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THE EULER-POISSON EQUATION AND OPTIMAL LINEAR CONTROL

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ABSTRACT

The calculus of variations is applied to the problem of finding the optimum control for a completely controllable n^{th} order stationary linear system with quadratic performance index. A simple procedure, which involves only factoring a $2n^{th}$ order even polynomial into a product of anti-Hurwitz and Hurwitz polynomials, emerges from the first variation. Moreover, an easily performed test for the sufficiency of such solutions as optimal is obtained from the second variation. Conditions under which the closed loop system is stable for the optimum control law are discussed. The examples illuminate the versatility of such an approach — among them being an application to the problem of obtaining singular solutions when the cost of control does not enter into the performance index.

INTRODUCTION

Since the introduction of the maximum principle, classical calculus of variations has been usually relegated to the secondary status of proving the maximum principle. However, the calculus of variations is still a powerful tool, and for problems to which it is applicable it can often yield solutions in an appreciably simpler manner than other methods. In particular, the Euler equations — the fundamental necessary condition arising from the first variation — can be used to determine simply the control laws for linear stationary plants. Moreover, the Legendre condition, a consequence of the second variation, may demonstrate the sufficiency of such solutions.

Certain advantages to the theoretical study of optimal systems behavior are also to be found in the calculus of variations approach. Because the system variable is always in evidence, it is often easy to assay the significance of certain terms in the performance index or the system equations themselves. To this end condition under which a closed loop optimum system will be stable, or when it will require a singular solution (i.e., not all of the boundary conditions can be independently specified) will be discussed.

FORMULATION OF THE PROBLEM

The stationary system to be considered will be assumed to be characterized by the n-dimensional state equation

$$\overset{\circ}{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u. \tag{1}$$

Moreover, under the assumption that the system is completely controllable, the matrices $\underline{A}$ and $\underline{b}$ in (1) can always be put in the canonical form[1]

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \tag{2a}$$

and

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} . \tag{2b}$$

Hence, (1) is completely equivalent to the n^{th} order scalar differential equation[2]

$$\sum_{i=0}^{n} a_i x^{(i)} = b u, \tag{3}$$

where $a_n = 1$ and the remaining coefficients are arbitrary.

The general problem to be treated is that of finding the control law u which takes the system (1) from a given arbitrary initial state

$$\underline{x}(0) = \underline{x}_o \tag{4a}$$

to the final state

$$\underline{x}(\infty) = \underline{0}, \tag{4b}$$

while simultaneously minimizing the performance index

$$I = \int_o^\infty (\underline{x}^T \underline{Q} \underline{x} + \rho u^2) dt. \tag{5}$$

Here, $\underline{Q}$ is a constant nonnegative definite matrix and ρ is a positive number.

THE CONTROL LAW FROM THE EULER-POISSON EQUATION

Before we can take the first variation of the performance index, it must be put in a form more amenable to such a treatment. We recognize from (1) and (2) that the state vector $\underline{x}$ can be written[2]

$$\underline{x} = \begin{bmatrix} x \\ x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(n-2)} \\ x^{(n-1)} \end{bmatrix}, \tag{6}$$

while, from (3),

$$u = \frac{1}{b}\left[\sum_{i=0}^{n} a_i \; x^{(i)}\right]. \tag{7}$$

Thus, if the elements of $\underline{Q}$ are denoted by q_{ij} $(q_{ij} = q_{ji})$, in view of (6) and (7) the performance index (5) can be rewritten

$$I = \int_o^\infty \left[\sum_{i,j=1}^{n} q_{ij} \, x^{(i-1)} x^{(j-1)} + \frac{\rho}{b^2} \sum_{i,j=1}^{n+1} a_{i-1} \, a_{j-1} \, x^{(i-1)} \, x^{(j-1)}\right] dt. \tag{8}$$

From the calculus of variations, the fundamental necessary condition for an extremum is that the integrand in (8),

$$F(x, x^{(1)}, \ldots, x^{(n)}) = \sum_{i,j=1}^{n+1} (q_{ij} + \frac{\rho}{b^2} a_{i-1} \, a_{j-1}) \, x^{(i-1)} x^{(j-1)} \tag{9}$$

(where

$$q_{i,n+1} = q_{n+1,j} = 0 \tag{10}$$

for all i or j), satisfies the Euler-Poisson equation[3]:

$$\sum_{k=0}^{n} (-1) \, \frac{d^k}{dt^k} \left(\frac{\partial F}{\partial x^{(k)}}\right) = 0. \tag{11}$$

