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TRANSIENT RESPONSES OF BUTTERWORTH AND CHEBYSHEV FILTER NETWORKS

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Introduction

Although the general method for obtaining the transient response of any filter network has been established by the use of transformation calculus, a few response curves of commonly used filter networks are available in existing literature. In this report[1], transient responses of the Butterworth and the Chebyshev filter networks are calculated for both suddenly impressed and suddenly removed sinusoidal input signals with angular frequency ω_c. The transient responses of these networks are treated in terms of the in-phase and the quadrature components, and these functions together with envelope functions are presented under the assumption of narrow bandwidth for the following cases:

1. For the n-th degree Butterworth filter networks with mid-band angular frequency ω_0 and half-power (or 3db) transmission band $\omega_0 \pm \omega_b$, the cases of n = 3 to 7, $\omega_c = \omega_0$ and the cases of n = 3 to 5, $(\omega_c - \omega_0)/\omega_b = 0.2, 0.4, 0.6, 0.8, 1.0$, and half-gain points.

2. For the n-th degree Chebyshev filter networks with mid-band angular frequency ω_0 and A_p db transmission band $\omega_0 \pm \omega_b$, the cases of n = 3 to 7, $\omega_c = \omega_0$, and $A_p = 0.01, 0.03, 0.1, 0.3$, 1db, and the cases of n = 3 to 5, $A_p = 0.03$db and $(\omega_c - \omega_0)/\omega_b = 0.2, 0.4, 0.6, 0.8, 1.0$.

Many words could be devoted to discussion of these response curves, but only a few general considerations will be described in this paper in order to suggest ways in which the curves might aid the designer.

Preliminaries

Before getting into details, we being with a summary of assumptions, definitions and formulas which will be used in this paper.

Transfer function: Let T(s) be a transfer function whose transient responses are to be considered. For the Butterworth and Chebyshev bandpass filter networks of the n-th degree, we can write T(s) in the following form:

$$T(s) = \frac{1}{H_{2n}(s)} \tag{1}$$

where $H_{2n}(s)$ is a Hurwitz polynomial of degree 2n. $H_{2n}(s)$ can be expressed in terms of its simple roots $\{s_k\}_{k=1}^{2n}$ as follows:

$$H_{2n}(s) = K \cdot \prod_{k=1}^{2n} (s-s_k) \tag{2}$$

where s_k satisfies

$$\mathrm{Re}\,[s_k] < 0, \quad k = 1,2,3,\ldots,2n. \tag{3}$$

Transient response: Sudden impression of a sinusoidal signal can be expressed as

$$f_1(t) = \begin{cases} 0, & t < 0 \\ e^{j\omega_c t} & t > 0 \end{cases} . \tag{4}$$

The input signal results output signal $g_1(t)$ of the network defined by (1), and $g_1(t)$ is given by

$$g_1(t) = \begin{cases} 0, & t < 0 \\ E(\omega_c, t) \cdot \dfrac{e^{j\omega_c t}}{H_{2n}(j\omega_c)} , & t > 0 \end{cases} , \tag{5}$$

where

$$E(\omega_c,t) = 1 + H_{2n}(j\omega_c) \sum_{k=1}^{2n} \frac{1}{s_k - j\omega_c} \cdot \frac{e^{j(s_k - j\omega_c)}}{H'_{2n}(s_k)} \tag{6}$$

is the response function normalized with its final value. The function $E(\omega_c,t)$ is complex valued and its real and imaginary parts are called in-phase and quadrature components, respectively.

If an input signal $f_2(t)$ expresses sudden removal of a sinusiodal signal, it can be written in

$$f_2(t) = \begin{cases} e^{j\omega_c t}, & t < 0 \\ 0, & t > 0 \end{cases} . \tag{7}$$

The signal is also written in

$$f_2(t) = e^{j\omega_c t} - f_1(t) . \tag{8}$$

Thus the output signal $g_2(t)$ caused by $f_2(t)$ is given by

$$g_2(t) = \begin{cases} \dfrac{e^{j\omega_c t}}{H_{2n}(j\omega_c)}, & t < 0 \\ [1 - E(\omega_c,t)] \cdot \dfrac{e^{j\omega_c t}}{H_{2n}(j\omega_c)}, & t > 0 \end{cases} . \tag{9}$$

It is observed that two envelope functions corresponding to turning-on and turning-off transients are generally different because of existence of the quadrature component, that is

$$1 - |E(\omega_c,t)| \neq |1 - E(\omega_c,t)| . \tag{10}$$

Narrow bandwidth approximation: If the bandwidth $\pm\, \omega_b$ of the bandpass filter network is small enough compared with its mid-band angular frequency ω_0, the roots $\{s_k\}_{k=1}^{2n}$ can be classified into two groups $\{s_k\}_{k=1}^{n}$ and $\{s_{n+k}\}_{k=1}^{n}$ satisfying

$$|s_k - j\omega_0| \ll |s_k + j\omega_0| \tag{11}$$

and

$$|s_{n+k} - j\omega_0| \gg |s_{n+k} + j\omega_0| \ . \tag{12}$$

Thus we obtain

$$\left| \frac{1}{s_k - j\omega_c} \right| \gg \left| \frac{1}{s_{n+k} - j\omega_c} \right| \tag{13}$$

unless large amount of detuning appears. Under this assumption, we can neglect one half of terms appearing in the summation in (6).

Normalization: For convenience, we adopt the following normalization formulas:

$$\left.\begin{aligned} & z = \frac{s - j\omega_0}{\omega_b} \ , \quad x = \frac{\omega - \omega_0}{\omega_b} \ , \\ & z_k = \frac{s_k - j\omega_0}{\omega_b} \ , \quad x_c = \frac{\omega_c - \omega_0}{\omega_b} \ , \\ & H_{2n}(s) \doteq D_n(z) \ , \\ & H'_{2n}(s) \doteq \frac{1}{\omega_b} D'_n(z) \ . \end{aligned}\right\} \tag{14}$$

Thus (6) can be written in

$$\left.\begin{aligned} E(\omega_c, t) &= 1 - D_n(jx_c) \sum_{k=1}^{n} \frac{1}{jx_c - z_k} \cdot \frac{e^{(-jx_c - z_m)\theta}}{D_n'(z_k)} \\ \theta &= \omega_b t \end{aligned}\right\} \qquad (15)$$

under the assumption of narrow bandwidth.

Transient Responses of Butterworth Filters

For the Butterworth filter network of n-th degree, $D_n(z)$ is given by

$$\left.\begin{aligned} D_n(z) &= \prod_{k=1}^{n} (z - z_k) \\ z_k &= -\sin \frac{2k-1}{2n} \pi + j \cos \frac{2k-1}{2n} \end{aligned}\right\} \qquad (16)$$

where the normalization factor ω_b gives the half-power (or 3db) transmission band. Thus the normalized transient response is easily obtained by the use of Eqs. (15) and (16). For the case of no detuning, that is $x_c = 0$, transient responses of networks of $n = 1$ to 7 are shown in Fig. 1. This set of curves is also available, for example, in [2] but slightly different in the normalization of the time axis. The curves show that the transient response of the Butterworth filter networks has considerable amount of overshoot and ringing even for the case of no detuning.

For the detuned case, in-phase and quadrature components for networks of degree $n = 3$ to 5 are shown in Fig. 2 to 7, each of which is corresponding to the detuning factor $x_c = 0.2$, 0.4, 0.6, 0.8, 1, and half-gain points, respectively. It can be observed that as the detuning factor x_c is increased from 0.2 to 0.8, positive smear in in-phase components becomes greater. This is closely correlated to the group delay characteristic of networks. It has been shown in [3] that

(1) the group delay characteristic $\tau_{Bn}(x)$ of the n-th degree Butterworth function has two maximum peaks at $\pm x_n$ and one local minimum at $x = 0$ wherever $n \geq 2$.

(2) x_n satisfies $0 < x_n < 1$.

(3) x_n satisfies $x_n > 0.8$ whenever $n \geq 3$.

Thus the increase in smear of in-phase components can be interpreted as the result of the fact that the group delay at the impressed frequency x_c is a monotone increasing function of x_c whenever $0 < x_c < 0.8$.

The last set of curves may be useful in the application of I.F. amplifier design for the television receiver. Approximate values of half-gain points for n = 3, 4, and 5 are x_c = 1.201, 1.147 and 1.116, respectively.

Turning-on and turning-off envelope functions, $|E(\omega_c,t)|$ and $|1 - E(\omega_c,t)|$, are shown in Fig. 8 to 13. The cross points of turning-on and turning-off envelope functions of the same network have the amplitude coordinate greater than 0.5 because of asymmetry caused by the existence of quadrature components. Considerable length of tails of turning-off envelope functions is an important feature of the set of curves. This is sometimes an essential defect in the practical application of the Butterworth filter networks. The synchronous detection can be used to improve the transient response, but it is not so powerful because of poor response in in-phase components.

Transient Responses of Chebyshev Filter

For the Chebyshev filter network of the n-th degree, $D_n(z)$ is given by

$$\left.\begin{aligned} D_n(z) &= \sqrt{a_p^2 - 1} \cdot 2^{n-1} \cdot \prod_{k=1}^{n} (z-z_k) \\ z_k &= j \cos (\zeta_k + j\eta) \\ \zeta_k &= -\frac{2k-1}{2n}\pi,\ \eta = \frac{1}{2n} \cosh^{-1} \frac{a_p^2+1}{a_p^2-1} \end{aligned}\right\} \quad (17)$$

where the normalization factor ω_b gives $A_p = 20 \log_{10} a_p$ db transmission band. Substituting (17) into (15), we can obtain in-phase and quadrature components.

For the case of no detuning, $x_c = 0$, transient responses of networks of $n = 3$ to 7 with $A_p = 0.01$, 0.03, 0.1, 0.3 and 1db are shown in Fig. 14 to 18. Part of the sets of curves is also available in [4], but different in the normalization of the amplitude scale for networks of even degree. It can be observed that as the pass-band ripple A_p is increased, time delay and ringing become greater. It is interesting that the responses of odd and even degree networks contain positive (or increasing) and negative (or decreasing) smear, respectively.

For the detuned case, in-phase and quadrature components for networks of $A_p = 0.03$db and $n = 3$ to 5 are shown in Fig. 19 to 23, each of which is corresponding to the detuning factor $x_c = 0.2$, 0.4, 0.6, 0.8 and 1, respectively. It has been shown in [3] that the group delay characteristic[2] $\tau_{Tn}(x)$ of the Chebyshev function of n-th degree is a monotone increasing function whenever $0 < x < 1$ and $n = 3$ to 5. So the increase in smear in in-phase components can be interpreted in the same way as the Butterworth filter.

The envelope function $|E(\omega_c,t)|$ and $|1 - E(\omega_c,t)|$ are shown in Fig. 24 to 28, each of which is corresponding to $x_c = 0.2, 0.4, 0.6, 0.8$ and 1, respectively. Remarkable smear in turning-on envelope functions and long masking tail in turning-off envelope functions are observed in these figures.

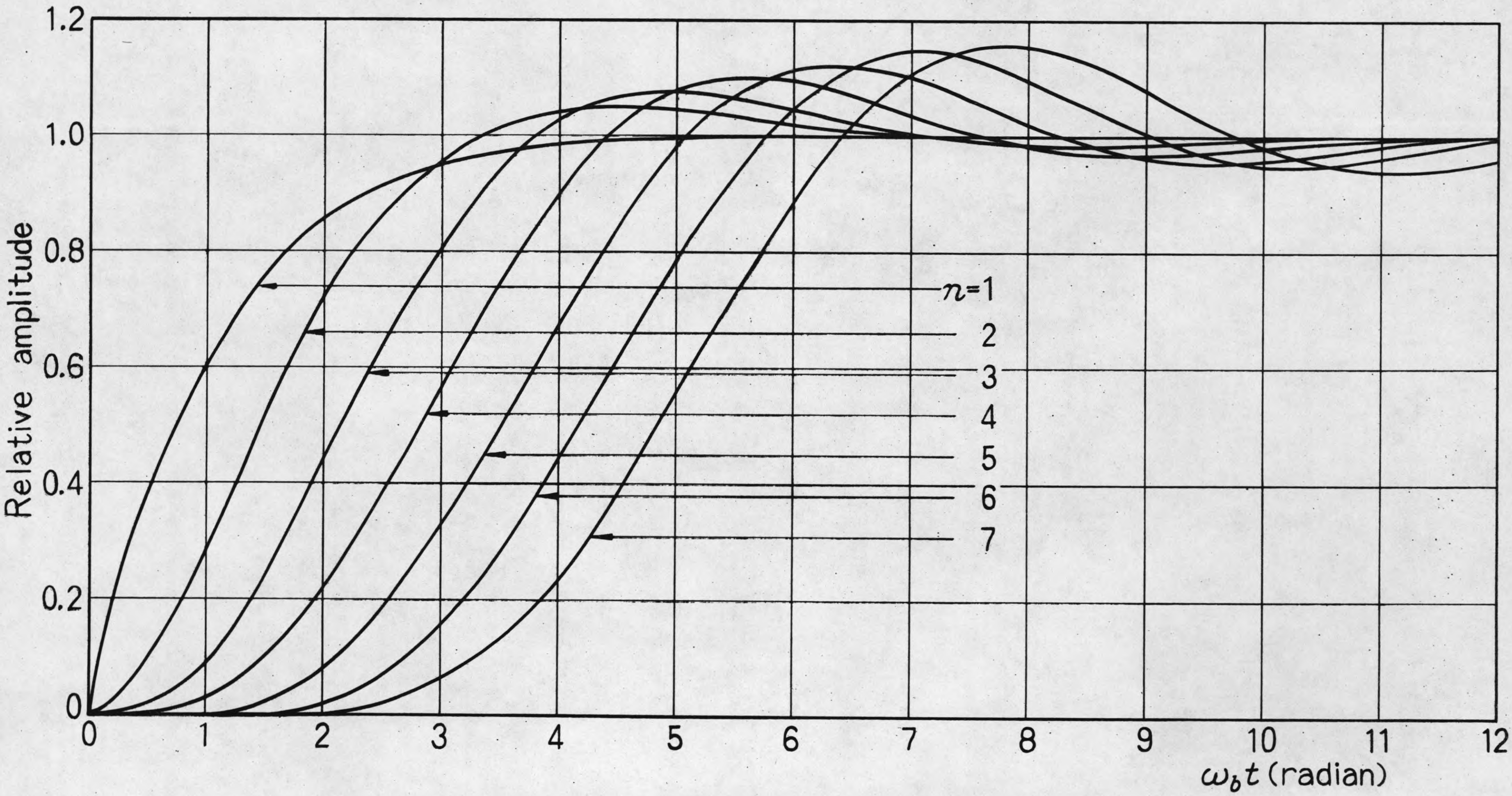

Fig. 1 Transient response of Butterworth filter, x_c = 0.

