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CONTROL SYSTEMS LABORATORY

DOPPLER FREQUENCY ANALYSIS BY A
STORAGE TUBE AND FILTER BANK ANALYZER.
THEORY OF THE SINUFLY COMPUTER.

Report R-86

November, 1956

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DOPPLER FREQUENCY ANALYSIS BY A STORAGE TUBE AND FILTER BANK ANALYZER. THEORY OF THE SINUFLY COMPUTER.

Report R-86

November, 1956

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I. Introduction

The basic function of all present day MTI radar is to examine the signal returned to the radar from a target or group of targets and to determine whether any doppler frequencies are present in that signal. Among the several devices which may be used to accomplish this end, are various types of delay and subtraction systems, which are rather well known both in practical details and in theory. Somewhat less widely known are MTI systems of the tuned filter variety. In this report we shall concern ourselves mainly with certain theoretical details of one particular system of this class, — the Sinufly computer.

The results reported here are somewhat mathematical in nature, and are not essentially bound to the Sinufly system. Rather, they are answers to questions motivated by a study of Sinufly. They are also answers to questions which could conceivably arise in other applications and it is therefore hoped that our presentation has been cast in a form which will not obscure the broader implications of our results. Nevertheless, these results do answer questions about sinufly, and since Sinufly is our principal interest here, the discussion will be centered about it.

A simplified description of the manner in which Sinufly works is the following. Video returns from a number of transmitted pulses are stored, as charge modulation, in successive traces on a barrier grid storage tube, each trace being displaced slightly from its predecessor, so as to form a B-Scope type raster on the storage surface. The wave packet which results when each range element of the storage tube is read out in a direction perpendicular to the direction of writing, is then fed into a balanced modulator which multiplies its amplitude by a certain weighting function. A spectral analysis of the weighted wave packet is then made by means of a tuned filter bank.[1]

The spectral analysis is performed in the following way: The weighted wave packet of length, say T_o, is fed into the filter bank. At time T_o the packet ends and the filter oscillations begin to decay. At time $T > T_o$ the output amplitude of each filter is sampled in order to determine to what extent the basic frequency, to which that filter is tuned, was present in the wave packet. For practical reasons, the sample time T is very soon after T_o.

Unfortunately it is found that various disturbances modify the wave packet which represents the radar return; disturbances such as thermal noise and storage tube noise for example. In addition the radar return itself has an undesired component which results from clutter. Both the noise disturbances and the clutter impart a random character to the output amplitude of the Sinufly filter. Noise disturbances are by nature random, and clutter is a function of terrain conditions which vary in a somewhat random manner. This means that any precise analysis of our system must be to some extent statistical.

The statistical part of the analysis given in this report consists in the use of some elementary probability theory. The adjective "elementary" is meant to express the fact that no conditional probabilities are used, and no powerful limit theorems are invoked.

In section II a theorem is proved which shows that all of our analysis can be concentrated, essentially, on examining the output spectrum of the storage tube.

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[1]For a more detailed description see Refs. 1 and 2.

Sections IV, V and VI deal with the problems of noise, signals in additive white noise, and signals in multiplicative noise, respectively. In these sections the probability density functions for the filter output amplitude are developed using the general forms for these density functions which are derived in Appendix A. The purpose of these sections is primarily to illustrate the method, which is in itself quite simple. Despite the conceptual simplicity however, some of the computations yield rather complex formuli. In fact one of the main features of this report is to show, by example, that certain types of analysis are impractical.

In section VII we introduce a notion of "enhancement" which, while it is not as meaningful or useful a criterion for the Sinufly system as detection probability, is still of value and is much more amenable to calculation.

The authors of this report have been assisted by every member of the Sinufly group at CSL, but they feel that some names must be mentioned explicitly. We have profited much from informative discussions with J. Robe, R. Swallow and W. Unruh; and, outside the Sinufly group, from discussions with D. Cooper, A. Nordsieck and J. Ruina. The numerical results of R. Swallow were an indispensable guide. The computations for graphs were done by Shirley Bailey and M. Martin.

II. A Method of Filtering and Detecting Signals in Finite Length Video Pulse Packets, and a Theorem on Packet Weighting and Sampling.

In this section we shall prove a simple theorem which will materially simplify our subsequent work of determing certain probability distributions. Let us first give a description of the problem considered.

We are given a linear filter, which we may consider as described by the differential operator

$$L = \frac{d^2}{dt^2} + 2\lambda \frac{d}{dt} + \omega_o^2.$$

Also we are given a certain stochastic process[1] of finite duration, which in our interpretation is some combination of signal and noise; let us denote it by $f(t)$ and choose our time origin so that $f(t) = 0$ for $t < 0$ and $f(t) = 0$ for $t > T_o$. We then weight this process by a factor $e^{-\lambda t}$, and pass the weighted process through our filter. Now let $F(t)$ be the output of our filter resulting from the input $f(t)\, e^{-\lambda t}$, i.e., $F(t)$ is the unique function satisfying the identities

$$L\, F(t) = f(t)\, e^{-\lambda t}, \qquad F(t) = 0 \text{ for } t < 0. \tag{2.1}$$

Of course, $F(t)$ is also a stochastic process. Next let us "sample" $F(t)$ at time $T > T_o$, i.e., we consider $F(T)$. The amplitude R of this sample $F(T)$ is a random variable[2] (as is $F(T)$ itself). Our main problem in this chapter is to determine the probability distribution

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[1] By a stochastic process we mean merely an indexed family of random variables. The indexing parameter is usually denoted by t and interpreted as time. (Ref. 4, p. 46)

[2] Ref. 4, p. 5

function for R. (Actually what we shall do is to determine the density function for this probability distribution.)

Our analysis can be materially simplified by utilizing a theorem which we can infer from the following discussion. If one solves the differential equation (2.1) he finds that the function F(t) can be represented in the form[1]

$$F(t) = \frac{e^{-\lambda t}}{\omega}\left\{\left(-\int_0^t \sin \omega s\, f(s)ds\right)\cos \omega t + \left(\int_0^t \cos \omega s\, f(s)ds\right)\sin \omega t\right\}. \quad (2.2)$$

where $\omega^2 = \omega^2_o - \lambda^2$. If we remember that $f(s) = 0$ for $s < 0$ or for $s > T_o$, and if we restrict t so that $t > T_o$, we see that we can write this solution in the more symmetric form

$$F(t) = \frac{e^{-\lambda t}}{\omega}\left\{\left(-\int_{-\infty}^{\infty} \sin \omega s\, f(s)ds\right)\cos \omega t + \left(\int_{-\infty}^{\infty} \cos \omega s\, f(s)ds\right)\sin \omega t\right\}. \quad (2.3)$$

If we write F(t) in the form

$$F(t) = X(t) \cos \omega t + Y(t) \sin \omega t,$$

(which is possible for any function), the amplitude of F(t), for $t > T_o$, is given by the expression

$$\left[X^2(t) + Y^2(t)\right]^{\frac{1}{2}},$$

where we are assuming that for $t > T_o$, X(t) and Y(t) are functions of slow variation.

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[1]This solution is readily found by any of the standard methods of solving second order linear differential equations with constant coefficients.

Before proceeding further, we should recall that $F(t)$ is, in fact, a function of λ and ω_o as well as of t. If we hold λ fixed, we may consider $F(t)$ as a function $F(\omega; t)$ of t and ω since the defining equation for ω (p. 9) is $\omega^2 = \omega_o{}^2 - \lambda^2$. All of this may be seen from either of the integral representations (2.2) or (2.3) given above. Henceforth in our discussion, λ is to be considered a constant.

Now let us fix $t = T > T_o$, and introduce the notation

$$R(\omega) = \left[X^2(T) + Y^2(T)\right]^{\frac{1}{2}} \tag{2.4}$$

for the amplitude of $F(T)$(remembering that $F(T)$ is a function of ω). Our representation (2.3) shows that

$$\begin{aligned} X(T) &= \frac{e^{-\lambda T}}{\omega}\left(-\int_{-\infty}^{\infty} \sin \omega s \; f(s)ds\right) \\ Y(T) &= \frac{e^{-\lambda T}}{\omega}\left(\int_{-\infty}^{\infty} \cos \omega s \; f(s)ds\right) \end{aligned} \tag{2.5}$$

These two integrals suggest a Fourier analysis of our function $f(t)$ which we can make as follows. We write the integrals as imaginary and real parts of a function

$$2\pi \, S(\omega) = \int_{-\infty}^{\infty} \cos \omega s \; f(s)ds - i\int_{-\infty}^{\infty} \sin \omega s \; f(s)ds.$$

Writing $\cos \omega s - i \sin \omega s = e^{-i\omega s}$, we have

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega s} \, f(s) \, ds.$$

The complex valued function $S(\omega)$ is called the <u>spectrum</u> of $f(s)$.

Actually we are interested in the modulus (absolute value) of $S(\omega)$, and this is given by

$$\left|S(\omega)\right| = \frac{1}{2\pi}\left\{\left(\int_{-\infty}^{\infty} \cos \omega s \, f(s)ds\right)^2 + \left(\int_{-\infty}^{\infty} \sin \omega s \, f(s)ds\right)^2\right\}^{\frac{1}{2}}. \quad (2.6)$$

If we now compare (2.6) with (2.4) and (2.5) we see that

$$R(\omega) = \frac{2\pi \, e^{-\lambda T}}{\omega}\left|S(\omega)\right|.$$

Let us now collect our results in the succinct form of a theorem. In the statement of this theorem we shall retain precisely the notation introduced in our discussion.

<u>Theorem</u>: <u>Let L, $f(t)$, and $F(t)$ be as introduced above i. e., $L\,F(t) = f(t)\,e^{-\lambda t}$, etc., let $R(\omega)$ be the amplitude of $F(T)$ where $T > T_o$, and let $S(\omega)$ be the spectrum of $f(t)$, then we have the relation</u>

$$R(\omega) = \frac{2\pi \, e^{-\lambda T}}{\omega}\left|S(\omega)\right|.$$

The importance of the weighting factor $e^{-\lambda t}$ as part of the driving function in (2.1) becomes obvious upon carrying out the details of its solution. It is just this weighting factor which is largely responsible for the simple form of the conclusion of our theorem. The condition on the sample time ($t = T > T_o$) must be specified in order for (2.3) to follow from (2.2) as a valid representation of $F(t)$. It is this representation of our solution (i.e. (2.3)) which can be directly related to a Fourier integral.

As mentioned earlier, $R(\omega)$ is (for fixed ω) a random variable, and our immediate problem is to determine its <u>probability density</u> function. The theorem established above allows us to do this directly, from the density function $p_\omega(s)$ for $|S(\omega)|$, by a simple substitution. Thus our real problem is now seen to consist in the determination of the density function for $|S(\omega)|$. We shall attempt to illustrate the procedures and difficulties involved in this by a number of examples in the following sections.

III. Spectra of a Class of Functions Associated with Signals

Let $\gamma_a(t)$ denote the "rectangle function" defined by

$$\gamma_a(t) = \begin{cases} 1, & \left| t - a \right| < \frac{T_o}{2} \\ 0, & \left| t - a \right| > \frac{T_o}{2} \end{cases}$$

The spectrum of $\gamma_a(t)$ is, therefore,

$$H_a(\omega) = \frac{1}{2\pi} \int_{a-\frac{T_o}{2}}^{a+\frac{T_o}{2}} e^{-i\omega t}\, dt = \frac{e^{-ia\omega}}{\pi} \left(\frac{\sin \frac{\omega T_o}{2}}{\omega} \right) \tag{3.1}$$

The functions $\gamma_a(t)$ play a considerable role in our further work, but, it can be shown that the final results do not depend on a, but only on T_o. We could thus work with any particular $\gamma_a(t)$ we chose in order to obtain our ultimate results. As the discerning reader will note there is a definite computational advantage in choosing $a = 0$, and we shall henceforth work with $\gamma(t) = \gamma_o(t)$.

Now let us consider the spectrum of $\gamma(t)\, e^{i\mu t}$. Proceeding as above, we find the spectrum to be

$$S_\mu(\omega) = \frac{1}{2\pi} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} e^{-i\omega t}\, e^{i\mu t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} e^{-i(\omega-\mu)t}\, dt = \frac{1}{\pi} \frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)}$$

We note that in particular,

$$S_o(\omega) = H_o(\omega) = H(\omega) = \frac{1}{\pi} \frac{\sin \frac{\omega T_o}{2}}{\omega}, \tag{3.2}$$

that is to say, we may consider $H(\omega)$ as defined by (3.1).

By using the spectra $S_\mu(\omega)$ properly, we may write down the spectra of $\gamma(t) \cos \mu t$ and $\gamma(t) \sin \mu t$. Let us denote their spectra by $U_\mu(\omega)$ and $V_\mu(\omega)$ respectively. Then we have

$$\begin{aligned} U_\mu(\omega) &= \frac{1}{2}\left[S_\mu(\omega) + S_{-\mu}(\omega)\right] \\ V_\mu(\omega) &= \frac{1}{2i}\left[S_\mu(\omega) - S_{-\mu}(\omega)\right] \end{aligned} \tag{3.3}$$

or more explicitly

$$\begin{aligned} U_\mu(\omega) &= \frac{1}{2\pi}\left\{\frac{\sin(\omega - \mu)\frac{T_o}{2}}{(\omega - \mu)} + \frac{\sin(\omega + \mu)\frac{T_o}{2}}{(\omega + \mu)}\right\} \\ V_\mu(\omega) &= \frac{1}{2\pi i}\left\{\frac{\sin(\omega - \mu)\frac{T_o}{2}}{(\omega - \mu)} - \frac{\sin(\omega + \mu)\frac{T_o}{2}}{(\omega + \mu)}\right\} \end{aligned} \tag{3.4}$$

This method of obtaining spectra from linear combinations of the $S_\mu(\omega)$ allows us to write down fairly general cases quite easily. The following examples, though interesting and worth noting at this point, will not be used in the subsequent sections.

For the first case, suppose that

$$g(t) = \sum_{k=1}^{n} c_k \, e^{i\mu_k t},$$

where $\mu_1, \ldots, \mu_n$ is <u>any</u> sequence (finite, of length n) of real numbers, and the c_k are constant (real or complex). Then the spectrum $S_g(\omega)$

of $\gamma(t)\, g(t)$ is given by

$$S_g(\omega) = \sum_{k=1}^{n} c_k \, S_{\mu_k}(\omega).$$

The next case is quite general. Let $G(t)$ be any function which is of bounded variation for $\left|t\right| \ll \frac{T_o}{2}$. Then $G(t)$ possesses a Fourier series which is convergent in the interval $\left|t\right| \ll \frac{T_o}{2}$, and which represents $G(t)$ (almost everywhere) in that interval.[1] Thus, for $\left|t\right| \ll \frac{T_o}{2}$

$$G(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{i\frac{2\pi kt}{T_o}},$$

where

$$c_k = \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} G(t)\, e^{-i\frac{2\pi kt}{T_o}} \, dt.$$

(If $G(t)$ is real, then $c_{-k} = \bar{c}_k$, the complex conjugate of c_k.) The spectrum of $\gamma(t)\, G(t)'$ is therefore

$$S_G(\omega) = \sum_{k=-\infty}^{\infty} c_k \, S_{\left(\frac{2\pi k}{T_o}\right)}(\omega).$$

[1]Ref. 5, p. 175

IV. Noise through a Filter

First let us discuss the notion of a stationary (wide sense) stochastic process.[1] The adjective stationary (wide sense) means that the expectations $E\left\{n(s)\right\}$ and $E\left\{n(s)\,n(s+t)\right\}$ are independent of s, where n(t) is our process. We shall henceforth assume that $E\left\{n(t)\right\} = 0$.

Now let us perform a purely formal analysis of n(t). We represent n(t) by

$$n(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, N(\omega)\, d\omega, \qquad (4.1)$$

then

$$N(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, n(t)\, dt. \qquad (4.2)$$

At this point it becomes convenient to calculate the covariance function $E\left\{N(\omega_1)\,\overline{N}(\omega_2)\right\}$ for subsequent use.[2] Using the facts that n(t) is real, and that

$$\overline{N}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(s+t)}\, n(s+t)\, dt,$$

we find that

$$E\left\{N(\omega_1)\,\overline{N}(\omega_2)\right\} =$$

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left\{n(s)\,n(s+t)\right\} e^{-i\omega_1 s}\, e^{i\omega_2(s+t)}\, ds\, dt.$$

---

[1] Ref. 4, pp. 8 and 95

[2] $\overline{N}(\omega_2)$ denotes the complex conjugate of $N(\omega_2)$.

