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THE ELECTROSTATIC INSTABILITY OF A BEAM OF CHARGED PARTICLES PENETRATING A PLASMA

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Abstract

The stability of an infinite beam of charged particles penetrating a plasma against small amplitude electrostatic perturbations is investigated.

In Section A the dispersion relation is derived, taking into account the velocity distribution of the beam particles.

In Section B is is shown that in the presence of collisions the electrostatic instabilities can be quenched if the beam particles have a velocity spread.

In Section C the results are applied to a beam of finite cross section held together by its own magnetic field. The criteria for the stability of such a beam against electrostatic perturbations is derived.

A. Derivation of the Dispersion Relation

We make the following assumptions and approximations:

1. In the steady state we treat beam and plasma as uniform over all space.

2. We neglect the effect of all magnetic forces on the plasma.

3. We neglect the thermal velocities of the plasma electrons.

4. We neglect the effect of the magnetic forces on the beam particles due to the steady state magnetic field produced by the unperturbed beam current; in short, we do not consider the pinch effect.

5. We assume that the changes in beam current density (or charge density) are small compared to the current (or charge) density produced in the plasma as a result of the perturbing fields.

6. We treat only small perturbations in the linear approximations.

7. We consider a single Fourier component, so that each first order quantity varies in space and time as:

$$\exp \left\{ i \vec{k} \cdot \vec{x} - i\omega t \right\}$$

If we suppose that we are interested in an initial value problem (in external forces after t = 0) we must then consider $\vec{k}$ to be real.

From these assumptions it follows that the first order plasma current density has the form:

$$\frac{4\pi i J_\alpha^{(\text{plasma})}}{\omega} = (\varepsilon - 1) E_\alpha$$

where the dielectric constant $\mathcal{E}$ of the plasma is given by:

$$\mathcal{E} = 1 = \frac{\omega_p^2}{\omega(\omega + i/\tau\ )} \tag{2}$$

$1/\tau$ is an effective collision frequency for the plasma electron (which may depend slightly on the frequency ω). ω_p is of course the usual "plasma frequency" for the electrons in the stationary plasma.

Similarly we can show that the first order change in the beam current has the form:

$$\frac{4\pi i J_\alpha^{(beam)}}{\omega} = K_{\alpha\beta}\ E_\beta \tag{3}$$

$\frac{K_{\alpha\beta}}{4\pi}$ may be called the susceptibility of the beam: the expression (3) is meant to include the effects of the first order magnetic field; we can include these effects in the form (3) by expressing $\vec{B}$ in terms of $\vec{E}$ via Faraday's law $\left[\vec{B} = \frac{c}{\omega}\vec{k} \times \vec{E}\right]$. An explicit calculation of the tensor $K_{\alpha\beta}$ will be given later. For the time being we shall make (5) more specific by the:

$$\text{assumption:}\ \left|K_{\alpha\beta}\right| << 1 \tag{4}$$

We can now write down Ampère's law:

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\left\{\vec{J}^{plasma} + \vec{J}^{beam}\right\}$$

for the first order quantities. Substituting (1) and (3) in the R. H. S. and $\vec{B} = \frac{c\vec{k} \times \vec{E}}{\omega}$ in the L. H. S., together with $\vec{\nabla} = i\vec{k}$; $\frac{\partial}{\partial t} = -i\omega$ gives immediately:

$$\left\{ (\delta_{\alpha\beta} \; k^2 - k_\alpha k_\beta) - \frac{\omega^2}{c^2} \; K_{\alpha\beta} \right\} \; E_\beta = (\mathcal{E} \; \frac{\omega^2}{c^2}) \; E_\alpha \tag{5}$$

The set of 3 equations (5) will then give us the eigenfrequencies and eigenmodes for our problem; it is convenient to think of $\lambda = \mathcal{E} \frac{\omega^2}{c^2}$ as the eigenvalue. To solve the eigenvalue problem we now make use of the assumption (4) and use standard perturbation methods to find the first order effect of the beam.

First we must find the eigenvalues and mode to zero-th order in $K_{\alpha\beta}$, i.e., without a beam. The three eigenvalues and modes are easily found: (a^o) $\lambda^o = \mathcal{E} \frac{\omega^2}{c^2} = 0$ with $\vec{E}$ parallel to $\vec{k}$. This mode corresponds to the familiar plasma oscillation. If the collision frequency $1/\tau$ is small compared to the plasma frequency ω_p the corresponding frequency is given by:

$$\omega = \pm \omega_p - \frac{i}{2\tau}$$

The mode is obviously damped in the absence of a beam.

