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Optimal Control of Systems with Delayed Observation Sharing Patterns

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In this paper we present an input-output point of view of certain optimal control problems with constraints on the processing of the measurement data. In particular, considering linear controllers and plant dynamics, we present solutions to the ℓ^1, $\mathcal{H}^\infty$ and $\mathcal{H}^2$ optimal control problems under the so-called one-step delay observation sharing pattern. Extensions to other decentralized structures are also possible under certain conditions on the plant. The main message from this unified input-output approach is that, linear structural constraints on the controller appear as linear constraints of the same type on the Youla parameter that parametrizes all controllers, as long as the part of the plant that relates controls to measurements possesses the same off-diagonal structure required in the controller. Under this condition, ℓ^1, $\mathcal{H}^\infty$ and $\mathcal{H}^2$ optimization transform to nonstandard, yet convex problems. Their solution can be obtained by suitably utilizing the Duality, Nehari and Projection theorems respectively.

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Abstract

In this paper we present an input-output point of view of certain optimal control problems with constraints on the processing of the measurement data. In particular, considering linear controllers and plant dynamics, we present solutions to the ℓ^1, $\mathcal{H}^\infty$ and $\mathcal{H}^2$ optimal control problems under the so-called one-step delay observation sharing pattern. Extensions to other decentralized structures are also possible under certain conditions on the plant. The main message from this unified input-output approach is that, linear structural constraints on the controller appear as linear constraints of the same type on the Youla parameter that parametrizes all controllers, as long as the part of the plant that relates controls to measurements possesses the same off-diagonal structure required in the controller. Under this condition, ℓ^1, $\mathcal{H}^\infty$ and $\mathcal{H}^2$ optimization transform to nonstandard, yet convex problems. Their solution can be obtained by suitably utilizing the Duality, Nehari and Projection theorems respectively.

1 Introduction

Optimal control under decentralized information structures is a topic that, although it has been studied extensively over the last forty years or so, still remains a challenge to the control community. The early encounters with the problem date back in the fifties and early sixties under the framework of team theory (e.g., [16,18].) Soon it was realized that, in general, optimal decision making is very difficult to obtain when decision makers have access to private information, but do not exchange their information [35]. Nonetheless, under particular decentralized information schemes such as the partially nested information structures [12] certain optimal control problems admit trackable solutions. Several results exist by now when exchange of information is allowed with a one-step time delay (which is a special case of the partially nested information structure.) To mention only a few we refer to [2,4,21] where LQG criteria are of interest, [3,24,25,26,27,28,29] where linear exponential-quadratic Gaussian (LEQG) problems are considered and certain connections to minimax quadratic problems are furnished. The interested reader may further refer to [1] which provides a very good reference guide on the topic.

In this paper, in contrast to the state-space view-point of the works previously referenced, we undertake an input-output approach to optimal control under the quasiclassical information scheme known as the one-step delay observation sharing pattern (e.g., [2]). Under this pattern measurement information can be exchanged between the decision makers with a delay of one time step. In the paper we define and present solutions to three optimal control problems: ℓ^1, $\mathcal{H}^\infty$ and $\mathcal{H}^2$ (or LQG) optimal disturbance rejection. The key ingredient in this approach is the transformation of the (linear) decentralization constraints on the controller to *linear* constraints on the Youla parameter used to characterize all controllers. Hence, the resulting problems in the input-output setting are, although nonstandard, convex. These problems resemble the ones appearing in optimal control of periodic systems when lifting techniques are employed [8,32], and can be solved by suitably utilizing the Duality, Nehari and Projection theorems respectively. Other structured control problems can also be dealt similarly *provided that the part of the plant that relates controls to measurements possesses the same off-diagonal structure required in the controller.* This condition is crucial in transforming linearly structural constraints. If it is satisfied, problems with n-step delay observation sharing patterns where $n > 1$, or with fully decentralized operation can be solved in a similar fashion.

The paper is organized as follows: section 2 provides background on input-output characterizations and on certain key theorems; section 3 defines precisely the problems of interest; section 4 provides their solution; section 5 is devoted to several concluding comments and discussions.

2 Mathematical Preliminaries

This section presents the notation and definitions to be used throughout the paper. Also, some important to our development mathematical results are provided. References are given to cover all of the needed mathematical background.

2.1 Generic Notation

In this subsection we give some generic notation that is used throughout the thesis.

$\rho(A)$ The spectral radius of the matrix A.

$\overline{\sigma}[A]$ The maximum singular value of the matrix A.

$|x|_p$ The p-norm of the finite dimensional vector $x = (x_1\ x_2\ \dots\ x_n)^T$ given as

$$|x|_p = (\sum_{i=1}^{n} |x_i|)^{1/p},\ p < \infty$$

$$|x|_p = \max_i |x_i|,\ p = \infty.$$

$|A|_1$ The 1-norm of the $m \times n$ matrix $A = (A_{ij})$ given as

$$|A|_1 = \max_{i=1,\dots,m} \sum_{j=0}^{n} |A_{ij}|.$$

$\hat{H}(\lambda)$ The λ-transform of a $m \times n$ real sequence $H = \{H(k)\}_{k=-\infty}^{\infty}$ defined as:

$$\hat{H}(\lambda) = \sum_{k=-\infty}^{\infty} H(k)\lambda^k.$$

X^* The dual space of the normed linear space X.

BX	The closed unit ball of X.
${}^{\perp}S$	The left annihilator of $S \subset X^*$.
$S^{\perp}$	The right annihilator of $S \subset X$.
Π_S	The projection operator onto the subset S of the Hilbert space X.
$\langle x, x^* \rangle$	The value of the bounded linear functional x^* at point $x \in X$.
T^*	The adjoint of the operator T.
Π_m^k	The k^{th}-truncation operator acting on a $m \times 1$ vector valued sequence $\{u(k)\}_{k=0}^{\infty}$ as $$\Pi_m^k : \{u(0), u(1), \ldots\} \longrightarrow \{u(0), \ldots, u(k), 0, 0, \ldots\}.$$
Λ_m	The right shift operator acting on a $m \times 1$ vector valued sequence $\{u(k)\}_{k=0}^{\infty}$ as $$\Lambda_m : \{u(0), u(1), \ldots\} \longrightarrow \{0, u(0), u(1) \ldots\}.$$

2.2 Some Basic Spaces

In this subsection we define certain important normed linear spaces that we very frequently refer to in the course of our development. These spaces are the following (for details look at [20,15,14,34]):

$\ell^1_{m\times n}$: The Banach space of all $m \times n$ matrices H each of whose entries is a right sided, absolutely summable real sequence $H_{ij} = \{H_{ij}(k)\}_{k=0}^{\infty}$. The norm is defined as:

$$\|H\|_{\ell^1_{m\times n}} := \max_i \sum_{j=1}^{n} \sum_{k=0}^{\infty} |H_{ij}(k)|.$$

ℓ^{∞}_m : The Banach space of real $m \times 1$ vectors u each of whose components is a magnitude bounded real sequence $\{u_i(k)\}_{k=0}^{\infty}$. The norm is defined as:

$$\|u\|_{\ell^{\infty}_m} := \max_i (\sup_k |u_i(k)|).$$

$\ell^{\infty,e}_m$: The extended ℓ^{∞}_m space: it is the space of all real right sided $m \times 1$ vector valued sequences.

ℓ^2_m : The Hilbert space of real $m \times 1$ vectors u each of whose components is an energy bounded real sequence $\{u_i(k)\}_{k=0}^{\infty}$. The norm is defined as:

$$\|u\|_{\ell^2_m} := (\sum_{i=1}^{m} \sum_{k=0}^{\infty} |u_i(k)|^2)^{1/2}.$$

$\mathcal{A}_{m\times n}$: The real Banach space of all $m \times n$ matrices $\hat{H}(\lambda)$ such that $\hat{H}(\lambda)$ is the λ-transform of an $\ell^1_{m\times n}$ sequence H. The norm is defined as

$$\left\|\hat{H}(\lambda)\right\|_{\mathcal{A}_{m\times n}} := \|H\|_{\ell^1_{m\times n}}.$$

This space is isometrically isomorphic to $\ell^1_{m\times n}$ i.e., $\ell^1_{m\times n} \simeq \mathcal{A}_{m\times n}$.

$\mathcal{A}^*_{m\times n}$: The Banach space of all $m \times n$ matrices H each of whose entries is a right sided, magnitude bounded real sequence $H_{ij} = \{H_{ij}(k)\}_{k=0}^{\infty}$. The norm is defined as:

$$\|H\|_{\mathcal{A}^*_{m\times n}} := \sum_{i=1}^{m} \max_j(\sup_k |H_{ij}(k)|).$$

$c^0_{m\times n}$: The subspace of $\mathcal{A}^*_{m\times n}$ consisting of all elements which converge to zero.

