January 1988

UILU-ENG-88-2204
ACT-88

COORDINATED SCIENCE LABORATORY
College of Engineering
Applied Computation Theory

COMPACT REPRESENTATION OF THE SEPARATING k-SETS OF A GRAPH

Arkady Kanevsky

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified

1b. RESTRICTIVE MARKINGS: None

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE:

3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S): UILU-ENG-88-2204 (ACT-88)

5. MONITORING ORGANIZATION REPORT NUMBER(S):

6a. NAME OF PERFORMING ORGANIZATION: Coordinated Science Lab, University of Illinois

6b. OFFICE SYMBOL (If applicable): N/A

7a. NAME OF MONITORING ORGANIZATION: NSF, ONR, and Semiconductor Research Corporation

6c. ADDRESS (City, State, and ZIP Code):
1101 W. Springfield Avenue
Urbana, IL 61801

7b. ADDRESS (City, State, and ZIP Code):
800 N. Quincy, Arlington, VA 22217 (ONR)
1800 G St., Washington, DC 20550 (NSF)
Research Triangle Park, NC 27709 (SRC)

8a. NAME OF FUNDING / SPONSORING ORGANIZATION: NSF, JSEP, and SRC

8b. OFFICE SYMBOL (If applicable):

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: NSF: ECS 8404866, JSEP: N00014-84-C-0149, SRC: 87-DP-109

8c. ADDRESS (City, State, and ZIP Code): see block 7b.

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification): Compact Representation of the Separating k-sets of a Graph

12. PERSONAL AUTHOR(S): Kanevsky, Arkady

13a. TYPE OF REPORT: Technical

13b. TIME COVERED: FROM ____ TO ____

14. DATE OF REPORT (Year, Month, Day): 1988 January

15. PAGE COUNT: 29

16. SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): graph theory, data structures, connectivity

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

We present an $O(n)$ space representation for the separating k-sets of an undirected k-connected graph G for fixed k, where n is the cardinality of the vertex set of G. Namely, the total space used by the representation is $O(k^2 n)$. We also improve the upper bound on the number of separating k-sets of G to $O(2^k \frac{n^2}{k})$, which has a matching lower bound.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION: Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL:

22b. TELEPHONE (Include Area Code):

22c. OFFICE SYMBOL:

Compact Representation of the Separating k-sets of a Graph

Arkady Kanevsky

Coordinated Science Laboratory
University of Illinois
Urbana, IL 61801

January 1988

ABSTRACT

We present an $O(n)$ space representation for the separating k-sets of an undirected k-connected graph G for fixed k, where n is the cardinality of the vertex set of G. Namely, the total space used by the representation is $O(k^2 n)$. We also improve the upper bound on the number of separating k-sets of G to $O(2^k \frac{n^2}{k})$, which has a matching lower bound.

1. Introduction

Connectivity is an important graph property and there has been a considerable amount of work on algorithms for determining connectivity of graphs [BeX,Ev2,EvTa,Ga,GiSo,LiLoWi]. An undirected graph $G=(V,E)$ is *k-connected* if for any subset V' of $k-1$ vertices of G the subgraph induced by $V-V'$ is connected [Ev]. A subset V' of k vertices is a *separating k-set* for G if the subgraph induced by $V-V'$ is not connected. For $k=1$ the set V' becomes a single vertex which is called an *articulation point*, and for $k=2,3$ the set V' is called a *separating pair* and a *separating triplet*, respectively. Efficient algorithms are available for finding all separating k-sets in k-connected undirected graphs for $k \leq 3$ [Ta,HoTa,MiRa,KaRa].

In [KaRa2,Ka] we addressed the question of the maximum number of separating pairs, triplets and k-sets in biconnected, triconnected and k-connected undirected graphs, respectively?

An undirected graph G on n vertices has a trivial upper bound of $\binom{n}{k}$ on the number of separating k-

---

This research was supported by NSF under ECS 8404866, the Semiconductor Research Corporation under 87-DP-109 and the Joint Services Electronics Program under N00014-84-C-0149.

sets, $k \geq 1$. The graph that achieves this bound for all k is a graph on n vertices without any edges. For $k=1$ the maximum number of articulation points in a *connected* graph is $(n-2)$ and a graph that achieves it is a path on n vertices. For $k=2$ the maximum number of separating pairs in an undirected biconnected graph is $\frac{n(n-3)}{2}$ and a graph that achieves it is a cycle on n vertices [KaRa2]. Further, we observed that there is an $O(n)$ representation for the separating pairs in any biconnected graph (although the number of such pairs could be $\Theta(n^2)$) [KaRa2]. For $k=3$ the maximum number of separating triplets in a triconnected graph is $\frac{(n-1)(n-4)}{2}$ and we presented a graph, namely the *wheel* [Tu], that achieves it [KaRa2]. The number of separating k-sets in a k-connected graph is $O(3^k n^2)$ and we show that the bound is tight up to the constant [Ka]. The lower bound on the number of separating k-sets in a k-connected undirected graph is $\Omega(2^k \frac{n^2}{k^2})$.

In this paper we present a linear representation of separating k-sets in k-connected undirected graphs. For $k=2$ representation is different from the one presented in [KaRa2]. We also give the alternative prove of the upper bound on the number of separating k-sets, which match the previous upper bounds for $k=2$ and $k=3$, and improves the upper bound for general k to $O(2^k \frac{n^2}{k})$. We will first present representation for $k=2$ and $k=3$ and then generalized the technique for general k.

2. Graph-theoretic definitions

An *undirected graph* $G=(V,E)$ consists of a *vertex set* V and an *edge set* E containing unordered pairs of distinct elements from V. A *path* P in G is a sequence of vertices $<v_0, \cdots, v_k>$ such that $(v_{i-1}, v_i) \in E, i=1, \cdots, k$. The path P *contains* the vertices $v_0, \cdots, v_k$ and the edges $(v_0, v_1), \cdots, (v_{k-1}, v_k)$ and has *endpoints* v_0, v_k, and *internal vertices* $v_1, \cdots, v_{k-1}$.

We will sometimes specify a graph G structurally without explicitly defining its vertex and edge sets. In such cases, $V(G)$ will denote the vertex set of G and $E(G)$ will denote the edge set of G. Also, if $V' \subseteq V$ and $v \in V$ we will use the notation $V' \cup v$ to represent $V' \cup \{v\}$.

An undirected graph $G=(V,E)$ is connected if there exists a path between every pair of vertices in V. For a graph G that is not connected, a *connected component* of G is an induced subgraph of G which is maximally connected.

A vertex $v \in V$ is an *articulation point* of a connected undirected graph $G=(V,E)$ if the subgraph induced by $V-\{v\}$ is not connected. G is *biconnected* if it contains no articulation point.

Let $G=(V,E)$ be a biconnected undirected graph. A pair of vertices $v_1, v_2 \in V$ is a *separating pair* for G if the induced subgraph on $V-\{v_1,v_2\}$ is not connected. G is *triconnected* if it contains no separating pair.

A triplet (v_1,v_2,v_3) of distinct vertices in V is a *separating triplet* of a triconnected graph if the subgraph induced by $V-\{v_1,v_2,v_3\}$ is not connected. G is *four-connected* if it contains no separating triplets.

Let $G=(V,E)$ be an undirected graph and let $V' \subseteq V$. A graph $G'=(V',E')$ is a *subgraph* of G if $E' \subseteq E \cap \{(v_i,v_j) \mid v_i,v_j \in V'\}$. The *subgraph of G induced by V′* is the graph $G''=(V',E'')$ where $E''=E \cap \{(v_i,v_j) \mid v_i,v_j \in V'\}$.

3. Representation for k=2

Let $G=(V,E)$ be an undirected biconnected graph with n vertices and m edges. We denote with $g(n)$ the upper bound on the size of a compact representation of separating pairs of a graph on n vertices. Let $\{v_1,v_2\}$ be a separating pair that divides G into nonempty G_1 and G_2. Let $\{w_1,w_2\}$ be a "cross" separating pair with $w_1 \in G_1$ and $w_2 \in G_2$. It divides G_1 into G'_1 and G''_1, and divides G_2 into G'_2 and G''_2 (see Figure 1).

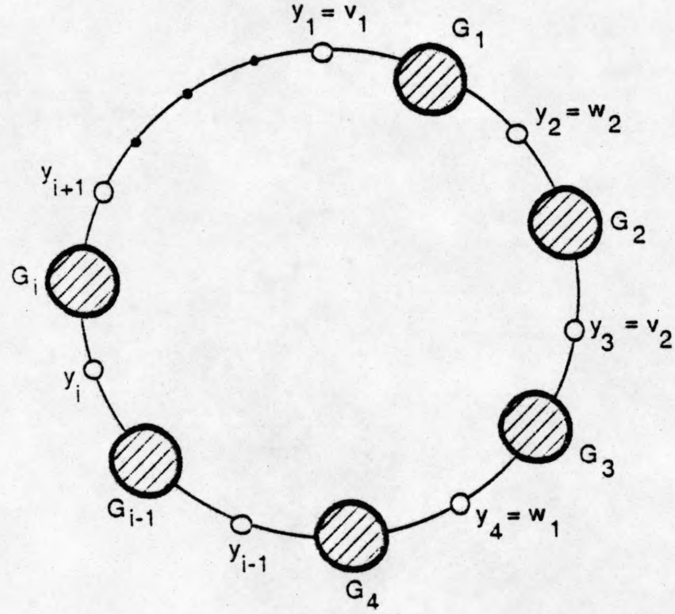

Figure 1.
Representation for k=2.

Consider a maximal set of vertices u in G_2 such that $\{w_1,u\}$ is a cross separating pair and, analogously, consider a maximal set of vertices x in G_1 such that $\{x,w_2\}$ is a cross separating pair. The set of u's is the set of articulation points in G_2. Moreover, the set of u's along with the subgraphs of G_2 between them is a path from v_1 to v_2. Analogously, the set x's is a set of articulation points of G_1 with additional condition that the x's along with the subgraphs of G_1 between them is a path from v_1 to v_2. Number the vertices v_1, u's, v_2, and x's by y_1, y_2 and so on going clockwise along the paths. We denote by G_i the subgraph of G between y_i and y_{i+1}. Note that some G_i can be empty (consists of a single edge). Thus, the graph G becomes a cycle with vertices y's and G_i's alternating on it. Every pair of vertices y's give a separating pair of G unless they are adjacent and the subgraph between them is empty. Hence, we can represent all of them by the following structure:

1) the cycle: the set of vertices y's

2) a vertex for every G_i with a flag to specify if G_i is empty. Edges between G_i and y_i, y_{i+1}.

Note that when there are no cross separating pairs then we get a trivial cycle with two vertices v_1 and v_2 and two edges connecting them. Since the sets x's and u's are maximal all other separating pairs are inside $G_i \cup y_i \cup y_{i+1}$. Note that G_i can be the union of disconnected components, but each of them is connected to y_i and y_{i+1}. Let the cardinality of set of vertices y's be l. Based upon the above observations we get the following recurrence relation

$$g(n) \le \sum_{i=1}^{l} g(n_i+2) + 4l ,$$

where $g(n_i+2)$ represent the upper bound for all separating pairs inside $G_i \cup y_i \cup y_{i+1}$. The cardinality of $G_i = n_i$, and $\sum_{i=1}^{l}(n_i+1) = n$. Any $g(n)$ that satisfy the recurrence will be an upper bound on the size of representation of separating pairs of G. Clearly, linear $g(n)$ is one of them (see Appendix).