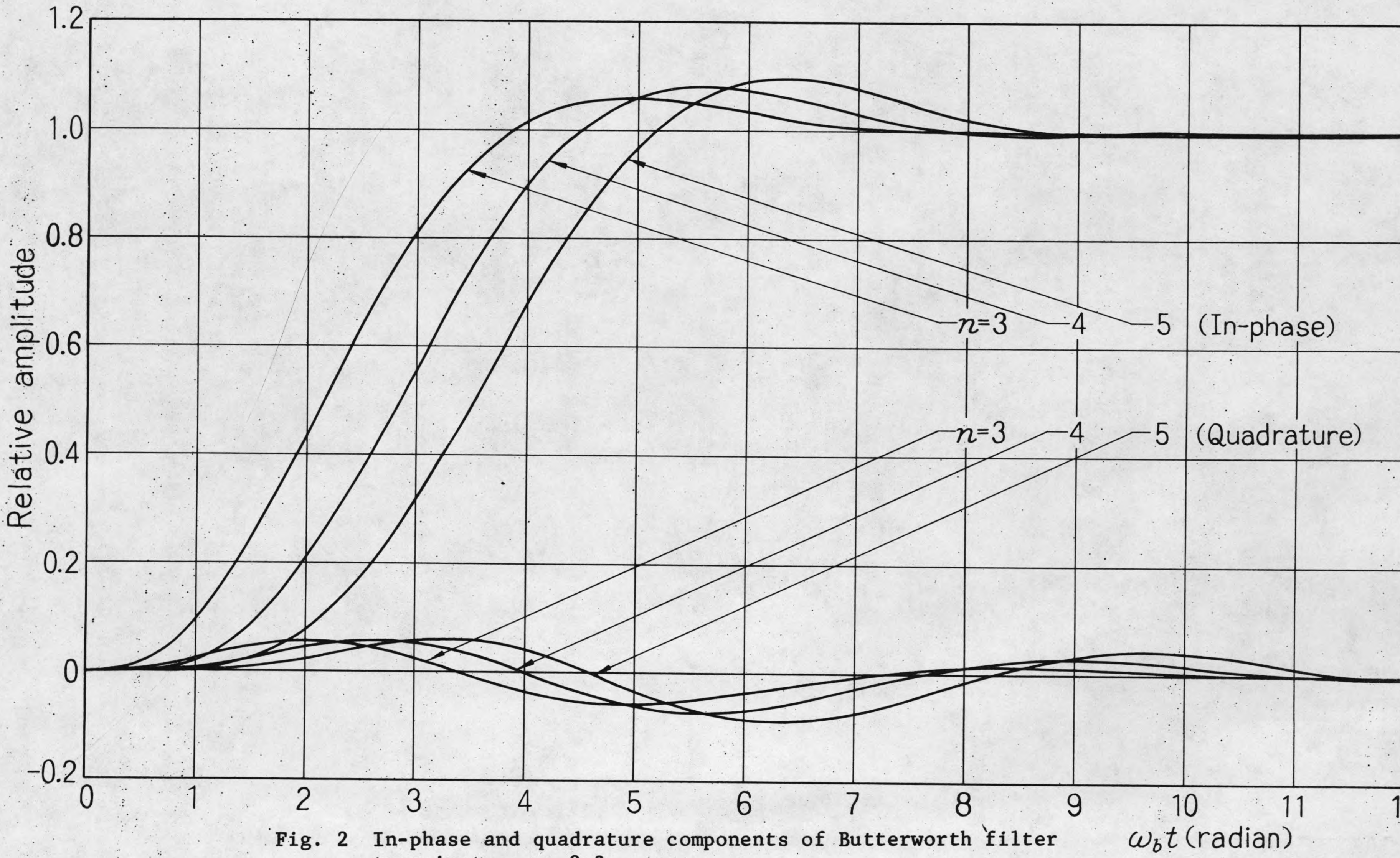

Fig. 2 In-phase and quadrature components of Butterworth filter transient, x_c = 0.2.

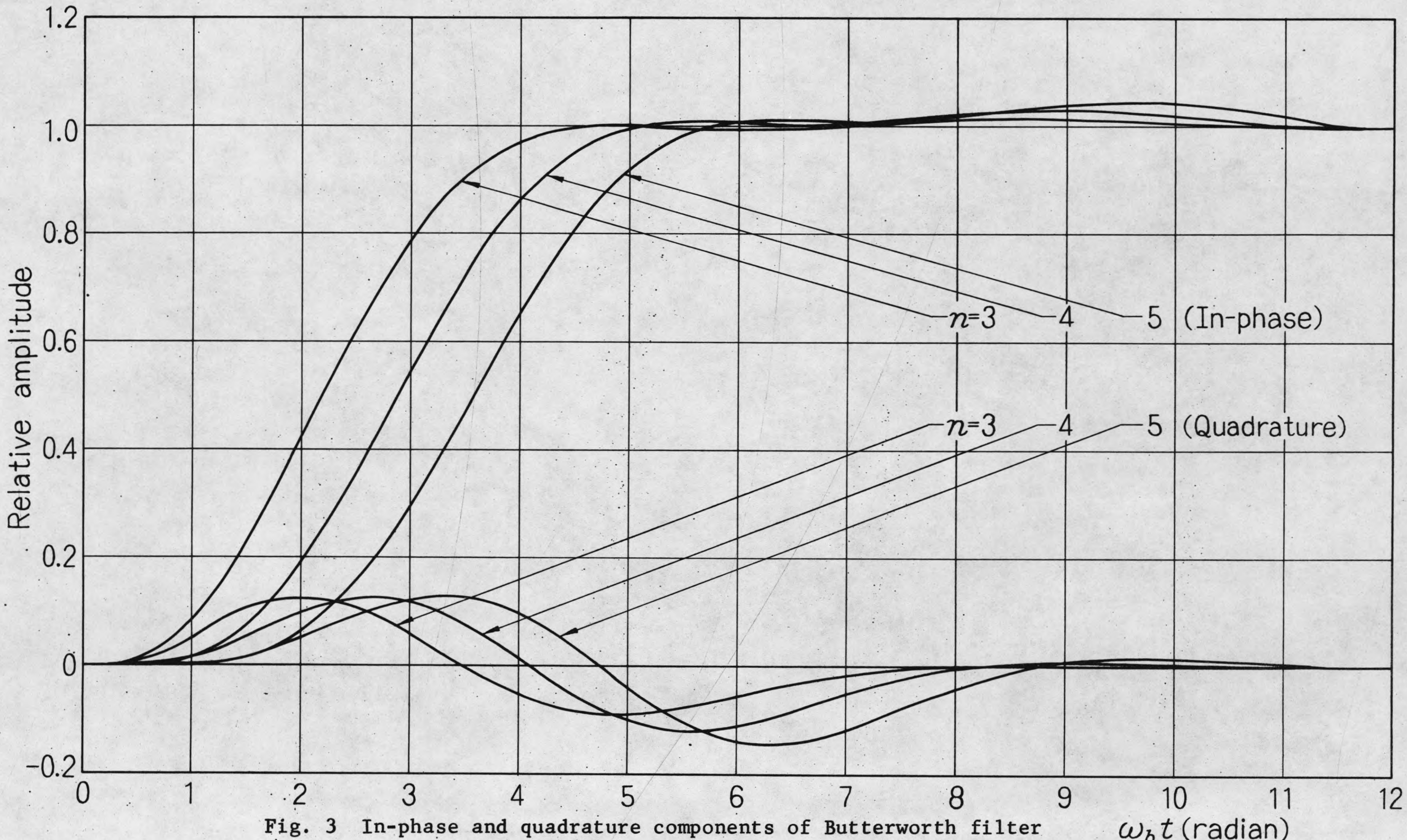

Fig. 3 In-phase and quadrature components of Butterworth filter transient, $x_c = 0.4$.

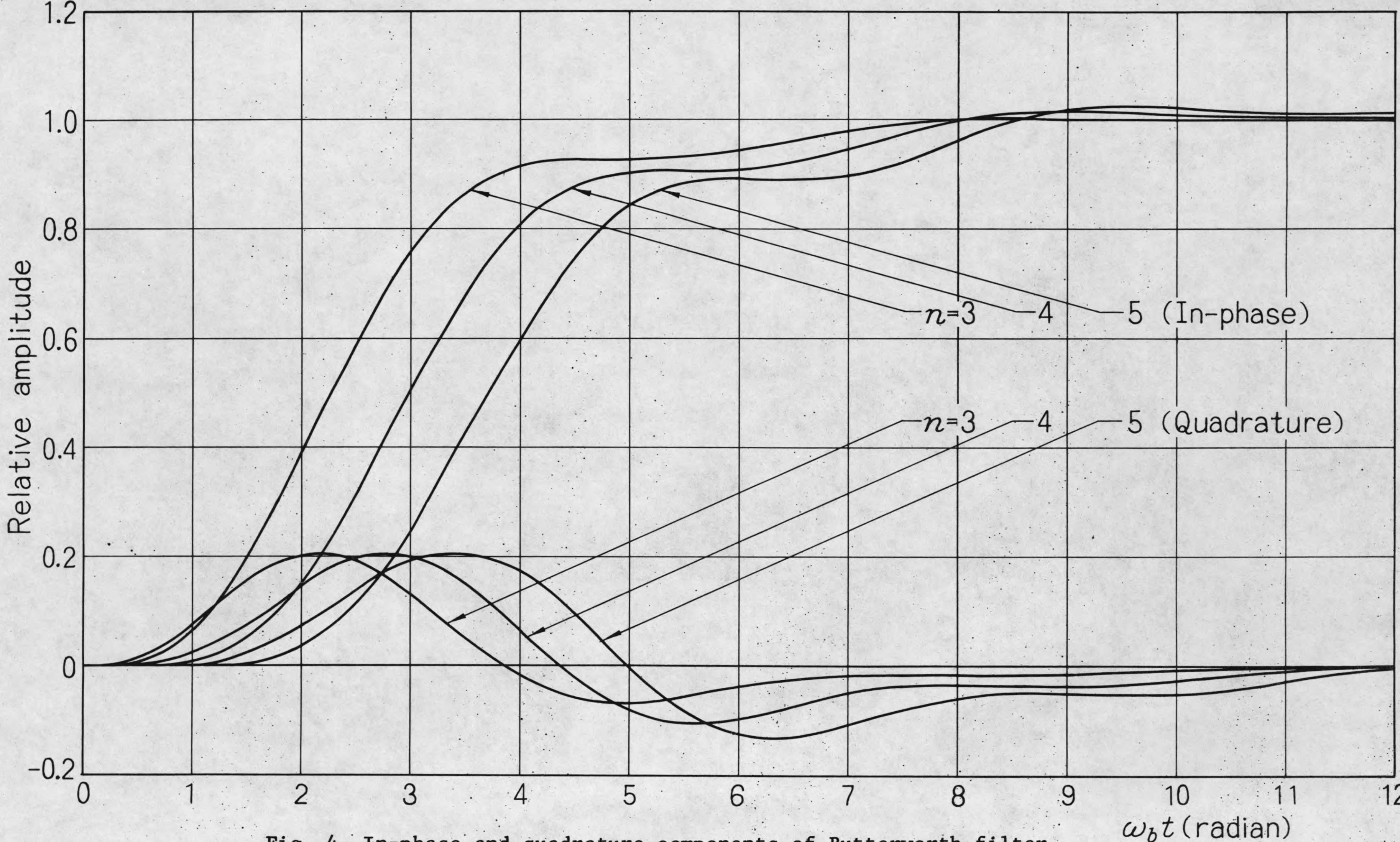

Fig. 4 In-phase and quadrature components of Butterworth filter transient, $x_c = 0.6$.

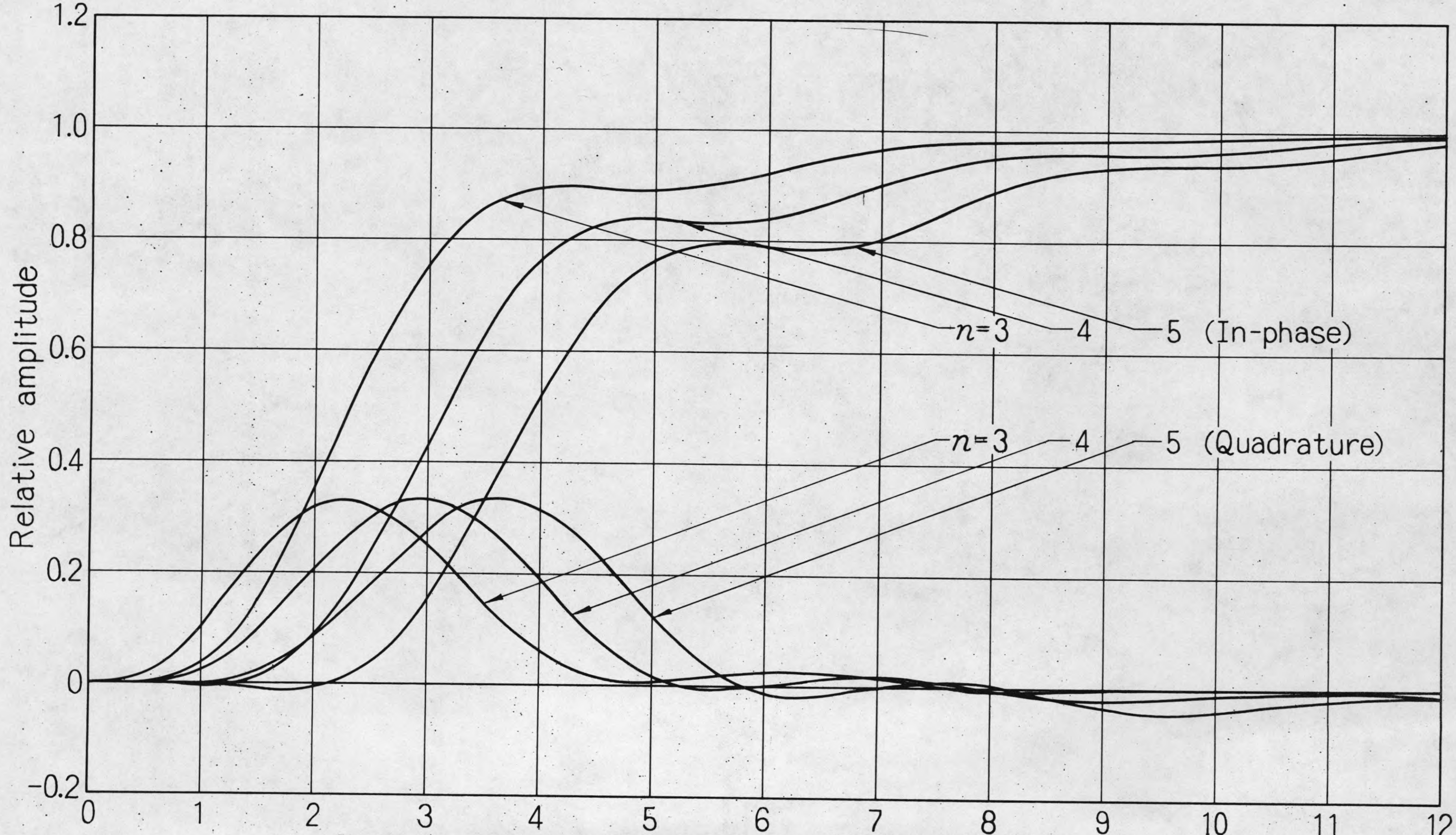

Fig. 5 In-phase and quadrature components of Butterworth filter transient, $x_c = 0.8$.

