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J. B. CRUZ, JR.
P. V. KOKOTOVIC
W. R. PERKINS

UNIVERSITY OF ILLINOIS – URBANA, ILLINOIS

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J. B. Cruz, Jr., P. V. Kokotovic, and W. R. Perkins
University of Illinois
Urbana, Illinois, U. S. A.

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J. B. Cruz, Jr., P. V. Kokotović, and W. R. Perkins
University of Illinois
Urbana, Illinois, U.S.A.

1. Introduction

In this paper we establish an approximation property of optimum controls expressible by Taylor series with respect to a parameter. We show that if an r-th order approximation is used instead of the exact optimum control, the optimum performance is approximated to order (2r+1). We also present several applications of this result to the design of near-optimum large systems.

2. Approximation Property of Near-Optimum Control

Consider a system

$$\dot{x} = f(x,u,q,t) \tag{1}$$

where the state x and the control u are n-dimensional and m-dimensional vectors respectively, $q \in [q_L, q_U]$ is a <u>scalar</u> time invariant parameter, $t \in [t_o, t_f]$ is time, and t_o and t_f are fixed time instants. The performance index is

$$J(u) = \pi[x(t_f,q),q] + \int_{t_o}^{t_f} V(x,u,q,t)\,dt, \tag{2}$$

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the initial state is

$$x(t_o,q) = x^o(q), \tag{3}$$

and the final state is free.

Definition 1. A control $u^* = u^*(t,q)$ continuous in t, $t\epsilon[t_o,t_f]$ for every $q\epsilon[q_L,q_U]$ is called the optimum control if it is the unique control u which minimizes (2) subject to (1) and (3), for every $q\epsilon[q_L,q_U]$.

Assume that $u^*(t,q)$ exists and that f, V, $x_o(q)$, and π are sufficiently smooth to insure the existence of $\partial^i u^*/\partial q^i$ for i = 0,1,..., i = 0,1,...,N at $q_o\epsilon[q_L,q_U]$. In what follows, these and other derivatives with respect to q are to be interpreted as evaluated at $q = q_o$.

Definition 2. A control $u^r = u^r(t,q)$ is called an r^{th} order near-optimum control if

$$\frac{\partial^i u^r}{\partial q^i} = \frac{\partial^i u^*}{\partial q^i},\ i = 0,1,\ldots,r \leq N\ . \tag{4}$$

Note that u^r is not unique because of the arbitrariness of higher order derivatives.

Theorem 1 (Main Theorem [1]): For the system (1) with initial state (3) and final state free, the performance indices $J(u^*)$ and $J(u^r)$ obtained with $u = u^*$ and $u = u^r$ have the property that

$$\frac{d^i J(u^*)}{dq^i} = \frac{d^i J(u^r)}{dq^i},\ i = 0,1,\ldots,(2r+1) \leq N. \tag{5}$$

Theorem 1 will be proved using the following two lemmas.

Lemma 1: For $0 \leq j \leq N$,

$$\frac{d^j[\delta J(u)]}{dq^j} = \sum_{k=0}^{j} \frac{j!}{(j-k)!k!} \left\{ \left[\frac{\partial^{j-k}}{\partial q^{j-k}} (\hat{\nabla}_x \pi - p)' \delta\xi_k \right] \Bigg|_{t=t_f} \right.$$

$$+ [\eta'_{j-k} \delta\xi_k]\Big|_{t=t_o} + \int_{t_o}^{t_f} \left[\frac{\partial^{j-k}}{\partial q^{j-k}} (\hat{\nabla}_u H)' \delta\omega_k + \right.$$

$$\left. \left. \frac{\partial^{j-k}}{\partial q^{j-k}} (\hat{\nabla}_x H + \dot{p})' \delta\xi_k + \frac{\partial^{j-k}}{\partial q^{j-k}} (\hat{\nabla}_p H - \dot{x})' \delta\eta_k \right] dt \right\} \tag{6}$$

where p is the costate, $H = V + p'f$ is the Hamiltonian, ω_k, ξ_k, and η_k denote the k^{th} order derivative with respect to q of u, x, and p, and $\hat{\nabla}_x \pi$, $\hat{\nabla}_x H$, and $\hat{\nabla}_u H$ denote $\nabla_x \pi$, $\nabla_x H$, $\nabla_p H$, and $\nabla_u H$ respectively, expressed as functions of t and q only.

