CONTROL SYSTEMS LABORATORY

THE GREATEST LOWER BOUND FOR THE
VARIANCE OF UNBIASED ESTIMATES

Report R-90

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Contract DA-36-039-SC-56695
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UNIVERSITY OF ILLINOIS · URBANA · ILLINOIS

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Prepared by

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I. Introduction

Consider a family of probability density functions $\left\{p(V,\alpha)\right\}$, where V is a point of E_n (Euclidean n-space) and α belongs to some real interval A. A classical result in the theory of parameter estimation is that the quantity

$$(1) \qquad D_*^2(\alpha_o) = \left[\int \left\{\left[\frac{\partial \log p(V,\alpha)}{\partial\alpha}\right]_{\alpha=\alpha_o}\right\}^2 p(V,\alpha_o)\, dV\right]^{-1}$$

is a lower bound for the quantity

$$(2) \qquad \int \left[F(V) - \alpha_o\right]^2 p(V,\alpha_o)\, dV$$

for all "regular" estimates F(V) satisfying

$$(3) \qquad \int F(V)\, p(V,\alpha)\, dV = \alpha \qquad \text{for all } \alpha\epsilon A.$$

That is, $D_*^2(\alpha_o)$ is a lower bound for the variance of regular unbiased estimates of α, when the true parameter value is α_o. A regular unbiased estimate whose variance equals $D_*^2(\alpha_o)$ for all α_o is called efficient. $D_*^2(\alpha_o)$ is not in general, however, the greatest lower bound for (2) subject to the constraint (3).

Let us call an estimate F(V) satisfying (3), which attains for all α_o the greatest lower bound of (2), a uniformly minimum variance unbiased estimate. Kendall[1] states that in the regular estimation case a necessary and sufficient condition for the existence of a uniformly minimum variance unbiased estimate is

$$(4) \qquad \frac{\partial \log p(V,\alpha)}{\partial\alpha} = \frac{H(V) - \alpha}{\lambda(\alpha)}, \qquad \text{all } \alpha\epsilon A$$

This is not correct--the condition is sufficient but not necessary, as is well known. Moreover, it might be possible for an estimate satisfying (3) to minimize (2) for one particular value of α_o but not for others. Kiefer[4], Barankin[5], and others have given stronger lower bounds than (1)--and in fact, Barankin[5] derived under certain conditions the greatest lower bound.

In the following, we develop a method for calculating the greatest lower bound of (2) for the class of all estimates satisfying (3)--i.e. with no regularity restrictions on the type of estimate considered. This method is actually an elaboration of Barankin's work, but with some of the restrictions removed. Also it provides in some cases a more explicit method of calculation than is given in Ref. 4 or Ref. 5. The method is applied to several examples, among which are: estimation of the standard deviation of a normal population with known mean and with unknown mean; estimation of the range of a rectangular distribution of known mean; and estimation of the mean of a Rayleigh distribution. (The results for these particular examples can be derived by other methods involving the properties of sufficient statistics, but the present method is applicable even in cases where sufficient statistics do not exist.)

II. A Method for Evaluating the Greatest Lower Bound

We suppose given a family $\left\{dP(V,\alpha)\right\}$ of probability measures over E_n (Euclidean n-space), where $V = (v_1, \ldots, v_n) \in E_n$ and α is in some real interval A. It will be assumed that the measurable sets in E_n are the same for all $\alpha \epsilon A$. By "function of V" we shall henceforth mean a function measurable on the sample space E_n for all $dP(V,\alpha)$.

By an "unbiased estimate of α," we mean any function F(V) satisfying

$$(5) \qquad \int F(V)\, dP(V,\alpha) = \alpha \quad , \qquad \text{all } \alpha \epsilon A.$$

(Integrals with respect to V always are taken over E_n.)

For each $\alpha_o \epsilon A$ and function F(V), consider

$$(6) \qquad \int \left[F(V) - \alpha_o \right]^2 dP(V,\alpha_o)$$

We shall assume henceforth that at least one function of V satisfying (5) identically for $\alpha \epsilon A$ exists for which (6) is finite.

We define

$$(7) \qquad D_m^{\,2}(\alpha_o) = \text{greatest lower bound of (6) for all F satisfying (5).}$$

Now, for any given α_o, consider the set $C^{(\alpha_o)}$ of functions of V consisting of all functions satisfying (5) identically for $\alpha \epsilon A$ and having finite second moment with respect to $dP(V,\alpha_o)$, and the limits in the mean of such functions with respect to $dP(V,\alpha_o)$.

The set $C^{(\alpha_o)}$ is clearly closed relative to convergence in mean with respect to $dP(V,\alpha_o)$; also it is clearly a convex set. Thus, there exists a unique (with probability one) function, which we shall denote by $F_m(V,\alpha_o)$, which minimizes (6) for all functions in $C^{(\alpha_o)}$. Clearly

(8) $$\int F_m(V,\alpha_o)\, dP(V,\alpha_o) = \alpha_o$$

(9) $$D_m^{\ 2}\ (\alpha_o) = \int \left[F_m(V,\alpha_o) - \alpha_o \right]^2 dP(V,\alpha_o)$$

If $F_m(V,\alpha_o)$ satisfies

(10) $$\int F_m(V,\alpha_o)\, dP(V,\alpha) = \alpha \quad , \quad \text{all } \alpha \epsilon A$$

then $F_m(V,\alpha_o)$ is the unbiased estimate of α having minimum variance when the true parameter value is α_o. Otherwise, there exists no unbiased estimate of α having minimum variance when the true value is α_o.

