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CONTROL SYSTEMS LABORATORY

THE TRANSIENT RESPONSE

OF

CASCADED BUTTERWORTH FILTERS

Report Number R-97

August, 1957

Contract DA-36-039-SC-56695
D/A Sub-Task 3-99-06-111

UNIVERSITY OF ILLINOIS · URBANA · ILLINOIS

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ABSTRACT

The impulse and step responses for cascaded, three or five pole Butterworth filters have been calculated on Illiac with an accuracy of three or more decimals. In this paper the dependence of the calculated step response upon the number of cascaded filters λ_2 is studied, both with no phase correction and with complete phase correction, with λ_2 in the range from 4 to 512.

It is found (for the three and five pole filters) that the rise time, period of ringing, and non-linear part of the time delay in the step response, for Butterworth filters without phase correction, are each approximately proportional to $\lambda_2^{+1/3}$, for large enough λ_2. The rise time and period of ringing are about proportional to $\lambda_2^{+1/2\,\lambda_1}$ for large λ_2 (λ_1 equals the number of poles) for the Butterworth filters with complete phase correction. Both asymptotic exponents are also derived analytically. In all cases studied the amount of overshoot seems to approach an upper limit and the logarithmic damping rate of the ringing seems to approach a lower limit as λ_2 increases.

The pronounced decrease in the damping rate of the ringing is the most important distortion accompanying cascading of the uncorrected Butterworth filters. This distortion is caused chiefly by the non-linear phase lag.

INTRODUCTION

In 1956-57 we developed an Illiac program which could be used to calculate the impulse and step response of a wide variety of cascaded Butterworth filters, these being the type of filters of principal interest to us. During the construction of this program we tried to keep in mind its potential usefulness for computing cosine and sine transformations of any reasonable function.

A Butterworth or flat-staggered tuned network[1] is characterized by the number of poles it contains. This parameter we call λ_1. Cascading of identical Butterworth filters introduces a second parameter, λ_2, the number of filters in cascade. In this paper we shall describe the results of calculations of transient response for $\lambda_1 = 3$ and 5 and λ_2 in the range 4 to 512. As an indication of the possible improvements to be achieved by phase correction networks, we also computed the transient response of "unrealizable" filters with no phase distortion but with the same amplitude response as the Butterworth filters.

The weight function or the response to unit impulse $w^o(t)$ may be defined[2] by the expression

$$w^o(t) = (1/2\pi) \int_{-\infty}^{\infty} Y(j\omega) \exp(j\omega t) \, d\omega \qquad (1)$$

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[1] Valley and Wallman, "Vacuum Tube Amplifiers". (Rad. Lab. Series, vol. 18.) McGraw-Hill (1948). Sect. 4.6.

[2] James, Nichols and Phillips, "Theory of Servomechanisms". (Rad. Lab. Series, vol. 25.) McGraw-Hill (1947). Sect. 2.13.

where $Y(j\omega) = \mathcal{F}(\omega) \exp\left[-j\mathcal{G}(\omega)\right]$ is the system function for the filter network. The response to unit step $u^{o}(t)$ is given by[3]

$$u^{o}(t) = \int_{o}^{t} w^{o}(t_1)\, dt_1 \qquad (2)$$

Because the amplitude response $\mathcal{F}(\omega)$ of the filter is an even function of frequency and the phase lag $\mathcal{G}(\omega)$ is an odd function of frequency, Eq. 1 can be rewritten as

$$w^{o}(t) = (1/\pi) \int_{o}^{\infty} \mathcal{F}(\omega) \cos\left[\omega t - \mathcal{G}(\omega)\right] d\omega \qquad (3)$$

For applications it is convenient to use dimensionless variables which may be introduced as follows:

$$f_o t = T$$

$$\omega = 2\pi f_o x$$

$$\mathcal{F}(\omega) = F(x)$$

$$\mathcal{G}(\omega) = G(x)$$

$$w^{o}(t) = w^{o}(T)$$

$$u^{o}(t) = u^{o}(T)$$

where f_o is the half-width of a filter at the half-power point.

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[3] Ref. 2, Sect. 2.7.

In terms of these quantities Eq. 3 becomes

$$w^{o}(T) = 2 \int_{o}^{\infty} F(x) \cos\left[2\pi xT - G(x)\right] dx \qquad (4)$$

and Eq. 2 becomes

$$u^{o}(T) = \int_{o}^{T} w^{o}(T)\, dT \qquad (5)$$

The amplitude response $F(x)\Big/_{\lambda_2=1}$ of one λ_1-pole Butterworth filter ($\lambda_2 = 1$), in terms of the reduced frequency variable x, is given by[4]

$$F(x)\Big/_{\lambda_2=1} = (1 + x^{2\lambda_1})^{-1/2} \qquad (6)$$

and the phase lag is given by[4]

$$G_1(x)\Big/_{\lambda_2=1} =$$

$$\sum_{j=1}^{k} tg^{-1}\left[2x\cos\overline{\theta}_j/(1 - x^2)\right], \quad \overline{\theta}_j = \pi(2j - 1)/2\lambda_1, \quad \lambda_1 \text{ even} \qquad (7)$$

$$tg^{-1}x + \sum_{j=1}^{k} tg^{-1}\left[2x\cos\overline{\theta}_j/(1 - x^2)\right], \quad \overline{\theta}_j = \pi j/2\lambda_1, \quad \lambda_1 \text{ odd}$$

where in each case k is chosen so that $0 < \overline{\theta}_j < \pi/2$. Therefore the amplitude and phase responses of λ_2 such identical filters in cascade

