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**ON THE SPATIAL  
STABILIZATION OF THE  
BEAM-PLASMA INSTABILITY**

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## 1. Introduction

It has been observed in many experiments<sup>(1,2,3)</sup> that when a low density energetic beam of electrons is injected into a plasma, oscillations are generated which increase in amplitude as one moves away from the injection point. At larger distances these oscillations saturate and subsequently decay. There are at least three mechanisms which can be responsible for this control of the linear instability. These are, in the order of increasing nonlinearity: (a) If the linearly unstable modes have a broad spectrum of wave numbers<sup>(4)</sup>,  $\Delta k$ , then the beam may be stochastically heated<sup>(5)</sup> as it passes into the plasma. Once the beam obtains a thermal spread,  $\Delta v_b$ , such that  $\Delta v_b \gtrsim |u_0 - (\omega/k)|$ , where  $u_0$  is the beam velocity, then collisional processes will cause stabilization, and possibly the subsequent decay;<sup>(3)</sup> (b) If the unstable modes have a narrow spectrum, it is possible that the generation of harmonics might induce stabilization and decay through their interaction back on the unstable modes.<sup>(6-8)</sup> It is this mode-mode coupling which is the primary subject of investigation in the present study; (c) If the mode-mode coupling does not stabilize the system then the oscillations may grow to an amplitude which is large enough to trap the beam particles. Since this trapping cuts off the beam movement over the waves, the instability is terminated.<sup>(9)</sup> The subsequent spatial behavior would presumably have some oscillatory features which have

not yet been experimentally observed. This may, of course, simply be due to the conditions of existing experiments rather than the universal effectiveness of (a) and/or (b).

In Section 2 we derive the nonlinear equations describing the mode-coupling in the case of a low density beam. In the present case, where we are concerned with spatial growth, it is necessary to retain the electron-ion collisions in the plasma component in order to obtain finite spatial growth for cold components (as this collision frequency tends to zero, the plasma thermal spread controls the spatial growth). The character of the linear dispersion relationship in that case is therefore reviewed. In Section 3 we discuss, and illustrate some of the difficulties involved in establishing the bounded character of solutions of nonlinear systems of equations - particularly those of the convolution type. The results of this section are largely negative in character. In Section 4 we obtain coupled mode equations for this system, and determine the modes which are most strongly coupled. A brief discussion of the three-mode coupling is also given. In Section 5 we present results of computer solutions of the equations obtained in Section 2 and discuss their implication concerning the spatial stabilization of the beam-plasma instability.

## 2. Basic Equations

To investigate the effectiveness of mode coupling we consider a semi-infinite plasma in the region  $x \geq 0$ , with a beam of density  $n_b^0$  injected with a velocity  $u_0 (> 0)$  at  $x=0$ . Since we are concerned with the case of an energetic beam,  $u_0 \gg \Delta v_p$  (where  $\Delta v_p$  is the plasma thermal spread), we will

base the analysis of mode coupling on the cold hydrodynamic equations. This approximation is valid provided that the phase velocities of the waves are outside the thermal spread of both the beam and plasma, i.e.  $\omega/k \gg \Delta v_p$  and  $|u_0 - (\omega/k)| \gg \Delta v_b$ . While this may be easily satisfied for the most unstable modes, it is not equally true for the harmonics, for which  $|u_0 - (\omega/k)|$  is very small. We will return to this point below. For the present we assume that the dynamics is governed by the equations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nu) = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{e}{m} E = \nu u \quad (2)$$

$$\frac{\partial E}{\partial t} = 4\pi e \sum (nu - n_0 u_0) \quad (3)$$

where (1) and (2) holds for both the plasma and beam components,  $\nu$  is the collision frequency with the background ions (and  $\nu_b = 0$ ), and the sum in (3) is over both components. The constant current is subtracted in (3) to account for the return current outside the system (otherwise  $E$  would increase linearly with time).

Since we are concerned with the stationary state, all functions are of the form  $f(x,t) = \sum_{\omega} f(\omega,x) e^{-i\omega t}$ , where  $\omega$  is real and  $f(-\omega,x) = f^*(\omega,x)$ . Equations (1-3) then become

$$\frac{d}{dx} [n(\omega')u(\omega-\omega')] - i\omega n(\omega) = 0 \quad (1')$$

$$\frac{1}{2} \frac{d}{dx} [u(\omega')u(\omega-\omega')] + (\nu - i\omega)u(\omega) + \frac{e}{m} E(\omega) = 0 \quad (2')$$

$$-i\omega E(\omega) = 4\pi e \sum n(\omega')u(\omega-\omega') \quad (3')$$

where the sum on  $\omega'$  is implied and the variable  $x$  is suppressed. The physical situation which will be considered is the case of a low density beam and small collision frequency  $\nu$  ( $\neq 0$ ). Specifically it will be assumed that

$$1 \gg \mu = \nu/\omega_p \gg \alpha \equiv n_b^0/n_p^0 > 0 \quad (4)$$

It should be emphasized that the collisions are necessary to obtain a finite spatial gradient when the plasma is cold. If  $\nu \rightarrow 0$ , then the assumption that  $\omega/k \gg \Delta v_p$  no longer holds, and the finite gradient is due to plasma thermal effects. Under the conditions (4) the spatial variation of the zero frequency components can be neglected. Moreover the plasma dynamics will be linear, even though the beam dynamics is nonlinear. To show this, and reduce the system of five equations (1') - (3'), we note that  $u_p(\omega=0) = 0$ . Then the linearized plasma equation (2') and (3') yields

$$u_p(\omega) = \frac{\omega_b^2}{\omega(\omega+i\nu) - \omega_p^2} n_b(\omega') u_b(\omega-\omega') \quad (\omega \neq 0) \quad (5)$$

where now, and below,  $n(\omega)$  is the fractional density variation (so  $n(\omega=0) = 1$ ). Similarly from equations (1') and (5)

$$n_p(\omega) = \frac{\omega_b^2}{\omega(\omega+i\nu) - \omega_p^2} n_b(\omega) \quad (\omega \neq 0) \quad (6)$$

so that  $|n_b(\omega)| > (\mu/\alpha) |n_p(\omega)|$ , showing that the fractional density variation is larger for the beam component under the conditions (4). Using (5) and (6) to eliminate the plasma component, equation (2') for the beam becomes

$$\frac{1}{2} \frac{d}{dx} u_b(\omega') u_b(\omega - \omega') - i\omega u_b(\omega) + \frac{i\omega_b^2(\omega + i\nu)}{\omega(\omega + i\nu) - \omega_p^2} n_b(\omega') u_b(\omega - \omega') = 0 \quad (7)$$

In the following the subscript on  $(n_b, u_b)$  can be dropped without confusion, since the plasma component has been eliminated.

It should be noted that, contrary to the more common nonlinear temporal problem, the equations (1') and (7) are nonlinear in the differential terms. To remove this feature we explicitly introduce the rapid spatial variation, setting

$$n(x, \omega) = N(x, \omega) e^{i(\omega/u_0)x}, \quad u(x, \omega) = u_0 U(x, \omega) e^{i(\omega/u_0)x} \quad (8)$$

and neglect the derivative of the nonlinear products  $U(\omega')U(\omega - \omega')$  and  $N(\omega')U(\omega - \omega')$  compared with  $(\omega/u_0)$  times these products.<sup>(10)</sup> Therefore (1') becomes

$$\frac{d}{dx} [N(\omega) + U(\omega)] + i(\omega/u_0) U(\omega) = -i(\omega/u_0) \Sigma' N(\omega') U(\omega - \omega') \quad (9)$$

where the prime on the sum indicates that  $\omega' \neq 0, \omega$ . Again using the conditions (4), the nonlinear portion of the last factor in (7) may be neglected (see below). Then (7) yields

$$\frac{d}{dx} U(\omega) + \frac{i\omega_b^2(\omega + i\nu)}{u_0[\omega(\omega + i\nu) - \omega_p^2]} [U(\omega) + N(\omega)] = -\frac{i\omega}{2u_0} \Sigma' U(\omega') U(\omega - \omega') \quad (10)$$

