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MULTIDIMENSIONAL SIGNAL RESTORATION AND BAND-LIMITED EXTRAPOLATION, I

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This technical report deals with signal restoration and extrapolation. First, we prove that time-limited deconvolution and time-limited time-variant linear restoration of multidimensional signals can be done by means of fixed-point solutions of contraction mappings.

Then, the band-limited multi-dimensional signal extrapolation problem is studied. (i) We provide several new algorithms for solving the discrete extrapolation problem and show the convergence of the discrete solutions so obtained to the

continuous extrapolation. (ii) We propose four basic models for the extrapolation problem and we present some new techniques for solving the continuous extrapolation problem when the known part of the signal is contaminated with noise. (iii) Several results concerning iterative algorithms for band-limited extrapolation are presented.

PREFACE

This technical report consists of four parts. The central problem is the extrapolation of band-limited signals.

In Part I, we show that time-limited signal restoration can be accomplished by computing fixed point solutions by means of a certain iterative procedure. An application to extrapolation is shown.

In Part II, a method is presented for obtaining discrete extrapolations of a finite extent sequence. The convergence of the discrete extrapolation to the continuous extrapolation when the sampling rate tends to infinity is shown.

In Part III, several results concerning the case where noise is present in the known part of the signal are presented. We also present 4 basic models for extrapolation and discuss their relationships.

In Part IV, the connection between the well-known iterative procedure for band-limited extrapolation (Papoulis-Gerchberg) is shown to be a special case of a more general procedure given by Landweber in 1951. The relationship between discretization of the Landweber procedure and some of the techniques given in Part II is discussed.

ITERATIVE TIME-LIMITED SIGNAL RESTORATION

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ABSTRACT

The purpose of this paper is to show that time-limited restoration of shift-invariant blurred signals can be done by means of fixed point solutions of contraction mappings, under rather general conditions for the distortion operator. All our results are valid for multidimensional signals. An application of these results to the iterative extrapolation of band-limited discrete images is shown.

I. INTRODUCTION

Many problems in signal restoration can be formulated in the following manner: Find a sequence $x(n)$, $-\infty < n < +\infty$, such that

$$y(n) = \sum_{m=-\infty}^{+\infty} h(n,m)x(m), \quad -\infty < n < +\infty \tag{1}$$

$$x(n) = (Cx)(n)$$

where C is a constraint operator, $y{:}y(n)$, $-\infty < n < +\infty$ is the observed signal and $h{:}h(m,n)$, $-\infty < m, n < +\infty$ is a known distortion operator.

An attempt to solve (1) is to seek solutions of the following equation:

$$x(n) = \lambda y(n) + \left((Cx)(n) - \lambda \sum_{m=-\infty}^{+\infty} h(n,m)(Cx)(m)\right) \tag{2}$$

or equivalently, if we choose functional notation, to look for the fixed points of the operator G :

$$G_\lambda(x) = \lambda y + (I - \lambda H)\, Cx \tag{2'}$$

where y denotes the observed sequence, and I and H denote the identity operator and the distortion operator, respectively. The motivation is that (2) prompts the iterative computation of the fixed points by means of the recursion formula:

$$x^{k+1} = \lambda y + (I - \lambda H)\, Cx^k \tag{3}$$

$$x^o = \text{initial guess}$$

(Note that the superscripts do <u>not</u> denote powers.)

Several authors have used this methodology for solving (1). Reference [1] provides an excellent review of this matter. The reader is referred also to [1] for a list of references. Another relevant paper is [2].

In solving the restoration problem by means of the iterative equation (3), three problems arise. The first is to find convergence properties for the iterative scheme. The second problem, which we consider, conceptually to be a much more important problem, is that of relating the fixed points of (2') with the original problem (1). The third problem is the effect of noise in the observed signal on the fixed points of equation (2).

It is clear that any solution of (1) is a fixed point of (2'), but there is no reason to believe that the converse holds in general; i.e., for some H and C there may be fixed points of (2') which are not solutions of (1) at all.

Let us give an example. If we put $(A < \pi)$

$$h(n,m) = \frac{\sin A(n-m)}{\pi(n-m)} \quad \text{if } n \neq m$$
$$h(n,n) = \frac{A}{\pi} \tag{4}$$

and if we consider a time-limiting operator as the constraint T:

$$T(x)(n) = 0 \text{ if } n \notin [n_a,n_b]$$
$$T(x)(n) = x(n) \text{ if } n \in [n_a,n_b] \tag{4'}$$

then a necessary condition for (1) to have a solution is that $\hat{y}(\omega)$ (the Fourier transform $\sum_{n=-\infty}^{+\infty} y(n)e^{in\omega}$ of the sequence y(n)) be of the form $\sum_{n=n_a}^{n_b} z(n)e^{in\omega}$ when $\omega \in (-A,A)$ and 0 when $|\omega| > A$. Now if we assume that y(n) does not have that property, then no solutions of (1) can exist. Nevertheless, as we will prove later, λ may be chosen such that (2') is a contraction mapping. This assures that the fixed point will exist no matter what y(n) is.

We think that this assumption about y(n) is very realistic since y may contain some noise, and in that case its Fourier transform might be very different from a finite trigonometrical polynomial (i.e. $\sum_{n=n_a}^{n_b} z(n)\cdot e^{in\omega}$) on (-A,A). This shows that by means of (2) we would be solving a rather different problem from that of (1).

In the case that (1) has at least one solution and λ may be chosen such that the operator (2') has a unique fixed point, then, of course, the fixed point is the unique solution of (1).

With respect to the first problem mentioned above, it is worth noting that Youla [3] has shown some interesting properties that assure the iterative procedure (3) to be convergent when $\lambda = 1$, H and C are projection operators onto closed linear manifolds in a Hilbert space.

The third problem to be considered is the stability of the method to be chosen for solving (1). In our case, solving (2) is the chosen technique. As we have already pointed out, (1) is frequently a very poorly conditioned problem since small amounts of noise added to the observation y(n) may make (1) not have any solution. It would be desirable that this was not the case with (2), and, if possible, it would be even better to be able to estimate how much the solutions of (2) will change when some noise is added to y(n).

A satisfactory solution to all these problems may be obtained when (2') is a contraction mapping, i.e., when there exists a real number d < 1 such that

$$\|G(x) - G(\bar{x})\| \leq d\,\|x - \bar{x}\|\ , \text{ for all } x, \bar{x}\ . \tag{5}$$

If (5) holds, then the iterative scheme (3) is convergent, the fixed point

(2) has to be unique, and it is also known that if we add some noise Δy to y, we will have:

$$\| z - \bar{z} \| \leq \frac{|\lambda|}{1 - d} \quad \| y \| \tag{6}$$

where z and $\bar{z}$ are the fixed points corresponding to the noise-free and noisy observations, respectively, i.e.:

$$z = \lambda y + (I - \lambda H)Cz$$

and

$$\bar{z} = \lambda(y + \Delta y) + (I - \lambda H)C\bar{z}$$

It is important to note that if d = 1 (i.e., G is a non-expansive mapping) then equation (6) will not hold. However, for some special G's, the iterative scheme (3) is convergent. (see [2])

In addition, we can obtain a noise-sensitivity analysis for the recursion (3). Let us put:

$$z^n = \lambda y + (I - \lambda H)\ Cz^{n-1},\ z^o = \text{initial guess}$$

and

$$\bar{z}^n = \lambda(y + \Delta y) + (I - \lambda H)C\,\bar{z}^{n-1},\ \bar{z}^o = \text{initial guess.}$$

Then, we get

$$\| \bar{z}^n - \bar{z}^n \| \leq |\lambda| \quad \| \Delta y \| \sum_{i=0}^{n-1} d^i + \| z^o - \bar{z}^o \| d^n \tag{7}$$

(where d^i denotes the ith power of d)

We think it is very important to point out that the constant d will depend on the parameter λ. Then, it is interesting to choose λ in order to minimize the noise-sensitivity ratio:

$$\frac{|\lambda|}{1 - d}$$

Sometimes, this amount is minimized by choosing $\lambda = 0$, and d approaches 1 when λ tends to 0; this means that there is a trade-off between noise sensitivity reduction and speed of convergence, since the smaller d is, the faster the convergence $z^n \to z$, as it follows from

$$\| z^n - z \| \leq d \| z^{n-1} - z \| \quad .$$

Two important classes of problems are:

A. H is a convolution operator:

$$(Hx)(n) = (h*x)(n) = \sum_{m=-\infty}^{+\infty} h(m,n)x(m) \quad ,$$

In this case, the composition of time-limiting operator defined in (4') and any non-linear non-expansive mapping will be the constraint operator.

B. H is a linear non-negative time-variant operator

$$(Hx)(m) = \sum_{n=-\infty}^{+\infty} h(m,n)x(n) \quad ,$$

$$h(m,n) \geq 0, \text{ for all } n,m$$

Here, the constraint to be considered is non-negative time-limited signals, i.e.:

$$(Px)(n) = \begin{cases} x(n) \text{ if } x(n) \geq 0 \text{ and } n \in [n_a, n_b] \\ 0 \text{ otherwise} \quad . \end{cases}$$

In either case A or B we can prove that there exists some real number such that

$$G_\lambda(x) = \lambda y + (I - \lambda H)Cx$$

satisfies

$$\| G_\lambda(x) \| \leq d \| x \| \tag{8}$$

where d is a constant which verifies $d < 1$, by imposing some additional, rather general conditions on H. Note that in case A, (8) implies (5). This is not true for case B. Therefore, in the remainder of this paper case B will not be considered. Nevertheless, a proof of (8) for case B is included in Appendix II, because we believe that the result may be useful in its own right.

Section II summarizes the main results. In section III, an application of the shift invariant case will be given. Specifically, we will show that the iterative procedure (3) can be used for obtaining an iterative extrapolation of a discrete band-limited multidimensional signal. This result extends those recently given in [5].

Section IV presents some discussions on the iterative restoration technique given by (3) when the constraint operator is a truncation to $[n_a, n_b]$ and the given signal is corrupted with noise. We also show how the results obtained in Section II provide theoretical support for some of the results in [1].

II. MAIN RESULTS

In this section we will state our main results. All the proofs are given in Appendix I.

Let T denote the time-limiting operator over $[n_a, n_b]$ as defined in (4') and δ be the unit sample sequence: $\delta(0) = 1$; $\delta(n) = 0$, $n \neq 0$.

Theorem 1

Let $h{:}h_n$, $-\infty < n < +\infty$ be a finite-energy sequence, and let Γ be a subset of $[-\pi,\pi]$ of non-zero measure. Let us suppose that h has the following property: there is a number K and there exists some real constant λ such that:

(a) $$|1 - \lambda\hat{h}(\omega)| \leq 1, \quad \text{for all } \omega \in [-\pi,\pi].$$

(b) $$|1 - \lambda\hat{h}(\omega)| \leq K < 1, \text{ for all } \omega \in \Gamma.$$

Then, it follows that we can find a constant $M < 1$ such that:

$$\|(\delta - \lambda h) * Tx\| \leq M \, \|Tx\|$$

for all infinite sequences x: $x(n)$, $-\infty < n < +\infty$.

Conditions (a) and (b) of Theorem 1 may be satisfied under rather general conditions for the kernel $h{:}h_n$, $-\infty < n < +\infty$ (Theorem 3). Since the operator $(\delta - \lambda h)*T$ is linear, Theorem 1 also ensures that $(\delta - \lambda h)*T$ is a contraction mapping. Therefore, we have the following

Corollary

With the same notation as in Theorem 1, and under the same conditions for h: $h(n)$, $-\infty < n < \infty$, there exists a real number λ such that the operator:

$$(\delta - \lambda h)*T$$

is a contraction mapping.

Another important result which can be immediately obtained from Theorem 1 is the following:

Theorem 2

With the same conditions and notations as in Theorem 1, if we assume in addition that C is any nonexpansive operator:

$$\| C x - C \bar{x} \| \leq \| x - \bar{x} \| ,$$

then, the operator

$$(\delta - \lambda h)*TC$$

is a contraction mapping, for some real number λ.

A particular case of Theorem 2 is when C is the positive constraint operator defined by

$$Px(n) = \begin{cases} x(n), \text{ if } x(n) \geq 0 \\ 0 \text{ otherwise.} \end{cases}$$

Since this operator is nonexpansive, Theorem 2 holds.

It is important to find what conditions h should satisfy in order to meet conditions (a) and (b) in Theorem 1.

Let us begin with an important example. We consider the frequency-limiting operator $SINC_A$ defined in (4). Since its Fourier transform is

$$\chi_A(\omega) = \begin{cases} 1 \text{ if } \omega \in (-A,A), \\ 0 \text{ otherwise.} \end{cases} \qquad \text{(We are looking at the interval } [-\pi,\pi] \text{ only.)}$$

we have

$$|1 - \lambda\chi_A(\omega)| = 1, \text{ if } \omega \notin (-A,A)$$

$$|1 - \lambda\chi_A(\omega)| = |1 - \lambda|, \text{ if } \omega \in (-A,A)$$

This means that any $\lambda \in (0,2)$ satisfies both requirements (a) and (b) of Theorem 1, with $K = |1 - \lambda|$ and $\Gamma = (-A,A)$. For any such λ, we have that

$$(\delta - \lambda\ \mathrm{SINC}_A) * TC$$

is a contraction mapping, for any nonexpansive operator C, as it has been stated in Theorem 2.

The next theorem shows that there exist many other sequences which satisfy conditions (a) and (b).

<u>Theorem 3</u>

Let H be a complex-valued function defined over $[-\pi,\pi]$.

(I) If H has the following properties:

Ia. ω: $\mathrm{Re}H(\omega) = 0 \rightarrow H(\omega) = 0$

Ib. ω: $\mathrm{Re}H(\omega) \neq 0 \rightarrow \mathrm{Re}H(\omega) > M > 0$.

Ic. $|H(\omega)|$ is bounded, when $\omega \in [-\pi,\pi]$.

or

(II) If H has the following properties:

IIa. $H(\omega) = \mathrm{Re}H(\omega) \geq 0$ for all $\omega \in [-\pi,\pi]$.

IIb. $H(\omega)$ is bounded, $\omega \in [-\pi,\pi]$.

IIc. There exist a subset S of $[-\pi,\pi]$ and a constant M such that $H(\omega) \geq M > 0$, when $\omega \in S$.

Then, in either case, there exist some real numbers λ and K, and a set $\Gamma \subset [-\pi,\pi]$:

(a) $|1 - \lambda H(\omega)| \leq 1$, for all $\omega \in [-\pi,\pi]$.

(b) $|1 - \lambda H(\omega)| \leq K < 1$, for all $\omega \in \Gamma$.

In Theorem 3, when H plays the role of the Fourier transform of a finite-energy sequence h_n, $-\infty < n < +\infty$, we get two new conditions under which Theorem 1, its corrollary, and Theorem 2 can be applied.

III. APPLICATIONS

First of all, we would like to point out some aspects related to the iterative restoration approach for the multidimensional case.

Basically, the main results given in Section II may be extended to the multidimensional case. Nevertheless, due to the nature of the problem, in multiple dimensions the time-limiting operator becomes much richer than in the 1-d case, even though all the results in Sections II and III concerning the 1-d problem can be extended to the case in which the time-limiting operator T is defined with respect to any bounded set of integer numbers.

We will consider the 2-d case without losing generality, and we will state the two-dimensional version of only Theorem 1. Later in this Section we will show an application of Theorem 1 for the multidimensional case. To establish some notation we state:

Theorem 4

Let h: $h(n_1,n_2)$, $-\infty < n_1,n_2 < +\infty$ be a finite-energy sequence, and Γ be a subset of $[-\pi,\pi] \times [-\pi,\pi]$ which contains, at least, an open set. Let us suppose that h has the following property: There is a number k and there exists some real constant λ such that:

(a) $\quad |1-\lambda \hat{h}(\omega_1,\omega_2)| \leq 1, \quad$ for all $\omega_1,\omega_2 \in [-\pi,\pi]$

(b) $\quad |1-\lambda \hat{h}(\omega_1,\omega_2)| \leq k < 1,$ for all $(\omega_1,\omega_2) \in \Gamma$.

Then, it follows that we can find a constant $M < 1$ such that:

$$\|(\delta_2 - \lambda h) * E_{\sigma} x\| \leq M \| E_{\sigma} x\|$$

where $E_{\mathcal{A}}$ is given by:

$$E_{\mathcal{A}}(x)(n_1,n_2) = \begin{cases} x(n_1,n_2) \text{ if } (n_1,n_2) \varepsilon \mathcal{A}, \\ 0 \text{ otherwise,} \end{cases}$$

x: $x(n_1,n_2)$ $-\infty < n_1, n_2 < +\infty$ is any 2-dimensional sequence,

δ_2 denotes the 2-d unit sample: $\delta_2(0,0) = 1$ and 0 otherwise.

The proof of this theorem is analogous to the 1-d case and will be omitted. However, it is worth saying that this 2-d result is not obtained by using the 1-d Theorem 1 with separability arguments or by stacking the 2-d sequences into vectors. It is also important to note that $\mathcal{A}$ need not be a square or a rectangle.

Similar conditions to those of Theorem 3 may be derived for more dimensions. As an example, let us write the 2-d low-pass operator

$$SINC_{A,B}(n_1,n_2) = SINC_A(n_1).\ \ SINC_B(n_2)$$

The frequency response is:

$$\chi_{A,B}(\omega_1,\omega_2) = \begin{cases} 1 \text{ if } (\omega_1,\omega_2) \in AxB \\ 0 \text{ otherwise} \end{cases}$$

In that case, given $\lambda \in (0,2)$ and $\Gamma = AxB$, we have the hypothesis of Theorem 4 satisfied, and therefore

$$(\delta_2 - \lambda\ SINC_{A,B}) * E_{\mathcal{A}}$$

is a contraction mapping.

This particular example, as well as its 1-d counterpart, has been shown to lead to iterative algorithms for the extrapolation of a band-limited discrete signal, given a finite number of known values of the signal. In our results, the known samples of the signal need not form a square (in the 2-d case) and need not be consecutive (in the 1-d case). These results extend those recently given in [5]. In addition, since the sequence h in Theorem 4 is more general than $SINC_{A,B}$ as defined above, this two-dimensional version of Theorem 1 is useful to provide some other band-limited extrapolations of the given image. For a more detailed discussion of this subject matter the reader is referred to [4], [5]. One basic tool in our extrapolation technique consists of solving the system of equations:

$$z(n_1,n_2) = (h*E_{\mathcal{A}}x)(n_1,n_2), \quad (n_1,n_2) \in \mathcal{A} \tag{9}$$

where $z(n_1,n_2)$, $(n_1,n_2) \in \mathcal{A}$ are the values of the known part of the image, and h is any two-dimensional sequence which satisfies the hypothesis of Theorem 4. It is also assumed that h is band-limited to $\Omega \subseteq [-\pi,\pi] \times [-\pi,\pi]$, i.e.,

$$h(\omega_1,\omega_2) = \sum_{n_1 \in Z} \sum_{n_2 \in Z} h(n_1,n_2)\, e^{in_1\omega_1}\, e^{in_2\omega_2} = 0$$

$$\text{if } (\omega_1,\omega_2) \notin \Omega$$

Theorem 4 provides an iterative procedure for solving (9) and a rationale for choosing λ independently of $\mathcal{A}$: the two-dimensional sequence

$$x^{k+1} = \lambda E_{\mathcal{A}}z + (\delta_2 - \lambda h)*E_{\mathcal{A}}\, x^k \tag{10}$$

Since $(\delta_2 - \lambda h)*E_{\mathcal{A}}$ is a contraction mapping, the convergence of (10) is ensured. If x denotes the limit of the procedure, it is easy to verify:

$$E_{\mathcal{A}}x = \lambda E_{\mathcal{A}}z + E_{\mathcal{A}}(E_{\mathcal{A}}x - \lambda h*E_{\mathcal{A}}x)$$

and since $E_{\mathcal{A}}E_{\mathcal{A}}x = E_{\mathcal{A}}x$, we conclude

$$z(n_1,n_2) = (h * E_{\mathcal{A}} x)(n_1,n_2), \quad (n_1,n_2) \in \mathcal{A}$$

Three important remarks are in order: 1. The low-pass operator h need not be the function $SINC_{A,B}$. 2. The pass-band of h need not be a rectangle (the same observation also applies to the set where the signal is known: $\mathcal{A}$). 3. The parameter λ can be chosen independently of $\mathcal{A}$.

To end this section, we would like to point out that different h's provide different extrapolations of the known data, $z(n_1,n_2)$, $(n_1,n_2) \in \mathcal{A}$. This flexibility is exploited in [4].

