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Abstract

Using as logic modules two-input one-output arbitrary logic gates, this paper considers the problem of the longest chain (number of levels) in a tree-type interconnection realizing a Boolean function of n variables. Specifically, we are interested in the minimum number of levels L(n) by which we can constructively realize all Boolean functions of n variables. It was previously shown that $L(n) \leq n$ for $n = 3,4$ and it was so conjectured for $n = 5$; in this paper we are able to show that this holds for $n = 5, 6, 7$ and conjecture that $L(8) \leq 8$.

On the Delay Required to Realize Boolean Functions*

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The problem of the delay required to realize switching functions has received some attention in the very recent past [1,2]. Specifically, given a class of logical modules to be used for network synthesis, the delay (sometimes called the computation time) of a network realizing a Boolean function is the largest number of modules in any network path connecting the input terminals to the output terminal. This largest number of modules will be called the number of levels. We denote by $L(n)$ the largest number of levels required to realize all Boolean functions of n arguments with a given class of modules. Upper and lower bounds have been obtained by rather straightforward counting arguments [1,2].

A more interesting problem, however, is the development of constructive procedures achieving a given $L(n)$. Following the model proposed in [1], we shall consider tree-type interconnections of arbitrary two-input one-output logic gates. We then recall that if the input variables are available in true and complemented form, only AND, OR, and EXCLUSIVE-OR gates need be considered. In [1] it was shown that, for this class of modules, $L(n) = n$ for $n = 3,4$; and it was conjectured that the same statement was true for $n = 5$. In this paper we show that $L(n) \leq n$ for $n = 5, 6, 7$ and conjecture that the same holds for $n = 8$. This then implies that

$$L(n) \leq n+1 \text{ for } n \leq 135$$

$$L(n) \leq n+2 \text{ for } n \leq 2^{135}+135$$

and so on.

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A Boolean function $f(x_1,\ldots,x_n)$ is said to be p-realizable if it can be realized by a tree network of the type described with at most p levels. The main result is based on the following theorem.

Theorem - Every function $f(x_1,x_2,x_3,x_4)$ admits one of the two following decompositions[1]:

$$(1)\quad f = \bar{x}_j f_o + x_j f_1$$

with f_o and f_1 both 2-realizable, or

$$(2)\quad f = \psi(x_1,\ldots,x_4)\varphi(x_1,\ldots,x_4)$$

with φ 3-realizable and ψ 2-realizable.

Proof: The argument is lengthy and detailed but very simple. We first recall [1] that all equivalence classes[2] of functions of three variables are 2-realizable except for the following three classes (Figure 1), which are 3-realizable:

$$h_1 = (z+\bar{x}y)(\bar{x}+y) \qquad h_2 = (z\oplus xy)(\bar{x}+y) \qquad h_3 = (x\oplus y\oplus z) + \bar{x}\bar{y}$$

$$\underset{y}{\underset{x}{\boxed{\begin{array}{cccc}0&1&0&0\\1&1&1&0\end{array}}}}\,z \qquad \boxed{\begin{array}{cccc}0&1&0&0\\1&0&1&0\end{array}} \qquad \boxed{\begin{array}{cccc}1&1&0&1\\1&0&1&0\end{array}}\ .$$

Figure 1

Let us now expand a function $f(x_1,x_2,x_3,x_4)$ as

$$f(x_1,x_2,x_3,x_4) = \bar{x}_j f_o + x_j f_1$$

---

[1] In the sequel, + and ⊕ denote OR and EXCLUSIVE-OR, respectively.

[2] By equivalence class of Boolean functions is meant the set of all functions which are obtained from a member of the class by permuting or complementing the input variables.

where f_o and f_1 are functions of three variables. If for some $j = 1,2,3,4$, both f_o and f_1 do not belong to the classes h_1, h_2 or h_3 then the statement is proved.

Assume the contrary. We must generate all the equivalence classes of 4-variable functions for which condition$^{(1)}$ is not verified. To this end we resort to the Karnaugh map representation of 4-variable functions (Figure 2) and use the following procedure:

1. Select the two lower rows of the map to represent either h_1 or h_2 or h_3. (Figure 1)
2. Select abcd so that the 1st and 4th rows form either h_1 or h_2 or h_3. (Select efgh so that the 2nd and 3rd rows form either h_1 or h_2 or h_3.)
3. For all choices of efgh (abcd) test whether decomposition (1) is possible with respect to either x or y.