For our purposes, the most important property of $F_m(V,\alpha_o)$ is the following: consider functions $f(\alpha)$ defined over A of the form

(11) $$f(\alpha) = \int H(V)\, dP(V,\alpha) \quad , \quad \text{all } \alpha \epsilon A$$

where H(V) has finite second moment with respect to $dP(V,\alpha_o)$; i.e. where

(12) $$\int H^2(V)\, dP(V,\alpha_o) < \infty$$

<u>Theorem 1</u>: Let $f(\alpha)$ be of the form (11); then

(13) $$\int F_m(V,\alpha_o)\, H(V)\, dP(V,\alpha_o)$$

is a constant for all H satisfying (12) and satisfying (11) for the given $f(\alpha)$.

<u>Proof</u>: Let $H_1(V)$ and $H_2(V)$ both satisfy (11) and (12). Then

$F^*(V) = F_m(V,\alpha_o) + \epsilon \left[H_1(V) - H_2(V) \right]$, ϵ real, is an element of $C^{(\alpha_o)}$. Now consider

$D(\epsilon) = \int \left[F^*(V) - \alpha_o \right]^2 dP(V,\alpha_o)$. If

$\int F_m(V,\alpha_o) \left[H_1(V) - H_2(V) \right] dP(V,\alpha_o) \neq 0$, then $\left.\frac{dD}{d\epsilon}\right|_{\epsilon=0} \neq 0$.

This contradicts the definition of $F_m(V,\alpha_o)$.

We shall use Theorem 1 to define a linear functional $\Lambda^{(\alpha_o)}$ on the set of functions f of form (11), as follows: if f is of form (11),

$$\Lambda^{(\alpha_o)} \left[f \right] = \int H(V) \left[F_m(V,\alpha_o) - \alpha_o \right] dP(V,\alpha_o) \tag{14}$$

where H(V) is any function of V satisfying both (11) and (12). By Theorem 1, this defines $\Lambda^{(\alpha_o)} \left[f \right]$ uniquely (no matter which H satisfying (11) and (12) is used in (14)).

Denote by ϕ_o the function whose values on A are $\phi_o(\alpha) = 1$; and denote by ϕ_1 the function whose values on A are $\phi_1(\alpha) = \alpha$. It is clear from the definition of $\Lambda^{(\alpha_o)}$ and from the properties of $F_m(V,\alpha_o)$ that

$$\Lambda^{(\alpha_o)} \left[\phi_o \right] = 0 \tag{15}$$

$$\Lambda^{(\alpha_o)} \left[\phi_1 \right] = D_m^2(\alpha_o) \tag{16}$$

Thus, evaluation of $D_m^2(\alpha_o)$ is equivalent to evaluation of $\Lambda^{(\alpha_o)} \left[\phi_1 \right]$. It is also clear that

(17) $$\Lambda^{(\alpha_o)}\left[\sum_{i=1}^{K} c_i f_i\right] = \sum_{i=1}^{K} c_i \Lambda^{(\alpha_o)}\left[f_i\right]$$

We must next prove some continuity properties of $\Lambda^{(\alpha_o)}$.

<u>Theorem 2</u>: Suppose

(18) $$f_i(\alpha) = \int H_i(V)\, dP(V,\alpha) \qquad \text{all } \alpha \epsilon A$$

i=1, 2, .

and i=∞

and suppose that for each i (including ∞) $H_i(V)$ can be selected among all functions of V satisfying (18) for the given i in such a way that H_i approaches H_∞ in mean with respect to $dP(V,\alpha_o)$.

Then

(19) $$\Lambda^{(\alpha_o)}\left[f_\infty\right] = \lim_{i\to\infty} \Lambda^{(\alpha_o)}\left[f_i\right]$$

<u>Proof</u>:

$$\Lambda^{(\alpha_o)}\left[f_\infty\right] - \Lambda^{(\alpha_o)}\left[f_i\right]$$

$$= \int \left[F_m(V,\alpha_o) - \alpha_o\right]\left[H_\infty(V) - H_i(V)\right] dP(V,\alpha_o)$$

and the conclusion follows immediately.

<u>Theorem 3</u>: Suppose

(20) $$f_i(\alpha) = \int H_i(V)\, dP(V,\alpha) \quad , \quad \text{all } \alpha \epsilon A$$

i=1, 2, . . .

and suppose that for each i, H_i can be selected among all functions satisfying (20) for the given i in such a way that the sequence $\left\{H_i\right\}$

converges in mean to the same function $H_\infty(V)$ with respect to $dP(V,\alpha)$ for all $\alpha\epsilon A$.

Then

(21) $$\lim_{i\to\infty} f_i(\alpha) = \int H_\infty(V)\, dP(V,\alpha) = f_\infty(\alpha) \quad , \quad \text{all } \alpha\epsilon A$$

(the last equality is a definition); and

(22) $$\Lambda^{(\alpha_o)}\left[f_\infty\right] = \lim_{i\to\infty} \Lambda^{(\alpha_o)}\left[f_i\right]$$

Proof: (21) is obvious from the hypothesis of Theorem 3 and (22) follows from Theorem 2.