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[4] Ref. 1, Sect. 7.5.

are given by

$$F(x) = \left[F(x) \Big/_{\lambda_2=1} \right]^{\lambda_2}$$

$$G_1(x) = \lambda_2 \left[G_1(x) \Big/_{\lambda_2=1} \right]$$

Complete phase correction is expressed by the equation

$$G(x) = G_o(x) = 0 \qquad (8)$$

Typical results of the Illiac calculation are shown in Fig. 1a, b for the impulse response. (The following discussion would also apply to the distortion by the filters of unit step input.) The filter whose characteristics are illustrated is a five pole filter. A unit impulse passed through the four, five-pole filters (either with full phase correction and with no phase correction) gives the responses shown in Fig. 1a. We note that the width of the impulse response is not significantly improved for this value of λ_2 by the phase correction but that the amplitude of the ringing is noticeably reduced by the phase correction. When the unit impulse has passed through 512 five-pole filters, its shape has been changed more drastically as is shown in Fig. 1b. With full phase correction $G(x) = 0$ the impulse response for $\lambda_2 = 512$ is widened by a factor of 1.5 over what it was for $\lambda_2 = 1$, and this change of shape can be attributed entirely to the slight narrowing of the band pass of the iterated filter owing to the repeated cascading. The situation for the 512-fold cascaded Butterworth filter with no phase correction is seen to be quite different. The impulse has set the system ringing with a long period and a very low rate of decay.

