REPORT R-506 MAY, 1971 UILU-ENG 71-2209

CSL COORDINATED SCIENCE LABORATORY

ON MULTITRANSMISSION NETWORKS

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by

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This work was supported in whole by the Joint Services Electronics Program (U. S. Army, U. S. Navy and U. S. Air Force) under Contract DAAB-07-67-C-0199.

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Abstract

This paper considers the problem of designing networks whose function is the simultaneous transmission of k independent signals over k vertex-disjoint paths (k-transmission). All paths are assumed to consist of two edges and to traverse an intermediate vertex (bus). Necessary and sufficient conditions are obtained for realizability of a k-transmission, which are then used for establishing a lower bound to the number of transmission edges for given numbers of stations and busses. The sufficient condition is also used for the design of optimal and suboptimal k-transmissions. Finally the problem of the realization of multitransmissions is considered under the hypothesis of edge failures: conditions, bounds and design methods are described for the special case of single edge failure.

ON MULTITRANSMISSION NETWORKS*

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1. Introduction

The problem of designing multiple transmission networks arises in complex information processing systems consisting of several functional units (stations) with a programmable interconnection. The function of these networks is the simultaneous transmission of a number of independent signals over separate paths. Typical examples are a communication network or a restructurable bus system of a large digital computer.

Formally, given two nonempty sets of vertices (stations), $A \equiv \{a_1, a_2, \ldots, a_n\}$ and $B \equiv \{b_1, b_2, \ldots, b_m\}$, and a (possibly empty) set of vertices (busses) $V = \{v_1, v_2, \ldots, v_r\}$, we must design a graph with vertex set $A \cup V \cup B$ such that $k \leq \min(n,m)$ arbitrary vertex-disjoint paths can be established between A and B. Such graph is said to realize a k-transmission.

This problem, or closely related ones, are certainly not new [1-3], and implementations of multiple transmission networks occur in many existing systems of the kinds mentioned above. It appears, however, that the structural properties of multitransmission networks are not fully understood. It is also readily realized that no simple, universal criterion of simplicity can

*This work was supported by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under contract DAAB-07-67-C-0199 and by the University of Illinois.

be offered as a design guideline. Rather, it is desirable to know which are the trade-offs existing among the various design parameters, so that specific choices can be guided by an adequate theoretical background. For example, the designer should be aware of how many transmission edges can be traded off for an extra bus.

In this framework, the purpose of this paper is to investigate the capabilities of an interesting class of connections, which were studied in [1]. These connections are characterized by the property that edges exist only from A to V and from V to B; therefore if the connection must transmit at least k signals, the cardinality r of V must be no less than k (Figure 1).

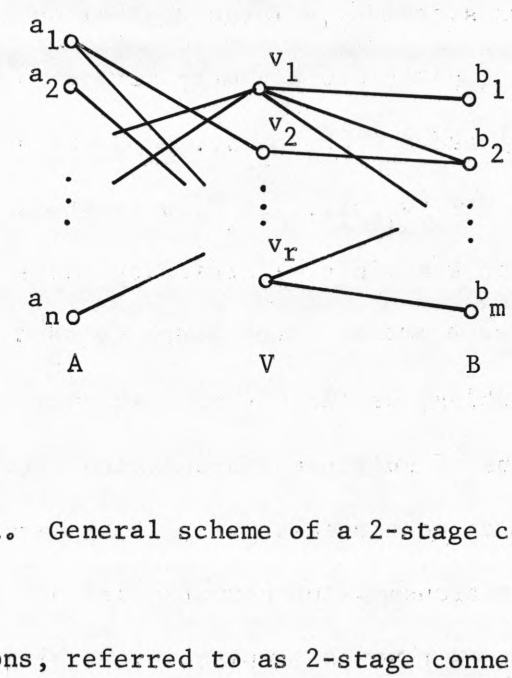


Figure 1. General scheme of a 2-stage connection.

This class of connections, referred to as 2-stage connections, offer considerable flexibility and advantages over the trivial connections for which V is empty and each vertex of A is connected to each vertex of B.

We shall first investigate necessary and sufficient conditions for 2-stage connections to realize a k-transmission: subsequently we shall use these conditions to establish a lower bound to the edge-complexity of the connection, and we shall show how closely we can approach the bounds in specific cases, by presenting some design procedures. Finally we shall address ourselves to the design of survivable connections under edge failure. In contrast to the formal methods used in [1], our approach is mainly combinatorial rather than graph-theoretic.

2. Necessary and sufficient conditions

Without loss of generality, we assume that $m \leq n$; if k denotes the transmission multiplicity, clearly $n \geq m \geq k$. The 1-stage connections between A and V and between V and B are referred to as the <u>left</u> and <u>right</u> connections, respectively.