From (9)

$$\frac{\partial F}{\partial x^{(k)}} = 2 \sum_{i=1}^{n+1} (q_{k+1,i} + \frac{\rho}{b^2} a_k \, a_{i-1}) \, x^{(i-1)}; \tag{12}$$

hence, the Euler-Poisson equation is

$$2 \sum_{k=0}^{n} \sum_{i=1}^{n+1} \left[(-1)^k \left(q_{k+1,i} + \frac{\rho}{b^2} a_k \, a_{i+1} \right) x^{(k+i-1)} \right] = 0 \tag{13}$$

or

$$2 \sum_{k,\ell=0}^{n} \left[(-1)^k \left(q_{k+1,\ell+1} + \frac{\rho}{b^2} a_k \, a_\ell \right) x^{(k+\ell)} \right] = 0. \tag{14}$$

We see that all terms whose indices sum to an odd number, i.e.,

$$k + \ell = 2\,\nu - 1, \quad \nu = 1,2,\ldots,n, \tag{15}$$

drop from consideration. This occurrence is not surprising since these terms were coefficients of exact differentials in (9), and hence do not figure in the optimal solution although they do figure in the ultimate value of the performance index[4]. Consequently, insofar as obtaining an optimum control law is concerned, the most general matrix $\underline{Q}$ might as well take the form[5]

$$\underline{Q} = \begin{bmatrix} q_{11} & 0 & q_{13} & 0 & q_{15} & \cdots \\ 0 & q_{22} & 0 & q_{24} & 0 & \cdots \\ q_{13} & 0 & q_{33} & 0 & q_{35} & \cdots \\ 0 & q_{24} & 0 & q_{44} & 0 & \cdots \\ - & - & - & - & - & - \end{bmatrix} . \tag{16}$$

Returning to the Euler-Poisson equation, we see that it takes the final form

$$\sum_{m=0}^{n} (-1)^m c_m \, x^{(2m)} = 0, \tag{17}$$

where the coefficients c_m are given by

$$c_m = 2 \sum_{\substack{k,\ell \\ k+\ell=2m}} \left(q_{k+1,\ell+1} + \frac{\rho}{b^2} a_k \, a_\ell \right), \ m=0,\ldots,n. \tag{18}$$

Since (17) is a linear differential equation, $x = e^{st}$ will yield an algebraic expression for s:

$$P(s^2) = \sum_{m=0}^{n} (-1)^m c_m s^{2m} = 0. \tag{19}$$

It is well-known in circuit theory that if this even polynomial in s has no purely imaginary roots (the exceptional case where imaginary roots occur will be discussed in the next section), its roots occur in quadrantal symmetry in the s-plane[6]; consequently, half of the solutions go to zero at infinity while the other half increase without bound. Because of the end-point boundary condition (4b), the unbounded solutions must be discarded. The polynomial $P(s^2)$, when it has no purely imaginary roots, can always be factored into a unique product of an anti-Hurwitz and a Hurwitz polynomial[7]:

$$P(s^2) = H(-s)\, H(s). \tag{20}$$

It is this Hurwitz polynomial which represents the desired solution; suppose

$$H(s) = \sum_{i=0}^{n} h_i s^i, \tag{21}$$

then the differential expression governing the optimal x is

$$\sum_{i=0}^{n} h_i x^{(i)} = 0 \tag{22}$$

or

$$x^{(n)} = -\frac{1}{h_n} \sum_{i=0}^{n-1} h_i x^{(i)}. \tag{23}$$

But, from (3),

$$x^{(n)} = -\sum_{i=0}^{n-1} a_i x^{(i)} + bu; \tag{24}$$

therefore, upon combining (23) and (24), we obtain an expression for the optimum control u in terms of x,

$$u = \frac{1}{b} \left[\sum_{i=0}^{n-1} \left(a_i - \frac{h_i}{h_n}\right) x^{(i)} \right], \tag{25}$$

or, from (6), in terms of the state variables,

$$u = \frac{1}{b}\left[\sum_{i=0}^{n-1} (a_i - \frac{h_i}{h_n})\ x_i\right]. \tag{26}$$