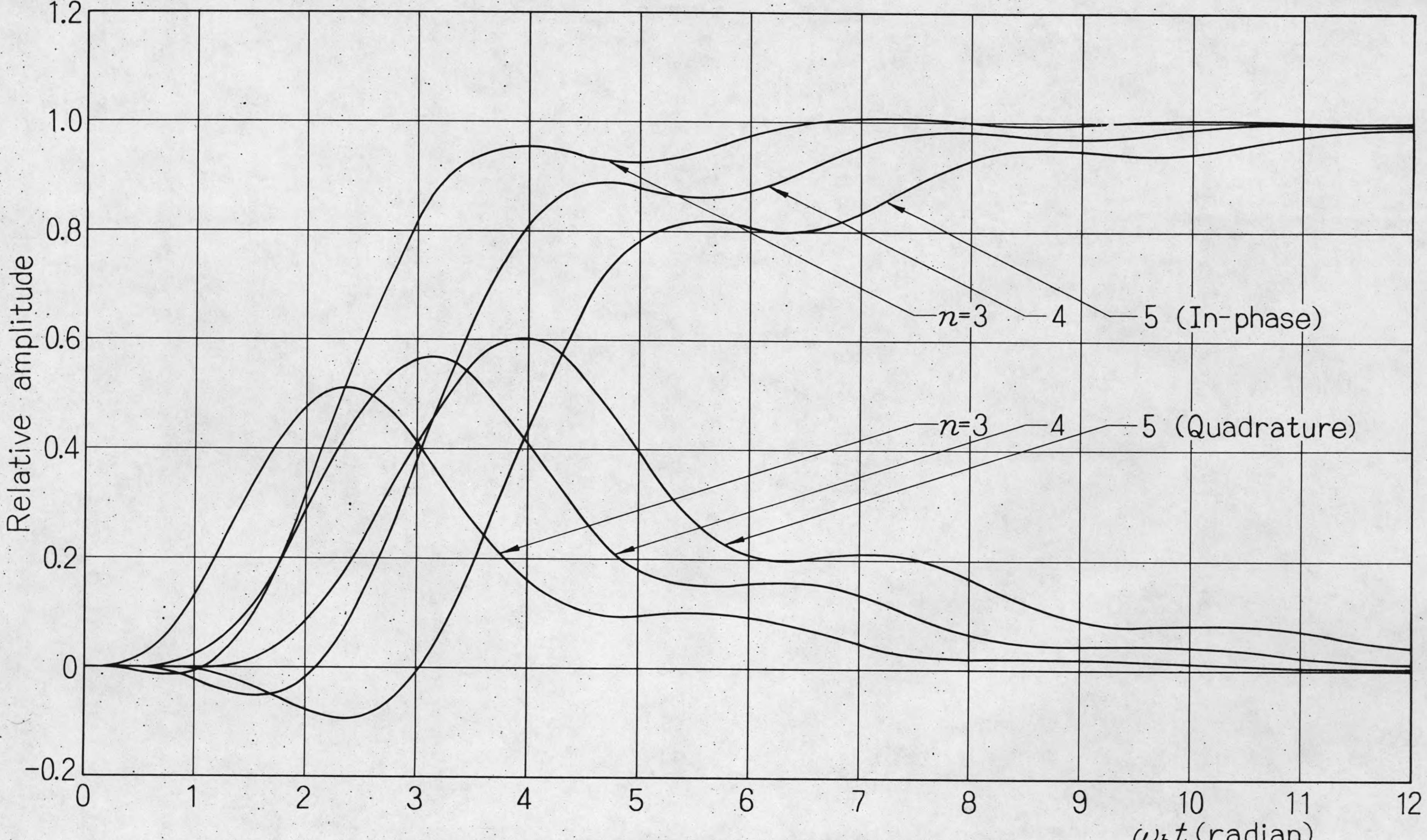

Fig. 6 In-phase and quadrature components of Butterworth filter transient, $x_c = 1$.

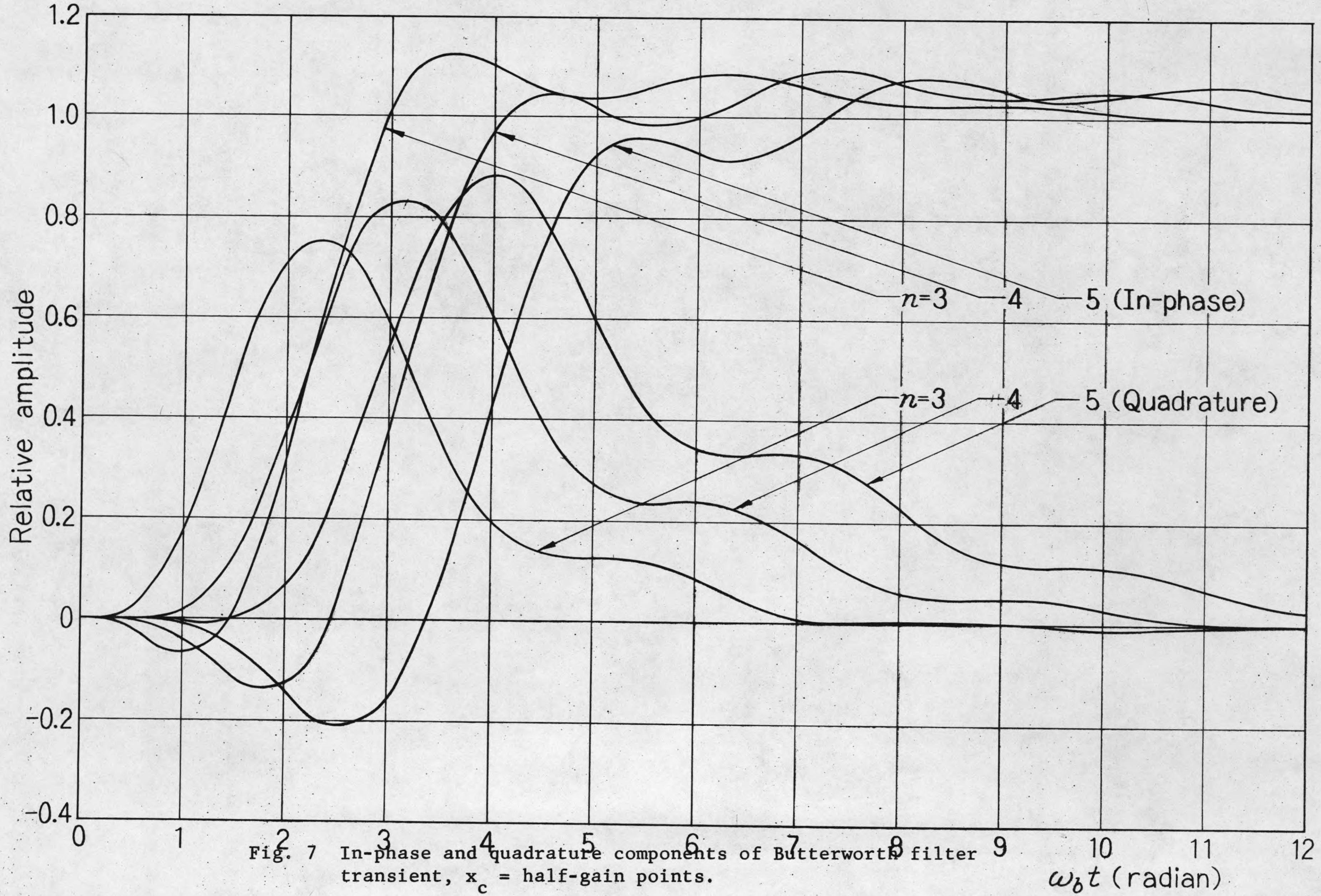

Fig. 7 In-phase and quadrature components of Butterworth filter transient, x_c = half-gain points.

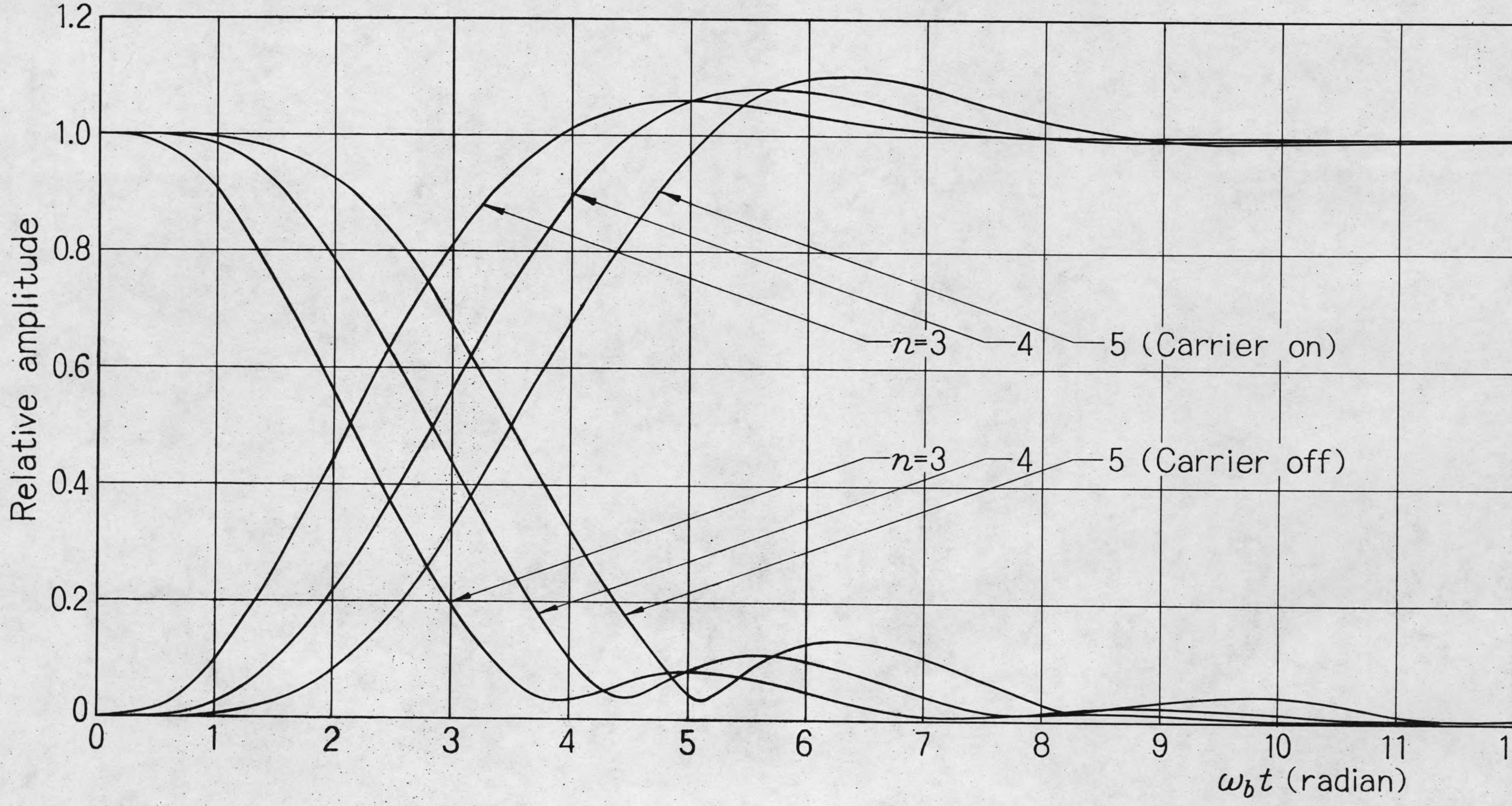

Fig. 8 Envelope function of Butterworth filter transient, x_c = 0.2.

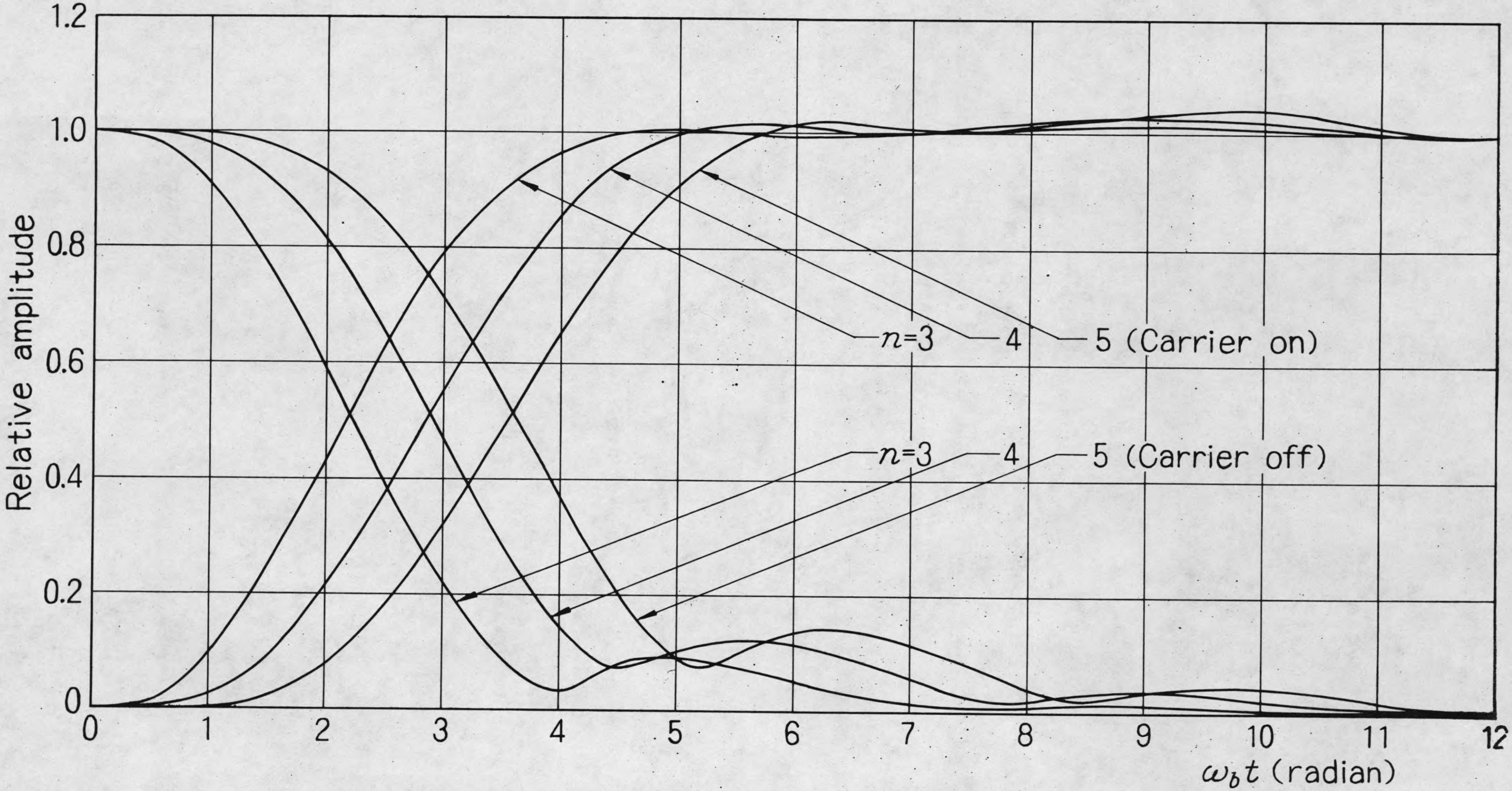

Fig. 9 Envelope function of Butterworth filter transient, $x_c = 0.4$.