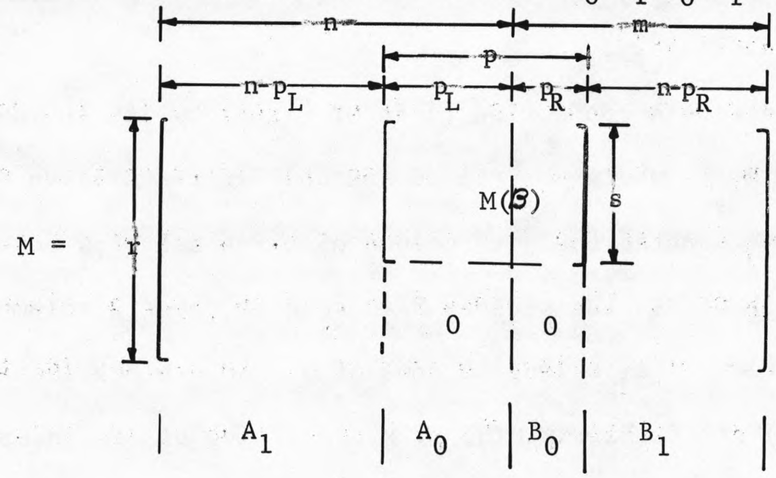
We describe a connection (left or right) by its incidence binary $r \times n$ matrix $M = \|m_{ij}\|$, where $m_{ij} = 1$ if and only if there is an edge between $a_j \in A$ and $v_i \in V$; $\underline{c}_j$ denotes the j-th column of M. A set of s rows of M is called an <u>s-block</u> of M. The s-block $\mathcal{B}$ is said to <u>cover</u> a column $\underline{c}_j$ if all the nonzero entries of $\underline{c}_j$ belong to rows of $\mathcal{B}$. An s-block identifies a (possibly empty) $s \times t$ submatrix $M(\mathcal{B})$ of M consisting of its intersection with the t columns it covers. An s-block is said to be <u>feasible</u> if it covers at most s columns. A feasible s-block is said to be <u>complete</u> if it covers exactly s columns. A complete s-block is <u>prime</u> if it does not contain any other complete s'-block with $s' < s$. A $p \times n$ binary matrix M $(p \geq k)$ is <u>k-feasible</u> if no column of M is 0 and every s-block of M is feasible for $s = 1,2,\ldots,k-1$.

Let M_L and M_R denote the matrices of the left and right connections, respectively, and let $M = [M_L, M_R]$. We have the following necessary condition:

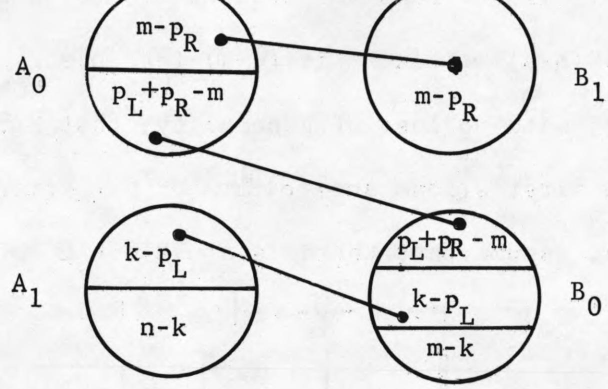
Theorem 1. An $r \times (n+m)$ matrix M realizes a k-transmission only if it is k-feasible $(r \geq k)$.

Proof: Assume that M is not k-feasible, i.e., there is an s-block $\mathcal{B}$ covering $\underline{c}_{j_1}, \underline{c}_{j_2}, \ldots, \underline{c}_{j_p}$ with $p > s$ and $s \leq k-1$. If $M(\mathcal{B})$ is contained in either M_L or M_R, the result is immediate. Indeed, assume $M(\mathcal{B}) \subset M_L$. If we select a set of k vertices of A containing q $(q \leq k)$ members $a_{i_1}, \ldots, a_{i_q}$ from $a_{j_i}, \ldots, a_{j_p}$ $(p \geq q > s)$, then $a_{i_1}, \ldots, a_{i_q}$ have only $s < q$ outlets on V and the k-transmission is irrealizable.

Assume now that $M(\mathcal{B})$ contains columns of both M_L and M_r. To within column permutations M can be put in the following form, where the columns of M are partitioned into four subsets A_0, A_1, B_0, B_1, as shown. If



either $p_L > s$ or $p_R > s$, by the preceding argument M does not realize a k-transmission. Therefore, we assume $p_L \leq s$ and $p_R \leq s$ $(s \leq k-1)$. We first claim that $p_L \leq m - p_R$. Indeed, if $p_L > m - p_R$, consider the transmission pictorially illustrated below, with the cardinalities of subsets of columns



indicated. This transmission consists of $(m-p_R) + (p_L+p_R-m) + (k-p_L) = k$ links, and is specifiable because all the cardinalities are nonnegative. Indeed, $m-p_R > 0$ (from $p_R \le s \le k-1$ and $m \ge k$), $p_L-(m-p_R) > 0$ by assumption, $k-p_L > 0$ (since $p_L \le s \le k-1$) and $m-k \ge 0$, $n-k \ge 0$ by hypothesis. The selected k-transmission, however, uses only $s \le k-1$ vertices of V, a contradiction.