IV. DISCUSSIONS

In connection with the problems we address in this paper, we would like to point out some loose ends found in Section III.C of [1]. In [1] it was proven that the operator

$$(\delta - \lambda h) * T \tag{11}$$

is nonexpansive, for some values of λ, when $\hat{h}$ satisfies

$$|1 - \lambda \hat{h}(\omega)| \leq 1 \text{, for all } \omega \in [-\pi, \pi] . \tag{12}$$

The authors pointed out that if Re $\hat{h}$ is non-negative, then λ may be chosen to meet the condition (12). We think that this assertion may not hold in general. Let us suppose that there exists a set $\theta \subseteq [-\pi, \pi]$ such that

$$\hat{h}(\omega) = j \cdot \mathrm{Im}\hat{h}(\omega) \text{ , } \omega \in \theta \text{ .}$$

In this case, there is no λ that satisfies (12), unless $\lambda = 0$.

With respect to the operator (11), the authors of [1] suggested that some other constraint should be added to make it a contraction mapping, or at least, to make the corresponding iterative procedure converge. We have proven in this paper that this is not necessary (Theorem 1).

In [1], it is proved that when the convolution operator is given by $SINC_A$, and when the constraint is a composition of T and P, the iterative scheme

$$x^k = \lambda y + (\delta - \lambda \; SINC_A) * TPx^{k-1} \text{, } x^o = \lambda y \tag{13}$$

converges, when $\lambda \in (0,2)$. We think that the proof is not rigorous even though the assertion is correct (immediate corollary of our Theorem 2 when C is the positivity constraint P, which assures that $(\delta - \lambda SINC_A) * TP$ is a contraction mapping). In [1], the argument used is that at each step of the iterative procedure (13), the x^k does not satisfy the constraint $TPx = x$ except, eventually, in the limit when $k \to \infty$. Therefore

$$\|x^{k+1} - x^k\| \le r_2 \|x^k - x^{k-1}\|$$

with $0 < r_2 < 1$. It is very important to point out that r_2 depends on k. Let us write $r_2(k)$ to make that dependency explicit. This means that

$$\|x^{k+1} - x^k\| \le \prod_{m=1}^{k} r_2(m) \|x^1 - x^0\|$$

There is no reason to believe that $\prod_{m=1}^{k} r_2(m) \to 0$ when $k \to \infty$.

A related point is in connection with the relation

$$\|Tx - Tz\|^2 = L\|x - z\|^2,$$

where $L = 1 - \dfrac{\sum_{n \notin [na,nb]} |x(n) - z(n)|^2}{\|x - z\|^2}$.

It has been asserted that $\|Tx - Tz\| = \|x - z\|$, if and only if z and x are zero outside $[n_a,n_b]$, which is not quite true since any two sequences x and z which coincide outside $[n_a,n_b]$ satisfies $L = 1$ (we assume $x \neq z$ outside $[na,nb]$).

We conclude this paper by pointing out some important issues related to the application of the iterative procedure

$$x^{k+1} = \lambda y + (I - \lambda H)Cx^k \tag{3}$$

to restoration. Let us assume that $C = DT$ (where T is the time-limiting constraint) then we can write (3) as

$$x^{k+1} = \lambda y + (I - \lambda H)DTx^k \tag{14}$$

A careful look at (14) shows that in each step, the new estimation x^{k+1} uses the values Tx^k of x^k only. This means that if the iteration converges: $x^k \to x$ the following iteration:

$$T\,x^{k+1} = \lambda Ty + T(I - \lambda H)DTx^k \qquad (15)$$

will also converge, and the limit will be Tx. Let us suppose that all the necessary conditions are satisfied so that Tx will be the solution to our problem:

$$\begin{aligned} y &= Hx \\ x &= Tx,\ x = Dx \end{aligned} \qquad (16)$$

In that case, it is clear from our analysis that the only information that (14) (or equivalently (15)) is using from the available data is Ty, that is $y(n)$, $n \in [n_a, n_b]$. However, this is a characteristic of the iterative approach (14) and the interval $[n_a, n_b]$ where the sought signal x is supported.

If the solution to (16) exists and is unique, this analysis shows that the portion Ty of the data y will be enough to recover x. However, as was pointed out in Section I, this will not be the case if some noise is present in the given data y. In that case, there is no a priori reason to believe that the portion $y(n)$, $n \in [n_a, n_b]$ is the "best" choice for applying any restoration algorithm.

There are many alternatives for dealing with this problem. The first natural attempt is to try to "shift" the data. Unfortunately, if one has in mind the restoration algorithm (15), this operation will introduce changes in the distortion operator. This might make those conditions on the distortion H which are needed for insuring convergence of (15) impossible.

Another alternative is to "enlarge" the interval of time where the original signal x is to be sought. This means that it might be possible to apply (15) for T_1 defined as

$$T_1 z(n) = \begin{cases} z(n), & n \in [n_a', n_b'], \\ 0 & \text{otherwise} \end{cases}$$

where $[n_a, n_b] \subseteq [n_a', n_b']$. In this case, one needs to study how the part of the restored signal which leaks out of $[n_a, n_b]$ depends on the noise in the data. Another related alternative is to use some constraint which incorporates into the restoration scheme (15) the information contained in $T_1 x^k - Tx^k$, for every k. For example, if D is not a point-by-point operator, it will not commute with T, that is to say $DT \neq TD$. In this case, D might do the desired job if we implement equation (15) by means of TD instead of DT. This also shows the importance of combining the constraints properly in the iterative procedure even though $DTx = TDx$ when x is the sought solution to the problem (16).

Another technique was given in [1]. It consists of low-passing the data y before applying any restoration technique. This process produces a signal $\tilde{y} = h'*y$. In this case, this operation merges the data y according to the nature of the noise and therefore, the window $\tilde{y}(n)$, $n \in [n_a, n_b]$ should contain more information about y than $y(n)$, $n \in [n_a, n_b]$. However, in applying iteration (15) (with $\tilde{y}$ as data and $h'*H$ as the distortion operator) it seems hard to explain why $\tilde{y}(n)$, $n \in [n_a, n_b]$ is the best part from $\tilde{y}$ which can be used in the restoration.

Appendix I

To prove Theorem 1 we will need to prove the following lemmas:

Lemma 1

Let $S = \{x(n),\ n_a \le n \le n_b : \sum_{n=n_a}^{n_b} x^2(n) = 1\}$, and let Λ be any subset of $[-\pi,\pi]$ such that $[-\pi,\pi]-\Lambda$ is a set of non-zero measure. Then, there exists a constant $c < 1$, such that:

$$\int_{\omega \in \Lambda} |\hat{x}(\omega)|^2 d\omega \le c, \text{ for all } x \in S.$$

Proof

It is clear that

$$\int_{\omega \in \Lambda} |\hat{x}(\omega)|^2 d\omega \le \int_{-\pi}^{\pi} |\hat{x}(\omega)|^2 d\omega \tag{AI.1}$$

Nevertheless, since $[-\pi,\pi]-\Lambda$ is a set of non-zero measure, and $\hat{x}(\omega) = \sum_{n=na}^{nb} x(n)\cdot e^{in\omega}$ cannot vanish over $[-\pi,\pi]-\Lambda$ (because $[-\pi,\pi]-\Lambda$ is an infinite set) then strict inequality must hold in (A1). Taking into account that $x \in S$, we have:

$$\int_{\omega \in \Lambda} |\hat{x}(\omega)|^2 d\omega < 1 . \tag{AI.2}$$

But the left-hand term in (AI.2) is a continuous real valued function of x, defined on the compact set S, then we may write

$$\max_{x \in S} \int_{\omega \in \Lambda} |\hat{x}(\omega)|^2 d\omega = c < 1 . \tag{AI.3}$$

(3) proves the lemma.

An immediate corollary is

Lemma 2

Using the same notation as in Lemma 1, there is a constant $C < 1$ such that

$$\int_{\omega \in \wedge} |\hat{x}(\omega)|^2 d\omega < C \, \|x\|^2,$$

for all sequences x: $x(n) = 0$ if $n \notin [n_a, n_b]$.

<u>Comments</u>

1. Let us suppose that $\wedge$ is also of positive measure. It is worth saying that using the same idea as in Lemma 1, it may be shown that there exists another constant, say d, $d > 0$ such that

$$\int_{\omega \in \wedge} |\hat{x}(\omega)|^2 d\omega \geq d \, \|x\|^2$$

for all x: $x(n) = 0$ if $n \notin [n_a, n_b]$.

2. The constant C in Lemma 2 as well as d depends on only the set $\wedge$ and the time interval $[n_a, n_b]$ under consideration.

<u>Theorem 1</u> (proof)

$$\|(\delta - \lambda h) * Tx\|^2 = \|(1 - \lambda \hat{h}) \hat{T} x\|^2 = \int_{-\pi}^{\pi} |1 - \lambda \hat{h}(\omega)|^2 \cdot |\hat{T}\hat{x}|^2 d\omega$$

Taking into account properties (a) and (b) of Theorem 1:

$$\|(\delta - \lambda h) * Tx\|^2 \leq K^2 \int_{\omega \in \Gamma} |\hat{T}x(\omega)|^2 d\omega + \int_{\omega \in \wedge} |\hat{T}x(\omega)|^2 d\omega$$

but $$\int_{\omega \in \Gamma} |\hat{T}x(\omega)|^2 d\omega = \|Tx\|^2 - \int_{\omega \in \wedge} |\hat{T}x(\omega)|^2 d\omega$$

then, we have

$$\|(\delta - \lambda h) * Tx\|^2 \leq K^2 \|Tx\|^2 + (1 - K^2) \int_{\omega \in \wedge} |\hat{T}x(\omega)|^2 d\omega,$$

and using Lemma 2:

$$\| (\delta - \lambda h) * Tx \|^2 \leq [K^2 + (1-K^2) C] \|Tx\|^2$$

where $C < 1$. If we denote

$$M = [K^2 + (1-K^2)\, C]^{\frac{1}{2}}$$

and realize that $M < 1$, we have completed the proof.

Theorem 3 (proof)

We will give only the proof for the condition II. We want to find λ:

$$|1-\lambda H(\omega)|^2 = 1-2\lambda \,\mathrm{Re}H(\omega) + |H(\omega)|^2 \lambda^2 \leq 1$$

Under the conditions stated by II, we get the inequality:

$$-2\lambda H(\omega) + H^2(\omega)\lambda^2 \leq 0, \text{ for all } \omega \in (-\pi,\pi) . \qquad \text{(AI.4)}$$

When $H(\omega) = 0$, (AI.4) becomes automatically true; when this is not the case, we must choose λ:

$$0 < \lambda \leq 2 \frac{H(\omega)}{H^2(\omega)} , \text{ for all } \omega \in [-\pi,\pi] .$$

This yields to the condition:

$$0 < \lambda \leq \frac{2}{\|H\|_\infty} , \ \|H\|_\infty = \sup_{\omega \in [-\pi,\pi]} H(\omega) . \qquad \text{(AI.5)}$$

When $\omega \in S$, we need λ to be such that:

$$|1-\lambda H(\omega)| \leq K < 1.$$

Let us put $(1-C^2)-2\lambda H(\omega) + \lambda^2 H^2(\omega) \leq 0$, $\omega \in S$, $C > 0$ for some λ and C to be chosen.

In making such a choice λ must satisfy:

$$\lambda \in \left(\frac{1-C}{H(\omega)} , \frac{1+C}{H(\omega)} \right) \text{ , for all } \omega \in S.$$

Since $\omega \in S$, then $\frac{1-C}{H(\omega)} < \frac{1-C}{M}$; this implies that C should be such that:

$$1 > C \geq \frac{1 - \frac{M}{\|H\|_\infty}}{1 + \frac{M}{\|H\|_\infty}} > 0 \tag{AI.6}$$

in order that λ can exist. λ should belong to $\left(\frac{1-C}{M} , \frac{1+C}{\|H\|_\infty}\right)$ which is non-empty because of (AI.6). Finally, λ must also satisfy (AI.5); by getting both conditions together, we have

$$\lambda \in \left(0, \frac{2}{\|H\|_\infty}\right) \cap \left(\frac{1-C}{M}, \frac{1+C}{M}\right) \tag{AI.7}$$

Since C is positive and less than 1, (AI.7) is equivalent to $\lambda \in \left(\frac{1-C}{M}, \frac{1+C}{M}\right)$.

Appendix II

We present here an extension of Theorem 1 to the case where H is a shift-variant linear operator. The constraint involved is TP where T is the truncation operator and P is the following positive operator:

$$(Px)(n) = \begin{cases} x(n), \text{ if } x(n) \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

Theorem

Let $h: h(m,n)$, $-\infty < m < +\infty$, $-\infty < n < +\infty$, be a sequence such that:

(a') $\quad \sum_{m=-\infty}^{+\infty} h(m,n) \cdot h(m,\bar{n}) < +\infty$, for all $n,\bar{n}$: $n_a \leq n,\bar{n} \leq n_b$

(b') $\quad h(m,n) \geq 0$, for all m,n.

(c') $\quad h(m,n) > 0$, for n,m: $n_a \leq n,m \leq n_b$.

Then, there exist some constants λ, and $C < 1$, such that:

$$\| (I - \lambda h) PTx \| \leq C \| PTx \|$$

for all sequences $x : x(n)$, $-\infty < n < +\infty$.

Proof

We will start by assuming that $\| TPx \| = 1$. Let us call $y = Px$.

$$\| (I - \lambda h) TPx \|^2 = \sum_{m=-\infty}^{-\infty} \left| Ty(m) - \lambda \sum_{n=n_a}^{n_b} h(m,n)\, y(n) \right|^2 \qquad \text{(AII.1)}$$

where λ is to be chosen properly. (AII.1) becomes:

$$\sum_{m=-\infty}^{+\infty} \{ (Ty(m))^2 - 2\lambda Ty(m) \cdot \sum_{n=n_a}^{n_b} h(m,n)y(n) + \lambda^2 (\sum_{n=n_a}^{n_b} h(m,n)y(n))^2 \}$$

$$= 1 - 2\lambda \sum_{m=n_a}^{n_b} y(m)\cdot \sum_{n=n_a}^{n_b} h(m,n)y(n) + \lambda^2 \cdot \sum_{m=-\infty}^{+\infty} \left(\sum_{n=n_a}^{n_b} h(m,n)y(n)\right)^2 .$$

Using the property (a') of h we get

$$+\infty > \sum_{m=-\infty}^{+\infty} \left(\sum_{n=n_a}^{n_b} h(m,n)y(n)\right)^2 = \sum_{n=n_a}^{n_b} \sum_{\bar{n}=n_a}^{n_b} \left(\sum_{m=-\infty}^{+\infty} h(m,n)h(m,\bar{n})\right)y(n)y(\bar{n})$$

Now, we have the function $\Psi_\lambda : \ z \to \|(I-\lambda h)z\|^2$ defined over the set

$S' = \{z : z(n), n_a \leq n \leq n_b : \|z\| = 1,\ z \geq 0\}$ which is a compact set. Taking into account that Ψ_λ is real-valued (and finite!) and continuous, Ψ_λ attains its maximum value over S':

$$\max_{z \in S'} \Psi_\lambda(z) = C_\lambda \tag{AII.2}$$

We want λ to be a real number (and positive) such that $C_\lambda < 1$. But (AII.2) says that it will be enough to find λ:

$$\Psi_\lambda(z) < 1, \text{ for all } z \in S' .$$

So, we want to choose λ such that

$$-2\lambda \sum_{m=n_a}^{n_b} \sum_{n=n_a}^{n_b} h(m,n)y(m)y(n) + \lambda^2 \sum_{m=-\infty}^{+\infty} \left(\sum_{n=n_a}^{n_b} h(m,n)y(n)\right)^2 < 0 \tag{AII.3}$$

for all $y : y(n),\ n_a \leq n_b \leq n_b : y \in S'$.

By properties (b') and (c') we know

$$\sum_{m=-\infty}^{+\infty} \left(\sum_{n=n_a}^{n_b} h(m,n)y(n)\right)^2 \text{ is non-zero}$$

for all $y \in S'$.

Then, getting (A6) is equivalent to choosing λ:

$$0 < \lambda < \frac{2 \sum_{m=n_a}^{n_b} \sum_{n=n_a}^{n_b} h(m,n)y(m)y(n)}{\sum_{n=na}^{nb} \sum_{\bar{n}=na}^{nb} \{\sum_{m=-\infty}^{+\infty} h(m,n)h(m,\bar{n})\} \quad y(n)y(\bar{n})} \qquad \text{(AII.4)}$$

for all $y \in S'$.

The condition (AII.4) is realized when we choose $\lambda > 0$:

$$\lambda \cdot \sum_{m=-\infty}^{+\infty} h(m,n) \cdot h(m,\bar{n}) < 2h(n,\bar{n}), \text{ for all } n_a \leq n,\bar{n} \leq n_b \ . \qquad \text{(AII.5)}$$

Such a number λ exists since $h(n,\bar{n}) > 0$, $n_a \leq n,\bar{n} \leq n_b$, and since (AII.5) must be satisfied for a finite set of indices only. The fact that (AII.5) implies (AII.4) is seen by using the positiveness of $y(n)$, $n_a \leq n \leq n_b$, and by observing that y is not identically zero. Now it is easy to see that the same λ can be to assure $\|(I - \lambda h)TPx\|^2 \leq C \|TPx\|^2$, $C < 1$ when $\|TPx\|^2 \neq 1$.

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Discrete and Continuous

Band-Limited Signal Extrapolation

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ABSTRACT

This paper has two main purposes. First, some new algorithms for the extrapolation of multidimensional band-limited sequences are presented. These algorithms extend those given in [1] in two ways: (1) We do not impose any restrictions on either the shape of the region containing the set of samples of the 2-dimensional signal to be extrapolated or the shape of its pass-band zone. (2) We obtain a class of sequences which are band-limited extrapolations of the given data (including the minimum norm solution of [1] as a special case). This is useful when some apriori information about the signal to be extrapolated is available. The second objective of our paper is to relate the discrete extrapolation problem to the continuous signal extrapolation problem [2]. Specifically, we prove that the solution obtained by using our approach for the discrete extrapolation problem tends to the solution of the continuous problem when the sampling rate used in the known part of the continuous signal to be extrapolated approaches infinity.

I. INTRODUCTION

Let us begin by recalling what is meant by continuous signal extrapolation and discrete signal extrapolation. Let $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ be a multidimensional signal of finite energy,

$$\left\{ \int_{\mathbb{R}^n} \left| f(x) \right|^2 dx \right\}^{1/2} = \| f \|_2 < \infty . \tag{1}$$

and let f be band-limited to some set Ω,

$$\hat{f}(\omega) = 0 \text{ if } \omega \in \Omega \subseteq \mathbb{R}^n , \tag{2}$$

where $\hat{f}$ denotes the continuous Fourier transform of f. Now, if we are given a piece of f only, (i.e.: we know the function $g: \mathcal{A}_c \longrightarrow \mathbb{C}$: $g(x) = f(x)$, $x \in \mathcal{A}_c$) then <u>the continous extrapolation problem is</u>

To find $f: \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

i) f is band-limited to Ω.

ii) $f(x) = g(x)$ for $x \in \mathcal{A}_c$.

It is well known that f is uniquely determined because f is an analytic function (We assume that $\mathcal{A}_c$ contains, at least, an open set). Several authors have addressed this problem and some algorithms were given ([2],[3],[4], among others). The technique suggested in [2] involved the computation of the so-called prolate spheroidal wavefunctions. These functions were shown to be very useful in understanding the extrapolation problem, as well as in proving the convergence of the iterative algorithm given in [3]. Ref. [4] presented an iterative procedure which is essentially equivalent to that of [3] but its convergence was proven without the use of the prolate functions. This equivalency was shown in [5]. In the same paper [4], another procedure was given which requires the solution of the integral equation:

$$\int_{\mathcal{A}_c} \mathrm{sinc}_{\Omega}(x-t).z(t)dt = f(x),\ x \in \mathcal{A}_c \tag{3}$$

where sinc_{Ω} is the operator whose Fourier transform is

$$x_{\Omega}(\omega) = \begin{cases} 1, & \omega \in \Omega \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

This approach has one main disadvantage: the integral equation (3) may not have a solution.

There does not exist any reason to restrict our attention to the sinc_{Ω} kernel only, since any other kernel, say h_{Ω}, band-limited to Ω, leads us to a new approach by solving the integral equation:

$$\int_{\mathcal{A}_c} h_{\Omega}(x-t)z(t)dt = f(x),\ x \in \mathcal{A}_c \tag{5}$$

provided that z(t) exists.

When any of these algorithms are applied numerically, some approximations are necessary, such as sampling functions, quadrature formulas for integrals, replacing infinite domains by finite ones, etc. As it was pointed out in [1], and as far as we know, no relationships were established between the approximations and the real solution to the problem.

To relate the continuous extrapolation problem with the discrete one, it might be useful to consider the iterative procedure ([4]):

$$f_0 = 0$$

$$f_{n+1}(x) = f_n(x) + \mathrm{sinc}_{\Omega} * (g - Tf_n)(x), \tag{6}$$

where $g(x) = f(x)$, if $x \in \mathcal{a}_c$ and 0 elsewhere, T denotes the time-limiting operation:

$$Tf(x) = \begin{cases} f(x), & x \in \mathcal{a}_c \\ 0, & \text{elsewhere} \end{cases} \tag{7}$$

and * denotes convolution.

