality. In general, retaining only the μ^j terms, $k < i$, in u_i, the resulting control also is $O(\mu^i)$ near-optimal in the sense of Theorem 3.

Repeating the derivation in Section IV, we can show that u_i stabilizes the full system (1) with similar stabilizing properties as u_c of (24). We first introduce the x, $\eta = z + \bar{A}_2^{-1}\bar{a}_2$ variables and consider U in (31) as a tentative Lyapunov function. The analysis is more cumbersome but results similar to Theorems 1 and 2 and Corollary 1 can be established.

VI. Discussion and Example

The computational advantage of the proposed procedure is that all the terms of V in (47), (51) are obtained from equations involving the slow variable x only. Moreover V_0^0 and V_2^0 are solved for independently. Explicit consideration of the initial boundary layer is avoided and it is optimally stabilized by the z variable feedback. Furthermore using the x,η variables an estimate of the domain of stability is easily obtained. Alternatively, for a stability domain to encompass a prescribed bounded set $\eta \in E \subset R^m$ a bound for μ can be determined.

Several aspects of the design procedure and the stability properties of the resulting feedback system are now illustrated by considering the optimal control problem of the second order system

$$\begin{aligned} \dot{x} &= xz \\ \mu\dot{z} &= -z + u \end{aligned} \tag{56}$$

with respect to the performance index

$$J = \int_0^\infty (x^4 + \frac{1}{2} z^2 + \frac{1}{2} u^2)dt. \tag{57}$$

Solving the reduced problem we obtain $L = V_0^0 = x^2$ and $u_o = -x^2$. The optimally controlled reduced system (12) is $\dot{x} = -x^3$ and its unique asymptotically stable equilibrium is $x = 0$. Note that the linearization of the reduced system fails to provide any stability information at $x = 0$. Let D be the interval $[-1,1]$, that is, $L = c_o = 1$ at $x = \pm 1$ by Assumption IV.

The pair $(A_2, B_2) = (-1,1)$ satisfies (3) and we can solve (22) for $V_2^0 = \frac{1}{2}(\sqrt{2}-1)$ such that $\bar{A}_2 = -\sqrt{2}$. Then the substitution of $V_0^0 = L = x^2$ and V_2^0 into (23) yields the following expressions for (24) and (16)

$$u_c = -(\sqrt{2}x^2 + (\sqrt{2}-1)z) \tag{58}$$

$$\bar{V} = x^2 + \mu\sqrt{2}x^2 z + \mu \frac{1}{2}(\sqrt{2}-1)z^2. \tag{59}$$

The resulting feedback system is

$$\begin{aligned} \dot{x} &= xz \\ \mu\dot{z} &= -\sqrt{2}x^2 - \sqrt{2}z. \end{aligned} \tag{60}$$

This result is essentially nonlinear since the linearization of (60) at $x=0$, $z=0$ does not provide any stability information. Using the change of variables $\eta = z + x^2$, system (60) becomes

$$\begin{aligned} \dot{x} &= -x^2 + x\eta \\ \mu\dot{\eta} &= -2\mu x^4 - (\sqrt{2} - 2\mu x^2)\eta. \end{aligned} \tag{61}$$

Since we require $|x| \leq 1$, μ is restricted to be less than $1/\sqrt{2}$. The tentative Lyapunov function (31) is

$$U(x,\eta,\mathcal{E}) = x^2 + \frac{1}{2}(\sqrt{2}-1)\mathcal{E}\eta^2. \tag{62}$$

If we require that the initial conditions of (61) be in $|x| \leq .8$, $|\eta| \leq 5$, then we must set $\mathcal{E}$ to be less than .0695 in order for the ellipse

$$S(x,\eta,\mathcal{E}) = \{x,\eta : U = x^2 + \frac{1}{2}(\sqrt{2}-1)\mathcal{E}\eta^2 = 1\} \tag{63}$$

to enclose these initial conditions. Plots of S in the x,η coordinates and the x,z coordinates for $\mathcal{E} = .06$ are shown in Figure 1. The time derivative of U with respect to (61) is

$$\dot{U} = -(g_1 - \frac{\mu}{\mathcal{E}}y^2) - \frac{\mathcal{E}}{4\mu}\xi^2 - \frac{\mathcal{E}}{\mu}M\eta^2 \tag{64}$$

where

$$g_1 = 2x^4, \quad y = 2(1 - \mathcal{E}(\sqrt{2}-1)x^2)x^2 \tag{65}$$

$$\xi = \eta - \frac{2\mu}{\mathcal{E}}y, \quad M = \frac{7}{4} - \sqrt{2} - 2\mu(\sqrt{2}-1)x^2. \tag{65}$$