Theorem 4: Let u be in some finite real interval U and let

(23) $$f(\alpha,u) = \int H(V,u)\, dP(V,\alpha) \quad \text{all } \alpha\epsilon A, \; u\epsilon U$$

where, for each u, H(V,u) has finite second moment with respect to $dP(V,\alpha)$ for all $\alpha\epsilon A$. Also suppose that $\lambda(u)$ is a function defined on U such that $\lambda(u)\, H(V,u)$ and $\lambda(u)\, f(\alpha,u)$ are integrable with respect to u for each $V\epsilon E_n$, $\alpha\epsilon A$. Define

(24) $$H^*(V) = \int_U \lambda(u)\, H(V,u)\, du$$

(25) $$f^*(\alpha) = \int_U \lambda(u)\, f(\alpha,u)\, du$$

Also suppose that $\lambda(u)\, H(V,u)$ is continuous in the mean over U with respect to $dP(V,\alpha_o)$, and is measurable over the product space $E_n \times U$ for each $\alpha\epsilon A$. Then

$$f^*(\alpha) = \int H^*(V)\, dP(V,\alpha) \quad , \quad \text{all } \alpha\epsilon A \tag{26}$$

and

$$\Lambda^{(\alpha_o)}\left[f^*\right] = \int_U \lambda(u)\, \Lambda^{(\alpha_o)}\left[f(.,u)\right] du \tag{27}$$

<u>Proof</u>: (26) follows from integrating both sides of (23) after multiplying by $\lambda(u)$, and exchanging the order of integration. To prove (27), consider the Riemann sums

$$\sum f(\alpha,\xi_i)\, \lambda(\xi_i)\, (u_i - u_{i-1}) \quad , \quad \text{where } u_{i-1} \leqq \xi_i < u_i \ .$$

Now,

$$\sum f(\alpha,\xi_i)\, \lambda(\xi_i)\, (u_i - u_{i-1}) = \int \sum \lambda(\xi_i)\, H(V,\xi_i)\, (u_i - u_{i-1})\, dP(V,\alpha)$$

Also, by the continuity in the mean hypothesis, the Riemann sums

$\sum \lambda(\xi_i)\, H(V,\xi_i)\, (u_i - u_{i-1})$ converge in mean with respect to $dP(V,\alpha_o)$ to $H^*(V)$. Thus, (27) follows from Theorem 2.

Theorems 2, 3, and 4 are admittedly tailored to the examples of the next section. Many other similar theorems can be stated--for example, giving sufficient conditions for $\Lambda^{(\alpha_o)}$ to commute with differential operators; or perhaps weakening the conditions, or giving slightly different conditions, for the above theorems.

From this point on we assume that the probability measures $dP(V,\alpha)$ have density functions $p(V,\alpha)$. We turn our attention to the following function, defined for every fixed $\alpha_o \epsilon A$:

(28) $$G(\alpha_o,\beta,\gamma) = \int \frac{p(V,\beta)\ p(V,\gamma)}{p(V,\alpha_o)}\ dV \quad , \quad \gamma \epsilon A,\ \beta \epsilon B^{(\alpha_o)}$$

The notation $\gamma \epsilon A$, $\beta \epsilon B^{(\alpha_o)}$ means that for the given α_o, $G(\alpha_o,\beta,\gamma)$ is assumed to exist for all $\gamma \epsilon A$, provided β is in some set $B^{(\alpha_o)} \subset A$. (Examples are given in the next section.) It is clear that $p(V,\alpha_o) = 0$ for some V must imply that $p(V,\beta) = 0$ for all $\beta \epsilon B^{(\alpha_o)}$. If both numerator and denominator of the integrand in (28) are zero for some V, the integrand is understood to be zero.

We also assume that

(29) $$\int \left(\frac{p(V,\beta)}{p(V,\alpha_o)}\right)^2 p(V,\alpha_o)\ dV < \infty \quad \text{for } \beta \epsilon B_1^{(\alpha_o)} \subset B^{(\alpha_o)}$$

We may thus for each $\beta \epsilon B_1^{(\alpha_o)}$ consider $\Lambda^{(\alpha_o)}$ acting on $G(\alpha_o,\beta,\gamma)$, considered as a function of γ. From the definition (14) of $\Lambda^{(\alpha_o)}$, it is clear that

(30) $$\Lambda^{(\alpha_o)} \left[G(\alpha_o,\beta,.) \right] = \beta - \alpha_o$$

for all $\beta \epsilon B_1^{(\alpha_o)}$ for which

(31) $$\int F_m(V,\alpha_o)\ p(V,\beta) dV = \beta$$

We denote the subset of $B_1^{(\alpha_o)}$ for which (31) is satisfied by $B_2^{(\alpha_o)}$.

<u>Theorem 5</u>: A sufficient condition that a particular β

belonging to $B_1^{(\alpha_o)}$ be in $B_2^{(\alpha_o)}$ is that any sequence of functions of V which converges in mean to a function $H_\infty(V)$ with respect to $p(V,\alpha_o)$ also converges in mean to the same function $H_\infty(V)$ with respect to $p(V,\beta)$.

<u>Proof</u>: $F_m(V,\alpha_o)$ is defined to be the limit in the mean with respect to $p(V,\alpha_o)$ of functions satisfying (5) identically in α. Then $F_m(V,\alpha_o)$ is also the limit in the mean with respect to $p(V,\beta)$ of functions satisfying (5), for any β for which the stated condition of Theorem 5 holds. Thus, (31) holds for any such β.