Now, if Δ is any positive real number, one possible sampled version of (6) is:

$$y_0(x_j) = 0$$

$$y_{n+1}(x_j) = y_n(x_j) + \Delta \sum_{x_k \in \mathcal{a}_c} \mathrm{sinc}_{\Omega}(x_j - x_k)[g(x_k) - y_n(x_k)] \tag{8}$$

where $x_j = j\Delta$, $j \in \mathbb{Z}$ (the set of integer numbers).

It is well known that the sequence of functions f_n defined by (6), constitutes a family of band-limited functions which converges to f in the energy norm, i.e.:

$$\| f_n - f \|_2 \longrightarrow 0, \text{ when } n \longrightarrow \infty,$$

and, therefore, f_n approaches f in the maximum norm, i.e.:

$$\max_x \left| f_n(x) - f(x) \right| = \left\| f_n - f \right\|_\infty \longrightarrow 0, \text{ when } n \longrightarrow \infty .$$

Nevertheless, there is no apriori reason to believe that the family of sequences defined in (8) converges. This problem was solved in [1], for $\Delta = 1$; it was shown that y_n converges to a sequence y in the energy norm, i.e.:

$$\sum_{k\in\mathbb{Z}} \left| y_n(k) - y(k) \right|^2 \longrightarrow 0, \ n \longrightarrow \infty \tag{9}$$

In addition, it was proven in [1] that the limit y is one solution to the discrete extrapolation problem:

Given $\left\{ g(k),\ k \in \mathcal{A} \right\}$, a set of discrete samples, find a sequence $z(k)$, $k \in \mathbb{Z}^n$, such that:

1. $z(k) = g(k)$, $k \in \mathcal{A}$
2. z is band-limited to Ω, i.e.:

$$F(z)(\omega) = \sum_{k\in\mathbb{Z}^n} z(k).e^{ik\omega} = 0, \text{ when } \omega \notin \Omega \subset [-\pi, \pi]^n \tag{10}$$

(F(z) will denote the Fourier series of the sequence z)

Since the discrete extrapolation problem does not have a unique solution, it was also shown in [1] that the sequence y of (9) has the minimum-norm property among those which satisfy 1. and 2. Due to this simple characterization of y, many other algorithms may be chosen to compute it. Nevertheless, [1] did not discuss the relationship between y and the continous extrapolation problem.

Taking into account that the discrete extrapolation problem has infinite many solutions, we think there is no reason to believe that the minimum-norm extrapolation y is the only useful sequence to approximate the solution of the continuous problem. In this paper, we will show that, in fact, there exist many other criteria to pick up solutions to the discrete extrapolation problem which yield approximations of the continuous problem. All the results are valid for multi-dimensional signals. Since we do not need any restriction on the sets $\mathcal{A}$ and Ω , our results extend those discrete techniques presented in [1] to 2-dimensional signals. We also present some iterative techniques, which generalize that of (8), which are shown to be convergent by using the results obtained in a recent paper ([6]). The connection between the discrete and continuous extrapolation problems will be made by means of a rather general theorem on interpolation by means of band-limited functions. This theorem indicates that many other techniques might be investigated in order to get approximations of the continuous extrapolation problem. In connection with the new algorithms presented here, we also include some numerical examples.

II. MAIN RESULTS

In this section we present our main results. Section II.1 is devoted to new techniques for the discrete extrapolation problem. Section II.2 is mainly concerned with the connection between the discrete and the continuous problems.

II.1 Let z: $z(n)$, $n \in \mathcal{a}$ be a finite sequence. We want to extrapolate z to a band-limited sequence. One possible technique to do such extrapolation is as follows: Let h: $h(m)$, $m \in \mathbb{Z}^n$ be a band-limited sequence. Then, if we define $y = h * x$, where x is any sequence, we will get another band-limited sequence y. Now, if we could find an x which makes $y(n) = z(n)$, $n \in \mathcal{a}$, then we would get one possible band-limited extrapolation of z. The next theorem states this result:

Theorem 1

Let Ω be any non-zero measure subset of $[-\pi, \pi]^n$ and $\mathcal{a}$ be any finite subset of $\mathbb{Z}^n$. Let $\hat{h}: [-\pi, \pi] \longrightarrow \mathbb{R}$ be a non-negative, bounded function, which satisfies $\hat{h}(\omega) = 0$, $\omega \notin \Omega$. Let $z(n)$, $n \in \mathcal{a}$ be a set of observations. Then, if we define $y = h * x$, where h is the sequence:

$$h(n) = \int_{\Omega} \hat{h}(\omega) e^{-in\omega} d\omega \tag{11}$$

and x is a sequence which satisfies:

$$x(m) = 0 \text{ if } m \notin \mathcal{a}.$$

$$z(n) = \sum_{m \in \mathcal{a}} h(n-m)x(m), \quad n \in \mathcal{a} \tag{12}$$

Then: y is an extrapolation of z, band-limited to Ω. Moreover, the sequence x is the solution of the following optimization problem:

$$\text{minimize} \quad \int_{\Omega} \hat{h}(\omega) \left| \sum_{k \in \mathbb{Z}^n} s(k) e^{ik\omega} \right|^2 d\omega \tag{13a}$$

subject to

$$\sum_{k \in \mathbb{Z}^n} \left| s(k) \right|^2 < \infty \tag{13b}$$

$$z(n) = \int_{\Omega} \hat{h}(\omega) \Big(\sum_{k \in \mathbb{Z}^n} s(k) e^{-i(n-k)\omega} \Big) d\omega, \qquad n \in \mathcal{a} \tag{13c}$$

It is convenient to remark that the matrix H: $h(n-m)$, $n,m \in \mathcal{a}$ (14)

is positive-definite; (see the appendix for a proof) in particular, this assures that the sequence x, defined by equation (12), exists.

It is worth pointing out the following property:

Corollary 1

If $z(n)$, $n \in \mathcal{a}$ is real, and if $\hat{h}(\omega) = \hat{h}(-\omega)$ for all ω, then extrapolation y given by theorem 1 is also real.

Now, we connect the result stated in Theorem 1 with those of [1].

Corollary 2

Under the same conditions of Theorem 1, if in addition $\hat{h}$ has the following properties:

(i) $\hat{h}(\omega) \neq 0 \quad \omega \in \Omega$

(ii) The function $\hat{k}(\omega)$ defined as follows:

$$\hat{k}(\omega) = \begin{cases} 0, \text{ if } \omega \notin \Omega \\ \dfrac{1}{\hat{h}(\omega)}, \text{ if } \omega \in \Omega \end{cases}$$

has finite energy: $\displaystyle\int_{[-\pi,\pi]^n} \left| \hat{k}(\omega) \right|^2 d\omega < \infty$

then, the sequence y given in (11) is the solution of the following optimization problem: Find $s(k)$, $k \in \mathbb{Z}^n$:

$$\text{minimize} \quad \int_{\Omega} \hat{k}(\omega) \left| F(s) \right|^2 (\omega)\, d\omega \tag{15}$$

subject to:

$$s(m) = z(m), \; m \in \mathcal{A} \tag{16a}$$

$$F(s) \text{ is bounded in } [-\pi, \pi]^n \tag{16b}$$

$$s \text{ is band-limited to } \Omega \tag{16c}$$

Corollary 3

If $\hat{h}$ satisfies

$$\hat{h}(\omega) = \begin{cases} 1, \; \omega \in \Omega \\ 0, \text{ otherwise} \end{cases} \tag{17}$$

then, the extrapolated sequence y is the minimum-norm solution of the discrete extrapolation problem, where $z(m)$, $m \in \mathcal{A}$ is the initial data.

Since the matrix H defined in (14) is positive-definite, well known algorithms may be applied to solve the system of equations (12). In the 1-dimensional case, if $\hat{h}(w) = \hat{h}(-w)$, $\omega \in [-\pi, \pi]$, then H is symmetric and since H is Toeplitz, the Levinson's algorithm can be applied. For the multi-dimensional case, more conditions related to the symmetry of $\hat{h}$ are necessary to assure H to be block Toeplitz.

Anyway, in the 1-dimensional case, as well as in the multidimensional case, we have an iterative algorithm to compute $x(m)$, $m \in \mathcal{A}$, and therefore, to compute $y(m)$, $m \in \mathbb{Z}^n$. This algorithm yields some generalizations of the interative approach (8).

<u>Theorem 2</u>

Under the same conditions as in Theorem 1, let λ be any real number such that:

(a) $\left| 1 - \lambda\hat{h}(\omega) \right| \leq 1$, for all $\omega \in [-\pi,\pi]^n$.

(b) $\left| 1 - \lambda\hat{h}(\omega) \right| \leq K < 1$, for some K and for all $\omega \in \Omega$.

Then, the sequence $x^k(m)$, $m \in \mathcal{A}$, $k \in \mathbb{Z}$, $k \geq 0$, given by the formulas: $m \in \mathcal{A}$.

$$x^0(m) = \text{any initial guess.}$$

$$x^{k+1}(m) = x^k(m) + \lambda\Big(z(m) - \sum_{n \in \mathcal{A}} h(m-n)x^k(n)\Big) \qquad (18)$$

converges to the solution of equation (12). Moreover, the sequence $y^k(m)$, $m \in \mathbb{Z}^n$, $k \in \mathbb{Z}$, $k \geq 0$, given by

y^0 any initial guess band-limited to

$$y^{k+1}(m) = y^k(m) + \lambda \sum_{n \in \mathcal{Q}} (z(n) - y^k(n))h(m-n) \tag{19}$$

converges to the extrapolation y, given in Theorem 1, in the energy norm, i.e.

$$\left\| y^k - y \right\|_2 \longrightarrow 0,\ k \longrightarrow \infty$$

It is clear that (19) becomes (8) when the dimension n is 1, $\Delta = 1$ and $h(m) = \text{sinc}_{\Omega}(m)$, $m \in \mathbb{Z}$.

The recursive formula (19) is well known. The interesting point is about the way λ was chosen. We have a set of admisible values for λ which depends on $\hat{h}$ only. This means that any λ in this set is useful to make the iteration (19) converges, no matter what $\mathcal{Q}$ is considered. In other words, λ does not depend on the size of the matrix H. The conditions imposed on $\hat{h}$, i.e.: boundnes and non-negativity, assure that the set of possible values of λ is non-empty (see [6]).

To end this section, we would like to point out that equation (12) might be understood as a sampled version of the integral equation

$$\int_{\mathcal{Q}_c} h_{\Omega}(s-t)x(t)dt = f(s),\ s \in \mathcal{Q}_c \tag{20}$$

where $z(n) = f(n\Delta)$ and where the integral is to be replaced by

$$\Delta \sum_{n \in \mathcal{Q}} h_{\Omega}[(m-n)\Delta]x(n\Delta)$$

Nevertheless, we think that this approach should be avoided since (20) might not have any solution. Even if this was not the case, the sampled approximation

$$\Delta \sum_{n \in \mathcal{Q}} h_{\Omega}[(m-n)\Delta]x(n\Delta) = z(m),\ m \in \mathcal{Q} \tag{21}$$

would be a non-reliable technique to approach the solution $x(t)$, $t \in \mathcal{A}_c$: though (21) has always a unique solution: $x_\Delta(n), n \in \mathcal{A}$, there is no evidence that $x_\Delta(n)$ approaches $x(t)$ when Δ tends to zero. On the other hand, we are interested in getting a good approximation of f beyond $\mathcal{A}_c$ and not in determining an approximation of $x(t)$, $t \in \mathcal{A}$.

II.2 In this section we discuss the connection between the discrete and the continuous extrapolation problems. We begin by stating the main result:

Theorem 3

Let f be band-limited to Ω. Let S_k: $k \in \mathbb{Z}$, $k \geqslant 0$ be a family of subsets of $\mathbb{R}^n$ such that:

(i) $\text{Mesh}(S_k) = \sup_{z \in S_k} \left\{ \inf_{\substack{y \in S_k \\ y \neq z}} |y-z| \right\}$ tends monotonically to zero.

ii) $S_k \subseteq \mathcal{A}_c$, for all k.

iii) $\bigcup_k S_k$ is dense in a certain open subset of $\mathcal{A}_c$.

Let $(f_k)_{k \in \mathbb{Z},\ k \geqslant 0}$ be a sequence of Ω-band-limited functions:

a) $f_k(x) = f(x)$, for all $x \in S_k$.

b) $\|f_k\|_2 \leqslant C$, for all k.

Then, there exists a subsequence f_{n_k} which converges to f uniformly over compact sets in $\mathbb{R}^n$, i.e.: for each $K \subseteq \mathbb{R}^n$ compact and given $\varepsilon > 0$, there exists $k_0 \in \mathbb{Z}$, $k_0 \geqslant 0$ such that

$$\sup_{x \in K} \left| f(x) - f_{n_k}(x) \right| \leqslant \varepsilon, \text{ for all } k \geqslant k_0.$$

We would like to point out that Theorem 3 may be understood as an interpolation theorem where each f_k is a Ω-band-limited interpolation of the data $f(x)$, $x \in S_k$. Property b) states that the energy of f_k must be bounded by some constant C, which is independent of k. Then, given any compact set K, any interpolation f_{n_k} is close to f over K when $k \gg k$. Nevertheless, since f_{n_k} is a subsequence of f_k, the theorem does not assure, in principle, that f_h will be close to f over K for all $h \geqslant n_{k_0}$.

Now, we give a complementary result which will be useful:

Theorem 4

Under the same conditions and notations as in Theorem 3, let us suppose that the constant C can be chosen as $\|f\|_2$, i.e.:

$$\|f_k\|_2 \leqslant \|f\|_2, \text{ for all } k \in \mathbb{Z},\ k \geqslant 0.$$

Then, the subsequence f_{n_k} converges to f uniformly over the whole space $\mathbb{R}^n$.

Now we will apply these results to relate the continuous and the discrete extrapolation problems: Let us suppose that $\hat{h}: \mathbb{R}^n \longrightarrow \mathbb{R}$ is non-negative, bounded and $\hat{h}(w) = 0$, $w \notin \Omega$. We assume $\hat{h}(w) \neq 0$, $w \in \Omega$. We are given $f(x)$, $x \in \mathcal{A}_c$, where f is Ω-band-limited with finite energy. We assume that f and $\hat{h}$ satisfy the following relationship:

$$\frac{\hat{f}}{\hat{h}} \text{ has finite energy in } \Omega: \int_\Omega \left| \frac{\hat{f}(\omega)}{\hat{h}(\omega)} \right|^2 d\omega < \infty. \tag{22}$$

Now, let $(T_k)_k$ be a sequence of positive numbers such that:

i) T_k tends monotonically to zero.

ii) $\frac{\pi}{T_k} \geqslant \|\Omega\| = \sup\left\{ \|x\|,\ x \in \Omega \right\}$, for all k.

and let us consider the following sample sets $S_k \subseteq \mathcal{A}_c$:

$$S_k = \left\{ m.\ T_k : m.\ T_k \in \mathcal{A}_c,\ m \in \mathbb{Z}^n \right\}$$

The family $\{S_k\}_k$ satisfy all the conditions stated in i), ii) and iii) of Theorem 3. Now, for each k we take samples of f over S_k: $Z_{T_k}(n) = f(nT_k)$, $nT_k \in \mathcal{a}_c$ and we look for a discrete extrapolation of $Z_{T_k}(n)$, $n \in \mathcal{a}_{T_k} = \{m: mT_k \in \mathcal{a}_c\}$, band-limited to $T_k.\Omega = \{T_k.x : x \in \Omega\}$. We know that we need a criterion to pick up one extrapolation. Let us suppose that we follow the technique given in Theorem 1:

$$y_{T_k}(m) = (h_{T_k} * x_{T_k})(m),\ m \in \mathbb{Z}^n, \tag{22'}$$

where $F(h_{T_k})(\omega) = \hat{h}(\frac{\omega}{T_k}), \omega \in [-\pi, \pi]^n$

We recall that

$$h(s) = \int_\Omega \hat{h}(\omega).e^{-i\omega s}\, d\omega,\ s \in \mathbb{R}^n.$$

Then, the function

$$f_k(s) = (T_k)^n . \sum_{m \in \mathcal{a}_{T_k}} h(s - mT_k) x_{T_k}(m)\ s \in \mathbb{R}^n \tag{23}$$

is band-limited to Ω ; in addition, since $h_{T_k}(n-m) = (T_k)^n.h(T_k(n-m))$, we also have $f_k(mT_k) = y_{T_k}(m)$, for all m and, in particular, $f_k(x) = f(x)$, when $x \in S_k$.

Theorem 1 and the relationship between f and h given by formula (22) yield to the following theorem (for a detailed proof, see the Appendix).

<u>Theorem 5</u>

There exists a constant C such that

$$\| f_k \|_2 \leqslant C, \text{ for all k.}$$

where f_k is given by formula (23).

It is worth pointing out that the function f_k given in (23) is the natural interpolation of $y_{T_k}(m)$, $m \in \mathbb{Z}^n$, by means of a Ω -band-limited waveform. In fact (23) is exactly the same as (22') replacing the discrete variable m by the continuous one S (properly scaled).

Now, taking into account Theorem 5, the other properties of f_k and Theorem 3, we obtain the following result:

Theorem 6

There exists a subsequence of T_k: $T_{n_k} \rightarrow 0$ such that f_{n_k} approaches f uniformly over any compact set in $\mathbb{R}^n$.

Note that when $\hat{h}$ is the indicator function of Ω (17), property (22) holds, since f has finite energy. In this particular case, it will be shown in the Appendix that the constant C in Theorem 5 can be chosen equal to $\|f\|_2$. Therefore, Theorem 4 applies and we have a subsequence which converges uniformly over $\mathbb{R}^n$ to f. In addition, it can be proved that $\|f_k\|_2 \rightarrow \|f\|_2$, where the limit is taken over the whole sequence and not only for n_k.

Two questions remain unanswered. The first one is related to the convergence of f_k to f where k is any index, when $\hat{h}$ is the indicator of Ω. The other question is about the convergence of f_{n_k} to f over the entire $\mathbb{R}^n$ when $\hat{h}$ is an arbitrary function which satisfies the properties discussed above, i.e.: bounded, positive and (22).

III. NUMERICAL EXAMPLES

The purpose of this section is to present some numerical examples related to the new extrapolation techniques given in Section II.1. We will solve the extrapolation problem following the ideas given in Theorem 1, Section II.1. We are given $f:[-1,1] \longrightarrow \mathbb{R}$ which is a piece of a band-limited function (here $\Omega = (-\frac{\pi}{2}, \frac{\pi}{2})$). The frequency response of the kernel h, which were used are plotted in Figures 1, 2 and 3. They will be called h_1, h_2 and h_3, respectively. The data f will be a piece of h_1, h_2 or h_3 sampled on $[-1,1]$ using $T = \frac{1}{8}$. All the extrapolations to be considered are on $[-3,3]$ and they are sampled using the same T as in the known interval $[-1,1]$.

We use two approaches. In the first approach, we solve the system of equations.

$$Z = Hx \tag{12}$$

by means of the computer algorithm given in [11]. We recall that H is given by the formula $H_{m,n} = T.h_i\left\{(m-n)T\right\}$ and $Z: Z_n = h_j(nT)$ where h_i and h_j are the filter and the data to be used. The extrapolation is given by $y = h * x$.

Tables 1, 2 and 3 summarize the results obtained with this approach. In the first case (Table 1), we use h_1 and the data, sampled with $T = \frac{1}{8}$ (see(a)) and h_1 as the filter (see (b)). The result is much better than that obtained with h_2 as the filter (c). Nevertheless, it is clear that the last combination (i.e. h_1 data and h_2 filter) does not satisfy the relationship (22). Also included was the solution of the system of equations (12) (see (d)). It is seen that (d) is far from the real solution of (12). That shows that the technique may not be reliable. Nevertheless, it is also seen that the extrapolation is of acceptable quality beyond (-1,1). The phenomenon, which was also observed in the other numerical examples given below, has not been fully explained yet. Although the solution x_0 given by the algorithm [11] is far from the actual solution, the extrapolated values on [-1,1] are close to the real samples. This effect may be explained as a consequence of the ill-conditioned nature of the matrix H. We think that for the examples considered here small errors inside [-1,1] (i.e.: $\| Hx_0 - z \|$ small) assure the extrapolation to be close to the real waveform on the sample points in [-3,3].

In the second case (Table 2) we used h_2 as the data. (a) is the list of samples on [-3,3] with $T = \frac{1}{8}$ (of course, only the points in [-1,1] will be used in the extrapolation procedure). (b), (c) and (d) show the extrapolation obtained using h_1, h_2 and h_3 as the filters, respectively. Last, Table (3) shows the corresponding results when h_3 provides the data.