4. Representation for k=3

The *wheel* W_n [Tu] is C_{n-1} together with a vertex v and an edge between v and every vertex on C_{n-1}. It is easy to see that W_n is triconnected and has $\frac{(n-1)(n-4)}{2}$ separating triplets.

Assume there exists a separating triplet $\{v_1,v_2,v_3\}$ in G, which separates G into nonempty G_1 and G_2 (see Figure 2).

Lemma 1: Only one of these three vertices has type 3 separating triplets $\{w_1,v_i,w_2\}$ such that $w_1 \in G_1$ and $w_2 \in G_2$ [KaRa2].

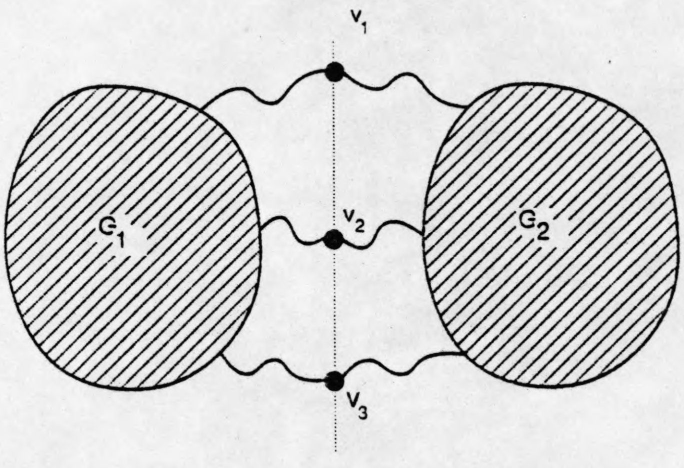

Figure 2.
Separating G into G_1 and G_2 by separating triplet $\{v_1,v_2,v_3\}$

Proof: Assume there is separating triplet $\{w_1,v_2,w_2\}$ of the third type in G, where $w_1 \in G_1$ and $w_2 \in G_2$. It separates G_1 into K_1 and K_2, and separates G_2 into K_3 and K_4. Vertices v_1 and v_3 must belong to the different components with respect to separating triplet $\{w_1,v_2,w_2\}$, otherwise either $\{w_1,v_2\}$ is a separating pair, or $\{w_2,v_2\}$ is a separating pair, or both.

Claim 1 Vertex v_2 has a direct edge to every nonempty subgraph K_1,K_2,K_3,K_4.

W.L.O.G. assume that K_1 is not empty and $\forall x \in K_1$, $(x,v_2) \notin E$. Then $\{v_1,w_1\}$ is a separating pair of G, which separates K_1 from the rest of the graph.

□

Now, we will prove that there are no separating triplets of the third type which use v_1 or v_3. We will prove this by contradiction. W.L.O.G. assume there is a separating triplet $\{u_1,v_1,u_2\}$, where $u_1 \in G_1$ and $u_2 \in G_2$ (u_1 may be equal to w_1 and u_2 may be equal to w_2).

Case 1: $u_1 \in K_2$, if K_2 is not empty (see Figure 3).

By Claim 1 for v_1 and the existence of separating triplet $\{u_1,v_1,u_2\}$, K_1, w_1, $K_2 - u_1$ belong to the same connected component with respect to separating triplet $\{u_1,v_1,u_2\}$. If v_2 belongs to the same component then $\{v_1,u_1\}$ is a separating pair which separates $K_3 \cup w_2 \cup K_4 \cup v_3$ from the rest of the graph. If v_2 does not belong to the same component then $\{v_1,u_1\}$ is a separating pair which separates $K_1 \cup w_1 \cup K_2 - u_1$ from the rest of the graph.

Analogously, $u_2 \notin K_4$.

Case 2: $u_1 = w_1$.

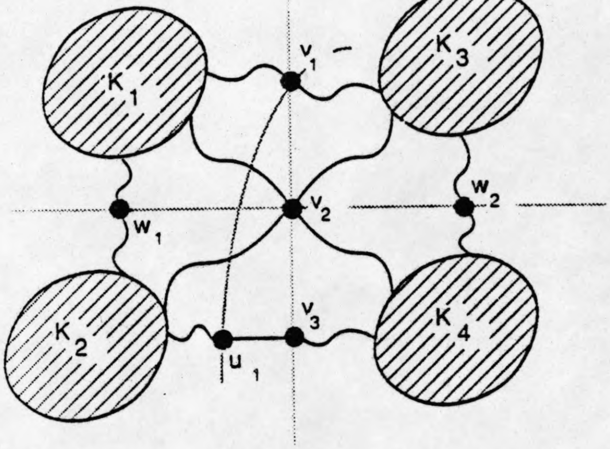

Figure 3.
Illustrating Case 1 in the proof of Lemma 1.

Since $\{u_1,v_1,u_2\}$ is a separating triplet then v_2 does not have any edges to K_1 and hence, K_1 is empty by Claim 1. But then $\{v_1,u_2\}$ is a separating pair, if $\{u_1,v_1,u_2\}$ is a separating triplet.

Analogously, $u_2 \neq w_2$.

Case 3: $u_1 \in K_1$ and $u_2 \in K_3$.

If $\{u_1,v_1,u_2\}$ is a separating triplet then either $\{u_1,u_2\}$, or $\{u_1,v_1\}$, or $\{v_1,u_2\}$ is a separating pair.

That means that if there is a separating triplet of the third type which uses one of the $v_i, i=1,2,3$ then there are no separating triplets of the third type that use the other $v_j, j=1,2,3,\ j\neq i$.

□

Let $\{v_1,v_0,v_2\}$ be a separating triplet of a graph G on n vertices, and v_0 be the only one of the three vertices of this separating triplet which might participate in a separating triplets of the third type with respect to $\{v_1,v_0,v_2\}$. Consider all separating triplets of the third type $\{w_1,v_0,w_2\}$ such that $w_1 \in G_1$ and $w_2 \in G_2$, together with $\{v_1,v_0,v_2\}$. All such separating triplets use v_0 as the "central" vertex. Rename the vertices w_1's, w_2's, v_1 and v_2 into $\{v_1, v_2, \cdots, v_l\}$ going clockwise, such that they form the wheel with v_0 in a center, where any two nonadjacent vertices form a separating triplet with v_0. The subgraphs between v_i and v_{i+1} are denoted with G_i, and some of them may be empty. Now, the graph G looks like a wheel with v_0 in a center v_i, and G_i $(i=1, \cdots, l)$ on a cycle.

Every pair of vertices on the cycle of the wheel form a separating triplet with v_0 unless they are adjacent (v_i and v_{i+1}) and the subgraph (G_i) between them is empty. Hence, we can represent these separating triplets by the following structure:

1) the wheel: $\{v_0, v_1, \cdots, v_k\}$ with edges of G

2) a vertex for every G_i with a flag to specify if G_i is empty. The edges between G_i and v_i, v_{i+1} and between v_0 and v_i, G_i with flags to specify if the edge is real.

Let us see where the rest of separating triplets of G lie.

Observation The remaining separating triplets belong to $G_i \cup v_0 \cup v_i \cup v_{i+1} \cup$ the neighbor of v_i in G_{i-1} if such a neighbor is unique $\cup$ the neighbor of v_{i+1} in G_{i+1} if such a neighbor is unique.

Let $\{w_1, w_2, w_3\}$ be a separating triplet with $w_1 \in G_1$ and $w_2, w_3 \in G_2$. The separating triplet $\{w_1, w_2, w_3\}$ separates G_1 into L_1 and L_2, and separates G_2 into L_3 and L_4 (Figure 4).

Let us see how the original separating triplet $\{v_1, v_2, v_3\}$ is separated by the separating triplet $\{w_1, w_2, w_3\}$.

The vertices $\{v_1, v_2, v_3$ cannot belong to the same connected component of G with respect to the separating triplet $\{w_1, w_2, w_3\}$, otherwise either w_1 would be an articulation point, or $\{w_2, w_3\}$ would be a separating pair, or both. W.L.O.G. assume that v_1 belongs to one connected component and v_2, v_3 to the other.

Subgraph L_1 must be empty, otherwise $\{w_1, v_1\}$ becomes a separating pair. Since the graph is triconnected, we have

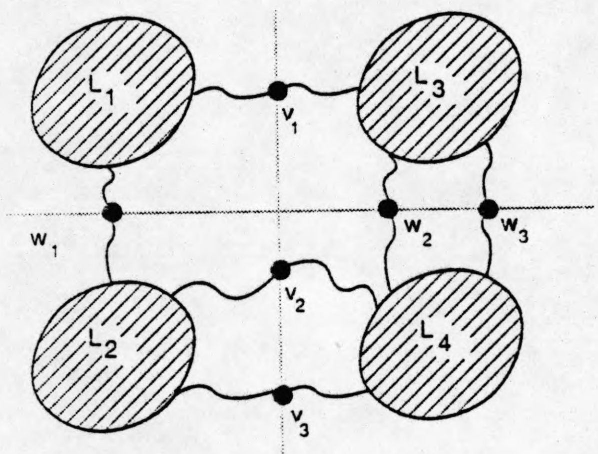

Figure 4.
Illustrating the proof of the Observation.

1) $(w_1, v_1) \in E$,

2) $\exists\, x,y \in L_3 \cup w_2 \cup w_3$: $(x,v_1) \in E$, $(y,v_1) \in E$ and

3) $\forall\, z \in L_2 \cup L_4 \cup v_2 \cup v_3$: $(z,v_1) \notin E$.

Hence, vertex w_1 is the unique neighbor of vertex v_1 in G_1. Moreover, if there are any separating triplets with one vertex in G_1 and two in G_2 which separate v_1 from v_0 and v_2, then w_1 is one of the vertices of the triplet.

A separating triplet cannot have all its three vertices in three different G_i's otherwise two of these vertices would form a separating pair. From the proof of the Lemma 1 and the fact that the set $\{v_1, v_2, \cdots, v_k\}$ is maximal, we know that if there is a separating triplet which involves a vertex from G_i, then the other two vertices belong to $\{v_i\} \cup \{v_{i+1}\} \cup \{v_0\} \cup G_i$ and the neighbor of v_i in G_{i-1}, if such a neighbor is unique, and symmetrically a 'unique' neighbor of v_{i+1} in G_{i+2}. This proves the Observation.

□

Let $g(n)$ be the size of a compact representation of the separating triplets in a graph on n vertices, and let the number of vertices in G_i be n_i. Then $\sum_{i=1}^{k}(n_i + 1) + 1 = n$, and we can write the following recurrence relation

$$g(n) = \sum_{i=1}^{l} g(n_i + 5) + (6l + 1),$$

where $(6l + 1)$ stands for the space used to store the wheel information including multiple edges. The solution to this recurrence is clearly linear (see Appendix). This proves that there is a succinct $O(n)$ size representation of the separating triplets.