We now have at our disposal the machinery to compute $D_m^{\ 2}(\alpha_o) = \Lambda^{(\alpha_o)}\left[\phi_1\right]$ in many cases. Eq. (30) gives the values of $\Lambda^{(\alpha_o)}$ on a certain family of functions of γ, each member of the family corresponding to some $\beta \epsilon B_2^{(\alpha_o)}$. If $B_2^{(\alpha_o)}$ gives a sufficiently large family of functions of γ, we may be able to express $\phi_1(\gamma) = \gamma$ as a linear combination of functions $G(\alpha_o,\beta,\gamma)$ for $\beta \epsilon B_2^{(\alpha_o)}$, or as a limit of such linear combinations. If this can be done in such a way as to satisfy the conditions of Theorems 2, 3, 4, or the like, $\Lambda^{(\alpha_o)}\left[\phi_1\right]$ can be calculated.

Since $G(\alpha_o,\beta,\gamma)$ is fundamental to this process, it is useful to state one more fact concerning it:

<u>Theorem 6</u>: If $p(V,\alpha) = \prod_{i=1}^{n} h_i\ (v_i,\alpha)$, all $\alpha \epsilon A$

(where h_i are density functions)

define

$$G_i^{(1)}(\alpha_o,\beta,\gamma) = \int \frac{h_i(v_i,\beta)\, h_i(v_i,\gamma)}{h_i(v_i,\alpha_o)}\, dv_i \qquad (32)$$

and assume that for all $\gamma \epsilon A$, $G_i^{(1)}$ exists for the same values of β for each i. Then

$$G(\alpha_o,\beta,\gamma) = \prod_{i=1}^{n} G_i^{(1)}(\alpha_o,\beta,\gamma) \qquad (33)$$

In particular, if $h_i(V,\alpha) = h(V,\alpha)$, all i; hence, if $G_i^{(1)}(\alpha_o,\beta,\gamma) = G^{(1)}(\alpha_o,\beta,\gamma)$ for all i, then

$$G(\alpha_o,\beta,\gamma) = \left\{ G^{(1)}(\alpha_o,\beta,\gamma) \right\}^n \qquad (34)$$

<u>Proof</u>: Let $E_\gamma \left\{ \right\}$ be the expected value of the quantity in braces with respect to $p(V,\gamma)$. Then,

$$G(\alpha_o,\beta,\gamma) = E_\gamma \left\{ \prod_{i=1}^{n} \frac{h_i(v_i,\beta)}{h_i(v_i,\alpha_o)} \right\} = \prod_{i=1}^{n} E_\gamma \left\{ \frac{h_i(v_i,\beta)}{h_i(v_i,\alpha_o)} \right\}$$

$$= \prod_{i=1}^{n} G_i^{(1)}(\alpha_o,\beta,\gamma)$$

III. <u>Examples</u>

Suppose

$$p(V,\alpha) = \left[c(k) \right]^n \frac{1}{\alpha^n} \exp \left[- \frac{1}{2\alpha^k} \sum_{i=1}^{n} v_i^{\,k} \right] \qquad (35)$$

k a positive even integer

$\alpha > 0$.

Here $c(k)$ is an appropriate normalizing constant. This corresponds to estimation of α, using a sample of n from the population

$$h(v,\alpha) = c(k)\,\frac{1}{\alpha}\ \exp\left[-\frac{1}{2}\left(\frac{v}{\alpha}\right)^k\right] \tag{36}$$

From Theorem 6, it is readily determined that

$$G(\alpha_o,\beta,\gamma) = \left[\frac{\alpha_o^{\,2}}{(\alpha_o^{\,k}-\beta^k)^{1/k}}\cdot\frac{1}{\left(\gamma^k+\dfrac{\alpha_o^{\,k}\beta^k}{\alpha_o^{\,k}-\beta^k}\right)^{1/k}}\right]^n \tag{37}$$

for $\gamma > 0$, $\beta \leqq \alpha_o$.

It is easily verified (applying Theorem 5) that $B_2^{(\alpha_o)}$ is the set $\beta \leqq \alpha_o$.

Now let

$$u = \frac{\alpha_o\beta}{(\alpha_o^{\,k}-\beta^k)^{1/k}} \tag{38}$$

$$\beta = \frac{\alpha_o u}{(\alpha_o^{\,k}+u^k)^{1/k}}$$

Then

$$G(\alpha_o,\beta,\gamma) = \left(\frac{\alpha_o^{\,k}+u^k}{\gamma^k+u^k}\right)^{\frac{n}{k}}\ ,\qquad 0 \leqq u < \infty \tag{39}$$

and (30) implies that

$$\Lambda^{(\alpha_o)}\left[\frac{1}{(\gamma^k+u^k)^{n/k}}\right] = \frac{-\alpha_o}{(\alpha_o^{\,k}+u^k)^{n/k}}\left[1-\frac{u}{(\alpha_o^{\,k}+u^k)^{1/k}}\right] \tag{40}$$

for $0 \leqq u < \infty$

This defines $\Lambda^{(\alpha_o)}$ on the family of functions $\left\{ (\gamma^k + u^k)^{\frac{-n}{k}} \right\}$, $0 \leqq u < \infty$. Now, it can be shown that for $\gamma > 0$,

(41) $$\gamma = \frac{k}{n} \frac{\Gamma(\frac{n+k}{k})}{\Gamma(\frac{n+1}{k}) \Gamma(\frac{k-1}{k})} \lim_{T \to \infty} \left\{ T - \int_0^T \frac{u^n \, du}{(\gamma^k + u^k)^{n/k}} \right\}$$