The main observation that can be drawn from the examples is that, in all cases, the best extrapolation is given when the filter is the same as the data. Another observation is that the h_1 filter performs better than the h_2 one where the data is h_3 and better than h_3 when the data is h_2. However, we tried some other examples using $T = \frac{1}{16}$ and the result was not the same as those obtained with $T = \frac{1}{8}$. It was observed in that case that when h_2 is the given data, h_3 performs better than h_1. On the other hand, h_1 performs still better than h_3 when the data is h_2.

The second approach used here is the iterative one given in Theorem 2. The filter and the data were chosen to be h_1 (therefore, the iterative technique coincides with that given in Reference [1]). It is clear that the iterative approach is unacceptably slow (Table 4(b))(the same conclusion was obtained from other examples not included here). However, the solution of (12) provided by the iterative technique ((C) Table 4) is much closer to the actual solution than that given by the exact algorithm [11]. But the extrapolation is of poor quality. The numerical properties of the various algorithms are being studied further, and we hope to present some of the results in a future paper.

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IV. APPENDIX

Theorem 1 (proof)

The formula which is to be minimized is

$$\int_{\Omega} h(\omega) \left| \sum_m x(m).e^{im.\omega} \right|^2 d\omega = G(x) \tag{A1}$$

But (A1) becomes:

$$\sum_m \sum_n h(m-n)x(m).x^*(n) = G(x) \tag{A2}$$

On the other hand, the constraint $\int \hat{h}(\omega) \sum_m x(m).e^{im\omega}.e^{-ik\omega} d\omega = z(k)$, becomes:

$$\sum_m x(m).h(k-m) = z(k),\ k \in \mathcal{A} \tag{A3}$$

It is easy to see that

$$\sum_{k\in\mathcal{A}} \sum_{n\in\mathcal{A}} h(n-k)x(k)x^*(n) = \int_{\Omega} \hat{h}(\omega). \left|\sum_m x(m)e^{im.\omega}\right|^2 d\omega \tag{A4}$$

Since $\hat{h}$ is non-negative and not identically zero. (A4) is always positive unless $x(m) = 0$, $m \in \mathcal{A}$ ($\mathcal{A}$ is finite!). This proves that H: $h(n-k)$, n, $k \in \mathcal{A}$ is positive definite. Then, we can write when $n \in \mathcal{A}$:

$$x(n) = \sum_{k\in\mathcal{A}} \ell(n,k)z(k) - \sum_{k\in\mathcal{A}} \ell(n,k). \sum_{m\notin\mathcal{A}} h(k-m)x(m) \tag{A5}$$

Since we have $h(n-k) = h(k-n)^*$, then the inverse $\ell(n,k)$ satisfies $\ell(n,k) = \ell(k,n)^*$. Taking conjugates in (A5):

$$x^*(n) = \sum_{k\in\mathcal{A}} \ell(k,n)z^*(k) - \sum_{k\in\mathcal{A}}\sum_{m\notin\mathcal{A}} \ell(k,n)h(m-k)x^*(m), n\in\mathcal{A} \qquad (A6)$$

On the other hand, equation (A3) is equivalent to:

$$\sum_{m\in\mathcal{A}} h(n-m)x(m) = z(n) - \sum_{m\notin\mathcal{A}} h(n-m)x(m),\ n \in \mathcal{A} \qquad (A7)$$

Now, replacing (A7) into (A2) and using (A5), (A6) we get:

$$G(x) = \sum_{n\in\mathcal{A}}\sum_{k\in\mathcal{A}} \ell(k,n)z(n)z^*(k) - \sum_{k\in\mathcal{A}}\sum_{m\notin\mathcal{A}}\sum_{n\in\mathcal{A}} z(n)\ \ell(k,n)h(m-k)x^*(m)$$

$$+\sum_{n\in\mathcal{A}}\sum_{m\in\mathcal{A}}\sum_{k\in\mathcal{A}} h(n-m)\ \ell(m,k)x^*(n)z(k) + \sum_{m\notin\mathcal{A}}\sum_{n\notin\mathcal{A}} h(n-m)x(m)x^*(n)$$

$$-\sum_{n\notin\mathcal{A}}\sum_{m\in\mathcal{A}} h(n-m)x^*(n)\sum_{k\in\mathcal{A}}\sum_{\bar{m}\notin\mathcal{A}} h(k-\bar{m})x(\bar{m})\ \ell(n,k) \qquad (A8)$$

By simple inspection, the second and the third terms in (A8) cancel out.

If we put the forth and the fifth terms together we will have:

$$J = \sum_{m\notin\mathcal{A}}\sum_{n\notin\mathcal{A}} h(n-m)x(m)x^*(n) - \sum_{\bar{m}\notin\mathcal{A}}\sum_{n\notin\mathcal{A}}\left(\sum_{m\in\mathcal{A}}\sum_{k\in\mathcal{A}} h(n-m)\ \ell(n,k)h(k-\bar{m})\right)x(m)x^*(n)$$

It is worth pointing out that J does not depend on $x(k)$, $k\in\mathcal{A}$ and it is a quadratic function for $x(k)$, $k\notin\mathcal{A}$. Then, we have

$$G(x) = \sum_{k\in\mathcal{A}}\sum_{n\in\mathcal{A}} \ell(k,n)z(k)z^*(n) + J(x) \qquad (A9)$$

In (A9), the first term on the right hand: I is non-negative and it does not depend on $x(m)$, $m\in\mathbb{Z}^n$. Since $G(x)$ is always non-negative, it remains to prove that $J \geqslant 0$, to conclude that

$$\min G(x) = \sum_{k\in\mathcal{Q}} \sum_{n\in\mathcal{Q}} \ell(k,n)z(k)z^*(n)$$

and that the optimum is reached when $x(m) = 0$, $m \notin \mathcal{Q}$. Then, equation (A7) will become:

$$z(n) = \sum_{m\in\mathcal{Q}} h(n-m)x(m), \ n \in \mathcal{Q}$$

Now, let us suppose that $J < 0$ for some $x(k)$, $k \notin \mathcal{Q}$. Then, we define a new sequence: $r(k)$, $k \in \mathbb{Z}^n$:

$$r(k) = \lambda x(k), \ k \notin \mathcal{Q}$$

and $\quad r(k)$, $k \in \mathcal{Q}$:

$$\sum_{k\in\mathcal{Q}} h(n-k)r(k) = z(n) - \sum_{k\notin\mathcal{Q}} h(n-k)r(k), \ n \in \mathcal{Q} \tag{A10}$$

where λ is some real number. Since (A10) is the constraint equation (A8), then formula (A9) holds:

$$G(r) = I + J(r) = I + \lambda^2 J(x) \tag{A11}$$

Now, choosing λ^2 large enough we get $I + \lambda^2 J(x) < 0$. This λ will lead to a sequence $r(k)$, $k \in \mathbb{Z}^n$: $G(r) < 0$. The last assertion is a contradiction because $G \geqslant 0$.

<u>Corollary 2</u> (proof)

Let us call M_1 the optimal value of the expression defined in (13a), subject to (13b) and (13c); let x_0 the optimum given by equation (12). If we define $y_0 = h * x_0$, then y_0 will satisfy conditions (16a), (16b) and (16c). Then the optimal value of the functional given in (15) M_2 satisfies $M_2 \leq M_1$. Now, let y be any feasible solution of the problem (15), i.e.: y is any sequence which satisfies conditions (16a), (16b) and (16c). We define the function: $\hat{x}: [-\pi, \pi]^n \longrightarrow \mathbb{C}$

$$\hat{x}(\omega) = \begin{cases} 0, \omega \in \Omega \\ \dfrac{\hat{y}(\omega)}{\hat{h}(\omega)}, \omega \in \Omega \end{cases}$$

$\hat{x}$ has finite energy because $\hat{y}$ is bounded and $\int_{\Omega} \frac{1}{|\hat{h}(\omega)|^2} d\omega$ is finite. In addition, $\hat{x}$ satisfies equation (13b) and (13c) (we consider $x(n) = \frac{1}{2\pi} \int_{\Omega} \hat{x}(\omega).e^{-i\omega.n} d\omega$). In conclusion,

$$\int_{\Omega} \frac{|\hat{y}|^2}{\hat{h}} d\omega = \int_{\Omega} \hat{h} |\hat{x}|^2 d\omega \geqslant M_1 \tag{A12}$$

Equation (A12) implies $M_2 \geqslant M_1$. Then $M_1 = M_2$ and $y_0 = h * x_0$ realizes the optimum value M_2.

Theorem 2 (proof)

Let δ denote the unit impulse-response: $\delta(m) = 1$ and $\delta(m) = 0$, $m \neq 0$. It was proven (see[6]) that the operator:

$$(\delta - \lambda h) * T_{\mathcal{A}} x \tag{A13}$$

defined over infinite sequences x: $x(m)$, $m \in \mathbb{Z}^n$, is a contraction mapping, i.e., there exists a constant $d < 1$ such that

$$\|(\delta - \lambda h) * T_{\mathcal{A}} x\|_2 \leqslant d \|T_{\mathcal{A}} x\|_2 \leqslant d \|x\|_2 \tag{A14}$$

In formulas (A13) and (A14), $T_{\mathcal{A}}$ denotes the truncation operator relative to the set $\mathcal{A}$. Now, it is clear that

$$x(n) - \lambda \sum_{m \in \mathcal{A}} h(n-m) x(m) \ : \ n \in \mathcal{A}$$

is also a contraction mapping defined over finite-length sequences: $x(m)$, $m \in \mathcal{A}$. But the recursive formula given in (18) is convergent to the fixed point: $x(n)$, $n \in \mathcal{A}$:

$$\sum_{n \in \mathcal{A}} h(m-n) x(n) = z(m) \ , \ m \in \mathcal{A}$$

Now, denoting $x^{k+1}(m) = x^k(m) + \lambda(z(m) - \sum_{n\in\mathcal{Q}} h(m-n)x^k(n))$,

since x^k approaches x when $k \longrightarrow \infty$ (i.e., $\sum_{n\in\mathcal{Q}} |x(n)-x^k(n)|^2$ tends to zero) we have:

$$h * x^k \longrightarrow h * x \text{ in the energy norm, } k \longrightarrow \infty .$$

Denoting $y^k = h * x^k$ and replacing x^k by its recursive form we get (19).

<u>Theorem 3</u> (proof)

We know that f_k can be extended as an entire function over $\mathbb{C}^n$ (the space of n complex variables) by means of the relation:

$$f_k(z) = \int_{\Omega} \hat{f}_k(\omega).e^{iz\omega}\, d\omega \quad z\in \mathbb{C}^n. \tag{A15}$$

We will prove that $f_k(z)$, $z \in \mathbb{C}$, for all k constitutes a sequence of normal function over $\mathbb{C}^n$ (see [7], [8]). We recall that a normal family of analytic function is uniformly bounded over any compact at K in $\mathbb{C}^n$, i.e., for each K there exists a constant c_K such that:

$$\max_{z\in K} |f_m(z)| \leqslant c_K, \ c_K \text{ does not depend on m.}$$

By means of (A15) we have, for all $z \in \mathbb{C}^n$,

$$|f_k(z)| \leqslant \left\{\int_{\Omega} |f_k(s)|^2 ds\right\}^{1/2} . \left\{\int_{\Omega} |e^{iz\omega}|^2 d\omega\right\}^{1/2} \tag{A16}$$

Now, the first factor on the right-hand of (A16) is bounded by hypothesis; the second factor is bounded in any set of the form:

$$\Delta : \left\{ z:(z_1,\dots,z_n)\in \mathbb{C}^n: \ \left|\mathrm{Im}\ z_i\right| \leqslant M_i, \text{ for all } i \right\} \tag{A17}$$

Of course, any bounded set in $\mathbb{C}^n$ is contained in a set of the form (17). Now, it is well known that any normal sequence of analytic functions in $\mathbb{C}^n$ contains a subsequence which is convergent uniformly over compact sets (see [7]). But, in that case, such a subsequence f_{n_k} has an analytic limit, i.e., for each K compact set in $\mathbb{C}^n$, $f_{n_k} \longrightarrow g$ uniformly over K, where g is an analytic function over $\mathbb{C}^n$.

We need to prove that $g(x) = f(x)$, when $x \in \mathbb{R}^n$. To this end, we recall that $f_{n_k(x)} = f(x)$, for $x \in S_{n_k}$. Since the sequence mesh (S_k) tends monotonically to zero and S_{n_k} is a subsequence of S_k, we have that $\bigcup_k S_{n_k}$ is also dense in an open subset of $\mathcal{A}_c$, say $\mathcal{B}_c$. Now, for any $x \in \mathcal{B}_c$ we can pick x_o arbitrarily close to x such that $f_{n_k}(x_o) = f(x_o)$ for some function f_{n_k}. Since the family $(f_{n_k})_k$ is equicontinuous (because the family is normal!) we have $f(x) = g(x)$. Now, both f and g are entire functions of the real variables $x \in \mathbb{R}^n$, and they coincide on an open set $\mathcal{B}_c$ (which is open set in $\mathbb{R}^n$, not in $\mathbb{C}^n$!) then, $f = g$ over $\mathbb{R}^n$ and the theorem is proved.

Theorem 4 (proof)

By Theorem 3, we know that there exists a sequence f_{n_k} which converges to f uniformly over compact sets of $\mathbb{R}^n$. Now, in addition:

$$\| f_{n_k} \|_2 \leqslant \| f \|_2 \text{, for all k.} \tag{A18}$$

Now, let us define $g_{n_k}(x) = |f_{n_k}(x)|^2$ for all k and $x \in \mathbb{R}^n$. g_{n_k} is a sequence of non-negative functions:

$$\lim_{k \longrightarrow \infty} g_{n_k}(x) = |f(x)|^2 \text{, for all } x \in \mathbb{R}^n.$$

Now, the well known Fatou's theorem applies [9]:

$$\int_{\mathbb{R}^n} \lim_k g_{n_k}(x)dx \leqslant \lim_k \inf \int_{\mathbb{R}^n} g_{n_k}(x)dx \tag{A19}$$

where lim inf denotes the lower limit of the sequence $\int_{\mathbb{R}^n} g_{n_k}(x)dx$.
Then, (A19) is equivalent to:

$$\int_{\mathbb{R}^n} |f(x)|^2 dx \leqslant \lim_k \inf \int_{\mathbb{R}^n} |f_{n_k}(x)|^2 dx \tag{A20}$$

Now, taking into account (A18), we conclude:

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \|f\|_2^2 \leqslant \lim_k \inf \|f_{n_k}\|_2^2 \leqslant \lim_k \sup \|f_{n_k}\|_2^2 \leqslant \|f\|_2^2$$

which means: $\lim_k \|f_{n_k}\|_2^2$ exists and equals $\|f\|_2^2$.

Now, we apply the well known result which states: $f_{n_k} \longrightarrow f$ in the energy norm if f_{n_k} approaches f in each point $x \in \mathbb{R}^n$, and $\|f_{n_k}\|_2 \longrightarrow \|f\|_2$ (see [10]).

Theorem 5 (proof)

Since $y_{T_k}(m)$, $m \in \mathbb{Z}^n$, are samples of f_k at the points mT_k, then

$$\hat{f}_k(\omega) = (T_k)^n . F(y_{T_k})(\omega T_k) \ , \ \omega \in \Omega \qquad \text{(A21)}$$

Replacing the expresion of y_{T_k} given by (22'), we get, from (A21):

$$\hat{f}_k(\omega) = (T_k)^n . \hat{h}_{T_k}(\omega T_k) . \sum_{n \in \mathcal{A}} x_{T_k}(n) . e^{in\omega T_k}.$$

Now, $\|\hat{f}_k\|_2^2$ becomes:

$$\int_{\Omega} \left| f_k(\omega) \right|^2 d\omega = (T_k)^{2n} . \int_{\Omega} \left| \hat{h}_{T_k}(\omega T_k) \right|^2 . \left| \sum_{n \in \mathcal{A}} x_{T_k}(n) . e^{in\omega T_k} \right|^2 d\omega \qquad \text{(A22)}$$

Changing variables in (A22): $\bar{\omega} = T_k . \omega$, we have:

$$\|\hat{f}_k\|_2^2 = (T_k)^n \int_{\Omega T_k} \left| \hat{h}_{T_k}(\bar{\omega}) \right|^2 . \left| \sum_{n \in \mathcal{A}} x_{T_k}(n) . e^{in\bar{\omega}} \right|^2 d\bar{\omega}$$

Now, $\left| \hat{h}_{T_k}(\bar{\omega}) \right| = \left| \hat{h}\left(\frac{\bar{\omega}}{T_k}\right) \right| \leq C_1$, for all $\bar{\omega} \in \Omega T_k$,

then: $\|\hat{f}_k\|_2^2 \leq C_1 (T_k)^n \int_{\Omega T_k} \left| \hat{h}_{T_k}(\omega) \right| . \left| \sum_{n \in \mathcal{A}} x_{T_k}(n) . e^{in\omega} \right|^2 d\omega$

We assure that the integral on the right hand term satisfies:

$$\int_{\Omega T_k} \left|\hat{h}_{T_k}(\omega)\right| \cdot \left|\sum_{n\in \mathcal{Q}} x_{T_k}(n).e^{i\omega n}\right|^2 d\omega \leqslant \int_{\Omega T_k} \hat{h}_{T_k}(\omega) \left|\frac{F(f\ (nT_k))(\omega)}{\hat{h}_{T_k}(\omega)}\right|^2 d\omega$$

Just because $\dfrac{F(f(mT_k))(\omega)}{\hat{h}_{T_k}(\omega)}$ satisfies all the requirements stated in properties (13b) and (13c). Therefore, the above inequality holds from (13a). Then, we obtain

$$\left\| \hat{f}_k \right\|_2^2 \leqslant C_1^2\ (T_k)^n \int_{\Omega T_k} \left| \frac{F(f(mT_k))}{\hat{h}_{T_k}(\omega)}\ (\omega) \right|^2 d\omega \tag{A23}$$

Now, since $F(fnT_k))(\omega) = \frac{1}{T_k}.\ \hat{f}(\frac{\omega}{T_k}),\ \omega\in\Omega T_k$, we have

$$\left\| f_k \right\|_2^2 \leqslant C_1^2.\ (T_k)^{-n} \int_{\Omega T_k} \left| \frac{\hat{f}(\frac{\omega}{T_k})}{\hat{h}(\frac{\omega}{T_k})} \right|^2 d\omega\ .$$

Changing variables once more, we obtain:

$$\left\| \hat{f}_k \right\|_2^2 \leqslant C_1^2 \int_{\Omega} \left| \frac{\hat{f}(\omega)}{\hat{h}(\omega)} \right|^2 d\omega\ ,\ \text{where } C_1 = \sup_{\omega\in\Omega} \hat{h}(\omega) \tag{A24}$$

The bound we have obtained in (A24) does not depend on k, therefore, the theorem is proved.

Remarks

1. In the particular case: $\hat{h}(\omega)$ is constant $\omega \in \Omega$, then equation (A24) yields to:

$$\| \hat{f}_k \|_2 \leqslant \| \hat{f} \|_2 ,$$

proving, in that way, our assertion in Section 2.2.

2. It is clear that the bounds and inequalities obtained for $\| \hat{f}_k \|_2$ hold for $\| f_k \|$ because of Parseval's theorem.