From (8) and (9) it is easily shown that in the linear approximation

$U = 0(\sqrt{\alpha/\mu}) N \ll N$ . Introducing the notation

$$z = \sqrt{\alpha/\mu} (x\omega_p/u_0) \quad , \quad \Omega = \omega/\omega_p \quad , \quad U(z,\omega) = i \sqrt{\frac{\alpha}{\mu}} V(z,\omega) \quad (11)$$

so that  $N$  and  $V$  are now of the same order, and using the conditions (4), equations (9) and (10) reduce to

$$\frac{d}{dz} N(\Omega) = \Omega V(\Omega) + \Omega \Sigma' N(\Omega') V(\Omega-\Omega') \quad (12)$$

$$\frac{d}{dz} V(\Omega) = \epsilon(\Omega) N(\Omega) + \frac{1}{2} \Omega \Sigma' V(\Omega') V(\Omega-\Omega') \quad (13)$$

where

$$\epsilon(\Omega) = \frac{-\mu(\Omega+i\mu)}{\Omega(\Omega+i\mu)-1} \quad (14)$$

and  $N(-\Omega) = N^*(\Omega)$ ,  $V(-\Omega) = -V^*(\Omega)$ . The spatial variation in (12) and (13) is now in terms of the slow variable  $z$ . The approximations which have been used to obtain (12) and (13) are such that the linear dispersion relation of these equations is exact. If we set  $N(z,\Omega)$  and  $V(z,\Omega)$  proportional to  $e^{\sigma z}$ , then the linear equations yield

$$\sigma_{\pm}(\Omega) = \pm \sqrt{\frac{\mu}{2a}} \left\{ \sqrt{-b \pm \sqrt{b^2 + \mu^2 \Omega^2}} + i \operatorname{sign}(\Omega) \sqrt{b \pm \sqrt{b^2 + \mu^2 \Omega^2}} \right\} \quad (15)$$

where  $a = (\Omega^2 - 1)^2 + \mu^2 \Omega^2$  and  $b = \Omega^2 (\Omega^2 + \mu^2) - \Omega^2$ . The root  $\sigma_+$  corresponds to the spatially growing mode in the positive  $z$  direction. The form of  $\sigma_+(\Omega)$  is shown in Figure 1 (note that the linear wave number,  $e^{ikx}$ , is related to  $\sigma$  by  $k = (\omega/u_0) - i(\alpha/\mu)^{1/2} (\mu_p/u_0)\sigma$ ). The unstable frequencies have a range  $\omega_p + \frac{1}{4} v \gtrsim \omega \gtrsim \omega_p - 2v$  in which the real part of  $\sigma$  is at least one half



its maximum value. Thus the spectrum is more narrow as  $v$  decreases, and the mechanism (a) discussed in the introduction is less effective. For the most unstable mode one finds  $\sigma_+ \equiv (\sigma_r + i\sigma_i)$

$$\Omega^2 = 1 - (\mu/3^{1/2}) ; \sigma_r = (3^{3/2}/8)^{1/2} \quad (16)$$

$$\frac{\omega}{k} - u_0 \simeq - (3^{1/2}\alpha/8\mu)^{1/2} u_0$$

Since it has been assumed that  $|\frac{\omega}{k} - u_0| \gg \Delta v_b$ , we need  $(3^{1/2}\alpha/8\mu) \gg (\Delta v_b/u_0)^2$ . On the other hand if  $\omega \gtrsim 2\omega_p$ , one finds that  $(\omega/k) - u_0 \simeq -\alpha^{1/2} (u_0/\Omega)$  which is much smaller than for the unstable modes. Thus the present treatment, in which all modes are treated as nonresonant, is erroneous for the harmonics of the unstable modes. To correct for this we might take as a reasonable approximation the dispersion relation for a cold plasma and a Lorentzian beam (with a velocity spread  $\Delta v_b$ )

$$1 = \frac{1}{\Omega(\Omega+i\mu)} + \frac{\alpha}{[\Omega - (ku_0/\omega_p) + i(k\Delta v_b/\omega_p)]^2}$$

or, in terms of  $\sigma$ ,

$$1 = \frac{1}{\Omega(\Omega+i\mu)} + \frac{\alpha}{[i\sigma(\alpha/\mu)^{1/2} + i|\Omega|(\Delta v_b/u_0) + \sigma(\alpha/\mu)^{1/2}(\Delta v_b/u_0)]^2}$$

Referring to Figure 1, it is clear that the last term in the denominator is negligible, so that

$$\sigma_{\pm} = (\sigma_{\pm})_{\Delta v_b=0} - |\Omega| (\mu/\alpha)^{\frac{1}{2}} (\Delta v_b/u_0) \quad (17)$$

For  $\Omega \simeq 1$  this correction is negligible, provided that  $(\alpha/\mu) \gg (\Delta v_b/u_0)^2$ , but for  $\Omega > 2$  the correction is the dominant real part of  $\sigma_{\pm}$  if  $\Delta v_b/v_0 \gg \mu \alpha^{\frac{1}{2}}/\Omega^4$ . Note that this correction can not be obtained from a hydrodynamic treatment (since it is the Landau damping generated by a Lorentzian beam). Nonetheless it can be appended to the hydrodynamic equations by changing the function  $\epsilon(\Omega)$  in equation (14), or by altering both linear terms in (12) and (13). Since this alteration is not unique we may presumably choose either method (see Section 5).

### 3. The Problem of Bounded Solutions

The mode coupling mechanism will produce saturation only if all solutions of (12)-(13), which are initially sufficiently small, are bounded (as functions of  $z$ ). The problem of establishing conditions for boundedness of all solutions of a system of differential equations is generally very difficult<sup>(11)</sup>. Before considering the system (12,13), for which the conditions are not known, we will first review some known results and some further simple examples.

One system of equations which arise from perturbation methods<sup>(7)</sup>, for which both necessary and sufficient conditions are known, is

$$\frac{dN_k}{dt} = \sum_{k', k''} |V_{kk'k''}|^2 s_k N_k N_{k'} N_{k''} \left( \frac{s_k}{N_k} + \frac{s_{k'}}{N_{k'}} + \frac{s_{k''}}{N_{k''}} \right) \quad (18)$$

where  $s_k = \pm 1$  and  $N_k \geq 0$  (real functions). The functions  $|V_{kk'k''}|^2$  are

symmetric under the interchange of  $(k, k', k'')$ . In this case the linear equations are neutral ( $N_k = \text{constant}$ ). The necessary and sufficient condition for the boundedness of all solutions such that  $N_k(t=0) > 0$ , is that there exist a set of positive finite constants  $\{c_k\} (\infty > c_{\max} \geq c_k \geq c_{\min} > 0)$  such that, for  $(k, k', k'')$  for which  $V_{k, k', k''} \neq 0$ ,  $c_k s_k + c_{k'} s_{k'} + c_{k''} s_{k''} = 0$ . If this condition can not be satisfied (e.g., if all  $s_k$  have the sign) then (18) generally has singular solutions ( $N_k \sim (1-at)^{-1}$ ) which diverge for finite times ("explosive instabilities"). The proof in this case rests heavily upon the fact that the (real)  $N_k$  are positive definite functions, which are linearly neutral. If either of these conditions are dropped, the problem of establishing boundedness becomes much more complicated.

To illustrate this point we consider a generalization of a common three mode model

$$\begin{aligned}\dot{x}_1 &= \alpha_1 x_1 + \beta_1 x_2^* x_3 \\ \dot{x}_2 &= \alpha_2 x_2 + \beta_2 x_1^* x_3 \\ \dot{x}_3 &= \alpha_3 x_3 + \beta_3 x_1 x_2\end{aligned}\tag{19}$$

where the star refers to the complex conjugate. Here the nonlinear term is of a convolution form (if  $x_{-k} = x_k^*$ ) similar to (12,13). First consider the case where the  $\alpha_k = i \omega_k$  are purely imaginary (linear stability), and let

$$x_k(t) = A_k(t) e^{i\omega_k t + \theta_k(t)} \quad (A_k(0) \geq 0)$$

One readily obtains from (19), if the  $\beta_k$  are all real

$$A_k = \beta_k A_{k'} A_{k''} \cos \theta ; \quad \theta = \theta_3 - \theta_2 - \theta_1 + \Delta\omega t\tag{20}$$

where ( $k \neq k' \neq k''$ ) and  $\Delta\omega = \omega_3 - \omega_2 - \omega_1$ . Also

$$\dot{\theta} = \Delta\omega - (\beta_1 A_1^{-1} A_2 A_3 + \beta_2 A_2^{-1} A_1 A_3 + \beta_3 A_3^{-1} A_1 A_2) \sin \theta \quad (21)$$

From (20) one obtains two constants of the motion

$$\beta_2 A_1^2 - \beta_1 A_2^2 = C_1 ; \beta_3 A_1^2 - A_3^2 = C_2 \quad (22)$$

This shows that if  $\beta_k \beta_{k'} < 0$ , then  $A_k$  and  $A_{k'}$  are bounded, and hence  $A_{k''}$  is bounded. Furthermore, multiplying (21) by  $A_1 A_2 A_3 \cos \theta$ , yields a third constant

$$A_1 A_2 A_3 \sin \theta = C_3 + \frac{\Delta\omega}{2\beta_1} A_1^2$$

If one sets  $N_k = A_k^2$ , these constants can be used in (20) (for  $k = 1$ ) to obtain

$$\begin{aligned} \dot{N}_1 &= \pm 2\beta_1 \left\{ N_1 N_2 N_3 - \left( C_3 + \frac{\Delta\omega}{2\beta_1} N_1 \right)^2 \right\}^{\frac{1}{2}} \\ &= \pm 2 \operatorname{sign}_1 (\beta_1) \left\{ N_1 (\beta_3 N_1 - C_2) (\beta_3 N_1 - C_2) - \beta_1^2 \left( C_3 + \frac{\Delta\omega}{2\beta_1} N_1 \right)^2 \right\}^{\frac{1}{2}} \quad (23) \end{aligned}$$

where  $\pm = \operatorname{sign}(A_1 A_2 A_3 \cos \theta)$ . Now, if  $\Delta\omega = 0$  (three resonant modes), and  $\theta(0) = 0$  or  $\pi$  then  $C_3 = 0$ . If all  $\beta_k \beta_{k'} > 0$  then one can readily see from (20) that unbounded solutions can be obtained by taking  $\theta(0) = 0, \pi$  such that  $\operatorname{sign}(\beta_k A_1 A_2 A_3 \cos \theta) > 0$ . From (22) it then follows that all solutions of (19) with  $\alpha_k = i\omega_k$  and  $\Delta\omega = 0$  are bounded if and only if  $\sum a_k \beta_k = 0$  for some set  $\{a_k > 0\}$ . Moreover the unbounded solutions are singular, going as  $N_k \sim (1 - a t)^{-2}$  near the singularity.