Since n(t) is stationary we may write

$$E\left\{n(s)\ n(s+t)\right\} = R(t) = \lim_{C\to\infty} \frac{1}{2C}\int_{-C}^{C} n(s)\ n(s+t)\ ds. \qquad (4.3)$$

R(t) is called the correlation function of the process, and we should emphasize that R(t) is a function, not a process.[1] The Fourier transform of R(t) is frequently referred to as the <u>power spectrum</u> of the process n(t).[2] Let

$$W(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\ R(t)\ dt \qquad (4.4)$$

be the power spectrum, then we have

$$E\left\{N(\omega_1)\ \overline{N}(\omega_2)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega_2-\omega_1)s} W(\omega_2)\ ds$$

$$= \frac{W(\omega_2)}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega_2-\omega_1)s}\ ds = W(\omega_2)\ \delta(\omega_2-\omega_1), \qquad (4.5)$$

where $\delta(\omega_2-\omega_1)$ is a Dirac δ-function.

We shall now interpret our process n(t) as noise. When we speak of <u>white noise</u>, we refer to a stationary Gaussian process whose power spectrum is constant. Let $W(\omega) = \sigma^2$ denote this constant in our case.[3]

From (4.3) and (4.4), it follows that, for white noise, we have

$$E\left\{n(s)\ n(s+t)\right\} = \int_{-\infty}^{\infty} e^{-i\omega t}\ W(\omega)\ d\omega = \sigma^2\ \delta(t),$$

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[1] Ref. 4, pp. 46 and 71

[2] Ref. 3, pp. 305 and 306

[3] This assumption is not a necessary one and it will be dropped in section seven.

or

$$E\left\{n(t_1)\, n(t_2)\right\} = \sigma^2\, \delta(t_2 - t_1).$$

This means the noise is totally uncorrelated.

A few words might be in order at this point concerning the occurrence of δ-functions and the corresponding infinite variances. This situation arises from representing a physically impossible situation by a mathematically non-existent process, and is further aggravated by using a nonexistent integral to obtain an equally nonexistent transform. This is the formal analysis of (4.1) and (4.2). Explicitly, we mean to say that $n(t)$, as a nontrivial, totally uncorrelated stochastic process is nonexistent, and $N(\omega)$ is also a nonexistent process. This lack of existence does not, however, void them from being useful, any more than the δ-function is obstructed from its usefulness by its nonexistence. In fact, $n(t)$ and $N(\omega)$ are useful in the same way and for the same reasons as the δ-function—namely, to shorten and facilitate computation.

Of course, the use of the nonexistent $n(t)$ and $N(\omega)$ could be avoided by a careful formulation of our noise process as arising from a process with orthogonal increments.[1,2] Since there would be little practical gain from this, we shall make free use of δ-functions.

Next we wish to investigate noise $f(t)$ which is given by $f(t) = \gamma(t)\, n(t)$, i.e., the noise exists only over the time from $-\frac{T_o}{2}$ to $\frac{T_o}{2}$. Thus it follows from (3.1) and (4.1) that

$$f(t) = \gamma(t)\, n(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\omega_1+\omega_2)t}\, N(\omega_1)\, H(\omega_2)\, d\omega_1\, d\omega_2$$

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[1] Ref. 4, pp. 425 - 436

[2] Ref. 3, pp. 314 - 322

or

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega t} S_*(\omega)\, d\omega$$

where

$$S_*(\omega) = \int_{-\infty}^{\infty} H(\alpha)\, N(\omega-\alpha)\, d\alpha.$$

Using this last expression we may calculate the covariance function[1]

$$E\left\{S_*(\omega_1)\, \overline{S}_*(\omega_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\alpha)\, H(\beta)\, E\left\{N(\omega_1-\alpha)\, \overline{N}(\omega_2-\beta)\right\} d\alpha\, d\beta$$

$$= \sigma^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\alpha)\, H(\beta)\, \delta(\omega_2-\omega_1+\alpha-\beta)\, d\alpha\, d\beta,$$

by referring to (4.5). Thus we have

$$E\left\{S_*(\omega_1)\, \overline{S}_*(\omega_2)\right\} = \sigma^2 \int_{-\infty}^{\infty} H(\alpha)\, H(\omega_2-\omega_1+\alpha)\, d\alpha. \tag{4.6}$$

Let us now introduce the notation

$$\rho(\omega_1, \omega_2) = E\left\{S_*(\omega_1)\, \overline{S}_*(-\omega_2)\right\} = E\left\{S_*(\omega_1)\, S_*(\omega_2)\right\}$$

and

$$\rho(\omega) = \rho(\omega, \omega) = E\left\{S_*(\omega)\, S_*(\omega)\right\},$$

and

$$\rho_o = \rho(0).$$

Introducing this notation into (4.6) gives

$$\rho(\omega) = \sigma^2 \int_{-\infty}^{\infty} H(\alpha)\, H(\alpha-2\omega)\, d\alpha = \sigma^2 \int_{-\infty}^{\infty} H(\alpha+\omega)\, H(\alpha-\omega)\, d\alpha. \tag{4.7}$$

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[1] Ref. 4, p. 95 (clearly $E\left\{S_*(\omega)\right\} = 0$)

and

$$E\left\{ S_*(\omega)\ \overline{S}_*\ (\omega) \right\} = \sigma^2 \int_{-\infty}^{\infty} H(\alpha)\ H(\alpha)\ d\alpha = \rho(0) = \rho_o.$$

From (4.7) and (3.2) we see that $\rho(\omega)$ is real, hence

$$E\left\{ \overline{S}_*(\omega)\ \overline{S}_*(\omega) \right\} = \rho(\omega).$$

Perhaps we should remark here that $\rho(\omega)$ is real only because we chose $\gamma(t) = \gamma_o(t)$; for of all the $H_a(\omega)$, $-\pi < a < \pi$, only $H_o(\omega)$ is real. In section VI we shall see the complication which arises when we do not force the imaginary part of $\rho(\omega)$ to vanish.

The variances $\sigma_1{}^2(\omega)$ and $\sigma_2{}^2(\omega)$ of the real and imaginary parts of $S_*(\omega)$ are given by

$$\sigma_1{}^2(\omega) = \frac{1}{4} E\left\{ \left[S_*(\omega) + \overline{S}_*(\omega)\right]^2 \right\}$$

$$\sigma_2{}^2(\omega) = -\frac{1}{4} E\left\{ \left[S_*(\omega) - \overline{S}_*(\omega)\right]^2 \right\}.$$

Thus we have

$$\sigma_1{}^2(\omega) = \frac{1}{2}\left[\rho_o + \rho(\omega)\right]$$

$$\sigma_2{}^2(\omega) = \frac{1}{2}\left[\rho_o - \rho(\omega)\right] \tag{4.8}$$

In order to make the formulas more explicit, we calculate $\rho(\omega)$ and ρ_o,

$$\rho(\omega) = \frac{\sigma^2}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin\,(\alpha + \omega)\frac{T_o}{2}\ \sin\,(\alpha - \omega)\frac{T_o}{2}}{\alpha^2 - \omega^2}\, d\alpha \tag{4.9}$$

$$= \frac{T_o\sigma^2}{2\pi} \cdot \frac{\sin\,\omega T_o}{\omega T_o}.$$

This makes (4.8) become

$$\begin{cases} \sigma_1^2(\omega) = \dfrac{\sigma^2 T_o}{4\pi}\left(1 + \dfrac{\sin \omega T_o}{\omega T_o}\right) \\ \sigma_2^2(\omega) = \dfrac{\sigma^2 T_o}{4\pi}\left(1 - \dfrac{\sin \omega T_o}{\omega T_o}\right). \end{cases} \tag{4.10}$$

Our primary interest is not in $\sigma_1^2(\omega)$ and $\sigma_2^2(\omega)$, but in the probability density function $p_\omega(s)$, of $\left|S_*(\omega)\right|$, which is expressed in terms of these variances.

In Appendix A the probability density function p(s) is derived for the modulus $Z = \left|X + iY\right|$ of a complex random variable. The most general result of this derivation (A.15) displays the explicit dependence of p(s) on the variances σ_1^2 and σ_2^2. Since these variances are functions of ω (4.10), we introduce the subscript ω on p(s) to indicate its dependence on the specific filter frequency under consideration. $p_\omega(s)$ can also be expressed simply in terms of ρ_o and $\rho(\omega)$. From (4.8) we see that

$$\begin{cases} \sigma_1^2 + \sigma_2^2 = \rho_o \\ \sigma_1^2 - \sigma_2^2 = \rho(\omega) \end{cases}$$

and that

$$\sigma_1^2 \, \sigma_2^2 = \frac{1}{4}\left[\rho_o^2 - \rho^2(\omega)\right].$$

Since the real and imaginary parts of $S_*(\omega)$ each have Gaussian distributions with zero means and variances $\sigma_1^2(\omega)$ and $\sigma_2^2(\omega)$ respectively,[1] the probability density function $p_\omega(s)$ of $\left|S_*(\omega)\right|$ is immediately obtainable from formula (A.19) which is derived in Appendix A. This

[1] Ref. 8, p. 209

formula gives

$$p_\omega(s) = \frac{s}{\sigma_1\sigma_2} \exp\left[-\frac{(\sigma_1^2+\sigma_2^2)}{4\sigma_1^2\sigma_2^2}s^2\right] I_o\left[\frac{(\sigma_1^2-\sigma_2^2)}{4\sigma_1^2\sigma_2^2}s^2\right].$$

In order to insure the validity of this formula we must be certain that the real and imaginary parts of $S_*(\omega)$ are independent. The real part of $S_*(\omega)$ is given by $\frac{1}{2}\left[S_*(\omega)+\overline{S}_*(\omega)\right]$ and the imaginary part by $\frac{1}{2i}\left[S_*(\omega)-\overline{S}_*(\omega)\right]$. Since $S_*(\omega)$ is Gaussian we only need to show that

$$E\left\{\frac{1}{2}\left[S_*(\omega)+\overline{S}_*(\omega)\right]\frac{1}{2i}\left[S_*(\omega)-\overline{S}_*(\omega)\right]\right\} = 0$$

in order to establish the independence of the real and imaginary parts. On expanding this expression we see that

$$E\left\{\frac{1}{2}\left[S_*(\omega)+\overline{S}_*(\omega)\right]\frac{1}{2i}\left[S_*(\omega)-\overline{S}_*(\omega)\right]\right\}$$

$$= \frac{1}{4i}E\left\{S_*(\omega)\,S_*(\omega)+\overline{S}_*(\omega)\,S_*(\omega)-S_*(\omega)\,\overline{S}_*(\omega)-\overline{S}_*(\omega)\,\overline{S}_*(\omega)\right\}$$

$$= \frac{1}{4i}E\left\{S_*(\omega)\,S_*(\omega)\right\} - \frac{1}{4i}E\left\{\overline{S}_*(\omega)\,\overline{S}_*(\omega)\right\}$$

$$= \frac{1}{4i}\rho(\omega) - \frac{1}{4i}\rho(\omega) = 0.$$

Thus the independence is established.

It is useful to have this density given in terms of our correlation (or covariance) functions ρ_o, and $\rho(\omega)$. In terms of these we have

$$p_\omega(s) = \frac{2s}{\sqrt{\rho_o^2-\rho^2(\omega)}} \exp\left[-\frac{\rho_o s^2}{\rho_o^2-\rho^2(\omega)}\right] I_o\left[\frac{\rho(\omega)s^2}{\rho_o^2-\rho^2(\omega)}\right]. \quad (4.11)$$

In this particular case we can be even more explicit, since ρ_o and $\rho(\omega)$ are known explicitly, from (4.9). Upon substitution we find (4.12)

$$p_\omega(s) = \frac{4\pi s}{\sigma^2 T_o\sqrt{1-\left(\frac{\sin \omega T_o}{\omega T_o}\right)^2}} \exp\left\{- \frac{2\pi s^2}{\sigma^2 T_o\left[1-\left(\frac{\sin \omega T_o}{\omega T_o}\right)^2\right]}\right\} I_o\left\{\frac{2\pi s^2 \frac{\sin \omega T_o}{\omega T_o}}{\sigma^2 T_o\left[1-\left(\frac{\sin \omega T_o}{\omega T_o}\right)^2\right]}\right\}.$$

Of course, all of these expressions are for $p_\omega(s)$ when $s \geqslant 0$; for $s \leqslant 0$ we have $p_\omega(s) = 0$ identically. The density function $p_o(s)$, for the special case $\omega = 0$, is given by

$$p_o(s) = \frac{2}{\sigma\sqrt{T_o}} \exp\left(- \frac{\pi s^2}{\sigma^2 T_o}\right) \qquad \text{for } s > 0.$$

In order that one may get a feeling for how the density function behaves as the parameter ω changes, we have included in Fig. 4-1 some graphs of $p_\omega(s)$ as a function of s for several values of ω. Graphs of $\sigma_1^2(\omega)$ and $\sigma_2^2(\omega)$ as a function of ω have been included in Fig. 4-2 to provide some idea of their behavior. For computational convenience we have chosen the constants $T_o = 2$ and $\sigma^2 = \pi$. These choices are unrealistic, but we are justified in making them since the graphs are not to be used quantitatively. Density functions were plotted for values of $\omega = 0, \frac{\pi}{12}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$. In Fig. 4-3 the distribution functions corresponding to the density functions plotted in Fig. 4-1 are plotted on Rayleigh probability paper. On this paper a Rayleigh distribution is represented by a straight line.

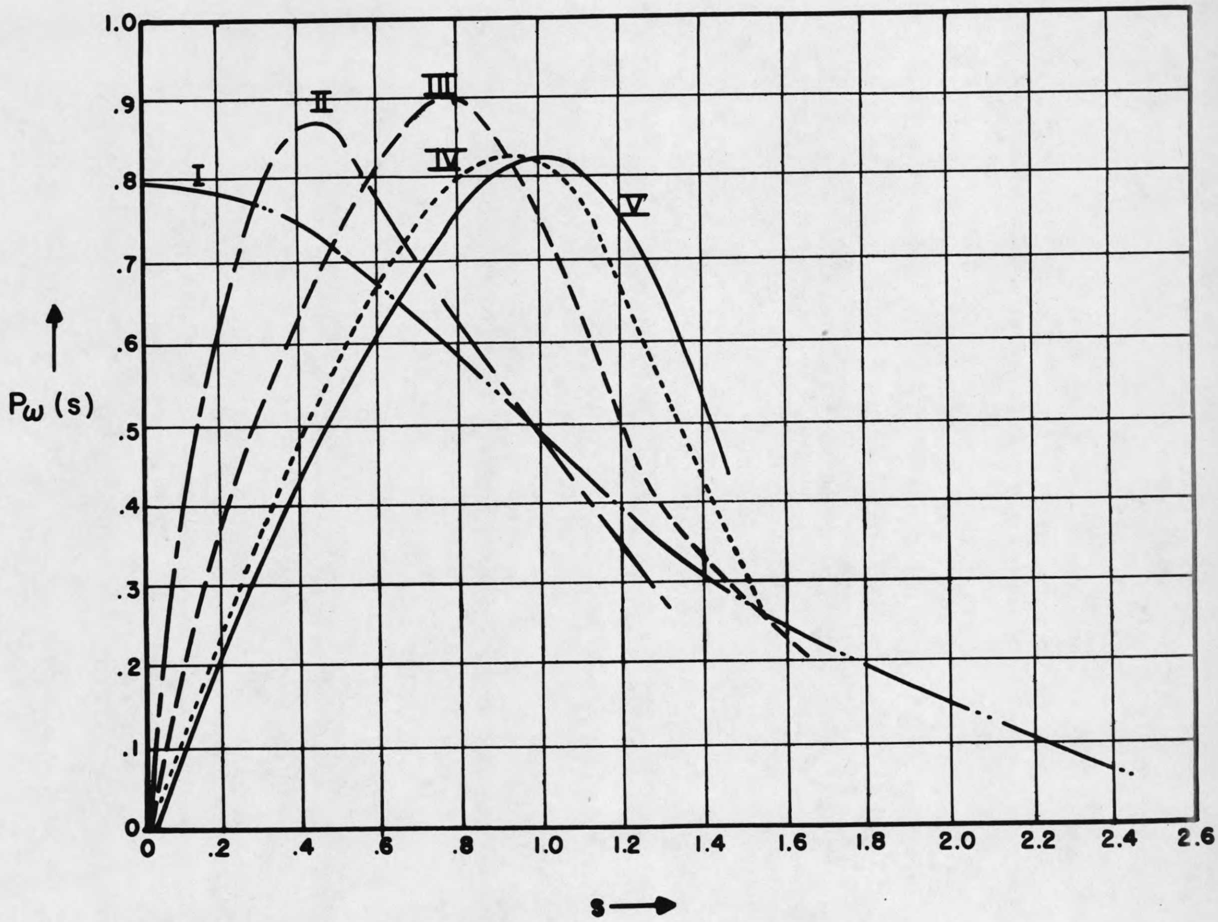

FIG. 4-1. Graphs of $p_\omega(s)$ (Eq. 4.12) with $T_o = 2$; $\sigma^2 = \pi$ for different values of ω.