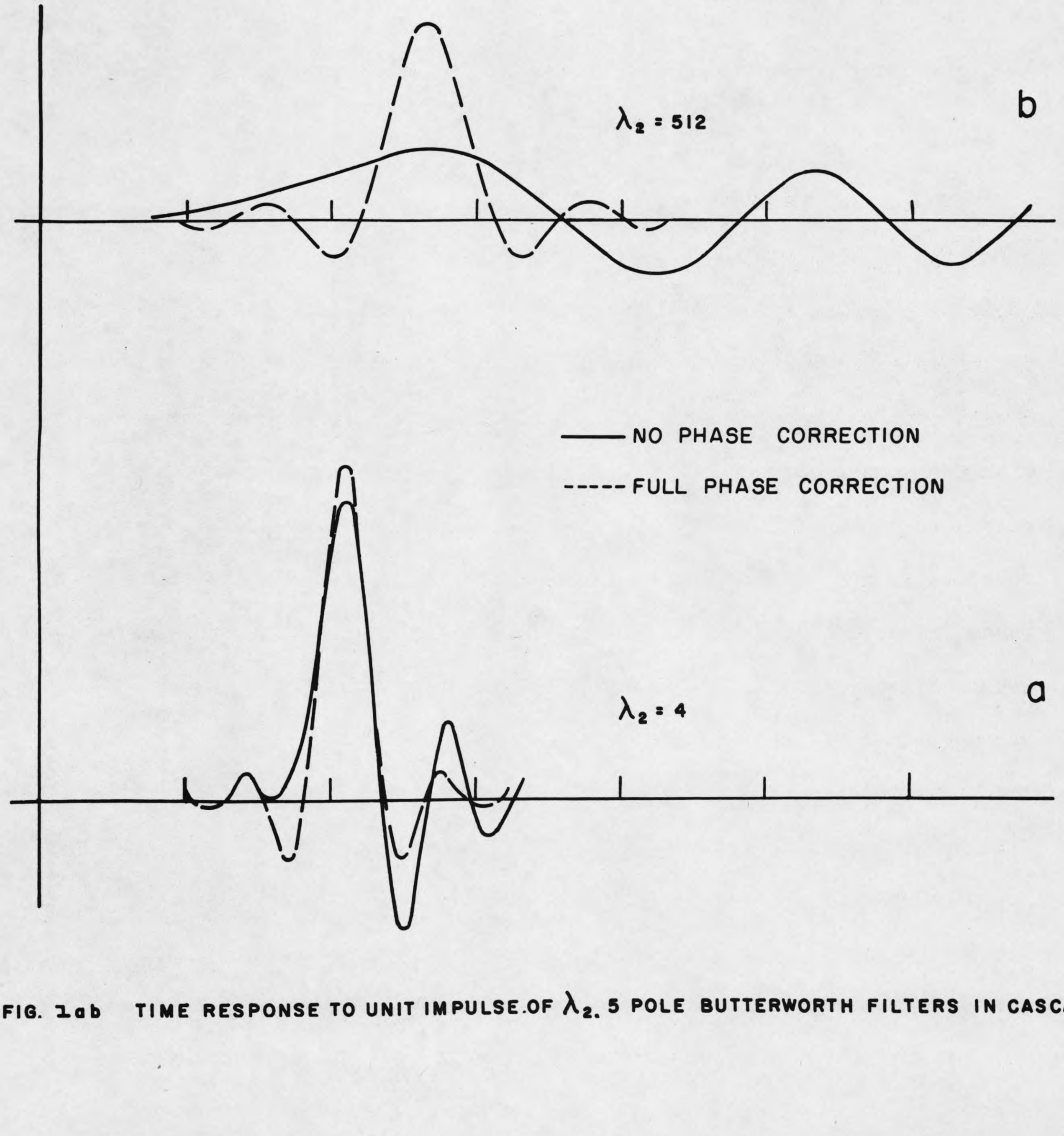

FIG. 1ab TIME RESPONSE TO UNIT IMPULSE OF λ_2. 5 POLE BUTTERWORTH FILTERS IN CASCADE.

DESCRIPTION OF CALCULATIONS

Two values of λ_1 were used, 3 and 5. The parameter λ_2 was varied between 4 and 512. In all cases, the impulse and step responses were calculated for no phase correction and with complete phase correction. In this report we describe only the step response, the data for which is given in Table I. (Complete tables of w(T) and u(T), which are the values of $w^o(T)$ and $u^o(T)$ obtained by numerical integration, are on file at CSL for the cases considered.)

We shall pick certain parameters characterizing these step responses and discuss these parameters in some detail. The parameters of interest are calculated from the coordinates of the six numbered points of a step response curve defined in Fig. 2. These coordinates are listed in Table I. The five parameters, listed in Table II, are derived from these coordinates and are defined as follows:

1. time delay $T_{0.1} = T_1$
2. rise time $\tilde{T} = (T_2 - T_1)$
3. overshoot $\Delta u^o = 100\,(u^o_3 - 1)$
4. period $T_d = 2(T_6 - T_5)$
5. damping factor $\sigma_d = 2 \ln \left| (u^o_5 - 1)/(u^o_6 - 1) \right|$

All of these characteristics refer to the response to unit step although some of them also have a direct relationship to the response to unit impulse. The time delay is the time required (measured from T = 0) for the step response to reach the value of 0.10 . The rise time is the interval of time required for the step response to increase from