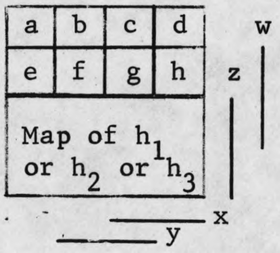

Figure 2

Clearly for the functions obtained in Step 1 and 2 decomposition (1) is not possible either with respect to z or w. Hence, the functions which fail the test of Step 3 must be analyzed.

Steps 1 and 2 yield the maps illustrated in figure 3.

$$\begin{bmatrix} 0&1&0&0\\ e&f&g&h\\ 0&1&0&0\\ 1&1&1&0 \end{bmatrix}$$

1)

$$\begin{bmatrix} 0&1&0&1\\ e&f&g&h\\ 0&1&0&0\\ 1&1&1&0 \end{bmatrix}$$

2)

$$\begin{bmatrix} a&b&c&d\\ 1&0&1&0\\ 0&1&0&0\\ 1&1&1&0 \end{bmatrix}$$

3)

$$\begin{bmatrix} a&b&c&d\\ 1&1&1&0\\ 0&1&0&0\\ 1&1&1&0 \end{bmatrix}$$

4)

$$\begin{bmatrix} 0&0&0&1\\ e&f&g&h\\ 0&1&0&0\\ 1&0&1&0 \end{bmatrix}$$

5)

$$\begin{bmatrix} 0&1&0&0\\ e&f&g&h\\ 0&1&0&0\\ 1&0&1&0 \end{bmatrix}$$

6)

$$\begin{bmatrix} 0&1&1&1\\ e&f&g&h\\ 0&1&0&0\\ 1&0&1&0 \end{bmatrix}$$

7)

$$\begin{bmatrix} a&b&c&d\\ 1&0&1&0\\ 0&1&0&0\\ 1&0&1&0 \end{bmatrix}$$

8)

$$\begin{bmatrix} a&b&c&d\\ 1&1&1&0\\ 0&1&0&0\\ 1&0&1&0 \end{bmatrix}$$

3A)

$$\begin{bmatrix} 1&1&0&1\\ e&f&g&h\\ 0&1&0&0\\ 1&0&1&0 \end{bmatrix}$$

7A)

$$\begin{bmatrix} 0&1&1&1\\ e&f&g&h\\ 1&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

9)

$$\begin{bmatrix} 1&1&0&1\\ e&f&g&h\\ 1&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

10)

$$\begin{bmatrix} a&b&c&d\\ 1&0&0&0\\ 1&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

11)

$$\begin{bmatrix} a&b&c&d\\ 1&0&1&0\\ 1&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

12)

$$\begin{bmatrix} 0&0&0&1\\ e&f&g&h\\ 1&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

7B)

$$\begin{bmatrix} 0&1&0&0\\ e&f&g&h\\ 1&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

7C)

Map 3A is equivalent to map 3, and maps 7A, 7B, 7C are equivalent to map 7; therefore they will not be considered in the following analysis. Maps 1, 2, 4, 6, 8, 9, 10, 11, 12 can be analyzed collectively. Let φ_j denote the function obtained by setting equal to 1111 the unspecified parameters in map j. Then any function f obtained from assigning these parameters can be expressed as $f = \psi \cdot \varphi_j$, where $\psi = \xi(x,y) + \chi(w,z)$ for some ξ and $\chi = z + \bar{w}$ for $j = 1,2,6,9,10$, and $\chi = z + w$ for $j = 4,8,11,$ 12. Since both χ and ξ are 1-realizable, $\chi + \xi$ is 2-realizable. Moreover the functions

$$\varphi_1 = (x\bar{y} + \bar{z}w) + \bar{w}z(\bar{x}+y)$$

$$\varphi_2 = [(x\oplus y)+(z\oplus w)][\bar{x}+y+\bar{z}]$$

$$\varphi_4 = (\bar{x}+y)(\bar{z}+\bar{w}) + \bar{x}y + \bar{z}\bar{w}$$

$$\varphi_6 = [(z\oplus w)+\bar{x}y][w+\bar{z}+(\bar{x}\oplus y)]$$

$$\varphi_8 = [(\bar{x}\oplus y)+(\bar{z}\oplus w)](\bar{w}+\bar{z}+\bar{x}y)$$

$$\varphi_9 = [x\oplus y\oplus z\oplus w] + \bar{x}w + x\bar{z}$$

$$\varphi_{10} = [\bar{x}+\bar{y}+(z\oplus w)][\bar{z}+w+(\bar{x}\oplus y)]$$

$$\varphi_{11} = [(\bar{x}\oplus y) + (\bar{z}\oplus w)](\bar{x}+\bar{y}+\bar{w})$$

$$\varphi_{12} = [x\oplus y\oplus z\oplus w] + \bar{x}\bar{y} + \bar{z}\bar{w}$$

are 3-realizable, whence decomposition (2) obtains.