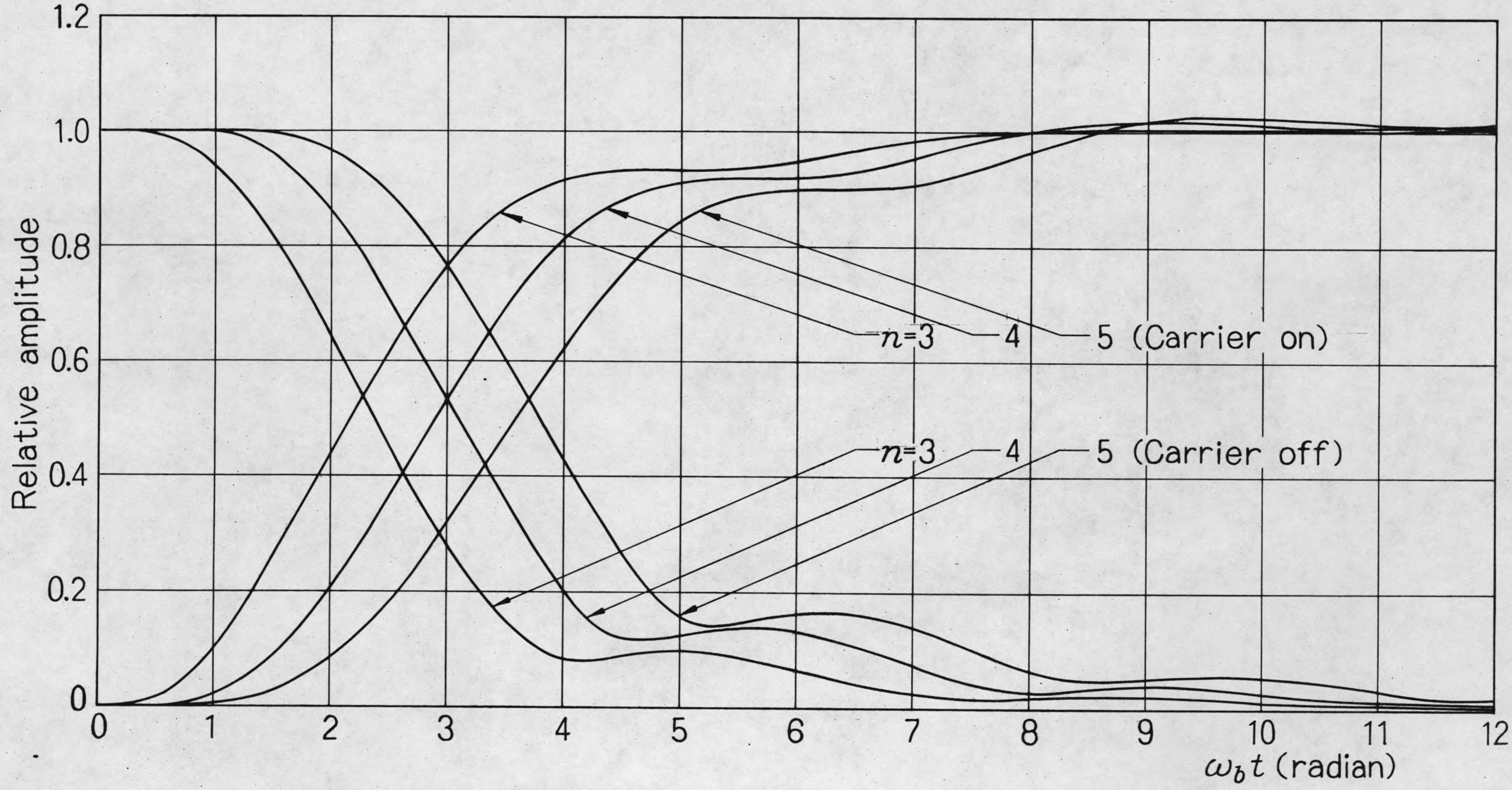

Fig. 10 Envelope function of Butterworth filter transient, $x_c = 0.6$.

18

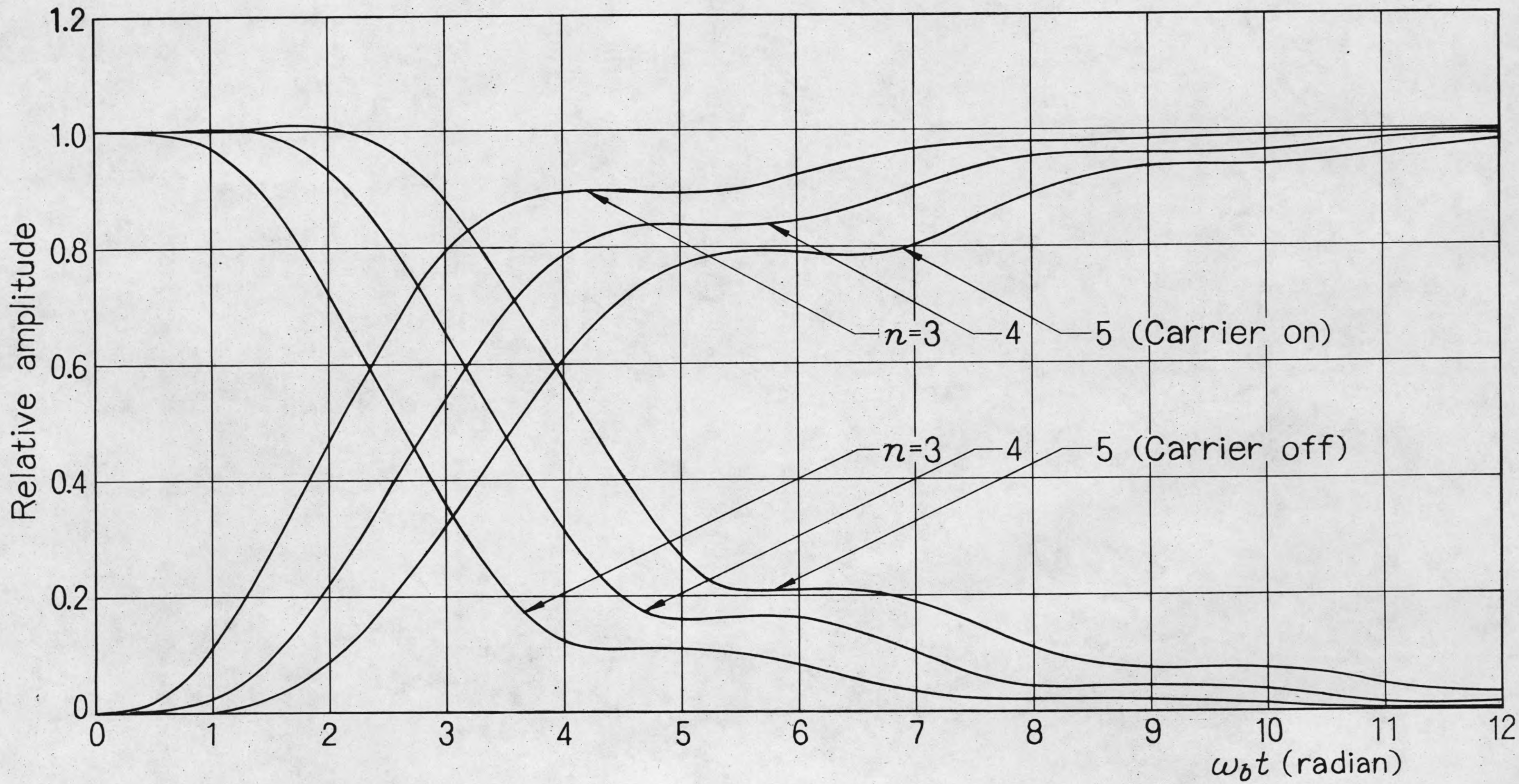

Fig. 11 Envelope function of Butterworth filter transient, x_c = 0.8.

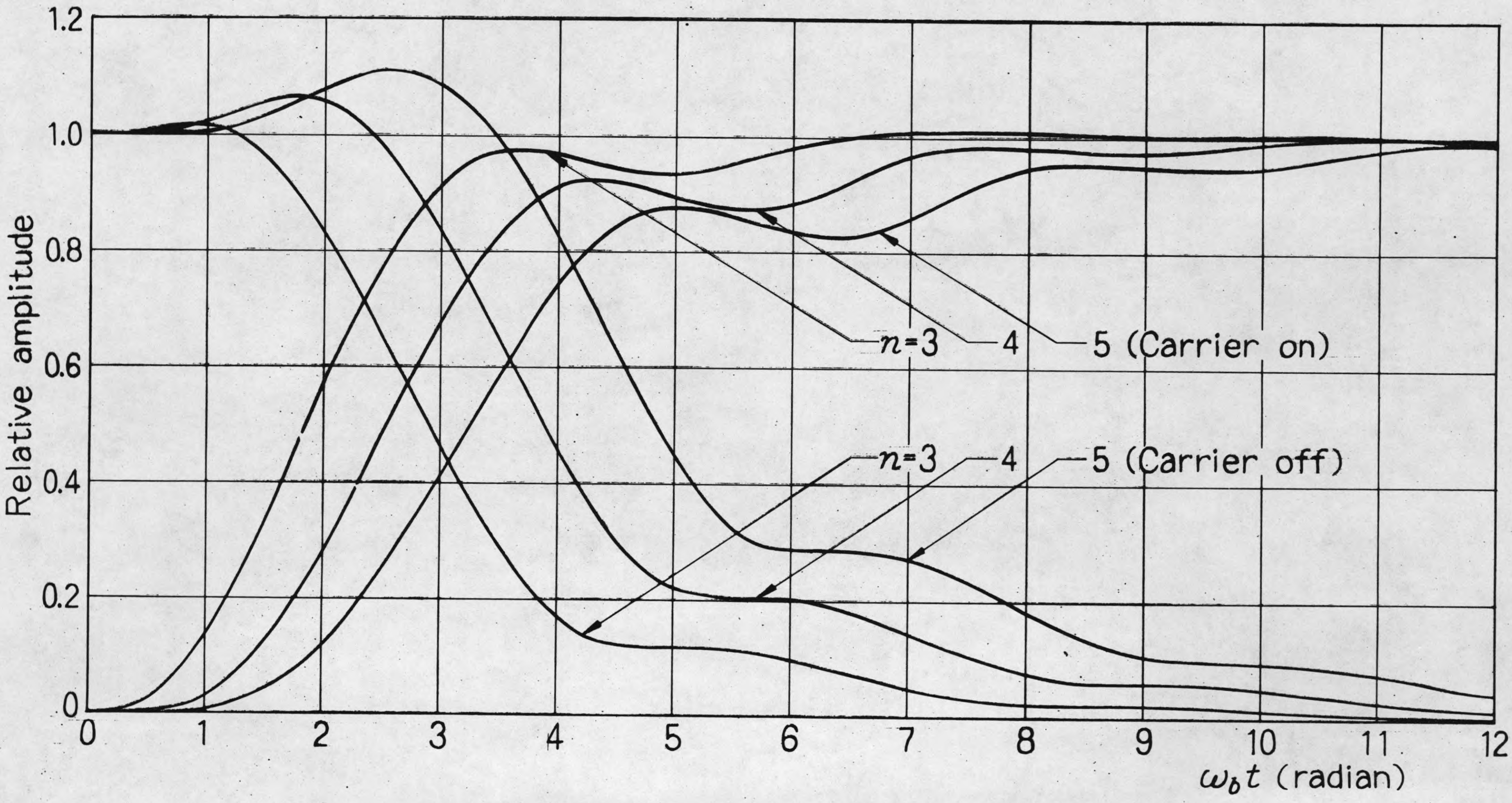

Fig. 12 Envelope function of Butterworth filter transient, $x_c = 1$.

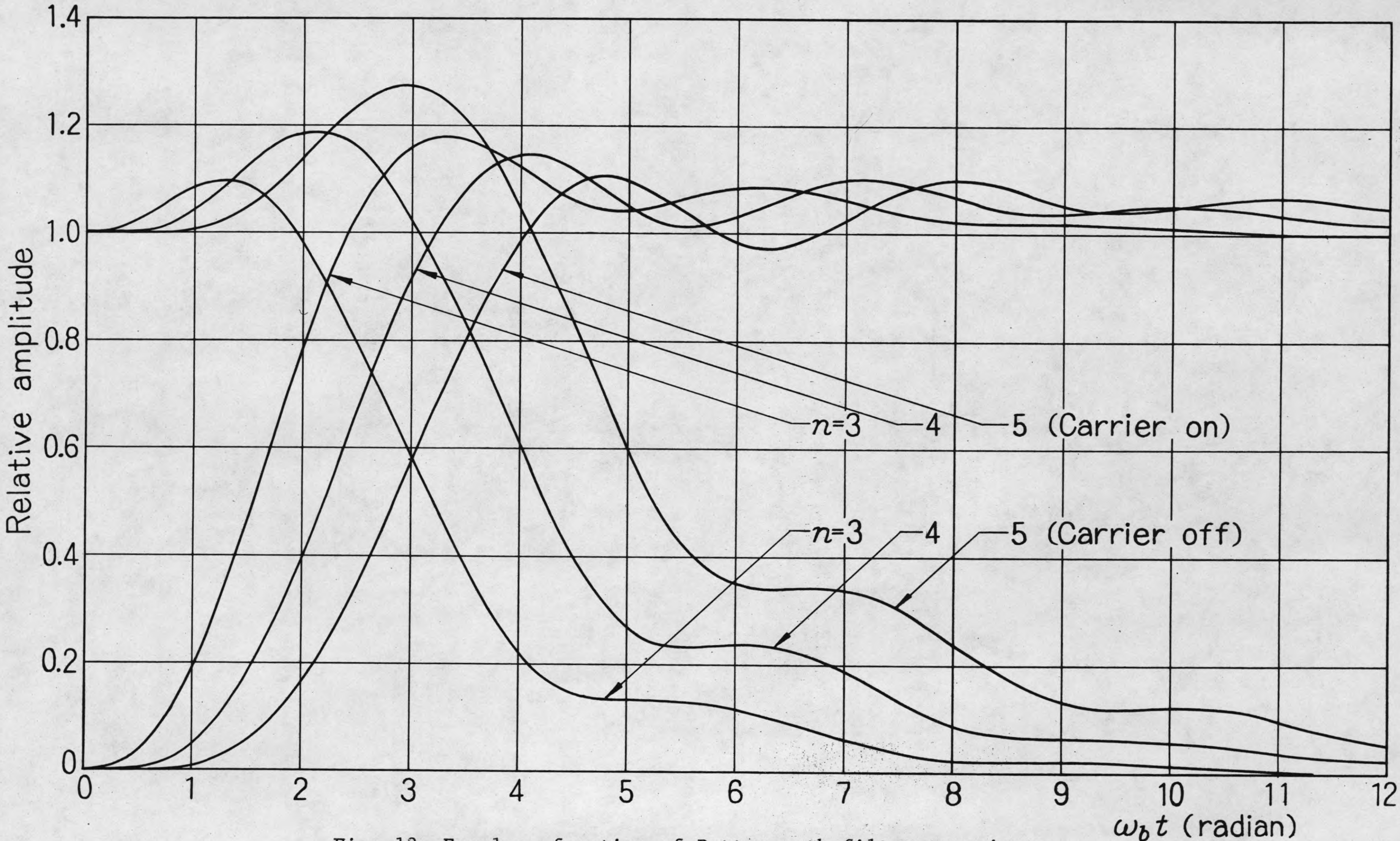

Fig. 13 Envelope function of Butterworth filter transient, x_c = half-gain points.