To this point a solution for u has been obtained which satisfies but one of the necessary conditions for an extremum. Before we continue to a simple sufficiency condition for minima, it is well to review the method of solution for u. For this solution, divorced from the accompanying theory, is of an extremely simple nature. Steps in the solution for $u(\underline{x})$:

1) Given

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{b}\ u \tag{27a}$$

in the form (2) and

$$I = \int_o^\infty (\underline{x}'\ \underline{Q}\ \underline{x} + \rho\ u^2)dt, \tag{27b}$$

solve for the coefficients

$$c_m = 2 \sum_{\substack{K,\ell \\ K+\ell=2m}} (q_{K+1,\ell+1} + \frac{\rho^2}{b^2} a_K\ a_\ell),\ m=0,\ldots,n; \tag{27c}$$

2) Form the even polynomial

$$P(s^2) = \sum_{m=0}^{n} (-1)^n\ c_m\ s^{2m}, \tag{27d}$$

and factor it into the unique product of an anti-Hurwitz and a Hurwitz polynomial,

$$P(s^2) = H(s)\ H(-s); \tag{27e}$$

3) Use the coefficients h_i of s^i in the Hurwitz polynomial H(s) to form the expression for u,

$$u = \frac{1}{b}\left[\sum_{i=0}^{n-1} (a_i - \frac{h_i}{h_n})\ x_i\right]. \tag{27f}$$

STABILITY AND SINGULARITY OF THE CLOSED LOOP SOLUTION

In the preceding section we saw that an optimum solution may be obtained so long as the polynomial $P(s^2)$ has no purely imaginary roots. If there are imaginary roots in $P(s^2)$, they given rise the terms of the form

$$K_1 \sin \omega t + K_2 \cos \omega t,$$

neither of which satisfy the boundary condition (4b) at $t = \infty$. Hence, the only way one might employ such a system would be by picking a very special set of initial conditions on $\underline{x}$ such that these oscillatory modes are not excited. In such a case since $\underline{x}_o$ is no longer completely arbitrary we might say that we have a singular solution (or none at all).

Although in general it requires a Sturm test to ascertain whether the polynomial $P(s^2)$ has any purely imaginary roots[8], there are some simple sufficient conditions under which the absence of imaginary roots can be guaranteed. To find these conditions we must investigate the polynomial $P(s^2)$, (19), for $s = j\omega$:

$$P(-\omega^2) = \sum_{m=0}^{n} c_m \omega^{2m}. \tag{28}$$

It is obvious that either

$$c_m > 0, \quad m=0,\ldots,n \tag{29a}$$

or

$$c_m < 0, \quad m=0,\ldots,n \tag{29b}$$

is a sufficient (though by no means necessary) condition for no imaginary roots of $P(s^2)$. It is under this condition – which does not necessarily exclude unstable plants – that the driven plant will be stable. The kind of stability about which we have been talking is of course strictly mathematical; the stability of the final physical system depends on the implementation of the control law. If right half s-plane poles of the original system have been "moved" into the left half plane in the final system, we might expect stability; if, on the other hand, right half s-plane poles have been cancelled in the final system by right half s-plane zeros, we would be naive to expect such a state of affairs to persist.

SUFFICIENT CONDITION FOR THE SOLUTION OF THE EULER-POISSON EQUATION TO PROVIDE A MINIMUM

Ordinarily, to guarantee that the solution to the Euler-Poisson equation (17 and 18) provides a local minimum, we would have to investigate the conditions of Jacobi and Weierstrass or Legendre[9]. However, there is a sufficient condition for a strong minimum (an extension of one usually found in elementary treatments of the calculus of variations[10,11]), which, although it is quite stringent, is easily tested. Moreover, this condition does hold in most cases of interest.

Consider the problem

$$\min I = \int_{t_1}^{t_2} F(t,x,x^{(1)},\ldots,x^{(n)})\,dt; \tag{30}$$

if $x_o(t)$ is the minimizing function, we take the varied function

$$\bar{x}(t) = x_o(t) + \epsilon\,\eta(t) \tag{31a}$$

where

$$\eta^{(i)}(t_1) = \eta^{(i)}(t_2) = 0,\ i=0,\ldots,n-1, \tag{31b}$$

and the second variation is given by

$$\delta^2 I = \epsilon^2 \left.\frac{d^2 I}{d\epsilon^2}\right|_{\epsilon=0} \tag{32}$$

or

$$\delta^2 I = \epsilon^2 \int_{t_1}^{t_2} \sum_{i,j=0}^{n} \left[\frac{\partial^2 F}{\partial x^{(i)}\,\partial x^{(j)}}\right]_{x=x_o} \eta^{(i)}\,\eta^{(j)}\,dt. \tag{33}$$

The second variation is a quadratic form in $\eta^{(i)}$, if it were a positive definite quadratic form, the second variation would be positive for any manner of variation $\eta^{(t)}$. This condition guarantees a strong minimum; however, it is quite stringent, and, if it does not hold, we must apply the more difficult tests mentioned above.