An important point to note is that the solutions of (20,21) can be made bounded by a small change in the linear term (if the  $A_k(0)$  are

sufficiently small). Thus, if  $\Delta\omega \neq 0$ , it can be seen from (23) that  $N_1$  will be stabilized if the square root vanishes as  $N_1$  increases. Clearly for large enough  $\Delta\omega(N_1^0, N_2^0, N_3^0)$  dependent on the initial conditions, all solutions will be bounded. The details are not of interest here, but only the fact that the linear terms can have a significant influence on the boundedness of the solutions.

This simple case also can be used to shed light on the necessary conditions for singular solutions ("explosive" instabilities) recently discussed by Wilhelmsson, Stenflo and Engelmann.<sup>(12)</sup> In the present context one looks for solutions of the form

$$A_k = \frac{\beta_k}{t_0 - t} + O(1) ; \theta = \theta_0 + O(t_0 - t)$$

and considers only the most singular terms of (20,21). One then readily obtains the conditions

$$\beta_k \beta_{k'} > 0 \text{ and } \tan \theta_0 = 0$$

This necessary condition is, however, not sufficient. Clearly, from above, if  $\Delta\omega$  is sufficiently large the solutions are bounded. However, the above necessary condition is based only on the most singular terms, which do not depend on the value of  $\Delta\omega$ . If one carries the analysis to the next order taking

$$A_k = \frac{\beta_k}{t_0 - t} + C_k + O(t_0 - t) ; \theta = \theta_0 + D(t_0 - t) + O((t_0 - t)^2)$$

one obtains (among other conditions)

$$\begin{aligned} -D = \Delta\omega - (\beta_1 \beta_1^{-1} \beta_2 \beta_3 + \text{etc.}) D \cos \theta_0 - (\beta_1 \beta_1^{-1} (\beta_2 C_3 + C_2 \beta_3) + \text{etc.}) \sin \theta_0 \\ + (\beta_1 \beta_1^{-2} C_1 \beta_2 \beta_3 + \text{etc.}) \sin \theta_0 \end{aligned}$$

where "etc." refers to two permutations. Since, from the necessary condition  $\sin\theta = 0$ , one obtains

$$-D = \Delta\omega - 3D$$

which yields a value for  $D$  even when  $\Delta\omega \neq 0$ . Clearly this approach will not indicate the conditions for nonsingular solutions when  $\Delta\omega \neq 0$ . This deficiency is related to the fact that  $t_0$  is never obtained by this method, and moreover  $t_0$  is related to the initial conditions, which determine whether the solution is singular. A simple example illustrates this fact. Thus the equation

$$\dot{x} = \alpha x + x^2$$

has a singular solution (for  $t \geq 0$ ) if and only if  $\alpha^{-1} \ln(1 + \alpha/x_0) > 0$ , in which case the singularity occurs at  $t_s = \alpha^{-1} \ln(1 + (\alpha/x_0))$ . Substituting the expression

$$x(t) = \sum_{k=0} \beta_k (t_0 - t)^{k-1}$$

and equating powers of  $(t_0 - t)$ , yields the correct values for  $\beta_k$  (and therefore a convergent series), but gives no expression for  $t_0$  - and, in particular, does not establish under what conditions  $t_0$  is positive.

Another interesting point should be noted about singular solutions of (19). If one assumes that  $x_k = z_k / (t_0 - t)$  (where  $z_k$  are complex) near the singularity, then it readily follows that a necessary condition is

$$\beta_k \beta_{k'} > 0 \quad (\text{complex } \beta_k)$$

If  $\beta_k = \rho_k e^{i\phi_k}$ , this implies that

$$\phi_k + \phi_{k'} = 0, \text{ or } 2\pi \text{ (all } k, k')$$

If there are three or more coupled modes, one concludes that all  $\phi_k = 0$  or all  $\phi_k = \pi$  in other words, all  $\beta_k$  are real and of the same sign. This is quite different than the result obtained for singular solutions of the  $A_k$ , namely

$$\sum_k \tan(\theta_0 + \theta_k) = 0$$

for some  $\theta_0$ . This can easily be satisfied even if all  $\phi_k \neq 0, \pi$ . However, when this latter condition is not satisfied, one finds from the equations for  $\dot{\theta}_k$  (not  $\dot{\theta}$ ), that  $\dot{\theta}_k \rightarrow \infty$  at the singularity of the  $A_k$ . This means that the  $x_k$  of equation (19) rotate infinitely fast in their individual complex planes as one approaches the singularity, hence  $x_k$  is not of the form  $z_k/(t_0 - t)$ , but rather  $C_k \exp(i\lambda_k/(t_0 - t))/(t_0 - t)$ . Hence a singularity in the variables  $A_k$  can correspond to two radically different types of singularities in the  $x_k$  variables - one being an essential singularity.

What is even more important to notice is the influence of a real part of  $\alpha_k$  in part of  $\alpha_k = i\omega_k + \gamma_k$  then (20) is replaced by

$$\dot{A}_k = \gamma_k A_k + \beta_k A_{k'} A_{k''} \cos\theta \quad (24)$$

where, again ( $k \neq k' \neq k''$ ), and (21) still applies. In this case (22) is replaced, for example, by

$$\frac{d}{dt} (\beta_2 A_1^2 - \beta_1 A_2^2) = 2(\beta_2 \gamma_1 A_1^2 - \beta_1 \gamma_2 A_2^2) \quad (25)$$

This equation implies that if either  $\gamma_k$  is positive (corresponding to linear instability), and  $\beta_1 \beta_2 < 0$ , then there are unbounded solutions ( $A_k \neq 0$ ). These solutions may or may not be singular (the sign of  $\beta_3$  is not specified), but in any case they are unbounded. Even if  $\beta_1 \beta_2 < 0$ , so that there is no singular solution, the solutions will be unbounded if both  $\gamma_k > 0$ . Thus the only possibility for (21,24) to have only bounded solutions when the system is linearly unstable is for only one mode to be unstable (say  $\gamma_1 > 0$ ) and  $\beta_1 \beta_2 < 0$ ,  $\beta_1 \beta_3 > 0$ . The important question then is whether these conditions are sufficient as well as necessary.

To show that these conditions are not sufficient, consider the equations

$$\frac{d}{dt} (A_1 A_2 A_3 \sin\theta) = \Delta\omega A_1 A_2 A_3 \cos\theta + \Gamma A_1 A_2 A_3 \sin\theta \quad (26)$$

$$\frac{d}{dt} (A_1 A_2 A_3 \cos\theta) = -\Delta\omega A_1 A_2 A_3 \sin\theta + \Gamma A_1 A_2 A_3 \cos\theta + \sum \beta_k A_k'^2 A_k''^2$$

where  $\Gamma = \sum \gamma_k$ , which are obtained from (21,24). Clearly, if  $\Delta\omega = 0$  first equation shows that there are unbounded solutions if  $\Gamma > 0$ . Characteristically one can not immediately conclude that the solutions  $A_k(t)$ , are bounded when  $\Gamma < 0$ .

What has been established about the solutions of (21) and (24) is only that if  $\gamma_1 > 0$ , there are unbounded solutions if any of the following conditions hold:

- a) if  $\beta_1 \beta_2 > 0$  or  $\beta_1 \beta_3 > 0$
- b) if either  $\gamma_2$  or  $\gamma_3$  is positive
- c) if  $\Delta\omega = 0$  and  $\sum \gamma_k > 0$



However, if  $\gamma_1 > 0$ , no sufficient condition for bounded solutions has been obtained.

To indicate the difficulty in establishing sufficiency, consider the simplest case of resonance ( $\Delta\omega=0$ ), and the particular solution  $\theta=0$ .