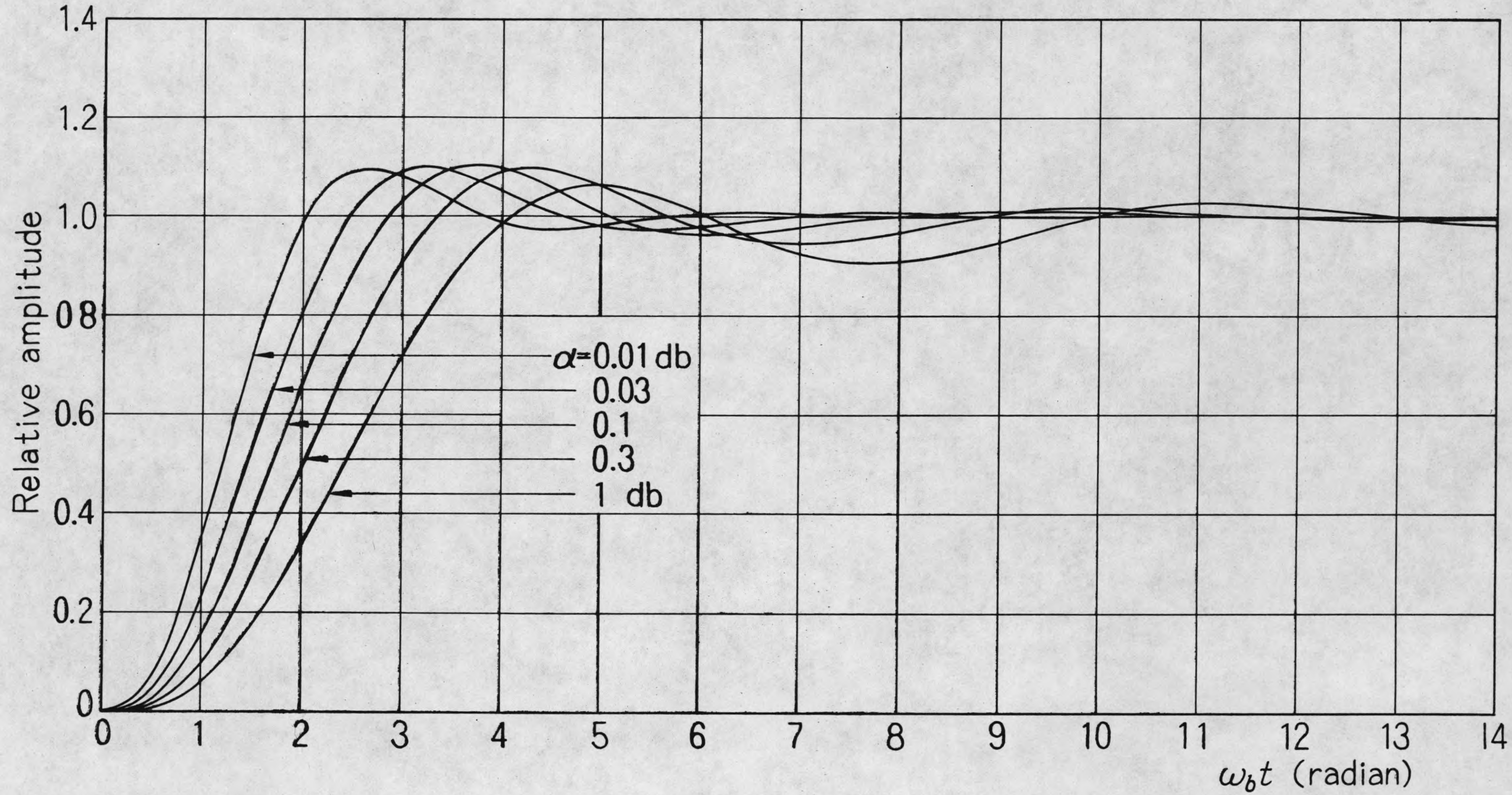

Fig. 14 Transient response of Chebyshev filter, n = 3, x_c = 0.

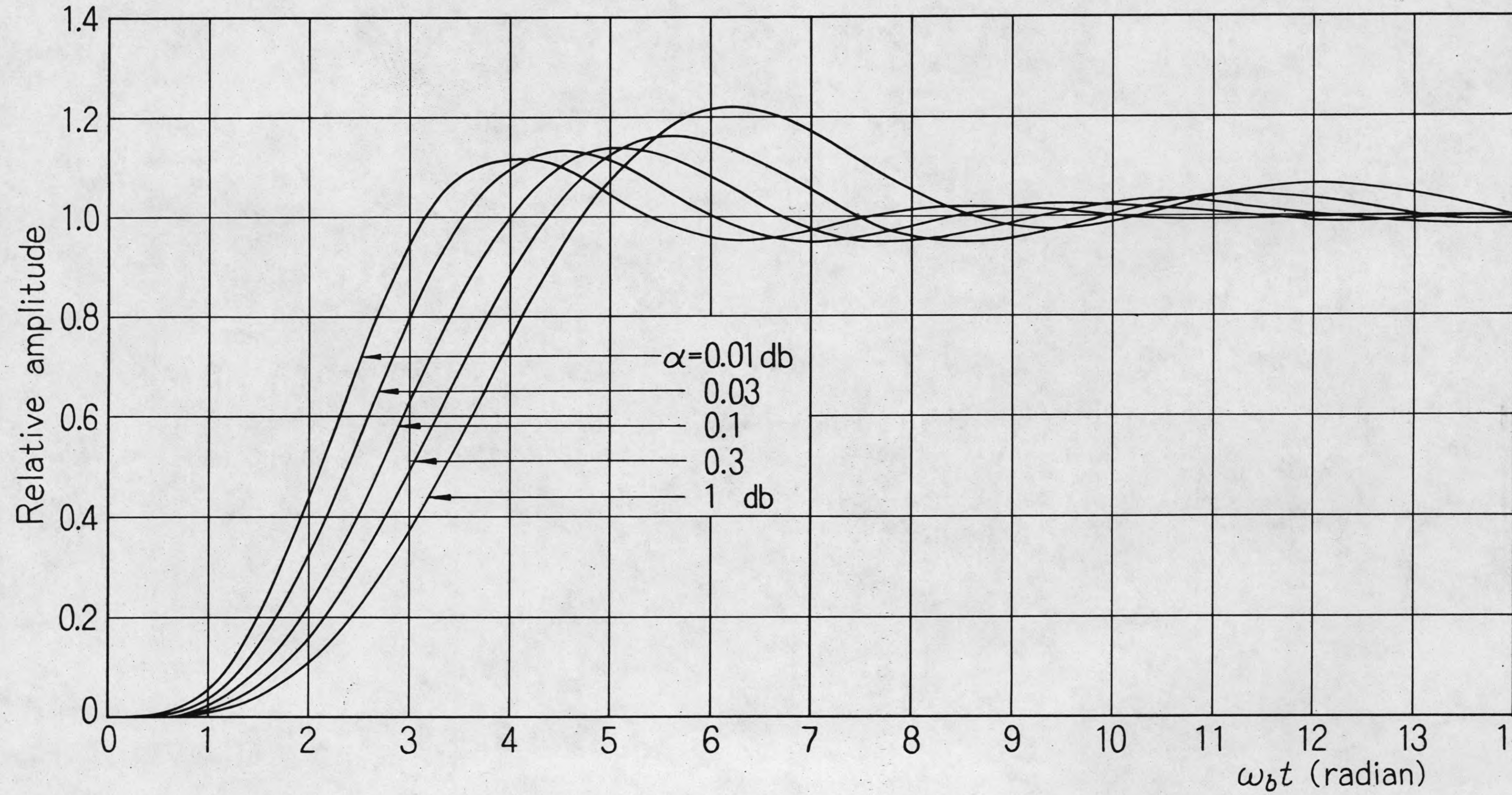

Fig. 15 Transient response of Chebyshev filter, n = 4, x_c = 0.

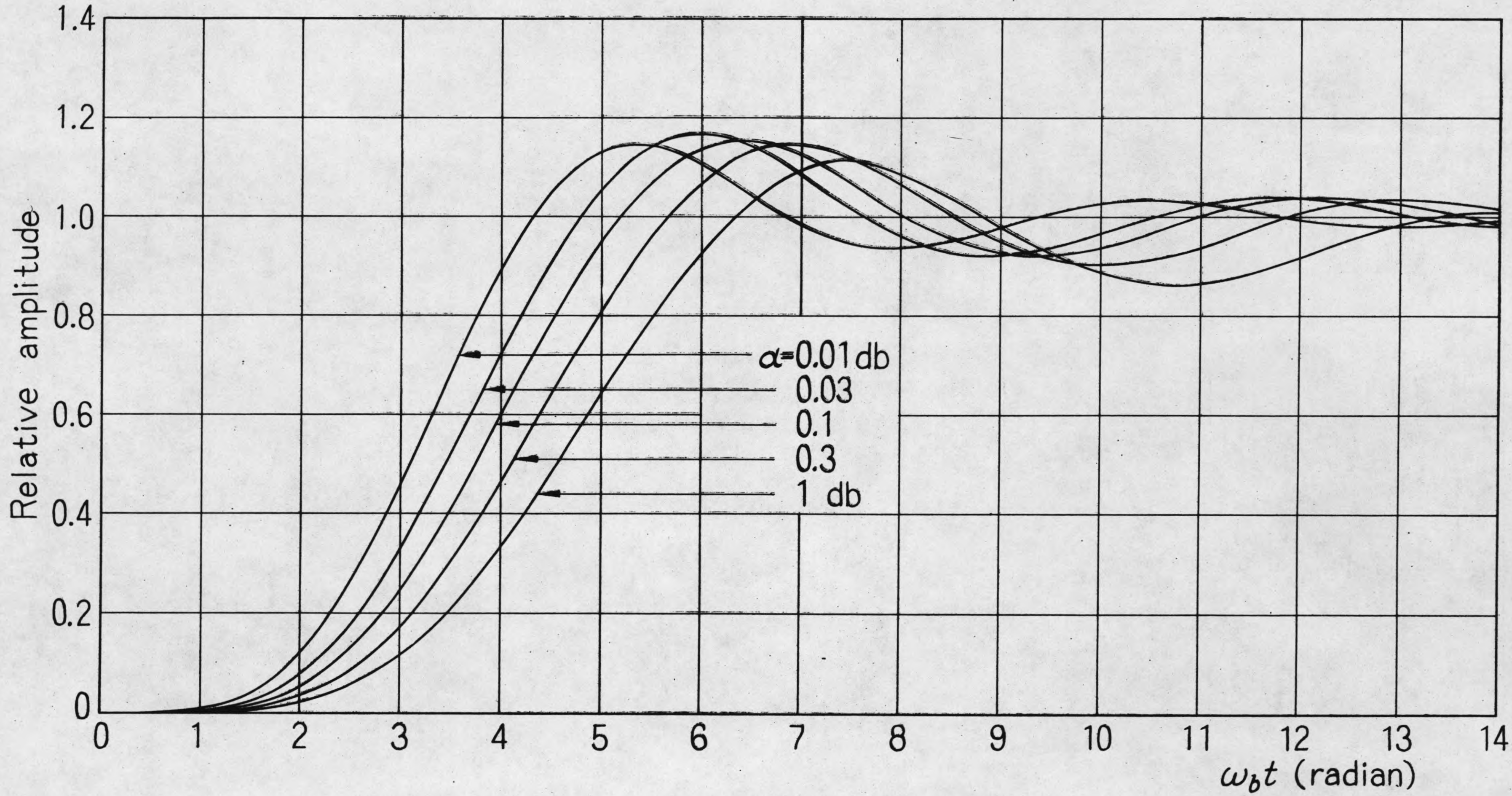

Fig. 16 Transient response of Chebyshev filter, n = 5, x_c = 0.

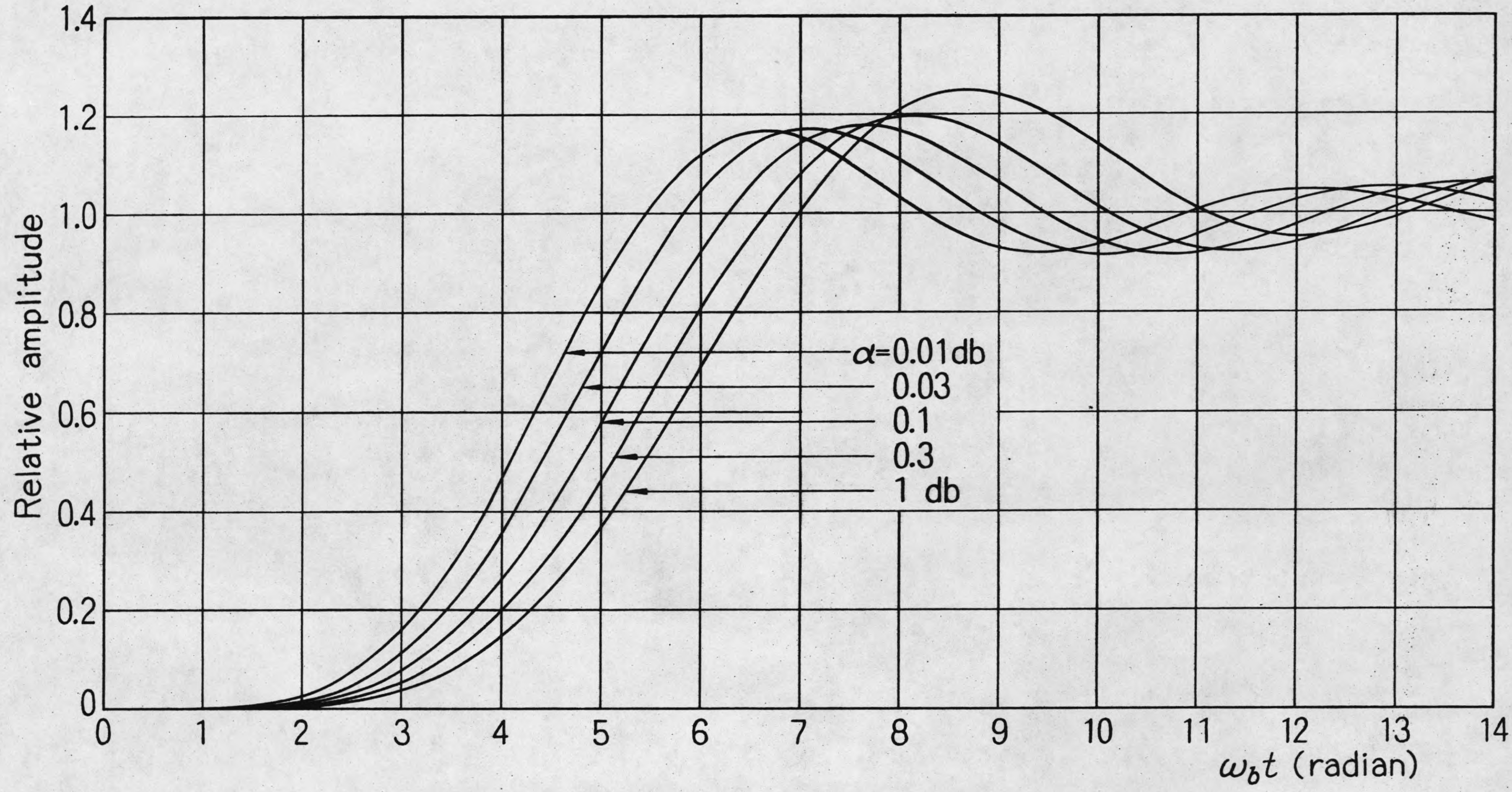

Fig. 17. Transient response of Chebyshev filter, n = 6, x_c = 0.