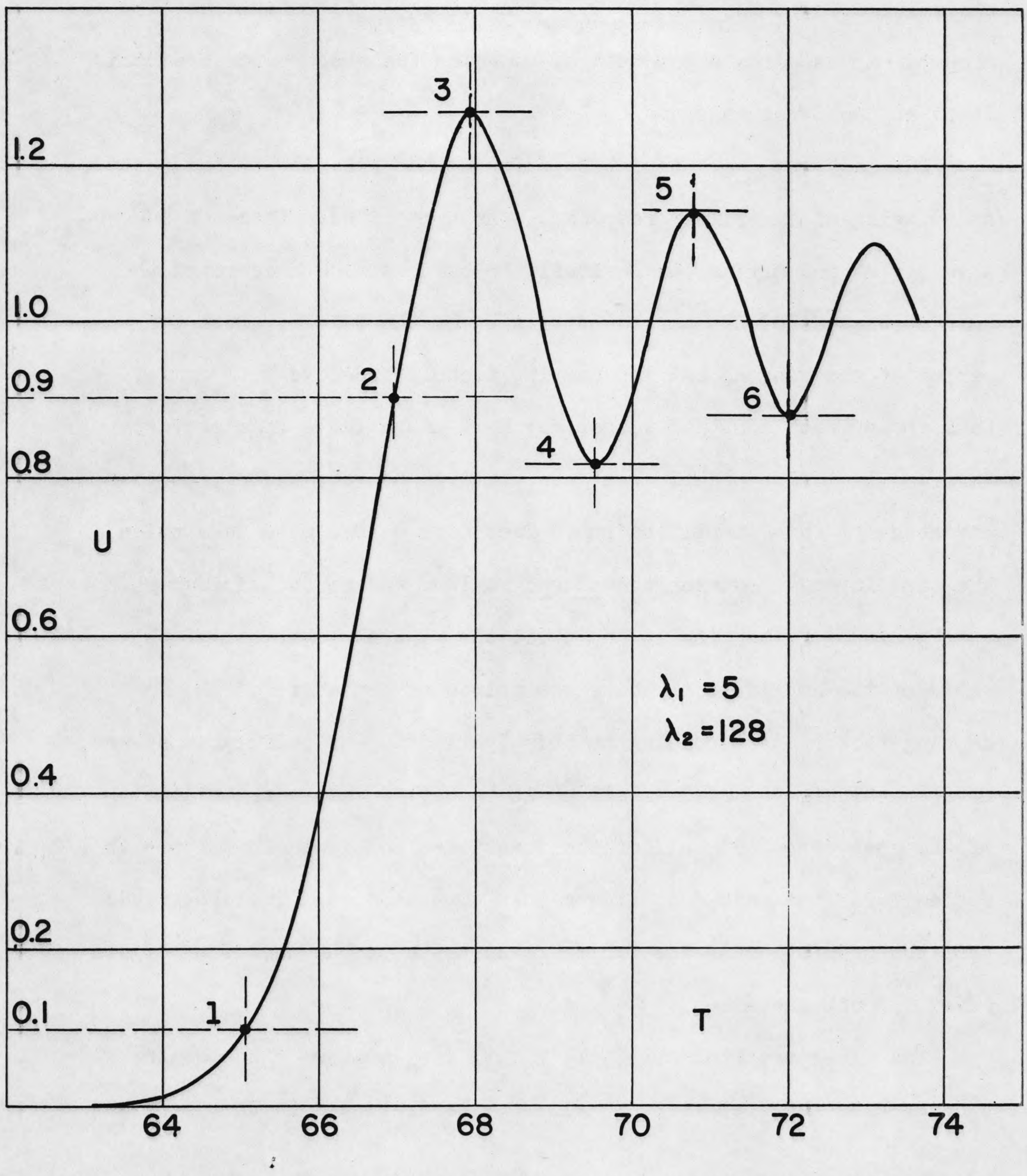

FIG. 2 RESPONSE TO UNIT STEP OF 128 5-POLE BUTTERWORTH FILTERS IN CASCADE.

the value 0.10 to the value 0.90. The overshoot is defined as the amount, expressed as a percentage, by which the step response exceeds unity at the first maximum.

The last two characteristics refer to the behavior of the decay and ringing of the filter response. The time of occurrence of successive minima and maxima can obviously be used as some measure of the characteristics of the period of ringing of the filter. Both the period of the ringing and its damping factor, from cycle to cycle, tend to approach constant values as the time variable becomes very large. It was not practicable, nor would it be very useful, to take advantage of this simplification. Therefore we chose to take twice the time interval between the <u>second</u> maximum and <u>second</u> minimum as the "period" of the ringing and twice the natural logarithm of the ratio of the amplitude of these two points as a measure of the damping factor corresponding to this "period". Our calculations were, correspondingly, carried in all cases to beyond the second minimum of the step response. All of the quantities mentioned are listed in Table II for the cascaded filters that were studied. It is believed that the quantities in the table are accurate to the number of digits given in each instance.*

The five parameters $T_{0.1}$, $\tilde{T}$, T_d, Δu^o, σ_d that we thus use to characterize the step response will now be discussed in this order.

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* Description of methods used in performing the Fourier transforms on the Illiac and in studying the error problems associated with this calculation will be postponed to future reports.

RESULTS

$T_{0.1}$ - (delay)

For $G = G_1$ (no phase correction), the delay time for $\lambda_2 \gg 4$ derived from the Illiac calculations agrees closely with the values calculated from the equation

$$T_{0.1} = \lambda_2 \left(b_{0.1} - a_{0.1} \lambda_2^{-0.6}\right) \tag{9}$$

where $a_{0.1}$, $b_{0.1}$ have the values shown in the first two columns of the following table:

λ_1	$a_{0.1}$	$b_{0.1}$	$b_{0.1}$
3	0.156	0.321	0.318
5	0.165	0.520	0.515
		(Illiac)	(stationary phase)

An elementary application of the method of stationary phase predicts the values of $b_{0.1}$ in the third column of the above table. The small difference between the values of $a_{0.1}$ for $\lambda_1 = 3$ and $\lambda_1 = 5$ suggests that possibly $a_{0.1}$ becomes independent of λ_1 as λ_2 increases indefinitely. It is clear that although the delay $T_{0.1}$ is not a linear function of λ_2, the non-linearity decreases with increase of λ_2.

For filters with $G = G_0$ (full phase correction), a delay $T_{0.1}$ has no meaning unless features are added to these "unrealizable" filters which are not considered in this report.

$\tilde{T}$ (rise time) Fig. 3.