This can be derived as follows: let us find $\lambda(u)$ such that

(42) $$\int_0^\infty \lambda(u) \, (\gamma^k + u^k)^{-(\frac{n+k}{k})} = - \frac{1}{n\gamma^{k-1}}$$

The expression on the left side of (42) can be transformed by suitable change of variable to a generalized Stieltjes transform[2]. It turns out[2] that

(43) $$\lambda(u) = - \frac{k}{n} \frac{\Gamma(\frac{n+k}{k})}{\Gamma(\frac{n+1}{k}) \Gamma(\frac{k-1}{k})} u^n$$

Now, Eq. (42) cannot be multiplied by γ^{k-1} and then integrated with respect to γ under the integral, since the result would diverge. However, for $0 < T < \infty$, let

(44) $$- n \int_0^T \gamma^{k-1} \lambda(u) \, (\gamma^k + u^k)^{-(\frac{n+k}{k})} du = f_T(\gamma)$$

where $f_T(\gamma) \longrightarrow 1$ as $T \longrightarrow \infty$ for each $\gamma > 0$. Eq. (44) may be integrated between zero and γ under the integral sign, giving

(45) $$\frac{k}{n} \frac{\Gamma(\frac{n+k}{k})}{\Gamma(\frac{n+1}{k}) \Gamma(\frac{k-1}{k})} \left[\int_0^T \left\{ 1 - \frac{u^n}{(\gamma^k + u^k)^{n/k}} \right\} du \right]$$

$$= \int_0^\gamma f_T(\xi) \, d\xi$$

Letting $T \to \infty$, we obtain (41). Therefore assuming for the moment the applicability of our method, and recalling that $\Lambda^{(\alpha_o)}\left[T\right] = 0$, all T, we get

$$(46) \qquad \Lambda^{(\alpha_o)}\left[\phi_1\right] = \frac{k\alpha_o}{n} \frac{\Gamma\left(\frac{n+k}{k}\right)}{\Gamma\left(\frac{n+1}{k}\right)\Gamma\left(\frac{k-1}{k}\right)} \int_o^{\infty} \frac{u^n\left[1 - \frac{u}{(\alpha_o^k + u^k)^{1/k}}\right] du}{(\alpha_o^k + u^k)^{n/k}}$$

which, after another transformation of variable, gives

$$(47) \qquad D_m^2(\alpha_o) = \Lambda^{(\alpha_o)}\left[\phi_1\right] = \frac{k\alpha_o^2}{n} \frac{\Gamma\left(\frac{n+k}{k}\right)}{\Gamma\left(\frac{n+1}{k}\right)\Gamma\left(\frac{k-1}{k}\right)} \int_o^{\infty} \frac{(1+v^k)^{1/k} - 1}{v^2(1+v^k)^{\frac{n+1}{k}}} dv$$

Two special cases of interest are:

a) k=2 (estimation of the standard deviation of a normal population)

In this case, $\int_o^{\infty} \frac{(1+v^2)^{1/2} - 1}{v^2(1+v^2)^{\frac{n+1}{2}}} dv$

can be evaluated in closed form (as $\lim_{t \to o} \int_t^{\infty} \left\{ \right\} dv$), and the result is:

$$D_m^2(\alpha_o) = \frac{\alpha_o^2}{2n} \left\{ \frac{n^2 \Gamma^2(\frac{n}{2})}{\Gamma^2(\frac{n+1}{2})} - 2n \right\} \qquad \text{(for } k=2\text{)} \tag{48}$$

Now, it so happens[3] that the estimate

$$\hat{\alpha} = \sqrt{\frac{n}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \left\{ \frac{1}{n} \sum_{i=1}^{n} v_i^2 \right\}^{\frac{1}{2}} \tag{49}$$

is unbiased and has variance equal to $D_m^2(\alpha_o)$ as given by (48) for all $\alpha_o > 0$. Thus, $\hat{\alpha}$ given by (49) is the uniformly minimum-variance unbiased estimate of the standard deviation of a normal population.

b) $k \to \infty$

As $k \to \infty$, one can evaluate $\lim_{k\to\infty} D_m^2(\alpha_o)$ with the aid of the formulas

$$\lim_{k\to\infty} (1 + v^k)^{1/k} = \max\,(1,v) \tag{50}$$

$$\lim_{k\to\infty} \frac{k}{\Gamma(\frac{n+1}{k})} = n + 1 \tag{51}$$

One obtains

$$\lim_{k\to\infty} D_m^2(\alpha_o) = \frac{\alpha_o^2}{n(n+2)} \tag{52}$$

As $k \to \infty$, the probability density function $h(v,\alpha)$ in (36) approaches

$$\begin{aligned} h(v,\alpha) &= \frac{1}{2\alpha} \quad , \quad |v| \leq \alpha \\ &= 0 \quad , \quad |v| > \alpha \end{aligned} \tag{53}$$