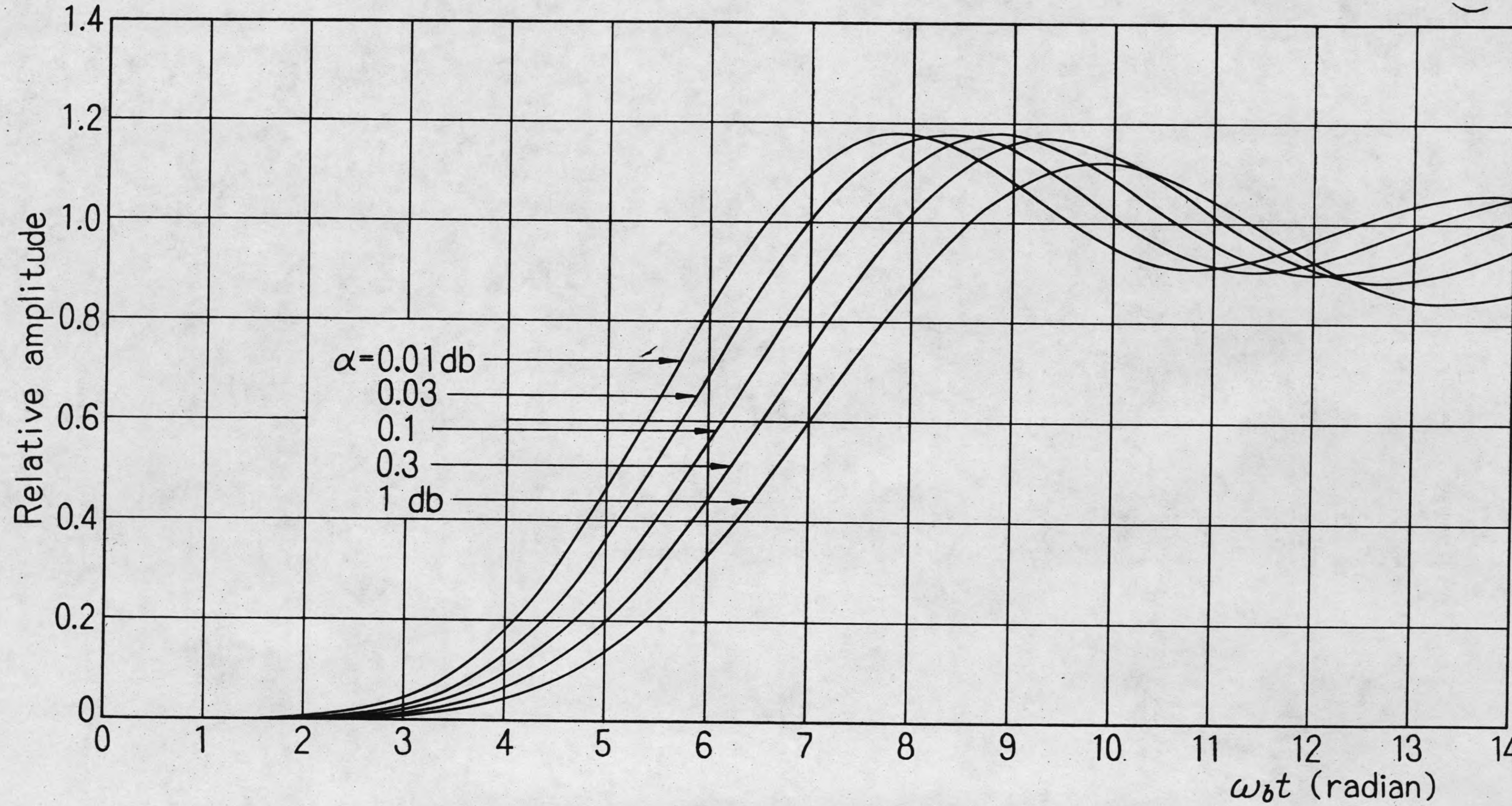

Fig. 18 Transient response of Chebyshev filter, $n = 7$, $x_c = 0$.

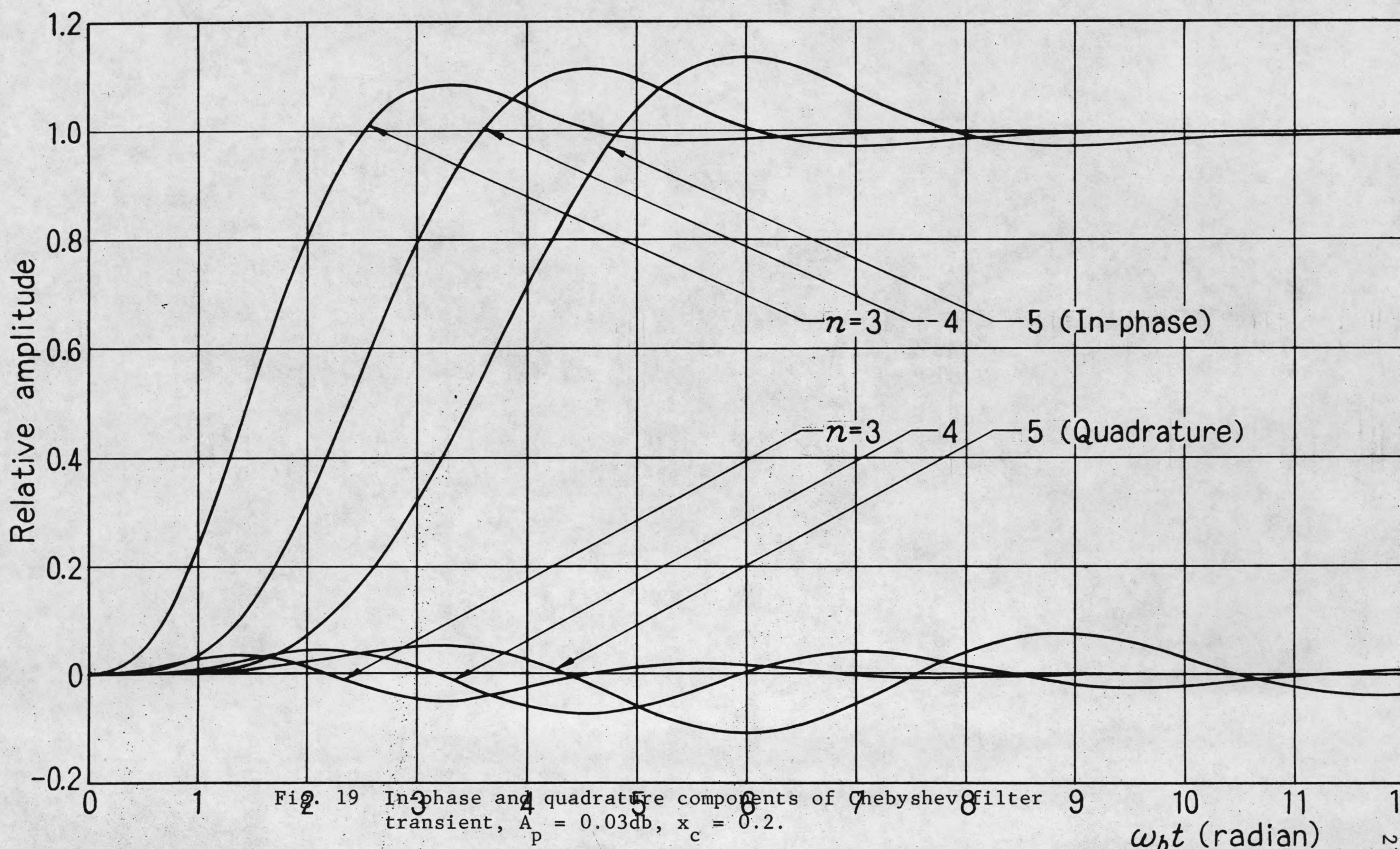

Fig. 19 In-phase and quadrature components of Chebyshev filter transient, A_p = 0.03db, x_c = 0.2.

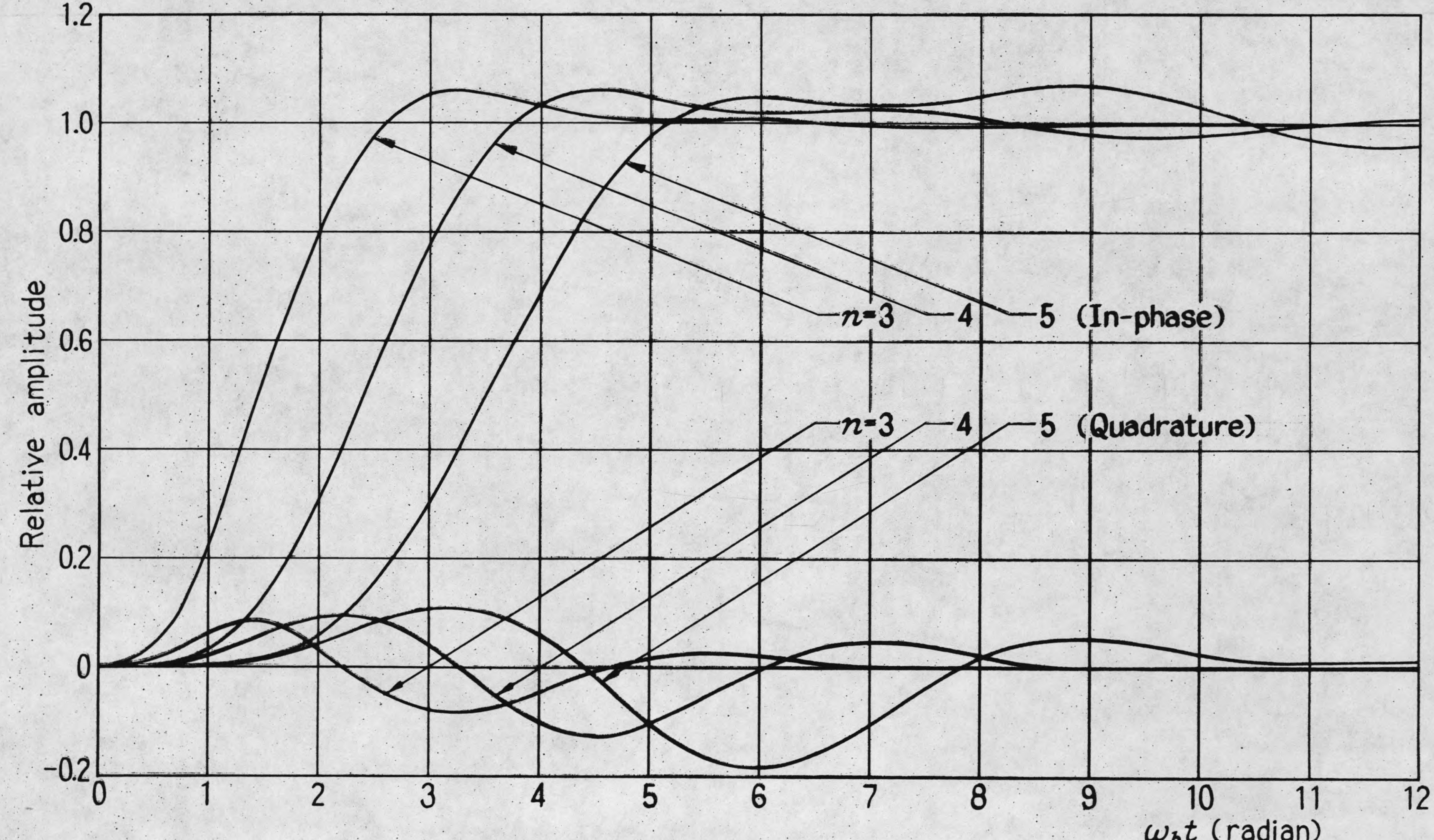

Fig. 20 In-phase and quadrature components of Chebyshev filter transient, A_p = 0.03db, x_c = 0.4.

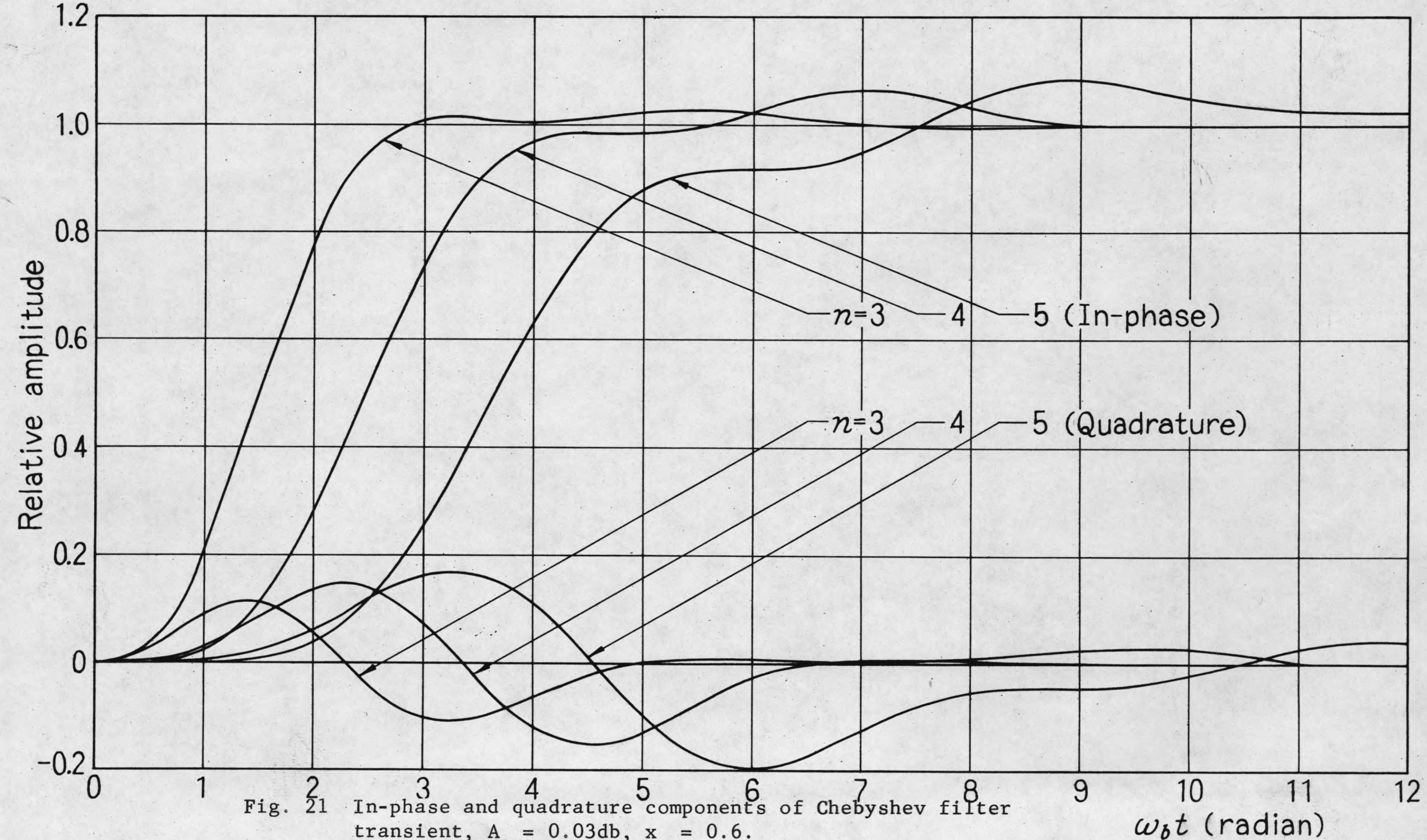

Fig. 21 In-phase and quadrature components of Chebyshev filter transient, A_p = 0.03db, x_c = 0.6.