(b^o) $\lambda^o = \mathcal{E} \frac{\omega^2}{c^2} = k^2$ with $\vec{E}$ perpendicular to $\vec{k}$. There are two independent modes, with--if we so choose--mutually orthogonal $\vec{E}$. These are ordinary transverse electromagnetic waves. The relationship between frequency and wavenumber (for $\frac{1}{\tau} \ll \omega_p$) is given by:

$$\omega = \pm \sqrt{\omega_p^2 + k^2c^2} - \frac{1}{2} \frac{i}{\tau} \frac{\omega_p^2}{\omega_p^2 + k^2c^2} \qquad *)$$

We note that in the electrostatic mode the phase velocity can take on any value, while in the transverse modes the phase velocity is always greater than the speed of light. We shall see later that the instability which can arise in the electrostatic mode is due to the fact that when the phase velocity is close to (slightly less than) the beam velocity energy can be extracted from the beam to supply the collision losses and to increase the amplitude of oscillations. Since this possibility is not present in the transverse modes we shall not discuss them in detail.

Having identified the unperturbed (no beam) eigenstates we can now proceed to calculate to first order the effect of the beam. The usual methods familiar from quantum mechanics give immediately the first order correction to the eigenvalue due to the perturbation $-\frac{\omega^2}{c^2} K_{\alpha\beta}$. The correction to λ is just $-\frac{\omega^2}{c^2} l_\alpha K_{\alpha\beta} l_\beta$, where l_α is a normalized (unit length) eigenvector to zero-th order. We thus find for the various modes:

$$\lambda^0 + \lambda^1 = 0 - \frac{\omega^2}{c^2} \frac{k_\alpha K_{\alpha\beta} k_\beta}{k^2} = \mathcal{E} \frac{\omega^2}{c^2} \qquad (a^1)$$

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*) The dispersion equation has a third root:

$$\omega = \frac{-k^2c^2}{k^2c^2 + \omega_p^2} \frac{i}{\tau}$$

or

$$\mathcal{E} + \frac{k_\alpha \, K_{\alpha\beta} \, k_\beta}{k^2} = 0.$$ Electrostatic mode to first order in $K_{\alpha\beta}$.

$$\lambda^0 + \lambda^1 = k^2 - \frac{\omega^2}{c^2} \; l_\alpha \; K_{\alpha\beta} \; l_\beta = \mathcal{E} \, \frac{\omega^2}{c^2} \qquad (b^1)$$

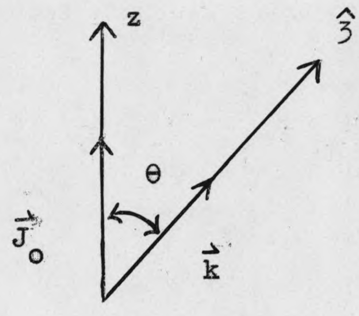
or:

$\mathcal{E} + l_\alpha \, K_{\alpha\beta} \, l_\beta = k^2$. Transverse mode to first order in $K_{\alpha\beta}$. l_α is a unit vector (polarization vector) perpendicular to the wave vector $\vec{k}$. We have the transparent result that the effect of the beam is accounted for by adding the appropriate diagonal term of the beam susceptibility to the susceptibility of the plasma.

We shall not consider the transverse modes further and rewrite the dispersion equation for the electrostatic mode in the form:

$$\mathcal{E} + K_{33} = 0 \qquad (6)$$

where the axis $\hat{3}$ points along the wave vector $\vec{k}$; we shall denote by $\hat{z}$ the axis which points along the unperturbed beam current, and by θ the angle between the beam and the wave vector:



We proceed now to calculate the tensor $K_{\alpha\beta}$ which relates the first order change in beam current to the electric field of the perturbation. We shall give a derivation only for the K_{33} component, which is very simple and is the only one needed for our purpose. In this case we only need to compute the (1[st] order) beam current density J_3 along k due to an electric field in the same direction, for in this particular case there is no magnetic field ($\vec{k} \times \vec{E} = 0$).

For simplicity, we carry out the derivation as though all beam particles had the same unperturbed velocity, and average over the velocity spectrum after we are done.