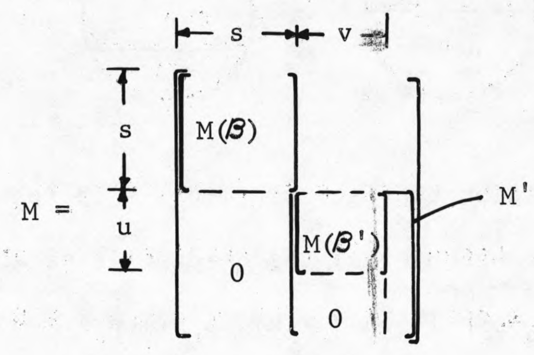
Therefore $p_L \le m-p_R$; this and $n \ge m$ imply $p_R \le n-p_L$. Let $\mathcal{L}$ be a set of p_R links between A_0 and B_1 and p_L links between B_0 and A_1, and consider a k-transmission containing q links of $\mathcal{L}$ $(s<q \le p_R+p_L=p)$. This k-transmission is clearly irrealizable since q links must use s<q vertices of V.

Q.E.D.

As we shall see below, the condition of theorem 1 is also sufficient for r=k; for r>k, it appears to be not very tight. If, however, we restrict ourselves to 2-stage connections whose right connection is complete (each vertex of V is connected to each vertex of B), we can obtain a necessary and sufficient condition on the left connection. With these hypotheses, we say that a (left) connection matrix M_L realizes a k-transmission if any set of k vertices of A has k distinct outlets in V. First we give the following two lemmas.

Lemma 1. Let M be a q-feasible $p \times q$ matrix ($p \geq q$) containing a complete s-block $\mathcal{B}$ for $s \leq q-1$. If the rows and columns of $M(\mathcal{B})$ are suppressed from M, the resulting $(p-s) \times (q-s)$ matrix M' is (q-s)-feasible.

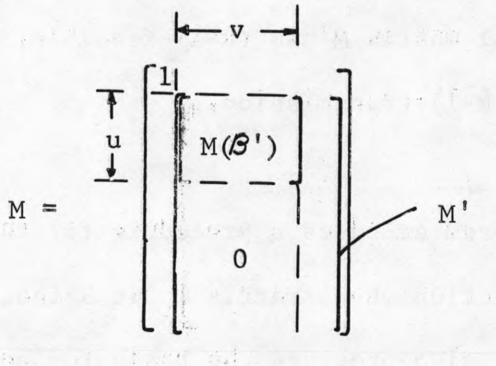
Proof: Assume, with no loss of generality, that $M(\mathcal{B})$ consist of the intersection of the first s rows and columns of M. After removal of the rows and columns of $M(\mathcal{B})$, assume that there is a u-block $\mathcal{B}'$ of M' covering $v>u$



columns (of M'), with $1 \leq u \leq q-1-s$. This means that there is an (s+u)-block of M covering $(s+v) > (s+u)$ columns (of M) with $(s+u) \leq q-1$, thereby violating the hypothesis that M be q-feasible. We conclude that for any $1 \leq u \leq (q-s)-1$ each u-block of M' covers at most u columns of M'. Q.E.D.

Lemma 2. Let M be a (q-feasible) $p \times q$ matrix containing no complete s-block for $s = 1,2,\ldots,q-1$. Then the $(p-1)\times(q-1)$ matrix M' obtained by removing from M a row and column whose intersection is 1, is (q-1)-feasible.

Proof: By definition, M is q-feasible, since any s-block of M ($s < q$) is feasible. Let the first column and row be those that are removed. After this removal, assume that there is a u-block $\mathcal{B}'$ of M' covering $v > u$ columns of M', with $1 \leq u \leq q-2$. This means that there is a (u+1)-block of M covering v columns (of M). Since



$u+1 \leq q-1$ and $v \geq (u+1)$, this violates the hypothesis that M contains no complete s-blocks for $s \leq q-1$, hence the thesis holds. Q.E.D.

We are now ready to prove the central theorem.

Theorem 2 - A necessary and sufficient condition for an $r \times n$ left connection matrix M_L to realize a k-transmission is that M_L be k-feasible.

Proof: Notice that M_L is k-feasible if and only if each column submatrix of M_L with p columns is p-feasible $(p=1,2,\ldots,k)$.

(Necessity): It follows directly from Theorem 1, since M_L is a column submatrix of $M = [M_L, M_R]$.