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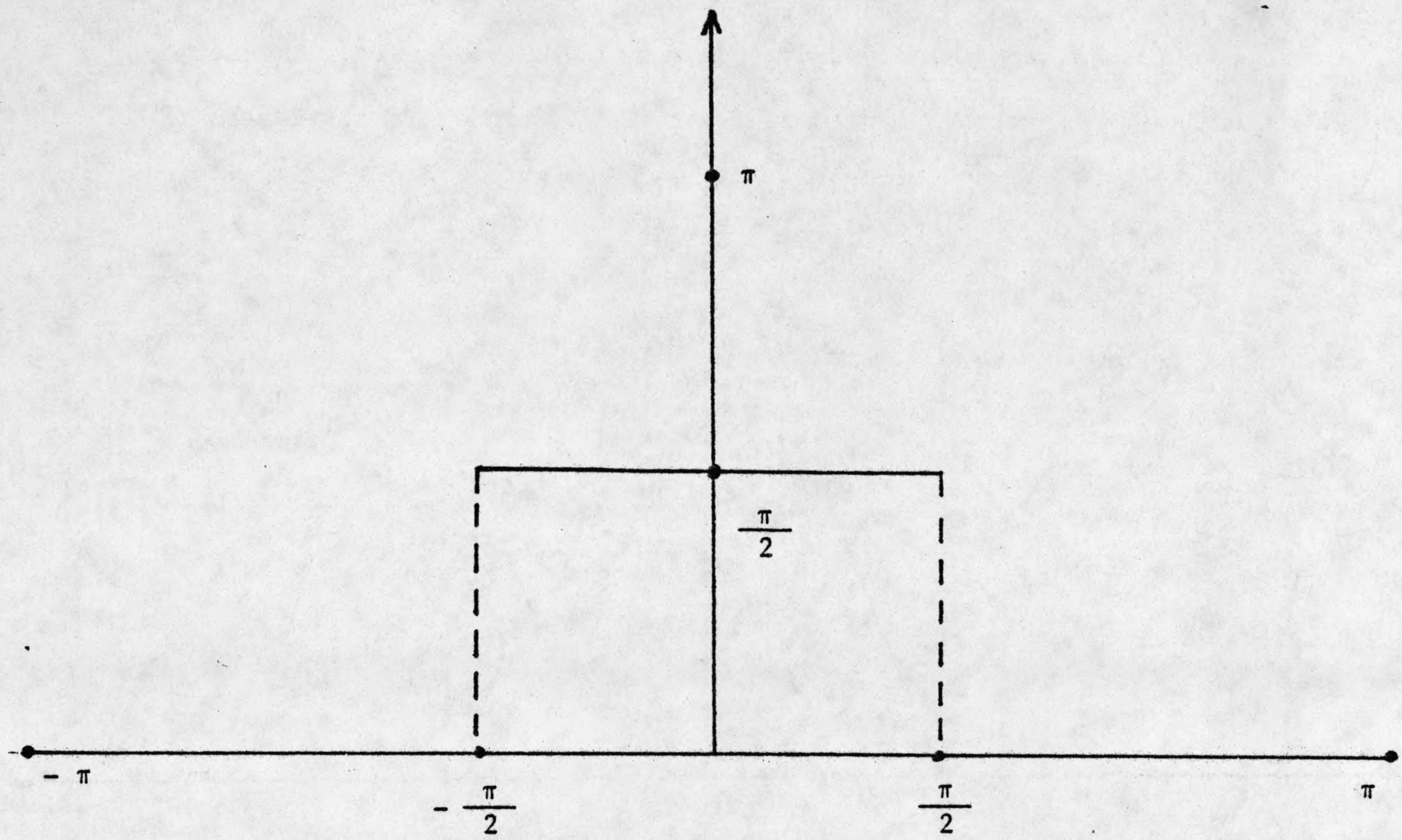

Figure 1: Frequency response of h_1.

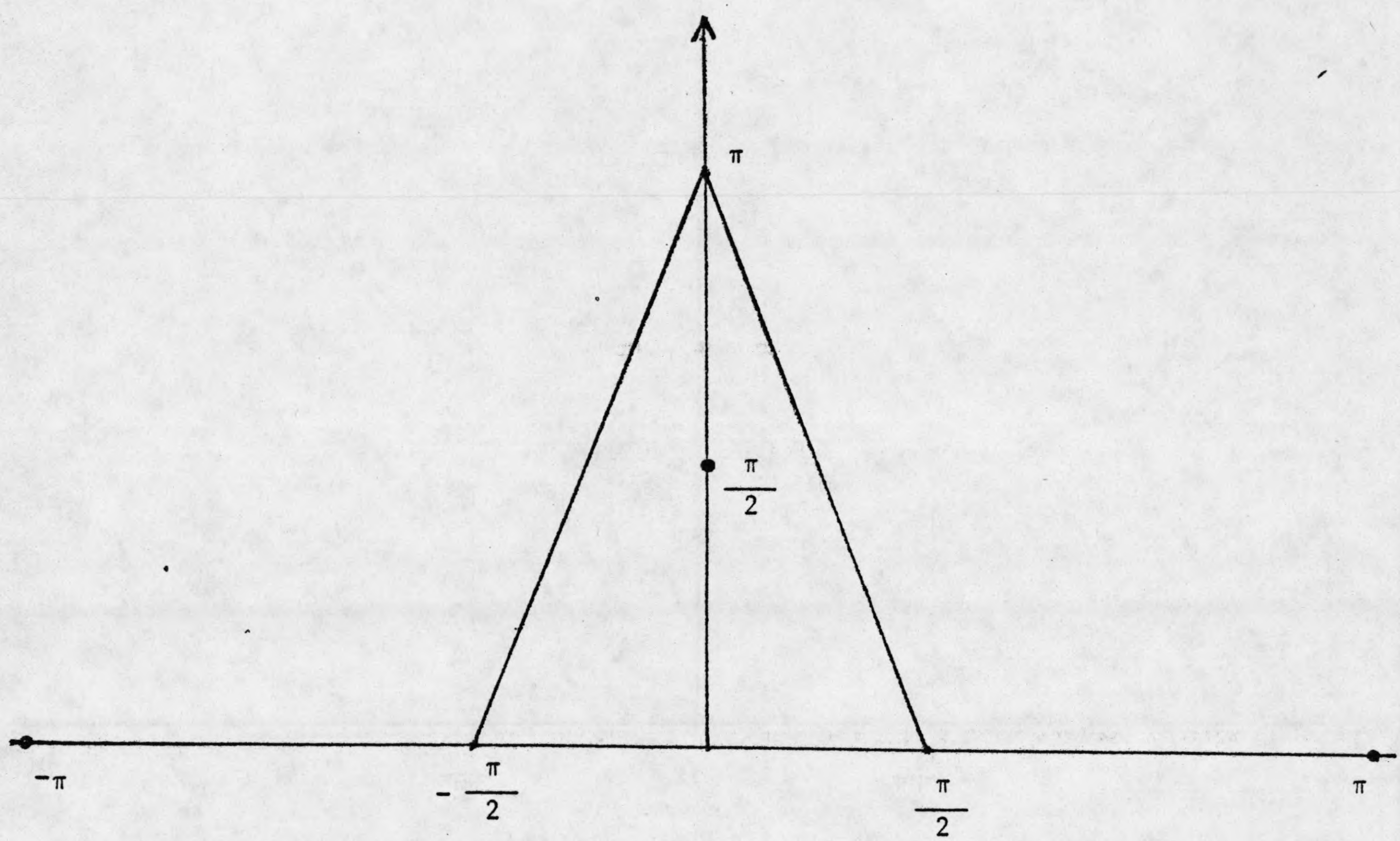

Figure 2: Frequency response of h_2

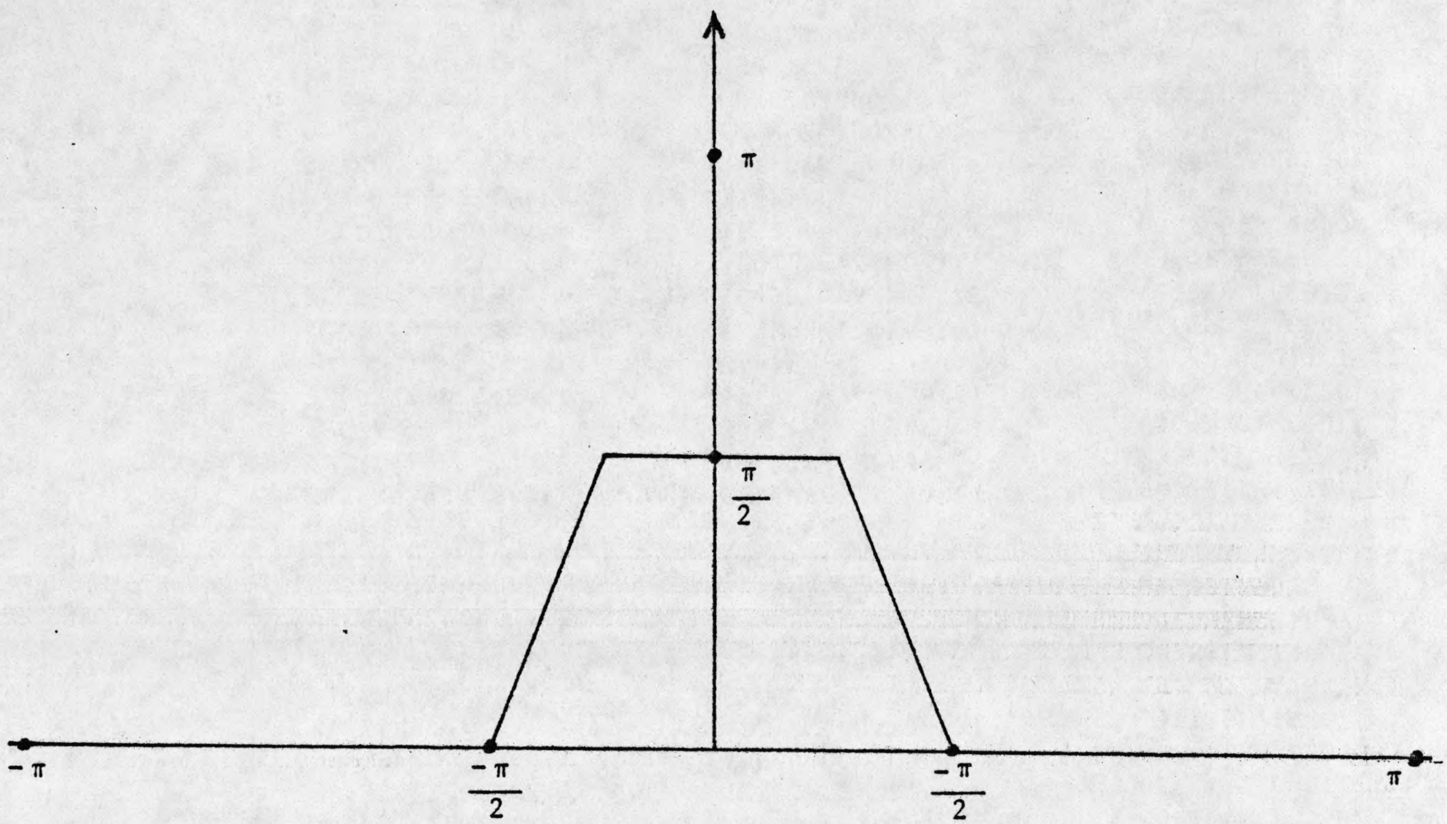

Figure 3: Frequency response of h_3.

-.33333333333333	-.33333333209066	-.33212189843471	
-.34114270622721	-.34114270540351	-.34009071699620	
-.33595619364047	-.33595619311943	-.33507094916422	
-.31675032849620	-.31675032818607	-.31602603162173	
-.28284271247462	-.28284271230575	-.28226641294896	
-.23392430863983	-.23392430856046	-.23347959440434	
-.17008152549560	-.17008152546893	-.16975023504347	
-.09180721036053	-.09180721036139	-.09156945254654	
-.00000000000000	-.00000000001236	.00016575946938	
.10404817174194	.10404817172734	.10415808018297	
.21867624706577	.21867624705373	.21874561719596	
.34188937416591	.34188937415853	.34193074656650	
.47140452079103	.47140452078774	.47142729954794	
.60470517258367	.60470517258377	.60471687419340	
.73910362600903	.73910362601175	.73910855688155	
.87180913813620	.87180913813916	.87181064300239	
1.00000000000000	1.00000000000366	1.00000052014366	
1.12089746331797	1.12089746332030	1.12089846981689	
1.23183937668171	1.23183937668426	1.23183893412352	
1.33035137968407	1.33035137968591	1.33035095315427	
1.41421356237309	1.41421356237472	1.41421401586966	
1.48152062138560	1.48152062138454	1.48152079340070	
1.53073372946036	1.53073372946118	1.53073390293866	
1.56072257612903	1.56072257612941	1.56072259601206	
1.57079632679490	1.57079632679486	1.57079639658332	
1.56072257612903	1.56072257612920	1.56072279624641	
1.53073372946036	1.53073372945978	1.53073380514979	
1.48152062138560	1.48152062138607	1.48152046650648	
1.41421356237309	1.41421356237287	1.41421278472990	
1.33035137968407	1.33035137968395	1.33035170286894	
1.23183937668171	1.23183937668212	1.23184034787118	
1.12089746331797	1.12089746331784	1.12089746817946	
1.00000000000000	1.00000000000009	1.00000002421439	.95068
.87180913813620	.87180913813851	.87180737033486	-2.14846
.73910362600903	.73910362601449	.73909564130008	-2.95774
.60470517258367	.60470517259601	.60468173027039	10.00635
.47140452079103	.47140452081793	.47135117556900	-.98298
.34188937416591	.34188937421544	.34178054705262	-11.44979
.21867624706577	.21867624715765	.21847426146269	1.99738
.10404817174194	.10404817189742	.10369591601193	8.28442
-.00000000000000	.00000000025609	-.00058276951313	7.63455
-.09180721036053	-.09180720995656	-.09272820223123	-8.06340
-.17008152549560	-.17008152487821	-.17148872371763	10.83133
-.23392430863983	-.23392430772389	-.23600788880140	-10.60697
-.28284271247462	-.28284271115028	-.28584258444607	8.85064
-.31675032849620	-.31675032662586	-.32096337247640	-13.40259
-.33595619364047	-.33595619105471	-.34174564853311	16.92903
-.34114270622721	-.34114270272198	-.34894366795197	-10.08683
-.33333333333333	-.33333332866595	-.34365388518199	2.21438
(a) samples h_1	(b) filter h_1	(c) filter h_2	(d) x_o

Table 1

(a) samples h_2	(b) filter h_1	(c) filter h_2	(d) filter h_3
.07073553026307	.07040772070576	.07073561794816	.07028323480517
.09204604960431	.09180706913696	.09204611672723	.09170373336019
.11639584951652	.11622514411965	.11639589996154	.11614054013444
.14371798895554	.14359875821901	.14371802613486	.14353076099906
.17388446887138	.17380323690804	.17388449568331	.17374963148723
.20670577598798	.20665195120273	.20670579484828	.20661061305600
.24193180247901	.24189724211601	.24193181537097	.24186599831114
.27925416178699	.27923275435751	.27925417032433	.27920980288991
.31830988618379	.31829716815264	.31830989165388	.31828083532127
.35868645669528	.35867926360697	.35868646003703	.35866810561129
.39992808164897	.39992424882212	.39992808359076	.39991677003854
.44154310728793	.44154120949497	.44154310834652	.44153665460544
.48301241353161	.48301156009438	.48301241405935	.48300886947254
.52379862090405	.52379828242238	.52379862114262	.52379684300104
.56335591165996	.56335580275618	.56335591174656	.56335524174210
.60114024988492	.60114022256630	.60114024990777	.60113994126732
.63661977236757	.63661977024259	.63661977237595	.63661985326326
.66928511474227	.66928511270976	.66928511475460	.66928518190980
.69865943602928	.69865943404147	.69865943602245	.69865957064030
.72430790934640	.72430790752060	.72430790934865	.72430798008281
.74584645715611	.74584645579853	.74584645716939	.74584649654571
.76294952571004	.76294952484386	.76294952571689	.76294959595543
.77535671496286	.77535671442820	.77535671496878	.77535678984714
.78287810661135	.78287810632355	.78287810661493	.78287819141406
.78539816339745	.78539816313787	.78539816340307	.78539823752362
.78287810661135	.78287810810980	.78287810661800	.78287819470279
.77535671496286	.77535671690748	.77535671496634	.77535685000475
.76294952571004	.76294952703938	.76294952571212	.76294958582730
.74584645715611	.74584645802315	.74584645715186	.74584647003212
.72430790934640	.72430790981525	.72430790935769	.72430789121427
.69865943602928	.69865943630793	.69865943604142	.69865936471615
.66928511474227	.66928511489823	.66928511473969	.66928516683402
.63661977236757	.63661977250740	.63661977236922	.63661980154575
.60114024988492	.60114022133348	.60114024986359	.60114073456498
.56335591165996	.56335578917424	.56335591155684	.56335678824689
.52379862090405	.52379823777665	.52379862061437	.52380041847937
.48301241353161	.48301144953075	.48301241289596	.48301547151641
.44154310728793	.44154097639148	.44154310602372	.44154766041902
.39992808164897	.39992380467993	.39992807935749	.39993480429985
.35868645669528	.35867847989903	.35868645278140	.35869552637450
.31830988618379	.31829586511594	.31830987984236	.31832146469969
.27925416178699	.27923068999189	.27925415195739	.27926802597358
.24193180247901	.24189410152030	.24193178772657	.24194743184489
.20670577598798	.20664733586273	.20670575453764	.20672173707862
.17388446887138	.17379665534827	.17388443953136	.17389837244991
.14371798895554	.14358961873313	.14371794706942	.14372617639310
.11639584951652	.11621275003426	.11639579289628	.11639329248283
.09204604960431	.09179061721829	.09204597452548	.09202539437683
.07073553026307	.07038630328481	.07073543248842	.07068720360985

Table 2

.02706931551123	.02724024788662	.02728325889484	.02707809650113
.04795625045259	.04808069670699	.04811376662349	.04796306318423
.07189743112649	.07198622347425	.07201122652987	.07190262548278
.09882629910364	.09888825957307	.09890682281275	.09883018280817
.12861522453723	.12865740733525	.12867088919211	.12861806695604
.16107504947415	.16110298453009	.16111251925213	.16107708072308
.19595600707915	.19597393674053	.19598048593070	.19595742079806
.23295003701988	.23296113969786	.23296549821316	.23295099057496
.27169448261153	.27170107683287	.27170387679143	.27169510315362
.31177712034439	.31178084849944	.31178254244151	.31177750725629
.35274243805386	.35274442293098	.35274539155216	.35274266819135
.39409904517670	.39410002622344	.39410053651227	.39409917216904
.43532806817473	.43532850755059	.43532874365337	.43532813333354
.47589235714680	.47589252927855	.47589262390102	.47589238688397
.51524630666027	.51524636078511	.51524638223054	.51524631728239
.55284607557846	.55284608727874	.55284609177033	.55284607917758
.58815997768240	.58815997756463	.58815998073260	.58815997722718
.62067880758574	.62067880756376	.62067880542600	.62067880764448
.64992586507037	.64992586508015	.64992585740401	.64992586522635
.67546644561665	.67546644572781	.67546645560651	.67546644655317
.69691657549290	.69691657563999	.69691658983356	.69691657633894
.71395078606598	.71395078619935	.71395078871865	.71395078622822
.72630874360551	.72630874398283	.72630874384777	.72630874384572
.73380057723727	.73380057743725	.73380057688337	.73380057744998
.73631077818511	.73631077833670	.73631078013568	.73631077833352
.73380057723727	.73380057766281	.73380057871691	.73380057722443
.72630874360551	.72630874380320	.72630874358583	.72630874366723
.71395078606598	.71395078611295	.71395078796195	.71395078649857
.69691657549290	.69691657549720	.69691657955991	.69691657573912
.67546644561665	.67546644558297	.67546645959374	.67546644542790
.64992586507037	.64992586503104	.64992586761946	.64992586471226
.62067880758574	.62067880756058	.62067879870301	.62067880631048
.58815997768240	.58815997763050	.58815997920465	.58815997730153
.55284607557846	.55284608753118	.55284605076304	.55284607319345
.51524630666027	.51524636063004	.51524623663863	.51524630332415
.47589235714680	.47589252798116	.47589221643284	.47589234811176
.43532806817473	.43532850357678	.43532784743002	.43532804765255
.39409904517670	.39410001672059	.39409874897683	.39409900314513
.35274243805386	.35274440389458	.35274212755030	.35274235400675
.31177712034439	.31178081358939	.31177694993676	.31177696779105
.27169448261153	.27170101751472	.27169476405834	.27169421974077
.23295003701988	.23296104422752	.23295131453779	.22294960591011
.19595600707915	.19597378989101	.19595913341618	.19595532824565
.16107504947415	.16110276700965	.16108138271375	.16107402072680
.12861522453723	.12865709531980	.12862672927440	.12861371393728
.09882629910364	.09888782433086	.09884571211296	.09882414218987
.07189743112649	.07198563124393	.07192846367252	.07189442586309
.04795625045259	.04807990855079	.04800384695409	.04795215837589
.02706931551123	.02723921979354	.02713990503980	.02706385478427
(a) samples h_3	(b) filter h_1	(c) filter h_2	(c) filter h_3

Table 3

-.29279990487287	
-.29441971273457	
-.28381571790791	
-.26011280408958	
-.22275094012770	
-.17151341580962	
-.10654578320493	
-.02836473105037	
.06214358290023	
.16373590021311	
.27483476394020	
.39356146344383	
.51777821793923	
.64513850363114	
.77314416915926	
-.89920776342964	
1.02071832608082	
1.13510876964443	
1.23992291808134	
1.33288026117779	
1.41193653875738	
1.47533838138218	
1.52167040188691	
1.54989334965484	
1.55937220033535	
1.54989334965485	
1.52167040188689	
1.47533838138218	
1.41193653875738	
1.33288026117779	
1.23992291808136	
1.13510876964443	
1.02071832608082	-.08713507494
.89920776342964	.111162110041
.77314416915926	.296715143258
.64513850363114	.464075988918
.51777821793923	.6083174559853
.39356146344382	.7251838889843
.27483476394019	.8112216050677
.16373590021310	.863884926879
.06214358290022	.8816144324287
-.02836473105037	.863884926879
-.10654578320493	.8112216050677
-.17151341580962	.7251838889843
-.22275094012770	.6083174559853
-.26011280408958	.4640759889181
-.28381571790791	.296715143258
-.29441971273457	.111162110041
-.29279990487287	-.087135074942

(a) filter h_1 after 50 iterations $\lambda = 1/3.14$

solution of the system

Table 4

SOME ASPECTS OF BAND-LIMITED SIGNAL EXTRAPOLATION: MODELS, DISCRETE APPROXIMATIONS, AND NOISE

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ABSTRACT

We present some theoretical results on the band-limited signal extrapolation problem. In section I we describe four basic models for the extrapolation problem. These models are useful in understanding the relationship between the continuous extrapolation problem and some discrete algorithms given in refs. [1] and [2]. One of these models was shown to approximate the continuous band-limited extrapolation problem ([3]). Another model is obtained when DFT is used to implement the well-known iterative algorithm given in refs. [4] and [5] which was designed for solving the continuous extrapolation problem; in section II this model is related to the continuous model. We also present some new techniques for solving the continuous extrapolation problem when the known part of the signal to be extrapolated is contaminated with noise $\eta(x)$: $\max_x |\eta(x)| \le \varepsilon$. In section III we show that the extrapolation problem is very sensitive to noise even in cases where only small amounts of extrapolation are desired. This result indicates that in the presence of noise extrapolation techniques should be used judiciously in order to obtain reasonable results.