There remains to examine maps 3, 5, and 7 of figure 3. We consider these three cases separately.

A) Map 3. Decomposition (1) cannot be obtained for the following choices of abcd: (0001, 0011, 0100, 0101, 0111, 1001, 1011, 1101, 1111). We denote each resulting function as φ_{3i} where i is the integer spelled by abcd. Then for i = 3,7,9,11,13,15 $\varphi_{3i} = \varphi_4 \circ \tau_{3i}$, where φ_4 has been defined above and τ_{3i} is 2-realizable. Specifically we have:

$$\tau_{3,3} = w\bar{y}+x+z \quad \tau_{3,7} = (\bar{y}\oplus \bar{w})+x+z \quad \tau_{3,9} = xw+\bar{y}+z$$

$$\tau_{3,11} = \bar{y}+x+z \quad \tau_{3,13} = (x\oplus \bar{w})+\bar{y}+z \quad \tau_{3,15} = x+\bar{y}+z+\bar{w}$$

Hence decomposition (2) is obtained. The remaining φ_{31}, φ_{34}, φ_{35} can be expressed as follows

$$\varphi_{3,1} = [(x\oplus y\oplus z\oplus w)\oplus \bar{x}y\bar{w}][\bar{x}+y+\bar{z}]$$

$$\varphi_{3,4} = [(\bar{x}\oplus y)(z\oplus w) + \bar{x}y(z+\bar{w})]\circ 1$$

$$\varphi_{3,5} = [(x\oplus y\oplus z\oplus w) + yz]\circ[\bar{x}+\bar{z}+\bar{w}]$$

i.e., they all admit of decomposition (2).

B) Map 5. Decomposition (1) cannot be obtained for (e,f,g,h) = (0010, 1000, 1010, 1011, 1110). Then for i = 2,8,10,14 $\varphi_{5i} = [(x\oplus y\oplus z\oplus w)\oplus \xi_i] \circ (\bar{x}+y+\bar{z})$, i.e., a decomposition of type (2) if ξ_i is at most 2-realizable. Indeed

$$\xi_2 = (w\oplus y)\bar{x}\bar{z} \qquad \xi_8 = (x\oplus \bar{w})y\bar{z}$$
$$\xi_{10} = \bar{x}y\bar{w}\bar{z}, \qquad \xi_{14} = \bar{x}y\bar{z} \;.$$

The remaining $\varphi_{5,11}$ admits of decomposition (2), that is

$$\varphi_{5,11} = [(x\oplus y\oplus z\oplus w) \oplus x\bar{y}w]\cdot[x+\bar{y}+w].$$

C) Map 7. Decomposition (1) cannot be obtained for (e,f,g,h) = (0010, 0011, 1010, 1100, 1110). For i = 12, 14 $\varphi_{7,i} = [(x\oplus y\oplus z\oplus w) + \bar{z}(x+y)]\circ \tau_{7,i}$ where $\tau_{7,i}$ is 2-realizable, thereby yielding decomposition (2). Indeed

$$\tau_{7,12} = \bar{x}+\bar{w} \quad , \quad \tau_{7,14} = \bar{x}+y+\bar{w}.$$

For i = 2,3,10 $\quad \varphi_{7,i} = [(x\oplus y\oplus z\oplus w) + \bar{z}x]\cdot \tau_{7,i}$ and

$$\tau_{7,2} = y+\bar{w} \quad , \quad \tau_{7,3} = y+\bar{w}+x\bar{z} \quad , \quad \tau_{7,10} = \bar{x}+y+\bar{w}.$$

This concludes the proof.