$$p_\omega(s) = \frac{2s}{\sqrt{1 - \left(\frac{\sin 2\omega}{2\omega}\right)^2}} \exp\left[- \frac{s^2}{1 - \left(\frac{\sin 2\omega}{2\omega}\right)^2} \right] I_o \left[\frac{s^2 \frac{\sin 2\omega}{2\omega}}{1 - \left(\frac{\sin 2\omega}{2\omega}\right)^2} \right]$$

$p_{\pi/2}(s) = 2s \exp(-s^2)$ is a Rayleigh distribution

It occurs periodically for every $\omega = \frac{n\pi}{2}$ and is also the limit density function towards which $p_\omega(s)$ tends as $\omega \longrightarrow \infty$

CURVE		
CURVE	I	$\omega = 0$
"	II	$\omega = \frac{\pi}{12}$
"	III	$\omega = \frac{\pi}{4}$
"	IV	$\omega = \frac{\pi}{3}$
"	V	$\omega = \frac{\pi}{2}$

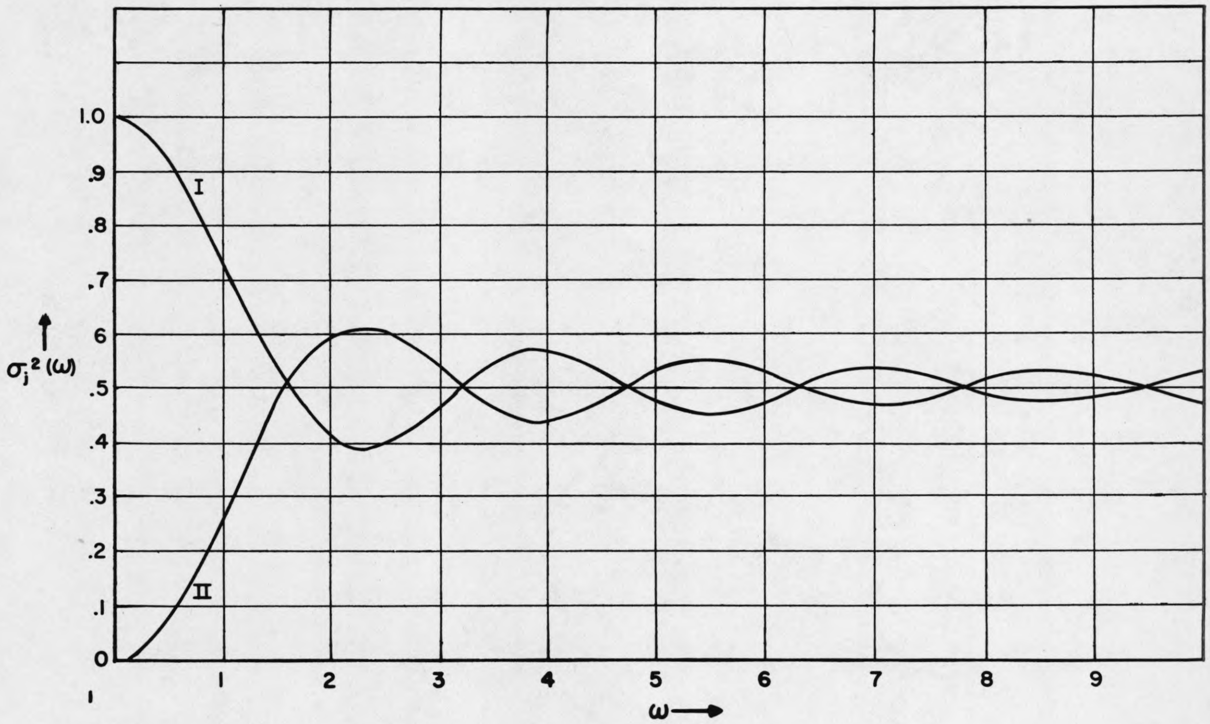

FIG. 4-2. Graphs of $\sigma_1^2(\omega)$ and $\sigma_2^2(\omega)$ (Eq. 4.10) with $T_o = 2$, $\sigma^2 = \pi$

Curve I $\quad \sigma_1^2(\omega) = \frac{1}{2}\left(1 + \frac{\sin 2\omega}{2\omega}\right)$

Curve II $\quad \sigma_2^2(\omega) = \frac{1}{2}\left(1 - \frac{\sin 2\omega}{2\omega}\right)$

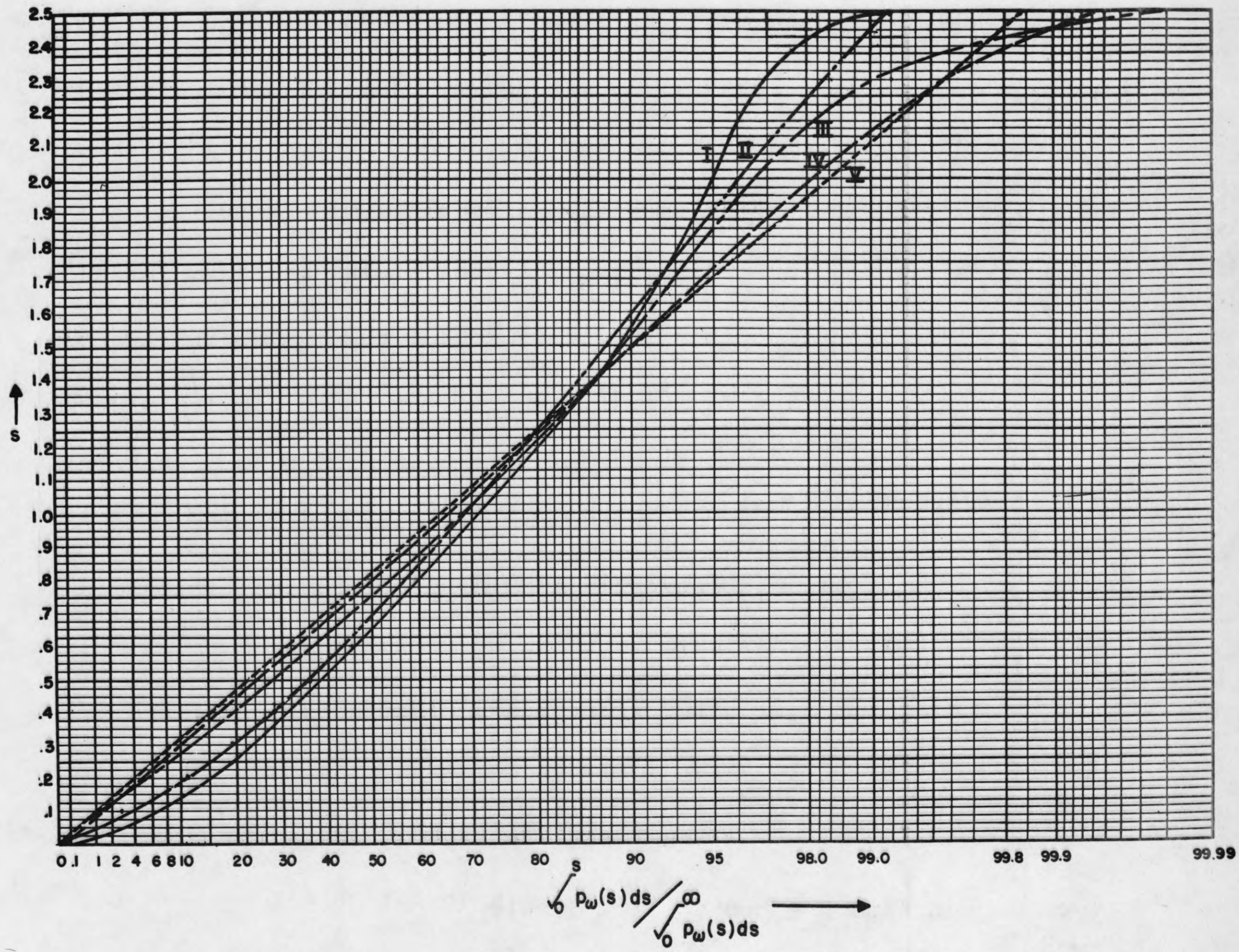

FIG. 4-3. Graphs of normalized probability distribution functions corresponding to the probability density functions plotted in FIG. 4-1.

These graphs are plotted on Rayleigh probability paper and consequently the Rayleigh distribution which results when $\omega = \frac{\pi}{2}$ is a straight line.

The points from which these curves were plotted were obtained by measuring areas under the probability density curves of FIG. 4-1.

CURVE		
CURVE	I	$\omega = 0$
"	II	$\omega = \frac{\pi}{12}$
"	III	$\omega = \frac{\pi}{4}$
"	IV	$\omega = \frac{\pi}{3}$
"	V	$\omega = \frac{\pi}{2}$

V. Signals Plus Additive Noise through the Sinufly Filter

Let us now consider the case in which we have a signal $s(t)$ present as well as Gaussian noise $n(t)$. Our input (before weighting) is now

$$f(t) = \gamma(t)\left\{ s(t) + n(t) \right\} .$$

Let the frequency spectrum of the signal $s(t)$ be

$$M(\omega) = \frac{1}{2\pi} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} e^{-i\omega t} s(t)\, dt,$$

then the spectrum $S_+(\omega)$ of $f(t)$ is given by

$$S_+(\omega) = M(\omega) + S_*(\omega).$$

Since $E\left\{ S_*(\omega) \right\} = 0$, we have

$$E\left\{ S_+(\omega) \right\} = M(\omega).$$

Let

$$M(\omega) = \mu(\omega) + i\ \nu(\omega), \tag{5.1}$$

then we can find the probability density

$$p_\omega(s) = \tag{5.2}$$

$$\frac{s}{\sigma_1\sigma_2} \exp\left[- \frac{(\sigma_1{}^2+\sigma_2{}^2)s^2+2(\sigma_2{}^2\mu^2+2\sigma_1{}^2\nu^2)}{4\sigma_1{}^2\sigma_2{}^2} \right] N\left[\frac{s\nu}{\sigma_2{}^2}, \frac{s\mu}{\sigma_1{}^2}, \frac{(\sigma_1{}^2-\sigma_2{}^2)}{4\sigma_1{}^2\sigma_2{}^2}s^2 \right] ,$$

from (A.18) in Appendix A. The μ and ν occurring in this formula are the $\mu(\omega)$ and $\nu(\omega)$ of (5.1). The $\sigma_1{}^2$ and $\sigma_2{}^2$ are the $\sigma_1{}^2(\omega)$ and $\sigma_2{}^2(\omega)$ of the previous section.

We must point out now, however, that the formula (5.2), although complete for the simple case under consideration, is actually too complex to be of much practical use. In order to illustrate what we mean, let us find $p_\omega(s)$ explicitly for a few simple examples.

First, suppose that $s(t)$ consists merely of a constant

$$s(t) = a,$$

then the spectrum of $s(t)\,\gamma(t)$ is

$$\frac{a}{\pi}\,\frac{\sin\frac{\omega T_o}{2}}{\omega}.$$

From (5.1) we see that

$$\mu(\omega) = \frac{a}{\pi}\,\frac{\sin\frac{\omega T_o}{2}}{\omega}$$

$$\nu(\omega) = 0.$$

Using the expressions for $\sigma_1{}^2(\omega)$ and $\sigma_2{}^2(\omega)$ from the previous section, we see that (5.2) becomes

$$p_\omega(s) = \frac{4\pi s}{\sigma^2 T_o\sqrt{1-\left(\frac{\sin\omega T_o}{\omega T_o}\right)^2}}\exp\left\{-\frac{s^2+\left(1-\frac{\sin\omega T_o}{\omega T_o}\right)\frac{a^2}{\pi^2}\left(\frac{\sin\frac{\omega T_o}{2}}{\omega}\right)^2}{\frac{\sigma^2 T_o}{2\pi}\left[1-\left(\frac{\sin\omega T_o}{\omega T_o}\right)^2\right]}\right\}$$

$$N\left\{0,\ \frac{4as\sin\omega\frac{T_o}{2}}{\sigma^2\omega T_o\left(1+\frac{\sin\omega T_o}{\omega T_o}\right)},\ \frac{2\pi s^2\sin\omega T_o}{\sigma^2\omega T_o\left[1-\left(\frac{\sin\omega T_o}{\omega T_o}\right)^2\right]}\right\}.$$

(The function $N(\alpha, \beta, \gamma)$ is discussed in Appendix B.)

For values of ωT_o sufficiently large, $\frac{1}{\omega T_o}$ is negligible compared to unity and $p_\omega(s)$ becomes

$$p_\omega(s) = \frac{4\pi s}{\sigma^2 T_o}\exp\left\{-\frac{2\pi s^2+\frac{2a^2}{\pi}\left(\frac{\sin\frac{\omega T_o}{2}}{\omega}\right)^2}{\sigma^2 T_o}\right\} I_o\left(\frac{2s\cdot 2a\,\frac{\sin\frac{\omega T_o}{2}}{\omega}}{\sigma^2 T_o}\right).$$

In order to see this we must make the approximation $\frac{\sin \omega T_o}{\omega T_o} = 0$ and apply (A.16) and (A.17) of Appendix A. I_o is the modified Bessel coefficient of order zero.[1]

This density is a familiar one, presented by many authors. It can be put in a more familiar form by setting $\sigma^2 T_o = 4\pi\psi_o$ and using

$$\mu = \frac{a}{\pi} \frac{\sin\frac{\omega T_o}{2}}{\omega}$$

for the mean, then the expression for $p_\omega(s)$ given above becomes

$$p_\omega(s) = \frac{s}{\psi_o} e^{-\left(\frac{s^2+\mu^2}{2\psi_o}\right)} I_o\left(\frac{s\mu}{\psi_o}\right).$$

This expression is a natural one to use if it can be assumed that T_o is large and ω is bounded away from zero.

Unfortunately, the approximation $\frac{1}{\omega T_o} << 1$ is not valid in our application. This is best appreciated by referring to one of the examples in section VII (say example 1, p. 50) where in a typical sinufly system we are interested in studying the range of values of ω from zero to $\frac{\omega_r}{2} = 2000\ \pi$ radians per second. In this example, $T_o = .03$ seconds and the approximation $\frac{1}{\omega T_o} << 1$ is good for $\omega T_o > 10$ or $\omega > 333$ radians per second and poor in the range of frequencies $\omega = 0$ to $\omega = 333$ radians per second.

In order to indicate further the great complexity which is manifest in even the simplest cases, we shall calculate $p_\omega(s)$ for $s(t) = a_o + a \cos \mu t + b \sin \mu t$, a simple sinusoidal signal plus a constant term. Again working from (3.2) and (3.3) we see that the spectrum of

[1]Ref. 5, p. 373

$\gamma(t)\ s(t)$ is

$$M(\omega) = \frac{a_o}{\pi}\,\frac{\sin\frac{\omega T_o}{2}}{\omega} + \frac{a}{2\pi}\left\{\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)} + \frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right\}$$

$$+ \frac{b}{2\pi i}\left\{\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)} - \frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right\}$$

$$= \mu(\omega) + i\ \nu(\omega). \tag{5.3}$$

With a little squaring of expressions here and there, we find, from (5.2) and (5.3) that $p_\omega(s)$ is given by

$$p_\omega(s) = \frac{4\pi s}{\sigma^2 T_o\sqrt{1-\left(\frac{\sin\omega T_o}{\omega T_o}\right)^2}}\ \exp\left\{-2\pi\left(s^2 + \frac{1}{\pi^2}\left(1-\frac{\sin\omega T_o}{\omega T_o}\right)\left[a_o^{\,2}\left(\frac{\sin\frac{\omega T_o}{2}}{\omega}\right)^2\right.\right.\right.$$

$$+ aa_o\,\frac{\sin\frac{\omega T_o}{2}}{\omega}\left(\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)} + \frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right) + \frac{a^2}{4}\left(\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)}\right)^2$$

$$\left.+ \frac{a^2}{2}\,\frac{\sin(\omega-\mu)\frac{T_o}{2}\ \sin(\omega+\mu)\frac{T_o}{2}}{(\omega^2-\mu^2)} + \frac{a^2}{4}\left(\frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right)^2\right] + \frac{b^2}{4\pi^2}\left(1+\frac{\sin\omega T_o}{\omega T_o}\right)\left[\left(\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)}\right)^2\right.$$

$$\left.\left.\left. - 2\,\frac{\sin(\omega-\mu)\frac{T_o}{2}\ \sin(\omega+\mu)\frac{T_o}{2}}{(\omega^2-\mu^2)} + \left(\frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right)^2\right]\right)\frac{1}{\sigma^2 T_o\left(1-\left(\frac{\sin\omega T_o}{\omega T_o}\right)^2\right)}\right\}.$$

$$N\left(\frac{-2sb\left(\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)} - \frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right)}{\sigma^2 T_o\left(1-\frac{\sin\omega T_o}{\omega T_o}\right)},\ \frac{4sa_o\,\frac{\sin\omega\frac{T_o}{2}}{\omega} + 2sa\left\{\frac{\sin(\omega-\mu)\frac{T_o}{2}}{(\omega-\mu)} + \frac{\sin(\omega+\mu)\frac{T_o}{2}}{(\omega+\mu)}\right\}}{\sigma^2 T_o\left(1+\frac{\sin\omega T_o}{\omega T_o}\right)},\right.$$

$$\left.\frac{2\pi s^2\,\frac{\sin\omega T_o}{\omega T_o}}{\sigma^2 T_o\left(1-\left(\frac{\sin\omega T_o}{\omega T_o}\right)^2\right)}\right).$$

A formula of this complexity is somewhat discouraging and one would attempt to make graphs from it only in a case of direst need. It is unlikely that we shall ever be so tempted since, in any case, the signal presented here is still far too simple to be realistic.