Therefore, it is reasonable to suppose that (52) gives $D_m^2(\alpha_o)$ for the estimation of α in a population with density function given by (53), for a sample of n. That this is in fact the case can be checked by applying the above methods directly to a population with density function given by (53). This is left as an exercise for the reader. (Here one may use the additional fact that in this case $\Lambda^{(\alpha_o)}\left[f\right] = 0$ for any $f(\alpha)$ which is constant for $0 < \alpha \leqq \alpha_o$.) For estimation of α in (53), $D_m^2(\alpha_o)$ is attained for all α_o by the unbiased estimate

$$(54) \qquad \hat{\alpha} = \frac{n+1}{n} \max_i \left| v_i \right|$$

To show that the conditions for applying this method of evaluation are satisfied (in the general case given by (35)), we will first point out that, from the definition of $G(\alpha_o,\beta,\gamma)$ and from the expression (35) for $p(V,\alpha)$, it is readily determined that

$$(55) \qquad \frac{1}{(\gamma^k + u^k)^{n/k}} = \int H(V,u)\, p(V,\gamma)\, dV$$

where

$$(56) \qquad H(V,u) = \frac{1}{(\alpha_o^{\,k} + u^k)^{n/k}} \frac{p(V,\beta)}{p(V,\alpha_o)}$$

$$= \frac{1}{u^n} \exp \left[-\frac{1}{2u^k} \sum_{i=1}^{n} v_i^{\,k} \right]$$

It is readily verified that the conditions of Theorem 4 (with $\lambda(u)$ given by (43), and $U = \left[0,T\right]$) and of Theorem 3 are satisfied.

Still another example is

$$(57)\qquad p(V,\alpha) = \prod_{i=1}^{n} \left\{ \frac{2v_i}{\alpha^2} \exp\left[\frac{-v_i^2}{\alpha^2}\right] \right\}$$

corresponding to a sample of n from a population with density function

$$(58)\qquad h(v,\alpha) = \frac{2v}{\alpha^2} \exp\left[\frac{-v^2}{\alpha^2}\right]$$

It turns out that

$$(59)\qquad G(\alpha_o,\beta,\gamma) = \left(\frac{\alpha_o^2 + u^2}{\gamma^2 + u^2}\right)^n \quad , \quad 0 \leqq u < \infty$$

where

$$(60)\qquad u = \frac{\alpha_o\beta}{(\alpha_o^2 - \beta^2)^{1/2}} \quad , \ \beta \leqq \alpha_o$$

and

$$(61)\qquad \Lambda^{(\alpha_o)}\left[\frac{1}{(\gamma^2 + u^2)^n}\right] = \frac{-\alpha_o}{(\alpha_o^2 + u^2)^n}\left[1 - \frac{u}{\sqrt{\alpha_o^2 + u^2}}\right]$$

This is exactly the same as (40) for $k = 2$, with n in place of $\frac{n}{2}$. Thus,

$$(62)\qquad D_m^2(\alpha_o) = \frac{\alpha_o^2}{n}\left[\frac{\Gamma^2(n+1)}{\Gamma^2(n+\frac{1}{2})} - n\right]$$

Since the mean $\overline{v}$ in (58) is given by

(63) $$\bar{v} = \sqrt{\frac{4}{\pi}}\ \alpha,$$

the minimum variance of unbiased estimates of the mean is just $\frac{4}{\pi}$ times $D_m^{\ 2}(\alpha_o)$.

There are some cases in which $\phi_1(\gamma) = \gamma$ must be expressed in terms of a differential operator acting on $G(\alpha_o, \beta, \gamma)$, e.g.

$$\gamma = \left\{ \frac{\partial G}{\partial \beta} \Big|_{\beta=\alpha_o} \right\} \text{ x constant}_1 \quad + \quad \text{constant}_2$$

As an example, the reader might try applying the above methods to the case of estimation of the mean of a normal population with known variance. (The classical results are readily obtained.)

IV. More than One Unknown Parameter

The present section will extend the above results to cases where there is more than one unknown parameter. Since the development parallels that of the above sections closely, and the theorems are proved in the sameway, we will not give detailed proofs of the various steps in the argument.

Consider a family $\{dP(V, \alpha)\}$ of probability measures over Euclidean n-space E_n, where $V = (v_1, \ldots, v_n) \in E_n$ and $\alpha = (\alpha_1, \ldots, \alpha_K)$ belongs to some parallelopiped A in E_K. It is assumed that the measurable sets are the same for all α in A. By "function of V" we mean a function measurable on the sample space E_n for all $dP(V, \alpha)$.

By "unbiased estimate of α_1" we mean any function $F(V)$ satisfying

(64) $$\int F(V)\, dP(V, \alpha) = \alpha_1 \quad , \quad \text{all} \quad \alpha \epsilon A$$

(Integrals with respect to V always are taken over E_n.)

Let $\alpha_o = (\alpha_{o1}, \ldots, \alpha_{oK})$ denote a particular point of A. For any given F and α_o, consider

(65) $$\int \left[F(V) - \alpha_{o1} \right]^2 dP(V,\alpha_o)$$

We assume henceforth that at least one function F(V) exists satisfying (64) for all $\alpha \epsilon A$ and for which (65) is finite.

Define

(66) $$D_m^2(\alpha_o) = \text{greatest lower bound of (65)}$$

for all F satisfying (64).