$\mathcal{L}^\infty_{m\times n}$: The Banach space of all $m \times n$ matrix valued functions F defined on the unit circle of the complex plain with

$$\|F\|_{\mathcal{L}^\infty} := \text{ess} \sup_{\theta\in[0,2\pi]} \overline{\sigma}[F(e^{j\theta})] < \infty.$$

$\mathcal{H}^\infty_{m\times n}$: The Banach space of all $m \times n$ matrix valued functions F analytic in the open unit disk of the complex plain with

$$\|F\|_{\mathcal{H}^\infty} := \sup_{r\in[0,1)} \max_{\theta\in[0,2\pi]} \overline{\sigma}[F(re^{j\theta})] < \infty.$$

This space can be considered as a closed subspace of $\mathcal{L}^\infty_{m\times n}$.

$\mathcal{L}^2_{m\times n}$: The Hilbert space of matrix valued functions F defined on the unit circle of the complex plain with

$$\|F\|_{\mathcal{L}^2} := [(2\pi)^{-1} \int_0^{2\pi} trace(F^T(e^{-j\theta})F(e^{j\theta}))d\theta]^{1/2} < \infty.$$

$\mathcal{H}^2_{m\times n}$: The Hilbert space of all $m \times n$ matrix valued functions F analytic in the open unit disk of the complex plane with

$$\|F\|_{\mathcal{H}^2} := \sup_{r\in[0,1)} [(2\pi)^{-1} \int_0^{2\pi} trace(F^T(re^{-j\theta})F(re^{j\theta}))d\theta]^{1/2} < \infty.$$

This space can be considered as a closed subspace of $\mathcal{L}^2_{m\times n}$. Moreover, $\ell^2_m \simeq \mathcal{H}^2_{m\times 1}$.

2.3 Input-Output Characterization of Linear Systems

In this subsection we consider the input-output characterization of systems by viewing them as linear operators.

2.3.1 Causality

We start with the notion of a causal operator

Definition 2.1 *Let $T : \ell_n^{\infty,e} \longrightarrow \ell_m^{\infty,e}$ be an operator. T is called causal if*

$$\Pi_m^k T u = \Pi_m^k T \Pi_n^k u, \quad \forall k = 0, 1, 2, \ldots,$$

T is called strictly causal if

$$\Pi_m^k T u = \Pi_m^k T \Pi_n^{k-1} u, \quad \forall k = 0, 1, 2, \ldots .$$

The class of all causal operators T on $\ell^{\infty,e}$ will be denoted by $\mathcal{L}^{m\times n}$. Such operators can be represented by infinite block triangular matrices of the form

$$\begin{pmatrix} T_0(0) & 0 & \cdots \\ T_1(1) & T_1(0) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $T_k(i)$ are $m \times n$ matrices for all i, k. This representation is another way to state that these operators are convolution operators; i.e., if $y = Tu$ then

$$y(k) = \sum_{i=0}^{k} T_k(k-i)u(i).$$

2.3.2 Stability

Next, we consider the notion of ℓ^α-stability where $\alpha = \infty, 2$.

Definition 2.2 *Let T be a causal operator in $\mathcal{L}^{m\times n}$. Then T is ℓ^α-stable if its induced norm over ℓ^α is bounded; i.e., if*

$$\|T\| = \sup_{u\in\ell_n^\alpha,\, u\neq 0} \frac{\|Tu\|_{\ell_m^\alpha}}{\|u\|_{\ell_n^\alpha}} < \infty.$$

The class of all ℓ^α-stable systems equipped with the induced norm, will be denoted by $\mathcal{B}_{TV}^{m\times n}(\ell^\alpha)$. This class is a Banach space and in particular it is a Banach algebra with multiplication defined as composition. In the case where $\alpha = \infty$ we refer to the space $\mathcal{B}_{TV}^{m\times n}(\ell^\infty)$ as the space of Bounded-Input-Bounded-Output (BIBO) stable, or simply, stable systems. Moreover, the following fact can be easily checked.

Fact 2.1 *The space $\mathcal{B}_{TV}^{m\times n}(\ell^\infty)$ can be represented as the space of all infinite $m \times n$ block lower triangular matrices of the form*

$$\begin{pmatrix} T_0(0) & 0 & \cdots \\ T_1(1) & T_1(0) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $T_k(i)$ are $m \times n$ matrices for all i, k such that

$$\sup_k |(T_k(k)\, T_k(k-1)\, \ldots\, T_k(0))|_1 < \infty.$$

In the case when a system T is given in terms of a finite dimensional state-space description of the form

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t),$$

$t = 0, 1, 2, \ldots$ with $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ being time varying matrices then

Definition 2.3 *The system T is exponentially stable if there are constants $c_1, c_2 > 0$ such that for all t_0 and $x(t_0)$ the following holds*

$$|x(t)|_2 \leq c_1 |x(t_0)|_2 e^{c_2(t-t_0)} \quad t = t_0, t_0+1, \ldots.$$

We also define the notion of stabilizability and detectability as follows

Definition 2.4 *The pair $(A(\cdot), B(\cdot))$ is called stabilizable if there exists a bounded matrix function $K(\cdot)$ such that the system $x(t+1) = (A(t) - B(t)K(t))x(t)$ is exponentially stable. Similarly, the pair $(A(\cdot), C(\cdot))$ is detectable if there is a bounded matrix function $L(\cdot)$ such that the system $x(t+1) = (A(t) - L(t)C(t))x(t)$ is exponentially stable.*

A finite dimensional system with $(A(\cdot), B(\cdot))$ stabilizable and $(A(\cdot), C(\cdot))$ detectable is called stabilizable.

2.3.3 Time (Shift) Invariance

An important subclass of the general time varying systems of the class $\mathcal{L}^{m \times n}$ is the class of time (shift) invariant systems. If $T : \ell_n^{\infty,e} \longrightarrow \ell_m^{\infty,e}$ then

Definition 2.5 *T is time invariant if it commutes with the shift operator; i.e.,*

$$\Lambda_m T = T \Lambda_n.$$

The space of time invariant operators that are also ℓ^α-stable is denoted by $\mathcal{B}_{TI}^{m \times n}(\ell^\alpha)$ where $\alpha = \infty, 2$.

2.4 Notational Convention

To avoid proliferation of notation we will often drop the m and n in the notation given in the previous subsections when the dimension is not important or when it is clear from the context. Also, subscripts on the norms are dropped when there is no ambiguity.

2.5 Fundamental Connections

In this subsection we provide the connections between the spaces of section 2.2 and $\mathcal{B}_{TI}(\ell^\alpha)$ where $\alpha = \infty, 2$. We start with BIBO time invariant operators. The following fact can be easily checked:

Fact 2.2 *Every element of $\ell^1_{m\times n}$ ($\mathcal{A}_{m\times n}$) defines an operator in $\mathcal{B}^{m\times n}_{TI}(\ell^\infty)$ via convolution (multiplication in the λ-domain) and vice versa. Moreover, the spaces $\ell^1_{m\times n}$, $\mathcal{A}_{m\times n}$, $\mathcal{B}^{m\times n}_{TI}(\ell^\infty)$ are isometrically isomorphic; i.e.,*

$$\ell^1_{m\times n} \simeq \mathcal{A}_{m\times n} \simeq \mathcal{B}^{m\times n}_{TI}(\ell^\infty).$$

The above fact means that a ℓ^∞-stable operator T is associated with a sequence $\{T(i)\}_{i=0}^\infty \in \ell^1$ which is the impulse response. This can be easily seen from the Toeplitz representation of T:

$$T = \begin{pmatrix} T(0) & 0 & \cdots \\ T(1) & T(0) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

The induced operator norm over ℓ^∞ is exactly the ℓ^1-norm of $\{T(i)\}_{i=0}^\infty$. The sequence $\{T(i)\}_{i=0}^\infty$ convolves with the input sequence $\{u(k)\}_{k=0}^\infty \in \ell^\infty$ to produce the output sequence of T. In the space $\mathcal{A}$ this translates to multiplication of the corresponding λ-transforms. Finally, the isomorphy establishes that composition (of operators) in $\mathcal{B}_{TI}(\ell^\infty)$ translates to convolution in ℓ^1 and, of course, multiplication in $\mathcal{A}$.