For the problem we are considering the coefficients in the quadratic form are constant and are given by

$$\frac{\partial^2 F}{\partial x^{(i)}\ \partial x^{(j)}} = q_{ij} + \frac{\rho}{b^2} a_{i-1}\ a_{j-1}, \quad i=1,\ldots,n+1, \tag{34a}$$

where

$$q_{i,n+1} = q_{n+1,j} = 0, \tag{34b}$$

and

$$a_n = 1. \tag{34c}$$

Moreover, because of the boundary conditions on $\eta(t)$ (31b), we can integrate out all of the terms for which $i \neq j$[12]. All terms for which the sum of the indices is an odd number integrate out entirely, while all terms for which the sum of the indices is an even number become the integrals of squares:

$$\int_{t_1}^{t_2} \eta^{(i)}\ \eta^{(j)}\ dt = (-1)^{\frac{|i-j|}{2}} \int_{t_1}^{t_2} \left[\eta^{(\frac{i+j}{2})}\right]^2 dt, \tag{35a}$$

when

$$i + j = 2\nu, \quad \nu = 0,\ldots,n. \tag{35b}$$

Thus, the integrated quadratic form (33) becomes, in our problem,

$$\delta^2\ I = \epsilon^2 \int_o^\infty \sum_{k=0}^{n} d_k (\eta^{(k)})^2\ dt, \tag{36a}$$

where

$$d_k = \sum_{\substack{i,j=1 \\ i+j=2k+2}}^{n+1} \left[(-1)^{\frac{|i-j|}{2}} (q_{ij} + \frac{\rho}{b^2}\ a_{i-1}\ a_{j-1})\right], k=0,\ldots,n. \tag{36b}$$

Consequently, the sufficient condition for a strong minimum becomes

$$d_k > 0, \tag{37}$$

where d_k is given by (36b).

EXAMPLES

To show the simplicity of the present approach, we will examine a second order example which has been previously treated in the literature by the maximum principle[13]. Consider the second order system characterized by the equation

$$\ddot{x} + \dot{x} = u \tag{38}$$

with the performance index

$$I = \frac{1}{2}\int_0^\infty (x^2 + \dot{x}^2 + u^2)\, dt. \tag{39}$$

consequently, we have

$$q_{11} = \frac{1}{2}, \tag{40a}$$

$$q_{12} = 0, \tag{40b}$$

$$q_{22} = \frac{1}{2}, \tag{40c}$$

and

$$a_0 = 0, \tag{41a}$$

$$a_1 = 1, \tag{41b}$$

$$a_2 = 1, \tag{41c}$$

and ρ=b=1.

Hence the coefficients c_m (18) are

$$c_0 = 1, \tag{42a}$$

$$c_1 = 2, \tag{42b}$$

$$c_2 = 1, \tag{42c}$$

and the polynomial $P(s^2)$ (19) is

$$s^4 - 2\, s^2 + 1 = 0. \tag{43}$$

The Hurwitz factor of this polynomial is

$$H(s) = s^2 + 2\ s + 1; \tag{44}$$

thus, the desired linear control law is

$$u = -\dot{x} - x. \tag{45}$$

The reader can easily verify that this example satisfies the sufficient condition for a strong minimum (37).