I. Introduction

Let us begin by recalling what is meant by continuous band-limited extrapolation. Let f be a finite energy function defined over the real line: $(f \in L^2)$

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, dt < \infty \tag{1}$$

Let us suppose that its Fourier transform f is zero outside $[-\Omega, \Omega]$: $\hat{f}(\omega) = 0$, $\omega \notin [-\Omega, \Omega]$. Since f is analytic, given a piece of f, say $g : [-T, T] \to C$: $g(t) = f(t)$, $t \in [-T, T]$, we can expect to recover $f(t)$, when $t \notin [-T, T]$ from g. Hence, the continuous extrapolation problem is (under the conditions stated above):

Find $f(t)$, $t \notin [-T, T]$

given $f(t)$, $t \in [-T, T]$.

(see fig. 1(a)). This model will be referred to as the continuous-continuous model. The finite-energy property of f ensures that $\hat{f}$ is a function. Moreover, it can be shown that (Parseval's identity)

$$\int_{-\infty}^{+\infty} |f|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}|^2 \, d\omega \tag{2}$$

The band-limited property assumed for f gives us the following inversion formula:

$$f(t) = \frac{1}{2\pi} \int_{\Omega} \hat{f}(\omega) \, e^{-i\omega t} \, d\omega \tag{3}$$

Nevertheless, $\hat{f}(\omega)$ may not be written as $\int_{-\infty}^{+\infty} f(t) \, e^{i\omega t} \, dt$, since $f(t)$ is not integrable on the real line.

One possible way of getting an extension of the Fourier transform for non-integrable functions is by means of the distribution theory [6]. (The Fourier transform $\hat{f}$ for $f \in L^2$ can be understood in the distributional sense). Probably, the most important mathematical consequence to signal analysis of such an extension is to get a rigorous definition of the Fourier transform for continuous periodic signals.

Let f be a P-periodic signal defined over the real line:

$$f(t) = f(t + P) \qquad \text{for all } t, \tag{4}$$

and let us suppose that f is Ω-band-limited. This means that its distributional Fourier transform $\hat{f}$ is supported in $[-\Omega, \Omega]$. In that case it can be proved that $\hat{f}$ is a linear combination of impulses. The weights in that linear combination are related to the DFT of samples of f. Now, we have another model for the band-limited extrapolation problem, which we call <u>continuous-discrete</u>: given a piece $f(t)$, $t \in [-T,T]$, of a Ω-band-limited P-periodic signal $f(t)$, we want to determine $f(t)$, $t \notin [-T,T]$ (see fig. 1(b)). The solution for this extrapolation model is unique since f is analytic. Moreover, f is a linear combination of exponential functions:

$$f(t) = \sum_{n=-k_o}^{k_o} c_n e^{-2\pi i n t/P} \tag{5}$$

where $\quad k_o = [\frac{\Omega P}{2\pi}] \quad$ (see [7])

The two models given in fig. 1(a) and 1(b) are both concerned with the continuous extrapolation problem; this means that the signals to be extrapolated are continuous in time. Nevertheless, in the first case the Fourier transform is also continuous in time, while in the second case the

Fourier transform is a finite array of impulses. There is a major difference in solving the extrapolation problems for the two models: the P-periodic Ω-band-limited function is completely determined by $2k_o + 1$ samples in $[-T, T]$. This assertion is a consequence of (5)([1]). Of course, this is not the case for model (1a).

An attempt to solve the extrapolation problem given in model (1a) is the following well-known iterative procedure [4], [5]:

$$f_o = 0$$

$$f_{n+1} = \mathrm{sinc}_{\Omega} * (g + (I - J_{[-T,T]})\, f_n) \tag{6}$$

where I is the identity operator and $J_{[-T,T]}$ is the truncation to $[-T,T]$. One possible way of getting a numerical implementation of the recursive formula (6) is based on the following identity

$$f_{n+1} = f_n + \mathrm{sinc}_{\Omega} * (g - J_{[-T,T]}\, f_n) \ , \ n > 0 \tag{6'}$$

In addition, it is necessary to sample the functions and the convolution in (6'). After doing that we get the following discrete recursion:

$$y_{n+1}(j) = y_n(j) + \Delta \sum_{\Delta m \in [-T,T]} \mathrm{sinc}_{\Omega}(\Delta(j-m))\,(g(m\Delta) - y_n(m)) \tag{7}$$

$$y_o(j) = 0$$

where Δ denotes the distance between two consecutive samples and j is any integer number. Now, it is clear that $y_{n+1}(j)$ may not be $f_{n+1}(j\Delta)$. It is well-known that $f_n \to f$ uniformly on the real line. On the other hand, it was shown in [2] that the sequence y_n tends to the minimum norm solution y of the following discrete-continuous extrapolation problem of $g(m\Delta) = z(m)$,

$-[\frac{T}{\Delta}] \leq m \leq [\frac{T}{\Delta}]$.

Given $z(m)$, $m \in A$ (A is a finite set of integer numbers) find a finite-energy sequence $y(m)$, $m \in Z$:

(i) y is band-limited to $(-\Omega, \Omega)$.

(ii) $y(m) = z(m)$, $m \in A$.

Under condition (i), the band-limited property should be understood in terms of the Fourier series of y, that is to say

$$F(y)(\omega) = \sum_{m=-\infty}^{+\infty} y(m) \cdot e^{im\omega} = 0 \qquad \text{when} \quad |\omega| > \Omega$$

This is the third model sketched in Fig. 1(c). This model appears as a consequence of the numerical implementation of the recursion (6'). However, it is also clear that the extrapolation problem which the recursive formula (7) approaches is completely different from the original problem (model 1(a)). In ref. [3] model 1(c) was fully connected with model 1(a).

Another technique of implementing the recursive relationship (6) is by means of the discrete Fourier transform (DFT). Equation (6) can be written in the following equivalent form:

$$f_{n+1} = \check{s}_{n+1} \tag{8a}$$

$$s_{n+1} = J_{[-\Omega,\Omega]} \cdot (g + (I-J_{[-T,T]})f_n)^{\wedge} \tag{8b}$$

where $J_{[-\Omega,\Omega]}$ is the truncation function to $[-\Omega,\Omega]$, $\vee$ and $\wedge$ denote inverse and direct continuous Fourier transform respectively. Replacing $\vee$ and $\wedge$ by their corresponding DFT and implementing the frequency cut-off operator in terms of samples of the frequency space, we get the following recursion

$$\ell_{n+1}(j) = \frac{1}{M}\sum_{m=-N}^{N} \alpha(m)\, e^{2\pi imj/M} \qquad j \in [-N, N] \tag{9a}$$

$$\alpha(m) = \begin{cases} 0 & N \geq |m| > k_o = [\frac{\Omega M \Delta}{2\pi}] \\ \sum_{j=-N}^{N} \beta(j)\, e^{-2\pi ijm/M} & |m| \leq k_o \end{cases} \tag{9b}$$

$$\beta(j) = \begin{cases} g(j\Delta), & |j| \leq [\frac{T}{\Delta}] \\ \ell_n(j), & N \geq |j| \geq [\frac{T}{\Delta}] \end{cases} \tag{9c}$$

In formulas (9b) and (9c) Δ denotes the distance between consecutive samples which is used to extract values of g inside [-T,T], and to approach the continuous Fourier transform; $M = 2N + 1$ is the length of the DFT. There is no a priori evidence that justifies the convergence of the procedure given by (9) because $\ell_n(j)$ is not $f_n(j\Delta)$. In ref. [1] it was proven that if $k_o = [\frac{T}{\Delta}] = L$ then ℓ_{n+1} converges to the solution of the following discrete-discrete extrapolation problem

Given the sequence $x(n)$, $-L \leq n \leq L$

Find a sequence $h(n)$, $-N \leq n \leq N$ such that

(i) $h(n)$ is band-limited to $[-k_o, k_o]$ (10a)

(ii) $h(n) = x(n)$, $n \in [-L, L]$ (10b)

Condition (i) should be understood in terms of the DFT,

$$\sum_{n=-N}^{N} h(n)\, e^{-2\pi inm/M} = 0 \qquad N \geq |m| \geq L, \quad M = 2N + 1 .$$

Fig. 1(d) sketches this new model for the extrapolation problem.

In ref. [1] it was assumed that the number of non-zero frequencies is the same as the number of given samples: $x(n)$, $n \in [-L,L]$. Nevertheless, as well as it was done for the discrete-continuous model (c), it should not be difficult to prove that when $k_o > [\frac{T}{\Delta}]$ the same algorithm (9) converges to the minimum norm sequence $h(n)$ which satisfies (10a), (10b). The case $k_o < [\frac{T}{\Delta}] = L$ may not yield any solution.

It is clear that the discrete-discrete model is completely different from the continuous-continuous model. Nevertheless, it naturally appears from another numerical implementation of equation (6). Unlike the discrete-continuous case, this model has not been connected to the continuous-continuous case. In other words, no one has shown how the extrapolation obtained by means of the discrete-discrete model is related to the original continuous-continuous problem. It is worth pointing out that the recursive equation (9) is just a method of getting the extrapolated sequence given by equations (10a) and (10b). Hence the solution provided by the algorithm (9) was not related to the continuous solution. Some relationships between models 1(a) and 1(d) will be given in Section II. Nevertheless, we did not get the same class of relationships we had obtained for the connection between models 1(a) and 1(c) ([3]). To end this section we would like to point out that the continuous-discrete 1(b) and discrete-discrete 1(d) models are closely related. This is because a continuous P-periodic Ω-band-limited function is completely determined by the samples $f(n\Delta)$ where Δ satisfies:

$$\frac{\pi}{\Delta} > \Omega \quad \text{and} \quad \frac{P}{\Delta} = 2N + 1$$

Then, if Δ is small enough so that

$$[\frac{T}{\Delta}] \geq [\frac{P}{2\pi}]$$

then the discrete-discrete model will apply and the solution given by this model will coincide with the samples of the P-periodic continuous extrapolation.

II. MAIN RESULTS

This section is divided into two parts. The first part (II.a) is devoted to the ϵ-tolerance band-limited extrapolation problem. The second part (II.b) is concerned with further relationships between the discrete-discrete model and the continuous-continuous model for the band-limited extrapolation problem. All proofs are relegated to the Appendix.

II.a

In this Section we address the ϵ-tolerance band-limited extrapolation problem. We present some theoretical results and their application to the extrapolation of noisy signals. These results show that the discrete-discrete model and the discrete-continuous model can be applied to obtain approximations for a continuous-continuous extrapolation problem in the presence of noise. Let g be a piece of an Ω-band-limited signal, $g : [-T, T] \to C$, and ϵ be any positive number. Then, the ϵ-tolerance band-limited extrapolation problem can be stated in the following way:

Find $h : R \to C$:

$$h \text{ is band-limited to } [-\Omega, \Omega] \tag{11a}$$

$$|h(t) - g(t)| \leq \epsilon \quad \text{for } t \in [-T, T]. \tag{11b}$$

It is clear that conditions (11a) and (11b) do not define h uniquely (see Section III). Theorem 1 and Theorem 2 present two techniques for getting a solution of this problem. The techniques are based on the approximation of the solution by means of the discrete-discrete model (adapted for the ϵ-tolerance case). On the other hand, Theorem 3 presents a different approach based on the discrete-continuous model (adapted for the ϵ-tolerance case).

In the latter case, we obtain a solution of (11a) and (11b) which is characterized as the minimum energy band-limited function among those which satisfy (11a) and (11b)

Theorem 1

Given $\epsilon > 0$ and $g : [-T, T] \to C$ a piece of a Ω-band-limited function f which is assumed to be absolutely summable, i.e., $\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty$. Let M be any positive integer and

$$\Delta = \sqrt{\frac{2\pi T}{\Omega M}}$$

Let us define the following sequence of functions:

$$\varphi_\Delta(x) = \frac{1}{M} \sum_{k=-k_o}^{k_o} z_\Delta(k) \cdot e^{2\pi i k x / M\Delta}, \quad k_o = \left[\frac{M\Delta\Omega}{2\pi}\right] \tag{12}$$

where $z_\Delta(k)$, $-k_o \leq k \leq k_o$ satisfies

$$\left| \frac{1}{M} \sum_{k=-k_o}^{k_o} z_\Delta(k)\, e^{2\pi i k n / M} - g(n\Delta) \right| \leq \epsilon, \quad n \in [-k_o, k_o] \tag{13a}$$

z_Δ minimizes $\sum_{k=-k_o}^{k_o} |z(k)|$ among those which satisfy condition (13a) (13b)

Then, there exists a subsequence of real numbers $\Delta_n \to 0$ and a Ω-band-limited function h such that:

$$\max_{x \in [-T,T]} |h(x) - g(x)| \leq \epsilon, \text{ and}$$

φ_{Δ_n} approaches h uniformly over compact sets in the real line, when $\Delta_n \to 0$.
($\forall K$ compact set, $\forall \delta \geq 0, \forall n > n_o(\delta)$: $\max_{x \in K} |\varphi_{\Delta_n}(x) - g(x)| \leq \delta$)

Observe that the definition of Δ ensures that $[\frac{T}{\Delta}] = k_o$.

It is clear that the sequence of functions φ_Δ constitutes a family of $M\Delta$-periodic Ω-band-limited functions. The condition (13a) merely says that φ_Δ satisfies $|\varphi_\Delta(n\Delta) - g(n\Delta)| \leq \epsilon$ when $n\Delta \in [-T, T]$. In addition, condition (13b) is a technical requirement to ensure that φ_Δ can be bounded on compact sets in the complex plane (a detailed proof is included in the Appendix).

It is also clear that conditions (13a) and (13b) can be put, for $M = 2N+1$, in this equivalent way:

We seek for a M-periodic sequence $y(n)$, $n \in (-\infty, +\infty)$ such that:

(a) y is band-limited to $[-k_o, k_o]$.

(b) $|y(n) - g(n\Delta)| \leq \epsilon$ for $n \in [-k_o, k_o]$.

(c) If z denotes the M-point DFT of y, then z minimizes

$\sum_{k=-N}^{N} |z(k)|$, where y satisfies (a) and (b).

Conditions (a) and (b) are the discrete-discrete counterpart of the problem which Theorem 1 solves. Condition (c) is an additional constraint ((a) and (b) do not define $y(n)$ uniquely) that makes the connection between both problems possible. There is another (c)-type constraint which also leads to a solution of the continuous extrapolation problem with ϵ-tolerance. That is the purpose of the next theorem.

Theorem 2

Under the same conditions and assumptions as in Theorem 1, if we replace condition (13b) by the following one:

$$z_\Delta \text{ minimizes } \sum_{k=-k_o}^{k_o} |z(k)|^2 \text{ among those which satisfy condition (13a)} \tag{13b'}$$

then, the same conclusion as in theorem 1 holds: There exists a subsequence $\Delta_m \to 0$ such that $\varphi_{\Delta_m} \to h'$ uniformly over compact sets and h' is a solution of (11a) and (11b).

We have already pointed out that the continuous-continuous extrapolation problem with ϵ-tolerance has no unique solution. Hence the extrapolated signal h given in Theorem 1 and its counterpart h' given in Theorem 2 may not coincide. The characterization of these solutions has not yet been found. However, it is our conjecture that the solution provided by Theorem 2 is the minimum noisy signal among those which satisfy conditions (11a) and (11b).

Another possible approach is obtained by means of the continuous-discrete model. In this case, the solution h" can be characterized as the minimum-energy Ω-band-limited function which satisfies

$$\max_{x \in [-T,T]} | g(x) - h''(x) | \leq \epsilon.$$

Nevertheless, its computation may be more involved.

Theorem 3

Let ϵ and Δ be any two positive numbers. Let $g : [-T, T] \to C$ a piece of a Ω-band-limited finite energy function f. For each $\Delta < 2T$ we define the function f_Δ as follows: $(L = [\frac{T}{\Delta}])$

$$f_\Delta(x) = \sum_{k\in[-L,L]} x_\Delta(k)\cdot \mathrm{sinc}_\Omega(x - k\Delta) \quad , \quad \text{where}$$

$x_\Delta(k)$ is the solution of the following optimization problem:

$$\text{minimize} \sum_{-L\leq k\leq L} x(k)\ z(k) \tag{14a}$$

subject to:

$$|z(k) - g(k\Delta)| \leq \epsilon \ , \quad -L \leq k \leq L \tag{14b}$$

$$\sum_{-L\leq k\leq L} x(k)\ \mathrm{sinc}_\Omega[\Delta(i-k)] = z(i) \ , \quad -L \leq i \leq L \tag{14c}$$

Then, there exists a sequence $\Delta_m \to 0$ such that $f_{\Delta_m} \to h''$ uniformly over compact sets in the real line. Here, h" denotes the minimum-energy function which is band-limited to $[-\Omega,\Omega]$ and satisfies

$$\max_{x\in[-T,T]} |g(x) - h''(x)| \leq \epsilon$$

Moreover, it can be shown that f_{Δ_m} approaches h" uniformly on the whole real line.

It is worth pointing out that the function f_Δ constructed in the theorem is just the minimum-energy Ω-band-limited function that satisfies

$$\max_{k\in[-L,L]} |g(k\Delta) - f_\Delta(k\Delta)| \leq \epsilon \tag{15}$$

It is also interesting to observe that Theorem 3 requires f to be of finite-energy only. On the other hand, Theorems 1 and 2 require absolute summability of f. This last property ensures that f is smooth (at least a continuous function). Hence, simple cases such as $f(x) = \text{sinc}_{\Omega}(x)$ are not included in the first two theorems given above.

The techniques presented here are potentially useful for getting solutions of the extrapolation problem in the presence of noise $\eta(x)$ in the observed portion of the signal. Let us assume that $|\eta(x)| \leq \delta$ and that we are given a noisy observation $\tilde{g}(x) = g(x) + \eta(x)$, when $x \in [-T, T]$. Theorems 1, 2 and 3 hold when $g(x)$ is replaced by $\tilde{g}$. This means that theorem 1 and theorem 2 give us two Ω-band-limited functions h and h' such that: $(\epsilon = 2\delta)$

$$|h(x) - \tilde{g}(x)| \leq \epsilon \quad \text{and} \quad |h'(x) - \tilde{g}(x)| \leq \epsilon$$

when $x \in [-T, T]$. Theorem 3 also constructs a Ω-band-limited function h" such that $|h''(x) - \tilde{g}(x)| \leq \delta$ with the minimum-energy property. The potential difference between techniques as presented in Theorems 1, 2 and 3 may lie in the numerical methods used for solving the optimization problems. Further research is being done on this problem.

II.b

In this section more theoretical relationships between the discrete-discrete model and the continuous-continuous model for the band-limited extrapolation problem are presented. The main goal is to try to construct an approximation of the continuous-continuous problem by means of the discrete-discrete model. The main result is given by the following theorem.