VI. Signals and Multiplicative Noise through the Sinufly Filter

There is a certain type of noise we encounter which is proportional to the strength of the detected signal. This noise can be represented by

$$f(t) = \gamma(t)\, s(t)\, n(t) \tag{6.1}$$

where $\gamma(t) \cdot s(t)$ is signal and $n(t)$ is a noise factor. Such noise is usually called multiplicative noise. Our method may be applied to deal with it as follows. Let

$$\gamma(t)\, s(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, M(\omega)\, d\omega, \tag{6.2}$$

and

$$n(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, N(\omega)\, d\omega \tag{6.3}$$

as before. Then

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, S(\omega)\, d\omega,$$

where

$$S(\omega) = \int_{-\infty}^{\infty} M(\alpha)\, N(\omega-\alpha) d\alpha. \tag{6.4}$$

As an illustration, let us consider the case where $n(t)$ is white Gaussian noise and we assume that $n(t)$ has zero mean for each value of t. As before, what we wish to know is the probability density function for $\left|S(\omega)\right|$. Since $N(\omega)$ is Gaussian, so is $S(\omega)$, consequently we need only to know the variances of the real and imaginary parts of $S(\omega)$ in order to know its distribution completely.

Proceeding as before, we have

$$E\left\{S(\omega_1)\ \overline{S}(\omega_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(\alpha)\ \overline{M}(\beta)\ E\left\{N(\omega_1-\alpha)\ \overline{N}(\omega_2-\beta)\right\} d\alpha\ d\beta,$$

and using the fact that

$$E\left\{N(\omega_1-\alpha)\ \overline{N}(\omega_2-\beta)\right\} = \sigma^2\delta(\omega_2-\omega_1+\alpha-\beta),$$

which comes from (4.5), we see that

$$E\left\{S(\omega_1)\ \overline{S}(\omega_2)\right\} = \sigma^2\int_{-\infty}^{\infty} M(\alpha+\omega_1)\ \overline{M}(\alpha+\omega_2)\ d\alpha.$$

We now define

$$\rho(\omega_1,\ \omega_2) = E\left\{S(\omega_1)\ S(\omega_2)\right\}.$$

Since $S(\omega) = \overline{S}(-\omega)$, we have

$$\begin{aligned}\rho(\omega_1,\ \omega_2) &= E\left\{S(\omega_1)\ \overline{S}(-\omega_2)\right\}\\ &= \sigma^2\int_{-\infty}^{\infty} M(\alpha+\omega_1)\ \overline{M}(\alpha-\omega_2)\ d\alpha.\end{aligned}$$

We further define

$$\rho(\omega) = \rho(\omega,\omega),\ \text{and}\ \rho_o = \rho(0)$$

and we clearly have

$$\rho(\omega,\ -\omega) = \rho(0) = \rho_o.$$

We observe that $M(\omega)$ is in general complex, in contrast to $H(\omega)$ which is real; consequently, $\rho(\omega)$ can no longer be assumed to be real. Let $\rho_1(\omega)$ and $\rho_2(\omega)$ be the real and imaginary parts of $\rho(\omega)$ so that

$$\rho(\omega) = \rho_1(\omega) + i\ \rho_2(\omega).$$

The real and imaginary parts of $S(\omega)$ are respectively $\frac{1}{2}\left[S(\omega) + \overline{S}(\omega)\right]$ and $\frac{1}{2i}\left[S(\omega) - \overline{S}(\omega)\right]$. Thus their respective variances $\sigma_1{}^2(\omega)$ and $\sigma_2{}^2(\omega)$ are given by

$$\sigma_1{}^2(\omega) = \frac{1}{2}\left[\rho_o + \rho_1(\omega)\right]$$

$$\sigma_2{}^2(\omega) = \frac{1}{2}\left[\rho_o - \rho_1(\omega)\right] .$$

The correlation coefficient of the real and imaginary parts of $S(\omega)$ is[1]

$$r(\omega) = \frac{\rho_2(\omega)}{\sqrt{\rho_o{}^2 - \rho_1{}^2(\omega)}} = \frac{E\left\{\frac{1}{2}\left[S(\omega)+\overline{S}(\omega)\right]\frac{1}{2i}\left[S(\omega)-\overline{S}(\omega)\right]\right\}}{\sigma_1\sigma_2}$$

From these relations we see that

$$\rho_1 = \sigma_1{}^2 - \sigma_2{}^2, \qquad \rho_2 = 2r\,\sigma_1\sigma_2, \qquad \rho_o = \sigma_1{}^2 + \sigma_2{}^2$$

and

$$4(1-r^2)\,\sigma_1{}^2\,\sigma_2{}^2 = \rho_o{}^2 - \left|\rho\right|^2 .$$

Since

$$W(0,0,\alpha,\beta) = I_o\left(\sqrt{\alpha^2+\beta^2}\right),$$

where $W(\cdot)$ is defined in (A.14), Appendix A, and if we use the formula (A.19) for the distribution, (the special case, with zero means) we see that the density function $p_\omega(s)$ of $\left|S(\omega)\right|$ is given by

$$p_\omega(s) = \frac{2s}{\sqrt{\rho_o{}^2-\left|\rho(\omega)\right|^2}}\exp\left(-\frac{\rho_o s^2}{\rho_o{}^2-\left|\rho(\omega)\right|^2}\right)I_o\left(\frac{\left|\rho(\omega)\right|s^2}{\rho_o{}^2-\left|\rho(\omega)\right|^2}\right).$$

This expression clearly reduces to the one given in (4.11) if $\rho_2(\omega) = 0$, where $\rho_1(\omega)$ remains general and is not the special case given in (4.10).

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[1]Ref. 8, p. 277

An alternate method of treating the multiplicative noise encountered in the output of the storage tube is the following. Under sufficiently ideal conditions such noise need not be considered as random. For example, suppose that under identical conditions of storage on a given range element of the tube, a specific output noise function $n(t)$ multiplies the desired storage signal. Specifically, if the stored signal is a D-C term of strength $\underline{a}$, the output would be $\underline{a} \cdot n(t)$ each time the entire process is carried through.

We can, therefore, treat the problem of multiplicative noise from the storage tube in the following way. Let the lines to be read out be numbered with an index j, and let the noise function, for a constant term of strength 1, be $n_j(t)$ for the jth line. Then the representation of the wave packet arising from the jth line can be given by

$$f_j(t) = \gamma(t)\ s(t)\ n_j(t),$$

which replaces (6.1). We represent the spectrum of $n_j(t)$ by $N_j(\omega)$, as in (6.3), and thus find the spectrum $S_j(\omega)$ of $f_j(t)$ by

$$S_j(\omega) = \int_{-\infty}^{\infty} N_j(\omega-\alpha)\ M(\alpha)\ d\alpha, \qquad (6.5)$$

the analogue of (6.4).

To carry out the analysis of the multiplicative noise arising from the storage tube, based on these latter assumptions we would examine each line (of the read out) separately, using (6.5).

In a practical sinufly system the multiplicative noise read out of the storage tube has both a random and a nonrandom component and a precise analysis of it would require the use of both of the techniques proposed in this section.

VII. Clutter and Enhancement

The analysis for white noise, given in section IV, can easily be extended to include other stochastic processes. The assumption of whiteness is the assumption that the power spectrum $W(\omega)$, occurring in (4.5), is a constant σ^2. This assumption is by no means necessary, and we shall now repeat part of the development without making use of it.

Let

$$c(t) = \int_{-\infty}^{\infty} e^{i\omega t} N(\omega)\, d\omega$$

and let

$$f(t) = \gamma(t)\, c(t) = \int_{-\infty}^{\infty} e^{i\omega t} S(\omega)\, d\omega, \tag{7.1}$$

then

$$S(\omega) = \int_{-\infty}^{\infty} H(\alpha)\, N(\omega-\alpha)\, d\alpha,$$

where $H(\omega)$ is the function defined in (3.2).

The covariance function for $S(\omega_1)$, and $\overline{S}(\omega_2)$ is given by

$$E\left\{S(\omega_1)\, \overline{S}(\omega_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\alpha)\, H(\beta)\, E\left\{N(\omega_1-\alpha)\, \overline{N}(\omega_2-\beta)\right\} d\alpha\, d\beta. \tag{7.2}$$

Using (4.5) we see that (7.2) above becomes

$$E\left\{S(\omega_1)\, \overline{S}(\omega_2)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\alpha)\, H(\beta)\, W(\omega_2-\beta)\, \delta(\omega_2-\omega_1+\alpha-\beta)\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{\infty} H(\alpha)\, H(\omega_2-\omega_1+\alpha)\, W(\omega_1-\alpha)\, d\alpha.$$

If we replace α by $\omega_1-\alpha$, we obtain the formula

$$E\left\{S(\omega_1)\ \bar{S}(\omega_2)\right\} = \int_{-\infty}^{\infty} W(\alpha)\ H(\omega_1-\alpha)H(\omega_2-\alpha)\ d\alpha.$$

More explicitly

$$E\left\{S(\omega_1)\ \bar{S}(\omega_2)\right\} = \frac{1}{\pi^2}\int_{-\infty}^{\infty} W(\alpha)\ \frac{\sin(\omega_1-\alpha)\frac{T_o}{2}}{(\omega_1-\alpha)}\ \frac{\sin(\omega_2-\alpha)\frac{T_o}{2}}{(\omega_2-\alpha)}\ d\alpha. \qquad (7.3)$$

If we assume a random ground model[1], the form of the power spectrum for clutter is shown in Fig. 7.1 and is given by the expression

$$W(\alpha) = (m\alpha_o)^2\ \delta(\alpha) + \xi(\alpha) \qquad (7.4)$$

where

$$\xi(\alpha) \begin{cases} = m^2(\alpha+\alpha_o) & \text{for } -\alpha_o \leqslant \alpha \leqslant 0 \\ = -m^2(\alpha-\alpha_o) & \text{for } 0 \leqslant \alpha \leqslant \alpha_o \\ = 0 & \text{otherwise} \end{cases}$$

where α_o is the width of the clutter spectrum.

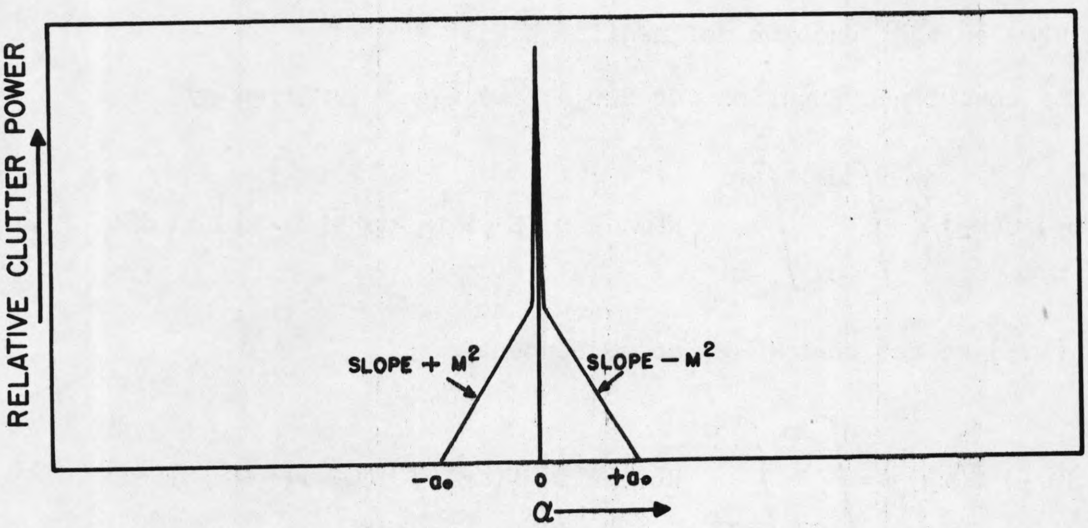

FIG. 7-1. Video power spectrum for clutter, assuming a non-coherent radar, a square law detector and the random ground model[1] to represent the terrain. (Eq. 7.4)

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[1] Ref. 7, p. 36

For this particular ground model, the power in the D-C part of the clutter spectrum is equal to the power in the A-C part of the spectrum. Later, when signal is introduced it will be seen that the power in the D-C part of the spectrum is increased.[1]

We now introduce the <u>mean</u> <u>power</u> <u>function</u>

$$\psi_c(\omega) = E\left\{ \left| S(\omega) \right|^2 \right\}. \tag{7.5}$$

In our case, using the $\xi(\alpha)$ introduced above we find $\psi_c(\omega)$ to be given by

$$\psi_c(\omega) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \left\{ m^2\alpha_o{}^2\, \delta(\alpha) + \xi(\alpha) \right\} \frac{\sin^2(\omega-\alpha)\frac{T_o}{2}}{(\omega-\alpha)^2}\, d\alpha.$$

This is easily seen to be

$$\psi_c(\omega) = \left(\frac{m\alpha_o}{\pi}\, \frac{\sin\frac{\omega T_o}{2}}{\omega} \right)^2 - \frac{m^2}{\pi^2}\int_0^{\alpha_o} (\alpha-\alpha_o) \left\{ \frac{\sin^2(\omega-\alpha)\frac{T_o}{2}}{(\omega-\alpha)^2} + \frac{\sin^2(\omega+\alpha)\frac{T_o}{2}}{(\omega+\alpha)^2} \right\} d\alpha.$$

The integral

$$\int_0^{\alpha_o} (\alpha-\alpha_o)\, \frac{\sin^2(\alpha-\omega)\frac{T_o}{2}}{(\alpha-\omega)^2}\, d\alpha \tag{7.6}$$

can be evaluated by setting $\beta = \alpha-\omega$, so that (7.6) becomes

$$\int_{-\omega}^{\alpha_o-\omega} (\beta+\omega-\alpha_o)\, \frac{\sin^2\frac{\beta T_o}{2}}{\beta^2}\, d\beta =$$

$$\int_{-\frac{\omega T_o}{2}}^{(\alpha_o-\omega)\frac{T_o}{2}} \frac{\sin^2 u}{u}\, du + \frac{T_o}{2}(\omega-\alpha_o)\int_{-\frac{\omega T_o}{2}}^{(\alpha_o-\omega)\frac{T_o}{2}} \frac{\sin^2 u}{u^2}\, du.$$

---

[1] Ref. 6

But integration by parts shows that

$$\int_0^x \frac{\sin^2 u}{u^2}\, du = \int_0^{2x} \frac{\sin u}{u}\, du - \frac{\sin^2 x}{x}, \tag{7.7}$$

hence

$$\int_0^{\alpha_o} (\alpha-\alpha_o) \frac{\sin^2(\alpha-\omega)\frac{T_o}{2}}{(\alpha-\omega)^2}\, d\alpha =$$

$$\left[Ss(x) + \frac{T_o}{2}(\omega-\alpha_o)\left(Si(2x) - \frac{\sin^2 x}{x} \right) \right]_{-\frac{\omega T_o}{2}}^{(\alpha_o-\omega)\frac{T_o}{2}} .$$

In like manner we see that

$$\int_0^{\alpha_o} (\alpha-\alpha_o) \frac{\sin^2(\alpha+\omega)\frac{T_o}{2}}{(\alpha+\omega)^2}\, d\alpha =$$

$$\left[Ss(x) - \frac{T_o}{2}(\omega+\alpha_o)\left(Si(2x) - \frac{\sin^2 x}{x} \right) \right]_{\frac{\omega T_o}{2}}^{(\alpha_o+\omega)\frac{T_o}{2}} .$$

Consequently, $\psi_c(\omega)$ is given explicitly by

$$\psi_c(\omega) = \left(\frac{m\alpha_o}{\pi} \frac{\sin\frac{\omega T_o}{2}}{\omega} \right)^2 - \frac{m^2}{\pi^2} \left\{ Ss\ (\omega-\alpha_o)\frac{T_o}{2} + Ss\ (\omega+\alpha_o)\frac{T_o}{2} - 2\ Ss(\frac{\omega T_o}{2}) \right.$$

$$- \frac{T_o}{2} \left[(\omega-\alpha_o)\left(Si\left[(\omega-\alpha_o)T_o \right] - \frac{\sin^2(\omega-\alpha_o)\frac{T_o}{2}}{(\omega-\alpha_o)\frac{T_o}{2}} \right) + (\omega+\alpha_o)\left(Si\ (\omega+\alpha_o)T_o \right.\right.$$

$$\left.\left.\left. - \frac{\sin^2(\omega+\alpha_o)\frac{T_o}{2}}{(\omega+\alpha_o)\frac{T_o}{2}} \right) \right] + \omega T_o \left(Si\ (\omega T_o) - \frac{\sin^2 \frac{\omega T_o}{2}}{\frac{\omega T_o}{2}} \right) \right\}. \tag{7.8}$$

The functions Ss(x) and Si(x) are defined by

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin u}{u}\,du\,, \qquad \mathrm{Ss}\,(x) = \int_0^x \frac{\sin^2 u}{u}\,du\,.$$

These functions are well known and well tabulated.[1]

The function $\psi_c(\omega)$ given by (7.8) is the mean power function for clutter alone. We shall also need the mean power function for signal plus clutter. We shall denote this as $\psi_{s+c}(\omega,\nu)$. As an example of our method we will consider a function f(t) which consists of a single non-random sinusoidal signal $s(t) = a \sin \nu t$ in addition to a random clutter signal c(t),

$$f(t) = \gamma(t)\left[c(t) + s(t)\right] = \gamma(t)\,c(t) + \gamma(t)\,s(t)$$

From (6.2) and (7.1) the spectrum of f(t) is given by

$$S(\omega) + M(\omega,\nu) = \frac{1}{2\pi}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} c(t)\,e^{-i\omega t}dt + \frac{1}{2\pi}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} s(t)\,e^{-i\omega t}dt\,.$$

$$\psi_{s+c}(\omega,\nu) = E\left\{\left|\left[S(\omega) + M(\omega,\nu)\right]\right|^2\right\} = E\left\{\left[S(\omega) + M(\omega,\nu)\right]\left[\overline{S}(\omega)+\overline{M}(\omega,\nu)\right]\right\}$$

$$= E\left\{S(\omega)\,\overline{S}(\omega)\right\} + E\left\{M(\omega,\nu)\,\overline{M}(\omega,\nu)\right\} = \psi_c(\omega) + \left|M(\omega,\nu)\right|^2$$

where

$$E\left\{S(\omega)\,\overline{M}(\omega,\nu)\right\} = E\left\{\overline{S}(\omega)\,M(\omega,\nu)\right\} = 0$$

because the signal and clutter are uncorrelated and $S(\omega)$ has zero mean.