Then the three equations (24) are of the form

$$\dot{A}_1 = \gamma_1 A_1 + \beta_1 A_2 A_3 \quad (27)$$

$$\dot{A}_2 = -\gamma_2 A_2 - \beta_2 A_1 A_3$$

$$\dot{A}_3 = -\gamma_3 A_3 - \beta_3 A_1 A_2$$

where now  $\gamma_k, \beta_k > 0$ ,  $\gamma_2 \neq \gamma_3$ , and  $A_k(0) > 0$ . It can be seen that the non-linear term initially makes  $A_1$  more unstable. The condition  $\gamma_2 \neq \gamma_3$  is required for boundedness, since otherwise  $A_2 = (\beta_2/\beta_3)^{\frac{1}{2}} A_3$  yields an unbounded solution. Equation (27) therefore represents the simplest three-mode, linearly unstable system which may have only bounded solutions. Clearly if  $A_2 A_3$  does not change sign,  $A_1$  is unbounded. However, from the equation

$$\frac{d}{dt} (A_2 A_3) = -(\gamma_2 + \gamma_3) A_2 A_3 - (\beta_2 A_3^2 + \beta_3 A_2^2) A_1$$

one can conclude that if  $A_2$  and  $A_3$  do not simultaneously go to zero then, since  $A_1$  is assumed to be increasing,  $A_2 A_3$  will change sign. This tends to stabilize  $A_1$ , so that the unbounded nature of the solution is not trivial even if true. Some weight can be given to unbounded solutions by examining the form of the stationary points of (27) around the origin,  $A_k = 0$ ,

namely

$$A_1^0 = \oplus (\gamma_2 \gamma_3)^{\frac{1}{2}}, \quad A_2^0 = \pm \oplus (\gamma_1 \gamma_3)^{\frac{1}{2}}, \quad A_3^0 = \mp (\gamma_1 \gamma_2)^{\frac{1}{2}}$$

where the signs are correlated in an obvious manner. If one linearizes about these points, setting  $A_k = A_k^0 + B_k$ , and looks for the normal modes  $C = \sum \alpha_k B_k$  such that  $\dot{C} = \nu C$ , one finds that for all stationary points.

$$\nu^3 + (\gamma_2 + \gamma_3 - \gamma_1)\nu^2 + 4\gamma_1 \gamma_2 \gamma_3 = 0$$

Since the roots must satisfy  $\nu_1 \nu_2 \nu_3 = -4\gamma_1 \gamma_2 \gamma_3$  and  $\nu_1 \nu_2 + \nu_1 \nu_3 + \nu_2 \nu_3 = 0$  one easily establishes that two roots must have positive real parts. Hence all stationary points are unstable. This fact does not establish the unbounded character of the solutions of (27), but it does show that any bounded solution can not be stationary.<sup>(13)</sup> The net result of these considerations is that no proof of the boundedness or unboundedness of solutions of (27) have been obtained.

We now summarize the results of this section:

- 1) The problem of establishing sufficient conditions for which all solutions are bounded is much more difficult than obtaining sufficient conditions for unbounded solutions. The difference arises from the fact that the former problem generally involves all equations of motion (in lieu of a Liapouov-type function), whereas the latter can be established from particular solutions.
- 2) The necessary conditions for a singular solution, obtained from a consideration of the most singular terms, is not

generally sufficient. Linear terms are capable of removing such singular solutions. Moreover the absence of singular solutions does not imply boundedness (as illustrated by (25), with  $\gamma_1 > 0$ ,  $\gamma_2 > 0$  and  $\beta_1 \beta_2 < 0$ ).

- 3) Singular solutions of the amplitudes  $A_k$  can correspond to two radically different singular solutions for the  $x_k$ , depending on whether the coefficients  $\beta_k$  are real or complex. In the latter case the  $x_k$  must rotate infinitely fast at the singular time (in their complex plane).

#### 4. Coupled Mode Equations

The nonlinear equations (12) and (13) are much more difficult to analyse than those discussed in the last section. Considering the limited success in treating even these simpler equations, one is forced to either greatly simplify (12-13), or else to revert to computer solutions of these equations (again, in a truncated form). Before discussing the results of some of these computer solutions (Section 5), we will consider here the coupled normal mode equations which arise from (12-13) - in the hope that they will shed some light on the most effective coupling between different frequency components.

If one introduced the functions

$$M_{\pm}(\Omega, z) = -(i/8)(\Omega^{-1} \sigma_{\pm}(\Omega) N(\Omega, z) + V(\Omega, z)) \quad (28)$$

where  $\sigma_{\pm}(\Omega)$  is given by (15), then equations (12-13) reduce to the simple form

$$\frac{d}{dz} M_{\pm}(\Omega, z) = \sigma_{\pm}(\Omega) M_{\pm}(\Omega, z) + i \sigma_{\pm}(\Omega) \sum'_{\pm, \pm} \left\{ \frac{\Omega'}{\sigma_{\pm}(\Omega')} + \frac{\Omega - \Omega'}{\sigma_{\pm}(\Omega - \Omega')} + \frac{\Omega}{\sigma_{\pm}(\Omega)} \right\} M_{\pm}(\Omega') M_{\pm}(\Omega - \Omega') \quad (29)$$

where the  $\pm$  signs are the same on all functions with the same frequency argument, and the sum is on both  $\Omega' (\neq \Omega)$  and the two independent sets of  $\pm$  signs.

Since  $\sigma(\Omega)$  is relatively small outside the unstable frequency range, it is clear from (29) that an unstable mode is most influenced by modes outside of this range. For large  $\Omega$  the coupling coefficient  $(\Omega/\sigma)$  approaches the value  $(\Omega/i\mu^{1/2})$  - a large imaginary value. For  $1 \gg \Omega$ , this coefficient obtains a maximum value of  $\mu^{-1/2}$  for  $\Omega=0(\mu^{1/2})$  - that is, a large real value. To determine the location of this maximum, assume that  $1-\mu \gg \Omega \gg \mu$ , and obtain for the real part of  $(\Omega/\sigma)$

$$\text{Re}(\Omega/\sigma) \simeq (2\mu)^{1/2} \left[ \frac{2\Omega^2(1-3\Omega^2) + \frac{1}{2}\mu^2}{\Omega^2(1-2\Omega^2) + \mu^2} \right]^{1/2} \simeq \mu^{-1/2} \left[ 1 - \Omega^2 - \frac{3}{4} \frac{\mu^2}{\Omega^2} \right]^{1/2} \quad (30)$$

This has its maximum value at

$$\Omega^2 = \frac{\sqrt{3}}{2} \mu \quad \text{or} \quad \Omega = .931 \mu^{1/2} \quad (31)$$

It has been assumed in this derivation that  $\mu^{1/2} \gg \mu$ . In this case the coupling of an unstable mode ( $\Omega \simeq 1$ ) to the mode (31) can only be done by a third mode whose frequency,  $\Omega' \sim 1/\sqrt{\mu}$ , lies outside the most unstable frequency range  $1 + \frac{1}{2}\mu \gtrsim \Omega \gtrsim 1 - 2\mu$ . This implies that two unstable modes do not couple to the low frequency mode (31). Instead they couple to a

mode of frequency  $\Omega = 0(\mu)$ . If, however, one sets  $\Omega = s\mu$ , ( $s \gg \mu^{-\frac{1}{2}}$ ) then one finds for the coupling constant

$$\frac{\Omega}{\sigma_+} \simeq \left( \frac{s}{2\mu(s^2+1)} \right)^{\frac{1}{2}} \left\{ [s + (s^2+1)^{\frac{1}{2}}]^{\frac{1}{2}} - i[-s + (s^2+1)^{\frac{1}{2}}]^{\frac{1}{2}} \right\} \quad (32)$$

which can still be comparable with the maximum value of  $\mu^{-\frac{1}{2}}$  (e.g., if  $\Omega = \mu$ ,  $\text{Re}(\Omega/\sigma) \simeq .77 \mu^{-\frac{1}{2}}$ ). Thus, while the coupling of two unstable modes to a low frequency mode is not as efficient as the coupling to a high frequency mode, it might play a role in stabilizing the unstable modes. We might finally note that for  $\Omega^2 = 1-s\mu$  ( $s \ll \mu^{-\frac{1}{2}}$ ), the coupling constant equals

$$\frac{\Omega}{\sigma_+} = 2^{-\frac{1}{2}} \left\{ [s + (s^2+1)^{\frac{1}{2}}]^{\frac{1}{2}} - i[-s + (s^2+1)^{\frac{1}{2}}]^{\frac{1}{2}} \right\}$$

which is relatively small.

Since the harmonic coupling is relatively strong, we next consider the probable influence of such coupling on an unstable mode. From (29) we obtain the simplified equations

$$\frac{d}{dz} M(\Omega) \simeq \sigma(\Omega) M(\Omega) + i \frac{4\Omega\sigma(\Omega)}{\sigma(2\Omega)} M(2\Omega)M^*(\Omega) \quad (33)$$

$$\frac{d}{dz} M(2\Omega) \simeq \sigma(2\Omega) M(2\Omega) + 2i \Omega M^2(\Omega)$$

where now  $\Omega \simeq 1$  (an unstable mode). For nonlinear equations of this form, it is not difficult to show that there is always a singular solution, regardless of the values of the (complex) nonlinear coefficients. Whether or not an initial state  $M(n\Omega, z=0) \simeq 0$  becomes singular apparently depends

on the influence of the linear terms (as in (23) and the first example in footnote (13)). Despite the apparent simplicity of the four equations (33), no conditions for bounded solutions have been found to date.