Since $\lim_{x\to 0} y^2/g_1 = 2$, Assumption V is satisfied. For all x,η in the interior of S and $\mathcal{E} = .06$, $\dot{U}$ is negative definite for all $\mu \in (0,.03]$. Hence $x=0$, $z=0$ is asymptotically stable for all $|x| \leq .8$, $|z+x^2| \leq 5$ and $\mu \in (0,.03]$. Furthermore, $\overline{V}$ satisfies the Hamilton-Jacobi equation (15) with an error of $\mu 2\sqrt{2}x^2z^2$.

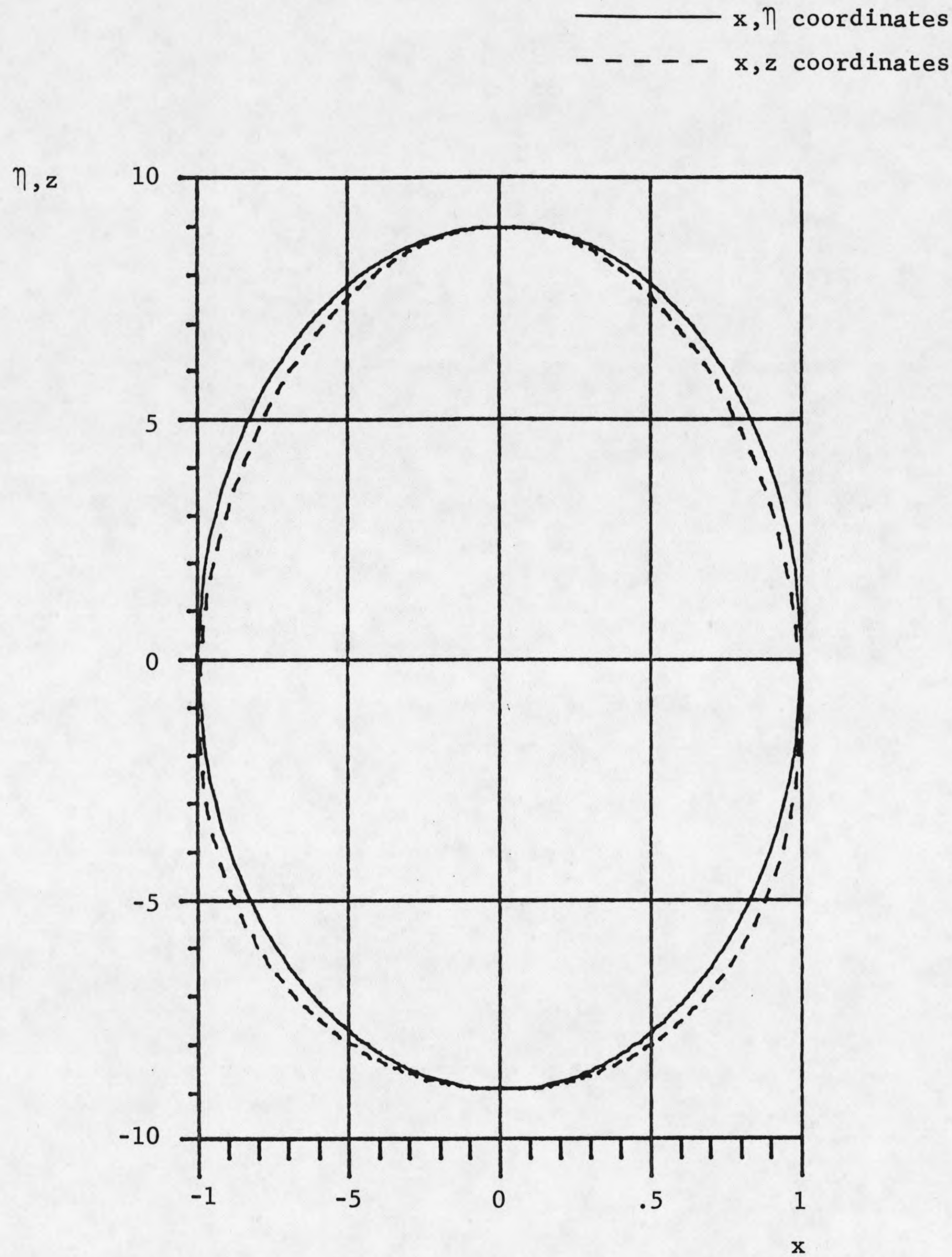

Figure 1. Plot of S in (63).