(Sufficiency): Due to the preceding remark, we only need consider an $r \times k$ matrix $M_L (r \geq k)$. The proof is by induction. For $k = 1$, a $r \times 1$ 1-feasible M_L contains at least one nonzero entry, hence it realizes a 1-transmission. Assume the theorem holds for p-feasible $r \times p$ matrices with $p \leq k-1$ and $r \geq p$, and let the $r \times k$ matrix M_L be k-feasible. Either M_L contains a complete s-block $\mathcal{B}$ or it does not. In the former case, by Lemma 1 we decompose M_L into an $s \times s$ s-feasible matrix $M(\mathcal{B})$ and an $(r-s) \times (k-s)$ $(k-s)$-feasible matrix M'; by the inductive hypothesis, $M(\mathcal{B})$ realizes an s-transmission $(s \leq k-1)$ and M' realizes a (k-s)-transmission ($k-s\leq k-1$), i.e., M_L realizes a k-transmission. In the latter case, by Lemma 2 a column of M_L is assigned to a row (a 1-transmission) and the residual $(r-1)\times(k-1)$ matrix M' is (k-1)-feasible; by the inductive hypothesis, M' realizes a (k-1)-transmission. Q.E.D.

The preceding theorem embodies a procedure for the construction of a k-transmission in a connection whose matrix M_L is k-feasible (briefly, a k-feasible connection). It also provides the basis for answering the following questions: 1) which is the minimum number of nonzero entries in an $r\times n$ k-feasible left connection matrix (i.e., the least number of edges)? 2) Given the maximum weight w of the columns, which is the maximum n admissible for an $r\times n$ connection matrix M to be k-feasible? These questions will be answered in the following section.

3. A Lower bound

Let an $r\times n$ incidence matrix M_L be given and assume that the weight of the columns of M_L does not exceed (k-1). For a given s ($s=1,2,\ldots,k-1$) we construct an $\binom{r}{s}\times n$ binary matrix $P^{(s)} = \|p_{ij}^{(s)}\|$, whose rows and columns are in a one-to-one correspondence with the s-blocks of M_L and with columns of M_L, respectively. $P^{(s)}$ is defined so that $p_{ij}^{(s)} = 1$ if and only if the s-block associated with the i-th row of $P^{(s)}$ covers column $\underline{c}_j$. Then, the k-feasibility condition requires that each row of $P^{(s)}$ contain no more than s 1's. Thus, counting by rows, $P^{(s)}$ contains at most

$$\binom{r}{s}s$$

1's. On the other hand, each column of M_L of weight j is covered by exactly

$\binom{r-j}{s-j}$ s-blocks; if n_j denotes the number of columns of weight j, then, counting by columns, $P^{(s)}$ contains exactly

$$\sum_{j=1}^{s} n_j \binom{r-j}{s-j}$$

1's. We therefore obtain the inequalities

$$(1) \qquad \sum_{j=1}^{s} n_j \binom{r-j}{s-j} \leq \binom{r}{s} s, \quad (s=1,2,\ldots,k-1).$$

It is convenient to rewrite these (k-1) inequalities as

$$(2) \qquad \sum_{j=1}^{k-1} c_j^{(s)} n_j \leq 1 \qquad (s-1,2,\ldots,k-1)$$

where $c_j^{(s)} = \binom{r-j}{s-j} / s \binom{r}{s}$, with the convention that $\binom{n}{m} = 0$ for $m < n$. We claim that if $(n_1, n_2, \ldots, n_{k-1})$ $(n_j \geq 0)$ satisfies (2) for s=k-1, then it satisfies (2) for every $s < k-1$. This follows immediately from the fact that $c_j^{(s)} < c_j^{(k-1)}$ for $s < k-1$. Therefore the inequality for s = k-1 is the most stringent one, that is, it implies all the others, and we may restrict ourselves to it. Our problem reduces to minimizing the linear function

$$C = \sum_{j=1}^{k-1} j\, n_j,$$

subject to the linear constraints

$$\begin{cases} \sum_{j=1}^{k-1} n_j = n \\ \sum_{j=1}^{k-1} \frac{1}{\nu_j} n_j \leq 1, \end{cases}$$

with

(3) $$\nu_j = (k-1)\binom{r}{j}\Big/\binom{k-1}{j}.$$

This is a simple integer L.P. problem. Since we are seeking a bound, we may consider is as a conventional L.P. problem (Notice that $\nu_i < \nu_j$ for $i < j$). Since there are two constraints, the general extremum solution needs contain only two nonzero variables n_i and n_j with

(4) $$n_i = \frac{\nu_i}{\nu_j-\nu_i}(\nu_j-n) \quad , \quad n_j = \frac{\nu_j}{\nu_j-\nu_i}(n-\nu_i).$$

For $\nu_i \leq n \leq \nu_{i+1}$, the solution relative to the basis n_i, n_{i+1} is certainly feasible. We claim it is also minimal. Indeed the well-known condition for minimality (see, e.g., [5] p. 61) becomes

$$[i,i+1]\begin{bmatrix} 1 & 1 \\ 1/\nu_i & 1/\nu_{i+1} \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1/\nu_p \end{bmatrix} - p \leq 0 \qquad (p = 1,2,\ldots,k-1)$$

or, equivalently,

(5) $$\psi_i(p) = \nu_{i+1}(\nu_p-\nu_i)+(i-p)\nu_p(\nu_{i+1}-\nu_i) \leq 0 \quad (p=1,2,\ldots,k-1).$$

With regard to $\psi_i(p)$ in appendix we prove the following lemma:

Lemma 3. The function $\psi_i(p)$, for $i = 1,2,\ldots,k-2$, satisfies the conditions: $(p = 1,2,\ldots,k-1)$:

i) $\psi_i(i) = \psi_i(i+1) = 0$,

ii) $\psi_i(p) \leq 0$ for $p \neq i, i+1$, with equality if and only if $r = k$.