We may now define an "enhancement function" $G(\nu)$ which will be used as a criterion for judging the quality of performance of a given system, and for comparing different systems.

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[1] Ref's. 10 and 11

$$G(\nu) = \frac{\psi_{s+c}(\omega,\nu)}{\psi_c(\omega)} = \frac{\psi_c(\omega) + \left|M(\omega,\nu)\right|^2}{\psi_c(\omega)} = 1 + \frac{\left|M(\omega,\nu)\right|^2}{\psi_c(\omega)}$$

We define signal strength for a signal $\gamma(t)\, s(t)$ by

$$\left\{ \int_{-\infty}^{+\infty} s^2(t)\, \gamma(t)\, dt \right\}^{1/2}$$

Applying Parseval's theorem we have

$$\left\{ \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} s^2(t)\, dt \right\}^{1/2} = \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|M(\omega,\nu)\right|^2 d\omega \right\}^{1/2} \tag{7.9}$$

as our expression for signal strength.

The corresponding expression for <u>mean</u> clutter strength is

$$\left\{ \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} E\left\{ c^2(t) \right\} dt \right\}^{1/2} = \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi_c(\omega)\, d\omega \right\}^{1/2} \tag{7.10}$$

The signal to clutter ratio is then obtained by dividing (7.9) by (7.10).

If one substitutes values of the parameters m, α_o, T_o, and ω into (7.8), $\psi_c(\omega)$ can be determined explicitly. Also $\left|M(\omega,\nu)\right|^2$ could be obtained from (3.4), and $G(\nu)$ could then be plotted.

Rather than proceed in this way however, we will turn our attention to a realistic example; one in which we do not make the assumption that the signal and clutter at the output of the second detector are independent random processes. Just such an example is furnished by the random ground model with a single moving target located in the center of the ground patch and moving with a velocity such that its center doppler angular frequency at the output of the radar second detector is ν. The power spectrum corresponding to this model has been derived by R. Swallow[1] and is illustrated in Fig. 7-2 and given by the expression

$$W_1(\alpha) = m^2\alpha_o^2(1 + X)^2\,\delta(\alpha) + \xi_1(\alpha) \tag{7.11}$$

where X is defined by[2]

$$X = a^2 \Big/ \sum_{j=1}^{J} E_j^{\,2} \;=\; a^2/JE^2$$

a = amplitude of the moving target return

E_j = amplitude of return from each of J random scatterers in the patch

and where

$$\xi_1(\alpha) \begin{cases} = m^2(\alpha+\alpha_o) & -\alpha_o \leqslant \alpha \leqslant 0 \\ = -m^2(\alpha-\alpha_o) & 0 \leqslant \alpha \leqslant \alpha_o \\ = m^2\,\alpha_o\,X & \nu - \frac{\alpha_o}{2} \leqslant \alpha \leqslant \nu + \frac{\alpha_o}{2} \\ = m^2\,\alpha_o\,X & -\nu - \frac{\alpha_o}{2} \leqslant \alpha \leqslant -\nu + \frac{\alpha_o}{2} \\ = 0 & \text{otherwise} \end{cases}$$

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[1]Ref. 6

[2]Ref. 7, pp. 4, 23, 28

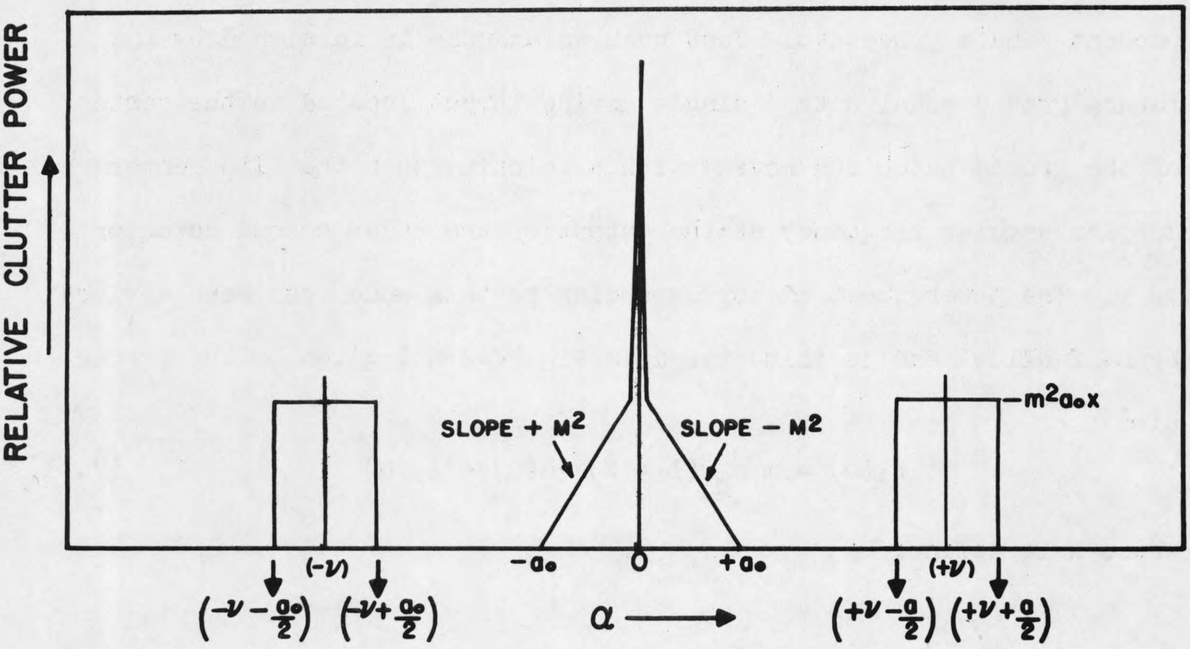

FIG. 7-2. Video power spectrum for clutter, plus a target having a center doppler angular freq. ν, assuming a non-coherent radar and a square law detector. (Eq. 7.11) The signal to clutter ratio, (X) is 1 in the example illustrated.

From (7.3) and (7.5) we see that

$$\Psi_{s+c}(\omega,\nu) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} W_1(\alpha)\,\frac{\sin^2(\omega-\alpha)\frac{T_o}{2}}{(\omega-\alpha)^2}\,d\alpha \tag{7.12}$$

$$\Psi_{s+c}(\omega,\nu) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\left\{m^2\alpha_o{}^2(1+2X+X^2)\,\delta(\alpha) + \xi_1(\alpha)\right\}\frac{\sin^2(\omega-\alpha)\frac{T_o}{2}}{(\omega-\alpha)\frac{T_o}{2}}\,d\alpha.$$

$$\Psi_{s+c}(\omega,\nu) =$$

$$\Psi_c(\omega) + (2X+X^2)\left(\frac{m\alpha_o}{\pi}\,\frac{\sin\frac{\omega T_o}{2}}{\omega}\right) + \frac{m^2\alpha_o X}{\pi^2}\int_{\nu-\frac{\alpha_o}{2}}^{\nu+\frac{\alpha_o}{2}}\frac{\sin^2(\omega-\alpha)\frac{T_o}{2}}{(\omega-\alpha)^2}\,d\alpha$$

$$+\frac{m^2\alpha_o X}{\pi^2}\int_{-\nu-\frac{\alpha_o}{2}}^{-\nu+\frac{\alpha_o}{2}}\frac{\sin^2(\omega-\alpha)\frac{T_o}{2}}{(\omega-\alpha)^2}\,d\alpha. \tag{7.13}$$

The second term in (7.13) represents the D-C power added to the spectrum due to the presence of the moving target, while the third and fourth terms represent its A-C power.

The integral

$$\int_{\nu-\frac{\alpha_o}{2}}^{\nu+\frac{\alpha_o}{2}} \frac{\sin^2(\alpha-\omega)\frac{T_o}{2}}{(\alpha-\omega)^2}\, d\alpha \tag{7.14}$$

can be evaluated by setting $\beta = \alpha-\omega$, so that (7.14) becomes

$$\int_{\nu-\frac{\alpha_o}{2}-\omega}^{\nu+\frac{\alpha_o}{2}-\omega} \frac{\sin^2\beta\frac{T_o}{2}}{\beta^2}\, d\beta = \frac{T_o}{2}\int_{(\nu-\frac{\alpha_o}{2}-\omega)\frac{T_o}{2}}^{(\nu+\frac{\alpha_o}{2}-\omega)\frac{T_o}{2}} \frac{\sin^2 u}{u^2}\, du .$$

It was shown previously in (7.7) that

$$\int_0^x \frac{\sin^2 u}{u^2}\, du = \int_0^{2x} \frac{\sin u}{u}\, du - \frac{\sin^2 x}{x} .$$

Therefore, (7.14) becomes

$$\frac{T_o}{2}\left\{ \mathrm{Si}\,\left(\nu+\frac{\alpha_o}{2}-\omega\right)T_o - \frac{\sin^2(\nu+\frac{\alpha_o}{2}-\omega)\frac{T_o}{2}}{(\nu+\frac{\alpha_o}{2}-\omega)\frac{T_o}{2}} - \mathrm{Si}\,\left(\nu-\frac{\alpha_o}{2}-\omega\right)T_o \right.$$

$$\left. + \frac{\sin^2(\nu-\frac{\alpha_o}{2}-\omega)\frac{T_o}{2}}{(\nu-\frac{\alpha_o}{2}-\omega)\frac{T_o}{2}} \right\}.$$

In a like manner

$$\int_{-\nu-\frac{\alpha_o}{2}}^{-\nu+\frac{\alpha_o}{2}} \frac{\sin^2(\alpha-\omega)\frac{T_o}{2}}{(\alpha-\omega)^2}\, d\alpha$$

becomes

$$\frac{T_o}{2}\left\{\mathrm{Si}\left(-\nu+\frac{\alpha_o}{2}-\omega\right)T_o-\frac{\sin^2\left(-\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(-\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}-\mathrm{Si}\left(-\nu-\frac{\alpha_o}{2}-\omega\right)T_o\right.$$

$$\left.+\frac{\sin^2\left(-\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(-\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}\right\}.$$

Thus (7.13) becomes

$$\psi_{s+c}(\omega,\nu)=\psi_c(\omega)+(2X+X^2)\left(\frac{m\alpha_o}{\pi}\,\frac{\sin\frac{\omega T_o}{2}}{\omega}\right)^2+\frac{m^2\alpha_o XT_o}{2\pi^2}\left\{\mathrm{Si}\left(\nu+\frac{\alpha_o}{2}-\omega\right)T_o\right.$$

$$-\mathrm{Si}\left(\nu-\frac{\alpha_o}{2}-\omega\right)T_o+\mathrm{Si}\left(-\nu+\frac{\alpha_o}{2}-\omega\right)T_o-\mathrm{Si}\left(-\nu-\frac{\alpha_o}{2}-\omega\right)T_o$$

$$-\frac{\sin^2\left(\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}+\frac{\sin^2\left(\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}-\frac{\sin^2\left(-\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(-\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}$$

$$\left.+\frac{\sin^2\left(-\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(-\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}\right\}.$$

From (7.12) we now get

$$G(\nu)=\frac{\psi_{s+c}(\omega,\nu)}{\psi_c(\omega)}=1+\frac{(2X+X^2)\left(\frac{m\alpha_o}{\pi}\,\frac{\sin\frac{\omega T_o}{2}}{\omega}\right)^2}{\psi_c(\omega)}$$

$$+\frac{m^2\alpha_o XT_o}{2\pi^2\psi_c(\omega)}\left\{\mathrm{Si}\left(\nu+\frac{\alpha_o}{2}-\omega\right)T_o-\mathrm{Si}\left(\nu-\frac{\alpha_o}{2}-\omega\right)T_o+\mathrm{Si}\left(-\nu+\frac{\alpha_o}{2}-\omega\right)T_o\right.$$

$$-\mathrm{Si}\left(-\nu-\frac{\alpha_o}{2}-\omega\right)T_o-\frac{\sin^2\left(\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}+\frac{\sin^2\left(\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}$$

$$\left.-\frac{\sin^2\left(-\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(-\nu+\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}+\frac{\sin^2\left(-\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}{\left(-\nu-\frac{\alpha_o}{2}-\omega\right)\frac{T_o}{2}}\right\}. \qquad (7.15)$$

In order to illustrate further these ideas we choose some realistic values for the constants in (7.8) and (7.15), and graph the resultant enhancement functions.

It might be worthwhile to point out here that the functions we have been discussing deviate significantly from the functions which appear at the input to the sinufly filter. The most important aspect of this deviation results from the fact that f(t) is read off a storage tube, in a manner described in the introduction, consequently it is of the form

$$f(t) = \gamma(t)\, c(t) \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} g(t-n\tau)$$

rather than

$$f'(t) = \gamma(t)\, c(t)$$

as previously assumed.

The function

$$\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} g(t-n\tau)$$

represents the modification of the signal $\gamma(t)\, c(t)$ resulting from reading the video information off of the storage tube in a direction orthogonal to the direction in which it had previously been stored. g(t) describes the profiles of the individual lines read off the tube and τ is the time interval between these lines.

The main effect of this "combing"[1] of the signal is that the resulting spectrum of f(t) is "folded" about a frequency $f_r = \frac{1}{\tau}$. Functions of the spectrum of f(t), such as $G(\nu)$ are also folded about

[1] Ref. 9, p. 28 (7)

the frequency f_r. Consequently it is necessary to plot $G(\nu)$ only over the range of frequencies from zero to $\frac{f_r}{2}$ at most.

Other modifications of the spectrum are dependent upon the exact form of the function $g(t)$.[1] If $g(t)$ is non zero only over a time interval $\tau' < \tau$, this means we have complete resolution of the radar range traces in the read out packet; the form of $g(t)$ can then be neglected and it can be approximated by a δ function.[1] With these assumptions

$$f(t) = \gamma(t)\, c(t) \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \delta(t-n\tau).$$

	X (Signal to clutter ratio)	T_o (Seconds)	$\omega_r = 2\pi f_r$ (radians per second)	ω/ω_r	α_o/ω_r	m
Example 1 Fig. 7-3	1	.03	$4000\,\pi$	.078	.043	$\frac{\pi}{\sqrt{8}}$
Example 2 Fig. 7-4	1	.005	$4000\,\pi$	.064	.032	$\frac{\pi}{\sqrt{10}}$

Table 7-1. Values of the constants used in (7.8) and (7.15) from which the curves for $G(\nu)$ in Figs. 7-3 and 7-4 were plotted. The radar pulse repetition frequency (f_r) is 2000 pulses per second. The filter angular frequency is ω radians per second and the clutter width is α_o radians per second.