5. Representation for general k

Let $G=(V,E)$ be an undirected k-connected graph with n vertices and m edges. We denote with $g(n)$ and $f(n)$ the upper bounds on the size of representation and the number of separating k-sets for k-connected graph on n vertices. Let $V' = \{v_1, v_2, \cdots, v_k\}$ be a separating k-set, whose removal separates G into nonempty G_1 and G_2 (see Figure 5). A separating k-set $\{w_1, w_2, \cdots, w_k\}$ of G is a *cross* separating k-set with respect to V' if $\exists\, i,j$: $w_i \in G_1$ and $w_j \in G_2$. Let the cardinalities of G_1 and G_2 be l and $n-l-k$, respectively. Let the upper bound on the size of the representation of the cross separating k-sets be $h(l, n-l)$, and the maximum number of cross separating k-sets be $r(l, n-l)$. Then any $g(n)$ and $f(n)$ that satisfy the recurrences

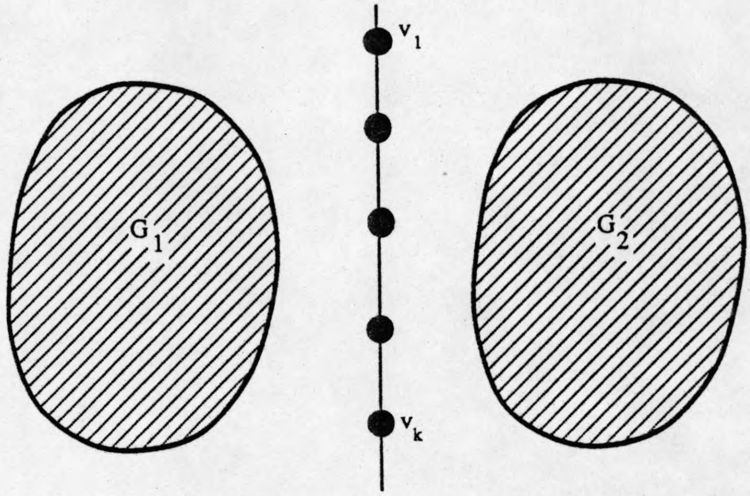

Figure 5.
Dividing G into G_1 and G_2 by separating k-set $\{v_1, \cdots, v_k\}$

$$g(n) = \left\lceil g(l+k) + g(n-l) + h(l, n-l) \right\rceil \quad ,$$
$$f(n) = \left\lceil f(l+k) + f(n-l) + r(l, n-l) + 1 \right\rceil \quad ,$$

are upper bounds on the size of representation and the number of separating k-sets in G. Now we will derive upper bounds for the functions h and r and tune up the recurrences.

Let $\{w_1, w_2, \cdots, w_k\}$ be a cross separating k-set with $\{w_1, \cdots, w_s\} \subset G_1$, $\{w_{s+t+1}, \cdots, w_k\} \subset G_2$ and $\{w_{s+1}, \cdots, w_{s+t}\} \subset \{v_1, \cdots, v_k\}$. The separating k-set $\{w_1, w_2, \cdots, w_k\}$ separates G_1 into G_3 and G_4, separates G_2 into G_5 and G_6, and divides $\{v_1, \cdots, v_k\}$ into $\{v_1, \cdots, v_r\}$, $\{v_{r+t+1}, \cdots, v_k\}$ and $v_{r+i} = w_{s+i}$, $i = 1, ..., t$. (see Figure 6)

Case 1 None of G_i, $i = 3,4,5,6$ are empty. (see Figure 6)

The sets $\{w_1, w_2, \cdots, w_{s+t}, v_1, \cdots, v_r\}$, $\{w_1, w_2, \cdots, w_{s+t}, v_{r+t+1}, \cdots, v_k\}$, $\{v_1, \cdots, v_{r+t}, w_{s+t+1}, \cdots, w_k\}$ and $\{v_{r+1}, \cdots, v_k, w_{s+t+1}, \cdots, w_k\}$ are separating sets of G that separate G_3, G_4, G_5 and G_6 respectively, so their cardinalities are greater than or equal to k. Then,

$$\begin{cases} s+t+r \ge k \\ r+t+k-s-t \ge k \\ s+t+k-r-t \ge k \\ k-r+k-s-t \ge k \end{cases} \Rightarrow \begin{cases} r+s+t \ge k \\ r \ge s \\ s \ge r \\ k \ge r+s+t \end{cases} \Rightarrow \begin{cases} r = s \\ r+s+t = k \end{cases}$$

From now on we replace the subscript r by s. Let $A = \{v_1, \cdots, v_s\}$, $B = \{v_{s+t+1}, \cdots, v_k\}$, $C = \{w_1, \cdots, w_s\}$, $D = \{w_{s+t+1}, \cdots, w_k\}$, and $T = \{v_{s+1}, \cdots, v_{s+t}\} = \{w_{s+1}, \cdots w_{s+t}\}$. For Case 1 $|A| = |B| = |C| = |D| = \dfrac{k-t}{2}$.

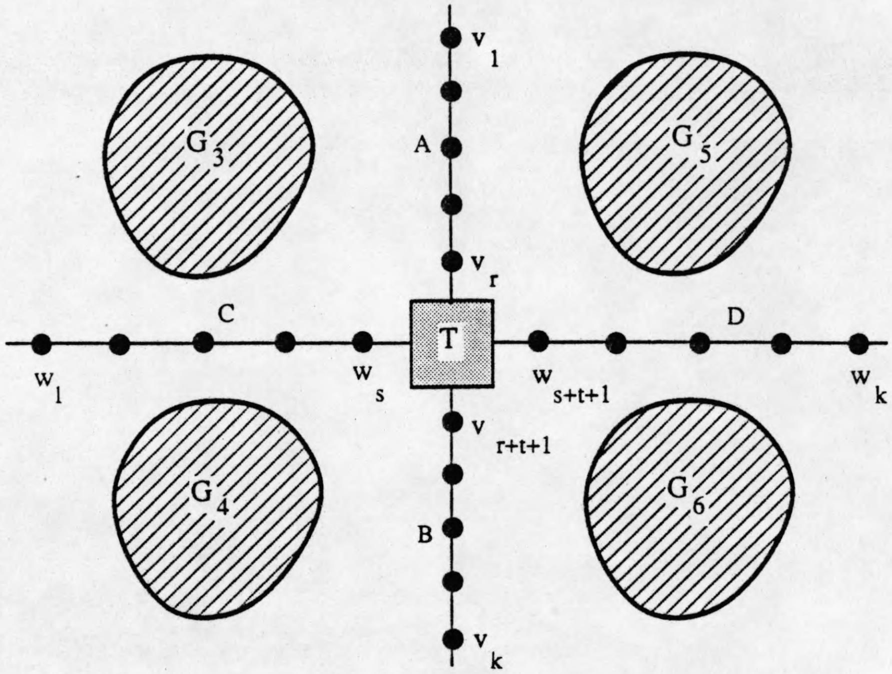

Figure 6.
Dividing G into nonempty components by separating k-sets
$\{v_1, \cdots, v_k\}$ and $\{w_1, \cdots, w_k\}$.

Claim 2 $\forall\, i \;\; i = s+1,...,t \; \exists\, x_j \in G_j,\, j = 3,4,5,6$: $(v_i, x_j) \in E$.

Proof: W.L.O.G. assume $\exists v_i$: $\forall x \in G_3$: $(x, v_i) \notin E$. Then $\{v_1, \cdots, v_{s+t}, w_1, \cdots, w_s\} - \{v_i\}$ is a separating (k-1)-set.

□

Claim 3 For every $x \in A$ there are $y \in G_3$ and $z \in G_5$, such that $(x,y) \in E$ and $(x,z) \in E$. Analogously, for every vertex x of B, C and D there are vertices y and z in appropriate neighboring G_i, i=3,4,5,6, which are adjacent to x.

Proof: W.L.O.G. assume there is $x \in A$ such that for every $y \in G_3$ $(x,y) \notin E$. Then $A \cup C \cup T$-$\{x\}$ is a separating (k-1)-set.

□

Lemma 2 All cross separating k-sets containing $C \cup T$ and at least one fixed vertex of D can be represented in $O((\frac{k-t}{2})^2)$ space, and their number is $O(2^{\frac{k-t}{2}})$.

Proof: Assume we have a separating k-set $\{w_1, \cdots, w_{s+t+a}, x_{s+t+a+1}, \cdots, x_{s+t+a+b}, y_{s+t+a+b+1}, \cdots, y_k\}$, where x's$\in G_5$, y's$\in G_6$, $a \geq 1$, and either b or $k-s-t-a-b$ is greater or equal to 1 (the new cross separating k-set is different from the old one) (see Figure 7).

Let $H = \{x_{s+t+a+1}, \cdots, x_{s+t+a+b}\}$ (x's) and $I = \{y_{s+t+a+b+1}, \cdots, y_k\}$ (y's), and let D be divided into $D' = \{w_{s+t+1}, \cdots, w_{s+t+a}\}$, E which is in the same connected component as G_3, A, and part of G_5, and F which is in the

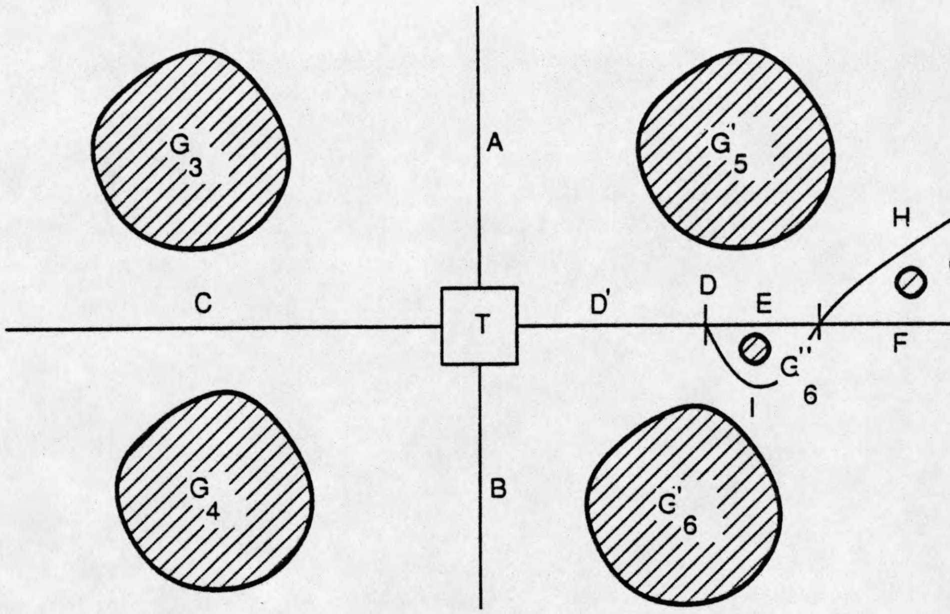

Figure 7.
Illustrating the proof of Lemma 2.

same connected component as G_4, B and part of G_6. Also let H divide G_5 into G'_5 and G''_5, and let I divide G_6 into G'_6 and G''_6 (see Figure 7).

Separating sets $T+D'+E+H$ and $T+D'+F+I$ separate G''_5 and G''_6, respectively. The cardinalities of these separating sets are less than k. Hence, G''_5 and G''_6 are empty. Moreover, since $C+T+D'+H+F$ and $C+T+D'+E+I$ are separating sets and $C+T+D$ and $C+T+D'+H+I$ are separating k-sets, $|E| = |H|$, and $|I| = |F|$. Note that the argument still holds if either H or I are empty.

Next, we will show that if we replace part of E and/or part of F we will necessarily use only vertices of H and/or I for it, regardless of whether we replace part of D' or not. In other words, H and I are unique for E and F. The proof is by contradiction.