To first order we have: $\vec{J} = \rho_o \ \delta\vec{v} + \vec{v}_o \ \delta\rho$, where $\vec{J}$ is the first order change in beam current, ρ_o and $\vec{v}_o$ are the unperturbed charge density of the beam velocity, and $\delta\rho$, $\delta\vec{v}$ are the corresponding first order changes. $\vec{J}$ and $\delta\rho$ are related because of conservation of charge by $\delta\rho = \dfrac{\vec{k} \cdot \vec{J}}{\omega}$ and $\delta\rho$ can be eliminated:

$$\vec{J} = \rho_o \left\{ \delta\vec{v} + \vec{k} \cdot \delta\vec{v} \, \frac{\vec{v}}{\Omega} \right\}; \qquad \Omega \equiv \omega - k \cdot v$$

For the component along $\vec{k}$ this is simply:

$$J_3 = \rho_o \ \frac{\omega}{\Omega} \ \delta v_3$$

The next step is to write Newton's second law (relativistic version):

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} \left[\frac{U}{c^2} \ \vec{v} \right] = \frac{U}{c^2} \ \frac{d\vec{v}}{dt} + \frac{\vec{v}}{c^2} \ (\vec{F} \cdot \vec{v})$$

With the usual notation $\frac{U}{c^2} = M \cdot \gamma$ this gives:

$$\frac{d\vec{v}}{dt} = \frac{1}{M\gamma} \left\{ \vec{F} - \frac{(\vec{F} \cdot \vec{v})\vec{v}}{c^2} \right\}$$

To obtain an integral, we note that--to lowest orders--the time variation of $\vec{F}$ along the path of a beam particle is given by a factor:

$$\exp\left\{ i\,\vec{k} \cdot \vec{x} - i\omega t \right\} \approx \exp\left\{ - i\Omega t \right\}.$$ The integral is therefore:

$$\delta\vec{v} = \frac{i}{M\gamma\Omega} \left\{ \vec{F} - \frac{(\vec{F} \cdot \vec{v})\,\vec{v}}{c^2} \right\}$$

In our special case $\vec{F} = e\,E_3\,\hat{3}$ $\left\{ \text{in general } \vec{F} = e \left\{ \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right\} = e \left\{ \vec{E} + \frac{\vec{v}}{\omega} \times (\vec{k} \times \vec{E}) \right. \right.$. Putting all of our results together we get the general result:

$$K_{\alpha\beta} = -\omega_T^2 \left\{ \frac{\delta_{\alpha\beta}}{\omega^2} + \frac{k_\alpha v_\beta + k_\beta v_\alpha}{\omega^2 \Omega} + \left(\frac{k^2c^2}{\omega^2} - 1\right) \frac{v_\alpha v_\beta}{c^2} \frac{1}{\Omega^2} \right\}$$

and the special result:

$$K_{33} = -\frac{\omega_T^2}{\Omega^2} \left(1 - \frac{v_3^2}{c^2}\right) = -\frac{\omega_B^2}{\Omega^2} \tag{7}$$

where:

$$\omega_T^2 = \frac{4\pi N e^2}{M\gamma} \; ; \quad \omega_B^2 = \omega_T^2 \left(1 - \frac{v_3^2}{c^2}\right)$$

with:

N = unperturbed particle density of the beam.

e = charge of one beam particle.

M = mass of one beam particle.

$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

v = speed of a beam particle.

v_3 = component of $\vec{v}$ along $\vec{k}$.

$\Omega = \omega - \vec{k} \cdot \vec{v} = \omega - kv_3.$

We must now modify (7) to allow for a spectrum of beam particle velocities. While no approximations are really necessary, we prefer to make the following work somewhat more transparent by noting that the principal effect of the velocity spread is taken into account if we allow for the variation of $\vec{v}$ only in the resonant denominator--provided of course the range of velocities (and of γ in the relativistic case) is reasonably narrow. Since Ω depends only on v_3 it is then sufficient to introduce a distribution function for v_3. We let the particle density for particles with v_3 in dv_3 be $N\,f\,(v_3)\,dv_3$ with $\int f\,(v_3)\,dv_3 = 1$. With this notation equation (7) is changed to:

$$K_{33} = -\,\omega_B^{\,2} \int \frac{f(v_3)\,dv_3}{(\omega - kv_3)^2} \equiv +\,\frac{\omega_B^{\,2}}{k} \int \frac{f'\,(v_3)\,dv_3}{\omega - kv_3} \tag{8}$$

where $\omega_B^{\,2}$ is a suitable average over the velocity spectrum, and the second form is obtained form the second upon integration by parts.

Our final form for the dispersion relation (6), using (2) and (8) is finally:

$$1 - \frac{\omega_p^{\;2}}{\omega(\omega + 1/\tau)} + \frac{\omega_B^{\;2}}{k} \int \frac{f'(v_3)\,dv_3}{\omega - kv_3} = 0 \qquad (9)$$

B. Energy Balance in the Electrostatic Instability

It is useful to write (9) as two separate equations by multiplying first by $-i\omega$ and then setting the real and imaginary parts separately equal to zero. The first of these two equations is then related directly to the energy balance in the process and has therefore a very direct physical interpretation. As a matter of fact we shall see that the second equation is not very important in practice.