### 5. Computer Solutions

In order to obtain information concerning the dynamics of more than three coupled modes, a number of computer calculations were performed. The basic equations used were (12,13), except that the terms

$$- \Omega \Gamma N(\Omega) \quad , \quad - \Omega \Gamma V(\Omega) \quad (34)$$

were appended to the right side of (12) and (13) respectively. The introduction of the function  $\Gamma$  is to simulate the damping of the harmonic modes (e.e.,  $\Omega \gg 1$ ) due to their resonant interaction with the beam particles. This modification of the equations (which, of course, is not unique), yields the linear result (17) provided that

$$\Gamma = (\mu/\alpha)^{\frac{1}{2}} (\Delta v_b/u_o)$$

For these computations the value  $\mu = 0.1$  was used, and values of  $\Gamma$  from zero to 2.5 were considered. Since distances are described by  $z$ , equation (11), the values of  $\alpha$  and  $(\Delta v_b/u_o)$  do not have to be specified. For  $\mu = 0.1$ , the most unstable mode occurs for  $\Omega \simeq 1 - \frac{1}{2}(\mu/3^{\frac{1}{2}}) \simeq .97$ , which has a growth rate  $\sigma = 3^{3/4}/8^{\frac{1}{2}} = .806$ . Equations (12-13) are solved, and the amplitudes

$$A(z, \Omega) \equiv |N(z, \Omega)|^2$$

are obtained and plotted in the following figures. It should be emphasized at the outset, that much of what is shown in these figures has nothing to do with physics, but only with mathematics. This is the case when any of the  $A(z, \Omega)$  have a value larger than unity, since this may correspond to negative densities.

Figures (2-4) illustrates the influence of the linear damping of the harmonics on the instability in the case where only five modes are considered:

$$\Omega_1 = .94 \quad , \quad \Omega_2 = .97 \quad (\text{two unstable modes})$$

$$\Omega_3 = 2\Omega_1 \quad , \quad \Omega_4 = 2\Omega_2, \quad \Omega_5 = \Omega_1 + \Omega_2$$

In Figure 2, the harmonic damping is zero ( $\Gamma=0$ ), and one clearly sees that the harmonic generation enhances the instability leading (probably) to an explosive instability. In Figure 3, the damping is taken to be  $\Gamma=1.5$ , and the harmonic waves grow more slowly. The less unstable mode ( $A_2$ ) saturates, and its harmonic and the sum mode ( $A_5$ ) saturate and decay. While the explosive nature of the instability is apparently removed, the unstable behavior of  $A_1$  and its harmonic,  $A_3$ , is not controlled. If the damping is further increased to  $\Gamma=2.5$ , as in Figure 4, the quantitative features are changed, but the instability is still not controlled. In particular, the linearly damped harmonic of  $A_1$  is still unstable.

From results such as shown in Figures (2-4) one can draw only limited conclusions. Namely one can not conclude that mode coupling does not control instabilities, but only that the present five mode system is

unstable. The natural question then is, does the inclusion of many modes help to produce stability? This has been examined with many examples, and the apparent answer is no. Thus, Figure 2 has been essentially reproduced in the case of five unstable modes

$$(\Omega_1, \dots, \Omega_5) = .88, .91, .94, .97, \text{ and } 1.0$$

coupled with all the sum frequencies (six more modes), which includes the harmonics of the most unstable modes.

The next case to be studied is when higher order modes are included, together with damping (which increases linearly with  $\Omega$ , according to (17)). An example of such a computation is shown in Figure 5. In this case two unstable modes are coupled to the harmonic and sum modes and also the second harmonic and sum modes - nine modes in all. Note that only the nonlinear region of  $z$  has been plotted, starting at  $z=3$ . In this case, even though  $\Gamma=2$  (so that the second harmonics have a strong linear damping, e.g.,  $A_9 \sim e^{-11.6z}$ ), the first ( $A_5$ ) and second ( $A_9$ ) harmonics of the most unstable mode ( $A_2$ ) do not saturate for  $z < 10$ . Moreover the second harmonic becomes larger than the first harmonic. This suggests that the possible shortcoming of these calculations is that a sufficient number of harmonics have not been included (to act as an energy sink).

To examine this point, a single unstable mode and five of its harmonics were considered. The result is shown in Figure 6. Since  $\Gamma=2$  again,  $A_6$  is linearly damped as  $e^{-23.5z}$ . Despite this fact one can see that, although the amplitudes of the first four harmonics are successively smaller, the last harmonic becomes abnormally large. While the system



appears to be bounded ( a point which can not be assured until larger distances are computed), the results appear to have little physical significance because of the large amplitudes (as noted above). Moreover the abnormal amplitude of the last harmonic is clearly not physically acceptable.

As a final exploratory example, the damping of the harmonics was changed to be  $\frac{1}{2} \Omega^3$  rather than  $\Gamma \Omega$ . The question was to see whether this increased damping would control the amplitude of the higher harmonic. Figure 7 illustrates the result of such a computation. In this case the same nine modes were considered as those in Figure 5 (the initial conditions, however, differed slightly). It can be seen that the highest harmonic is still driven to large amplitudes despite the increased damping.

While the results of computations of the present type cannot prove that mode coupling is ineffective in stabilizing the linear instability. All examples which have been examined indicate that this is the case. Obviously it would be of great interest to obtain some analytic method to substantiate this conclusion. None has been found to date.

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instability grows sufficiently rapidly (e.g., an "explosive" instability), the spatial variation of nonlinear terms will become of dominant importance in this spatial region.

- 11) For example see: "Boundedness and Stability", by H. A. Antosiewicz, in Nonlinear Differential Equations and Nonlinear Mechanics, Edited by J. P. LaSalle and S. Lefschetz (Academic Press, 1963), p. 259.
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$$\dot{x}_1 = x_1(1-x_2^2) - x_2 \quad ; \quad \dot{x}_2 = x_2(1-x_2^2) + x_1$$

which is easily solved using the variables  $x_1 = r \cos\theta$ ,  $x_2 = r \sin\theta$ .

Note that in the region  $|x_2| < 1$  the motion is "unstable", i.e.  $\dot{r} > 0$ ,

but the stabilizing region  $|x_2| > 1$  always controls this instability.

This boundedness is caused by an interplay between the nonlinear terms and the linear terms (which cause  $\dot{\theta} = 1$ , hence forcing the trajectory to periodically enter the stabilizing region).

Solvable examples in more than two variables are more difficult to concoct. One example is

$$\dot{x}_1 = x_1 - x_2 - \beta x_1 x_3 \quad ; \quad \dot{x}_2 = x_2 - x_1 - \beta x_2 x_3$$

$$\dot{x}_3 = x_3 + \beta(x_1^2 + x_2^2) - c \beta x_3^2$$

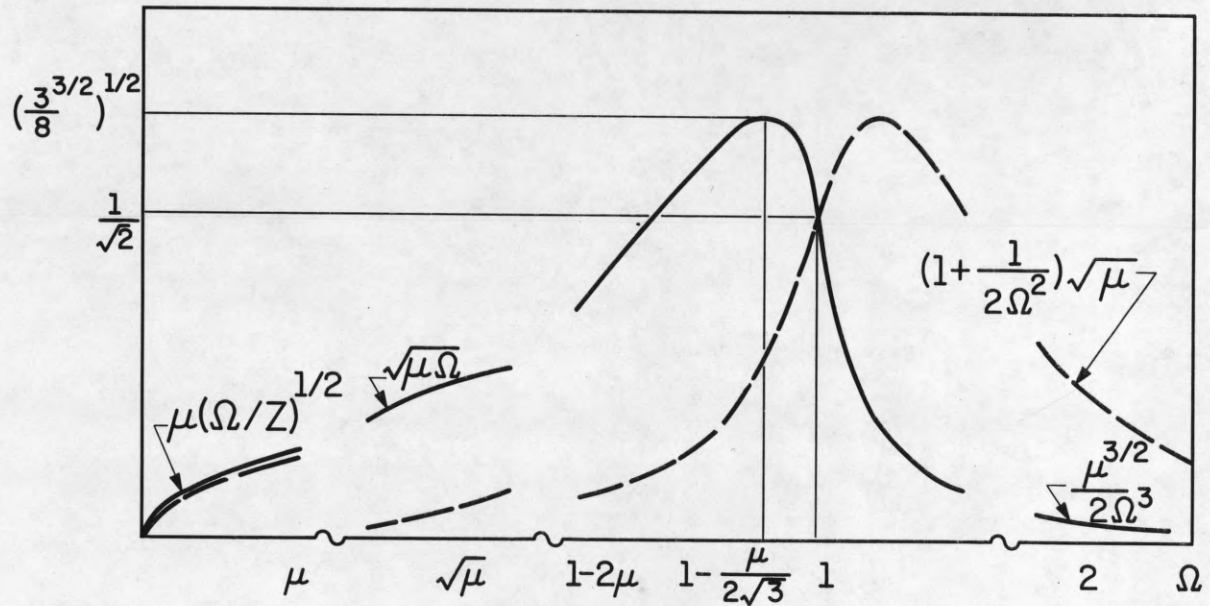
If one introduces the variables

$$x_1 = r \cos\theta \cos\phi, \quad x_2 = r \sin\theta \cos\phi, \quad x_3 = r \sin\phi$$