Using the constants in Table 7-1 and (7.8) and (7.15), $G(\nu)$ vs. target angular frequency ν has been plotted in Figs. 7-3 and 7-4.

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[1] Ref. 2

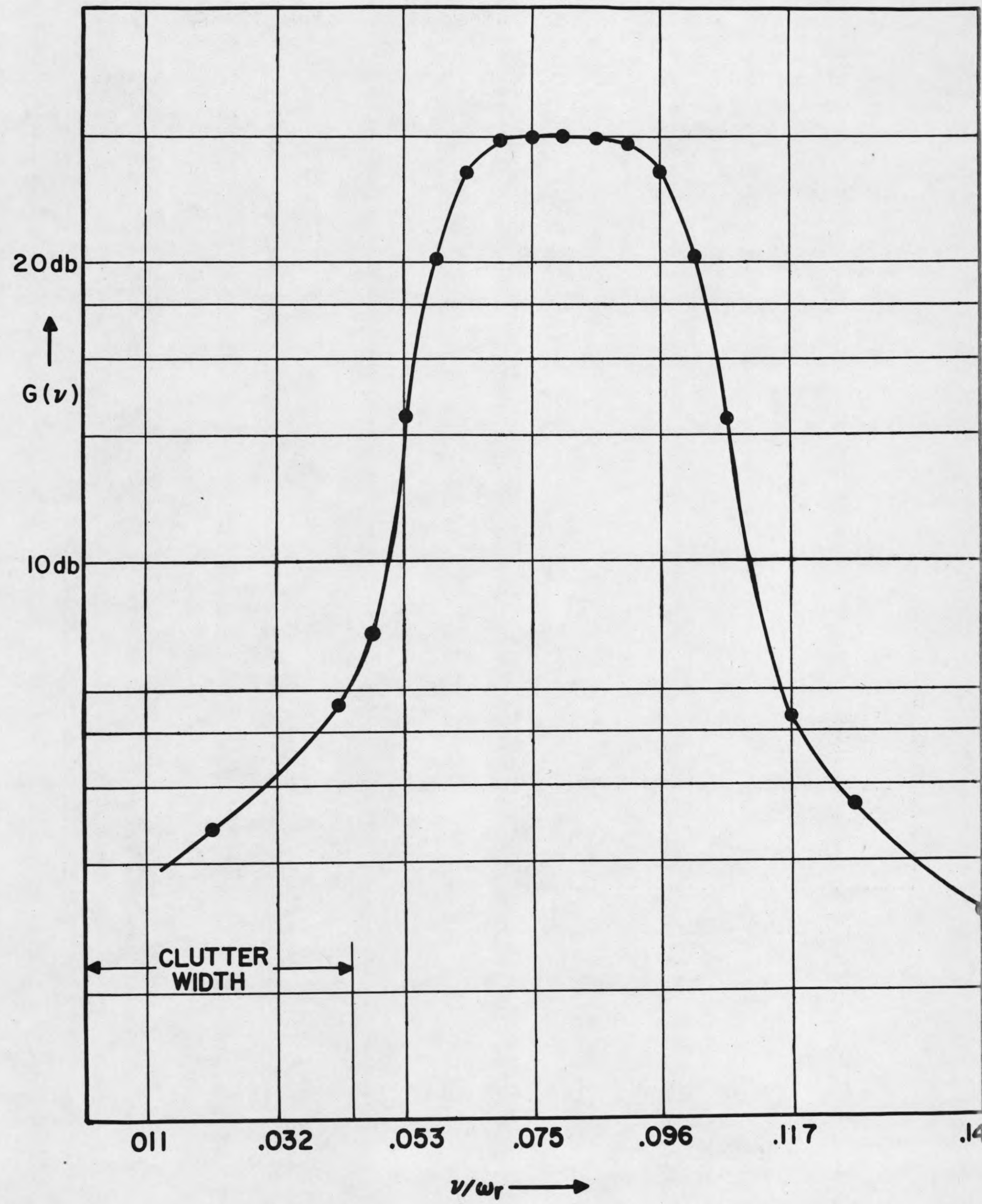

FIG. 7-3. Graph of a single filter enhancement function G(ν) for a Sinufly system. The pulse packet derived from each range element consists of samples from 61 video range traces.

$$X = \frac{\text{Signal power}}{\text{Clutter power}} = 1; \quad T_o = \text{Time duration of } f(t) = .03 \text{ sec.}$$

$$\alpha_o = \text{Width of clutter spectrum} = .043\ \omega_r; \quad \omega = \text{Filter freq.} = .078\ \omega_r$$

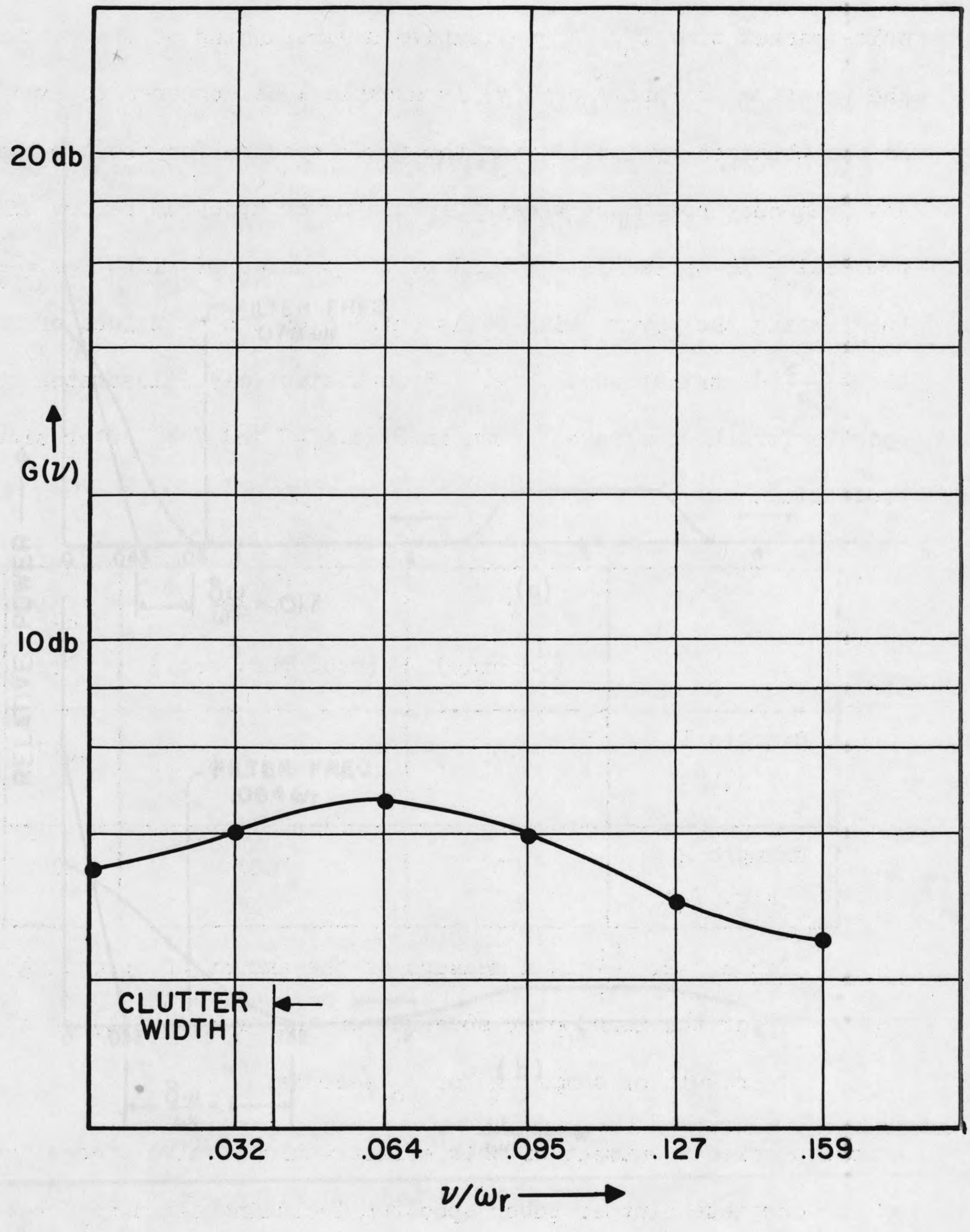

FIG. 7-4. Graph of a single filter enhancement function G(ν) for a Sinufly system. The pulse packet derived from each range element consists of samples from 11 video range traces.

$$X = \frac{\text{Signal power}}{\text{Clutter power}} = 1; \quad T_o = \text{Time duration of } f(t) = .005 \text{ sec.}$$

$$\alpha_o = \text{Width of clutter spectrum} = .038\ \omega_r; \quad \omega = \text{Filter freq.} = .076\ \omega_r$$

The essential difference between examples 1 and 2, which gives rise to the change in the enhancement function $G(\nu)$ is the difference in the pulse packet time T_o. A qualitative understanding of the reason for the relative sharpness of $G(\nu)$ in example 1 as compared to example 2 may be obtained by considering the power spectra for the two cases. Any frequency component present in the power spectrum before sinufly processing is spread into a band of frequencies of width $2\delta\omega = \frac{4\pi}{T_o}$ (neglecting the power which falls outside of the main lobe of the $\frac{\sin^2 x}{x^2}$ distribution). Fig. 7-5 qualitiatively illustrates the power spectra for the two cases shown in Figs. 7-3 and 7-4, after sinufly processing.

	T_o (seconds)	$\delta\omega = \frac{2\pi}{T_o}$ (rad. per. sec.)	$\frac{\delta\omega}{\omega_r}$
Example 1 Fig. 8-3	.03	67π	.017
Example 2 Fig. 7-4	.005	400π	.1

Table 7-2. $\frac{\delta\omega}{\omega_r}$ is a measure of the amount of spreading of the mean power spectrum of the process $f(t)$ as a result of sampling for T_o seconds.

A more precise treatment of this effect would involve a consideration of the complete clutter power spectrum including all side lobes of all frequency components present in the initial spectrum. The function $G(\nu)$ is in fact a consequence of just such a complete treatment. It is the power in the side lobes (neglected in the preceding qualitative

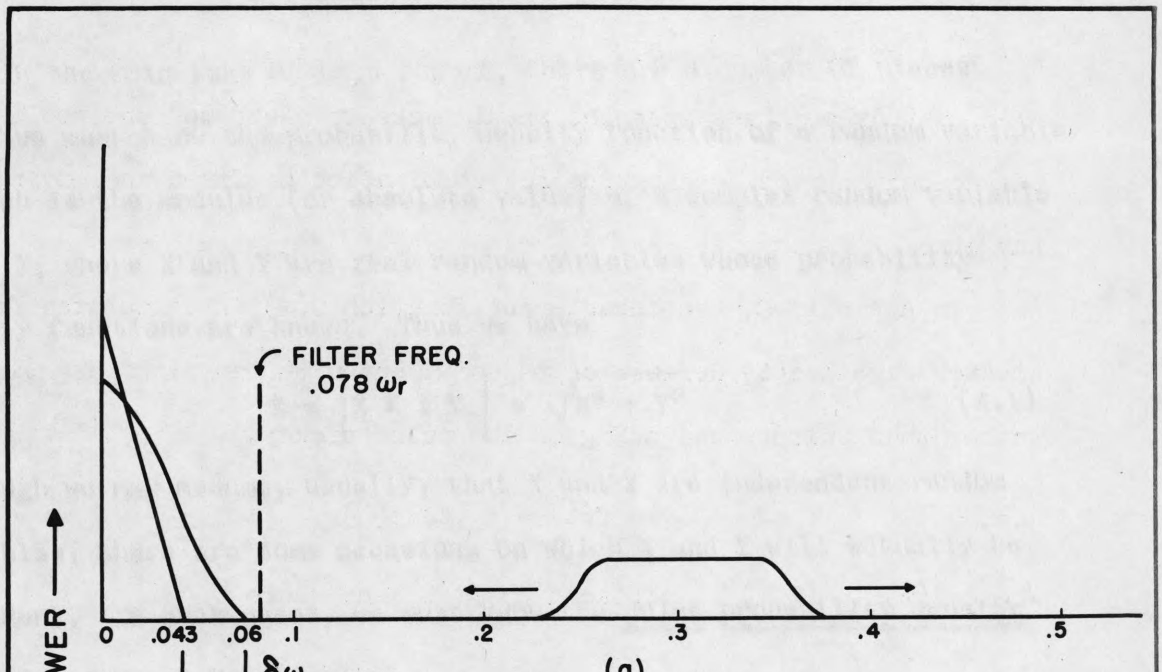

FIG. 7-5. Mean power spectrum for signal plus clutter after Sinufly processing.
(a) Example 1 (b) Example 2

discussion) which prevents $G(\nu)$ from becoming infinite outside the main lobe of the clutter spectrum over the band of frequencies from $\nu - \frac{\alpha_o}{2} - \delta\omega$ to $\nu + \frac{\alpha_o}{2} + \delta\omega$.

A study of these examples should make it clear how one would obtain enhancement curves for any desired combination of fixed and moving targets.

In Appendix C, the enhancement function $G(\nu)$ is calculated for the clutter signal $c(t)$ defined on page 39, but with Gaussian weighting rather than uniform weighting of the pulse packet.

i.e.
$$f(t) = f_1(t)\, c(t)$$

where $\gamma(t)$ in (7.1) is replaced by $f_1(t)$ and where

$$f_1(t) = e^{-1/2\left(\frac{6t}{T_o}\right)^2}$$

The results of this calculation are plotted in Figs. C-1 and C-2 and the $G(\nu)$ functions for $f(t) = \gamma(t)\, c(t)$ are replotted on these same graphs in order to facilitate direct comparison.

Appendix A. Probability Density Function for the Modulus of a Complex Random Variable.

In the main text of this report, there are a number of places where we must know the probability density function of a random variable Z which is the modulus (or absolute value) of a complex random variable X + i Y, where X and Y are real random variables whose probability density functions are known. Thus we have

$$Z = \left| X + i\,Y \right| = \sqrt{X^2 + Y^2} \tag{A.1}$$

Although we may assume, usually, that X and Y are independent random variables, there are some occasions on which X and Y will actually be dependent. In such cases, we must know the joint probability density[1] function of X and Y.

Our interest in this report is limited to the case for which X and Y are both Gaussian and, in case they are dependent, have a Gaussian joint distribution. In the special case where X and Y are independent, with zero means and equal variances, we get the well known Rayleigh distribution.

We shall use the following notation. The probability density function for Z will be denoted by p(s). It is defined by

$$p(s)\,ds = P\left\{ s < Z \leqslant s + ds \right\}, \tag{A.2}$$

where $P\left\{\cdot\right\}$ is the probability measure. The joint probability density function for X and Y will be denoted by q(x, y), and it is defined by

$$q(x, y)\,dx\,dy = P\left\{ x < X \leqslant x + dx,\ y < Y \leqslant y + dy \right\}. \tag{A.3}$$

We shall denote the mean of X by μ, and its variance by σ_1^2; and the mean of Y will be denoted by ν, and its variance by σ_2^2. The correlation

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[1] Ref. 4, pp. 6 and 7

coefficient κ of X and Y is defined by[1]

$$\kappa = \frac{E\left\{(X - \mu)(Y - \nu)\right\}}{\sigma_1 \sigma_2}, \tag{A.4}$$

it is, of course, zero if X and Y are independent.