For a second example we consider the third-order system where

$$\dddot{x} + \ddot{x} + \dot{x} + x = u \tag{46}$$

and

$$I = \frac{1}{2}\int_o^\infty (3x^2 + 10\dot{x}^2 + 7\ddot{x}^2 + u^2)dt. \tag{47}$$

The polynomial $P(s^2)$ (19) is

$$-s^6 + 6\ s^4 - 9\ s^2 + 4 = 0, \tag{48}$$

and its Hurwitz factor is

$$H(s) = s^3 + 4\ s^2 + 5\ s + 2. \tag{49}$$

Consequently, the optimum control law is

$$u = -3\ddot{x} - 4\dot{x} - x. \tag{50}$$

THE SINGULAR CASE: AN EXAMPLE

To show how the classical calculus of variations might be applied to find the so-called singular solutions of the optimal control problem, we will reconsider an example for which the singular solutions have been partially obtained by Johnson and Gibson[14]. The system state equations as given are

$$\dot{x}_1 = x_2 + u, \tag{51a}$$

$$\dot{x}_2 = u, \tag{51b}$$

and the performance index is

$$I = \int_o^T \frac{1}{2} x_1^2 \, dt. \tag{52}$$

To use these equations, we must convert them to the canonical form (2) via the transformation

$$y = y_1 = x_1 + x_2, \tag{53a}$$

and

$$\mathring{y} = \dot{y}_1 = y_2 = x_2; \tag{53b}$$

hence, in the new coordinates the system equation is

$$\mathring{\mathring{y}} = - u \tag{54}$$

and the performance index is

$$I = \int_o^T \frac{1}{2} (y-\mathring{y})^2 \, dt. \tag{55}$$

So long as u is within its prescribed bounds (here $|u| \leq 1$) it does not append constraint relations to the performance index. Consequently, the singular arc can be solved for directly by a well-known theorem of the calculus of variations which states that so long as the integrand

$$F(t,y,\mathring{y}) = \frac{1}{2} (y-\mathring{y})^2 \tag{56}$$

is explicitly independent of t, the Euler equation has a first integral given by

$$F - \mathring{y} \frac{\partial F}{\partial \mathring{y}} = \frac{1}{2} (y^2 - \mathring{y}^2) = K, \tag{57}$$

where K is a constant[15]. In the original coordinates (57) becomes

$$x_1 (x_1 + 2x_2) = 2 K = C; \tag{58}$$

this is the equation of the singular arcs, the constant K depends on the initial point on the singular arc (note: K need not be zero). Moreover, on these singular arcs we have

$$u = - x_1 - x_2. \tag{59}$$

A graphical presentation of the family of singular arcs and the limitations on x_1 and x_2 through u appears in Figure 1.

This has been but a prelude to what can be done with the calculus of variations in the investigation of singular solutions[16].

CONCLUSIONS

The calculus of variations has provided an extremely simple technique for finding the optimum control law for a linear, stationary plant under quadratic performance index. Moreover, it has provided a degree of insight sometimes unobtainable when more powerful techniques are employed. The examples have indicated that the classical calculus of variations might be a tool well worth resurrecting for the investigation of some of the subtle points of control theory.