Proof of Lemma 1: Append $p'(f-\dot{x})$ to the integrand in (2), form δJ, integrate $p'\dot{x}$ by parts, apply Leibnitz rule for the j^{th} derivative of a product, and set $q = q_o$.

Lemma 2: For r and j satisfying $0 \leq j \leq (2r+1) \leq N$, and for any control u,

$$\frac{d^j J(u)}{dq^j} = \sum_{k=r+1}^{j} \frac{j!}{(j-k)!k!} \left\{ \left[\frac{\partial^{j-k}}{\partial q^{j-k}} (\hat{\nabla}_x \pi - \eta_{j-k})' \xi_k \right] \Bigg|_{t=t_f} \right.$$

$$+ [\eta'_{j-k} \xi_k]\Big|_{t=t_o} + \int_{t_o}^{t_f} \left[\left(\frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_u H \right)' \omega_k + \right.$$

$$\left. \left. \left(\frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_x H + \dot{\eta}_{j-k} \right)' \xi_k + \left(\frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_p H - \dot{\xi}_{j-k} \right)' \eta_k \right] dt \right\}$$

$$+ g_o\Big|_{t=t_f} + \int_{t_o}^{t_f} e_o \, dt \tag{7}$$

where g_o is a function of $\xi_o, \ldots \xi_r$, and e_o is a function of $\xi_o, \ldots, \xi_r$, $\dot{\xi}_o, \ldots, \dot{\xi}_r$, $\eta_o, \ldots, \eta_r$, $\omega_o, \ldots, \omega_r$.

Proof of Lemma 2: In Lemma 1, for $j-k < k$,

$$\frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_u H, \frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_x H, \frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_p H, \text{ and } \frac{\partial^{j-k}}{\partial q^{j-k}} \hat{\nabla}_x \pi$$

do not involve ω_i, ξ_i, or η_i for $i \geq k$. Hence d^iJ/dq^j is linear in ω_k, ξ_k, and η_k for $j-k < k$, and the coefficients of ω_k, ξ_k, and η_k are the same as the coefficients of $\delta\omega_k$, $\delta\xi_k$, and $\delta\eta_k$ in (6). Denote by $(r+1)$ the smallest k for which $(j-k) < k$. Hence $j \leq (2r+1)$. Take the j^{th} derivative of $p'\dot{x}$, evaluate at $q = q_o$, and integrate

$$\sum_{k=r+1}^{j} \frac{j!}{(j-k)!k!} \eta'_{j-k} \dot{\xi}_k$$

by parts.

Proof of Theorem 1. The proof consists of evaluating (7) for (a) $u = u^*$ and (b) for $u = u^r$. The two expressions for the j^{th} derivative of $J(u^*)$ and $J(u^r)$ are then compared.

(a) For u to be optimum, $u = u^*$, it is necessary that

$$\frac{d^j}{dq^j} [\delta J(u^*)] = 0, \; j=0,1,\ldots,N. \tag{8}$$

Thus, from Lemma 1, it is necessary that

$$\frac{\partial^i}{\partial q^i} (\hat{\nabla}_p H - \dot{x}) = 0 \tag{9}$$

$$\frac{\partial^i}{\partial q^i} (\hat{\nabla}_x H + \dot{p}) = 0 \tag{10}$$

and

$$\frac{\partial^i}{\partial q^i}(\hat{\nabla}_u H) = 0 \tag{11}$$

with boundary conditions

$$\left.\frac{\partial^i x}{\partial q^i}\right|_{t=t_o} = \frac{\partial^i x^o}{\partial q^i} \tag{12}$$

and

$$\left.\left[\frac{\partial^i}{\partial q^i}(\hat{\nabla}_x \pi - p)\right]\right|_{t=t_f} \tag{13}$$

where $i = 0,1,\ldots,N$. The conditions (9), (10), (11), (12), and (13), define the derivatives ω_i^*, ξ_i^*, and η_i^*, $i = 0,1,\ldots,N$. Using these derivatives (7) becomes

$$\frac{d^j J(u^*)}{dq^j} = \sum_{k=r+1}^{j} [\eta^{*'}_{j-k}\xi^*_k]\Big|_{t=t_o} + g_o(\xi^*_o,\ldots,\xi^*_r)\Big|_{t=t_f} + \int_{t_o}^{t_f} e_o(\xi^*_o,\ldots,\xi^*_r,\ \dot{\xi}^*_o,\ldots,\dot{\xi}^*_r,\ \omega^*_o,\ldots,\omega^*_r,\ \eta^*_o,\ldots,\eta^*_r)\,dt \tag{14}$$