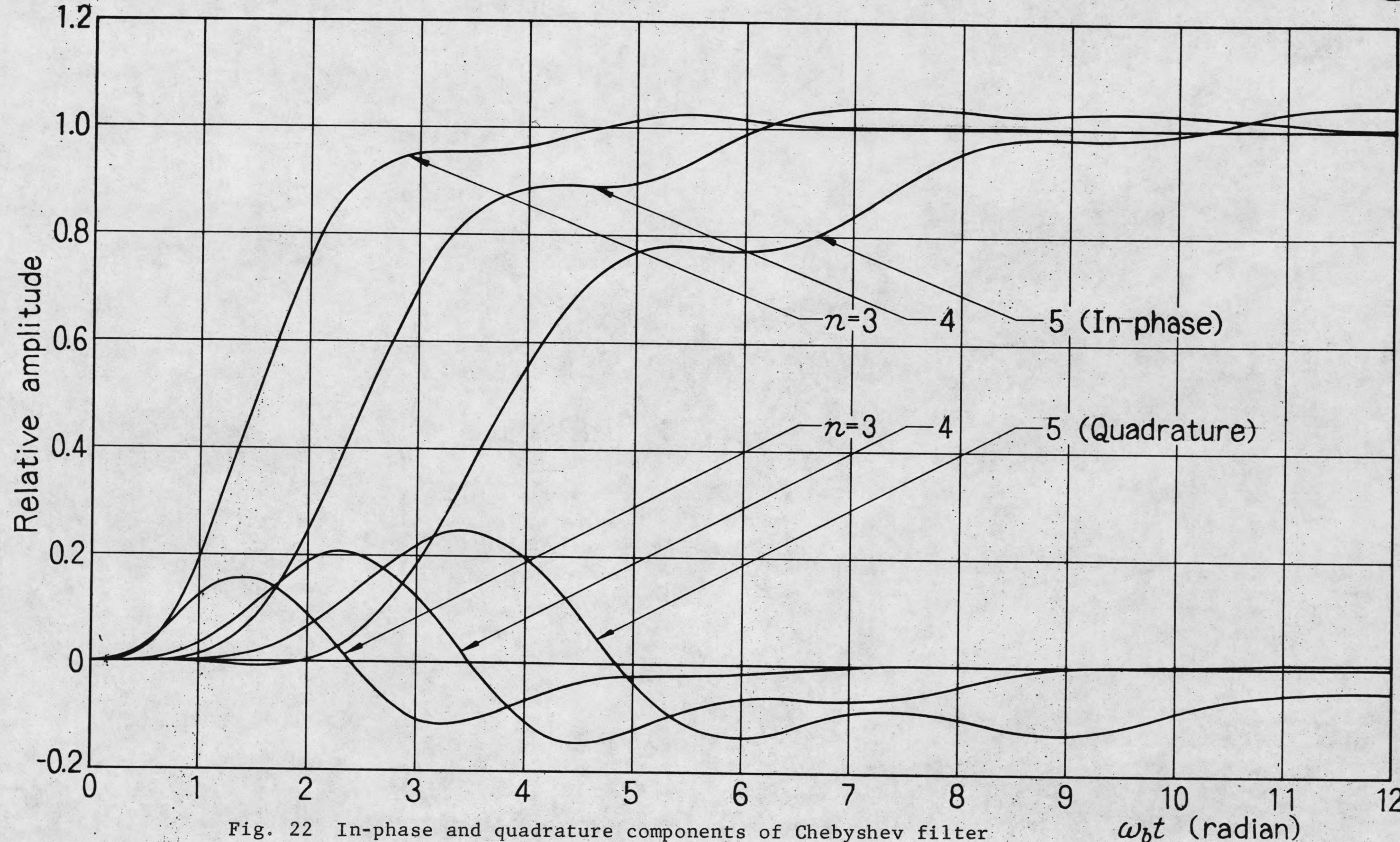

Fig. 22 In-phase and quadrature components of Chebyshev filter transient, A_p = 0.03db, x_c = 0.8.

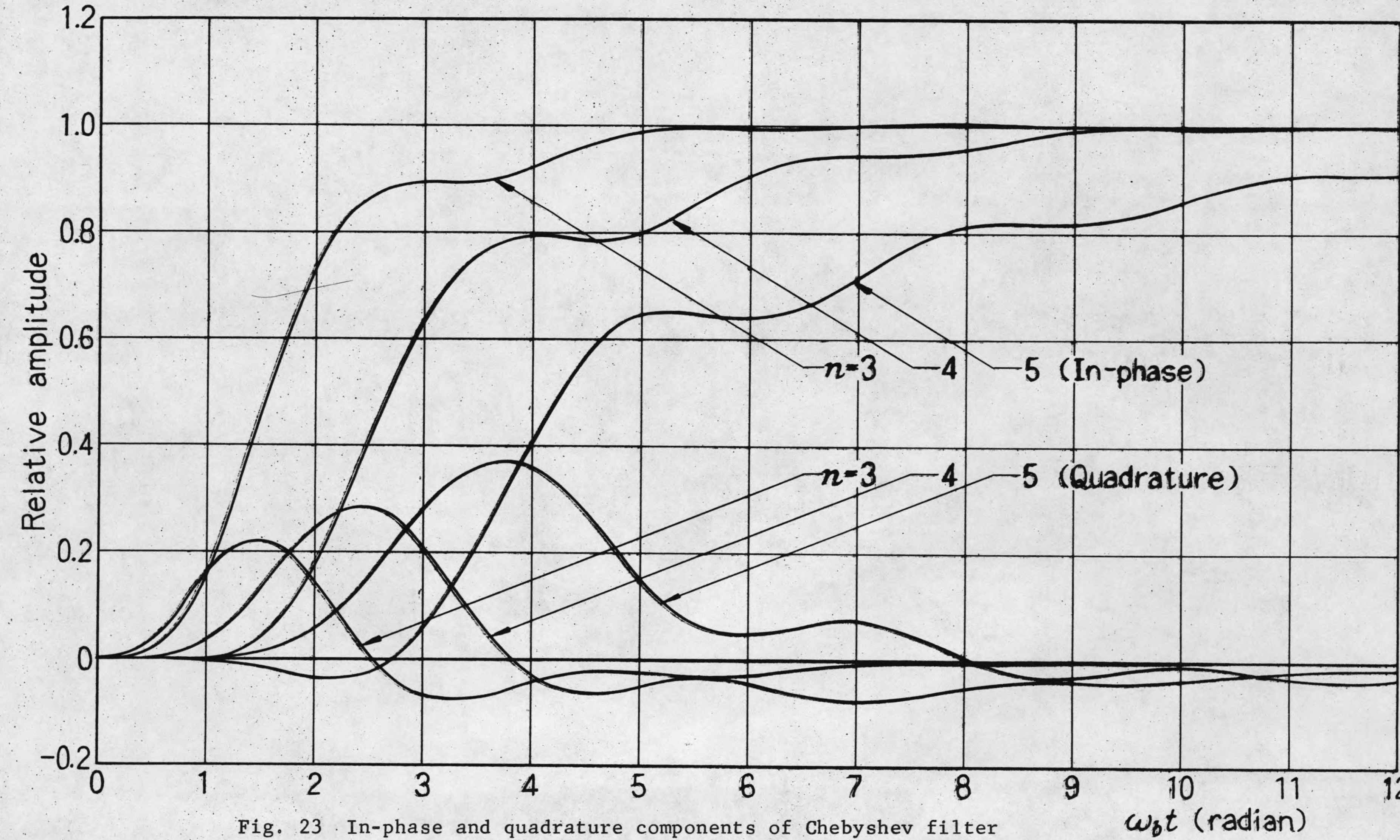

Fig. 23 In-phase and quadrature components of Chebyshev filter transient, A_p = 0.03db, x_c = 1.

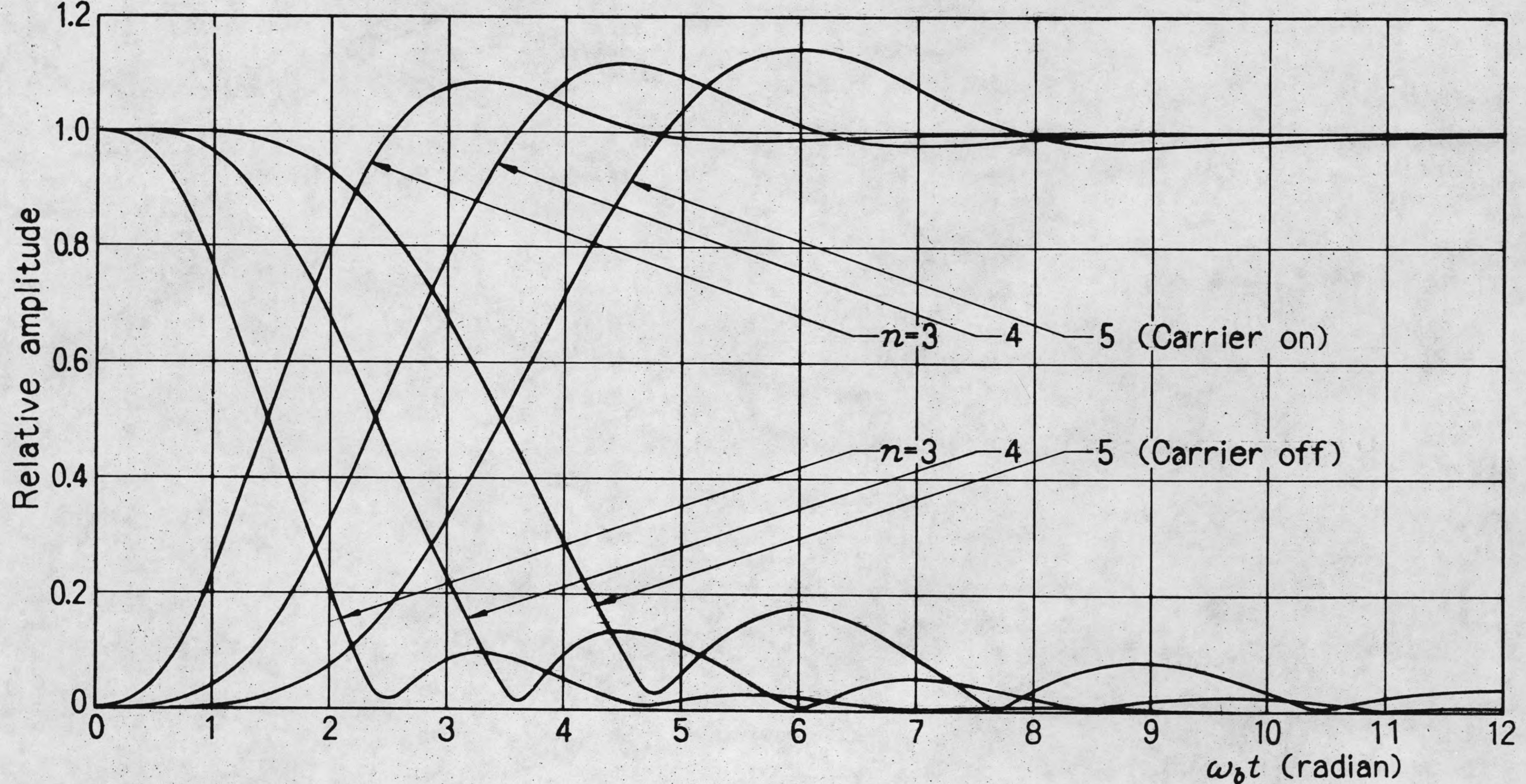

Fig. 24 Envelope function of Chebyshev filter transient, A_p = 0.03db, x_c = 0.2.