Assume that there exist $I_1+H_1 \neq I+H$, such that $C+T+D'+H_1+I_1$ is a separating k-set. Let $H_1 \subseteq G_5$ and $I_1 \subseteq G_6$. Also, let I_1+H_1 divide E into E_1 and E_2, and divide F into F_1 and F_2 (see Figure 8).

Let H_1 be separated into two parts, H'_1 adjacent to E and E''_1 adjacent to F. By the above arguments H'_1 is adjacent to E_1, H''_1 is adjacent to F_2, and I_1 is adjacent to E_2+F_1. Since all neighbors of E in G_6 are also in I, and all neighbors of F in G_5 are also in H, $H''_1 \subset H$ and I_1 is divided into $I'_1 = I \cup I_1$ and $I''_1 = I_1 - I'_1$. Let $H' = H - H''_1$ and let $I' = I - I'_1$.

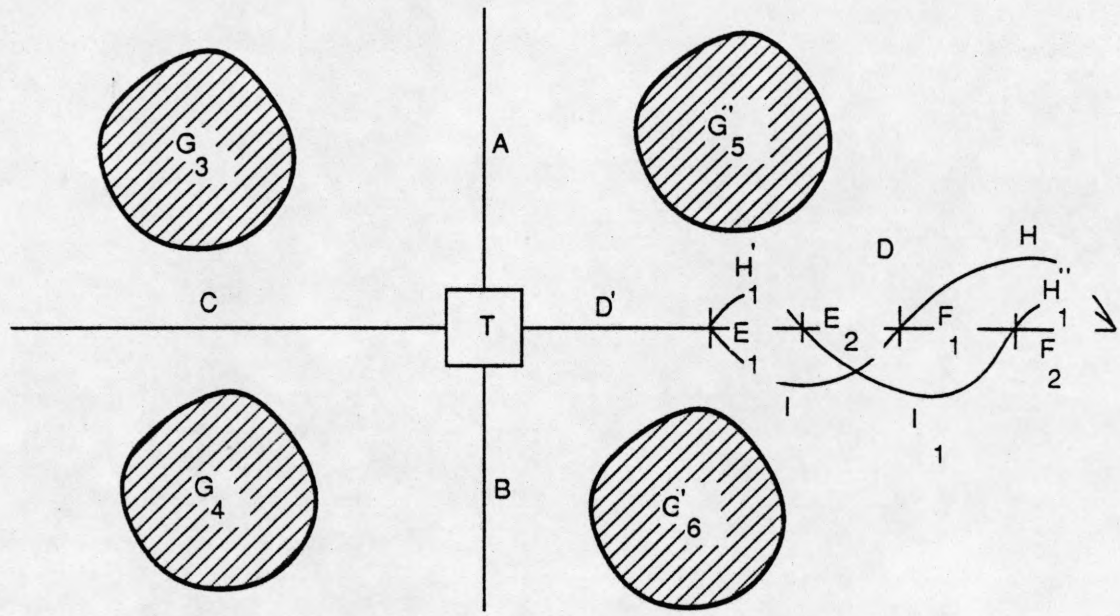

Figure 8.
Illustrating the uniqueness of a replacement for a part of cross separating k-set.

The separating set $T+D'+H'_1+H$ separates E_1 from the rest of the graph and has cardinality is less than k. Hence, E_1 is empty and we have $I = I'_1$, $E = E_2$ and $H_1 = H''_1$. Analogously, the separating set $T+D'+I_1+H$ separates F_1 from the rest of the graph and has cardinality is less than k. Hence, F_1 is empty and we have $F = F_2$, $E = E_1$, $H = H_1$ and $I = I_1$. This contradict the assumptions.

Note that the arguments still hold if either H or I are empty, or if we replace only parts of E and F. If part of D' is replaced as well, then we will not replace it, so that we will look only at the replacements for E and F. Also, if there exists a separating k-set that replaces F by H, then there is no $I_1 \subseteq G_6$ that replaces any part of F for any cross separating k-set described in Lemma 2.

Thus, any replacement of any part of F for any cross separating k-set specified by Lemma 2 lies in H. The set of vertices which is used for all possible replacement of any part of D for a cross separating k-sets specified by Lemma 2 will be called the *fringe* of D, where H is the fringe of F and I is the fringe of E. Note that there could be parts of D which do not have any replacements. The cardinality of the fringe of D is less than $\frac{k-t}{2} = |D|$. Hence, the representation of all cross separating k-sets with $C+T$ fixed along with at least one vertex from D takes $O((\frac{k-t}{2})^2)$ space, where $O((\frac{k-t}{2})^2)$ space is needed to specify all edges between D and its fringe. This proves the space complexity for the representation.

The number of different subsets of D is $2^{|D|}$. Since for every subset $E+F$ of D there is a unique replacement, (if it exists) that a separating k-set specified by Lemma 2, the number of separating k-sets with $C+T$ fixed along with at least one vertex from D is upper bounded by $O(2^{\frac{k-t}{2}})$. This proves the second part of the Lemma.

□

Corollary All cross separating k-sets containing $T+D$ and at least one vertex from C can be represented in $O((\frac{k-t}{2})^2)$ space, and their number is $O(2^{\frac{k-t}{2}})$.

Take the maximal set X of disjoint $C \in G_1$ such that C_i+T+D is a separating k-set. Analogously, take the maximal set Y of disjoint $D \in G_2$ such that $C+T+D_i$ is a separating k-set. For T fixed, all cross separating k-sets are upper bounded by $O(2^{\frac{k-t}{2}} |X| 2^{\frac{k-t}{2}} |Y|) = O(2^{k-t} |X| |Y|)$, and are represented in $O((\frac{k-t}{2})^2 (|X| + |Y|))$ space. Next we will see how many different T's we need to consider.

Take the smallest $T = T_1$ such that a cross separating k-set will have nonempty G_i i=3,4,5,6, if it exist. If there exist a separating k-set with different $T = T_2$, $T_1 \neq T_2$, then it can be of four different types:

Type 1). $T_2 \cap A \neq \emptyset$ and $T_2 \cap B \neq \emptyset$,

Type 2). $\left[T_2 \cap A = \emptyset \text{ or } T_2 \cap B = \emptyset \right]$ and $T_1 \cap T_2 \neq \emptyset$,

Type 3). $\left[T_2 \cap A = \emptyset \text{ or } T_2 \cap B = \emptyset \right]$ and $T_1 \cap T_2 = \emptyset$,

Type 4). $T_2 \cap A = \emptyset$ and $T_2 \cap B = \emptyset$.

Let us first consider type 4 cross separating k-sets. Since T_2 must lie completely inside T_1 and T_1 has the smallest cardinality, then $T_2 = T_1$. Let the cardinality of X, the maximal disjoint set of C's, be l_1, and let the cardinality of the maximal disjoint set Y be l_2, where $l_1 + l_2 = l$. Let us number A, the set X, B and the set Y. So A becomes A_1, the "nearest" D from Y becomes A_2, and so on going clockwise. The cardinality of this set is $l + 2$. From the proof of the Lemma 2 we know that all cross separating k-sets of type 4 consist of three parts: T_1, C which is inside G_1 and is inside some C's from set X and its fringe, and D which is inside G_2 and is inside some D's from set Y and its fringe. Note that $T \cup$ any two $A_i, i=1, \cdots, l+2$ are also separating k-sets if the parts of the graph between them are nonempty. We can also replace parts of A_i by its fringe as long the above condition will be true. Let the part of the graph G between A_i and $A_{i+1}, i=1, \cdots, l+2$ be $G_i, i=1, \cdots, l+2$ (i in this case taken mod $l+2$). Let $G_i -$ the fringe of A_i in $G_i -$ the fringe of A_{i+1} in G_i be $G'_i, i=1, \cdots, l+2$. The only case when $T \cup A_i \cup A_j$ (or parts of the fringe of A_i and A_{i+1}) $i<j$ is not a separating k-set when $i=j-1$ and $G'_i = \emptyset$.

Based upon above observations the structure (structure 1) which covers all cross separating k-sets of type 4 will be the following:

1) A_i with its fringes for all $i=1, \cdots, l+2$,

2) For every nonempty $G'_i, i=1, \cdots, l+2$ we fill all nonexistent edges of the complete graph on the neighbors of G'_i as real edges. If $G'_i, i=1, \cdots, l+2$ is empty for some i then we fill these edges as virtual edges. All of the edges of G between A_i and $G_{i+1}, i=1, \cdots, l+2$ are in the structure as real edges.

Let us see where the rest of the separating k-sets lie assuming there are no cross separating k-sets of type 1 and type 2. Note that we allow separating k-sets of type 3. Let us first the definition of the exceptional separating k-sets. The separating k-set is *exceptional* if it separates only part of A_i an nothing else for $i=1, \cdots, l+2$.

Lemma 3: All separating k-sets which are not covered by the structure 2 and not of type 1 and 2 and not exceptions are inside $G_i \cup A_i$ and its fringes inside $G_{i-1} \cup A_{i+1}$ and its fringes inside G_{i+1}.
Proof: Since there are no type 1 and type 2 and no exceptions in separating k-sets, no separating k-set is using T. There are also no cross separating k-set which are not covered by the structure 1. Let us see what happens if a separating k-set crosses some $A_i, i=1, \cdots, l+2$ (see Figure 9).

W.L.O.G. let $E \cup F \cup H$ is this separating k-set, which crosses A_i, where $E \subset G_5$, $F \subset G_6$ and $H \subset A_i$. It divides A_i into A'_i, A''_i and H. It also divides G_5 into G'_i and G''_i, and it divides G_6 into G'_6 and G''_6. Both A''_i and A'_i are nonempty, otherwise the set Y is not maximal, or there is no cross separating k-sets. If G''_5 and G''_6 are nonempty then $E \cup H \cup A''_i$ and $F \cup H \cup A''_6$ are separating sets with cardinalities bigger or equal to k. But both of them can not have cardinality bigger or equal to k, hence, one of G''_5 or G''_6 must be empty. W.L.O.G. let G''_6 be empty. Since $A_{l+1} \cup T \cup A_i$ and $A_{l+1} \cup T \cup A'_i \cup H \cup F$ are separating k-set and separating set, respectively, $|F| \geq |A''_i|$. Since $E \cup H \cup A''_i$ is a separating set, since both G''_5 and G''_6 can not be empty (exception), $|A''_i| \geq |F|$. Hence, $A''_i| = |F|$, and F is part of the fringe of A_i.