Next we encounter the ℓ^2-stable operators. First, we have the following [11] concerning the linear bounded operators on $\mathcal{L}^2$:

Fact 2.3 *An element $R \in \mathcal{L}^\infty_{m\times n}$ defines an operator from $\mathcal{L}^2_{n\times 1}$ to $\mathcal{L}^2_{m\times 1}$ via multiplication. Moreover, any operator from $\mathcal{L}^2_{n\times 1}$ to $\mathcal{L}^2_{m\times 1}$ can be represented with some $R \in \mathcal{L}^\infty_{m\times n}$; the induced operator norm is exactly $\|R\|_{\mathcal{L}^\infty_{m\times n}}$.*

The bounded operators on the subspace $\mathcal{H}^2$ of $\mathcal{L}^2$ are characterized by the next fact [11]:

Fact 2.4 *An element $X \in \mathcal{H}^\infty_{m\times n}$ defines an operator from $\mathcal{H}^2_{n\times 1}$ to $\mathcal{H}^2_{m\times 1}$ via multiplication and vice versa. Moreover, any operator from $\mathcal{H}^2_{n\times 1}$ to $\mathcal{H}^2_{m\times 1}$ can be represented with some $X \in \mathcal{H}^\infty_{m\times n}$; the induced operator norm is exactly $\|X\|_{\mathcal{H}^\infty_{m\times n}}$.*

Having in mind that $\ell^2_m \simeq \mathcal{H}^2_{m\times 1}$ it is not hard to establish that

Fact 2.5 *An element in $\mathcal{H}^\infty_{m\times n}$ defines an operator in $\mathcal{B}^{m\times n}_{TI}(\ell^2)$ via multiplication in the λ-domain and vice versa. Moreover, any operator in $\mathcal{B}^{m\times n}_{TI}(\ell^2)$ can be represented with an element in $\mathcal{H}^\infty_{m\times n}$ and also the spaces $\mathcal{H}^\infty_{m\times n}$, $\mathcal{B}^{m\times n}_{TI}(\ell^2)$ are isometrically isomorphic; i.e.,*

$$\mathcal{H}^\infty_{m\times n} \simeq \mathcal{B}^{m\times n}_{TI}(\ell^2).$$

Again, the previous fact means that any function $\hat{T}(\lambda) \in \mathcal{H}^\infty$ defines an operator T in $\mathcal{B}_{TI}(\ell^2)$ by multiplication and vice versa. The induced operator norm over ℓ^2 is exactly $\left\|\hat{T}(\lambda)\right\|_{\mathcal{H}^\infty}$. The input sequence $\{u(k)\}_{k=0}^\infty \in \ell^2$ transforms to a function $\hat{u}(\lambda) \in \mathcal{H}^2$ that multiplies $\hat{T}(\lambda)$ to produce the transform $\widehat{Tu}(\lambda) \in \mathcal{H}^2$ of the output sequence; then, by inverse transform we obtain the output $Tu \in \ell^2$.

2.6 Important Theorems

Here, we present without proofs some theorems from mathematical analysis [20,15,14] which play a central role in our development.

2.6.1 Duality

We start with the two Duality theorems which we use in section 4.

Let X be some normed linear space and let X^* denote its dual. That is, X^* is the space of all bounded linear functionals r on X. X^* is a normed linear space equipped with the induced norm; i.e., if $r \in X^*$ then

$$\|r\| = \sup_{x \in BX} |\langle x, r\rangle|$$

where $\langle x, r\rangle$ means the value of the functional r at x. The right annihilator (or orthogonal complement) space of a subspace S of X is defined as

$$S^\perp = \{r \in X^* : \langle x, r\rangle = 0 \quad \forall x \in S\}.$$

Suppose that Q is a subspace in the dual space X^* of X. Then the left annihilator of Q is defined as

$$^\perp Q = \{x \in X : \langle x, q\rangle = 0, \quad \forall q \in Q\}.$$

Finally, we say that $r \in X^*$, $x \in X$ are aligned if $\langle x, r\rangle = \|x\|\,\|r\|$. We now state the two main theorems:

Theorem 2.1 *Let x be an element in a real normed linear space X and let μ denote its distance from the subspace S. Then*

$$\mu = \inf_{k \in S} \|x - k\| = \max_{r \in S^\perp, \|r\| \le 1} \langle x, r\rangle$$

where the maximum is achieved for some r_o in $S^\perp$. Moreover, if the infimum on the left is achieved for some $k_o \in S$, then r_o is aligned with $x - k_o$.

Theorem 2.2 *Let S be a subspace of a real normed linear space X. Let $x^* \in X^*$ be a distance μ from $S^\perp$. Then*

$$\mu = \min_{r^* \in S^\perp} \|x^* - r^*\| = \sup_{x \in BS} \langle x, x^*\rangle$$

where the minimum on the left is achieved for some $r_o^* \in S^\perp$. Moreover, if the supremum on the right is achieved for some $x_o \in BS$ then $x^* - r_o^*, x_o$ are aligned.

In our development we use the second theorem although the first can be used equally well.

We also present a basic fact that shows the relations between the spaces $\mathcal{A}^*_{m\times n}$, $\ell^1_{m\times n}$ and $c^0_{m\times n}$.

Fact 2.6 *Every linear functional on $\ell^1_{m\times n}$ is representable uniquely in the form*

$$f(H) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{\infty} Y_{ij}(k) H_{ij}(k)$$

where $Y = (Y_{ij}) \in \mathcal{A}^*_{m\times n}$ and $H = (H_{ij}) \in \ell^1_{m\times n}$. Moreover the converse holds; hence, $(\ell^1_{m\times n})^* = \mathcal{A}^*_{m\times n}$. It can be also shown that $(c^0_{m\times n})^* = \ell^1_{m\times n}$ where the linear functionals are defined as above.

With this fact in mind and the Duality theorems we can turn distance problems in ℓ^1 to distance problems in $\mathcal{A}^*$ or in c^0.

2.6.2 Projection Theorem

As a corollary to the Duality theorems we present the Projection theorem when the space X happens to be a Hilbert space. In particular, we have:

Theorem 2.3 *Let X be a Hilbert space and M a closed subspace of X. Corresponding to any $x \in X$ there is a unique $m_o \in M$ such that $\|x - m_o\| \leq \|x - m\|$ for all $m \in M$. Furthermore a necessary and sufficient condition that m_o be the unique minimizer is that $x - m_o \in M^{\perp}$.*

This theorem is used in section 4 to solve a minimization problem in $\mathcal{L}^2$.

2.6.3 Nehari's Theorem

Nehari's theorem solves an important distance problem in the space $\mathcal{L}^{\infty}$. This theorem will be used in section 4 to solve a $\mathcal{H}^{\infty}$ problem. First, we define the Hankel operator associated with a function $R \in \mathcal{L}^{\infty}$ (e.g., [11]).

Definition 2.6 *Let $R \in \mathcal{L}^{\infty}$ and $u \in \mathcal{H}^2$. Then the Hankel operator with symbol R is the map $\Gamma_R : \mathcal{H}^2 \to (\mathcal{H}^2)^{\perp}$ with*

$$\Gamma_R u = \Pi_{(\mathcal{H}^2)^{\perp}} R u$$

where Ru is the function $R(\lambda)u(\lambda) \in \mathcal{L}^2$.

The induced norm $\|\Gamma_R\|$ of this operator measures the distance from an element R in $\mathcal{L}^{\infty}$ and the space $\mathcal{H}^{\infty}$.

Theorem 2.4 *Let $R \in \mathcal{L}^{\infty}$. Then*

$$\inf_{X \in \mathcal{H}^{\infty}} \|R - X\|_{\mathcal{L}^{\infty}} = \|\Gamma_R\|.$$

Moreover, there exists a $X_o \in \mathcal{H}^{\infty}$ that achieves the infimum.

3 Problem Definition

The standard block diagram for the disturbance rejection problem is depicted in Figure 1. In this figure, P denotes some fixed linear causal plant, C denotes the compensator, and the signals w, z, y, and u are defined as follows: w, exogenous disturbance; z, signals to be regulated; y, measured plant output; and u, control inputs to the plant. P can be thought as a four block matrix each block being a linear causal system. In what follows we will assume that both P and C are LTI systems; we comment on this restriction on C later in section 5. Furthermore, we assume that there is a predefined information structure that the controller C has to respect when operating on the measurement signal y. The particular information structure is precisely defined in the sequel.

3.1 The one-step delay observation sharing pattern

To simplify our analysis we will consider the case where the control input u and plant output y are partitioned into two (possibly vector) components u_1, u_2 and y_1, y_2 respectively, i.e., $u = (u_1\ u_2)^T$ and $y = (y_1\ y_2)^T$. Let $Y_k := \{y_1(0), y_2(0), \ldots, y_1(k), y_2(k)\}$ represent the measurement set at time

k. The controllers that we are considering (henceforth, admissible controllers) are such that $u_1(k)$ is a function of the data $\{Y_{k-1}, y_1(k)\}$ and $u_2(k)$ is a function of the data $\{Y_{k-1}, y_2(k)\}$. We refer to this particular information processing structure imposed on the controller as the *one-step delay observation sharing pattern.* Alternatively, partitioning the controller C accordingly as

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix},$$

we require that both C_{12} and C_{21} be strictly causal operators. Let now

$$\mathcal{S} := \{C \text{ stabilizing and LTI} : \; C_{12}, \; C_{21} \text{ strictly causal}\}$$

and let T_{zw} represent the resulting map from w to z for a given compensator $C \in \mathcal{S}$. The problems of interest are as follows.