(note  $\pi/2 \geq \phi \geq -\pi/2$ ) then

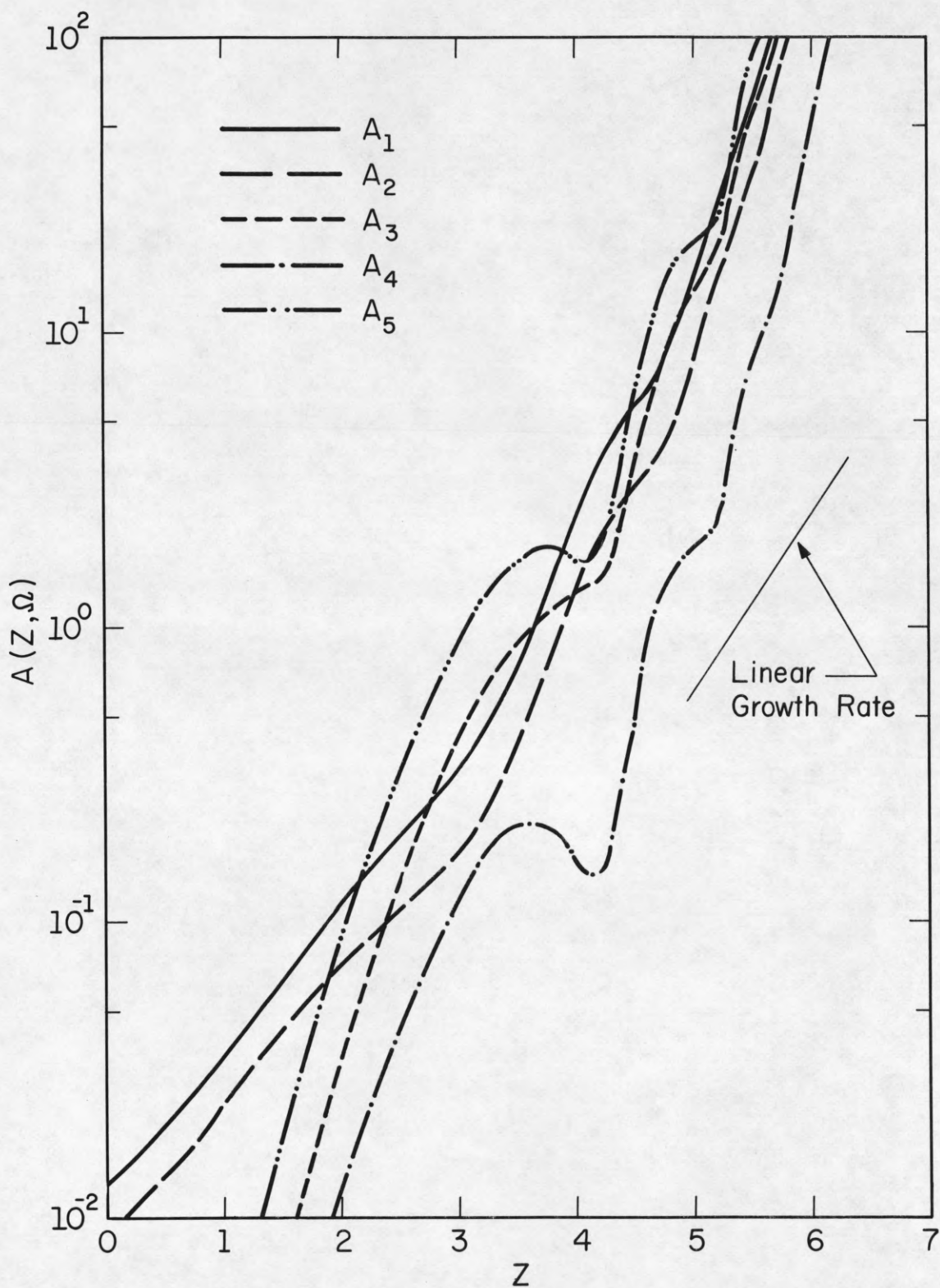
$$\dot{r} = r(1 - c\beta r \sin^3\phi), \quad \dot{\phi} = \beta r \cos\phi(1 - c \sin^2\phi), \quad \dot{\theta} = 1$$

These equations are, readily integrable. It is clear that, if  $c > 0$ , all solutions are bounded regardless of the sign of  $\beta$ , and if  $c > 1$  there are no stable stationary points (Note: if  $1 > c > 0$ ,  $\phi \rightarrow \text{sign}(\beta) \pi/2$  whereas, if  $c > 1$ ,  $\phi \rightarrow \text{sign}(\beta) \sin^{-1}(c^{-1/2})$ ).



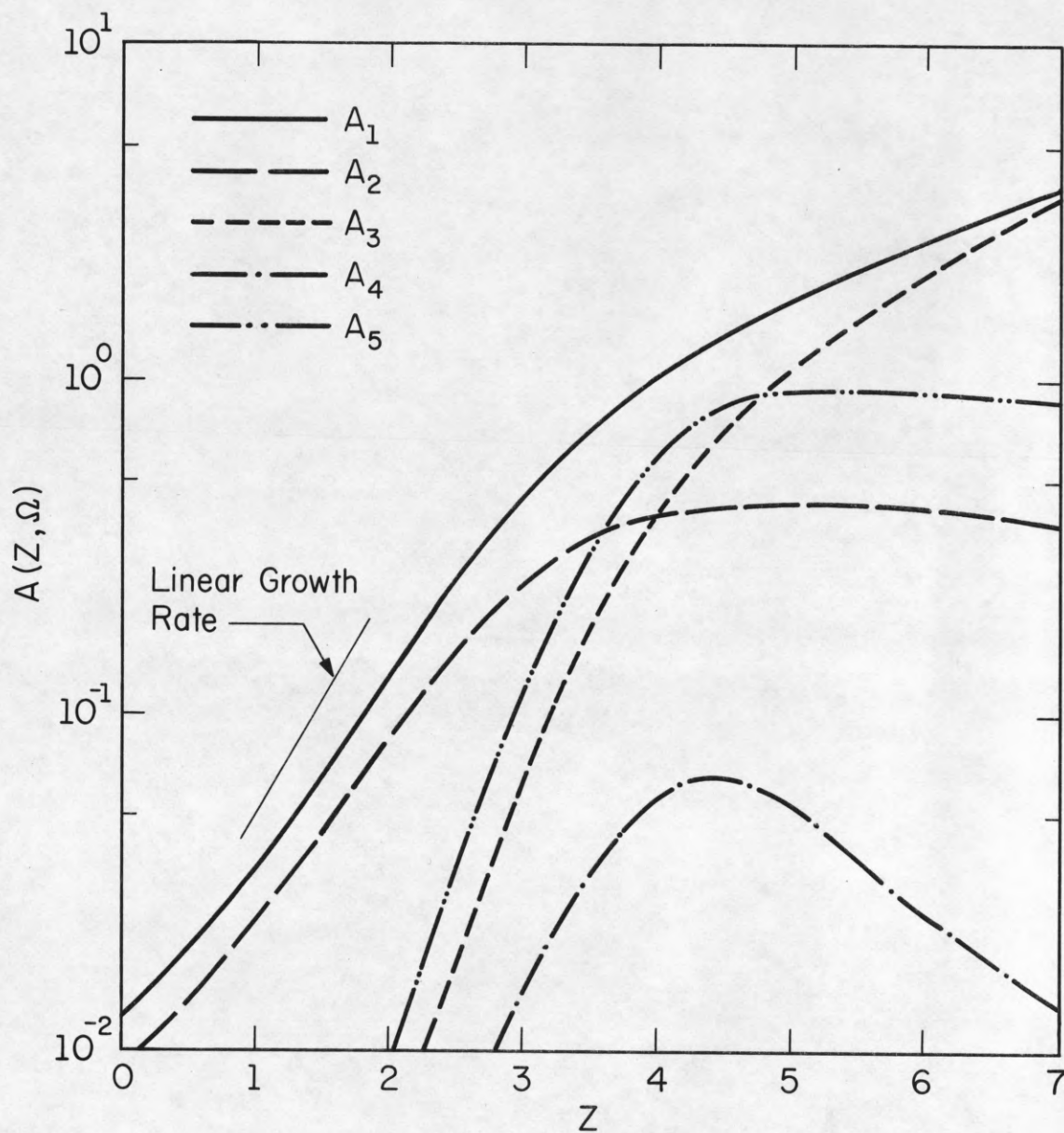
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Figure 1. The real (—) and imaginary (---) part of  $\sigma_+(\Omega)$ , equation (15), for four regions of  $\Omega$ .