Let us see if a cross separating k-set crosses two adjacent A_i's. W.L.O.G. $E \cup H_1 \cup F \cup H_2 \cup I$ is a separating k-set, which divides A_i into A'_i, H_1, and A''_i, and divides A_{i+1} into A'_{i+1}, H_2, and A''_{i+1}. It separates G_{i-1} into G'_{i-1} and G''_{i-1}, it separates G_i into G'_i and G''_i, it separates G_{i+1} into G'_{i+1} and G''_{i+1}. By the above argument,

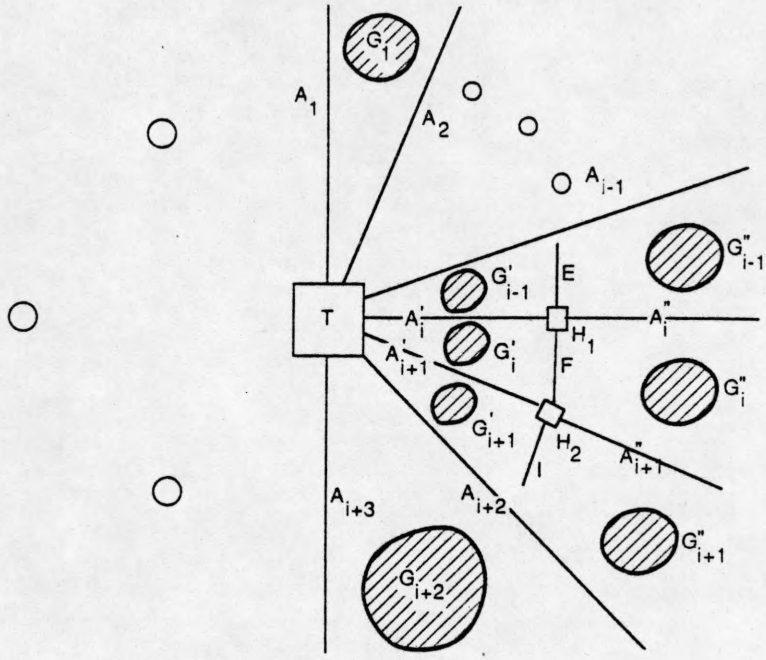

Figure 9.
Illustrating the proof of Lemma 3.

G''_{i-1} and G''_{i+1} are empty, and E belongs to the fringe of A_i, and I belongs to the fringe of $A_{i=1}$. Note that we don't need to use the assumption that there are no exceptions. A cross separating k-set can not cross three adjacent A_i's, since with respect to the middle A_i non of G''_5 and G''_6 can not be empty. Hence, all other separating k-set, except exceptions, belong to $G_i \cup A_i \cup$ its fringes in $G_{i-1} \cup A_{i+1} \cup$ its fringes in G_{i+1}.

□

Let us now consider exceptions. W.L.O.G. let there exist an exceptional separating k-set, which separates part of A_i. In other words, there is a separating k-set which separates part of A_i (A'_i), such that all of the vertices not in $A_i \cup T$ are neighbors of A'_i. The number of the neighbors of A'_i in $G_{i-1} \cup A_{i-1} \cup G_i \cup A_{i+1}$ is less than k. Consider the minimal set of subsets of A_i that covers all vertices of A_i which can be separated by some exceptional separating k-set. The number of subsets in this set is less than or equal to the cardinality of A_i, whence is at most $\frac{k-t}{2}$. The number of neighbors of A_i that are used for separating these subsets is less than or equal to k vertices per subsets, so their total is at most $\frac{k^2}{2}$. Note that $\frac{k^2}{2} - k$ such vertices can be inside either $G_{i-1} \cup A_{i-1}$ or $G_i \cup A_{i+1}$. Moreover, if $v \in A_i$ participates in some subset of A_i, that can be separated by an exceptional separating k-set, then v has less than k vertices in $G_{i-1} \cup A_{i-1} \cup G_i \cup A_{i+1}$. Hence, if we take the union of the following sets

1) $G_i \cup A_i \cup A_{i+1}$

2) the neighbors of A_i in $G_{i-1} \cup A_{i-1}$, that are used for exceptional separating k-sets

3) the fringe of A_i

4) the neighbors of A_{i+1} in $G_{i+1} \cup A_{i+2}$, that are used for exceptional separating k-sets

5) the fringe of A_{i+1} for all i's,

will contain all separating k-sets which are not covered by the structure.

The number of exceptional separating k-set for A_i is bounded by the number of different subsets of A_i. Hence, it is less than or equal to $2^{\frac{k-t}{2}}$. Thus, the number of exceptional separating k-sets is at most $(l+2)2^{\frac{k-t}{2}}$.

Based upon this Lemma and the above observation about exceptions, and using structure 1, we can write the following recurrence, which is valid if there are no type 1 or type 2 separating k-sets:

$$g(n) = \sum_{i=1}^{l+2} g(n_i + k(k-t) + t) + (l+2)(\frac{k-t}{2})k + t ,$$

where every term inside the sum covers one of the G_i's, and $(l+2)(\frac{k-t}{2}) + t$ is the upper bound on the size of the structure 1. Note that $\sum_{i=1}^{l+2} n_i + \frac{(l+2)(k-t)}{2} + t = n$. The solution to this recurrence is $O(kn + k^3)$ (see Appendix). Note that each $(n_i + k(k-t)+t)$ is less than n itself.

Analogously, the recurrence on the upper bound on the number of separating k-sets become

$$f(n) = \sum_{i=1}^{l+2} f(n_i + k(k-t)+t) + 2^{k-t} l \frac{l+2}{2} + 2^{\frac{k-t}{2}}(l+2) .$$

The solution to this recurrence is $O(2^k \frac{n^2}{k})$. Note that all cross separating k-set of type 3 are covered by these recurrences.

Now we will look at type 1. Let $T_2 \cap A = T'_2$, $T_2 \cap B = T''_2$, and $T_1 \cap T_2 = \overline{T}_2$. With respect to a new cross separating k-set which uses T_2 some G_i i=3,4,5,6 could be empty. Let us first look at a harder case when none of G_i i=3,4,5,6 are empty with respect to a new cross separating k-set.

A new cross separating k-set must cross C and D of the old cross separating k-set which uses T_1, otherwise the Claim 2 with respect to the new cross separating k-set will be violated (see Figure 10).

Second, $\overline{T}_2 = T_1$, otherwise Claim 2 will be contradicted for the old cross separating k-set.

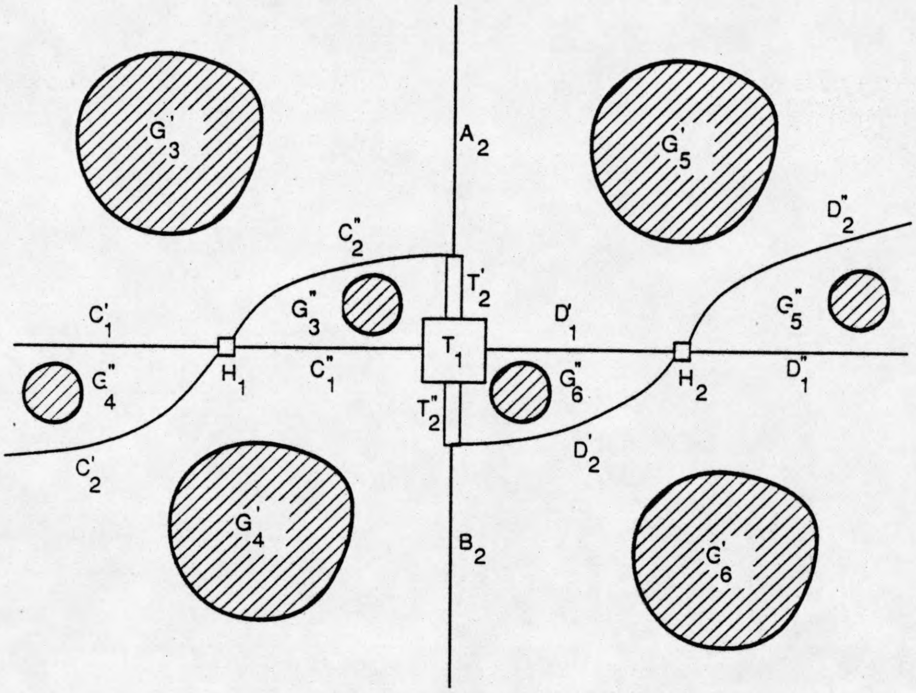

Figure 10.
Illustrating the configuration between two cross separating k-sets which use different T's.

Third, $C'_1+C'_2+H_1+T_1+T''_2$, $C''_1+C''_2+H_1+T_1+T'_2$, $D'_1+D'_2+H_2+T_1+T''_2$, and $D''_1+D''_2+H_2+T_1+T''_2$ are separating sets with cardinalities less than k, which separate $G''4$, G''_3, G''_6, and G''_5, respectively. Hence, G''_3, G''_4, G''_5, and G''_6 are empty.

Fourth, $C'_1+H_1+C''_2+T_2+D'_2+H_2+D''_2$, $C'_2+H_1+C''_2+T_2+D'_2+H_2+D''_1$, $C'_2+H_1+C''_1+T_2+D'_2+H_2+D''_2$, and $C'_2+H_1+T_2+D'_1+H_2+D''_2$ are separating sets. Hence, $|C'_1| \ge |C'_2|$, $|D'_1| \ge |D'_2|$, $|C''_1| \ge |C''_2|$, and $|D''_1| \ge |D''_2|$. Also, $C'_1+H_1+C''_2+T'_2+T_1+D'_1+H_2+D''_1$, $C'_2+T''_2+H_1+C''_1+T_1+D'_1+H_2+D''_1$, $C'_1+H_1+C''_1+T_1+T''_2+D'_2+H_2+D''_1$, and $C'_1+H_1+C''_1+T_1+T'_2+D'_1+H_2+D''_2$ are separating sets. Hence,

$$\begin{cases} |C'_2| + |T''_2| \ge |C'_1| \ge |C'_2| > 0 \\ |C''_2| + |T'_2| \ge |C''_1| \ge |C''_2| > 0 \\ |D'_2| + |T''_2| \ge |D'_1| \ge |D'_2| > 0 \\ |D''_2| + |T'_2| \ge |D''_1| \ge |D''_2| > 0 \end{cases}$$

Also since we are still in a Case 1 with respect to both old and new cross separating k-sets, we have the following equalities

$$\begin{cases} |T'_2| = |T''_2| \\ |A_2| = |B_2| = |D'_2| + |H_2| + |D''_2| = |C'_2| + |H_1| + |C''_2| \end{cases}$$

Note that the set T'_2 has edges to the set D''_1, the set T''_2 has edges to the set D'_1, the set T''_2 has edges to the set C'_1, and the set T'_2 has edges to the set C''_1, because of the Claim 2 with respect to the new cross separating k-set. Hence, the maximal disjoint sets for C's and D's (X and Y) will have cardinalities equal to 1.

Let us take a maximal T_2, and let us take the fringes of A_2, B_2, C and D (see Figure 11).

C'_1 does not have the fringe in G_4, otherwise part of C'_1 which has a fringe becomes a part of I'_1. If C'_1 has the fringe in G_3 then the part of C'_1 which has the fringe can be separated from the rest of the graph by a separating set C'_2+T''_2+T_1+ the fringe of C'_1 in G_3, whose cardinality is less than k. Hence, C'_1 does not have the fringe. Analogously, C''_1, D'_1, and D''_1 do not have the fringes. Symmetrically, T'_2 and T''_2 do not have the fringes.

Let $\hat{T}_2$ be the union of vertices which are used for all possible T_2 which create a cross separating k-sets with nonempty G_i i=3,4,5,6. Let $\hat{D}'_1$ be the union of all possible D'_1, $\hat{D}''_1$ be the union of all possible D''_1, $\hat{C}'_1$ be the union of all possible C'_1, $\hat{C}''_1$ be the union of all possible C''_1, $\hat{C}'_2$ be the union of all possible C'_2, $\hat{C}''_2$ be the union of all possible C''_2, $\hat{D}'_2$ be the union of all possible D'_2, and $\hat{D}''_2$ be the union of all possible D''_2. Let us show that all of these sets are disjoint.