We note first that if we restore a factor E_3 into (9) it reads:

$$E_3 + (\mathcal{E} - 1)\,E_3 + K_{33}\,E_3 = 0.$$

If we supply an additional factor $\frac{-i\omega}{4\pi}$ and compare with (1) and (3) we obtain:

$$\frac{-i\omega}{4\pi}\,E_3 + J_3^{\,plasma} + J_3^{\,beam} = 0$$

If we finally multiply by the complex conjugate $\overline{E}_3$ and take the real part we have:

$$\mathrm{Re}\left\{-\frac{i\omega}{4\pi}\,E_3\,\overline{E}_3\right\} + \mathrm{Re}\left\{J_3^{\,plasma}\,\overline{E}_3\right\} + \mathrm{Re}\left\{J_3^{\,beam}\,\overline{E}_3\right\} = 0$$

The first term is clearly the rate of change of the electrostatic field energy, the second term the rate at which energy is put into the plasma and the last term the rate at which energy is put into the beam. Obviously the last term must be negative and the first two positive if the perturbation is to grow. The terms in the equation are obtained by multiplying equation (9) by $-i\omega$ and taking the real part to correspond to the above rates of energy transfer aside from a factor $\left|\frac{E_3}{4\pi}\right|^2$.

It is convenient at this point to change notation:

$$\left\{\begin{array}{lcl} \text{Old notation} & & \text{New notation} \\ \omega & \longrightarrow & \omega + i\,\alpha \end{array}\right. \tag{10}$$

ω, α, are now real. In this notation we get from (9) the two equations:

$$\alpha + \frac{\omega_p^2}{\omega^2 + (\alpha + 1/\tau)^2}(\alpha + 1/\tau) - \omega_B^2 \int \frac{\alpha\, v_3\, f'(v_3)\, dv_3}{(\omega - kv_3)^2 + \alpha^2} = 0 \tag{11}$$

$$-\omega + \frac{\omega_p^2\ \omega}{\omega^2 + (\alpha + 1/\tau)^2} - \omega_B^2 \int \frac{(\omega - kv_3)\, v_3\, f'(v_3)\, dv_3}{(\omega - kv_3)^2 + \alpha^2} = 0 \tag{12}$$

We shall now explain the cryptic remark made previously that equation (12) is unimportant. This is true because k is after all a free parameter; in the last term in (11) ω enters only in the combination $\omega - kv_3$ so that if we treat $\omega - k\bar{v}_3$ (where v_3 is, say, the mean value of v_3) as a free parameter instead, we need not know ω at all. The other place where ω enters in (11) is clearly not too critical, and from equation (12) we know that as long as our assumption (5) in A is true (our dispersion relations are only correct if it is true) we know that $\omega^2 + (\alpha + 1/\tau)^2 = \omega_p{}^2$ to a very good approximation. It is therefore justified to rewrite (11) as:

$$\alpha + \alpha + 1/\tau \;\; - \omega_B{}^2 \int \frac{\alpha v_3 \; f'\,(v_3)\; dv_3}{(\omega - kv_3)^2 + \alpha^2} = 0 \qquad (11')$$

We now repeat and amplify slightly the remarks made about the physical interpretation of the various terms. Apart from a factor $\frac{|E_3|^2}{4\pi}$, α is the rate of increase of electrostatic energy; $\alpha + 1/\tau$ is the rate at which energy goes into the plasma; the first term, α, representing the rate of increase of kinetic energy of the plasma and $1/\tau$ the rate of energy loss by collisions; the last term represents the rate at which energy is extracted from the beam.

We make the obvious remark now--and discuss it more fully later--that if we can prove that in a given situation the last term is always smaller than $1/\tau$, the energy extracted from the beam is too small to compensate the losses and there is no instability.

We discuss first--briefly--the limiting case where the distribution function is infinitely sharp. We get the appropriate limit from (11') if we undo the integration by parts in the last term and then put $f(v_3) = \delta(v_3 - \bar{v}_3)$:

$$2\alpha + 1/\tau \;\; = \omega_B{}^2 \;\; \mathrm{Im}\left(\frac{\omega + i\alpha}{\Omega^2}\right);$$

$$\Omega = \omega + i\alpha - k\bar{v}_3 \; ; \quad \text{if } f(v_3) = \delta(v_3 - \bar{v}_3).$$

The term on the R. H. S. is $\omega_B^2 \left\{ \alpha \operatorname{Re} 1/\Omega^2 + \omega \operatorname{Im} 1/\Omega^2 \right\}$