The previous proposition has a very interesting consequence. Let f be a function of 7 variables $x_1,x_2,\ldots,x_7$. We expand it as follows:

$$(3) \quad f = \bar{x}_1\bar{x}_2\bar{x}_3f_o + x_1\bar{x}_2\bar{x}_3f_1 + \bar{x}_1x_2\bar{x}_3f_2 + x_1x_2\bar{x}_3f_3 + \bar{x}_1\bar{x}_2x_3f_4 + x_1\bar{x}_2x_3f_5$$
$$+ \bar{x}_1x_2x_3f_6 + x_1x_2x_3f_7.$$

Consider now the generic f_r, $r = 0,1,\ldots,7$. Either f_r admits of the disjunctive decomposition (1) or of the conjunctive decomposition (2). In the former case, let $j = 7$ for illustrative purposes. Then

$$x_1x_2x_3f_7 = x_1x_2x_3\bar{x}_jf_{70} + x_1x_2x_3x_jf_{71}$$

and since both f_{70} and f_{71} are 2-realizable, $x_1x_2x_3\bar{x}_jf_{70}$ is 3-realizable, and $x_1x_2x_3f_7$ is 4-realizable as shown by the network in Figure 4.

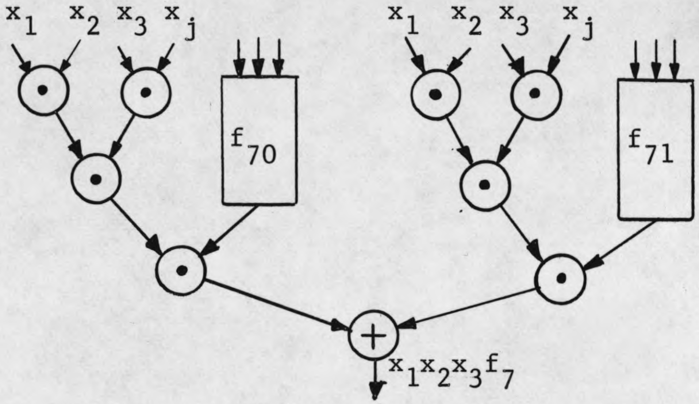

Figure 4. Network realizing the disjunctive decomposition of $x_1x_2x_3f_7$.

In the second case we have

$$x_1x_2x_3f_7 = x_1x_2x_3\ \psi\varphi$$

with φ and ψ 3- and 2-realizable, respectively. The network realizing $x_1x_2x_3f_7$ is given in Figure 5 and contains four levels. In all cases then

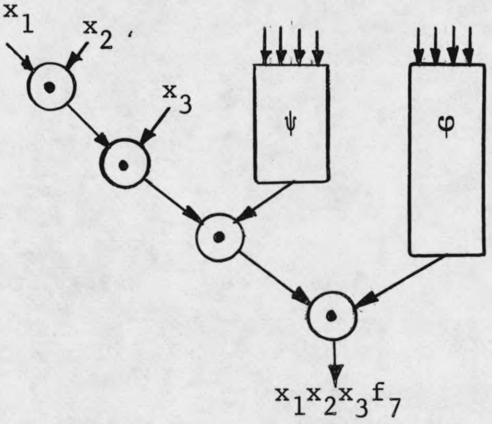

Figure 5. Network realizing the conjunctive decomposition of $x_1x_2x_3f_7$.

any term of the expansion (3) is 4-realizable. The subsequent OR-network required for the synthesis of f is 3-realizable. Clearly, the expansion[3] can be given for functions of 5 and 6 variables, which leads to the following conclusion:

Corollary: Every Boolean function of n variables is n-realizable for $n \leq 7$.

It must be noted that decomposition (1) is more restrictive than decomposition (2). In fact, if all 4-variable functions admitted of decomposition (2), then, as is readily obtained from Figure 5, n-realizability would be assured for $n \leq 8$. In all examples considered, no 4-variable function was encountered which did not admit of the conjunctive decomposition. For this reason, it is conjectured that also every function of 8 variables be 8-realizable. Exhaustive but tedious inspection of the 402 Harvard classes of the functions of 4 variables could finally settle this argument.

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Using as logic modules two-input one-output arbitrary logic gates, this paper considers the problem of the longest chain (Number of levels) in a tree-type interconnection realizing a Boolean function of n variables. Specifically, we are interested in the minimum number of levels $L(n)$ by which we can constructively realize all Boolean functions of n variables. It was previously shown that $L(n) \leq n$ for $n = 3,4$ and it was so conjectured for $n = 5$; in this paper we are able to show that this holds for $n = 5, 6, 7$ and conjecture that $L(8) \leq 8$.

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