The first two problems are deterministic: w is assumed to be any ℓ^α disturbance with $\alpha = \infty, 2$ and we are interested in minimizing the worst case ℓ^α norm of z. Namely, our objective can be stated as
(OBJ_α): Find C such that the resulting closed loop system is stable and also the induced norm $\|T_{zw}\|$ over ℓ^α for $\alpha = \infty, 2$ is minimized.

The third problem we want to solve is stochastic: we assume that w is a stationary zero mean Gaussian white noise with $E[ww^T] = I$ and we seek to minimize the average noise power in z. This is nothing else but a LQG problem. So our objective is stated as
(OBJ_{LQG}): Find C such that the resulting closed loop system is stable and also

$$\lim_{M \to \infty} (1/2M) \sum_{k=-M}^{M-1} trace(E[z(k)z^T(k)])$$

is minimized.

To solve the above problems the following assumptions are introduced. Let $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ then.

Assumption 3.1 *P is finite dimensional and stabilizable.*

Assumption 3.1 means that P has a state space description

$$P \sim (A, (B_1 \; B_2), \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix})$$

with the pairs (A, B_2) and (A, C_2) being stabilizable and detectable respectively. In addition we assume that

Assumption 3.2 *The subsystem P_{22} is strictly causal, i.e., $D_{22} = 0$.*

This assumption has the implication that the system of Figure 1 is well-posed [11,9]. More important than this, however, is the fact that it allows for a convenient characterization of the structural constraints on the controller as we shall see in the following section.

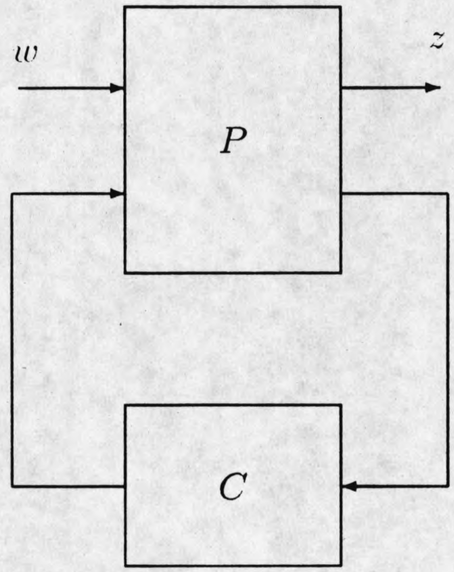

Figure 1: Block Diagram for Disturbance Rejection.

4 Problem Solution

The problems defined in the previous section can be related to problems in periodic systems where additional constraints that ensure causality appear in the so-called lifted system [8,32]. These constraints are of similar nature as with the problems at hand. The methods of solutions we develop herein are along the same lines with [8,32]. A common step in the solution of all of the problems defined earlier is the convenient characterization of all controllers that are in $\mathcal{S}$. This is done in the sequel.

4.1 Parametrization of all stabilizing controllers and feasible maps

Since we have assumed that P is finite dimensional with a stabilizable and detectable state space description we can obtain a doubly coprime factorization (dcf) of P_{22} using standard formulas (e.g., [11,34]) i.e., having P_{22} associated with the state space description $P_{22} \sim (A, B_2, C_2, D_{22})$ the coprime factorization such as in [11,34] is $P_{22} = N_l D_l^{-1} = D_r^{-1} N_r$ with

$$\begin{pmatrix} X_r & -Y_r \\ -N_r & D_r \end{pmatrix} \begin{pmatrix} D_l & Y_l \\ N_l & X_l \end{pmatrix} = I$$

where

$$N_l \sim (A_K, B_2, C_K, D_{22}),\ D_l \sim (A_K, B_2, K, I)$$

$$N_r \sim (A_M, B_M, C_2, D_{22}),\ D_r \sim (A_M, M, C_2, I)$$

$$X_l \sim (A_K, -M, C_K, I),\ Y_l \sim (A_K, B_2, -M, K, 0)$$

$$X_r \sim (A_M, -B_M, K, I),\ Y_r \sim (A_M, -M, K, 0)$$

with K, M selected such that $A_K = A + B_2K$, $A_M = A + MC_2$ are stable (eigenvalues in the open unit disk) and $B_M = B_2 + MD_{22}$, $C_K = C_2 + D_{22}K$. Note that the above formulas indicate that the coprime factors of P_{22} have as feedforward terms the matrices D_{22} or I or 0 which are all block diagonal. The following is a well-known result (e.g.,[11,34]):

Fact 4.1 *All ℓ^α-stabilizing LTI controllers C (possibly not in $\mathcal{S}$) of P are given by*

$$C = (Y_l - D_lQ)(X_l - N_lQ)^{-1} = (X_r - QN_r)^{-1}(Y_r - QD_r).$$

where $Q \in \mathcal{B}_{TI}(\ell^\alpha)$.

The above fact characterizes the set of all stabilizing controllers in terms of the so-called Youla parameter Q. The set $\mathcal{S}$ of interest is clearly a subset of the set implied by Fact 4.1 and is characterized by the constraint that the feedforward term of C should be block diagonal i.e.,

$$C(0) = \begin{pmatrix} C_{11}(0) & 0 \\ 0 & C_{22}(0) \end{pmatrix}.$$

However, a simple characterization is possible as the following lemma indicates

Lemma 4.1 *All ℓ^α-stabilizing controllers C in $\mathcal{S}$ of P are given by*

$$C = (Y_l - D_lQ)(X_l - N_lQ)^{-1} = (X_r - QN_r)^{-1}(Y_r - QD_r).$$

where $Q \in \mathcal{B}_{TI}(\ell^\alpha)$ and $Q(0)$ is block diagonal.

Proof It follows from the particular structure of the doubly coprime factors of P_{22} since $C(0) = -Q(0)(I - D_{22}Q(0))^{-1}$ with D_{22} block diagonal (in fact equal to zero) and hence $C(0)$ is block diagonal if and only if $Q(0)$ is block diagonal. ∎

Using the above lemma it is easy to show that all the feasible closed-loop maps are given as $T_{zw} = H - UQV$ where $H, U, V \in \mathcal{B}_{TI}(\ell^\alpha)$ and $Q \in \mathcal{B}_{TI}(\ell^\alpha)$ with $Q(0)$ block diagonal. Moreover, H, U, V are determined by P. Hence, we obtain in a straightforward manner the following lemma which shows how the objectives defined earlier transform to distance problems.

Lemma 4.2 *The objective (OBJ_α) with $\alpha = \infty$ or 2 is equivalent to the problem (OPT_α):*

$$\inf_{Q \in \mathcal{B}_{TI}(\ell^\alpha)} \|H - UQV\|_{\mathcal{B}_{TI}(\ell^\alpha)}$$

subject to $Q(0)$ is block diagonal. The objective (OBJ_{LQG}) is equivalent to the problem (OPT_{LQG}):

$$\inf_{Q \in \mathcal{H}^\infty} \|H - UQV\|_{\mathcal{H}^2}$$

subject to $Q(0)$ is block diagonal. Moreover, if Q_o is an optimal solution to any of the above problems then the corresponding optimal compensator is given as

$$C_o = (Y_l - D_lQ_o)(X_l - N_lQ_o)^{-1} = (X_r - Q_oN_r)^{-1}(Y_r - Q_oD_r).$$

It should be noted that all of the problems in Lemma 4.2 are, although infinite dimensional, minimizations of convex functionals over convex domains. In fact, all of these are distance problems in appropriate spaces and the main tools for their solution have been given in section 2 as we shall see in the sequel.

It is also important to note that Assumption 3.2 plays a central role in transforming the (linear) structural constraints on C to *linear* constraints on Q as indicated in Lemma 4.1. In fact, as it can be seen from the proof of Lemma 4.1, the constraint on Q remains unchanged even if we relax Assumption 3.2 to requiring a block diagonal D_{22} instead of $D_{22} = 0$ (which is of course block diagonal.) More generally, as long as P_{22} has the same off-diagonal structure as the one required on C, then the Youla parameter Q will have to have the same structure. Hence, linear structural constraints on C transform to the same linear structural constraints on Q via the parametrization of Fact 4.1 provided P_{22} satisfies the same constraints. If on the other hand Assumption 3.2 is completely relaxed allowing for fully populated D_{22}, then, the constraints on Q will no longer be linear or convex and hence the resulting optimization problem is hard to solve.

4.2 Equivalent problem formulation

We start solving the problems stated in Lemma 4.2 by first trying to transform the constraints on $Q(0)$ to constraints on the closed loop. As a first step, we perform an inner outer factorization [11] for U, V to obtain

$$U = U_i U_o, \quad V = V_o V_i$$

where the subscript i stands for "inner" and o for "outer"; i.e., $\hat{U}_i^T(\lambda^{-1})\hat{U}_i(\lambda) = I$ and $\hat{V}_i(\lambda^{-1})\hat{V}_i^T(\lambda) = I$. We will also make the simplifying technical assumption that $\hat{U}(\lambda)$, $\hat{V}(\lambda)$ do not lose rank on the unit circle and hence U_o, V_o have stable right and left inverses respectively. Note that the various factors in the inner outer factorization do not possess necessarily the block diagonal structure at $\lambda = 0$. Then we proceed by reflecting the constraints of $Q(0)$ on $U_o Q V_o$. Towards this end let $Z = U_o Q V_o$; the following proposition shows how Z is affected due to the constraints on Q.