For any given α_o, consider the set $C^{(\alpha_o)}$ of functions of V consisting of all functions satisfying (64) for all $\alpha \epsilon A$ and having finite second moment with respect to $dP(V,\alpha_o)$, and the limits in the mean of such functions with respect to $dP(V,\alpha_o)$. There is a unique (with probability one) function, denoted by $F_m(V,\alpha_o)$, which minimizes (65) for all functions in $C^{(\alpha_o)}$. Clearly

(67) $$\int F_m(V,\alpha_o)\, dP(V,\alpha_o) = \alpha_{o1}$$

(68) $$D_m^2(\alpha_o) = \int \left[F_m(V,\alpha_o) - \alpha_{o1} \right]^2 dP(V,\alpha_o)$$

Now consider functions $f(\alpha)$ defined over A of the form

(69) $$f(\alpha) = \int H(V)\, dP(V,\alpha) \quad , \quad \text{all } \alpha \epsilon A$$

where

(70) $$\int H^2(V)\, dP(V,\alpha_o) < \infty$$

We define a linear functional $\Lambda^{(\alpha_o)}$ on the set of functions f of form (69) by

(71) $$\Lambda^{(\alpha_o)}\left[f\right] = \int \left[F_m(V,\alpha_o) - \alpha_{o1}\right] H(V)\, dP(V,\alpha_o)$$

where H(V) is any function of V satisfying (70) and satisfying (69) for the given f. This can be shown to define $\Lambda^{(\alpha_o)}\left[f\right]$ uniquely (regardless of which particular H satisfying (69) is used) by the same reasoning as employed in Section II.

Denote by ϕ_o the function of α whose values on A are $\phi_o(\alpha) = 1$; and denote by ϕ_1 the function of α whose values on A are $\phi_1(\alpha) = \alpha_1$. Then

(72) $$\Lambda^{(\alpha_o)}\left[\phi_o\right] = 0$$

(73) $$\Lambda^{(\alpha_o)}\left[\phi_1\right] = D_m^{\,2}(\alpha_o)$$

Linearity and continuity properties can be shown for $\Lambda^{(\alpha_o)}$ similar to those proved in Section II.

Also, as in Section II, we now suppose that the probability measures $dP(V,\alpha)$ have density functions $p(V,\alpha)$ and we consider the function

(74) $$G(\alpha_o,\beta,\gamma) = \int \frac{p(V,\beta)\, p(V,\gamma)}{p(V,\alpha_o)}\, dV \quad , \quad \gamma\epsilon A, \ \beta\epsilon B^{(\alpha_o)}$$

Also assume that for $\beta \epsilon B_1^{(\alpha_o)} \subset B^{(\alpha_o)}$,

(75) $$\int \left[\frac{p(V,\beta)}{p(V,\alpha_o)}\right]^2 p(V,\alpha_o)\, dV < \infty$$

Thus, we may consider $\Lambda^{(\alpha_o)}$ acting on $G(\alpha_o,\beta,\gamma)$ considered as a function of γ for each $\beta \epsilon B_1^{(\alpha_o)}$. From (71)

(76) $$\Lambda^{(\alpha_o)}\left[G(\alpha_o,\beta,\ .\ )\right] = \beta_1 - \alpha_{o1}$$

for all $\beta \epsilon B_1^{(\alpha_o)}$ for which

(77) $$\int F_m(V,\alpha_o)\, p(V,\beta)\, dV = \beta_1 .$$

We denote by $B_2^{(\alpha_o)}$ the subset of $B_1^{(\alpha_o)}$ for which (77) is satisfied.

Also, we may state that if

(78) $$p(V,\alpha) = \prod_{i=1}^{n} h(v_i,\alpha) \quad , \quad \text{all } \alpha \epsilon A$$

(where $h(v_i,\alpha)$ is a density function), then

(79) $$G(\alpha_o,\beta,\gamma) = \left[G^{(1)}(\alpha_o,\beta,\gamma)\right]^n$$

where

(80) $$G^{(1)}(\alpha_o,\beta,\gamma) = \int \frac{h(v,\beta)\, h(v,\gamma)}{h(v,\alpha_o)}\, dv$$

We will apply this to the case of estimation of the standard deviation of a normal population with unknown mean. In this case, let

(81) $$h(v,\alpha) = \frac{1}{\alpha_1 \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{v - \alpha_2}{\alpha_1} \right)^2 \right]$$

where $-\infty < \alpha_2 < \infty; \quad \alpha_1 > 0.$

Also, for $n \geqq 2$, let

(82) $$p(V,\alpha) = \prod_{i=1}^{n} h(v_i,\alpha)$$

It is readily determined that, for β such that $\beta_1 \leqq \alpha_{o1}$,

(83) $$G(\alpha_o,\beta,\gamma) = \frac{\alpha_{o1}^{2n}}{(\alpha_{o1}^2\gamma_1^2 + \alpha_{o1}^2\beta_1^2 - \beta_1^2\gamma_1^2)^{n/2}}$$

$$\times \exp \left\{ -n \left[\frac{\alpha_{o1}^2\gamma_1^2\beta_2^2 + \alpha_{o1}^2\beta_1^2\gamma_2^2 - \beta_1^2\gamma_1^2\alpha_{o2}^2}{2\alpha_{o1}^2\beta_1^2\gamma_1^2} \right] + n \frac{\left[\alpha_{o1}^2\gamma_1^2\beta_2 + \alpha_{o1}^2\beta_1^2\gamma_2 - \beta_1^2\gamma_1^2\alpha_{o2} \right]^2}{2\alpha_{o1}^2\beta_1^2\gamma_1^2 \left[\alpha_{o1}^2\gamma_1^2 + \alpha_{o1}^2\beta_1^2 - \beta_1^2\gamma_1^2 \right]} \right\}$$

It is clear that $D_m^2(\alpha_o)$ is independent of the true value of the mean--i.e. of α_{o2}--so no generality is lost by setting $\alpha_{o2} = 0$. Then (83) simplifies to