The first term is unimportant if $\alpha << |\omega| \sim \omega_p$ and the real and imaginary parts of $1/\Omega^2$ are comparable (which is the case). In this approximation:

$$2\operatorname{Im}\Omega + 1/\tau = \omega_B^2 \omega_p \quad (1/\Omega^2); \text{ for } f(v_3) = \delta(v_3 - \bar{v}_3) \qquad (11'')$$

To solve this equation we must in general solve a cubic. However the solution is trivial in two extreme cases:

Case I: $\operatorname{Im}\Omega >> 1/\tau$. In this case if we write $\Omega = (\omega_B^2 \omega_p)^{1/3} z$; (11'') is approximately:

$2\operatorname{Im} z = \operatorname{Im} 1/z^2$ which has the solution $z = (-\cos\phi)^{1/3} \; e^{i\phi}$. The largest growth rate occurs for $\phi = 120^\circ$ which gives:

$$\alpha_{max} = \frac{3^{1/2}}{2^{4/3}} \; (\omega_B^2 \omega_p)^{1/3} \; ; \quad (\omega_B^2 \omega_p)^{1/3} >> 1/\tau$$

Case II: $\operatorname{Im}\Omega << 1/\tau$. In this case we write $\Omega = \omega_B(\omega_p \tau)^{1/2}$; (11'') is approximately:

$1 = \operatorname{Im} 1/z^2$, with solution $z = (-\sin^2\phi)^{1/2} \; e^{i\phi}$. The largest growth rate again occurs for $\phi = 120^\circ$ and:

$$\alpha_{max} = (3/4)^{3/4} \; \omega_B \, (\omega_p \tau)^{1/2} \; ; \quad (\omega_B^2 \omega_p)^{1/3} << 1/\tau \, .$$

In general the greatest growing rate give by (11'') can be found by finding the positive root of:

$$2\,\alpha_{max}^{3} + \frac{\alpha_{max}^{2}}{\tau} = \omega_B^{2}\,\omega_p\,\frac{\sqrt{3}\cdot 3}{8}$$

In any practical case the growth rate calculated with neglect of velocity spread is so fast that it is of great importance to argue about the precise rate. We must therefore return to equation (11') to see whether taking into account the velocity spread could suppress the instability altogether. It is clear from the structure of the last term in (11') that the width of the velocity distribution has little effect unless:

$$\left(\frac{\Delta v_3}{v_3}\right)^2 \gtrsim \frac{\alpha^2}{k^2 v_3^{2}} \approx \frac{\alpha^2}{\omega_p^{2}} \quad .$$

We proceed directly to the extreme case where the width is large:

$$\left(\frac{\Delta v_3}{v_3}\right)^2 \gg \frac{\alpha^2}{\omega_p^{2}} \quad .$$

In this case we obtain for the rate at which energy is transferred from the beam:

$$\lim_{\alpha \to 0} \omega_B^{2} \int \frac{\alpha\, v_3\, f'(v_3)}{(\omega - kv_3)^2 + \alpha^2}\, dv_3 = \omega_B^{2}\,\pi\,\frac{v_3\, f'(v_3)}{k} \cong$$

$$\pi\omega_B^{2}\left[\frac{v_3^{2}\, f'(v_3) v_3}{\omega_p}\right]_{v_3 = \frac{\omega}{k}}$$

Since the wave velocity $\frac{\omega}{k}$ is an adjustable parameter we have in the worst case:

$$\lim_{\alpha \to 0} \omega_B^2 \int \frac{\alpha\, v_3\, f'(v_3)\, dv_3}{(\omega - kv_3)^2 + \alpha^2} = \pi \frac{\omega_B^2}{\omega_p} \left[v_3^2\, f'(v_3) \right]_{max} \qquad (12)$$

We note that if the collision rate is small compared to this, the growth rate is then given by:

$$\alpha \approx \frac{\pi}{2} \frac{\omega_B^2}{\omega_p} \left[v_3^2\, f'(v_3) \right]_{max} \gg \frac{1}{2\tau} .$$

In most practical cases a growth rate comparable to the collision frequency is still so high that this case is also of no great interest. The only case of practical interest is then the one in which the rate of energy loss by the beam is insufficient to supply the collision loss in the plasma. From (11') and (12) we obtain the following condition for stability:

$$\frac{1}{\tau} > \pi \frac{\omega_B^2}{\omega_p} \left[v_3^2\, f'(v_3) \right]_{max} \qquad \text{for stability.}$$

We remark that this result is exact if the assumptions (1-5) listed at the beginning of A are correct, except for the approximation involved in taking ω_B^2 outside the integral sign in the equation (8).

The result (13) can be evaluated for any distribution in v_3. For the sake of having an explicit result we evaluate it for the special case of a gaussian distribution with R. M. S. derivation Δv_3 in v_3.