PP-776

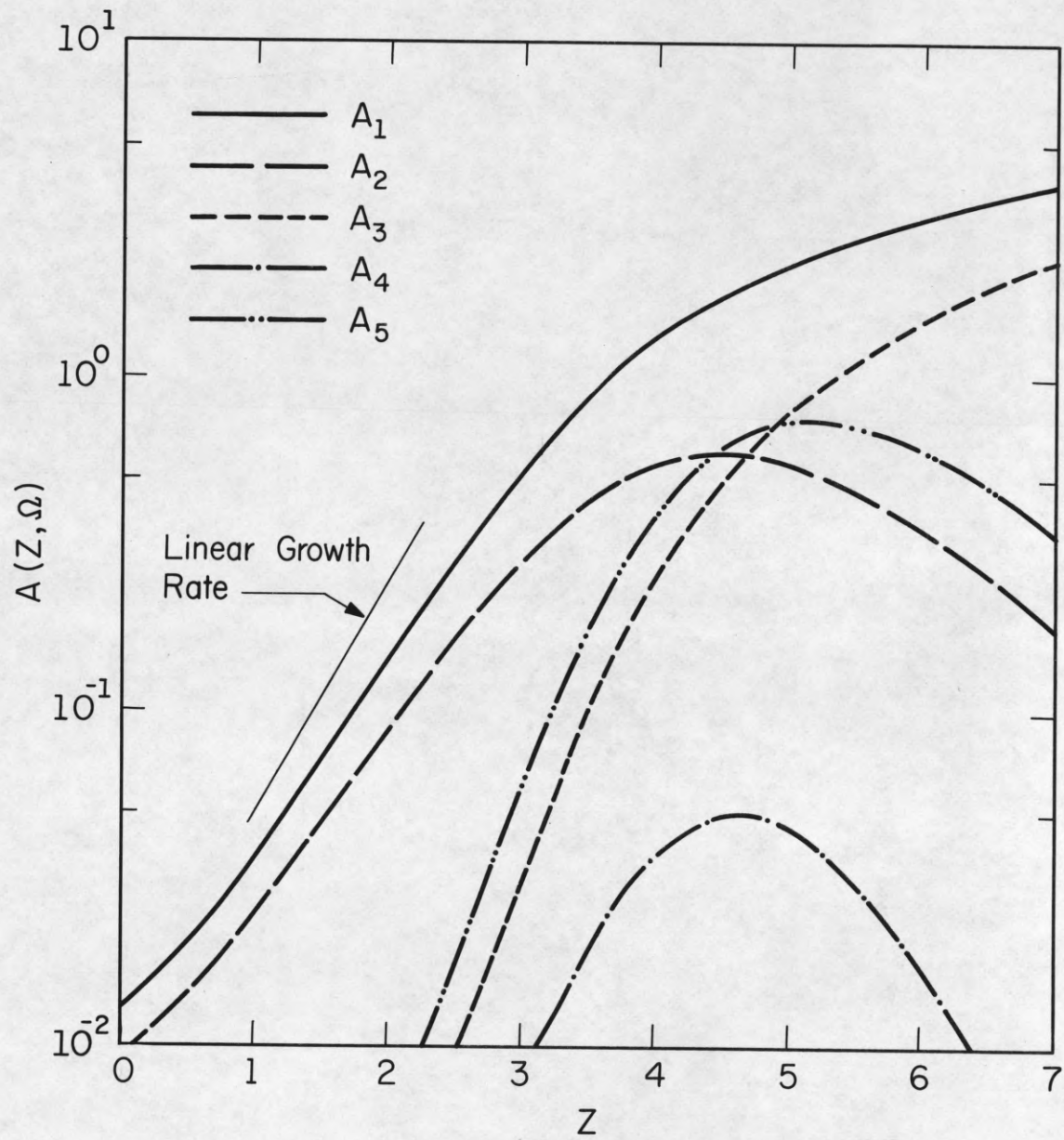
Figure 2. A solution of  $A(\Omega, z) = |N(\Omega, z)|^2$ , from equations (12-13, for the frequency components  $\Omega_1 = .94$ ,  $\Omega_2 = .97$ ,  $\Omega_3 = 2\Omega_1$ ,  $\Omega_4 = 2\Omega_2$ ,  $\Omega_5 = \Omega_1 + \Omega_2$ . The solution has the character of an "explosive instability" (singular solution).



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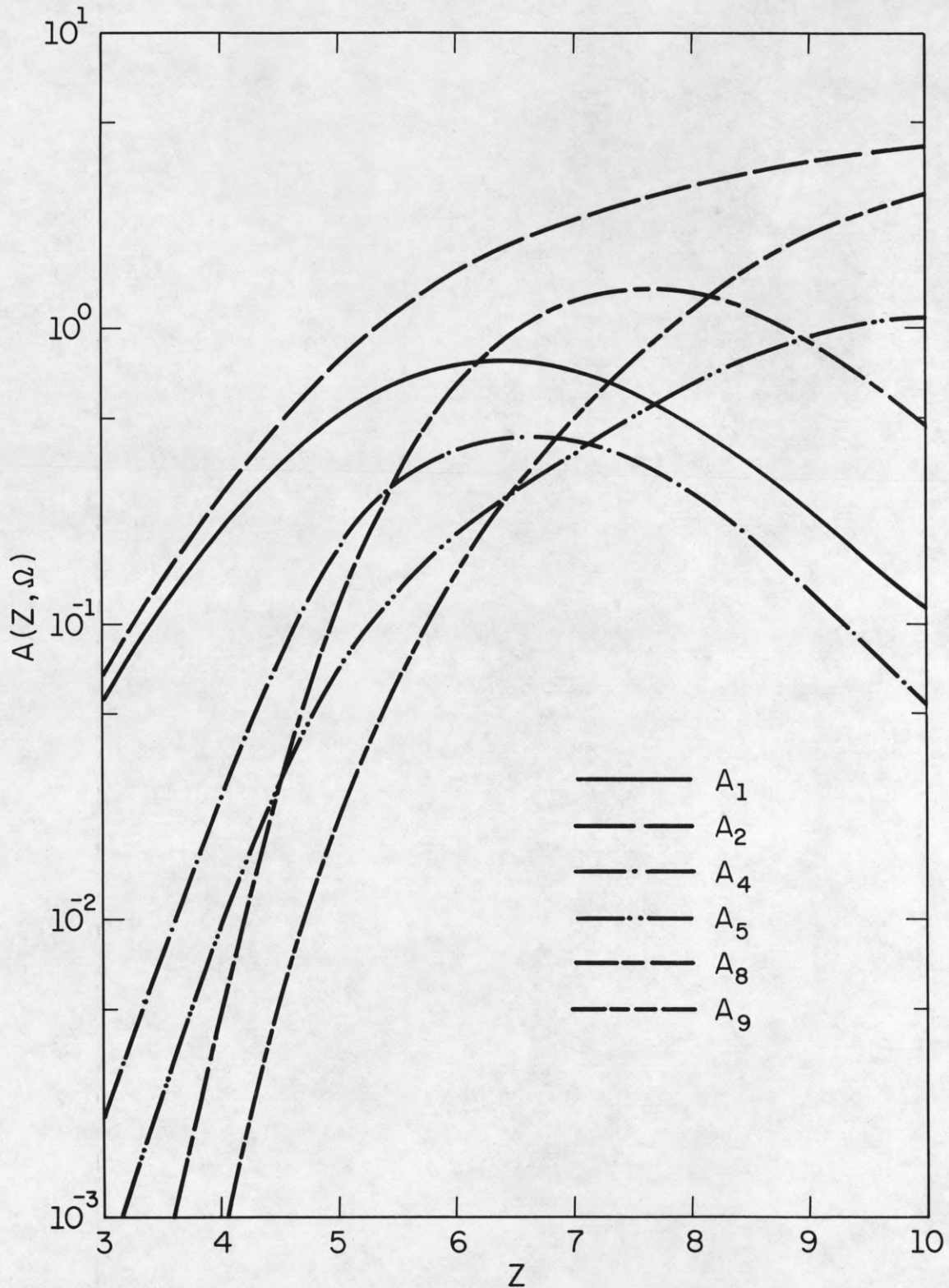
Figure 3. The same case as in Figure 1, except that the harmonics are damped according to (34), with  $\Gamma = 1.5$ . The singular character is apparently removed, but the most unstable mode and its harmonic are still essentially unbounded.