For $G = G_1$ (no phase correction) the calculated rise time of the

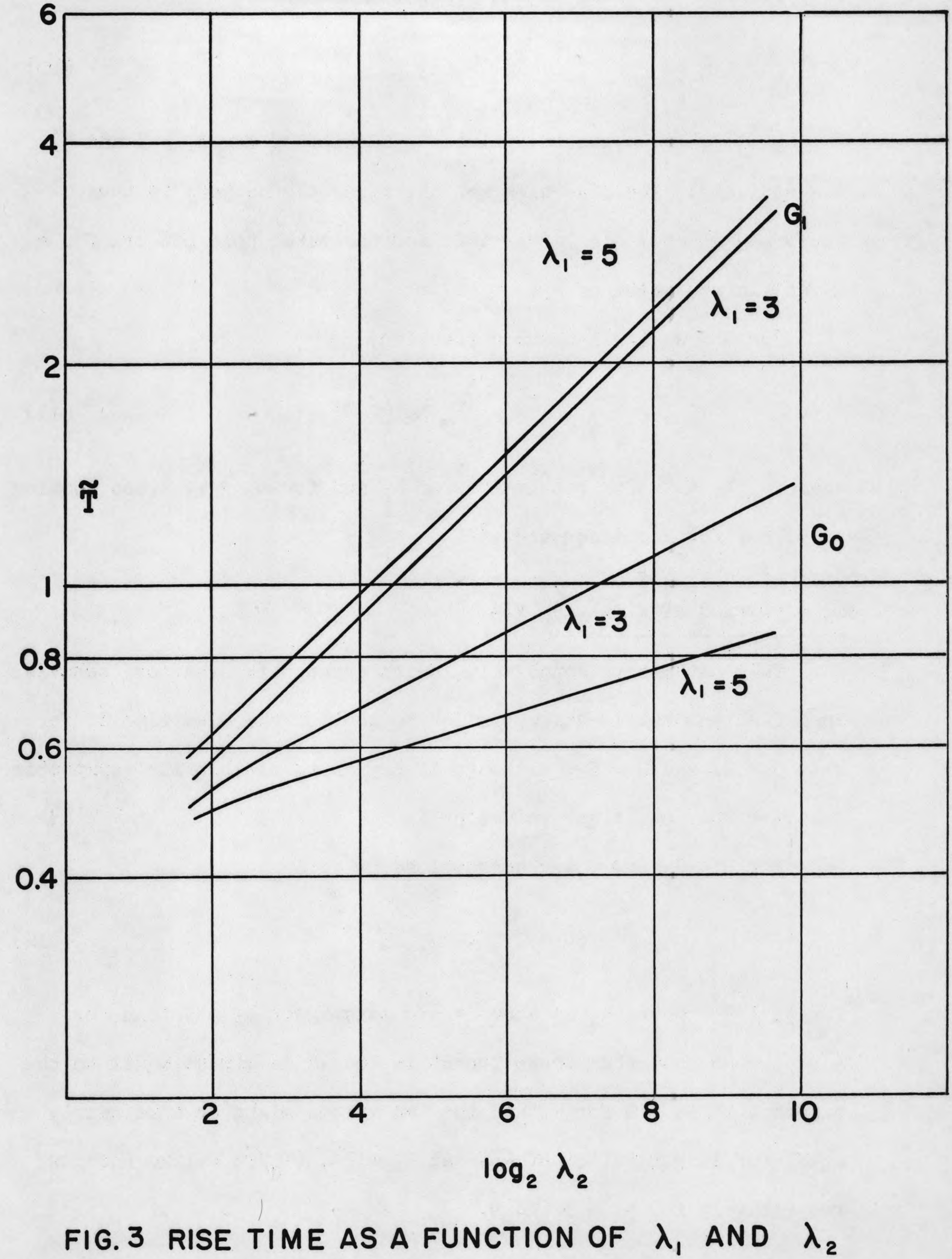

FIG. 3 RISE TIME AS A FUNCTION OF λ_1 AND λ_2

step response conforms to the formula

$$\tilde{T} \cong \tilde{a}_1 \lambda_2^{0.33} \qquad (10)$$

for $\lambda_2 > 8$ where $\tilde{a}$ has the values 0.36 and 0.39 for $\lambda_1 = 3$ and 5, respectively. The rise time for the five pole network is thus uniformly about 7 o/o larger than for the three pole network for any given (large) value of λ_2.

For $G = G_0$ (full phase correction)

$$\tilde{T} = \tilde{a}_0 \lambda_2^{1/2} \lambda_1 \qquad (11)$$

where $\tilde{a}_0 \cong 0.44$ for both values of λ_1 and for $\lambda_2 > 4$. (See section on T_d for further discussion.)

T_d - (period of ringing) Fig. 4.

This parameter, especially in its asymptotic behavior, behaves in a fashion that is quite similar to that of the rise time $\tilde{T}$, for both $G = G_0$ and $G = G_1$, although it begins to exhibit its asymptotic behavior only at larger values of λ_2.

For $G = G_1$ (no phase correction)

$$T_d \sim a_d \lambda_2^{0.3} \qquad (12)$$

for $\lambda_2 > 2^7$, where a_d is about 4 o/o larger for $\lambda_1 = 5$ than for $\lambda_1 = 3$. We note that the exponent in Eq. 12 is almost equal to the exponent in Eq. 10 for $\tilde{T}$, and the two values might be more nearly equal for larger values of λ_2. At $\lambda_2 = 2^9$, $(T_d/\tilde{T}) = 1.20$ and 1.23 respectively for $\lambda_1 = 3$ and 5.