If we put:

$$f(v_3)\,dv_3 = \frac{1}{\sqrt{2\pi}\,\Delta v_3}\, e^{\dfrac{-(v_3 - \bar{v}_3)^2}{2(\Delta\, v_3)^2}}\, dv_3$$

we get:

$$\frac{1}{\tau} > \sqrt{\frac{\pi}{2e}} \quad \frac{\omega_B^{\,2}}{\omega_p}\left(\frac{v_3}{\Delta v_3}\right)^2 = 0.76\ \frac{\omega_B^{\,2}}{\omega_p}\left(\frac{v_3}{\Delta v_3}\right)^2 \qquad (13)$$

For stability of a Gaussian distribution in v_3 with $< (v_3 - \bar{v}_3)^2 = (\Delta\, v_3)^2$.

It may be worthwhile to make one remark about this result. The result could have been obtained by the following argument. A particle moving with velocity $\bar{v}_3$ sees a perturbation with frequency $\omega - k\bar{v}_3$, while a particle with velocity $\bar{v}_3 \pm \Delta v_3$ sees a frequency $\omega - k(\bar{v}_3 \pm \Delta v_3)$. If the first particle keeps in phase (i.e., if $\omega - k\bar{v}_3 = 0$) the second particle in a time Δt will get out of phase by $k\ \Delta v_3\ \Delta t$ radians. If we set Δt equal to the growth time of the perturbation $(\Delta t \sim (\omega_B \sqrt{\omega_p \tau}\,)^{-1}$ if collisions are important) we might expect that the instability disappears for $k\ \Delta v_3\ \Delta t \sim 1$. This simple argument leads to the condition: $k\ \Delta v_3\ \Delta t \sim \frac{\omega_p}{v_3}\ \Delta v_3\ \ \frac{1}{\omega_B \sqrt{\omega_p \tau}} \gtrsim 1$ for stability; which coincides with (13). The argument is nevertheless spurious and leads to an incorrect conclusion in the case where collisions are unimportant (in the absence of velocity spread in the beam). The reason is simply that even if the velocity spread is such that it is impossible for all particles to respond coherently to the perturbation--so that some particles gain energy while others lose energy--it is always possible to choose a wave-number such that the beam does lose energy (on the average) and this is sufficient to guarantee instability, unless there is sufficient damping due to collisions in the plasma. In most applications--where the collision frequency is fairly high--a reduction in growth rate is of no practical importance.

C. Applications

To apply our results we must now specify the velocity distribution of particles, obtain from this the distribution function along the wave-vector $\vec{k}$, and see how the results concerning the instability depend on the angle θ between the wave-vector $\vec{k}$ and the direction of motion of the beam. The first point we must follow up to is that in fact we have in mind the application to a beam of finite cross-section held together by its own magnetic field. We must recognize this explicitly since the transverse velocities of the beam particle are related directly to it. We shall consider the connection between transverse velocity and lateral structure in a very crude and even not self-consistent manner.

We consider first a beam of uniform density within a cylinder of radius a. We state without proof that inside a beam the particles oscillate with a (radian) frequency ω_β which is connected to ω_T^2 (defined after equation (7) by:

$$\omega_\beta^2 = \frac{1}{2}\,\omega_T^2\,\frac{v^2}{c^2}.$$

In such a beam the transverse velocity of particles are distributed isotropically in direction but the magnitude varies with radius according to:

$$v_T^2 = \omega_\beta^2 (a^2 - r^2)$$

If we disregard the radial dependence and average over the cross-section we find:

$$\overline{v}_T^2 = \frac{1}{2} \omega_\beta^2 \; a^2 = \frac{1}{4} \frac{\omega_T^2 a^2}{c^2} v^2.$$

The distribution function in this case is a most peculiar one and we shall refuse to consider it seriuosly, mainly on the grounds that no real beam will ever look like this. For lack of a definite model we choose the simplest smooth distribution, i.e., a Gaussian, i.e., we assume:

$$\begin{cases} f(v_x \, v_y) \, dv_x \, dv_y = \dfrac{1}{\pi} \exp\left\{ - \dfrac{v_x^2 + v_y^2}{\overline{v}_T^2} \right\} \dfrac{dv_x \, dv_y}{\overline{v}_T^2} \\ \overline{v}_T^2 = \dfrac{1}{4} \dfrac{\omega_T^2 a^2}{c^2} v^2 \end{cases} \tag{14}$$

A precise definition of the beam radius is lacking.