If we are only interested in the optimality of the reduced problem and consider the z-part as due to "system parasitics," we can apply the reduced control u_o to (56) as $A_2 = -1$ is stable. System (56) controlled by u_o is

$$\dot{x} = xz$$
$$\mu\dot{z} = -x^2 - z. \tag{66}$$

Transforming z to $\eta = z + x^2$, system (66) becomes

$$\dot{x} = -x^3 + x\eta$$
$$\mu\dot{z} = -2\mu x^2 - (1 - 2\mu x^2)\eta. \tag{67}$$

We use U in (62) as a Lyapunov function for (67) and the time derivative of U with respect to (67) is

$$\dot{U} = -[2 - \frac{\mu}{\varepsilon} 2(\sqrt{2}-1)(\sqrt{2}+1 - \varepsilon x^2)^2]x^4 - \frac{\varepsilon}{\mu}\frac{\sqrt{2}-1}{2}[\eta - \frac{\mu}{\varepsilon} 2(\sqrt{2}+1 - \varepsilon x^2)x^2]$$
$$- \frac{\varepsilon}{\mu}(\sqrt{2}-1)(\frac{1}{2} - 2\mu x^2)\eta^2. \tag{68}$$

Thus for all x,η enclosed in S and $\epsilon = .06$, $\dot{U}$ is negative definite for all $\mu \epsilon (0,.02]$. Hence $x = 0$, $z = 0$ of (66) is asymptotically stable for all $|x| \leq .8$, $|z+x^2| \leq 5$, $\mu \epsilon (0,.02]$.

To obtain an $0(\mu^2)$ approximation of V in the sense of Theorem 3, we solve (53) for higher order terms of V_i^j and obtain

$$u_2 = u_c - \mu 2x^2 z \tag{69}$$

$$V^2 = \overline{V} + \mu \frac{x^4}{\sqrt{2}} + \mu^2 x^2 z^2. \tag{70}$$

System (56) controlled by u_2 becomes

$$\dot{x} = xz$$
$$\mu\dot{z} = -\sqrt{2}\,x^2 - (\sqrt{2}+\mu 2x^2)z, \qquad (71)$$

or, in the $x, \eta = z + x^2$ variables,

$$\dot{x} = -x^3 + x\eta$$
$$\mu\dot{\eta} = -\sqrt{2}\,\eta \qquad (72)$$

which is globally asymptotically stable for all $\mu > 0$. Furthermore, V^2 satisfies (15) with an error of $\mu^2(8x^4z^2 + 2x^2z^3)$.

VII. Conclusions

A nested power series expansion method has been proposed for solving the optimal control problem of a class of nonlinear singularly perturbed systems. The terms in the expansion V are obtained from equations involving only the slow variable x. In addition, V_0^0 and V_2^0 are solved for independently. Explicit consideration of the initial boundary layer is avoided and it is optimized by the z variable feedback. Sufficient conditions are obtained such that feedback controls using truncated series stabilize the nonlinear systems and the stability domain can encompass large initial conditions of z. These truncated controls can achieve near-optimality of $O(\mu)$, $O(\mu^2)$, etc. In particular, an $O(\mu)$ near-optimal feedback control can be implemented without knowing the value of the small parameter μ. The results apply to essentially nonlinear problems.

Appendix

Substituting (23) into (20) and rearranging yields

$$0 = X_1 + V_{0x} X_2 - \frac{1}{4} V_{0x} X_3 V_{0x}'$$

where

$$X_1 = p - (s' + 2a_2' V_2)\bar{A}_2^{-1} a_2 - (\frac{1}{2} s' + a_2' V_2)\bar{A}_2^{-1} B_2 R^{-1} B_2' \bar{A}_2'^{-1} (\frac{1}{2} s + V_2 a_2)$$

$$X_2 = \tilde{a}_o + \tilde{B}_o R^{-1} B_2' \bar{A}_2'^{-1} (\frac{1}{2} s + V_2 a_2)$$