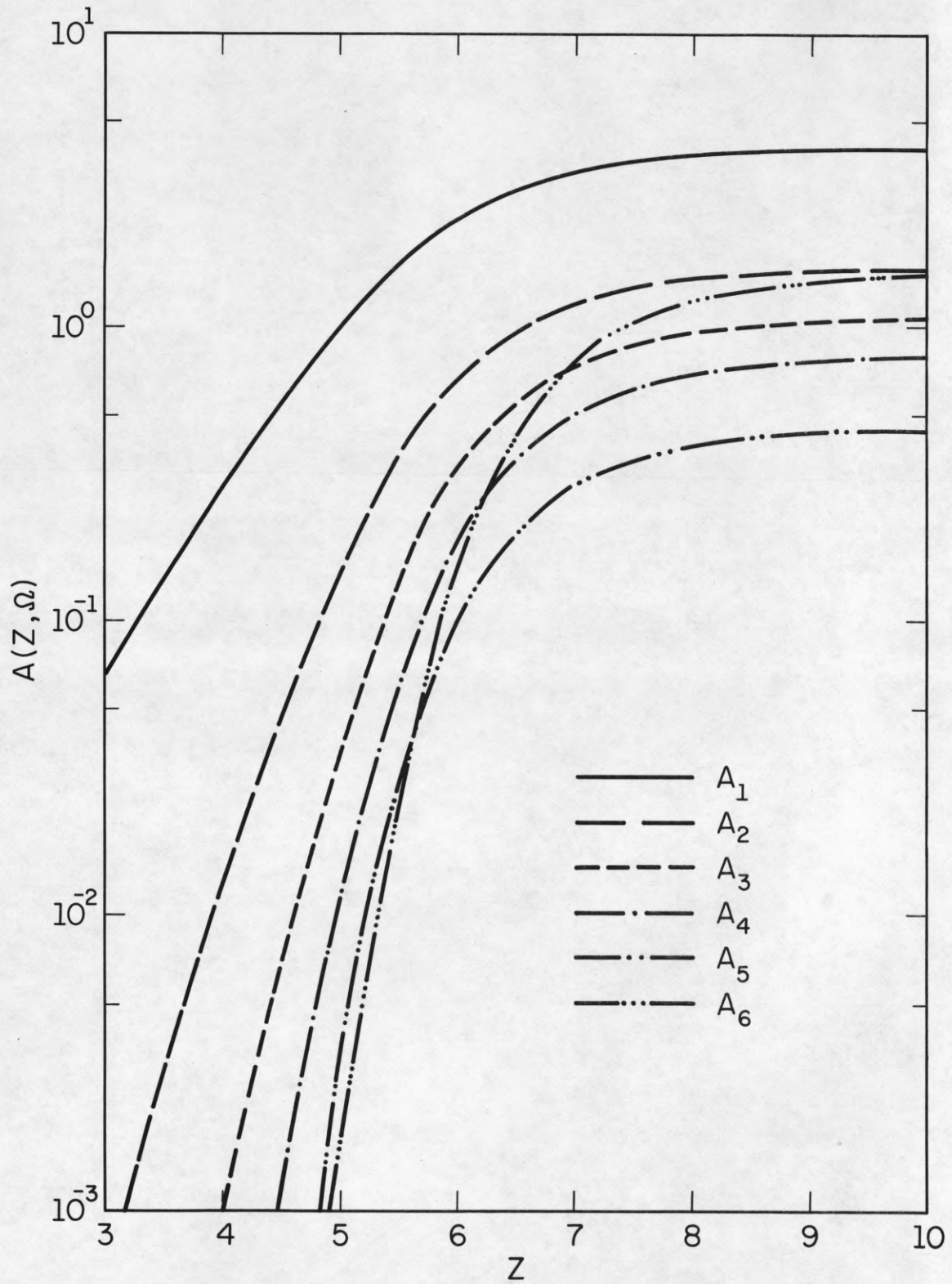
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Figure 4. The same as in Figure 2, except that  $\Gamma = 2.5$ . The unbounded nature of the solution persists even with this increased damping.



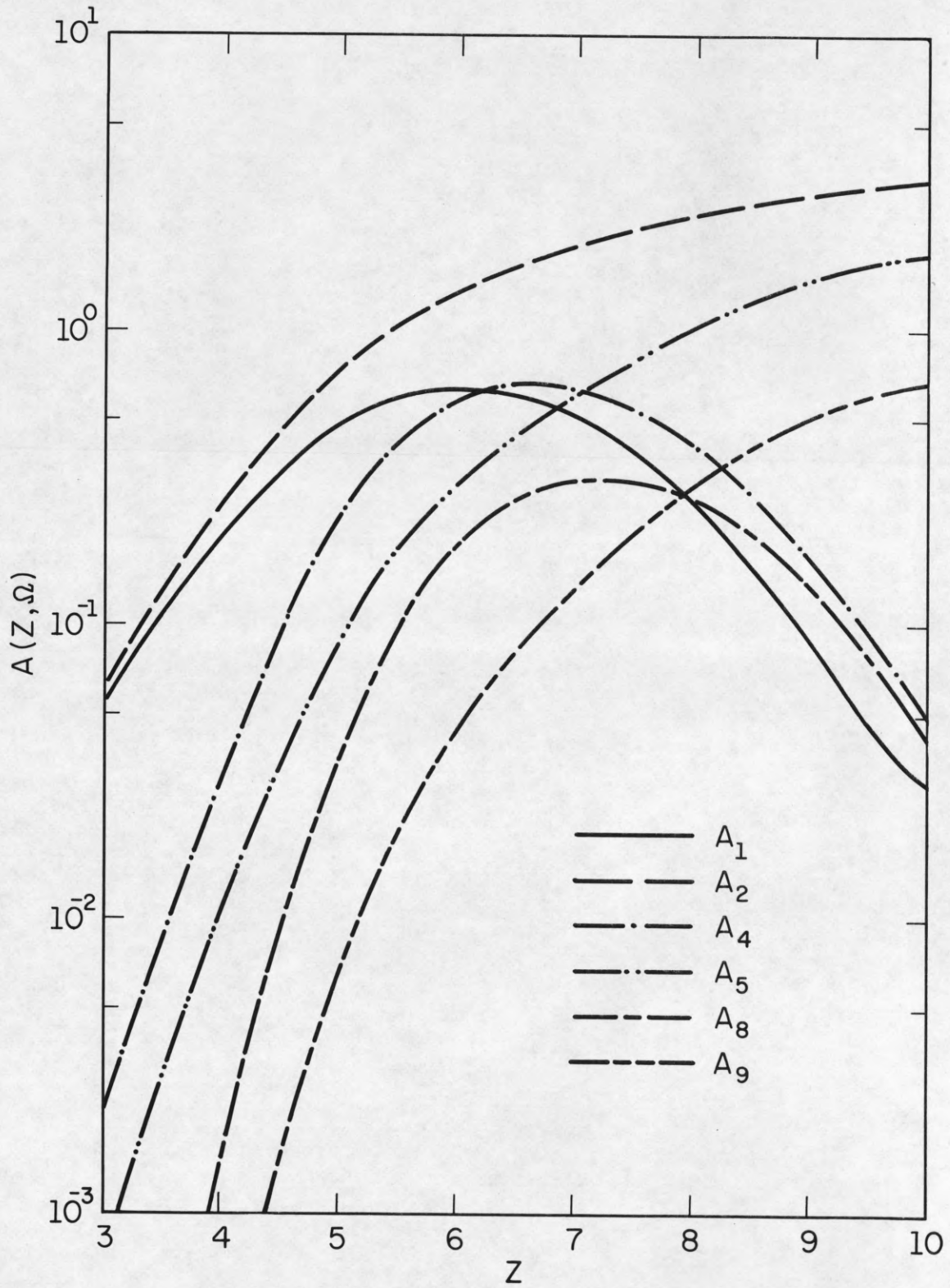
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Figure 5. A solution of (12,13,34) with  $\Gamma = 2$ , for the case of nine frequency components:  $\Omega_1 = .94$ ,  $\Omega_2 = .97$ ,  $\Omega_3 = 2\Omega_1$ ,  $\Omega_4 = \Omega_1 + \Omega_2$ ,  $\Omega_5 = 2\Omega_2$ ,  $\Omega_6 = 3\Omega_1$ ,  $\Omega_7 = 2\Omega_1 + \Omega_2$ ,  $\Omega_8 = \Omega_1 + \Omega_2$ ,  $\Omega_9 = 3\Omega_2$ . The inclusion of the second harmonics does not produce a bounded solution.



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Figure 6. An unstable mode coupled to five of its harmonics ( $\Omega_1 = 97$ ,  $\Omega_k = k\Omega_1$ ,  $k = 2, \dots, 6$ ) with  $\Gamma = 2$ . The last harmonic becomes abnormally large despite the strong linear damping.



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Figure 7. The same components as in Figure 5, but with  $\Gamma = \frac{1}{2} \Omega^2$ , producing a stronger damping of the higher harmonics. The second harmonic ( $A_9$ ) still obtains a large amplitude.

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