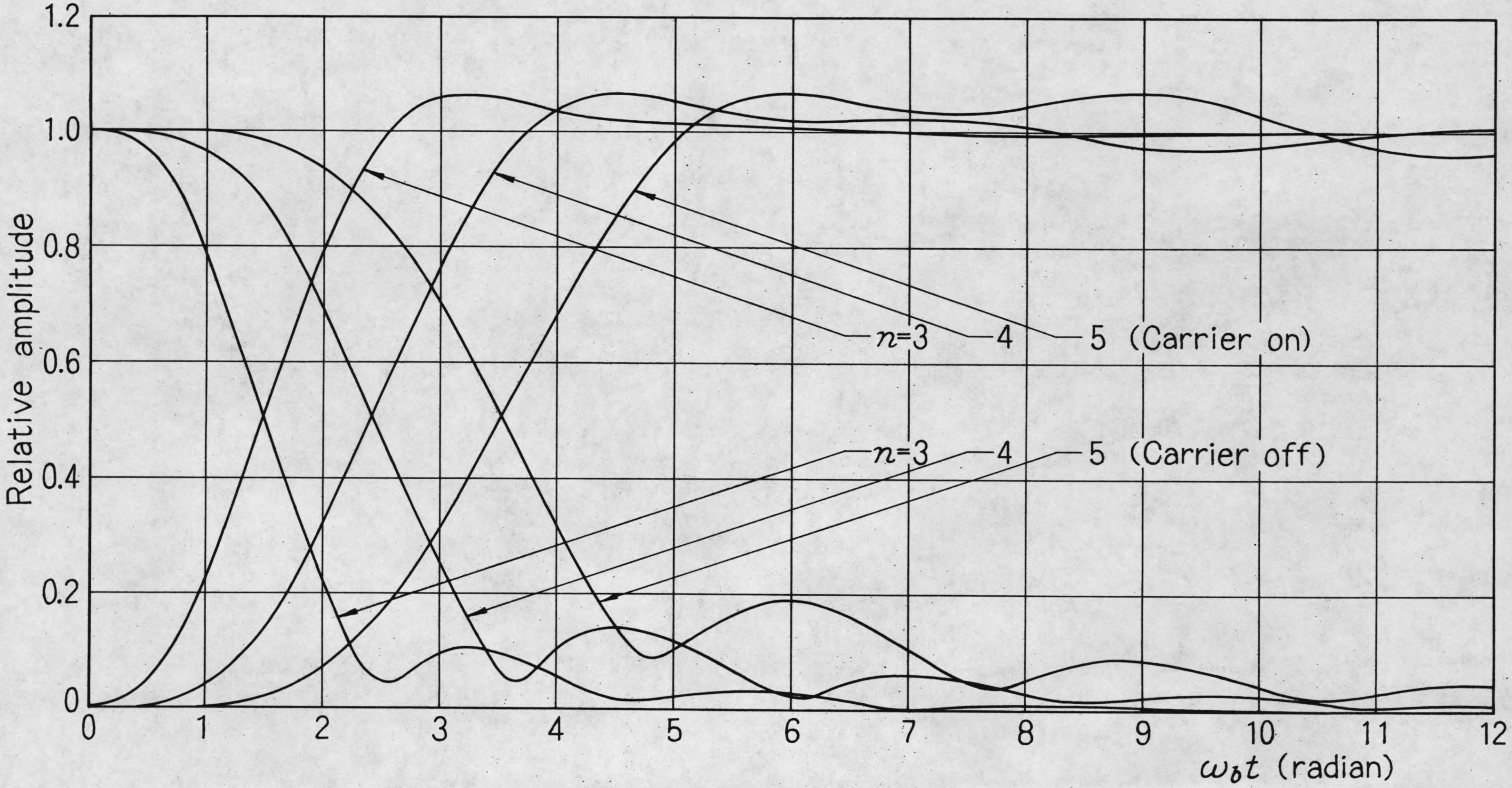

Fig. 25 Envelope function of Chebyshev filter transient, A_p = 0.03db, x_c = 0.4.

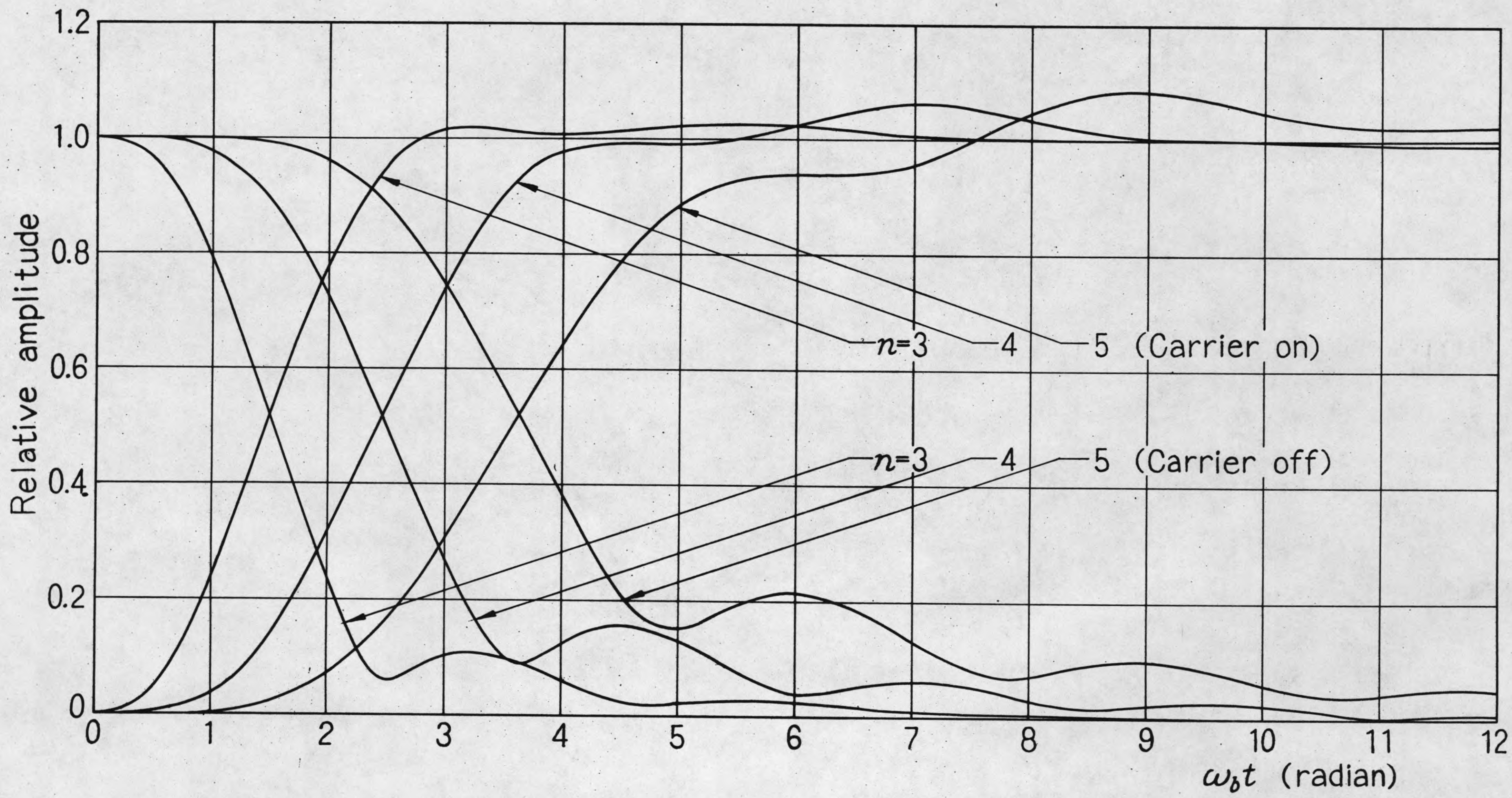

Fig. 26 Envelope function of Chebyshev filter transient, A_p = 0.03db, x_c = 0.6.

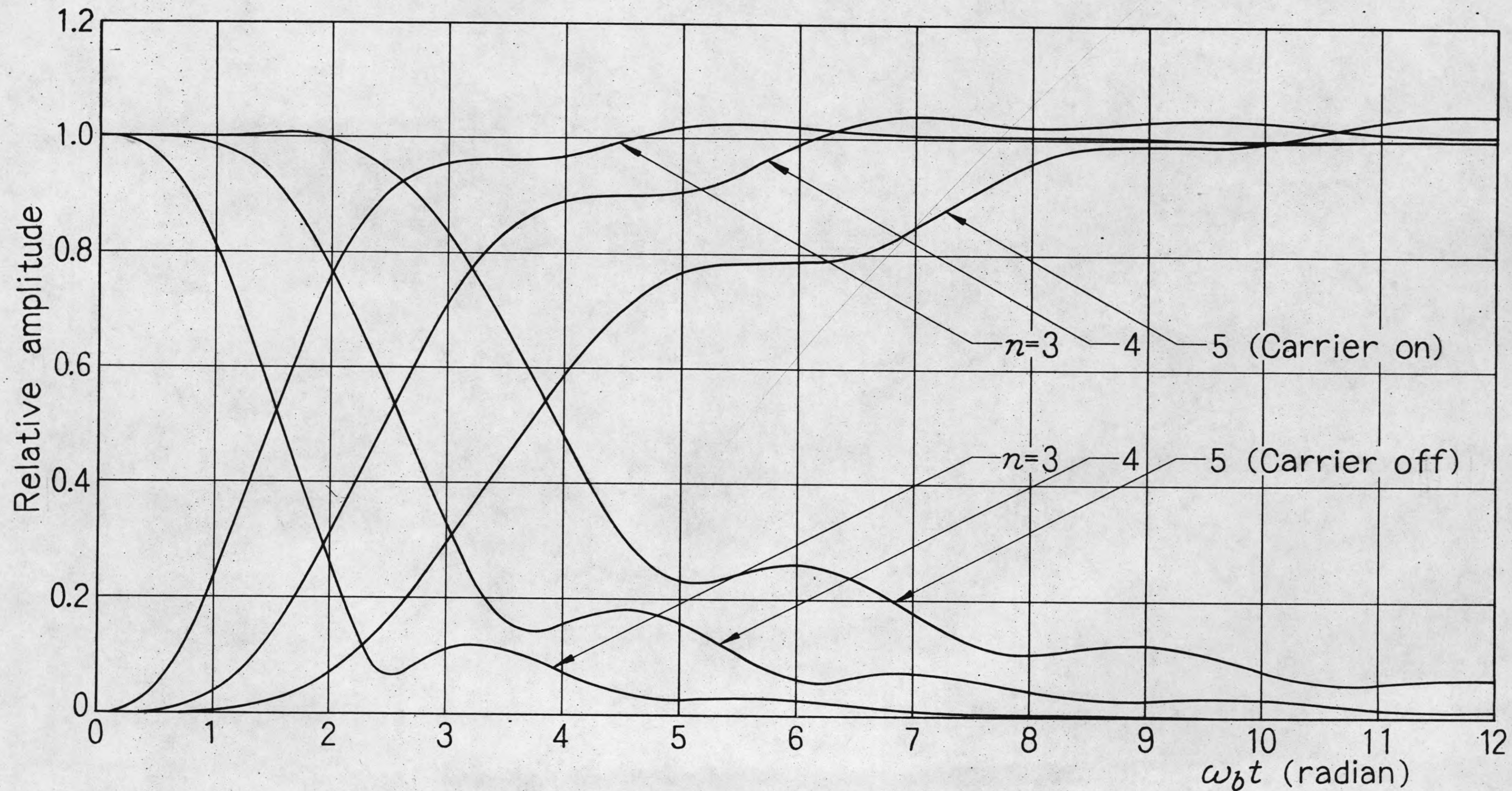

Fig. 27 Envelope function of Chebyshev filter transient, $A_p = 0.03$db, $x_c = 0.8$.

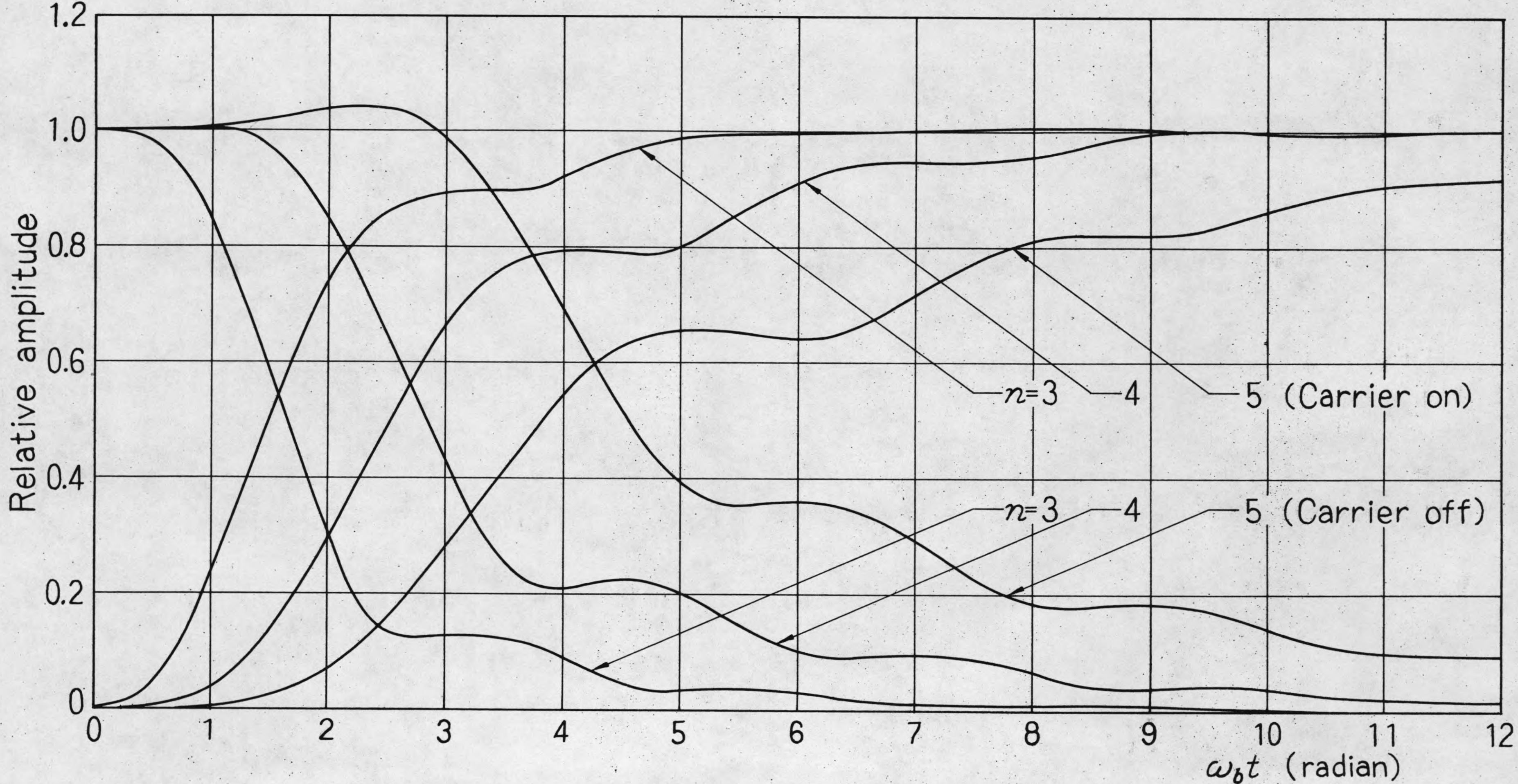

Fig. 28 Envelope function of Chebyshev filter transient, A_p = 0.03db, x_c = 1.

ACKNOWLEDGMENT

The author is grateful to Professor M. E. Van Valkenburg for many valuable suggestions and criticisms.

Footnotes

1. Part of this report is published in Japanese. See [1].

2. A set of group delay curves have been published recently. See [5].

REFERENCES

1. Kishi, G., "Transient Responses of Butterworth and Chebyshev Filter Networks", Jour. Inst. Elec. Comm. Engr., Japan, Vol. 41, pp. 979-984, October 1958.

2. Valley, G. E. and Wallman, H., Vacuum Tube Amplifiers, McGraw-Hill Book Co., Inc., New York, 1948.

3. Kishi, G., "Group Delay Characteristics of Butterworth and Chebyshev Filter Networks", Jour. Inst. Elec. Comm. Engrs., Japan, Vol. 38, pp. 691-696, September 1955.

4. Henderson, K. W. and Kautz, W. H., "Transient Responses of Conventional Filters", IRE Trans. Circuit Theory, Vol. CT-5, pp. 333-347, December 1958.

5. Lim, Jim-Twan and Scanlan, J. O., "Group Delay Characteristics of Chebyshev Filters", IEEE Trans. on Circuit Theory, Vol. CT-11, pp. 427-430, September 1964.