Proposition 4.1 *Let $Z \in \mathcal{B}_{TI}(\ell^\alpha)$ then*

$$\exists Q \in \mathcal{B}_{TI}(\ell^\alpha) \quad \textit{with} \quad Q(0) \quad \textit{block diagonal and} \quad Z = U_o Q V_o$$

if and only if

$$Z(0) \in S_A = \{U_o(0) A V_o(0) : A \quad \textit{block diagonal matrix}\}.$$

■

Proof The "if" direction goes as follows: Let U_{or}, V_{ol} denote any right and left stable inverses of U_o, V_o respectively. Then $U_{or}, V_{ol} \in \mathcal{B}_{TI}(\ell^\alpha)$. Let A be a block diagonal matrix such that $Z(0) = U_o(0) A V_o(0)$; define $Q_A = \{A, 0, 0, \dots\}$ and let $\tilde{Z} = Z - U_o Q_A V_o$ then $\tilde{Z} \in \mathcal{B}_{TI}(\ell^\alpha)$ and $\tilde{Z}(0) = 0$. Define now Q as $Q = U_{or} \tilde{Z} V_{ol} + Q_A$. It then follows that $Q \in \mathcal{B}_{TI}(\ell^\alpha)$, $Q(0)$ is block diagonal and $Z = U_o Q V_o$.

The "only if" direction is immediate. ■

Proposition 4.1 shows that only $Z(0)$ is constrained to lie in a certain finite dimensional subspace (i.e., S_A) otherwise Z can be arbitrary in $\mathcal{B}_{TI}(\ell^\alpha)$. Note that the characterization of this subspace is independent of the choice of right and left inverses for U_o, V_o respectively; hence it is exact. One can easily find a basis for this subspace by considering $S_A{}^\perp$ and finding a basis for this subspace. This is done as follows: For each element j of $Q(0)$ with indices (l_j, m_j) that has to equal 0 (i.e., the elements that are not in the block diagonal portion of $Q(0)$) we associate a matrix R_j with the same dimension as $Q(0)$ that has all its entries but one equal to 0. The nonzero entry is taken to equal 1 and its indices are precisely the ones that correspond to j i.e., (l_j, m_j). If r is the number of the elements in $Q(0)$ that are necessarily equal to 0 then we have the following proposition.

Proposition 4.2 *Let*

$$S_B := \{B : U_o^T(0) B V_o^T(0) \in span(\{R_j\}_{j=1}^r)\}.$$

Then,

$$S_A^\perp = S_B.$$

Proof Let A be a block diagonal matrix and $B \in S_A{}^\perp$; then since

$$\langle U_o(0) A V_o(0), B\rangle = \langle A, U_o^T(0) B V_o^T(0)\rangle \quad \forall A$$

it follows that $B \in S_B$. Conversely, it also follows that if $B \in S_B$ then $\langle U_o(0) A V_o(0), B\rangle = 0$. Hence,

$$S_A{}^\perp = S_B$$

or equivalently

$$S_B{}^\perp = S_A$$

which proves the proposition. ∎

A basis $\{B_1, B_2 \ldots, B_{j_B}\}$ for this subspace can be found in a routine way and is given in the Appendix. When U_o, V_o are square the computation of the basis is immediate. Namely,

$$B_j = U_o^{-T}(0) R_j V_o^{-T}(0) \quad j = 1, \ldots r.$$

In view of the previous developments we have

$$Z(0) \in S_A \text{ if and only if } \langle Z(0), B_j\rangle = 0 \quad \forall j = 1, \ldots j_B.$$

Summarizing, the optimization problems become

$$(\mathrm{OPT}_\alpha): \quad \inf_{Z \in \mathcal{B}_{TI}(\ell^\alpha), Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{B}_{TI}(\ell^\alpha)}, \quad \alpha = \infty, 2$$

and

$$(\mathrm{OPT}_{\mathrm{LQG}}): \quad \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^2}.$$

where S_A is characterized in terms of the basis $\{B_1, B_2 \ldots, B_{j_B}\}$ of $S_A{}^\perp = S_B$ of Proposition 4.2

4.3 Optimal ℓ^1 control

In this subsection we present the solution to the problem of optimal rejection of bounded persistent disturbances. In the previous section we defined precisely this problem i.e., (OBJ_∞) and demonstrated that this problem transforms to (OPT_∞). Having in mind the characterizations of the various spaces introduced in section 2, we can equivalently state (OPT_∞) as

$$(\mathrm{OPT}_\infty): \quad \inf_{Z\in\ell^1,\ Z(0)\in S_A} \|H - U_i Z V_i\|_{\ell^1}$$

The unconstrained problem i.e., when $Q(0)$ and hence $Z(0)$ is not constrained is solved in [6,7]. In [6,7] the problem is transformed to a tractable linear programming problem, via duality theory. In this subsection, we show that the same approach can be extended to yield the optimal solution for the constrained problem. In particular, we show that the constraints on $Q(0)$ can be transformed as linear constraints on the closed loop map of interest i.e., T_{zw}. Once this is done, duality theory can be used to provide the solution.

First, we consider the 1-block case by assuming that $\hat{U}(\lambda), \hat{V}(\lambda)$ have full row and column rank, respectively, for almost all λ; we will come back to the general case later on. Also assume that $\hat{U}(\lambda), \hat{V}(\lambda)$ have no zeros on the unit circle. Let now $\{P_n\}_{n=1}^{N_s}$ be as in [6,7] the basis for the functionals in $c^0_{m\times n}$ that annihilate the space

$$S_s = \{K : K = UQV\ , Q \in \ell^1_{m\times n}\}$$

i.e.,

$$\langle UQV, P_i\rangle = 0 \quad \forall i = 1,2,\ldots,N_s, \quad Q \in \ell^1_{m\times n}.$$

These functionals are attributed to the unstable zeros of U and V. Suppose now that we are able to find functionals $\{X_j\}_{j=1}^{J}$ in $c^0_{m\times n}$ having the following property (PROP):
if $K \in S_s$ then

$$\langle K, X_j\rangle = 0 \quad \forall j = 1,2,\ldots,J$$

if and only if

$$\exists Q \in \ell^1_{m\times n} \quad \text{with} \quad K = UQV \quad \text{and} \quad Q(0) \quad \text{block diagonal.}$$

Next, define S as

$$S = \{K : K = UQV\ , Q \in \ell^1_{m\times n}\ , Q(0) \quad \text{block diagonal}\}.$$

The following lemma, given without proof, stems from standard results in functional analysis (for example [15]):

Lemma 4.3 *Let $\{P_n\}_{n=1}^{N_s} \in c^0_{m\times n}$ as above and let $\{X_j\}_{j=1}^{J}$ in $c^0_{m\times n}$ satisfy (PROP) as above. Then the annihilator subspace ${}^\perp S$ of S can be characterized as*

$${}^\perp S = span(\{P_n\}_{n=1}^{N_s} \cup \{X_j\}_{j=1}^{J})$$

∎

All that the above lemma says is that the functionals $\{X_j\}_{j=1}^{J}$ add the extra constraints of causality of $Q(0)$ required to solve (OPT_∞) by enlarging the subspace ${}^\perp S_s$ to ${}^\perp S$. Since we now have a complete characterization of ${}^\perp S$ we can proceed exactly as in [6,7] to solve (OPT_∞). Namely, using duality we can transform (OPT_∞) to a maximization problem inside $B({}^\perp S)$:
By Fact 2.6 $(c^0_{m\times n})^* = \ell^1_{m\times n}$: moreover if M is the subspace in $c^0_{m\times n}$ defined as $M = span(\{P_n\}_{n=1}^{N_s} \cup \{X_j\}_{j=1}^{J})$ then from the definitions of $\{X_j\}_{j=1}^{J}$ and $\{P_n\}_{n=1}^{N_s}$ it is easy to verify as in [6,7] that $M^\perp = S$ which implies that S is weak * closed. Hence

$$\inf_{K\in S} \|H - K\| = \min_{K\in({}^\perp S)^\perp} \|H - K\| = \sup_{G\in B({}^\perp S)} \langle G, H\rangle$$

but since ${}^\perp S$ is finite dimensional

$$\sup_{G\in B({}^\perp S)} \langle G, H\rangle = \max_{G\in B({}^\perp S)} \langle G, H\rangle$$

therefore

$$\inf_{K\in S} \|H - K\| = \max_{G\in B({}^\perp S)} \langle G, H\rangle.$$

The right-hand side of the above equality can be turned into a finite dimensional linear programming problem [6,7] and hence we obtain the optimal $G = G_0$. The optimal K_0 is found by using the alignment conditions [6,7]

$$\langle G_0, H - K_0\rangle = \|H - K_0\|.$$

In the sequel we show how to obtain these $\{X_j\}_{j=1}^{J}$. Towards this end define the following functionals in c^0:

$$R_{z_j} = \{B_j, 0, 0, \ldots\} \quad \forall j = 1, 2, \ldots, j_B.$$

where $\{B_j\}_{j=1}^{j_B}$ be a basis for the finite-dimensional Euclidean space

$$S_A^\perp = S_B = \{B : U_o^T(0) B V_o^T(0) \in span(\{R_j(0)\}_{j=1}^{r})\}.$$