Not only does this lemma state the optimality of the mentioned solution, but it also tells us that the minimal solution is unique for $r > k$. Using relations (4), we then have the following theorem:

Theorem 2. Let n be the number of columns of M_L of weight at most $(k-1)$. For fixed r and k, a lower bound to the number of edges of a k-feasible left connection is given by

$$C \geq ni + (n-\nu_i)\frac{r-i}{r-k+1}, \tag{6}$$

where i is the largest integer for which $n \geq \nu_i$.

Notice that when $n = \nu_i$ the minimal solution vector contains the only nonzero component $n_i=\nu_i=n$; notice also that the minimal solution is a continuous piecewise linear function of n.

If we upper bound by $w \leq k-1$ the maximum weight of the columns of M_L, we automatically place an upper bound to the value of n for which a k-feasible connection is realizable, that is $n \leq \nu_w$. We observe that we may adjoin to the connection matrix M_L as many columns of weight k as we please and still preserve k-feasibility; thus, since $\nu_k = \infty$, for maximum weight $w \leq k$, the general upper-bound to the number n' of vertices of A for which a k-feasible connection may exist is given by

$$(7) \qquad n' \leq \nu_w \qquad (w = 1,2,\ldots,k).$$

There is an interesting special case to be examined. For r=k inequality (6) becomes (notice that in this case $\nu_i = k(k-1)/(k-i)$)

$$(8) \qquad C \geq ni + \left(n - \frac{k(k-1)}{k-i}\right)(k-i) = k(n-k+1).$$

This represents the number of edges of the left connection; adding to it the number mk of the edges of the right connection, a lower bound to the number N of edges of our design for r=k is

$$(9) \qquad N \geq k(n+m-k+1).$$

It must be pointed out that (9) has been obtained in the hypothesis that M_L be k-feasible and M_R be complete; however, we would obtain the same result using (the necessary condition of) Theorem 1, i.e., replacing n with (n+m) in (8). This shows, as mentioned before, that Theorem 1 embodies a tight condition for r=k; moreover it proves a conjecture by Aggarwal-Mayeda-Ramamoorthy [1], who also exhibited designs meeting bound (9), i.e., optimal. Hence (9) is also an upper bound. Other optimal designs exist for $r \geq k$ as will be shown in the following section.

4. Synthesis of Multitransmissions

The designs considered in this section are such that M_R is complete and M_L contains columns of at most two distinct weights.

First we obtain two simple corollaries to Theorem 2.

Corollary 1 - If the columns of an $r \times n$ matrix M have weight (k-1) and at most (k-1) columns are identical, M is k-feasible.

Proof: Indeed k-feasibility reduces to testing s-blocks for s = k-1. Then, by hypothesis, each (k-1)-block covers at most (k-1) columns, that is, it is feasible. Q.E.D.

As an example, assume that $n \geq \nu_{k-1} = (k-1)\binom{r}{k-1}$. It is possible to construct an $r \times n$ k-feasible matrix M_L by repeating (k-1) times each of the $\binom{r}{k-1}$ distinct columns of weight (k-1)(corollary 2) and adding $(n-\nu_{k-1})$ columns of weight k. The number of entries of the resulting matrix is

$$\nu_{k-1}(k-1) \quad + \quad (n-\nu_{k-1})k = nk-\nu_{k-1}$$

i.e., it coincides with the value offered by (9) for i = k-1. Thus, these designs are optimal.

Corollary 2 - If the columns of an $r \times n$ matrix M have weight (k-2) and are distinct, then M is k-feasible $(k \geq 3)$.

Proof: Indeed k-feasibility reduces to testing s-block for s=k-2,k-1. Each (k-2)-block covers exactly one column, and, due to the fact that $k-2 \geq 1$, the condition is satisfied; each (k-1)-block covers at most (k-1) columns since all columns are distinct and have weight (k-2). Q.E.D.