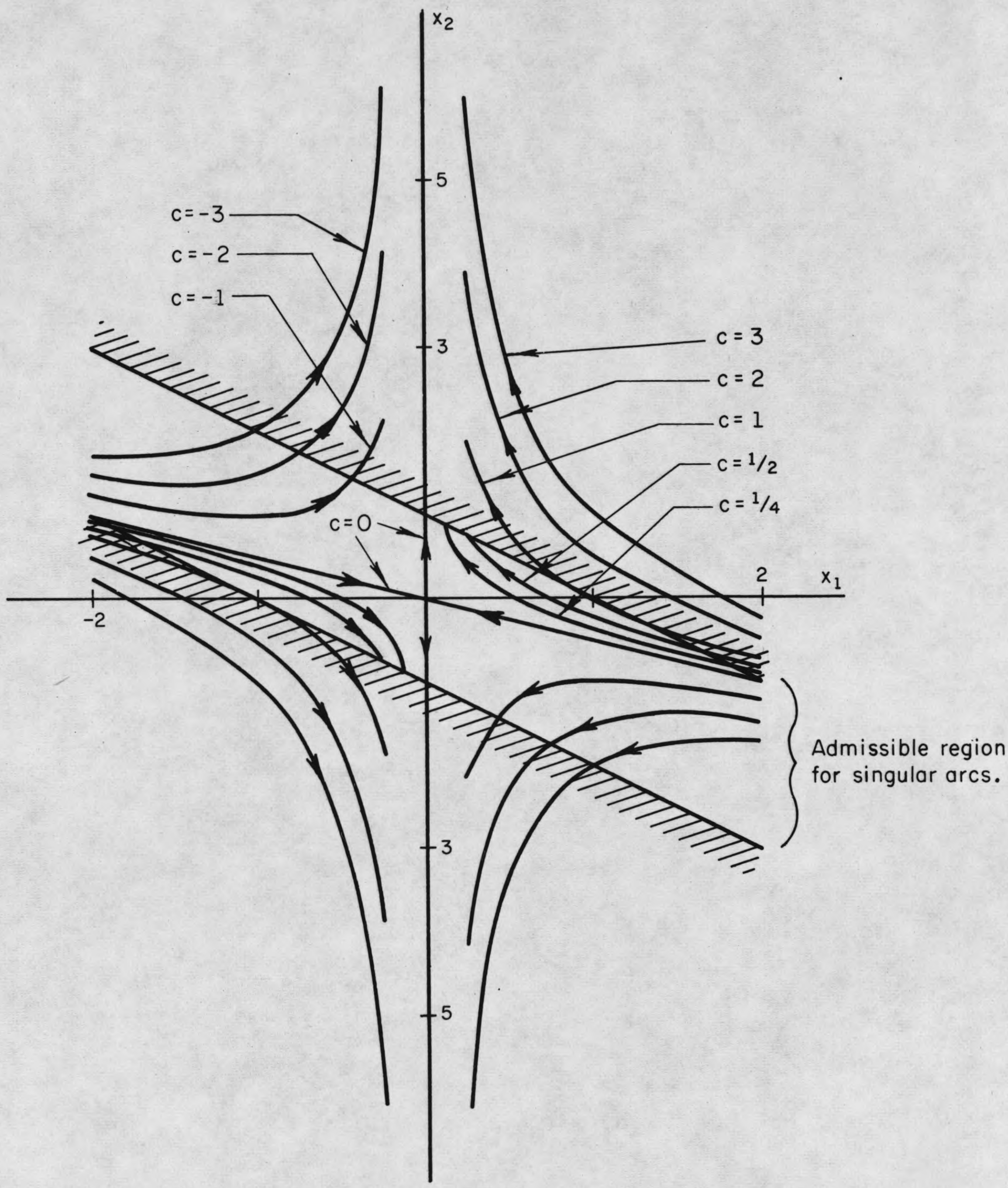

Figure 1. The Singular Arcs with Limits $|u| \leq 1$; in the Admissible Region $u = x_1 + x_2$

FOOTNOTES

1. W.M. Wonham and C.D. Johnson, "Optimal Bang-Band Control with Quadratic Performance Index," Proc. Fourth Joint Automatic Control Conference, pp. 101-112; 1963.

2. Throughout this paper $x^{(i)}$ will denote $\frac{d^i}{dt^i}$ x.

3. L.E. Elsgolc, Calculus of Variations, Addison-Wesley Publishing Co., Inc., Reading, Mass., p. 44; 1962.

4. Elsgolc, op. cit, pp. 33-34.

5. W.M. Wonham and C.D. Johnson, op. cit., have shown that in general for this problem Q can be taken as a diagonal matrix, but this can only be done at the expense of altering the diagonal entries with the off-diagonal entries.

6. E.A. Guillemin, Synthesis of Passive Networks, John Wiley & Sons, Inc., New York, pp. 279-290; 1957.

7. Guillemin, op. cit., pp. 279-290.

8. Guillemin, op. cit., pp. 24-31.

9. Elsgolc, op. cit., pp. 103-125.

10. R. Courant and D. Hilbert, Methods of Mathematical Physics, Interscience Publishers, Inc., New York, pp. 214-216; 1953.

11. H.S. Wilf, Mathematics for the Physical Sciences, John Wiley & Sons, Inc., New York, pp. 225-226.

12. See also Wonham and Johnson, op. cit., where a similar scheme is employed on Q.

13. G.L. Collina and P. Dorato, "Application of Pontryagin's Maximum Principle: Linear Control System," Research Report, MRI, Polytechnic Institute of Brooklyn, pp. 15-17; June, 1962.

14. C.D. Johnson and J.E. Gibson, "Singular Solutions in Problems of Optimal Control," IEEE Trans. on Automatic Control, vol. AC-8, pp. 4-15; Jan., 1963.

15. Elsgolc, op. cit., pp. 36-37.

16. M. Sobral, Jr., and R.A. Rohrer, "Optimal Singular Control Surfaces in Phase Space," to appear.

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