Let $Z \in \ell^1$ then clearly

$$Z(0) \in S_A$$

if and only if

$$\langle Z, R_{z_j}\rangle = 0 \quad \forall j = 1, 2, \ldots, j_B.$$

In view of the above, (OPT_∞) can be stated as

$$\inf_{Z} \|H - U_i Z V_i\|$$

with

$$Z \in \ell^1, \langle Z, R_{z_j}\rangle = 0 \quad \forall j = 1, \ldots, j_B.$$

We now show how to obtain the functionals $\{X_j\}_{j=1}^{J}$ that have the property (PROP) mentioned in the beginning of this subsection.

Theorem 4.1 *The functionals*

$$X_j = U_i R_{z_j} V_i \qquad j = 1, 2 \ldots j_B$$

satisfy (PROP).

Proof Consider the bounded operators T_{U_i}, T_{V_i} on ℓ^1 defined as

$$(T_{U_i}X)(t) = \sum_{\tau=0}^{t} U_i(\tau)X(t-\tau)$$

$$(T_{V_i}X)(t) = \sum_{\tau=0}^{t} X(\tau)V_i(t-\tau)$$

where $X \in \ell^1$. Their (weak*) adjoints $T^*_{U_i}, T^*_{V_i}$ on c^0 which are given by

$$(T^*_{U_i}Y)(t) = \sum_{\tau=0}^{\infty} U_i^T(\tau)Y(\tau+t)$$

$$(T^*_{V_i}Y)(t) = \sum_{\tau=0}^{\infty} Y(\tau+t)V_i^T(\tau)$$

where $Y \in c^0$

Notice, that since U_i, V_i inner then $\hat{U}_i^T(\lambda^{-1})\hat{U}_i(\lambda) = I$ and $\hat{V}_i(\lambda^{-1})\hat{V}_i^T(\lambda) = I$. Note also that $T^*_{U_i}, T_{U_i}$ represent multiplication from the right whereas $T^*_{V_i}, T_{V_i}$ represent multiplication from the left. Hence, it follows that $T^*_{U_i}T_{U_i} = I$ and $T^*_{V_i}T_{V_i} = I$

Interpreting

$$U_iZV_i = T_{U_i}(T_{V_i}(Z))$$

and

$$U_iR_zV_i = T_{U_i}(T_{V_i}(R_z))$$

with

$$R_z \in span(\{R_{z_j}\}_{j=1}^{j_B})$$

we can verify that

$$\langle U_iZV_i, U_iR_zV_i\rangle = \langle Z, R_z\rangle$$

Hence, if $X = U_iR_zV_i$ then

$$\langle UQV, X\rangle = 0 \text{ if and only if } \langle Z, R_z\rangle = 0$$

which completes the proof. ∎

Hence, the additional (finitely many) functionals due to the structural constraints are completely characterized in Theorem 4.1 and the solution to the 1-block problem follows the duality approach described earlier. Note that since the additional functionals X_j of Theorem 4.1 are in c^0 (in fact, in ℓ^1), the optimal solution has a finite impulse response (FIR) as in the unconstrained case.

So far in this subsection we assumed that $\hat{U}(\lambda), \hat{V}(\lambda)$ have full row and column rank respectively. However, there is no loss of generality since in the "bad" rank case, or the 4-block problem, (i.e., when the above assumption does not hold [6,7,17]) it is shown in [17] that in order to solve the unconstrained problem it is necessary to satisfy the feasibility conditions of a square subproblem. In particular, we can partition U, V as

$$U = \begin{pmatrix} \bar{U} \\ U_2 \end{pmatrix}, V = (\bar{V} \quad V_2)$$

where $\bar{U}, \bar{V}$ are square and invertible. Let $K = UQV$ then

$$K = \begin{pmatrix} \bar{K} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

A necessary condition for the existence of a solution is that $\bar{K}$ interpolates $\bar{U}, \bar{V}$ which is the aforementioned subproblem. For the constrained problem, in addition to the interpolation conditions on K (e.g., [5]) we can embed the constraints on $Q(0)$ in $\bar{K}$. This embedding relates to the 1-block subproblem and can be done as before and thus the additional constraints on the closed loop can be completely characterized. Thus the standard 4-block procedures [5] can be applied to solve the problem.

4.4 Optimal $\mathcal{H}^\infty$ control

In this subsection we present the solution to the problem of optimal rejection of energy bounded disturbances i.e.,

$$(\mathrm{OPT}_2): \quad \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^\infty}.$$

This is a constrained $\mathcal{H}^\infty$ problem. We solve this $\mathcal{H}^\infty$ problem, by suitably modifying the standard Nehari's approach [11] in order to account for the additional constraint on the parameter Q (and hence on Z.) This modification yields to a finite dimensional convex optimization problem over a convex set that needs to be solved before applying the standard solution to the Nehari problem. The solution to the above convex finite dimensional problem can be obtained using standard programming techniques. Once this is done, we obtain the optimal LTI controller by solving a standard Nehari's problem.

First we assume that U_i, V_i are square. We will come back to the general 4-block problem later. The solution to the 1-block case is as follows: Let $R = U_i^* H V_i^*$ where $\hat{U}_i^*(\lambda) = \hat{U}_i^T(\lambda^{-1}), \hat{V}_i^*(\lambda) = \hat{V}_i^T(\lambda^{-1})$ and define for each $J \in S_A$ the system R_J as:

$$\hat{R}_J(\lambda) = \lambda^{-1}(\hat{R}(\lambda) - J).$$

For each $J \in S_A$, let Γ_{R_J} represent the Hankel operator [11] with symbol R_J. Before we present the solution we will need to compute $\|\Gamma_{R_J}\|$ using state space formulae. In particular we are going to compute the controllability and observability grammians [11] associated with Γ_{R_J}. Towards this end let R correspond via the Fourier transform to the double-sided (since R is not necessarily causal) sequence $(R(i))_{i=-\infty}^{\infty}$ then R_J will correspond to $(R_J(i))_{i=-\infty}^{\infty}$ with $R_J(i) = R(i+1) \quad \forall i \neq -1$ and $R_J(-1) = R(0) - J$. Let now $\overline{G}$ represent the stable (causal) system associated with the pulse response $\{0, R(-1), R(-2), \ldots\}$ and let $(\overline{A}, \overline{B}, \overline{C}, 0)$ be a minimal state space description of it. Let also G represent the stable system associated with the pulse response $\{0, R_J(-1), R_J(-2), \ldots\}$ i.e., G is the anticausal part of R_J but viewed as a causal (one-sided) system. Then it easy to check that G has the state space description $(A, B, C, 0)$ with

$$A = \begin{pmatrix} \overline{A} & \overline{B} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad C = (\,\overline{C} \quad \overline{J}\,)$$

where $\overline{J} = R(0) - J$. Finally, let W_c, W_o be the controllability and observability grammians for G i.e.,

$$W_c = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

$$W_o = \sum_{k=0}^{\infty}(A^T)^k C^T C A^k.$$

Then W_c and W_o are the solutions to the Lyapunov equations :

$$W_c - AW_cA^T = BB^T, \quad W_o - A^T W_o A = C^T C.$$

Similarly, let $\overline{W}_c, \overline{W}_o$ be the controllability and observability grammians for $\overline{G}$.
Following [11] we have that $\|\Gamma_{R_J}\| = \rho^{1/2}(W_c^{1/2} W_o W_c^{1/2})$. Using the state space description we compute

$$W_c = \begin{pmatrix} \overline{W}_c & 0 \\ 0 & I \end{pmatrix}$$

$$W_o = \begin{pmatrix} \overline{C}^T\overline{C} & \overline{C}^T\overline{J} \\ \overline{J}^T\overline{C} & \overline{J}^T\overline{J} \end{pmatrix} + K$$

where

$$K = \sum_{k=1}^{\infty}(A^T)^k C^T C A^k = \begin{pmatrix} \overline{W}_o - \overline{C}^T\overline{C} & \overline{A}^T\overline{W}_o\overline{B} \\ \overline{B}^T\overline{W}_o\overline{A} & \overline{B}^T\overline{W}_o\overline{B} \end{pmatrix}.$$