Consider the following class of designs of M_L for $\nu_{k-2} \leq n < \nu_{k-1}$, 1) we generate the set $\mathcal{R}$ of the $\nu_{k-2} = \binom{r}{k-2}$ columns of weight (k-2); 2) we select $\lceil (n-\nu_{k-2})/(r-k+1) \rceil$ (*) columns from this set; 3) from any selected column, at most (r-k+2) extensions are obtained by changing from 0 to 1 a different entry in each extension. (This step is applicable to all selected columns except possibly one for which only $\left((n-\nu_{k-2}) + 1 - (\lceil (n-\nu_{k-2})/(r-k+1) \rceil - 1)(r-k+1)\right)$ extensions are formed.) 4) The columns of M_L are the unselected members of $\mathcal{R}$ and the extensions of the selected members. The number of "1" entries of M_L is found to be

$$n(k-2) + (n-\nu_{k-2}) + \left\lceil \frac{n-\nu_{k-2}}{r-k+1} \right\rceil ,$$

which, for integer $(n-\nu_{k-2})/(r-k+1)$, coincides with the value given by the lower bound (9) for i = k-2. We claim that the just constructed matrix M is k-feasible. Indeed, we only have to test (k-1)-blocks since, by construction, all the columns of weight (k-2) are distinct. Assume now that a (k-1)-block covers a set of $p \geq k$ columns and let q be the number of the columns of weight (k-1) in this set. Since no two extensions of the same original column are identical, the q columns of weight (k-1) originate from q distinct columns of weight (k-2): but this entalis that there are $p > (k-1)$ distinct columns of weight (k-2), whose union has weight (k-1), which is impossible. Thus, the given designs are also optimal.

(*) $\lceil \; \rceil$ denotes "smallest integer not less than".

<u>Example 1</u> - For r=5, k=4 and n=12 we have $10=\nu_2<n<\nu_3=30$. Therefore we select $\left\lceil\frac{12-10}{2}\right\rceil = 1$ column of weight 2 and replicate it three times. The resulting 5×12 matrix M, shows below, describes the design (the replicated column is $[00011]'$).

$$\begin{array}{cc} & \begin{array}{cc} \longleftarrow\!\!-\!\!-\!\!-\!\!-\ 9\ -\!\!-\!\!-\!\!-\!\!\longrightarrow & \longleftarrow\ 3\ \longrightarrow \end{array} \\ M = \ 5\updownarrow & \left[\begin{array}{ccccccccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Unfortunately no other general method can be presently offered for designs corresponding to $\nu_i \leq n \leq \nu_{i+1}$ for $i < k-2$, although some isolated near-optimal designs can be produced. For example for k = 5 and even r, consider the following designs. Suppose to enumerate the rows of M from 0 through (r-1); all the columns are distinct and each column has two 1's but not both in rows having the same parity. The resulting matrix is 5-feasible: indeed each 4-block can be classified, according to the parity of the rows it contains, as (EEEE) (EEEO), (EEOO), (EOOO), (OOOO), where E and O stand for "even" and "odd", respectively. For obvious reasons of symmetry, we only need consider the first three cases: by virtue of the stated property of the columns, in the first case the 4-block covers no column, in the second case at most three columns, in the third case at most four columns. With a similar argument we consider 3-blocks; finally a 2-block covers at most a single column, and the claim is proved. The largest n for which such a 5-feasible design is possible is given by the number of ways of selecting an even and an odd row among r rows, i.e., $r^2/4$. Clearly since $r \geq 5$,

$$\frac{r^2}{4} < \nu_2 = \frac{2}{3}\binom{r}{2}$$

which shows that the given designs are not optimal.

5. Failure-Tolerant Multitransmissions

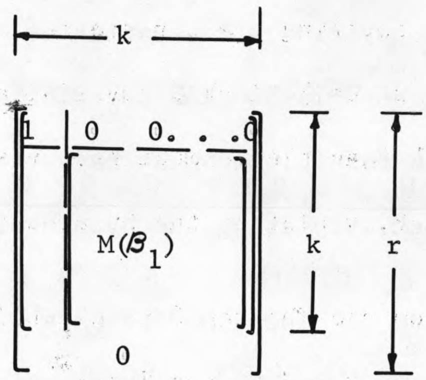
So far we have analyzed connections in relation to their capabilities to realize k-transmissions. It is highly desirable, however, (as pointed out by other authors [1,2]) to characterize those connections which maintain their transmitting capabilities even in the presence of edge failures, that is, when a number of their edges are not usable in any transmission.

In this paper we shall consider only the case of a single edge failure. A connection which allows an arbitrary k-transmission in the presence of one edge failure is said to be (k,1)-feasible (the same denotation applies to its incidence matrix M). Clearly, (k,1)-feasibility implies k-feasibility. Hereafter we shall assume M_R to be complete and we shall concentrate on M_L.

We then have the following necessary condition for the matrix M_L:

Theorem 3 - An $r \times n$ k-feasible matrix M_L is (k,1)-feasible only if each column has weight at least 2 and no complete k-block covers any complete (k-1)-block.