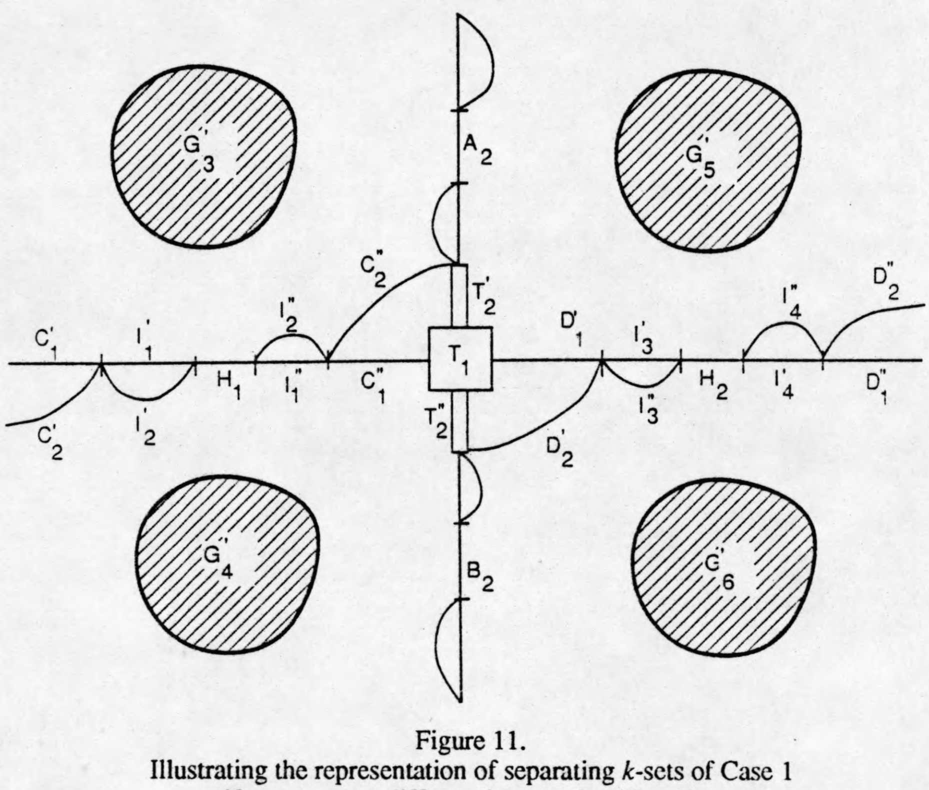

Figure 11.
Illustrating the representation of separating k-sets of Case 1
if two or more different intersecting T's exist.
(Structure 2).

Since all of them are symmetric we will prove it only for $\hat{C}'_1$ and $\hat{C}''_1$. Assume there are T_3 and T_4 such that C''_1 for T_3 is not disjoint from C'_1 for T_4. Then nonempty intersection of C''_1 for T_3 and C'_1 for T_4 is separated from the rest of the graph by a separating set C''_2 for $T_3 \cup T'_3 \cup T_1 \cup T''_4 \cup C'_2$ for T_4, whose cardinality is less than k. This contradiction proves the statement.

The cardinality of the union $\hat{D}''_2 \cup \hat{D}'_2 \cup I''_4 \cup I'_4$ is less than $\frac{k-t}{2}$, and analogously, the cardinality of $\hat{C}''_2 \cup \hat{C}'_2 \cup I'_1 \cup I''_2$ is less than $\frac{k-t}{2}$. Let us call $\hat{C}'_2$, $\hat{C}''_2$, $\hat{D}'_2$, and $\hat{D}''_2$ the *pseudofringe*. Note that A and B might have fringes , but by the symmetry $\hat{T}_2 - T_1$ does not have any fringes.

The structure which represent all separating k-sets for all possible T's will the following (structure 2):

1) the original separating k-set with its fringes,

2) the cross separating k-set with minimum cardinality T_1 with its fringes and pseudofringes,

3) for every nonempty G'_i i=3,4,5,6 we will fill all nonexistent edges of the complete graph on the neighbors of G'_i, if G'_i is empty for any i=3,4,5,6 we will fill these nonexistent edges of this complete graph by the virtual edges. (For G'_3 we fill the edges between the vertices of the fringe of A in G_3, T_1, $\hat{T}'_2$, part of A_2 which does not have any fringes, $\hat{C}'_1, I'_1, H_1, I''_2$ and $\hat{C}''_2$).

From the construction of the structure it is easy to see that this structure cavers all cross separating k-sets for all possible T's, of type 1. Let us see now where the rest of the separating k-sets lie, if we have separating k-sets of type 1.

If there exists T_2 with at least one of the G_i empty i=3,4,5,6, assuming it is not exception, such that there is another T_2 with $T_2 \cap T_1$ is nonempty along with nonempty $T_2 \cap B$ and $T_2 \cap A$, then all cross separating k-sets of this T_2 are covered by the above structure. (They belong to the fringes of A and/or B in G_1 or G_2 and the rest belong to the original cross separating k-set with its fringes or pseudofringes). So all cross separating k-sets are covered by this structure, assuming there are no exceptions, hence, all separating k-sets are either inside $G_1 \cup A \cup B \cup T_1 \cup$ the fringes of A and B in G_2, or $G_2 \cup A \cup B \cup T_1 \cup$ the fringes of A and B in G_1, or cross separating k-sets covered by the structure. Since the structure is symmetric, we can look at the cross separating k-sets where the original separating k-set is $C \cup D \cup T_1$. Then the pseudofringes of C and D become the pseudofringes of A and B. With respect to this separation of G all separating k-sets are either inside $G_3 \cup G_5 \cup C \cup D \cup T_1 \cup$ the fringe of C in G_4 and the fringe of

D in G_6, or inside $G_4 \cup G_6 \cup C \cup D \cup T_1 \cup$ the fringe of C in G_3 and the fringe of D in G_5, or separating k-sets covered by the structure. But since in both cases they are the same separating k-sets, all separating k-sets are either inside $G_3 \cup A \cup T_1 \cup C \cup$ the fringe of C in $G_4 \cup$ the fringe of A in G_5, or inside $G_4 \cup B \cup C \cup T_1 \cup$ the fringe of B in G_6, or inside $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6, or inside $G_6 \cup B \cup D \cup T_1 \cup$ the fringe of B in $G_4 \cup$ the fringe of D in G_5, or the separating k-sets covered by the structure. To cover all exceptions we will do what we did for types 3 and 4 separating k-sets, we will add $k(k-t)$ neighbors of A, B, C and D to each of G_3, G_4, G_5 and of G_6 which can participate in exceptional separating k-sets. Hence, the size of representation is

$$g(n) = \sum_{i=1}^{4} g(n_i + k(k-t)+t) + 8\frac{(k-t)}{2}k + t ,$$

where every term inside the sum covers one of G_i i=3,4,5,6 along with its appropriate neighbors and fringes, and $8\frac{(k-t)}{2}k + t$ is the upper bound on the size of the structure. Note that $\sum_{i=1}^{4} n_i + 2k - t = n$, hence the solution to the above recurrence is $O(nk + k^3)$ (see Appendix). The number of exceptional separating k-sets is upper bounded by $4\,2^{\frac{k-t}{2}}$. The upper bound on the number of separating k-sets become

$$f(n) = \sum_{i=1}^{4} f(n_i + k(k-t)+t) + \binom{4}{2} \cdot 2^{k-t} + 4 \cdot 2^{\frac{k-t}{2}} .$$

The solution to it is $O(2^k n + 2^k k^2)$ (see Appendix).

Let us now see what happens if we are in type 2 and no separating k-sets of type 1 exist. W.L.O.G. assume there is a separating k-set which uses $T_2 = T'_2 \cup \bar{T}_2$, where $T'_2 \in A$ and $\bar{T}_2 \in T_1$, and no separating k-set of type 1 exist (see Figure 12).

If G_i's i=3,4,5,6 are nonempty with respect to a new cross separating k-set then we become in the Case 1 with respect to a new cross separating k-set, hence $|A_2| = |B|$ which is impossible. Hence, one of the G_i i=3,4,5,6 with respect to a new cross separating k-set must be empty. W.L.O.G. let the empty G_i be either G_3 or G_4 with respect to the new cross separating k-set. If G_4 is empty then G_5 with respect to the new cross separating k-set must be empty, otherwise $T_1 \cup T'_2 \cup A_2 \cup D_2$ of the new cross separating k-set becomes a separating set with cardinality less than k. Hence, if G_4 is empty then all cross separating k-set of type 2 belong to the original separating k-set with its fringes. Then all separating k-set are either inside $G_1 \cup A \cup B \cup T_1 \cup$ the fringe of A in $G_5 \cup$ the fringe of B in G_6, or $G_2 \cup A \cup B \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of B in G_4, or they belong to the union of $A \cup B \cup T_1 \cup$ the fringes of A and B. Note that the latter separating k-sets are covered by the structure 2. We can write the recurrences

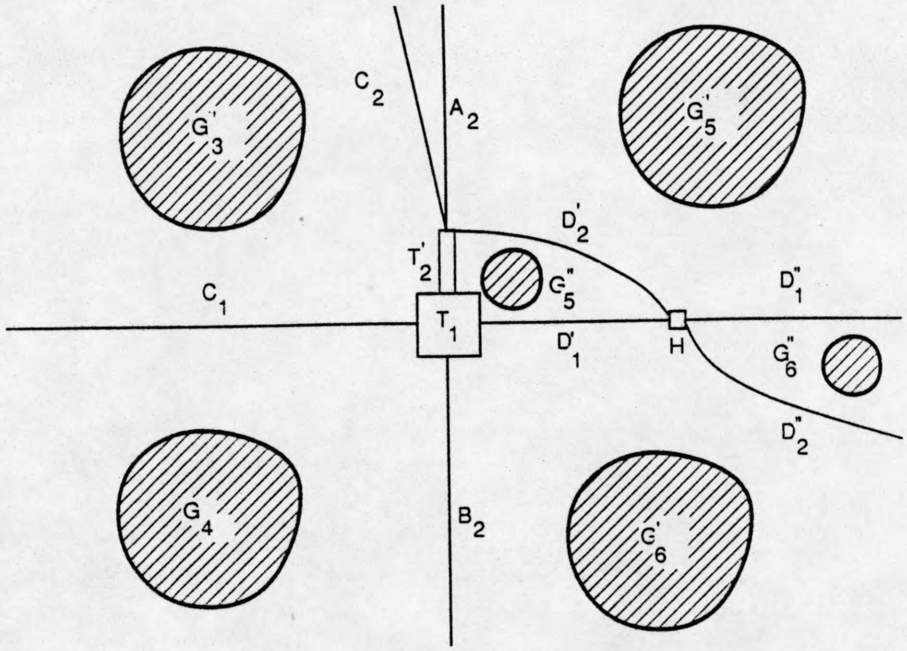

Figure 12.
Illustrating type 2 separating k-set when no type 1 separating k-set exist.

similar to the above ones except for the sum which will be up to 2 instead of up to 4. The solution will be still of the same order. If G_3 is empty then $|C_2| \ge |A_2|$, otherwise $C_2 \cup T'_2 \cup T_1 \cup B$ is a separating set with cardinality less than k. If D_2 crosses D_1 (see Figure 12) then $A_2 \cup T'_2 \cup T \cup D_2$ is a separating set, so $|C_2| = |A_2|$. $C \cup T_1 \cup D'_1 \cup H \cup D''_2$ is a separating set, so $|D''_2| \ge |D''_1|$. Also $C_2 \cup T_2 \cup D'_2 \cup H \cup D''_1$ is a separating set, so $|D''_1| \ge |D''_2|$. Combining these two we get $|D''_1| = |D''_2|$. Since, $C \cup T_1 \cup T'_2 \cup D'_2 \cup H \cup D''_1$ and $C_2 \cup T'_2 \cup T_1 \cup D'_1 \cup H \cup D''_2$ are separating sets, so $|T'_2 \cup D'_2| \ge |D'_1| \ge |D'_2|$. Since $T_1 \cup D''_1 \cup H \cup D''_2$ separates G''_6 from the rest of the graph, and since the cardinality of this separating set is less than k, G''_6 is empty. Hence, D''_2 belongs to the fringe of D in G_6. $\bar{T}_2 = T_1$ in order for the Claim 2 with respect to the old cross separating k-set to be true. And since $|C_2| + |T'_2| = |A|$ and since the cardinality of the new cross separating k-set is k, $|D'_2| = |D'_1|$. So, all cross separating k-sets of this type belong to $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6, if there are no exceptional separating k-sets. Also in the maximal set of disjoint D's (Y) all of D's except D_1 belong to G_6. If G_5 with respect to the new cross separating k-set is nonempty, then by the above argument C_2 will belong to the fringe of A. Hence, all cross separating k-sets belong to the set mentioned above, namely, $G_4 \cup A \cup T \cup D_1 \cup$ the fringe of A in $G_1 \cup$ the fringes of D_1 in G_5.