The distribution in v_Z is more straightforward, but we must clearly distinguish two cases. If the beam is not extremely relativistic there is likely to be a small but significant spread in the particle speeds. In this case it is sufficient to identify v_Z with the speed v for each particle, i.e., to neglect in this connection the small angle between the velocity vector of an individual particle and the beam axis. The distribution in v_Z is thus identified with the distribution in speed v. This clearly is adequate provided the spread Δv in v is large compared to $v(\overline{1 - \cos\theta})$ which corresponds to assuming:

$$\frac{\Delta v}{v} \gtrsim \frac{1}{2} \; \frac{\overline{v_x^2 + v_y^2}}{v^2} \sim \frac{1}{8} \; \frac{\omega_T^2 a^2}{c^2} \; .$$

In this case the spread in the components is v_3 if given by:

$$(v - \bar{v}_3)^2 = \overline{(v_Z - \bar{v}_Z)^2} \cos^2\theta + \overline{(v_x - \bar{v}_x)^2} \sin^2\theta =$$

$$v^2 \left\{ \left(\frac{\Delta v}{v}\right)^2 \cos^2\theta + \frac{1}{8} \left(\frac{\omega_T a}{c}\right)^2 \sin^2\theta \right. .$$

Therefore we have:

$$\left(\frac{\Delta v_3}{v_3}\right)^2 = \left(\frac{\Delta v}{v}\right)^2 + \frac{1}{8} \left(\frac{\omega_T a}{c}\right)^2 \tan^2\theta \qquad (15)$$

The result (15) applies to the extremely relativistic case as well only if we put $\left(\frac{\Delta v}{v}\right)^2 = 0$ in this case and refrain from applying it too very small angles θ. We assume now a Gaussian distribution of velocity and apply the result (13).

First, however, we must evaluate ω_B^2:

$$\omega_B^{\,2} = \omega_T^{\,2}\left(1 - \frac{v_3^{\,2}}{c^2}\right) = \omega_T^{\,2}\left(1 - \frac{v^2}{c^2}\cos^2\theta\right) =$$

$$\omega_T^{\,2}\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\sin^2\theta\right).$$

We shall--for definitiveness--consider only the case of at least moderately relativistic particles and neglect $1 - \frac{v^2}{c^2}$ compared to $\frac{v^2}{c^2}\sin^2\theta$.

Therefore we put:

$$\omega_B^{\,2} \doteq \omega_T^{\,2}\sin^2\theta \tag{16}$$

We finally substitute (15) and (16) in (13). In the extremely relativistic case we put $\left(\frac{\Delta v}{v}\right)^2 = 0$ in (13) and obtain:

$$\frac{1}{\omega_p\tau} \gtrsim 6\ \frac{c^2}{a^2\omega_p^{\,2}}\cos^2\theta \text{ for stability.}$$

Since this is harder to satisfy for $\cos\theta \approx 1$ we end up with:

$\frac{\omega_p a}{c} \gtrsim 2.5\sqrt{\omega_p\tau}$. Stability condition for (see however an extremely relativistic beam (19) below).

If the contribution $\left(\frac{\Delta v}{v}\right)^2$ to (15) cannot be neglected we obtain instead:

$$\frac{1}{\omega_p\tau} \gtrsim 6\ \frac{c^2}{\omega_p^{\,2}a^2}\ \frac{\sin^2\theta}{\tan^2\theta + \frac{8(\Delta v/v)^2}{\left(\frac{\omega_T a}{c}\right)^2}}$$

The worst case obtains for $\tan^4 \theta = \dfrac{8\left(\frac{\Delta v}{v}\right)^2}{\left(\frac{\omega_T a}{c}\right)^2}$

which gives:

$$\frac{\omega_p a}{c} \gtrsim \frac{2.5 \sqrt{\omega_p \tau}}{1 + \dfrac{2\sqrt{2}\;(\Delta v/v)}{\left(\frac{\omega_T a}{c}\right)}} \qquad \text{Condition for stability.} \qquad (18)$$

Which can be easier to satisfy than (17)

$$\frac{\Delta v}{v} \gtrsim \left(\frac{1}{8} \quad \frac{\omega_T^2 a^2}{c^2}\right)^{1/2}$$

We must now remark that there is one case in which the above treatment is clearly incorrect, physically. This case occurs for modes in which there are no nodes in the perturbation (except near or at the boundary). For such a mode the transverse distribution of velocities can obviously have no effect so that we must consider only the contribution of the longitudinal velocity distribution in evaluating (13). We need only to consider the extremely relativistic case in this connection (for the other case the $\frac{\Delta v}{v}$ term already included explicitly is dominant in such a mode). For an extremely relativistic beam the speed of all particles is essentially c and the only spread in v_Z arises from the angles between the particles' velocity and the z-axis.