From (A.1) and (A.2) we have

$$\begin{aligned} p(s)\, ds &= P\left\{s < \sqrt{X^2 + Y^2} \leqq s + ds\right\} \qquad \text{(A.5)} \\ &= P\left\{s^2 < X^2 + Y^2 \leqq s^2 + 2s\, ds\right\} \end{aligned}$$

$$= \sum_{y^2 \leqq s^2} P\left\{y^2 < Y^2 < y^2 + 2y\, dy,\ s^2 - y^2 < X^2 \leqq s^2 - y^2 + d(s^2 - y^2)\right\}.$$

But this last expression is equal to (A.6)

$$\sum_{y^2 \leqq s^2} P\left\{y < Y \leqq y + dy,\ \pm\sqrt{s^2 - y^2} < X \leqq \pm\sqrt{s^2 - y^2} + d\sqrt{s^2 - y^2}\right\}.$$

Using (A.3) we can express (A.5) in more explicit form by using the equivalent integral expression for (A.6).

$$p(s)\, ds = \int_{-s}^{s} q\left(\sqrt{s^2 - y^2}, y\right) \frac{s\, ds\, dy}{\sqrt{s^2 - y^2}} + \int_{-s}^{s} q\left(-\sqrt{s^2 - y^2}, y\right) \frac{s\, ds\, dy}{\sqrt{s^2 - y^2}} \tag{A.7}$$

Since we are restricting our interest to the case in which X and Y have a Gaussian joint density function, we have[2]

$$q(x, y) = \frac{1}{2\pi\, \sigma_1 \sigma_2 \sqrt{1 - \kappa^2}} \exp\left\{\frac{-1}{2(1 - \kappa^2)}\left(\frac{(x - \mu)^2}{{\sigma_1}^2} - 2\kappa \frac{(x - \mu)}{\sigma_1}\frac{(y - \nu)}{\sigma_2} + \frac{(y - \nu)^2}{{\sigma_2}^2}\right)\right\}. \tag{A.8}$$

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[1] Ref. 8, p. 265

[2] Ref. 8, p. 287

Therefore we have

$$p(s) = \frac{s}{2\pi\,\sigma_1\sigma_2\sqrt{1-\kappa^2}}\left[\int_{-s}^{s} \exp\left\{\frac{-1}{2(1-\kappa^2)}\left(\frac{(\sqrt{s^2-y^2}-\mu)^2}{\sigma_1{}^2}\right.\right.\right.$$

$$\left.\left.- 2\kappa\,\frac{(\sqrt{s^2-y^2}-\mu)}{\sigma_1}\,\frac{(y-\nu)}{\sigma_2} + \frac{(y-\nu)^2}{\sigma_2{}^2}\right)\right\}\frac{dy}{\sqrt{s^2-y^2}}$$

$$+ \int_{-s}^{s} \exp\left\{\frac{-1}{2(1-\kappa^2)}\left(\frac{(-\sqrt{s^2-y^2}-\mu)^2}{\sigma_1{}^2}\right.\right.$$

$$\left.\left.\left. + 2\kappa\,\frac{(\sqrt{s^2-y^2}+\mu)}{\sigma_1}\,\frac{(y-\nu)}{\sigma_2} + \frac{(y-\nu)^2}{\sigma_2{}^2}\right)\right\}\frac{dy}{\sqrt{s^2-y^2}}\right]. \tag{A.9}$$

If we make the substitutions

$$y = s\cos\theta, \quad \sqrt{s^2-y^2} = s\sin\theta, \quad dy = -\,s\sin\theta\,d\theta\,, \tag{A.10}$$

then our integrals become

$$\int_0^{\pi} \exp\left\{\frac{-1}{2(1-\kappa^2)}\left(\frac{(s\sin\theta-\mu)^2}{\sigma_1{}^2} - 2\kappa\,\frac{(s\sin\theta-\mu)}{\sigma_1}\,\frac{(s\cos\theta-\nu)}{\sigma_2}\right.\right.$$

$$\left.\left. + \frac{(s\cos\theta-\nu)^2}{\sigma_2{}^2}\right)\right\}d\theta, \tag{A.11}$$

and

$$\int_0^{\pi} \exp\left\{\frac{-1}{2(1-\kappa^2)}\left(\frac{(s\sin\theta+\mu)^2}{\sigma_1{}^2} + 2\kappa\,\frac{(s\sin\theta+\mu)}{\sigma_1}\,\frac{(s\cos\theta-\nu)}{\sigma_2}\right.\right.$$

$$\left.\left. + \frac{(s\cos\theta-\nu)^2}{\sigma_2{}^2}\right)\right\}d\theta\;. \tag{A.12}$$

Taking the periodicity of $\sin\theta$ and $\cos\theta$ into account, we may therefore write p(s) in the form

$$p(s) = \frac{s}{2\pi\,\sigma_1\sigma_2\sqrt{1-\kappa^2}}\left[\int_0^{2\pi} \exp\left\{\frac{-1}{2(1-\kappa^2)}\left(\frac{(s\sin\theta-\mu)^2}{\sigma_1^2}\right.\right.\right.$$

$$\left.\left.\left. -\,2\kappa\frac{(s\sin\theta-\mu)}{\sigma_1}\,\frac{(s\cos\theta-\nu)}{\sigma_2}+\frac{(s\cos\theta-\nu)^2}{\sigma_2^2}\right)\right\}d\theta\right]. \qquad (A.13)$$

In anticipation of the final result, let us now introduce the function

$$W(\alpha_1, \beta_1, \alpha_2, \beta_2) = \frac{1}{2\pi}\int_0^{2\pi} \exp\ \alpha_1\cos\theta + \beta_1\sin\theta$$

$$+\ \alpha_2\cos 2\theta + \beta_2\sin 2\theta\ d\theta\ . \qquad (A.14)$$

With a little algebraic manipulation on (A.13), combined with a use of the function introduced in (A.14), we can now write (A.15)

$$p(s) = \frac{s}{\sigma_1\sigma_2\sqrt{1-\kappa^2}}\exp\left\{-\frac{(\sigma_1^2+\sigma_2^2)s^2 + 2(\sigma_2^2\mu^2 - 2\kappa\sigma_1\sigma_2\mu\nu + \sigma_1^2\nu^2)}{4(1-\kappa^2)\sigma_1^2\sigma_2^2}\right\}$$

$$W\left(\frac{\nu\sigma_1^2-\kappa\mu\sigma_1\sigma_2}{(1-\kappa^2)\sigma_1^2\sigma_2^2}\,s,\ \frac{\mu\sigma_2^2-\kappa\nu\sigma_1\sigma_2}{(1-\kappa^2)\sigma_1^2\sigma_2^2}\,s,\ \frac{(\sigma_2^2-\sigma_1^2)s^2}{4(1-\kappa^2)\sigma_1^2\sigma_2^2},\ \frac{2\kappa\sigma_1\sigma_2 s^2}{4(1-\kappa^2)\sigma_1^2\sigma_2^2}\right).$$

This is the most general expression which we shall derive here. A number of important cases occur upon specializing various of the parameters σ_1, σ_2, κ, μ, ν; we shall tabulate some of these below. But first we must make the following observations about $W(\alpha_1, \beta_1, \alpha_2, \beta_2)$. We see that

$$W(\alpha_1, \beta_1, 0, 0) = I_0\left(\sqrt{\alpha_1^2+\beta_1^2}\right)$$

$$W(0, 0, \alpha_2, \beta_2) = I_0\left(\sqrt{\alpha_2^2+\beta_2^2}\right) \qquad (A.16)$$

where $I_o(\alpha)$ is the modified Bessel's function of order zero[1]. We also introduce

$$N(\alpha, \beta, \gamma) = W(\alpha, \beta, \gamma, 0), \tag{A.17}$$

which proves to be convenient.

With this notation we now make our specializations. Let us assume first that X and Y are independent, that is that $\kappa = 0$, then

$$p(s) = \frac{s}{\sigma_1\sigma_2} \exp\left\{- \frac{(\sigma_1^2 + \sigma_2^2)s^2 + 2(\sigma_2^2\mu^2 + \sigma_1^2\nu^2)}{4\sigma_1^2\sigma_2^2}\right\}$$

$$N\left(\frac{\nu s}{\sigma_2^2}, \frac{\mu s}{\sigma_1^2}, \frac{(\sigma_2^2 - \sigma_1^2)}{4\sigma_1^2\sigma_2^2} s^2\right). \tag{A.18}$$

Next we shall add the assumption that $\mu = \nu = 0$, i.e., zero means, then

$$p(s) = \frac{s}{\sigma_1\sigma_2} \exp\left\{- \frac{(\sigma_1^2 + \sigma_2^2)}{4\sigma_1^2\sigma_2^2} s^2\right\} I_o\left(\frac{(\sigma_1^2 - \sigma_2^2)}{4\sigma_1^2\sigma_2^2} s^2\right). \tag{A.19}$$

Now let us assume that $\kappa = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then we have

$$p(s) = \frac{s}{\sigma^2} \exp\left\{- \frac{s^2 + \mu^2 + \nu^2}{2\sigma^2}\right\} I_o\left(\frac{s}{\sigma^2} \sqrt{\mu^2 + \nu^2}\right). \tag{A.20}$$

If we now add the assumption that $\mu = \nu = 0$ also, we get the simple Rayleigh distribution

$$p(s) = \frac{s}{\sigma^2} \exp\left\{- \frac{s^2}{2\sigma^2}\right\}. \tag{A.21}$$

As a final specialization of (A.15) we retain the assumptions $\sigma_1^2 = \sigma_2^2 = \sigma^2$, and $\mu = \nu = 0$ but allow κ to be non zero (i.e. X and Y are dependent). Then we have

$$p(s) = \frac{s}{\sigma^2 \sqrt{1 - \kappa}} \exp\left\{- \frac{s^2}{2\sigma^2(1 - \kappa^2)}\right\} I_o\left(\frac{\kappa s^2}{2\sigma^2(1 - \kappa^2)}\right). \tag{A.22}$$

---

[1] Ref. 5, p. 373

While it is possible to enumerate other cases, it seems pointless to do so here; all such cases can be derived from (A.15). On the other hand, it seems worthwhile to offer a few comments on the functions $N(\alpha, \beta, \gamma)$ and $W(\alpha_1, \beta_1, \alpha_2, \beta_2)$ introduced above. This discussion of $N(\alpha, \beta, \gamma)$ and $W(\alpha_1, \beta_1, \alpha_2, \beta_2)$ is the subject of Appendix B.

Appendix B. Probability Density Function for the Case of Approximately Equal Variances.

It is clear from the text of chapter five that (A.18) is an important expression. We are particularly interested in this expression for cases where σ_2^2 is nearly equal to σ_1^2, thus it is worthwhile to obtain an expression for $N(\alpha, \beta, \gamma)$ which is valid(and simple) for small γ.

Observe first that

$$e^{(\alpha \cos \theta + \beta \sin \theta + \gamma \cos 2\theta)} = e^{(\alpha \cos \theta + \beta \sin \theta)} \left[1 + \gamma \cos 2\theta + O(\gamma^2)\right]^{1} \quad \text{(B.1)}$$

where $O(\gamma^2)$ signifies terms of order γ^2 and higher.

Consequently we have

$$N(\alpha, \beta, \gamma) = \frac{1}{2\pi} \int_0^{2\pi} e^{(\alpha \cos \theta + \beta \sin \theta)} d\theta + \frac{\gamma}{2\pi} \int_0^{2\pi} e^{(\alpha \cos \theta + \beta \sin \theta)} \cos 2\theta \, d\theta + O(\gamma^2). \quad \text{(B.2)}$$

Now let

$$\delta = \text{Arc tan } \frac{\beta}{\alpha}, \quad \text{(B.3)}$$

then

$$\alpha \cos \theta + \beta \sin \theta = \sqrt{\alpha^2 + \beta^2} \cos (\theta - \delta), \quad \text{(B.4)}$$

so that (B.2) becomes

$$N(\alpha, \beta, \gamma) = I_o \left(\sqrt{\alpha^2 + \beta^2}\right) + \frac{\gamma}{2\pi} \int_0^{2\pi} e^{\sqrt{\alpha^2 + \beta^2} \cos(\theta - \delta)} \cos 2\theta \, d\theta + O(\gamma^2). \quad \text{(B.5)}$$

---

[1] Ref. 5, p. 11

Upon replacing cos 2θ by $\cos 2\theta = \cos(2\theta - 2\delta)\cos 2\delta - \sin(2\theta - 2\delta)\sin 2\delta$, and letting $r = \sqrt{\alpha^2 + \beta^2}$, we get

$$\int_0^{2\pi} e^{r\cos(\theta - \delta)}\cos 2\theta \, d\theta = \int_0^{2\pi} e^{r\cos(\theta - \delta)}\cos(2\theta - 2\delta)\cos 2\delta \, d\theta$$

$$+ \int_0^{2\pi} e^{r\cos(\theta - \delta)}\sin(2\theta - 2\delta)\sin 2\delta \, d\theta \quad . \tag{B.6}$$

On letting $\psi = \theta - \delta$ we see that

$$\int_0^{2\pi} e^{r\cos\psi}\sin 2\psi \, d\psi = -2\int_0^{2\pi} e^{r\cos\psi}\cos\psi \, d(\cos\psi). \tag{B.7}$$

Since however,

$$e^{r\cos\psi}\cos\psi \, d(\cos\psi) = d\left(\frac{1}{r}e^{r\cos\psi}(\cos\psi - \frac{1}{r})\right) , \tag{B.8}$$

which is periodic of period 2π, we also see that

$$\int_0^{2\pi} e^{r\cos\psi}\sin 2\psi \, d\psi = 0. \tag{B.9}$$

Applying this to (B.6) we see that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{r\cos(\theta - \delta)}\cos 2\theta \, d\theta = \frac{1}{2\pi}\cos 2\delta\int_0^{2\pi} e^{r\cos\psi}\cos 2\psi \, d\psi$$

$$= \cos 2\delta \; I_2(r), \tag{B.10}$$

where $I_2(r)$ is the modified Bessel's function of order two.

Since

$$\cos 2\delta = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} , \quad \text{and} \quad r = \sqrt{\alpha^2 + \beta^2} , \tag{B.11}$$

we have

$$N(\alpha, \beta, \gamma) = I_o\left(\sqrt{\alpha^2 + \beta^2}\right) + \gamma \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} I_2\left(\sqrt{\alpha^2 + \beta^2}\right) + O(\gamma^2). \quad (B.12)$$

For the sake of completeness we shall give the power series expansion of $N(\alpha, \beta, \gamma)$ about the origin. Let us define

$$b_{jkmn} = (-1)^j \left\{1 + (-1)^{(2n - 2j + k)} + (-1)^{(2j + m)} + (-1)^{(2n + k + m)}\right\},$$

and define

$$a_{kmn} = \frac{1}{k!m!n!} \sum_{j=0}^{n} b_{jkmn} \frac{\Gamma\left(\frac{2j + m + 1}{2}\right)\Gamma\left(\frac{2n - 2j + k + 1}{2}\right)}{\Gamma\left(\frac{2n + m + k + 2}{2}\right)},$$

then

$$N(\alpha, \beta, \gamma) = \frac{1}{4\pi} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{kmn}\, \alpha^k \beta^m \gamma^n.$$

In this expression $\Gamma(\cdot)$ denotes the ordinary Γ function.[1]

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[1]Ref. 5, p. 235

Appendix C. Enhancement Function for Signal in Clutter, with Gaussian Packet Weighting

Let us replace the function $\gamma(t)$ of (7.1) with the function $\gamma(t)\ g(t)$

i.e.
$$f(t) = \gamma(t)\ g(t)\ c(t)$$
where
$$g(t) = \exp\left[-\frac{1}{2}\left(\frac{6t}{T_o}\right)^2\right].$$

Since $g(\frac{T_o}{2}) = g(-\frac{T_o}{2}) \approx .01\ g(t)_{max}$, and in view of the definition of $\gamma(t)$ we can write as a good approximation

$$f(t) = g(t)\ c(t).$$

The spectrum of $g(t)$ is given by

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\ e^{-i\omega t}\ dt$$

$$= \frac{T_o}{6\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\omega T_o}{6}\right)^2\right] = \frac{T_o}{6\sqrt{2\pi}} \exp\left[-.014\ \omega^2\ T_o^{\ 2}\right].$$

We now define

$$S(\omega) = \int_{-\infty}^{\infty} G(\alpha)\ N(\omega-\alpha)\ d\alpha$$

and form the covariance function

$$E\left\{S(\omega_1)\ \overline{S}(\omega_2)\right\} = \tag{C.1}$$

$$\frac{T_o^{\ 2}}{72\pi}\int_{-\infty}^{\infty} W(\alpha)\ e^{-.014\left[(\omega_1-\alpha)T_o\right]^2}\ e^{-.014\left[(\omega_2-\alpha)T_o\right]^2}\ d\alpha$$

which is the counterpart of (7.3). Then using the form of the clutter power spectrum given by (7.4) we get the following expression for the mean power spectrum of clutter alone, after Sinufly processing:

$$\psi_c(\omega) = E\left\{\left|S(\omega)\right|^2\right\} = \frac{T_o^{\,2}}{72\pi}\int_{-\infty}^{\infty}\left\{m^2\alpha_o^{\,2}\delta(\alpha) + \xi(\alpha)\right\} e^{-.028\left[(\omega-\alpha)T_o\right]^2} d\alpha$$

$$= \frac{(T_o m\alpha_o)^2}{72\pi} e^{-.028(\omega T_o)^2} + \frac{T_o^{\,2}}{72\pi}\int_{-\infty}^{\infty} \xi(\alpha)\, e^{-.028\left[(\omega-\alpha)T_o\right]^2} d\alpha .$$

Using (7.4) this becomes

$$\psi_c(\omega) =$$

$$\frac{(T_o m\alpha_o)^2}{72\pi} e^{-.028(\omega T_o)^2} - \frac{(T_o m)^2}{72\pi}\int_0^{\alpha_o} (\alpha-\alpha_o)\left\{e^{-.028\left[(\omega+\alpha)T_o\right]^2}\right.$$

$$\left. + e^{-.028\left[(\omega-\alpha)T_o\right]^2}\right\} d\alpha . \qquad (C.2)$$