Let us take the maximal set of C's and D's (X and Y). We know that all cross separating k-sets of type 2 with nonempty G_5 belong to $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6. Since we need to consider all symmetric cases, and since we don't have any cross separating k-sets of type 1, all cross separating k-sets of the type 2 belong to $G_3 \cup A \cup C \cup T_1 \cup$ the fringe of A in $G_5 \cup$ the fringe of C in G_4, or $G_4 \cup B \cup C \cup T_1 \cup$ the fringe of B in $G_6 \cup$ the fringe of C in G_3, or $G_5 \cup A \cup D \cup T_1 \cup$ the fringe of A in $G_3 \cup$ the fringe of D in G_6, or $G_6 \cup B \cup D \cup T_1 \cup$ the fringe of B in $G_4 \cup$ the fringe of D in G_5. Note that C's and D's are not the same in these sets. In case of G_3 C is "nearest" to A, in case of G_4 C is "nearest" to B, in case of G_5 D is "nearest" to A, and in case of G_6 D is "nearest" to B. Let us see where the rest of separating k-sets must lie. First, if there are no cross separating k-sets with G_5 nonempty (or same other appropriate symmetric G_i i=3,4,5,6) then it is still possible to have a cross separating k-sets.

All cross separating k-sets consist of three parts: part one is in G_1, part two is in G_2 and part three is T_1. Part one belongs to some C from the set X or its fringe or the fringe of A in G_3 or the fringe of B in G_4. Part two belongs to some D from the set Y or its fringe or the fringe of A in G_5 or the fringe of B in G_6. That covers all cross separating k-sets which use T_1, otherwise either set X or set Y is not maximal. We don't have any cross separating k-sets of type 1. All cross separating k-sets of type 2 with nonempty appropriate G_i with respect to them belong to the part of the graph between A and the nearest D in G_2 along with A and its fringe and D and its fringe. Hence, all other separating k-sets belong to $G_1 \cup A \cup B \cup T_1$ with its fringes, or $G_2 \cup A \cup B \cup T_1$ with its fringes.

Hence, all cross separating k-sets of type 2, except exceptions are covered by the structure 2 or inside the the subgraphs associated by G_1, G_{l_1+1}, G_{l_1+2} and G_{l+2}. As for the exceptions the upper bounds we got for types 3 and 4 still hold, since no part of T_1 can be separated by them (otherwise Claim 2 is contradicted). So, the recurrence which were written for the type 3 and 4 separating k-sets covers type 2 cross separating k-sets also, including exceptions. That conclude Case 1.

$\square$

Case 2 For any separating k-set every cross separating k-set will have one of the G_i i=3,4,5,6 empty. Not every vertex in both G_1 and G_2 can be used for cross separating k-sets.

W.L.O.G. let G_3 will be empty (see Figure 13).

Since G_4 is nonempty by assumption, and G_5 is nonempty since there are no exception, $C \cup T \cup B$ and $A \cup T \cup D$ are separating sets. So their cardinalities are bigger or equal to k, hence, $|C| = |A|$ and $|B| = |D|$. So, C is part of the fringe of A in G_1. Since this true for every T, all cross separating k-sets belong to $G_1 \cup A \cup T \cup B \cup$ the fringes of

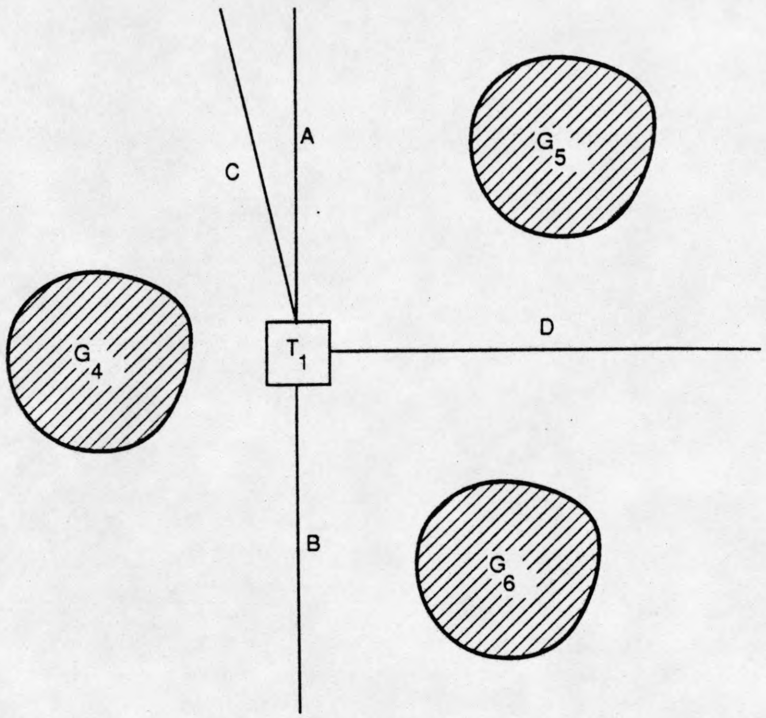

Figure 13.
Illustrating Cases 2 and 3.

A and B in G_2, or $G_2 \cup A \cup T \cup B \cup$ the fringes of A and B in G_1, except for exceptions. So all separating k-sets including the exceptions are either inside $G_1 \cup A \cup B \cup T \cup$ appropriate at most k^2 neighbors of $A \cup T \cup B$ in G_2 or inside $G_2 \cup A \cup B \cup T \cup$ appropriate at most k^2 neighbors of $A \cup T \cup B$ in G_1 which are used in exceptional separating k-sets. Hence,

$$g(n) = g(n_1 + k(k-1)) + g(n_2 + k(k-1)) + 4k^2,$$

where n_1 and n_2 are the cardinalities of G_1 and G_2. We still have that $n_1 + n_2 + k = n$, and the solution to this recurrence is $O(k^2 + n)$ (see Appendix). Note that $n_i + k(k-1) < n$ for i=1,2.

For the upper bound on the number of separating k-sets we get the following equality

$$f(n) = f(n_1 + 2k) + f(n_2 + 2k) + 2^k,$$

where 2^k covers all exceptional separating k-sets. And its solution is clearly smaller than $O(2^k \frac{n^2}{k})$ (see Appendix). That conclude Case 2.

□

Case 3 For every separating k-set all cross separating k-sets are lopsided (one of the G_i i=3,4,5,6 will be empty). And either G_1 or G_2 are such that every vertex of them is used for some cross separating k-set.

W.L.O.G. let G_3 be empty and the smallest G_1 every vertex of G_1 is used for some cross separating k-set (see Figure 13). There are two subcases: either G_5 or G_6 are empty, otherwise we will be in Case 2. Take C as large as possible.

If G_6 is empty then $A \cup B \cup C \cup D \cup T$ with all edges between them and filling real edges for nonempty G_5 and G_4 and virtual otherwise (analogous to the structure 1) will specify all cross separating k-sets. If G_5 is empty then $C \cup T \cup D$ separate A from the rest of the graph. Hence, $C \cup T \cup D$ is an exceptional separating k-set. So the third structure will be the following:

1) A, B and T - the original separating k-set,

2) All the neighbors of $A \cup B \cup T$ that are used for a cross separating k-sets with edges between them and the original separating k-set.

since the remaining separating k-sets are inside $G_2 \cup A \cup B \cup T$, we derive the following recurrence relation:

$$g(n) = g(n-1) + k^2,$$

whose solution is $f(n) = O(k^2 n)$. Analogously, we have the following recurrence relation for the upper bound on the number of separating k-sets

$$f(n) = f(n-1) + 2^k,$$

whose solution is $O(2^k n)$.

□

That conclude the proof of all cases. Our final result is that all separating k-sets have $O(k^2 n)$ space representation, and their number is $O(2^k \frac{n^2}{k})$.

REFERENCES

[BeX] M. Becker et al., "A probabilistic algorithm for vertex connectivity of graphs," *Information Processing Letters*, vol. 15, no. 3, 1982, pp. 135-136.

[Ev] S. Even, *Graph Algorithms*, Computer Science Press, Rockville, MD, 1979.

[Ev2] S.Even, "An algorithm for determining whether the connectivity of a graph is at least k," *SIAM J. on Computing*, vol. 4, 1975, pp. 393-396.

[EvTa] S.Even,R.E.Tarjan, "Network flow and testing graph connectivity," *SIAM J. on Computing*, vol. 4, 1975, pp.507-518.

[Ga] Z.Galil, "Finding the vertex connectivity of graphs," *SIAM J. on Computing*, vol. 9, 1980, pp. 197-199.

[GiSo] M.Girkar,M.Sohoni, "On finding the vertex connectivity of graphs," *Tech. Report ACT-77*, Coordinated Science Laboratory, University of Illinois, Urbana, Il, May, 1987

[HoTa] J.E.Hopcroft,R.E.Tarjan, "Dividing a graph into triconnected components," *SIAM J. on Computing*, vol. 2, 1973, pp. 135-158.

[Ka] A.Kanevsky, "On the number of minimum size separating vertex sets in a graph," *Tech. Report ACT-80*, Coordinated Science Laboratory, University of Illinois, Urbana, IL, July, 1987.

[KaRa1] A.Kanevsky,V.Ramachandran, "Improved algorithms for graph four-connectivity," *Proc. 28th IEEE ann. Symp. on Foundation of Computer Science*, Oct, 1987, pp. 252-259.

[KaRa2] A.Kanevsky,V.Ramachandran, "A characterization of separating pairs and triplets in a graph," *Tech. Report ACT-79*, Coordinated Science Laboratory, University of Illinois, Urbana, IL, July, 1987.

[LiLoWi] N.Linial,L.Lovasz,A.Wigderson, "A physical interpretation of graph connectivity, and its algorithmic applications," *Proc. 27th IEEE Ann. Symp. on Foundations of Comp. Sci.*, 1986, pp. 39-48.

[MiRa] G.L.Miller,V.Ramachandran, "A new graph triconnectivity algorithm and its parallelization," *Proc. 19th ACM Ann. Symp. on Theory of Computing*, New York, NY, May 1987, pp. 335-344.