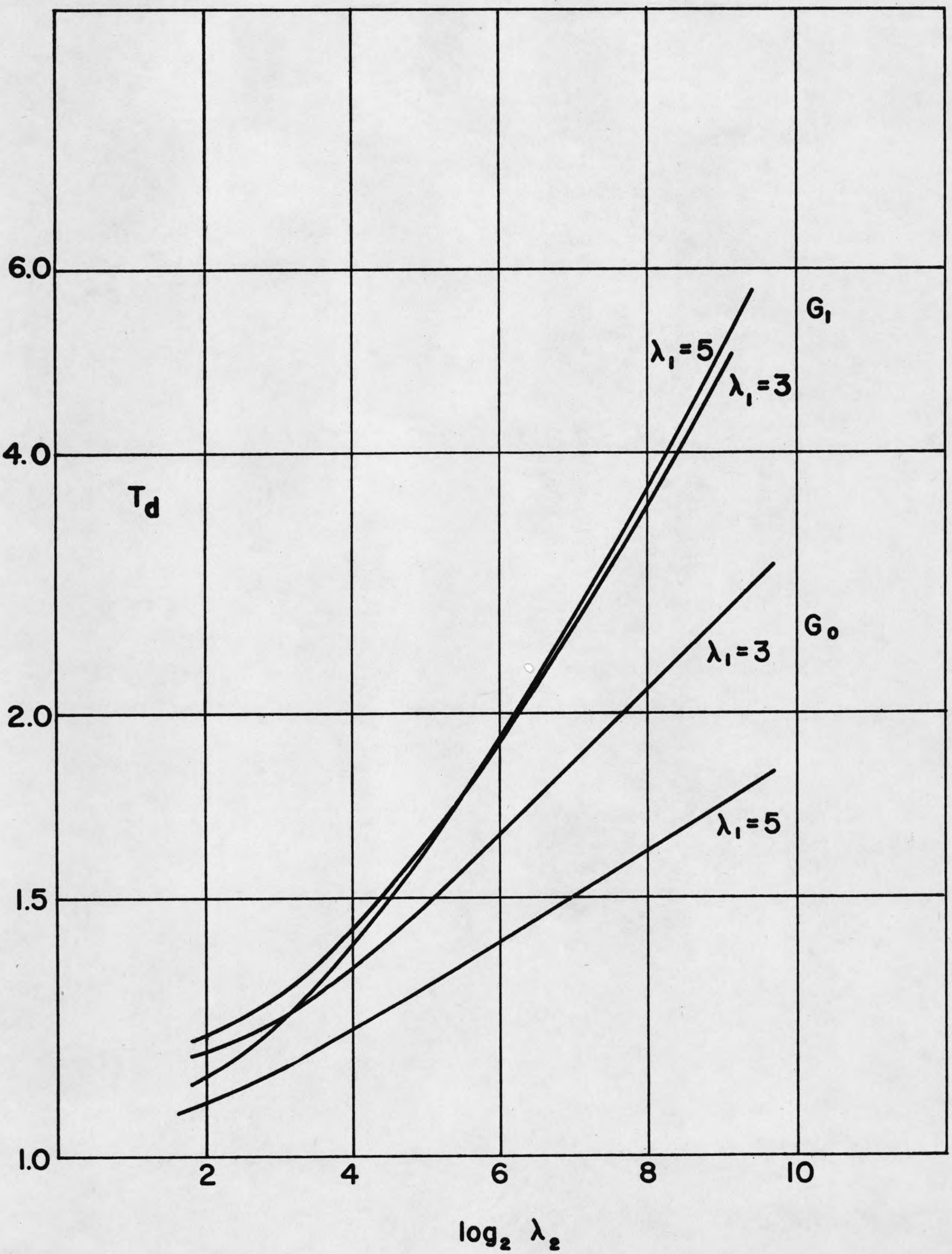

FIG. 4 PERIOD OF RINGING AS A FUNCTION OF λ_1 AND λ_2

For $G = G_0$ (full phase correction)

$$T_d = a_d \lambda_2^{1/2} \lambda_1 \tag{13}$$

for $\lambda_2 > 2^6$ where $a_d = 0.83, 0.93$ for $\lambda_1 = 3, 5$ respectively. Comparing with the results for $\tilde{T}$ we find that, asymptotically, $(T_d/\tilde{T}) \sim 1.88, 2.11$ for $\lambda_1 = 3, 5$ respectively.

The half width x_1 of the function $F(x)$ when its value is $2^{-1/2}$ (half-power point) is accurately proportional to $\lambda_2^{-1/2} \lambda_1$ for values of $\lambda_2 \gg 1$. If we replace $F(x)$ by a square band pass with band-width equal to x_1, then the time scale of the transient response of this square filter (with $G = G_0$) will be proportional to x_1^{-1} or to $\lambda_2^{1/2} \lambda_1$ which is just the variation we have found, for $G = G_0$, for the parameters T_d and $\tilde{T}$ which are measures of the time scale. Apparently for $G = G_1$, the effect of the phase distortion is such as to nearly cancel this dependence of the time scale upon the number of poles, λ_1, for, as we have seen, T_d and $\tilde{T}$ behave very much the same for the two values of λ_1.

Δu - (overshoot)

For $G = G_1$ (with $\lambda_2 > 8$) and for $G = G_0$ (with $\lambda_2 > 4$), the calculated overshoot data can be well represented by an equation of the form

$$\Delta u = b_0 - a_0 \lambda_2^{-p_0} \tag{14}$$

where the coefficients a_0, b_0, p_0 have the following values:

	λ_1	100 a_0	100 b_0	p_0
$G = G_1$	3	26.10	28.50	0.40
	5	20.0	27.55	0.62
$G = G_0$	3	2.7	7.06	1.02
	5	1.5	8.20	1.16

It is perhaps worth emphasizing that for each value of λ_1 the overshoot seems to have an upper limit which it approaches asymptotically for large λ_2.