Proof. The condition on the weight is trivial. Assume now that a complete k-block $\mathcal{B}$ covers a complete (k-1)-block $\mathcal{B}_1$. Without loss of generality, the relation between $M(\mathcal{B})$ and $M(\mathcal{B}_1)$ is as illustrated below:



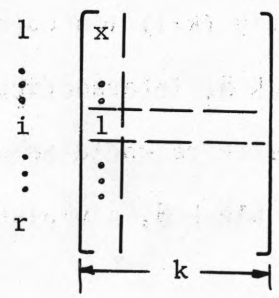
Notice that the first row of $M(\mathcal{B})$ is 10...0: if the edge corresponding to the single 1 fails, the k vertices of A corresponding to the columns of $M(\mathcal{B})$ have only (k-1) outlets on V and the k-transmission is irrealizable. Q.E.D.

A sufficient condition is offered by the following:

Theorem 4. An r×n k-feasible matrix M_L is (k,1)-feasible if each column has weight 2 and there is no complete s-block (s = 2,3,...,k-1).

Proof: Consider an r×k submatrix M' of M_L and assume that the entry marked x below identifies a failing edge.

$M' =$



By hypothesis there is at least another 1 entry in the same column (let it be the i-th entry so that the first column can be assigned to the ith row). We remove the first column and the i-th row and claim that the resulting matrix is (k-1)-feasible. Indeed, assume that in the resulting matrix there is a u-block covering $v > u$ patterns. This means that in the original matrix M' there is a (u+1)-block $\mathcal{B}$ covering at least v patterns; since $v \geq u+1$ and M_L is k-feasible we must have $v = u+1$. It follows that $\mathcal{B}$ is a complete (u+1)-block, violating the hypothesis. Q.E.D.

We can now use Theorem 3 to establish a lower bound to the number of edges for given n, r and k, and Theorem 4 as a guide in the development of design procedures.

The lower bound is established by means of the $\binom{r}{k-1} \times n$ matrix P, whose rows correspond to the (k-1)-blocks of M_L and whose columns correspond to the set of columns of M_L of weight not exceeding (k-1). With the usual meaning of symbols, $\sum_{2}^{k-1} n_j = n$; counting by columns, P contains exactly

$$\sum_{j=2}^{k-1} n_j \binom{r-j}{k-1-j}$$

1's. We now want to establish an upper-bound to the number of 1's of P counting by rows. By k-feasibility, each row may contain at most (k-1) 1's. However a row containing exactly (k-1) 1's corresponds to a complete (k-1)-block $\mathcal{B}$; each (k-1)-block $\mathcal{B}_1$ intersecting $\mathcal{B}$ in (k-2) rows must have no entry equal to 1, otherwise there would be a complete k-block $\mathcal{B} \cup \mathcal{B}_1$ containing the complete (k-1)-block $\mathcal{B}$, a violation of the condition of

Theorem 3. Now, assuming that M_L contains $t > 0$ complete (k-1)-blocks, we want to determine the largest number of possible incomplete (k-1)-blocks of M_L. To this end, we form an $\binom{r}{k-1} \times t$ binary matrix A, whose rows correspond to the (k-1)-blocks of M_L and whose columns correspond to the t complete (k-1)-blocks of M_L. An entry of A is 1 if and only if the corresponding (k-1)-blocks intersect in exactly (k-2) rows. Counting by columns, A contains exactly $t(k-1)(r-k+1)$ 1's. Each row of P may have at most $\min[t,(k-1)(r-k+1)]$ 1's, whence the number ν of rows of P which have at least one nonzero entry is bounded by

$$\nu \geq \left\lceil \frac{t(k-1)(r-k+1)}{\min(t,(k-1)(r-k+1))} \right\rceil = \max\left(t,(k-1)(r-k+1)\right).$$

Therefore, for $t > 0$, M_L contains t complete (k-1)-blocks and at most $[\binom{r}{k-1} - t - \max(t,(k-1)(r-k+1))]$ incomplete (k-1)-blocks. It follows that, counting by rows, P contains at most

$$(10) \qquad \binom{r}{k-1}(k-2) - \{\max(t,(k-1)(r-k+1))\cdot(k-2) - t\} \qquad (t>0)$$

1's. On the other hand, for t=0, M_L contains $\binom{r}{k-1}$ incomplete (k-1)-blocks and therefore, counting by columns, P contains at most

$$(11) \qquad \binom{r}{k-1}(k-2) \qquad (t=0)$$

1's. Notice that the value of (10) never exceeds the value of (11) for $k \geq 3$. It follows that (11) is the upper bound to the number of 1's of P and we have the inequality

$$\sum_{j=2}^{k-1} n_j \binom{r-j}{k-1-j} \leq (k-2)\binom{r}{k-1} \tag{12}$$

Notice that here again, and a fortiori, (12) is more stringent than any of the inequalities (2) for $s = 1,2,\ldots,k-2$. Defining

$$\mu_i = (k-2)\frac{\binom{r}{i}}{\binom{k-i}{i}},$$

by arguments similar to those developed in section 3, a lower bound to the number C of edges of the left connection is found to be

$$C \geq ni + (n-\mu_i)\,\frac{r-i}{r-k+2} \tag{13}$$

where i is the largest integer for which $n \geq \mu_i$.