[Ta] R.E.Tarjan, "Depth-first search and linear graph algorithms," *SIAM J. on Computing*, vol. 1, 1972, pp. 146-159.

[Tu] W. T. Tutte, *Connectivity in Graphs*, University of Toronto Press, 1966.

[Wh] H. Whitney, "Non-separable and planar graphs," *Trans. Amer. Math. Soc.* 34, 1932, pp.339-362.

APPENDIX

$$\sum_{i=1}^{l}(n_i + 1) = n \qquad 2 \le l \le n \qquad n_i \ge 0$$

$$g(n) \le \max_l(\sum_{i=1}^{l} g(n_i + 2) + 4l)$$

Let $g(n) = 4n - 16$,

$$g(n) \le \max_l(\sum_{i=1}^{l} g(n_i + 2) + 4l) = \max_l(\sum_{i=1}^{l}(4(n_i + 2) - 16) + 4l) =$$

$$\max_l(4\sum_{i=1}^{l}(n_i + 1) + 4l - 16l + 4l) = \max_l(4n - 8l) \le 4n - 16$$

$$\sum_{i=1}^{l}(n_i + 1) + 1 = n \qquad 2 \le l \le n-1 \qquad n_i \ge 0$$

$$g(n) \le \max_l(\sum_{i}^{l} g(n_i + 5) + 6l + 1)$$

Let $g(n) = 6n - 55$,

$$g(n) \le \max_l(\sum_{i=1}^{l} g(n_i + 5) + 6l + 1) = \max_l(\sum_{i=1}^{l}(6(n_i - 55) + 6l + 1) =$$

$$\max_l(6(\sum_{i=1}^{l}(n_i + 1) + 1) - 31l + 6l + 1) = \max_l(6n - 25l - 5) \le 6n - 55$$

$$\sum_{i=1}^{l}(n_i + \frac{k-t}{2}) + t = n \qquad 0 \le t \le k-2 \qquad 2 \le l \le 2\frac{n-t}{k-t} \qquad n_i \ge 0$$

$$g(n) \le \max_l(\sum_{i=1}^{l} g(n_i + (k-t)k + t) + lk\frac{(k-t)}{2} + t$$

Let $g(n) = 2nk - 4k^3 + 2k^2t + \frac{1}{2}k^2 - 3kt - t$,

$$g(n) \le \max_l \left(\sum_{i=1}^{l} g(n_i + (k-t)k + t) + lk\frac{k-t}{2} + t\right) \le$$

$$\max_l \left(\sum_{i=1}^{l} 2k(n_i + k(k-t) + t) - 4k^3 l + 2k^2 tl + \frac{1}{2}k^2 l - ktl - tl + lk\frac{k-t}{2} + t\right) =$$

$$\max_l \left(2k\left(\sum_{i=1}^{l}\left(n_i + \frac{k-t}{2}\right) + t\right) - 2kl\frac{k-t}{2} - 2kt + 2k^2 l(k-t) + 2ktl - 4k^3 l + 2k^2 tl + \frac{1}{2}k^2 l - 3ktl - tl + lk\frac{k-t}{2} + t\right) =$$

$$\max_l \left(2kn + 2k^3(l-2l) + 2k^2 t(-l+l) + k^2\left(\frac{1}{2}l + \frac{l}{2} - l\right) + kt\left(l - 2 + 2l - \frac{l}{2} - 3l\right) + t(-l+1)\right) \le$$

$$2kn - 4k^3 - 3kt + t \le 2kn - 4k^3 + 2k^2 t + \frac{1}{2}k^2 - 3kt - t$$

Hence, $g(n) = O(nk + k^3)$.

$$\sum_{i=1}^{l}\left(n_i + \frac{k-t}{2}\right) + t = n \qquad 2 \le l \le 2\frac{k-t}{k-t} \qquad 0 \le t \le n-2$$

$$f(n) \le \max_l \left(\sum_{i=1}^{l} f(n_i + k(k-t) + t) + 2^{k-t}\frac{l(l-2)}{2} + 2^{\frac{k-t}{2}} l\right)$$

Let

$$f(n) = 2^{k-t}nl - 2^{k-t}k^2 l + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - \frac{3}{2}2^{k-t}tl + 2^{k-t}kt + \frac{1}{2}2^{k-t}k - 2\,2^{k-t}k^2 - 2^{k-t}t - \frac{1}{2}2^{k-t}l - 2\,2^{\frac{k-t}{2}},$$

$$f(n) \le \max_l \left(\sum_{i=1}^{l}(n_i k(k-t) + t)2^{k-t}l - 2^{k-t}k^2 l^2 + 2^{k-t}ktl^2 + \frac{1}{2}2^{k-t}kl^2 - \frac{3}{2}2^{k-t}tl^2 + 2^{k-t}ktl +\right.$$

$$\left.\frac{1}{2}2^{k-t}kl - 2\,2^{k-t}k^2 l - 2^{k-t}tl - \frac{1}{2}2^{k-t}l^2 - 2\,2^{\frac{k-t}{2}} + \frac{1}{2}2^{k-t}l^2 - \frac{1}{2}2^{k-t}l + 2^{\frac{k-t}{2}} l\right) = \max_l (2^{k-t}\ln -$$

$$\frac{1}{2}2^{k-t}kl^2 + \frac{1}{2}2^{k-t}tl^2 - 2^{k-t}tl + 2^{k-t}k^2 l^2 - 2^{k-t}ktl^2 + 2^{k-t}tl^2 - 2^{k-t}k^2 l^2 + 2^{k-t}ktl^2 + \frac{1}{2}2^{k-t}kl^2 -$$

$$\frac{3}{2}2^{k-t}tl^2 + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - 2\,2^{k-t}k^2 l - 2^{k-t}tl - \frac{1}{2}2^{k-t}l^2 - 2\,2^{\frac{k-t}{2}} l + \frac{1}{2}2^{k-t}l^2 - \frac{1}{2}2^{k-t}l + 2^{\frac{k-t}{2}} l) =$$

$$\max_l \left(2^{k-t}\ln - 2\,2^{k-t}k^2 l + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - 2\,2^{k-t}kl - 2\,2^{k-t}tl - \frac{1}{2}2^{k-t}l - 2^{\frac{k-t}{2}} l\right) \le$$

$$\max_l \left(2^{k-t}\ln - 2^{k-t}k^2 l + 2^{k-t}ktl + \frac{1}{2}2^{k-t}kl - \frac{3}{2}2^{k-t}tl + 2^{k-t}kt + \frac{1}{2}2^{k-t}k - 2\,2^{k-t}k^2 - 2^{k-t}t - \frac{1}{2}2^{k-t}l - 2\,2^{\frac{k-t}{2}}\right)$$

Hence, $f(n) = O\left(2^k \frac{n^2}{k} + 2^k nk\right)$.

$$\sum_{i=1}^{4} n_i + 2k - t = n \qquad 0 \le t \le k-2$$

$$g(n) \le \sum_{i=1}^{4} g(n_i + k(k-t) + t) + 8k\frac{k-t}{2} + t$$

Let $g(n) = 4nk - \frac{16}{3}k^3 + \frac{16}{3}k^2 t + \frac{4}{3}k^2 - \frac{16}{3}kt - \frac{1}{3}t$,

$$g(n) \le \sum_{i=1}^{4} g(n_i + k(k-t) + t) + 4(k-t)k + t \le$$

$$\sum_{i=1}^{4}(4(n_i + k(k-t) + t)k - \frac{16}{3}k^3 + \frac{16}{3}k^2 t + \frac{4}{3}k^2 - \frac{16}{3}kt - \frac{1}{3}t) + 4(k-t)k + t =$$

$$4k(\sum_{i=1}^{4} n_i + 2k - t) - 8k^2 + 4kt + 16k^3 - 16k^2 t + 16kt - \frac{64}{3}k^3 + \frac{64}{3}k^2 t + \frac{16}{3}k^2 - \frac{64}{3}kt - \frac{4}{3}t + 4k^2 - 4kt + t =$$

$$4kn + k^3(16 - \frac{64}{3}) + k^2 t(\frac{64}{3} - 16) + k^2(\frac{16}{3} - 8 + 4) + kt(4 + 16 - \frac{64}{3} - 4) + t(1 - \frac{4}{3}) =$$

$$4kn - \frac{16}{3}k^3 + \frac{16}{3}k^2 t + \frac{4}{3}k^2 - \frac{16}{3}kt - \frac{1}{3}t$$

Hence, $g(n) = O(nk + k^3)$.

$$\sum_{i=1}^{4}(n_i + \frac{k-t}{2}) + t = n \qquad 0 \le t \le n-2$$

$$f(n) \le \sum_{i=1}^{4} f(n_i + k(k-t) + t) + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}}$$

Let $f(n) = 2^{k-t}n - \frac{4}{3}2^{k-t}k^2 + \frac{4}{3}2^{k-t}kt - \frac{5}{3}2^{k-t}t + \frac{2}{3}2^{k-t}k - 2\,2^{k-t} - \frac{4}{3}2^{\frac{k-t}{2}}$,

$$f(n) \le \sum_{i=1}^{4} f(n_i + k(k-t) + t) + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}} \le \sum_{i=1}^{4}(2^{k-t}(n_i + k(k-t) + t) - \frac{4}{3}2^{k-t}k^2 + \frac{4}{3}2^{k-t}kt -$$

$$\frac{5}{3}2^{k-t}t + \frac{2}{3}2^{k-t}k - 2\,2^{k-t} - \frac{4}{3}2^{\frac{k-t}{2}}) + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}} = 2^{k-t}n - 2^{k-t}k + 2\,2^{k-t}t - 2^{k-t}t +$$

$$4\,2^{k-t}k^2 - 4\,2^{k-t}kt + 4\,2^{k-t}t - \frac{16}{3}2^{k-t}k^2 + \frac{16}{3}2^{k-t}kt - \frac{20}{3}2^{k-t}t + \frac{8}{3}2^{k-t} - \frac{16}{3}2^{\frac{k-t}{2}} + 6\,2^{k-t} + 4\,2^{\frac{k-t}{2}} =$$

$$2^{k-t}n - \frac{4}{3}2^{k-t}k^2 + \frac{4}{3}2^{k-t}kt - \frac{5}{3}2^{k-t}t + \frac{2}{3}2^{k-t}k - 2\,2^{k-t} - \frac{4}{3}2^{\frac{k-t}{2}}$$

$$n_1 + n_2 + k = n \qquad n_1, n_2 \geq 0$$

$$g(n) \leq g(n_1 + k(k-1)) + g(n_2 + k(k-1)) + 4k^2$$

Let $g(n) = n - 6k^2 + 3k$,

$$g(n) \leq n_1 + k^2 - k - 6k^2 + 3k + n_2 + k^2 - k - 6k^2 + 3k + 4k^2 = n - 6k^2 + 3k$$

$$n_1 + n_2 + k = n \qquad n_1, n_2 \geq 0$$

$$f(n) \leq f(n_1 + 2k) + f(n_2 + 2k) + 2^k$$

Let $f(n) = 2^k n - 3\,2^k k - 2^k$,

$$f(n) \leq 2^k n_1 + 2k2^k - 3\,2^k k - 2^k + 2^k n_2 + 2k2^k - 3\,2^k k - 2^k + 2^k = 2^k n - 3\,2^k k - 2^k$$