σ_d - (damping ratio) Fig. 5.

The accuracy of the values of σ_d is both variable and questionable in our calculations, partly because one to four significant digits are lost when $(u - 1)$ is formed from u. Like Δu, σ_d appears to approach a limiting value, for each given λ_1, as λ_2 increases. Within the accuracy of our data, this asymptotic approach may be represented (for $\lambda_2 > 128$),

$$\sigma_d \sim B_d + A_d \lambda_2^{-1} \qquad (15)$$

where A and B have the following values:

	λ_1	A_d	B_d
$G = G_1$	3	43	0.43
	5	7.8	0.30
$G = G_0$	3	1.5(?)	2.4(?)
	5	1.75	1.37

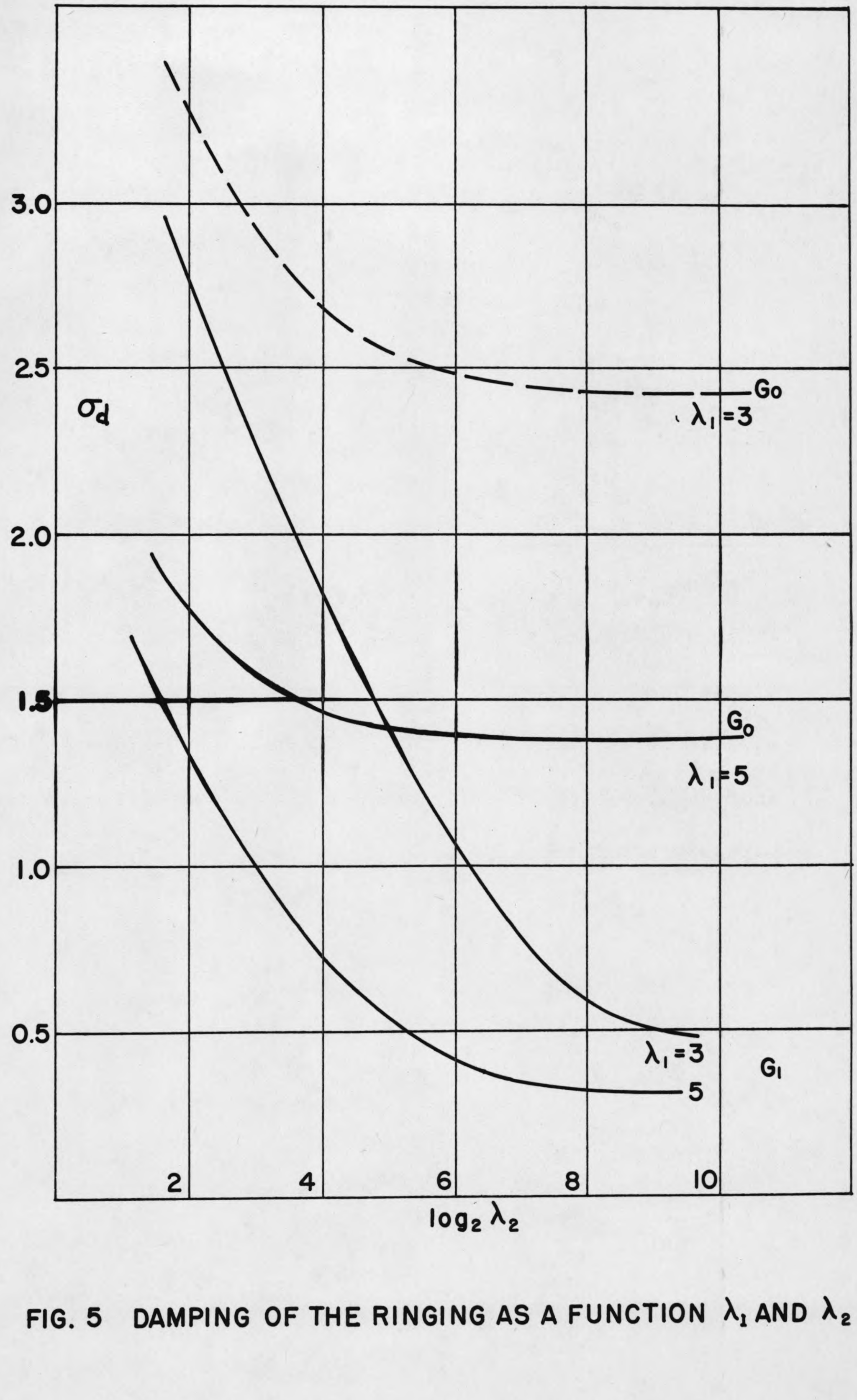

FIG. 5 DAMPING OF THE RINGING AS A FUNCTION λ_1 AND λ_2

For $G = G_1$, $\lambda_1 = 5$, the data supports the exponent one; for other cases, this value is assumed.