Theorem 4

Let $g : [-T, T] \to C$ be a piece of a Ω-band-limited function f which is supposed to be absolutely summable. There exist two sequences of positive real numbers $\epsilon_n \to 0$ and $\Delta_n \to 0$,

$$\frac{2\pi T}{\Omega\Delta_n^2} = M_n \quad \text{an integer number}$$

and a sequence of functions

$$\Psi_{\Delta_n,\epsilon_n}(x) = \frac{1}{M_n} \sum_{k=-k_o}^{k_o} z_{\epsilon_n}(k)\; e^{-\frac{2\pi i k x}{M_n \Delta_n}}$$

where $z_{\epsilon_n}(k)$, $-k_o \leq k \leq k_o$,

$$k_o = \left[\frac{\Omega M_n \Delta_n}{2\pi}\right],$$

is the solution of the following optimization problem:

$$\left| \frac{1}{M_n} \cdot \sum_{k=-k_o}^{k_o} z_{\Delta_n}(k) \cdot e^{-2\pi i k m/M_n} - g(m\Delta_n) \right| \leq \epsilon_n, \quad m \in [-k_o, k_o] \tag{16a}$$

$$\text{minimize} \sum_{k=-k_o}^{k_o} |z_{\epsilon_n}(k)| \tag{16b}$$

such that $\Psi_{\Delta_n,\epsilon_n} \xrightarrow[\substack{\Delta_n \to 0 \\ \epsilon_n \to 0}]{} f$ uniformly over compact sets in the real line.

This theorem asserts that it is possible to get an approximation of f by means of the discrete-discrete model. Nevertheless, conditions (16a) and (16b) are not equivalent to matching the given samples $g(m\Delta_n)$ via DFT of a sequence $z(k)$ which is zero if $k \in [-k_o, k_o]$. That is what the discrete-discrete version of Papoulis' algorithm carries out, as was pointed out in Section I. This problem is as far as we know, an open problem. We would like to point out that this problem, far from being a merely theoretically interesting question, is the only result which might ensure that the DFT implementation technique of Papoulis' algorithm does approach the real extrapolation. Hence, we would like to state the following:

Conjecture

Given $g : [-T, T] \to C$ a piece of a Ω-band-limited function f (which should be, at least, absolutely summable), let Δ be a real positive number such that $2\pi T/\Omega\Delta^2$ is an integer number M. Let

$$\psi_\Delta(x) = \frac{1}{M} \sum_{k=-k_o}^{k_o} z(k) \cdot e^{-2\pi ikx/M\Delta}$$

be a family of functions where $z(k)$ is defined by the condition ($k_o = [\frac{\Omega\Delta M}{2\pi}]$)

$$\psi_\Delta(n\Delta) = g(n\Delta) = \frac{1}{M} \cdot \sum_{k=-k_o}^{k_o} z(k) \cdot e^{-2\pi ikn/M},$$

$$-k_o \leq n \leq k_o$$

Then, there exists a subsequence $\Delta_m \to 0$ such that $\psi_{\Delta_m} \xrightarrow{\Delta_m \to 0} f$ uniformly over compact sets in the real line.

III. NOISE SENSITIVITY OF THE BAND-LIMITED SIGNAL EXTRAPOLATION PROBLEM

It is well-known that the band-limited signal extrapolation problem is a very ill-conditioned problem since a small amount of noise implies that there may be no solution. A related problem is how to decide if some approximation of the extrapolated signal is sufficiently close to the real extrapolation. Since we are only given a piece g (eventually noisy) of the signal to be extrapolated, all the criteria should lie on comparisons between the approximations and g. It is clear that it should not be expected that closeness to g implies closeness to f over the whole real line. Nevertheless it might be reasonable to believe that if only a small amount of extrapolated information was desired, (i.e.: up to an interval $(-T', T')$ where $T < T' < \infty$) then there would be a constant C (which does not depend on the signal) such that

$$\|f - \Psi\|_{\infty}^{[-T',T']} \leq C.\epsilon$$

when

$$\|g - \Psi\|_{\infty}^{[-T,T]} \leq \epsilon$$

where Ψ is the approximation to the extrapolation and ϵ is any positive number.

Unfortunately, this is not the case in general. The function Ψ may be a Ω-band-limited function which is close to f inside $[-T,T]$ and might differ from f in $[-T',T']$ as much as we want. The next theorem states this result more explicitly.

Theorem 5

Given any positive numbers ϵ and Θ, and given f and g as above, let P be any point in [-T', T'] such that $P \notin [-T, T]$. Then, there exists a Ω-band-limited function $\Psi_{\epsilon,P}$ that satisfies:

$$\left\| \Psi_{\epsilon,P} - g \right\|_{\infty}^{[-T,T]} \leq \epsilon$$

$$\left| \Psi_{\epsilon,P}(P) - f(P) \right| \geq \Theta$$

Theorem 5 shows that future research on this subject should take into account that some small noise in the observation or small errors in [-T, T] in the approximation of the signal to be extrapolated may mean large errors outside the known range.

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IV. APPENDIX

We will need a lemma for proving theorems 1, 2 and 4. This lemma states a well-known result related to the approximation of the continuous Fourier transform by means of the discrete Fourier transform (DFT). We include our proof here because it follows the same idea as the theorems' proofs.

Lemma 1

Let $f : R \to C$ be a Ω-band-limited function which is absolutely summable. Let M be a positive integer number $M = 2N + 1$ and k_o be $\left[\frac{\Omega M \Delta}{2\pi}\right]$ where Δ is a positive real number given by

$$\frac{2\pi T}{\Omega M} = \Delta^2 .$$

If we put

$$\gamma_\Delta(k) = \sum_{n=-N}^{N} f(n\Delta) \cdot e^{-2\pi i k n/M} \qquad k \in [-N, N]$$

and if we define

$$\phi_\Delta(x) = \frac{1}{M} \cdot \sum_{k=-k_o}^{k_o} \gamma_\Delta(k)\, e^{2\pi i k x/M\Delta}, \quad x \in R$$

we will obtain a sequence $\Delta_m \to 0$ such that

$$\phi_{\Delta_m} \to f \text{ uniformly on compact sets in } R.$$

Proof

First, we prove that ϕ_Δ converges to f in the weak sense, i.e., given any function L such that L is a smooth function that also has compact support:

$$\int_{-\infty}^{\infty} \phi_\Delta(x) \cdot L(x)\, dx \xrightarrow{\Delta \to 0} \int_{-\infty}^{\infty} f(x)\, L(x)\, dx .$$

$$\int_{-\infty}^{\infty} \phi_\Delta(x)\, L(x)\, dx = \sum_{k=-k_o}^{k_o} \frac{1}{M} \cdot \sum_{n=-N}^{N} f(n\Delta)\, e^{2\pi ikn/M} \cdot \hat{L}\left(\frac{2k\pi}{M\Delta}\right) \tag{A1}$$

Now if we call $Q = M\Delta$ (A1) will become

$$\frac{1}{2\pi} \frac{2\pi}{Q} \cdot \Delta \sum_{k=-k_o}^{k_o} \sum_{n=-N}^{N} f(n\Delta)\, e^{i(2\pi k/Q)\cdot(n\Delta)}\, \hat{L}\left(\frac{2\pi}{Q} \cdot k\right) \tag{A2}$$

Since $\Delta \to 0$ implies $Q \to \infty$, formula (A2) becomes

$$\frac{1}{2\pi} \int_{-\Omega}^{\Omega} \int_{-\infty}^{+\infty} f(x)\, e^{ix\omega}\, \hat{L}(\omega)\, dx\, d\omega \tag{A3}$$

But since $\int_{-\infty}^{+\infty} f(x)\, e^{ix\omega} dt = 0$ if $\omega \notin [-\Omega, \Omega]$, then (A3) is identical to

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x)\, e^{ix\omega}\, \hat{L}(\omega)\, dt\, d\omega \tag{A4}$$

(A4) is exactly

$$\int_{-\infty}^{+\infty} f(x)\, L(x)\, dx.$$

Now we prove that there exists an analytic function $\tilde{f} : R \to C$ and a sub-sequence $\Delta_m \to 0$ such that

$$\phi_{\Delta_m} \to \tilde{f} \text{ uniformly on compact sets.}$$

Since ϕ_{Δ_m} converges weakly to f then $f = \tilde{f}$ almost everywhere. But both f and $\tilde{f}$ are smooth functions. This implies $f(x) = \tilde{f}(x)$ for all $x \in R$.

Let z be any complex number. Let us extend the definition of ϕ_Δ to complex arguments z as follows:

$$\phi_\Delta(z) = \frac{1}{M} \cdot \sum_{k=-k_o}^{k_o} \left(\sum_{n=-N}^{N} f(n\Delta)\, e^{-2\pi ikn/M} \right) e^{2\pi ikz/M\Delta} \tag{A5}$$

Now it is easily seen that

$$|\phi_\Delta(z)| \leq \frac{2k_o}{M} \cdot \sum_{n=-N}^{N} |f(n\Delta)| \cdot e^{|\mathrm{Im}z|\Omega} \tag{A6}$$

But $\frac{2k_o}{M} \leq \frac{\Omega\Delta}{\pi}$, therefore equation (A6) becomes

$$|\phi_\Delta(z)| \leq \frac{\Omega}{\pi} \cdot \Delta \sum_{n=-N}^{N} |f(n\Delta)| \cdot e^{|\mathrm{Im}z|\Omega} \tag{A7}$$

When $\Delta \to 0$ the right-hand term in (A7) tends to

$$\int_{-\infty}^{+\infty} |f(x)|\, dx$$

Then there exists a constant C such that

$$|\phi_\Delta(z)| \leq C \cdot e^{|\mathrm{Im}z|\Omega} \tag{A8}$$

ϕ_Δ is a family of analytic functions on the complex plane. In addition, (A8) says that this family is uniformly bounded on compact sets in C. Such a family is called a normal family. Hence, there exists a subsequence $\Delta_m \to 0$ and an analytic function f such that

$$\phi_{\Delta_m} \to f \text{ uniformly on compact sets of the complex plane.} \quad [8]$$

We will work only with the noisy case as it was pointed out at the end of Section II.a. The non-noise case can be shown by exactly the same proofs.

Theorem 1 (proof)

We are given $\tilde{g}(t)$, a noisy piece of $f(t)$, for $t \in [-T, T]$. We know $\tilde{g}(t) = f(t) + \eta(t)$ where η is a continuous function $|\eta(t)| \leq \delta$. We are given $\epsilon \geq 2\delta$ and we know by lemma 1 given above that there exists $m_o(\delta)$ such that

$$|\phi_{\Delta_m}(x) - f(x)| \leq \delta \quad \text{if } x \in [-T, T], \quad m \geq m_o(\delta)$$

In particular, we have

$$|\phi_{\Delta_m}(n\Delta_m) - f(n\Delta_m)| \leq \delta, \quad -\left[\frac{T}{\Delta_m}\right] \leq n \leq \left[\frac{T}{\Delta_m}\right] \quad \text{(A9)}$$

Recall that

$$\phi_{\Delta_m}(n\Delta_m) = \frac{1}{M_m} \sum_{k=-k_o}^{k_o} \gamma_{\Delta_m}(k)\, e^{2\pi ikn/M_m} \quad ; \text{ by (A9)}$$

and since

$$|f(n\Delta_m) - \tilde{g}(n\Delta_m)| \leq \delta$$

$$\left| \frac{1}{M_m} \cdot \sum_{k=-k_o}^{k_o} \gamma_{\Delta_m}(k)\, e^{2\pi ikn/M_m} - \tilde{g}(n\Delta_m) \right| \leq \epsilon, \; n \in [-k_o, k_o] \quad \text{(A10)}$$

(A10) implies that

$$\sum_{k=-k_o}^{k_o} |z_{\Delta_m}(k)| \leq \sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)| \quad \text{(A11)}$$

because of properties (13a) and (13b) which z_{Δ_m} satisfies. Now, if we extend formula (12) for complex arguments z we will get

$$|\varphi_{\Delta_m}(z)| \leq \frac{1}{M_m} \sum_{k=-k_o}^{k_o} |z_{\Delta_m}(k)| \, |e^{2\pi ikz/M_m \Delta_m}| \qquad \text{(A12)}$$

By (A11) we have

$$|\varphi_{\Delta_m}(z)| \leq \frac{1}{M_m}\left(\sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)| \right) e^{|\mathrm{Im}z|\Omega} \qquad \text{(A13)}$$

It was shown in lemma 1 that

$$\frac{1}{M_m} \sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)| \leq C$$

Hence,

$$|\varphi_{\Delta_m}(z)| \leq C_1 \, e^{|\mathrm{Im}z|\Omega} \qquad \text{(A14)}$$

Then, φ_{Δ_m} is a normal family of analytic functions in the complex plane. This means that there exists a subsequence Δ_{m_k} of Δ_m : $\Delta_{m_k} \to 0$ and an analytic function h

$$\varphi_{\Delta_{m_k}} \to h \text{ uniformly on compact sets in the complex plane.}$$

In particular, the convergence is also pointwise. Then

$$|h(z)| \leq C \, e^{|\mathrm{Im}z|\Omega} \qquad \text{(A15)}$$

(A15) shows that $h(x)$, $x \in R$ defines a bounded function of Ω-exponential-type. In that case, the distributional Fourier transform is known to be supported in $[-\Omega, \Omega]$ (see [9]).

It can be easily proven that $|h(x) - \tilde{g}(x)| \leq \epsilon$, for $x \in [-T, T]$ because $\tilde{g}$ is a continuous function and φ_{Δ_m} is a family of equicontinuous functions (see [8]).

Theorem 2 (proof)

The proof of this theorem is very similar to that of theorem 1. We will just point out the differences with respect to the proof of theorem 1. Formula (A11) should be replaced by the following one:

$$\sum_{k=-k_o}^{k_o} |z_{\Delta_m}(k)|^2 \leq \sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)|^2 \qquad \text{(A11')}$$

In this case the family of functions φ_{Δ_m} should be bounded in the following way:

$$|\varphi_{\Delta_m}(z)| = \frac{1}{M} \left| \sum_{k=-k_o}^{k_o} z_{\Delta_m}(k) \cdot e^{2\pi ikz/M_m\Delta_m} \right|$$

$$\leq C \cdot \frac{\Delta_m^{1/2}}{M_m^{1/2}} \left[\sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)|^2 \right]^{1/2} e^{\Omega|\mathrm{Im}z|} \qquad \text{(A13')}$$

Then, since

$$\left[\sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)|^2 \right]^{1/2} \leq M_m^{1/2} \left[\sum_{n=-N_m}^{N_m} |f(n\Delta_m)|^2 \right]^{1/2}$$

(A13') becomes

$$|\varphi_{\Delta_m}(z)| \leq C \cdot \Delta_m^{1/2} \left[\sum_{n=-N_m}^{N_m} |f(n\Delta_m)|^2 \right]^{1/2} e^{|\mathrm{Im}z|\Omega}$$

Since f has finite energy (because it is continuous and absolutely summable) we get

$$|\varphi_{\Delta_m}(z)| \leq C_2\, e^{\Omega|\mathrm{Im} z|} \tag{A14'}$$

and therefore φ_{Δ_m} is a normal family of analytic functions. The theorem is proven.

In order to keep the conceptual relationship among the therems' proofs we would like to prove theorem 4 before theorem 3.

Theorem 4 (proof)

It was proven in lemma 1 that there exists a sequence $\Delta_m \to 0$ such that $\phi_{\Delta_m} \to f$ uniformly on compact sets. Let ϵ_m be the following real number

$$\max_{x \in [-T,T]} |\varphi_{\Delta_m}(x) - f(x)| = \epsilon_m$$

Now, taking into account properties (16a) and (16b) we conclude

$$\sum_{k=-k_o}^{k_o} |z_{\epsilon_n}(k)| \leq \sum_{k=-k_o}^{k_o} |\gamma_{\Delta_m}(k)| \tag{A16}$$

It can be easily shown from (A16) that $\Psi_{\Delta_m,\epsilon_m}$ is a family of normal analytic functions on the complex plane. Hence $\Psi_{\Delta_{m_k},\epsilon_{m_k}} \to K$ where K is an analytic function and the convergence is uniform on compact sets in the complex plane. It remains to prove that $f(x) = K(x)$. In doing that it is sufficient to show that $f(x) = K(x)$ when $x \in [-T, T]$ since both functions are analytic.

Recall that a normal family of analytic functions is equicontinuous ([8]). Then, let x_o be any point in $[-T, T]$, and c be any positive number. There exists $\delta > 0$ such that

$$|x - x_o| < \delta \rightarrow |f(x) - f(x_o)| \leq c, \tag{A17}$$

and

$$|x - x_o| < \delta \rightarrow |\Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x) - \Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x_o)| \leq c.$$

This last property is due to the equicontinuous property of the family $\Psi_{\Delta_{m_k}, \epsilon_{m_k}}$. In addition, there exists m_{k_o} such that

$$m_k \geqslant m_{k_o} \rightarrow |\Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x_o) - K(x_o)| \leq c$$

Then, Δ_{m_k} can be chosen small enough such that $|n\Delta_{m_k} - x_o| < \delta$ for certain n, and $\epsilon_{m_k} < c$. If we put $n\Delta_{m_k} = x_1$, we will get

$$|f(x_o) - K(x_o)| \leq |K(x_o) - \Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x_o)| + |\Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x_o) - \Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x_1)|$$

$$+ |\Psi_{\Delta_{m_k}, \epsilon_{m_k}}(x_1) - f(x_1)| + |f(x_1) - f(x_o)|$$

Hence, $|f(x_o) - K(x_o)| \leq 4c$. Since c is any positive number, $f(x_o) = K(x_o)$.

We will need the following lemma to prove theorem 3. This lemma is a well known result which comes from the general Hilbert space theory and it will not be proven here. For a proof see [10].

<u>Lemma 2</u>

Let H be a Hilbert space and let $x_1, \ldots, x_n$ be n-linearly independent elements of H. Let us consider the following optimization problem:

$$\min \langle x, x \rangle \tag{A19}$$

subject to

$$\langle x, x_i \rangle = \alpha_i, \quad i = 1, \ldots, n \tag{A20}$$

where α_i are arbitrary numbers and $\langle\, ,\, \rangle$ denotes the inner product defined in H. Then, the solution of such optimization problem is given by

$$x = \sum_{i=1}^{n} \beta_i x_i \,, \tag{A21}$$

where β_i, i = 1, ..., n are complex numbers which satisfy the following system of equations:

$$\sum_{i=1}^{n} \beta_i \langle x_i, x_j \rangle = \alpha_j \,, \quad j = 1, \ldots, n \tag{A22}$$

We will use the result stated in lemma 2 in the following particular case:

$$H = L^2(-\Omega, \Omega) = \left\{ \hat{f} : \int_{-\Omega}^{\Omega} |\hat{f}|^2 \, d\omega < \infty \right\} ; \; \langle \hat{f}, \hat{g} \rangle = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}\, \overline{\hat{g}}$$

$$x_k = x_k(\omega) = e^{i\omega k\Delta}, \quad k \in [-L, L] \,, \quad \omega \in (-\Omega, \Omega)$$

α_k is any complex number: $|\alpha_k - g(k\Delta)| \leq \epsilon$.

Now, the optimization problem given by (A19) and (A20) becomes

$$\min \int_{-\Omega}^{\Omega} |\hat{f}|^2 \, d\omega \tag{A21}$$

subject to

$$\frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{-i\omega k\Delta} \, d\omega = \alpha_k \,, \quad k \in [-L, L] \tag{A22}$$

In other words, given α_k, $k \in [-L, L]$ we seek for the minimum-norm band-limited function K such that $K(k\Delta) = \alpha_k$. Equations (A21) and (A22) describe this problem in terms of K (its Fourier transform). On the other hand, the solution of (A21) and (A22) is given by

$$\hat{K}(\omega) = \sum_{k\in[-L,L]} \beta_k e^{i\omega k\Delta} \tag{A23}$$

where β_k satisfies

$$\sum_{k\in[-L,L]} \beta_k \operatorname{sinc}_{\Omega}[(i-k)\Delta] = \alpha_i, \quad i \in [-L, L] \tag{A24}$$

This means that

$$K(x) = \sum_{k\in[-L,L]} \beta_k \operatorname{sinc}(x - k\Delta)$$

and β_k satisfies (A24). Now, theorem 3 readily follows.