(b) Next we evaluate (7) for $u = u^r$. Denote the state and costate obtained by using $u = u^r$ in (9) and (10) by x^r and p^r and denote the derivatives of u^r, x^r, and p^r by ω_i^r, ξ_i^r, and η_i^r, $i = 0,1,\ldots,r$. By construction, $\omega_i^r = \omega_i^*$ for $i = 0,1,\ldots,r$. Hence $\xi_i^r = \xi_i^*$, and $\eta_i^r = \eta_i^*$ for $i = 0,1,\ldots,r$. Moreover, ω_i^r, ξ_i^r, and η_i^r, for $i = 0,1,\ldots,r$, satisfy (9), (10), (11), and (13) for $i = 0,1,\ldots,r$. Finally, since the initial condition function $x^o(q)$ in (3) is specified, x^r satisfies (12) for all i.

Thus, $d^iJ(\bar{u}^r)/dq_j$ is equal to the right hand side of (14) which proves (5).

Theorem 1 can be generalized to the case when q is an s-dimensional vector.

Definition 3: Given an s-dimensional vector r whose components are integers $r_1, r_2, \ldots, r_3$, $\bar{u}^r$ is said to be an rth order near optimal control, if

$$\frac{\partial^k \bar{u}^r}{\partial q_1^{k_1} \partial q_2^{k_2} \ldots \partial q_s^{k_s}} = \frac{\partial^k u^*}{\partial q_1^{k_1} \partial q_2^{k_2} \ldots \partial q_s^{k_s}} \qquad (15)$$

for all $k_1, \ldots, k_s$ such that $0 \leq k_1 \leq r_1$, $0 \leq k_2 \leq r_2, \ldots, 0 \leq k_s \leq r_s$, $k_1 + k_2 + \cdots + k_s = k$, $r_1 + r_2 + \cdots + r_s \leq N$.

Theorem 2 [1]. For the system in (1) with initial state (3), final state free, performance index (2), and time invariant parameter vector $q \in Q \in E^s$, where Q is compact and convex,

$$\frac{\partial^j J(\bar{u}^r)}{\partial q_1^{j_1} \partial q_2^{j_2} \ldots \partial q_s^{j_s}} = \frac{\partial^j J(u^*)}{\partial q_1^{1_1} \partial q_2^{j_2} \ldots \partial q_s^{j_s}} \qquad (16)$$

for all $j_1, j_2, \ldots, j_s$, such that $0 \leq j_1 \leq (2r_1+1)$, $0 \leq j_2 \leq (2r_2+1), \ldots$, $0 \leq j_s \leq (2r_s+1)$, $j_1+j_2+ \cdots + j_s = j$, and $\sum_{i=1}^{s} (2r_s+1) \leq N$.

The proof of Theorem 2 is similar to that of Theorem 1.

The following sections survey some applications of Theorem 1 together with related perturbation and imbedding methods [1-15] to expansions of optimum controls. A perturbation method for the synthesis of systems with uncertain parameters [16] and imbedding methods for the synthesis of large scale systems [15,17] are presented below.

3. Perturbation of Uncertain Parameters

A first order expansion is used here to approximate the optimum control in the presence of a small perturbation in some parameters $q_1, \ldots, q_s$ which cannot be determined prior to the operation of the system. We now consider q as an s-dimensional vector and Δq as its deviation from q_o. Define the first order near-optimum control (see Definition 2) as

$$u^1 = u^*(t,q_o) + \frac{\partial u^*}{\partial q} \Delta q. \tag{17}$$

This control may be expressed in feedback form as [15]

$$u^1 = u^*(t,q_o) - H_{uu}^{-1}[H'_{ux} + f'_u P + (H_{qu} + f'_u G)(\xi_1^{*\prime}\xi_1^*)^{-1}\xi_1^{*\prime}]\Delta x(t,q) \tag{18}$$

where P and G satisfy

$$\dot{P} + PA + A'P - PSP + Q = 0; \quad P(t_f) = \pi_{xx}\Big|_{t=t_f} \tag{19}$$

and

$$\dot{G} - PSG + A'G + PW - Z = 0; \quad G(t_f) = \pi_{qx}\Big|_{t=t_f} \tag{20}$$