(84)
$$G(\alpha_o,\beta,\gamma) = \frac{\alpha_{o1}^{2n}}{(\alpha_{o1}^2\gamma_1^2 + \alpha_{o1}^2\beta_1^2 - \beta_1^2\gamma_1^2)^{n/2}}$$

$$\times \exp\left\{- n \left[\frac{\gamma_1^2\beta_2^2 + \beta_1^2\gamma_2^2}{2\beta_1^2\gamma_1^2}\right] + n \frac{\alpha_{o1}^2\left[\gamma_1^2\beta_2 + \beta_1^2\gamma_2\right]^2}{2\beta_1^2\gamma_1^2\left[\alpha_{o1}^2\gamma_1^2 + \alpha_{o1}^2\beta_1^2 - \beta_1^2\gamma_1^2\right]}\right\}$$

Now consider a function $\lambda(\beta_2)$ defined by

(85)
$$\lambda(\beta_2) = \frac{1}{\sqrt{2\pi}} \left(\frac{n}{\alpha_{o1}^2 - \beta_1^2}\right)^{1/2} \exp\left[\frac{-n\beta_2^2}{2(\alpha_{o1}^2 - \beta_1^2)}\right]$$

We have

(86)
$$\int_{-\infty}^{\infty} \lambda(\beta_2)\, G(\alpha_o,\beta,\gamma)\, d\beta_2 = \left[\frac{\alpha_{o1}^2}{\sqrt{\alpha_{o1}^2\gamma_1^2 + \alpha_{o1}^2\beta_1^2 - \beta_1^2\gamma_1^2}}\right]^{n-1}$$

and

(87)
$$\int_{-\infty}^{\infty} \lambda(\beta_2)\,(\beta_1 - \alpha_{o1})\, d\beta_2 = \beta_1 - \alpha_{o1}$$

Thus, integrating both sides of (76), (which holds for all β such that $\beta_1 \leqq \alpha_{o1}$), with respect to $\lambda(\beta_2)d\beta_2$, we obtain

$$\Lambda^{(\alpha_o)}\left[\left\{\frac{\alpha_{o1}^2}{\sqrt{\alpha_{o1}^2\gamma_1^2+\alpha_{o1}^2\beta_1^2-\beta_1^2\gamma_1^2}}\right\}^{n-1}\right] \tag{88}$$

$$= \beta_1 - \alpha_{o1}$$

But this is of precisely the same form as Eq. (40) of Section III (for $k = 2$), with $n - 1$ in place of n. Thus, we can now apply exactly the same methods as applied in Section III to obtain

$$D_m^{\ 2}(\alpha_o) = \frac{\alpha_{o1}^{\ 2}}{2(n-1)}\left\{\frac{(n-1)^2\,\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma^2\left(\frac{n}{2}\right)} - 2(n-1)\right\} \tag{89}$$

Consider the unbiased estimate

$$\hat{\alpha}_1 = \sqrt{\frac{n}{2}}\,\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left\{\frac{1}{n}\sum_{i=1}^{n}(v_i-\bar{v})^2\right\}^{\frac{1}{2}} \tag{90}$$

where

$$\bar{v} = \frac{1}{n}\sum_{i=1}^{n} v_i \tag{91}$$

According to Cramer[3], $\hat{\alpha}_1$ has variance equal to $D_m^{\ 2}(\alpha_o)$ as given by (89).

Using (89) and the results of Section III, we can compare the results for unbiased estimation of the standard deviation of a normal population a) with known mean, and b) with unknown mean. It is seen that $D_m^2(\alpha_o)$ for case b), for a sample of n, is equal to $D_m^2(\alpha_o)$ for case a), for a sample of n-1.

We may apply the same methods to obtain $D_m^2(\alpha_o)$ for the unbiased estimation of the mean of a normal population of unknown variance. It is convenient to retain the notation of Eq's. (81), (83), and (84), and to substitute the subscript 2 for the subscript 1 in Eq's. (64)-(71), (76), and (77). Also, we may again without loss of generality set $\alpha_{o2} = 0$. It is then easy to verify that

$$\gamma_2 = \frac{\alpha_{o1}^2}{n} \left[\frac{\partial}{\partial\beta_2} G(\alpha_o,\beta,\gamma) \right]_{\substack{\beta_2 = \alpha_{o2} = 0 \\ \beta_1 = \alpha_{o1}}} \tag{92}$$

Thus, (where now $\phi_1(\alpha) = \alpha_2$ for $\alpha \epsilon A$),

$$\begin{aligned} D_m^2(\alpha_o) &= \Lambda^{(\alpha_o)} \left[\phi_1 \right] = \frac{\alpha_{o1}^2}{n} \left[\frac{\partial}{\partial\beta_2} (\beta_2 - \alpha_{o2}) \right]_{\beta_2 = 0} \\ &= \frac{\alpha_{o1}^2}{n} \end{aligned} \tag{93}$$

which is the classical result.

References

1. Kendall, M. G., The Advanced Theory of Statistics, v. 2, 1951.

2. Erdelyi, et al, Tables of Integral Transforms, v. 2, 1954.

3. Cramer, H., Mathematical Methods of Statistics, Princeton University Press, 1946.

4. Kiefer, J., "On Minimum Variance Estimators," Annals of Math. Stat., v. 23, pp. 627-629, 1952.

5. Barankin, E. W., "Locally Best Unbiased Estimates," Annals of Math. Stat., v. 20, pp. 477-501, 1949.

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