Note that K does not depend on J. Also, since $K \geq 0$ then $K = K^{1/2}K^{1/2}$ with $K^{1/2} \geq 0$. Now, proceeding with the computations and rearranging certain terms we obtain:

$$W_c^{1/2} W_o W_c^{1/2} = \begin{pmatrix} I & 0 \\ 0 & \overline{J}^T \end{pmatrix} M^T M \begin{pmatrix} I & 0 \\ 0 & \overline{J} \end{pmatrix} + L^T L$$

where

$$M = \begin{pmatrix} \overline{C}\overline{W}_c^{1/2} & I \\ 0 & 0 \end{pmatrix}, \quad L = K^{1/2} W_c^{1/2}.$$

Hence

$$\|\Gamma_{R_J}\| = \rho^{1/2}(W_c^{1/2} W_o W_c^{1/2}) = \overline{\sigma}[\begin{pmatrix} M\overline{H} \\ L \end{pmatrix}]$$

with $\overline{H} = \begin{pmatrix} I & 0 \\ 0 & \overline{J} \end{pmatrix}$. The following lemma shows that $\|\Gamma_{R_J}\|$ is convex in J

Lemma 4.4 $\bar{\mu} = \inf_{J \in S_A} \|\Gamma_{R_J}\|$ *is a finite dimensional optimization of a convex and continuous functional on a convex closed set.*

Proof From the preceding discussion we have

$$\inf_{J \in S_A} \|\Gamma_{R_J}\| = \inf_{\overline{H} \in \overline{S}} \overline{\sigma}[\begin{pmatrix} M\overline{H} \\ L \end{pmatrix}].$$

where

$$\overline{S} = \{\begin{pmatrix} I & 0 \\ 0 & \overline{J} \end{pmatrix} : \quad \overline{J} = R(0) + J, \quad J \in S_A\}.$$

Clearly, since S_A is a subspace then $\overline{S}$ is a convex set. Moreover if $\overline{H}_1, \overline{H}_2 \in \overline{S}$, and $t \in [0,1]$ we have

$$\overline{\sigma}[\begin{pmatrix} M(t\overline{H}_1 + (1-t)\overline{H}_2) \\ L \end{pmatrix}] = \overline{\sigma}[\begin{pmatrix} tM\overline{H}_1 \\ tL \end{pmatrix} + \begin{pmatrix} (1-t)M\overline{H}_2 \\ (1-t)L \end{pmatrix}]$$

or

$$\overline{\sigma}[\begin{pmatrix} M(t\overline{H}_1 + (1-t)\overline{H}_2) \\ L \end{pmatrix}] \leq t\overline{\sigma}[\begin{pmatrix} M\overline{H}_1 \\ L \end{pmatrix}] + (1-t)\overline{\sigma}[\begin{pmatrix} M\overline{H}_2 \\ L \end{pmatrix}]$$

which shows that $\overline{\sigma}[\begin{pmatrix} M\overline{H} \\ L \end{pmatrix}]$ is convex in $\overline{H}$ and consequently in J. Also, continuity of $\overline{\sigma}[\begin{pmatrix} M\overline{H} \\ L \end{pmatrix}]$ with respect to J is apparent and therefore our claim is proved.

∎

We are now ready to show how to obtain the optimal solution to

$$\mu_{\mathcal{H}^\infty} = \inf_{Z\in\mathcal{H}^\infty, Z(0)\in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^\infty}.$$

Theorem 4.2 *The following hold:*

1. $\mu_{\mathcal{H}^\infty} = \inf_{Z\in\mathcal{H}^\infty, Z(0)\in S_A} \|R - Z\|_{\mathcal{H}^\infty} = \bar{\mu}$,
2. *A minimizer J_o of the preceding convex programming problem of Lemma 4.4 always exists. Moreover, if X_o is the solution to the standard Nehari problem*

$$\inf_{X\in\mathcal{H}^\infty} \|R_{J_o} - X\|$$

then the optimal solution Z_o is given by

$$\hat{Z}_o(\lambda) = J_o + \lambda \hat{X}_o(\lambda).$$

Proof For the first part note that since $\hat{U}_i(\lambda), \hat{V}_i(\lambda), \lambda I$ are inner then

$$\|H - U_i Z V_i\| = \|U_i^* H V_i^* - Z\| = \|R - Z\|$$

Writing $\hat{Z}(\lambda) = Z(0) + \lambda\hat{\overline{Z}}(\lambda)$ with $\overline{Z}$ arbitrary in $\mathcal{H}^\infty$ we have

$$\|R - Z\| = \left\|\hat{R}(\lambda) - Z(0) - \lambda\hat{\overline{Z}}(\lambda)\right\| = \left\|\lambda^{-1}(\hat{R}(\lambda) - Z(0)) - \hat{\overline{Z}}(\lambda)\right\|.$$

Now, if $J \in S_A$ then from Nehari's theorem we have:

$$\inf_{\overline{Z}\in\mathcal{H}^\infty} \left\|R_J - \overline{Z}\right\| = \|\Gamma_{R_J}\|$$

hence the first part of the proof follows.

The second part of the theorem is immediate given that a bounded minimizer J_o of the convex minimization in Lemma 4.4 can be found in S_A (note S_A is unbounded). In fact, this is always the case and the proof of it follows from the fact that the optimal solution Z has to be bounded: Clearly the selection $Z = 0$ is a legitimate one since $Z(0) \in S_A$. Hence $\mu_{\mathcal{H}^\infty} \leq \|H\|$; if now $\|Z\| > 2\|H\|$ then

$$\|H - U_i Z V_i\| \geq \|Z\| - \|H\| > \|H\| \geq \mu_{\mathcal{H}^\infty}.$$

Therefore for Z to be a minimizer it is necessary that $\|Z\| \leq 2\|H\|$ and hence $\overline{\sigma}[Z(0)] \leq 2\|H\|$ which implies that the search for the optimal J can be constrained in a closed and bounded subset $\overline{S}_A$ of S_A. Namely,

$$\overline{S}_A = \{J \in S_A : \quad \overline{\sigma}[J] \leq 2\|H\|\}.$$

But then the continuity of the cost implies that an optimal J_o can be found in $\overline{S}_A$ which is bounded.

∎

The previous theorem indicates what is the additional convex minimization problem that has to be solved in order to account for the constraint on $Q(0)$. The following corollary is a direct consequence from the proof of the previous analysis.

Corollary 4.1 *The convex minimization problem of Lemma 4.4 is*

$$\mu_{\mathcal{H}^\infty} = \bar{\mu} = \min_{J \in \overline{S}_A} \rho^{1/2}(W_c^{1/2} W_o W_c^{1/2})$$

with

$$W_c = \begin{pmatrix} \overline{W}_c & 0 \\ 0 & I \end{pmatrix}, \quad W_o = \begin{pmatrix} 0 & \overline{C}^T \overline{J} \\ \overline{J}^T \overline{C} & \overline{J}^T \overline{J} \end{pmatrix} + \begin{pmatrix} \overline{W}_o & \overline{A}^T \overline{W}_o \overline{B} \\ \overline{B}^T \overline{W}_o \overline{A} & \overline{B}^T \overline{W}_o \overline{B} \end{pmatrix}$$

and $\overline{J} = R(0) + J$.

∎

The above convex programming problem can be solved with descent algorithms. In [23] the authors treating a problem of $\mathcal{H}^\infty$ optimization with time domain constraints arrive at a similar finite dimensional convex programming problem. As they indicate the cost might not be differentiable at all points and therefore methods of non-differentiable optimization are called for. Although these generalized descent methods might be slow they have guaranteed convergence properties. In [23] and the references therein alternatives are given to improve the convergence rate. Several methods to solve the standard Nehari problem implied in Theorem 4.2 exist. One may refer to [11,19] to mention only a few.

The full 4-block problem i.e., when U_i and/or V_i are not square is treated analogously as in the standard Nehari approach [11] with the so-called γ-iterations. In particular, using exactly the same arguments as in [11] the same iterative procedure can be established where at each iteration step a 1-block (square) problem with the additional causality constraints on the free parameter Q needs to be solved. Hence, the aforementioned procedure of solving the $\mathcal{H}^\infty$ constrained problem is complete.