where

$$A = f_x - f_u H_{uu}^{-1} H'_{ux}$$

$$S = f_u H_{uu}^{-1} f'_u$$

$$Q = H_{xx} - H_{ux} H_{uu}^{-1} H'_{ux}$$

$$W = f_q - f_u H_{uu}^{-1} H_{qu}$$

$$Z = -H_{qx} + H_{ux} H_{uu}^{-1} H_{qx}.$$

$\Delta x(t,q)$ is

$$\Delta x(t,q) = x(t,q) - x^*(t,q_o). \tag{21}$$

It is assumed that $(\xi_1^{*'}\xi_1^*)$ has an inverse for $t \in (t_o, t_f]$. Otherwise, $(\xi_1^{*'}\xi_1^*)^{-1}$ is taken as the pseudo inverse. The case where the measurement of x is noisy is discussed in [17]. Similar expansion for initial state uncertainty is discussed in [2,3,10].

From Theorem 2, it is seen that this first order near optimal control yields a performance index which approximates the optimum one to third order.

4. Application to Linear State Regulator

The following corollary is an application of Theorem 1 to the optimum linear state regulator. Let (1) and (3) be

$$\dot{x} = Ax + Bu \tag{22}$$

$$J = \frac{1}{2}\int_o^T (x'Qx + u'Ru)\,dt \tag{23}$$

where A, B, Q, and R are continuously differentiable with respect to a parameter q in the neighborhood of $q = q_o$ up to order (N+1). With the usual assumption of the linear state regulator problem [18] the optimum feedback control is

$$u_p = -R^{-1}B'Px_p \tag{24}$$

where P satisfies

$$\dot{P} = -PA - A'P + PSP - Q; \quad P(T,q) = 0 \tag{25}$$

$S = BR^{-1}B'$, and x_p is the state of the optimum system

$$\dot{x}_p = (A - SP)x_p. \tag{26}$$

Define the control

$$u_m = -R^{-1}B'Mx_m \tag{27}$$

where

$$M(t,q) = \sum_{i=0}^{r} \frac{\partial^i P(t,q_o)}{\partial q_i} \frac{(q - q_o)^i}{i!} \tag{28}$$

and x_m is the state of the system

$$\dot{x}_m = (A - SM)x_m. \tag{29}$$

Corollary 1. The control u_m is an r^{th} order near optimum control and hence

$$\frac{d^i J(u_p)}{dq^i} = \frac{d^i J(u_m)}{dq^i}, \quad i = 0,1,\ldots,(2r+1). \tag{30}$$

Proof of Corollary 1. Differentiating (26) and (29), it is seen that

$$\frac{\partial^j x_m}{\partial q^j} = \frac{\partial^j x_p}{\partial q^j}, \quad j = 0,1,\ldots,r. \tag{31}$$

Differentiating (24) and (27), and comparing, it is seen that u_m is an r^{th} order near-optimum control, and hence (30) holds.

This corollary and a proof independent of Theorem 1 appears in [13].

5. Regular Perturbation of a Coupling Parameter

In this application the control expansion is used to simplify the design of a large scale linear system in which q plays the role of a coupling parameter ϵ

$$\dot{x} = Ax + Bu = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & \epsilon A_{12} \\ \epsilon A_{21} & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & \epsilon B_{12} \\ \epsilon B_{21} & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{32}$$

where x_1 and x_2 are n_1- and n_2-dimensional substates, u_1 and u_2 are r_1- and r_2-dimensional subcontrols, ϵ is a scalar coupling parameter, and A_1, A_{12}, A_{21}, A_2, B_1, B_{12}, B_{21}, B_2 do not depend on ϵ.

The performance index is (16) with

$$A = \begin{bmatrix} Q_1 & \epsilon Q_{12} \\ \epsilon Q_{12}' & Q_2 \end{bmatrix} \; ; \; R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} . \tag{33}$$

Instead of the optimum control (24), we use the near-optimum control (27) where P is the solution of (25), and M is given by (28) where q is replaced by the coupling parameter ϵ. When the coupling parameter is zero, the system decouples into two lower order subsystems. By expanding P at $\epsilon = 0$ the coefficients of the series (28) can be obtained from decoupled equations:

Theorem 3: [14]

The even order partial derivations of P are block diagonal

$$\frac{\partial^{2i} P}{\partial \epsilon^{2i}} = \begin{bmatrix} P_1^{2i} & 0 \\ 0 & P_2^{2i} \end{bmatrix} \tag{34}$$

and the odd order partial derivatives of P are block antidiagonal

$$\frac{\partial^{2i+1}}{\partial\epsilon^{2i+1}} P = \begin{bmatrix} 0 & P_{12}^{2i+1} \\ (P_{12}^{2i+1})' & 0 \end{bmatrix} . \tag{35}$$

Theorem 4 [14].