Notice that if M contains no complete s-block $(s=2,\ldots,k-1)$, particularly, no complete (k-1)-block, we may add to M an arbitrary number of columns of weight at least k without affecting the (k,1)-feasibility. This then leads to demonstrate the sufficiency of the following designs.

Assume $n \geq \mu_{k-1} = (k-2)\binom{r}{k-1}$. We repeat (k-2) times each distinct column of weight (k-1) and add $(n-\mu_{k-1})$ columns of weight k. By theorem 4 these designs are (k,1)-feasible; the number of edges coincides with the lower bound (13), hence the designs are optimal.

As a coincidence, an $r\times n$ matrix M whose columns are distinct and have weight w is at the same time (w+2)-feasible and (w+1,1)-feasible. This parallelism, however, does not seem to be further extensible.

Appendix

Lemma 3. Letting $\nu_j = (k-1)\binom{r}{j}/\binom{k-1}{j}$, for $i = 1,2,\ldots,k-2$ and $p = 1,2,\ldots,k-1$, we define:

$$\psi_i(p) = \nu_{i+1}(\nu_p - \nu_i) + (i-p)\nu_p(\nu_{i+1} - \nu_i) \tag{5}$$

and claim that

1) $\psi_i(i) = \psi_i(i+1) = 0$
2) $\psi_i(p) \leq 0$ for $p \neq i, i+1$, with equality if and only if $r=k$.

Proof: Part 1) follows directly from the definition of $\psi_i(p)$. To prove part 2), we assume at first that $r > k$. Notice that $\nu_{i+1} = \nu_i\left(1+\frac{r-k+1}{k-1-i}\right)$; letting $a \triangleq r-k+1$ we have for $1 \leq p < i$:

$$\varphi_i(p) \triangleq \frac{\psi_i(p)}{\nu(p)} = \left[1 + (i-p)\frac{a}{r-i}\right] - \prod_{j=k-i}^{k-1-p}\left(1+\frac{a}{j}\right);$$

a recursive relation between $\varphi_i(p)$ and $\varphi_i(p-1)$ is now easily obtained:

$$\begin{aligned}\varphi_i(p-1) &= [1+(i-p+1)\frac{a}{r-i}] - \prod_{j=k-i}^{k-p}(1+\frac{a}{j})\\ &= \varphi_i(p) + \frac{a}{r-i} - \prod_{j=k-i}^{k-p}(1+\frac{a}{j})\frac{a}{k-p} = \\ &= \varphi_i(p) + \frac{a}{r-i} - \frac{a}{k-p}[(1+(i-p)\frac{a}{r-i}) - \varphi_i(p)]\end{aligned}$$

that is,

$$\varphi_i(p-1) = \varphi_i(p)\frac{r-p+1}{k-p} - (r-k+1)\frac{(r-k)(i-p+1)}{(r-i)(k-p)} .$$

Since $r \geq k > i > p$, the second term in the right side is ≥ 0, with equality if and only if $r=k$. Notice now that for $p=i$, since $\varphi_i(i) = 0$, we have:

$$\varphi_i(i-1) = - \frac{(r-k)(r-k+1)}{(r-i)(k-i+1)} \leq 0$$

with equality if and only if $r=k$. It follows that $\varphi_i(p) \leq 0$ for $p = 1,2,\ldots,$ $i-1$, with equality if and only if $r=k$.

For $i+1 < p \leq k-1$ the argument is exactly parallel and we obtain the recursive relation

$$\varphi_i(p+1) = \varphi_i(p)\frac{k-1-p}{r-p} - (r-k+1)\frac{(r-k)}{(r-i)}\frac{(p-i)}{(r-p)};$$

here again, since $r \geq k > p > i$, the second term in the right side is ≥ 0, with equality if and only if $r=k$. Furthermore for $p = i+1$, since $\varphi_i(i+1) = 0$, we have:

$$\varphi_i(i+2) = - \frac{(r-k+1)(r-k)}{(r-i)(r-i-1)} \leq 0$$

with equality if and only if $r=k$. It follows that $\varphi_i(p) \leq 0$ for $p = i+2,$ $i+3, \ldots, k-1$, with equality if and only if $r=k$.

Q.E.D.

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Coordinated Science Laboratory University of Illinois Urbana, Illinois 61801	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

ON MULTITRANSMISSION NETWORKS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Franco P. Preparata

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May, 1971	23	4

8a. CONTRACT OR GRANT NO. DAAB-07-67-C-0199 b. PROJECT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) R-506
c. d.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) UILU-ENG 71-2209

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Joint Services Electronics Program through U. S. Army Electronics Command, Fort Monmouth, New Jersey, 07703

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DD FORM 1 NOV 65 1473

Security Classification

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Multitransmission Networks						
Lower Bounds						
Optimal Designs						

Security Classification

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