4.5 Optimal $\mathcal{H}^2$ control

The problem of interest is

$$(\mathrm{OPT_{LQG}}): \qquad \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^2}.$$

The solution to this nonstandard $\mathcal{H}^2$ problem is obtained by utilizing the Projection theorem as follows: Let again $R = U_i^* H V_i^*$ and let $Y = \{Y(0), Y(1), Y(2), \ldots\}$ represent the projection of R onto $\mathcal{H}^2$ i.e., $Y = \Pi_{\mathcal{H}^2}(R)$. We note that U_i and V_i need not be square. Consider now the finite dimensional Euclidean space E of real matrices with dimensions equal to those of $Y(0)$ and let Π_{S_A} represent the projection operator onto the subspace S_A of E. Then

Theorem 4.3 *The optimal solution Z_o for the problem*

$$\mu_{\mathcal{H}^2} = \inf_{Z \in \mathcal{H}^\infty, Z(0) \in S_A} \|H - U_i Z V_i\|_{\mathcal{H}^2}$$

is given by

$$Z_o = \{\Pi_{S_A}(Y(0)), Y(1), Y(2), \ldots\}.$$

Proof The proof follows from a direct application of the Projection theorem in the Hilbert space $\mathcal{L}^2$. Let $\mathcal{H}_S = \{Z : Z \in \mathcal{H}^2, Z(0) \in S_A\}$ then $\mathcal{H}_S$ is a closed subspace of $\mathcal{L}^2$. Also, let $\langle \bullet, \bullet \rangle$ denote the inner product in $\mathcal{L}^2$. Viewing U_i and V_i as operators on $\mathcal{L}^2$ we have that Z_o is the optimal solution if and only if

$$\langle H - U_i Z_o V_i, U_i Z V_i \rangle = 0 \quad \forall Z \in \mathcal{H}_S$$

or equivalently

$$\langle U_i^* H V_i^* - Z_o, Z \rangle = 0 \quad \forall Z \in \mathcal{H}_S.$$

or equivalently

$$\Pi_{\mathcal{H}_S}(U_i^* H V_i^* - Z_o) = 0.$$

But

$$\Pi_{\mathcal{H}_S}(U_i^* H V_i^*) = \{\Pi_{S_A}(Y(0)), Y(1), Y(2), \ldots\} \in \mathcal{H}^\infty$$

which completes the proof. ∎

The above theorem states that only the first component of the classical solution Y is affected. The computation of $\Pi_{S_A}(Y(0))$ is routine; for example having an orthonormal basis $\{B_j\}_{j=1}^r$ for $S_A^\perp$ we have that

$$\Pi_{S_A}(Y(0)) = Y(0) - \sum_{j=1}^{r} \langle Y(0), B_j \rangle B_j.$$

5 Concluding Remarks

In this paper we presented the solutions to the optimal ℓ^1, $\mathcal{H}^\infty$ and $\mathcal{H}^2$ disturbance rejection problems in the case of a one-step delay observation sharing pattern. We took an input-output point of view that enabled us to convert linear structural constraints on the controller to linear constraints on the Youla parameter characterizing all possible controllers. In the optimal ℓ^1 disturbance rejection problem, the key observation was that we can obtain a finite number of linear constraints (functionals) to account for the constraint on the Youla parameter. These functionals combined with the functionals of the unconstrained problem can be used exactly as in the standard ℓ^1 problem to yield a tractable linear programming problem. The $\mathcal{H}^\infty$ problem was solved using the Nehari's theorem whereas in the $\mathcal{H}^2$ problem the solution was obtained using the Projection theorem. In particular, the $\mathcal{H}^\infty$ problem was solved by modifying the standard Nehari's approach in order to account for the additional constraint on the compensator. This modification yielded a finite dimensional convex optimization problem over a convex set that needs to be solved before applying the standard solution to the Nehari problem. The solution to the above convex finite dimensional problem can be obtained easily using standard programming techniques. In the $\mathcal{H}^2$ case the solution was obtained from the optimal (standard) unconstrained problem by projecting only the feedforward term of the standard solution to the allowable subspace.

It should be realized that the key element in obtaining convex problems through the Youla parametrization approach was the assumption that, the part of plant that connects controls to measurements, i.e., P_{22} in Figure 1, has the same structure as the one that is required in the controller, i.e., a block diagonal feedthrough term. This is what makes the key Lemma 4.1 work. Note also that the other parts of the plant, i.e., P_{11}, P_{12} and P_{21} can have arbitrary structure. More generally, if the off-diagonal structure of P_{22} is the same as the structure required on the controller C then, the same methods presented herein are applicable. For example, if P_{22} is of the form

$$P_{22} = \begin{pmatrix} P_{22}^{11} & P_{22}^{12}\Lambda^{m_{12}} \\ P_{22}^{21}\Lambda^{m_{21}} & P_{22}^{22} \end{pmatrix}$$

where Λ is the unit right shift operator (i.e., the unit delay) and m_{ij} are nonnegative integers, then any imposed controller structure of the form

$$C = \begin{pmatrix} C_{11} & C_{12}\Lambda^{n_{12}} \\ C_{21}\Lambda^{n_{21}} & C_{22} \end{pmatrix}$$

can be dealt similarly as long as $n_{12} \leq m_{12}$ and $n_{21} \leq m_{21}$. This can be for a example the case of observation patterns with multiple delays. For this type of problems one can use lifting techniques (e.g., [13,8,32]) as a preliminary step to transform the problem to an equivalent one that imposes linear structural constraints only on the feedthrough term of the lifted controller. This latter problem is similar to the one-step delay observation sharing pattern dealt in this work. Also, the fully decentralized case where $n_{ij} = m_{ij} = \infty$ for some i, j is a convex problem; yet, one needs to resort to approximating schemes since the number of structural constraints in now infinite. One such approach will be to solve the problem for $n_{ij} = N < \infty$ to obtain a (super)optimal Youla parameter Q_N. Then let N grow sufficiently to get arbitrarily close to the optimal performance. This generates an increasing sequence of lower bounds on the optimal performance. Moreover, one can get a sequence of upper bounds by using a truncated Q_N, i.e., $\Pi^N Q_N$, which completely satisfies the structural constraints. Hence, one can get arbitrarily close to the optimal with ariori accuracy. However, the convergence details of such an approach need to be further investigated.

In the development herein we assumed that the admissible controllers were LTI. This may or may not be restrictive depending on the particular measure of interest. For the $\mathcal{H}^2$, or more precisely LQG, problem for example, it is well known (e.g., [12]) that it admits a linear optimal solution. A similar result has not established so far for the $\mathcal{H}^\infty$ case; yet there are related results (e.g., [10,29,28]) that can possibly lead to such a conclusion. For the ℓ^1 problem it can be shown that, in the unconstrained case, nonlinear controllers may [30] outperform linear ones. Thus, it seems likely that this will be the case for the constrained problem as well. Nonetheless, in both the $\mathcal{H}^\infty$ and ℓ^1 problems one can show using the exact same arguments as in [22] that linear time varying controllers do not outperform LTI ones.

In addition to purely discrete time problems, sampled data (i.e., continuous plant-discrete controller) problems with the same information patterns can also be dealt in a similar fashion using generalized lifting methods [33]. Also, problems involving decentralized constraints on more than two control and output components can be considered in an analogous manner.

The input-output approach presented herein for optimal control design provides an interesting and unifying point of view of certain standard problems with quasiclassical information patterns. However, it is not clear whether it can lead to a trackable synthesis method in the case of nonclassical information structures. The work in [31] for the fully decentralized case using such an approach is promising and could be useful in computing (near) optimal performance levels. More work is still needed to investigate the benefits, if any, of input-output methods.

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7 Appendix

Computation of $\{B_j\}_{j=1}^{j_B}$

Herein, we indicate how to find all matrices B (of appropriate dimensions) such that there are real numbers c_i with

$$U_o^T(0)BV_o^T(0) = \sum_{i=1}^{r} c_i R_i(0).$$

First, we consider the following problem (PA):
Let M be a given matrix and $\{E_i\}_{i=1}^{l}$ be a given set of linearly independent matrices. Find all matrices W such that

$$MW = \sum_i c_i E_i$$

for some real numbers c_i.

The solution to (PA) goes as follows:
Let E_i be partitioned to column vectors as $E_i = (e_1^i, e_2^i, \ldots, e_n^i)$ and $W = (w_1, w_2, \ldots, w_n)$ Also, define $P_k = -(e_k^1, e_k^2, \ldots, e_k^l)$ for $k = 1, 2, \ldots, n$ and $c^T = (c_1, c_2, \ldots c_l)$. Then,

$$(M \quad P_k)\begin{pmatrix} w_k \\ c \end{pmatrix} = 0.$$

Define $P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}$ and $\tilde{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$. Let $N = (diag(M, M, \ldots, M) \quad P)$ then $N\begin{pmatrix} \tilde{w} \\ c \end{pmatrix} = 0.$
Therefore, we can compute a basis for W by computing a basis for the null-space of the matrix N which completes the solution to (PA).

In view of the above construction if $C = BV_o^T(0)$ then we can compute a basis $\{C_j\}_{j=1}^{j_C}$ for the space

$$S_C = \{C : U_o^T(0)C = \sum_{i=1}^{r} c_i R_i(0)\}.$$

Then we can compute a basis $\{D_j\}_{j=1}^{j_D}$ for the space

$$S_D = \{D : V_o(0)D \in span(\{C_j\}_{j=1}^{j_C})\}.$$

Now, by defining $j_B = j_D$ and $B_j = D_j^T \quad j = 1, \ldots, j_B$ we obtain the required basis for S_B.

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