<u>Theorem 3</u> (proof)

We consider the following additional optimization problem

$$\min \|\hat{K}\|_2$$

where $\hat{K}$ satisfies (A23), β_k satisfies (A24) and α_k has the following property:

$$|\alpha_k - g(k\Delta)| \leq \epsilon, \quad k \in [-L, L]$$

Since

$$\|\hat{K}\|_2^2 = \sum_{k\in[-L,L]} \alpha_k \beta_k ,$$

we obtain the following equivalent optimization problem:

$$\text{minimize} \sum_{k\in[-L,L]} \beta_k \alpha_k \tag{A25}$$

subject to

$$\left| \alpha_k - g(k\Delta) \right| \le \epsilon \ , \quad k \in [-L, L] \tag{A26}$$

and

$$\sum_{k\in[-L,L]} \beta_k \, \mathrm{sinc}_\Omega[\Delta(i - k)] = \alpha_h \ , \quad h \in [-L, L] \tag{A27}$$

The last three conditions, (A25), (A26) and (A27), show that the optimization problem (14a), (14b) and (14c) does solve the minimum-norm Ω-band-limited function K such that

$$\left| K(k\Delta) - \tilde{g}(k\Delta) \right| \le \epsilon \ , \quad k \in [-L, L] \ .$$

where $\tilde{g}(x) = f(x) + \eta(x)$, $x \in [-T, T]$, η is a continuous function which satisfies $|\eta(x)| \le \epsilon$. Let us call f_Δ this optimal function which given by (A 23)
Since $|f(k\Delta) - \tilde{g}(k\Delta)| \le \epsilon$, it turns out that

$$\| f_\Delta \|_2 \le \| f \|_2 \tag{A28}$$

Now, we follow the same technique given in theorems 1, 2 and 4. This technique was also the same as that of [3]. The family of functions

$$f_\Delta(z) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}_\Delta(\omega) \, e^{i\omega z} \, d\omega \ ,$$

where z is a complex parameter, constitutes a normal family of analytic functions. This readily follows from (A28) and from the Schwartz inequality

$$|f_\Delta(z)| \leq \|\hat{f}\|_2 \cdot \left\{ \int_{-\Omega}^{\Omega} |e^{i\omega z}|^2 \, d\omega \right\}^{1/2} \tag{A29}$$

Then, (A29) becomes

$$|f_\Delta(z)| \leq c \, \|f\|_2 \cdot e^{|\mathrm{Im}\, z|\Omega} \tag{A30}$$

Now, there exists a subsequence $\Delta_m \to 0$ such that f_{Δ_m} approaches an analytic function S uniformly on compact subsets in the complex plane. From (A30) we obtain

$$|S(z)| \leq c \, \|f\|_2 \cdot e^{|\mathrm{Im}\, z|\Omega} \tag{A31}$$

(A31) shows that S is band-limited to $(-\Omega, \Omega)$. This assertion is a consequence of the Payley-Wiener theorem (see [9]). In addition, since f_{Δ_m} is a family of equicontinuous functions it is readily shown that

$$|S(x) - \tilde{g}(x)| \leq \epsilon, \quad x \in [-T, T] . \tag{A32}$$

Now, we will show that f_{Δ_m} converges to S uniformly on the whole real line. First, S also has finite energy. This is a consequence of Fatou's lemma (see [11]) which states

$$\int_{-\infty}^{+\infty} |S(x)|^2 \, dx \leq \underline{\lim} \int_{-\infty}^{+\infty} |f_{\Delta_m}(x)|^2 \, dx \leq \|f\|_2^2 \tag{A33}$$

Since S has finite energy and satisfies (A32), we have $\|f_{\Delta_m}\|_2 \leq \|S\|_2$. Putting (A33) and this last inequality together we will get

$$\|S\|_2 \leq \underline{\lim} \, \|f_{\Delta_m}\|_2 \leq \overline{\lim} \, \|f_{\Delta_m}\|_2 \leq \|S\|_2$$

This shows

$$\lim_{\Delta_m \to 0} \| f_{\Delta_m} \|_2 = \| S \|_2 .$$

But, since f_{Δ_m} converges to S pointwise we also have convergence in the mean,

$$\int_{-\infty}^{\infty} \left| f_{\Delta_m}(x) - S(x) \right|^2 dx \underset{\Delta_m \to 0}{\to} 0 \qquad \text{(see [12])}$$

Now, it is well-known that if a family of Ω-band-limited functions converges in the energy-norm to another Ω-band-limited function $S(x)$, then it converges to $S(x)$ uniformly on the whole real line. Let us suppose that L is any finite-energy Ω-band-limited function such that $|L(x) - \tilde{g}(x)| < \epsilon$, for all $x \in [-T, T]$. Then, it is clear that $\| f_{\Delta_m} \|_2 \leq \| L \|_2$. Therefore, $\| S \|_2 \leq \| L \|_2$. That shows that S has the minimum-energy property among all the Ω-band-limited functions which are ϵ apart from $\tilde{g}$ on $[-T, T]$.

<u>Theorem 5</u> (proof)

It is clear that the problem we need to solve is linear. That is to say, it will be enough to find $\psi_{\epsilon,P}$ such that

$$\| \psi_{\epsilon,P} \|_{\infty}^{[-T,T]} \tag{A34a}$$

and

$$| \psi_{\epsilon,P}(P) | \geq \theta \tag{A34b}$$

Let f be a smooth function (say C^{∞}) which vanishes outside $(-\Omega,\Omega)$. In that case

$$f(z) = \int_{-\Omega}^{\Omega} \hat{f}(\omega)\, e^{i\omega z}\, d\omega$$

gives a Ω-band-limited function which, in addition, satisfies

$$|f(z)| \leq C_N(1 + |z|)^N \cdot e^{|\mathrm{Im}z|\Omega} \tag{A35}$$

where N is any negative integer number and C_N is a constant which depends only on N and f but not on z. (This is a special case of the Payley-Wiener theorem which also assures that the converse property holds,[9]). Property (A35) shows that f(x) goes to zero at infinity, $x \in R$ even when it is multiplied by any polynomial: $(1 + |x|)^{-N}$. Now, it is obvious that we can assume

$$|f(P)| \geq |f(x)| \quad , \quad x \in [-T, T].$$

and therefore we can suppose that

$$|f(P)| > 1 \text{ and } |f(x)| < 1 \ , \quad x \in [-T, T].$$

Now, we put $f_n(z) = \left(\frac{z}{T_1}\right)^n \cdot f(z)$, with $P > T_1 = T + \delta > T$. It is clear that

$$\| f_n \|_{\infty}^{[-T,T]} \to 0 \ , \quad n \to \infty$$

and

$$| f_n(P) | \to \infty \ , \quad n \to \infty$$

Now, we just need to pick up n adequately to satisfy conditions (A34a) and (A34b). In addition, it is seen that $\psi_{\epsilon,P}$ is not only Ω-band-limited but also has the same property (A34) as f(z).

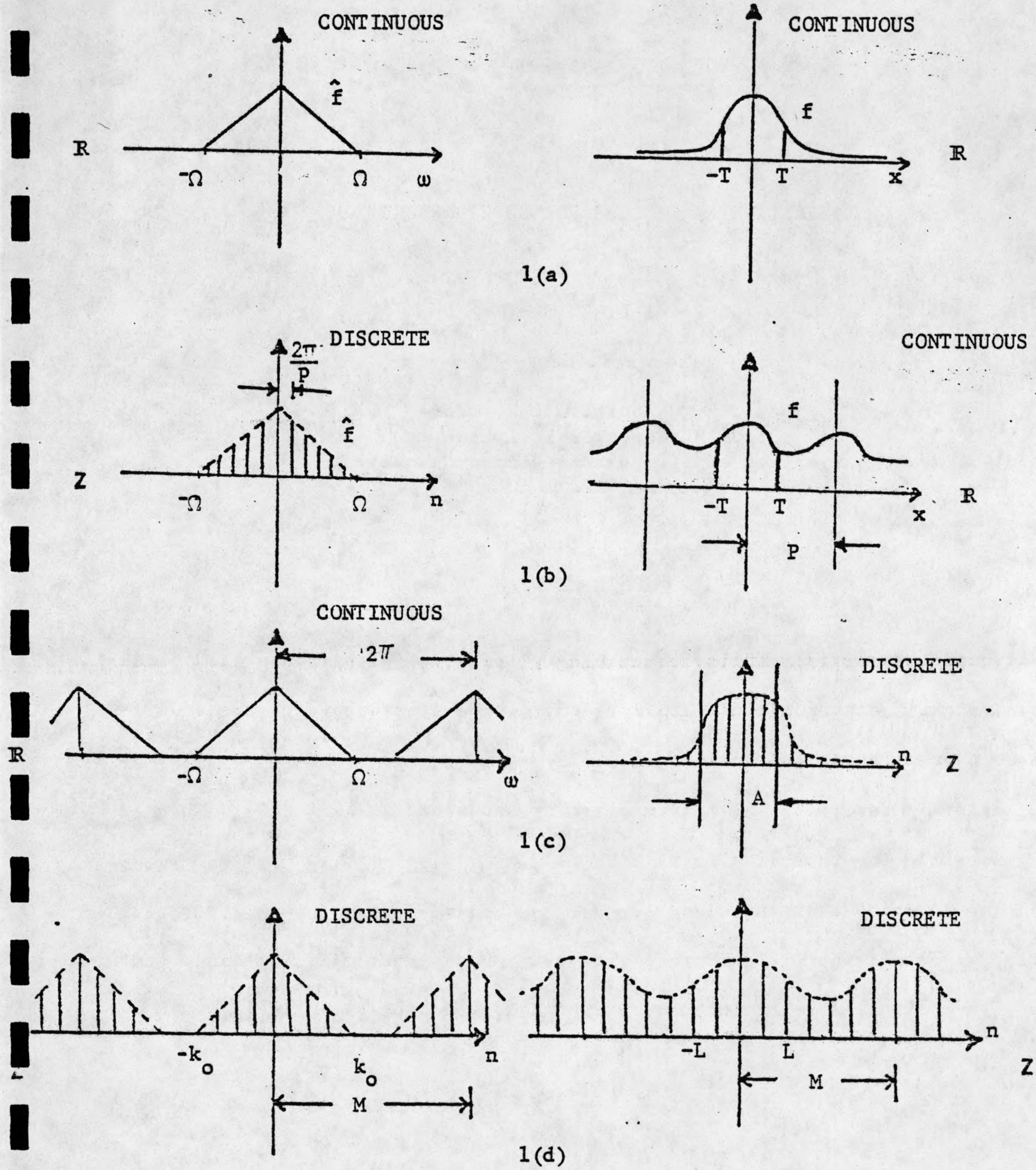

Figure 1 Four basic models for the extrapolation problem.

ON ITERATIVE PROCEDURES FOR SUPER-RESOLUTION

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ABSTRACT

In this paper, several results concerning iterative algorithms for band-limited signal extrapolation are presented. We first prove that the well-known Gerchberg-Papoulis algorithm ([1],[2]): $g_n = \text{sinc}_\Omega * (g + (I - T)g_{n-1})$ is a special case of an algorithm given by Landweber in 1951 ([3]). Then we will generalize the recursive formula $g_n = \text{sinc}_\Omega * (g + (I-T)g_{n-1})$ to cases where the low-pass operator sinc_Ω is replaced by some other low-pass filters. Finally, we show the relationship between this generalization and some discrete techniques for solving the extrapolation problem given in [4].

I. Gerchberg-Papoulis Algorithm and Landweber's Iteration

The band-limited signal extrapolation problem was addressed by several authors and several algorithms were given ([1], [2], [4], [5], [6], [7] and [10] among others). Probably the best known technique for solving the problem is the Gerchberg Papoulis algorithm:

$$g_o = 0$$

$$g_n = \text{sinc}_\Omega * (g + (I - J_A)g_{n-1}) \ , \quad n \geq 0 \tag{1}$$

where $g : (-A,A) \to C$ is a piece of a Ω-band-limited signal, J_A denotes the truncation operator to $(-A,A)$, I denotes the identity, sinc_Ω the function whose Fourier transform is the indicator of $(-\Omega,\Omega)$ and $*$ denotes convolution. It is well known that g_n converges to g in the energy norm,

$$\int |g_n(x) - g(x)|^2 \, dx \to 0$$

$$n \quad \infty \; n \to \infty$$

In particular, g_n will approach g uniformly over the real line because these functions are band-limited to $(-\Omega,\Omega)$. Equation (1) can be written in this equivalent form

$$g_o = 0$$

$$g_n = g_{n-1} + \text{sinc}_\Omega * J_A(g - g_{n-1}), \; n \geq 0 \tag{2}$$

Since $g_n \underset{n\to\infty}{\to} g$ in the energy norm, $\hat{g}_n \underset{n\to\infty}{\to} \hat{g}$ in the energy norm. (^ indicates Fourier transform.) We now take the Fourier transform of both terms in equation (2) to obtain:

$$f_o = 0$$

$$f_n = f_{n-1} + J_\Omega \cdot (J_A(g - \check{f}_n))^\wedge \ , \ n \geq 0 \tag{3}$$

where $^\vee$ denotes the inverse Fourier transform. Since $f_n(\omega) = 0$, if $\omega \notin (-\Omega,\Omega)$ we can write

$$f_n(\omega) = f_{n-1}(\omega) + \int_{-A}^{A} (g(z) - \int_{-\Omega}^{\Omega} e^{2\pi ixz} \ f_{n-1}(z) \ dz) \ e^{-2\pi ix\omega} \ dx \tag{4}$$

where $\omega \in (-\Omega,\Omega)$.

Since $\hat{g}_n = f_n$ then it is clear that f_n converges to f in $L^2(\Omega)$, where f satisfies

$$\int_{-\Omega}^{\Omega} f(\omega) \ e^{-2\pi ix\omega} \ d\omega = g(x), \quad x \in (-A,A) \tag{5}$$

Equation (5) defines f uniquely, and the solution exists because we assumed g to be band-limited to $(-\Omega,\Omega)$. If we denote $K(x,\omega) = e^{-2\pi ix\omega}$, $x \in (-A,A)$, $\omega \in (-\Omega,\Omega)$, we will be able to write the following equation which is equivalent to (5):

$$\int_{-\Omega}^{\Omega} K(x,\omega) \ f(\omega) \ d\omega = g(x), \ x \in (-A,A) \tag{6}$$

We recognize equation (6) to be a Fredholm integral equation of the first kind, where K defines a compact operator, $K : L^2(\Omega) \to L^2(A)$. In 1951, Landweber ([3]) proposed the following iterative procedure for computing the solution f to equation (6) where K is any compact integral operator:

$$f_o = 0$$

$$f_n = f_{n-1} + K^*(g - Kf_{n-1}) \ , \ n \geq 0 \tag{7}$$

Here, K^* denotes the adjoint of $K(K^* : L^2(A) \rightarrow L^2(\Omega))$. Landweber proved that if equation (6) has a solution and the solution is unique : f, then f_n will approach f in the energy norm over $(-\Omega,\Omega)$. For the case $K(x,\omega) = e^{-2\pi i x\omega}$, K^* is also given by the integral kernel $e^{2\pi i x\omega}$. Therefore, equation (7) becomes

$$f_o = 0$$

$$f_n(\omega) = f_{n-1}(\omega) + \int_{-A}^{A} e^{2\pi i x\omega}(g(x) - \int_{-\Omega}^{\Omega} f(z)\ e^{-2\pi i z x}\ dz), \ n \geq 0 \tag{8}$$

It turns out that (8) and (4) define the same iterative formula.

In conclusion, we have proven that the Gerchberg-Papoulis technique is obtained from Landweber's iteration by replacing $K(x,\omega) = e^{-2\pi i x\omega}$ in equation (7). In fact, Landweber also proved ([3]) that iteration (7) converges to the minimum norm solution in cases where (6) has multiple solutions. However, this extension is not useful for our situation because the solution to equation (5) is unique.

Some interesting consequences can be drawn from our result. Perhaps the most important observation from a theoretical point of view is related to the role played by the Prolate Spheroidal Wave functions ([10]). The proof of the convergence of iteration (1) given in [2] uses the completeness of the prolates in the space of Ω-band-limited functions. However, this property does not hold for all the kernels K. Therefore our result shows that the convergence of (2) can be proved by using a conceptually simpler approach. This remark shows that iteration (2) does not use any particular or special property which is satisfied by $e^{-2\pi ix\omega}$ only. On the other hand, in order to prove the equivalence of (1) and (2) we do need the property

$$\mathrm{sinc}_\Omega * g = g$$

where g is any Ω-band-limited function. However, the convenience of using (2) instead of (1) was shown by several authors through theoretical and practical evidence (see for example, [5], [8], [9]).

Since iteration (7) converges for any compact integral operator, the iterative procedure (2) can be generalized for getting the continuation of g when it is given by equation (6). Some other regularity conditions are required for K (for a detailed discussion, see [11]). This means that if g is an analytic function given by (6), g can be extrapolated by means of the following iterative procedure:

$$g_o = 0$$
$$g_n = g_{n-1} + KK^*(g - g_{n-1}) \;, \quad n \geq 0 \tag{9}$$

The convergence of g_n to g, uniformly over compact sets in C, is easily proved. It is worth pointing out that if $K(x,\omega) = e^{-2\pi ix\omega}$ then $KK^*(g - g_{n-1}) = \mathrm{sinc}_\Omega * J_A(g - g_{n-1})$ and therefore (9) will be the same iteration as that of formula (2).

Equation (9) allows us to consider new iterative procedures when some more <u>a priori</u> information about g is available. The purpose of the next section is to relate some of these generalizations to some discrete procedures given in [4].

II. Discrete and Continuous Iterations

If we sample equation (2) as follows:

$$S_o(k) = 0 \ , \quad \text{for all } k \in Z$$
$$S_n(k) = S_{n-1}(k) + \Delta \sum_{j\Delta \in (-A,A)} \text{sinc}_\Omega[(k-j)\Delta](g(j\Delta) - S_{n-1}(j)) \tag{10}$$

we will obtain a very well-known iterative procedure for getting a solution of the discrete extrapolation problem ([4], [6] and [7]). This algorithm was generalized in [4] to cases where the low-pass operator is replaced by some other low-pass filters. Specifically, if $h : \mathbb{R} \rightarrow C$ is a finite-energy function which is band-limited to $(-\Omega,\Omega)$ and satisfies $0 \leq \hat{h}(\omega) \leq 1$ for all $\omega \in (-\Omega,\Omega)$, then the following iteration converges to a certain solution of the discrete extrapolation problem:

$$t_o(k) = 0 \ , \quad \text{for all } k \in Z$$
$$k \in Z, \ t_n(k) = t_{n-1}(k) + \Delta \sum_{j\Delta \in (-A,A)} h[(k-j)\Delta](g(j\Delta) - t_{n-1}(j)) \tag{11}$$

It is clear that equation (10) is a particular case of (11). It was also shown in [4] that the limit of the iterative equation (11) approaches the continuous extrapolation when Δ tends to zero and when f and h satisfy the following relationship:

$$\int_{-\Omega}^{\Omega} \left|\frac{f(\omega)}{\hat{h}(\omega)}\right|^2 d\omega < \infty \tag{12}$$

(f is the Fourier transform of g.) It is worth pointing out that equation (10) was obtained by sampling a continuous iterative equation. On the other hand, iteration (11) was not obtained by sampling any recursion.

The purpose of the remainder of this section is to show that iteration (11) does come from sampling a continuous iterative equation similar to (2) which also computes the continuous extrapolation of g. This result will also help us understand the origin of condition (12). Let us assume that f and h satisfy

$$\int_{-\Omega}^{\Omega} \frac{|f(\omega)|^2}{\hat{h}(\omega)}\, d\omega < \infty \tag{13}$$

Condition (13) is weaker than (12) because $0 \leq h(\omega) \leq 1$. We now write

$$H(x,\omega) = e^{-2\pi i\omega x}\, h^{\frac{1}{2}}(\omega) \;,\; x \in (-A,A) \qquad \omega \in (-\Omega,\Omega) \tag{14}$$

It is easy to verify that H defines a compact integral operator from $L^2(\Omega)$ into $L^2(A)$. It is also seen that the integral equation

$$\int_{-\Omega}^{\Omega} H(x,\omega)\;\ell(\omega)\; d\omega = g(x) \;,\; x \in (-A,A) \tag{15}$$

has one and only one solution in $L^2(\Omega)$. Moreover, the solution is just $f(\omega)/h^{\frac{1}{2}}(\omega)$ which belongs to $L^2(\Omega)$ because property (13) holds. We are now able to apply Landweber's iteration to equation (15) for getting the solution ℓ and to apply extrapolation procedure (9) for obtaining the extrapolation of g. If we use the latter technique, we will get

$$\begin{aligned} g_o &= 0 \\ g_n &= g_{n-1} + HH^*(g - g_{n-1}) \;,\; n \geq 0 \end{aligned} \tag{9}$$

It is easy to verify that $HH^*(g - g_{n-1}) = h * J_A(g - g_{n-1})$ and therefore (9) becomes

$$g_o = 0$$
$$g_n = g_{n-1} + h * J_A(g - g_{n-1}) \;, \quad n \geq 0 \tag{16}$$

In this particular situation, g_n will approach g uniformly over the real line. This fact readily follows from the form of the kernel H and the convergence in $L^2(\Omega)$ of the corresponding Landweber's iteration. Equation (16) is an extension of iterative procedure (2) where sinc_Ω is replaced by h. It turns out that the discrete iterative algorithm defined by equation (11) can be obtained by sampling the continuous recursion (16). This gives a full explanation of the conceptual origin of the discrete technique (11).

As was pointed out above, condition (13) is weaker than (12) but it is strong enough to prove the convergence of (16) to the sought function g. This suggests that the discrete technique given by (11) should also approach the extrapolation if

$$\int_{-\Omega}^{\Omega} \frac{|f(\omega)|^2}{\hat{h}(\omega)} \, d\omega < \infty \tag{13}$$

We proved this proposition in a more general setting in ref. [11].

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