(a) The zero order submatrices $P_1(t,0)$ and $P_2(t,0)$ are obtained from

$$\dot{P}_1 = -P_1A_1 - A_1'P_1S_1P_1 - Q_1; \quad P_1(T,0) = 0 \tag{36}$$

and
$$\dot{P}_2 = -P_2A_2 - A_2'P_2 + P_2S_2P_2 - Q_2; \quad P_2(T,0) = 0 \tag{37}$$

where

$$S = \begin{bmatrix} S_1 & S_{12} \\ S_{12}' & S_2 \end{bmatrix} = BR^{-1}B' . \tag{38}$$

(b) The even order submatrices P_1^{2i} and P_2^{2i} for $i = 1,2,...$ are obtained from

$$\dot{P}_1^{2i} = -P_1^{2i}G_1 - G_1'P_1^{2i} - F_1^{2i-1}; \quad P_1^{2i}(T,0) = 0 \tag{39}$$

$$\dot{P}_2^{2i} = -P_2^{2i}G_2 - G_2'P_2^{2i} - F_2^{2i-1}; \quad P_2^{2i}(T,0) = 0 \tag{40}$$

where

$$G = \begin{bmatrix} G_1 & G_{12} \\ G_{21} & G_2 \end{bmatrix} = A - SP \tag{41}$$

F_1^{2i-1} and F_2^{2i-1} do not depend on P_1^{2i} and P_2^{2i}.

(c) The odd order submatrices $P_{12}^{2i+1} = (P_{21}^{2i+1})'$ for $i = 0,1,...$ are obtained from

$$\dot{P}_{12}^{2i+1} = -P_{12}^{2i+1}G_2 - G_1'P_{12}^{2i+1} - F_{12}^{2i}; \quad P_{12}^{2i+1}(T,0) = 0 \tag{42}$$

where F_{12}^{2i} does not depend on P_{12}^{2i+1}.

In applications the parameter ϵ may be a physical parameter or it may be introduced for computational simplification. In view of Theorem 3, an r^{th} order near-optimum control for a large scale system may be obtained by solving only two lower order uncoupled Riccati equations and a few uncoupled lower order linear equations. By Corollary 1, this control yields a near-optimum performance index which approximates the optimum performance index up to order $(2r+1)$.

6. Singular Perturbation of a Coupling Parameter

In this application the system reduces to a lower order dynamic subsystem and a nondynamic subsystem when the coupling parameter λ is zero. Although the subsystems are still coupled, the lowering of the dynamic order reduces computation in the design. For this case, the system description is

$$\frac{d\tilde{x}}{dt} = \tilde{A}_1\tilde{x} + \tilde{A}_2\tilde{z} + \tilde{B}_1 u \tag{43}$$

$$\lambda \frac{d\tilde{z}}{dt} = \tilde{A}_3\tilde{x} + \tilde{A}_4\tilde{z} + \tilde{B}_2 u \tag{44}$$

where λ is a given small positive constant and where $\tilde{x}$ is an n-dimensional vector, $\tilde{z}$ is an m-dimensional vector and u is an r-dimensional vector. The performance index is

$$J = \tfrac{1}{2}\int_{o}^{T} (\tilde{x}'Q\tilde{x} + u'Ru)\,dt \tag{45}$$

where Q is positive semidefinite and R is positive definite. Under the assumptions that A_1, A_2, A_3, A_4, B_1, and B_2 are continuous functions of t for any given $\lambda \geq 0$, the optimum matrix $\tilde{P}$, for $\lambda > 0$ may be obtained from

$$\dot{\tilde{P}} = -\tilde{P}\tilde{A} - \tilde{A}'P + \tilde{P}\tilde{B}R^{-1}\tilde{B}'\tilde{P} - Q_o, \quad \tilde{P}(T,\lambda) = 0 \tag{46}$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \frac{\tilde{A}_3}{\lambda} & \frac{\tilde{A}_4}{\lambda} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \frac{\tilde{B}_2}{\lambda} \end{bmatrix}, \quad Q_o = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}. \tag{47}$$