The integral

$$\int_0^{\alpha_o} (\alpha-\alpha_o)\, e^{-.028\left[(\omega-\alpha)T_o\right]^2} d\alpha \qquad (C.3)$$

is evaluated by setting $\beta = \alpha - \omega$ so that (C.3) becomes

$$\int_{-\omega}^{\alpha_o-\omega} (\beta+\omega-\alpha_o)\, e^{-.028(\beta T_o)^2} d\beta = \int_{-\omega}^{\alpha_o-\omega} \beta\, e^{-.028(\beta T_o)^2} d\beta$$

$$+ (\omega-\alpha_o)\int_{-\omega}^{\alpha_o-\omega} e^{-.028(\beta T_o)^2} d\beta .$$

Now letting $u = \beta T_o$ we get

$$\int_{-\omega}^{\alpha_o-\omega} \beta\, e^{-.028(\beta T_o)^2}\, d\beta + (\omega-\alpha)\int_{-\omega}^{\alpha_o-\omega} e^{-.028(\beta T_o)^2}\, d\beta$$

$$= \frac{1}{T_o^{\,2}}\int_{-\omega T_o}^{(\alpha_o-\omega)T_o} u\, e^{-.028\, u^2}\, du + \left(\frac{\omega-\alpha_o}{T_o}\right)\int_{-\omega T_o}^{(\alpha_o-\omega)T_o} e^{-.028\, u^2}\, du$$

$$= \frac{-1}{.056\, T_o^{\,2}}\left\{\exp\left[-.028\left((\alpha_o-\omega)T_o\right)^2\right] - \exp\left[-.028\,\omega^2\, T_o^{\,2}\right]\right\}$$

$$+ \frac{\sqrt{\pi}}{\sqrt{(2)\,(.056)}}\left(\frac{\omega-\alpha_o}{T_o}\right)\left\{\operatorname{erf}\left[\sqrt{.056}\,(\alpha_o-\omega)T_o\right] + \operatorname{erf}\left[\sqrt{.056}\,\omega T_o\right]\right\} \quad (C.4)$$

where $\operatorname{erf} X \equiv \sqrt{\frac{2}{\pi}}\int_0^X e^{-t^2/2}\, dt.$

The integral

$$\int_0^{\alpha_o} (\alpha-\alpha_o)\, e^{-.028\left[(\omega+\alpha)T_o\right]^2}\, d\alpha$$

can be evaluated by substituting $(-\omega)$ for ω in (C.3)

i.e.
$$\int_0^{\alpha_o} (\alpha-\alpha_o)\, e^{-.028\left[(\omega+\alpha)T_o\right]^2}\, d\alpha$$

$$= \frac{-1}{.056\, T_o^{\,2}}\left\{\exp\left[-.028\left((\alpha_o+\omega)T_o\right)^2\right] - \exp\left[-.028\,\omega^2\, T_o^{\,2}\right]\right\}$$

$$- \frac{\sqrt{\pi}}{\sqrt{(2)\,(.056)}}\left(\frac{\omega+\alpha_o}{T_o}\right)\left\{\operatorname{erf}\left[\sqrt{.056}\,(\alpha_o+\omega)T_o\right] - \operatorname{erf}\left[\sqrt{.056}\,\omega T_o\right]\right\}$$

Therefore (C.2) becomes

$$\psi_c(\omega) = \frac{(T_o m\alpha_o)^2}{72\pi} \; e^{-.028\,\omega^2 T_o^{\,2}}$$

$$+ \frac{m^2}{(.056)(72\pi)}\left\{\exp\left[-.028\left((\alpha_o-\omega)T_o\right)^2\right] + \exp\left[-.028\left((\alpha_o+\omega)T_o\right)^2\right]\right.$$

$$\left.- 2\exp\left(-.028\,\omega^2 T_o^{\,2}\right)\right\} - \frac{\sqrt{\pi}\; m^2\, T_o}{\sqrt{(2)(.056)}\;(72\pi)}\left\{\omega\left[\mathrm{erf}\left(\sqrt{.056}\,(\alpha_o-\omega)T_o\right)\right.\right.$$

$$\left. + \,\mathrm{erf}\left(\sqrt{.056}\,(\alpha_o+\omega)T_o\right)\right] - \alpha_o\left[\mathrm{erf}\left(\sqrt{.056}\,(\alpha_o-\omega)T_o\right)\right.$$

$$\left.\left. - \,\mathrm{erf}\left(\sqrt{.056}\,(\alpha_o+\omega)\right)T_o + 2\,\mathrm{erf}\,\sqrt{.056}\,\omega\, T_o\right]\right\} \qquad (C.5)$$

$\psi_c(\omega)$ (C.5) is the expression for the mean power spectrum for clutter alone. If we let $W(\alpha)$ of (C.1) be the power spectrum for signal plus clutter, i.e. $W_1(\alpha)$ of (7.11), then we get an expression for $\psi_{s+c}(\omega,\nu)$ which is the counterpart of (7.13).

$\psi_{s+c}(\omega,\nu)$

$$= \frac{T_o^{\,2}}{72\pi}\int_{-\infty}^{\infty}\left\{m^2\alpha_o^{\,2}(1+2X+X^2)\delta(\alpha) + \xi_1(\alpha)\right\} e^{-.028\left[(\omega-\alpha)T_o\right]^2}\, d\alpha$$

$$= \psi_c(\omega) + (2X+X^2)\,\frac{(m\alpha_o T_o)^2}{72\pi}\, e^{-.028\,\omega^2 T_o^{\,2}} + \frac{T_o^{\,2} m^2\alpha_o}{72\pi}$$

$$\left\{\int_{\nu-\frac{\alpha_o}{2}}^{\nu+\frac{\alpha_o}{2}} e^{-.028\left[(\omega-\alpha)T_o\right]^2}\, d\alpha + \int_{-\nu-\frac{\alpha_o}{2}}^{-\nu+\frac{\alpha_o}{2}} e^{-.028\left[(\omega-\alpha)T_o\right]^2}\, d\alpha\right\}$$

$$= \psi_c(\omega) + (2X+X^2)\,\frac{(m\alpha_o T_o)^2}{72\pi}\,e^{-.028\,\omega^2\,T_o^{\,2}} + \frac{m^2\alpha_o T_o}{72\sqrt{(2\pi)(.056)}}$$

$$\left\{\mathrm{erf}\left[\sqrt{.056}\,(\nu+\frac{\alpha_o}{2}-\omega)T_o\right] - \mathrm{erf}\left[\sqrt{.056}\,(\nu-\frac{\alpha_o}{2}-\omega)T_o\right]\right.$$

$$\left. + \mathrm{erf}\left[\sqrt{.056}\,(-\nu+\frac{\alpha_o}{2}-\omega)T_o\right] - \mathrm{erf}\left[\sqrt{.056}\,(-\nu-\frac{\alpha_o}{2}-\omega)T_o\right]\right\}. \quad (C.6)$$

Then

$$G(\nu) = \frac{\psi_{s+c}(\omega,\nu)}{\psi_c(\omega)} = 1 + \frac{(2X+X^2)\frac{(m\alpha_o T_o)^2}{72\pi}\,e^{-.028\,\omega^2\,T_o^{\,2}}}{\psi_c(\omega)} \quad (C.7)$$

$$+ \frac{m^2\alpha_o T_o}{72\sqrt{(2\pi)(.056)}\;\psi_c(\omega)}\left\{\mathrm{erf}\left[\sqrt{.056}\,(\nu+\frac{\alpha_o}{2}-\omega)T_o\right]\right.$$

$$- \mathrm{erf}\left[\sqrt{.056}\,(\nu-\frac{\alpha_o}{2}-\omega)T_o\right] + \mathrm{erf}\left[\sqrt{.056}\,(-\nu+\frac{\alpha_o}{2}-\omega)T_o\right]$$

$$\left. - \mathrm{erf}\left[\sqrt{.056}\,(-\nu-\frac{\alpha_o}{2}-\omega)T_o\right]\right\}.$$

This function $G(\nu)$ has been plotted in Figs. C-1 and C-2 using the same values for the constants m, α_o,T_o and X, as were used in section VII (examples 1 and 2, Figs. 7-3 and 7-4). The examples of section VII have been replotted in Figs. C-1 and C-2 to facilitate a direct comparison of the enhancement function for the weighted and unweighted cases.

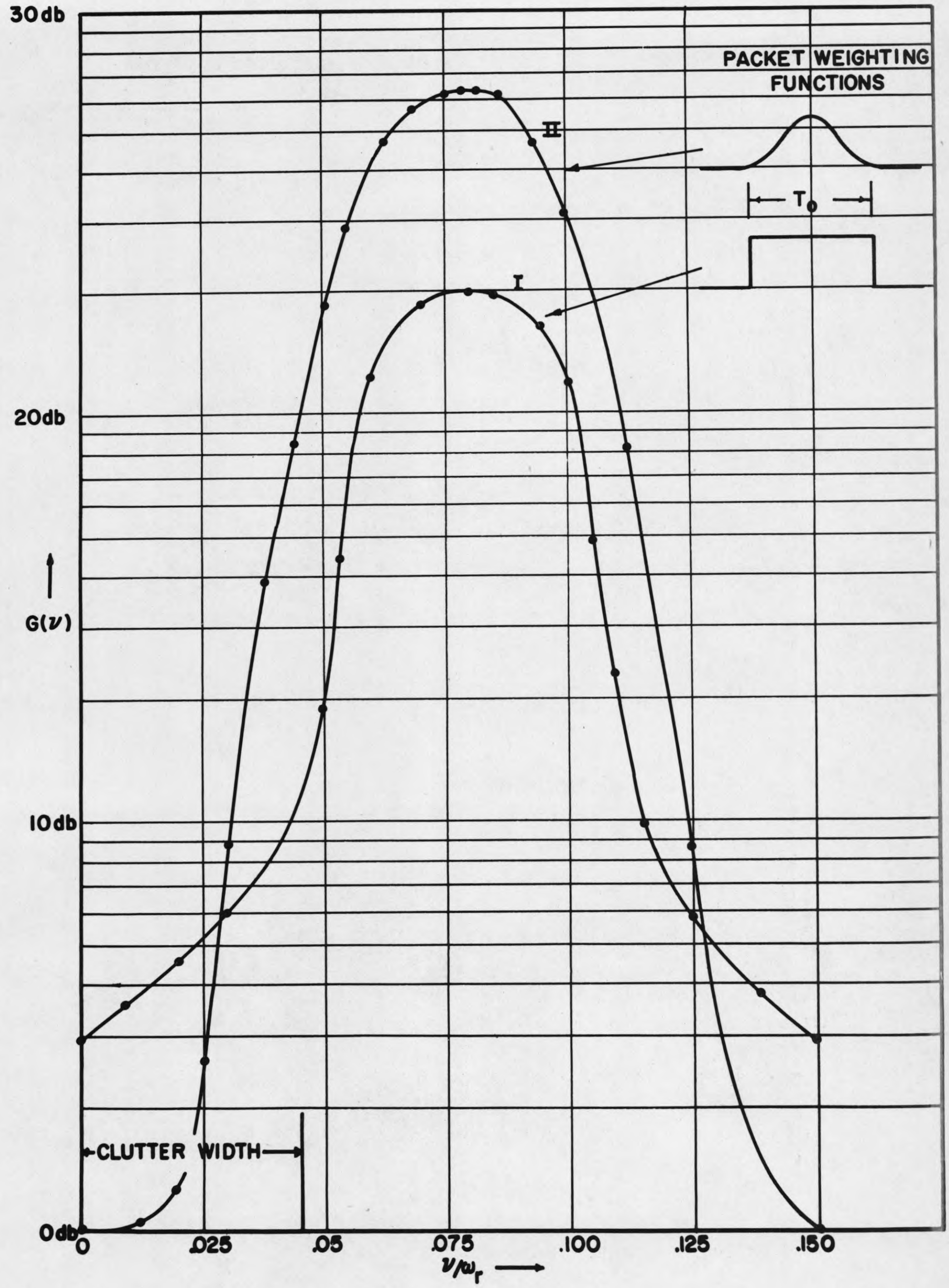

FIG. C-1. Graph of single filter enhancement functions G(ν) corresponding to uniform packet weighting and Gaussian packet weighting. The pulse packet derived from each range element consists of samples from 61 video range traces.

$$X = \frac{\text{Signal power}}{\text{Clutter power}} = 1; \quad T_o = \text{Time duration of } f(t) = .03 \text{ sec.}$$

$$\alpha_o = \text{Width of clutter spectrum} = .043\ \omega_r; \quad \omega = \text{Filter freq.} = .078\ \omega_r$$

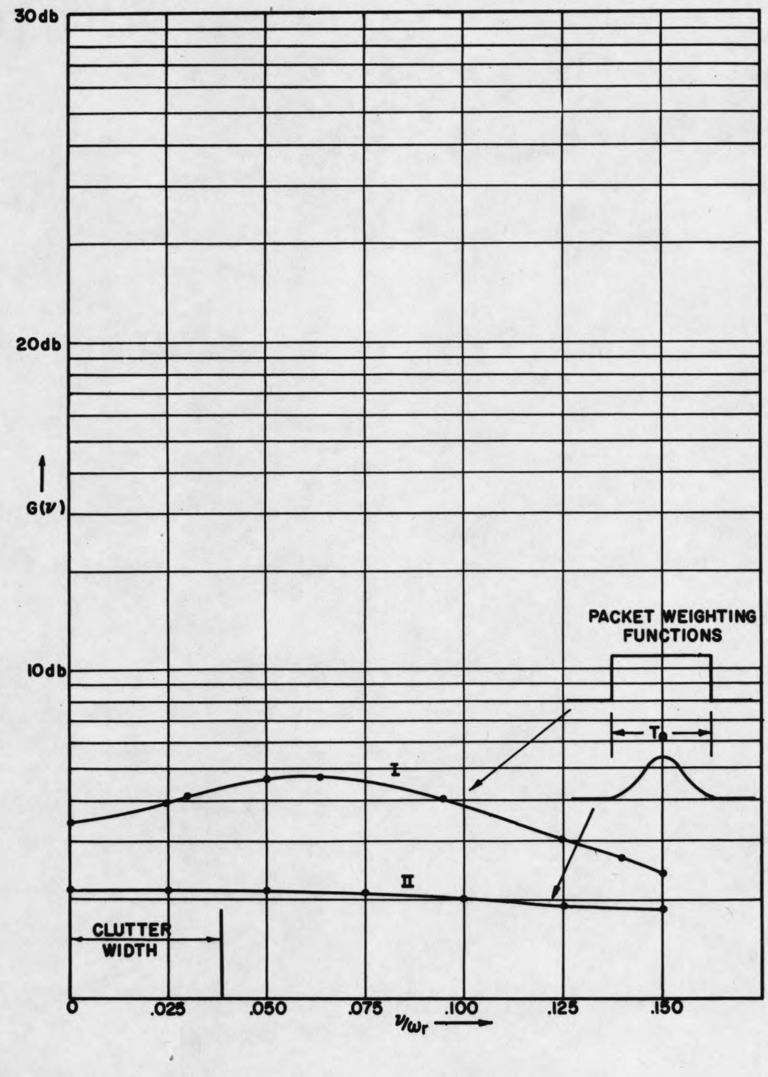

FIG. C-2. Graph of single filter enhancement functions G(ν) corresponding to uniform packet weighting and Gaussian packet weighting. The pulse packet derived from each range element consists of samples from 11 video range traces.

$$X = \frac{\text{Signal power}}{\text{Clutter power}} = 1; \quad T_o = \text{Time duration of } f(t) = .005 \text{ sec.}$$

$$\alpha_o = \text{Width of clutter spectrum} = .038\ \omega_r; \ \omega = \text{Filter freq.} = .076\ \omega_r$$

References

1. Control Systems Laboratory report R-29. — E. M. Lyman

2. Control Systems Laboratory report R-34 — G. S. Newell

3. Time Series and Harmonic Analysis, Berkeley Symposium on Mathematical Statistics and Probability. Pp. 303-43. California Press, 1949. — J. L. Doob

4. Stochastic Processes. Wiley, 1953. — J. L. Doob

5. Course of Modern Analysis, Cambridge Press, 1946. — Whittaker and Watson

6. Control Systems Laboratory report R-85 (Unpublished as of date of this report) — R. Swallow

7. Summary Engineering Report on Airborne Radar Detection of Moving Ground Targets. Volume 1. Contract AF-33 (038)-12473. H-1008. — Philco Corporation

8. Mathematical Methods of Statistics Princeton, 1946. — H. Cramer

9. Probability and Information Theory McGraw Hill Book Co., London, Pergamon Press, 1955. — Woodward

10. Table of generalized sine and cosine integral functions - Part I. Harvard University Press, 1949. — Staff of Computation Laboratory, Harvard University

11. Table of sine and cosine integrals. National Bureau of Standards, 1942. — Federal Works Agency W.P.A. City of New York

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