$$X_3 = \tilde{B}_o R^{-1} \tilde{B}_o'$$

$$\tilde{a}_o = a_1 - (A_1 - B_1 R^{-1} B_2' V_2)\bar{A}_2^{-1} a_2$$

$$\tilde{B}_o = B_1 - (A_1 - B_1 R^{-1} B_2' V_2)\bar{A}_2^{-1} B_2$$

$$\bar{A}_2 = A_2 - B_2 R^{-1} B_2 V_2$$

and the superscript 0 in V_{0x}^0 and V_2^0 has been dropped. Let $H = I + R^{-1} B_2' V_2 \bar{A}_2^{-1} B_2$. Then $H^{-1} = I - R^{-1} B_2' V_2 A_2^{-1} B_2$ and $H'^{-1} R H^{-1} = R + B_2' A_2'^{-1} Q A_2^{-1} B_2 = R_o$. Thus $\tilde{B}_o = B_1 H - A_1 \bar{A}_2^{-1} B_2 = B_o H$. Hence $X_3 = B_o R_o^{-1} B_o'$. Also,

$$X_2 = a_o + B_o R_o^{-1} [(R + B_2' A_2'^{-1} Q A_2^{-1} B_2) R^{-1} B_2' V_2 \bar{A}_2^{-1} + B_2' A_2'^{-1} V_2] a_2 + \frac{1}{2} B_o R_o^{-1} B_2' A_2'^{-1} s$$

$$= a_o + B_o R_o^{-1} B_2' A_2'^{-1} (A_2' V_2 + Q A_2^{-1} B_2 R^{-1} B_2' V_2 + V_2 A_2 - V_2 B_2 R^{-1} B_2' V_2)\bar{A}_2^{-1}$$

$$+ \frac{1}{2} B_o R_o^{-1} B_2' A_2'^{-1} s$$

$$= a_o - B_o R_o^{-1} s_o.$$

Furthermore, $\bar{A}_2^{-1} B_2 R^{-1} B_2' \bar{A}_2'^{-1} = A_2^{-1} B_2 H R^{-1} H' B_2' A_2'^{-1} = A_2^{-1} B_2 R_o^{-1} B_2' A_2'^{-1}$ and

$$\bar{A}_2^{-1} = A_2^{-1} + A_2^{-1}B_2R^{-1}B_2'V_2\bar{A}_2^{-1}$$

$$= A_2^{-1} + A_2^{-1}B_2R_o^{-1}B_2'(V_2 + A_2'^{-1}QA_2^{-1}B_2R^{-1}B_2'V_2)\bar{A}_2^{-1}$$

$$= A_2^{-1} - A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}QA_2^{-1} - A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}V_2.$$

Thus X_1 becomes

$$X_1 = p - s'A_2^{-1}a_2 + s'A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}QA_2^{-1} - \frac{1}{4}s'A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}s$$

$$+ a_2'V_2A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}V_2a_2 - a_2'(V_2\bar{A}_2^{-1} + \bar{A}_2'^{-1}V_2)a_2.$$

But

$$V_2\bar{A}_2^{-1} + \bar{A}_2'^{-1}V_2 = -V_2A_2^{-1} - A_2'^{-1}V_2 + V_2A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}QA_2^{-1} + A_2'^{-1}QA_2^{-1}B_2R_o^{-1}B_2'A_2^{-1}V_2$$

$$+ 2V_2A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}V_2$$

$$= A_2'^{-1}QA_2^{-1} - A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1} + (V_2 + A_2'^{-1}Q)A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}(V_2 + QA_2^{-1})$$

$$+ V_2A_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}V_2 - A_2'^{-1}QA_2^{-1}B_2R_o^{-1}B_2'A_2'^{-1}QA_2^{-1},$$

and

$$A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1} = [-(V_2 + A_2'^{-1}Q_2)A_2^{-1} + A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1}]B_2R^{-1}B_2'V_2A_2^{-1},$$

that is,

$$A_2'^{-1}V_2B_2R^{-1}B_2'V_2A_2^{-1} = -(V_2 + A_2'^{-1}Q_2)A_2^{-1}B_2R^{-1}B_2'V_2\bar{A}_2^{-1}$$

$$= (V_2 + A_2'^{-1}Q)A_2^{-1}B_2R^{-1}B_2'A_2'^{-1}(QA_2^{-1} + V_2),$$

implying $X_1 = p_o - s_o'R_o^{-1}s_o$. Hence elimination of V_1 from (20) yields the Hamilton-Jacobi equation (11) of the reduced problem.

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