The optimum control is

$$\tilde{u} = -\tilde{K}_1 x - \tilde{K}_2 z \tag{48}$$

where

$$\tilde{K} = [\tilde{K}_1 \quad \tilde{K}_2] = R^{-1}\tilde{B}'\tilde{P}. \tag{49}$$

It is shown in [16] that it is possible to expand K in a series in λ about $\lambda = 0$. Conditions which guarantee the existence and differentiability of K as $\lambda \to 0+$ are given in Theorem 5 below. It is assumed that symbols without the tilde (~) are evaluated at $\lambda = 0$:

<u>Theorem 5</u>. [16]. If in the system (43) and (44) with the performance index (45),

(a) A_4 is negative definite

(b) $\tilde{A}_1, \tilde{A}_2$, and $\tilde{B}_1$ have continuous partial derivatives with respect to λ and

(c) $\tilde{A}_3, \tilde{A}_4$, and $\tilde{B}_2$ have continuous second partial derivatives with respect to λ

then

$$\lim_{\lambda \to 0+} K \quad \text{and} \quad \lim_{\lambda \to 0+} \frac{\partial K}{\partial \lambda}$$

exists and given by

$$K_1(0+) = R^{-1}(B_1 - A_2A_4^{-1}B_2)'M \tag{50}$$

$$K_2(0+) = 0 \tag{51}$$

$$\left.\frac{\partial K_1}{\partial \lambda}\right|_{\lambda=0+} = R^{-1}(\beta_1'M + B_1'W_1 + \beta_2'L_2' + B_2'W_2') \tag{52}$$

and

$$\left.\frac{\partial K_2}{\partial \lambda}\right|_{\lambda=0+} = R^{-1}(B_1'L_2 + B_2'W_3). \tag{53}$$

M is the solution of the Riccati equation

$$\frac{dM}{dt} = -MF - F'M + MGR^{-1}G'M - Q; \; M(T) = 0 \tag{54}$$

and L_2 is given by

$$L_2 = -MA_2A_4^{-1} \, . \tag{55}$$

W_1, W_2, and W_3 are the solutions of

$$\frac{dW_1}{dt} = Z_1'W_1 + W_1Z_1 + Z_2'W_2' + W_2Z_2 + Y_1, \; W_1(T) = 0 \tag{56}$$

$$0 = -W_1A_2 - W_2A_4 + (MN + L_2N_2 - A_3')W_3 - \left(\frac{dL_2}{dt} + M\alpha_2 + L_2\alpha_4 + A_1'L_2 - MNL_2 - L_2N'L_2\right) \tag{57}$$

$$0 = W_3A_4 + A_4'W_3 + L_2'A_2 + A_2'L_2 \tag{58}$$

$$Z_1 = (-A_1 + N_1M + NL_2') \tag{59}$$

$$Z_2 = (-A_3 + N'M + N_2L_2') \tag{60}$$

$$Y_1 = -M\alpha_1 - \alpha_1'M - L_2\alpha_3 - \alpha_3'L_2' + M\Gamma_1M + M\Gamma L_2' + L_2\Gamma'M + L_2\Gamma_2L_2' \tag{61}$$

where

$$\tilde{\alpha}_i = \partial\tilde{A}_i/\partial\lambda,\ i = 1,2,3,4,\ \tilde{\beta}_i = \partial\tilde{B}_i/\partial\lambda,\ i = 1,2,$$

$$\tilde{N}_i = (\tilde{B}_iR^{-1}\tilde{B}_i'),\ i = 1,2,\ \tilde{N} = \tilde{B}_1R^{-1}\tilde{B}_2',\ \tilde{\Gamma}_i = \partial\tilde{N}_i/\partial\lambda,\ i = 1,2,$$

$$\tilde{\Gamma} = \partial\tilde{N}/\partial\lambda,\ F = A_1 - A_2A_4^{-1}A_3,\ G = B_1 - A_2A_4^{-1}B_2.$$

For details of the proof, see [16]. Note that all the calculations are for a reduced order system.

It can be shown that a result similar to Corollary 1 applies to the singular perturbation case. For example, if we use an approximation

$$K(\lambda) \cong K(0) + \frac{\partial K}{\partial\lambda}\lambda$$

the near optimum performance index approximates the optimum performance index to third order.

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13. ABSTRACT

In this paper we establish an approximation property of optimum controls expressible by Taylor series with respect to a parameter. We show that if an r-th order approximation is used instead of the exact optimum control, the optimum performance is approximated to order (2r+1). We also present several applications of this result to the design of near-optimum large systems.

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