$$v_Z = c \cos \theta \doteq c - \frac{1}{2} \; \frac{v_x^2 + v_y^2}{c}$$

using (14) we have in this case

$$f(v_3)\,dv_3 \simeq \exp\left\{-\frac{v_x^{\,2}}{\overline{v_T^{\,2}}}\right\}\frac{dv_T^{\,2}}{v_T^{\,2}} = \simeq \exp\left\{\frac{2(c-v_3)c}{\overline{v_T^{\,2}}}\right\}\frac{2c\,dv_3}{v_T^{\,2}}$$

From this:

$$f'(v_3)\,v_3^{\,2}\ \max \sim \frac{4\,c^4}{(\overline{v_T^{\,2}})^2} \simeq \frac{64}{\left(\frac{\omega_T\,a}{c}\right)^4}$$

We must also estimate:

$$\omega_B^{\,2} \simeq \omega_T^{\,2}\ \sin^2\theta$$

For the mode we have:

$$\sin\theta = \frac{k_\perp}{k} = \frac{k_\perp\,a}{ka} = \frac{k_\perp\,a}{\left(\frac{\omega_p\,a}{c}\right)}$$

A crude guess is $k_\perp\,a \approx 2.4$, which is the first root of $J_o(k_\perp\,r)$. With this guess we have $\omega_B^{\,2} \sim \dfrac{6\,\omega_T^{\,2}}{\left(\frac{\omega_p\,a}{c}\right)^2}$

Equation (13) then gives:

$$\frac{1}{\omega_p\tau} \gtrsim \pi\cdot\frac{6\,\omega_T^{\,2}}{\left(\frac{\omega_p\,a}{c}\right)^2}\ \frac{1}{\omega_p^{\,2}}\ \frac{64}{\frac{\omega_T^{\,4}\,a^4}{c^4}} \approx \frac{1200}{\frac{\omega_p^{\,4}\,a^4}{c^4}\ \frac{\omega_T^{\,2}\,a^2}{c^2}}$$

or:

$$\frac{\omega_p a}{c} \gtrsim \frac{6(\omega_p \tau)^{1/4}}{\left(\frac{\omega_T a}{c}\right)^{1/2}}$$ Condition for stability of an extremely relativistic beam.

This condition is usually much harder to fulfill than (17). If (19) is not fulfilled the growth rate of this mode will be:

$$\alpha \approx 1.3 \left[\frac{\left(\frac{\omega_T a}{c}\right)^2}{\frac{\omega_p a}{c}}\right]^{1/3} \frac{c}{a} \qquad \text{or:}$$

$$\alpha \approx 4.8 \frac{\omega_T}{\omega_p} \sqrt{\omega_p \tau} \frac{c}{a}$$

whichever is smaller. For typical choice of parameters this gives:

$$\alpha \approx 10^{-2} \frac{c}{a}$$

i.e., exponential growth is about 10^2 beam radii.

D. Discussion

Our results have been obtained under a number of assumptions and approximations which are not very well fulfilled in the practical case of a beam of finite size moving through a plasma produced by ionization of a gas by the beam particles themselves. In particular, the following points remain to be settled before one can claim that our analysis is applicable.

1. Boundary effects have been ignored. It seems improbable to us that these have any great effect. In the practical case (uniform plasma, uniform beam within a cylinder, sharp velocity distribution) one finds essentially the same dispersion relation except that the transverse component of the wave vector $k_\perp$ (the perturbation in this case varies as $e^{i\phi} J_n(k_\perp a)$ inside the beam) is not necessarily real but is determined by the boundary conditions. Numerical analysis of the simplest modes indicates that this has very little effect.

2. The beam current density presumably will not be uniform over the beam but will decrease more or less smoothly. The energy arguments on which our analysis is based, should not be affected appreciably and it seems very unlikely that this is important, even though an analysis has not been carried out.

3. We have considered the effect of the particle motion due to the pinch magnetic field only in a very crude way, by supposing that the effect is the same as with particles moving without such a field but with a distribution of transverse velocities. It has certainly not been proved that the two situations are equivalent. It seems very probable that they are when both effects are small; but it is not clear whether the "Landau damping" due to the transverse velocities is really appreciable.

In the principal mode (no nodes) discussed briefly at the end of C it is clear physically that the transverse velocities play no important role, so that this uncertainty does not apply.

4. Perhaps the most obscure effect is that due to variation in plasma densities, which results in the plasma frequency (of the stationary

plasma) being different at different radii. If the instability mode persists in this case it cannot be described in terms of the Fourier decomposition that we have used here. Until such an analysis has been carried out, one must necessarily regard the results presented here with considerable caution.