Even though the damping per cycle, σ_d, asymptotically approaches a lower limit, the damping per unit time, (σ_d/T_d) decreases indefinitely as λ_2 increases because T_d, as we have seen, appears to increase proportionally to a positive power of λ_2.

Analysis for large values of λ_2

When λ_2 is large, the amplitude function $F(x)$ becomes negligibly small for all $x \gg x_1$ where $x_1 \sim \lambda_2^{-1/2\ \lambda_1}$ as shown before and illustrated in Fig. 6. If λ_2 is large enough, $x_1 \ll 1$ and the phase function $G_1(x)$ is well approximated by the first two terms of its power series

$$G_1(x) \sim \lambda_2 \, (g_1 \, x + g_3 \, x^3) \tag{16}$$

where

$$g_1 = 1 + 2 \sum_{j=1}^{k} \cos \bar{\theta}_j$$

$$g_3 = -\frac{1}{3} \left(1 + 8 \sum_{j=1}^{k} \cos^3 \bar{\theta}_j\right) + 2 \sum_{j=1}^{k} \cos \bar{\theta}_j \tag{17}$$

for Butterworth filters with an odd number of poles. The angles $\bar{\theta}_j$ give the positions, relative to the negative x axis, of those poles lying in the upper half plane. The angles and coefficients have the following values:

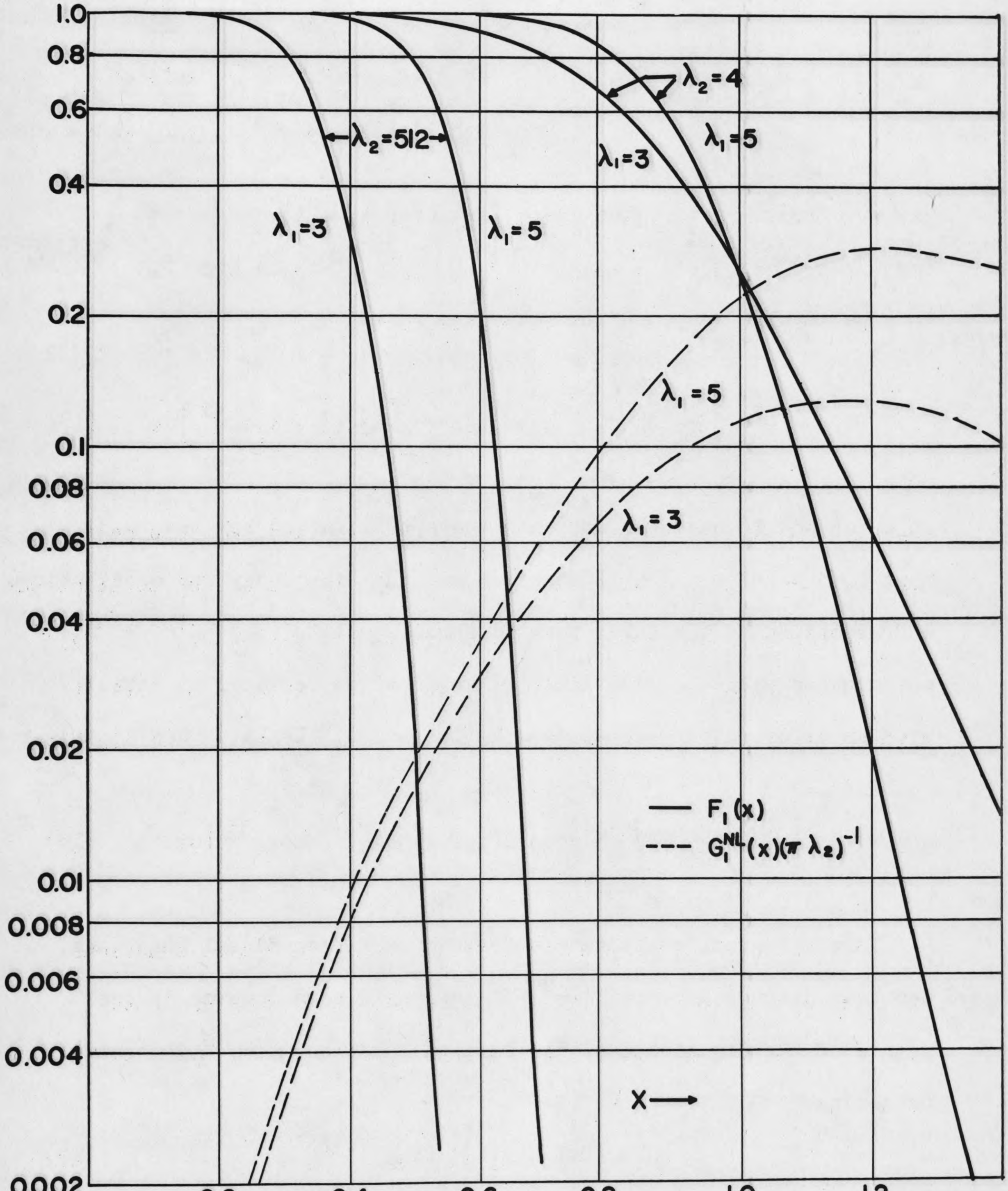

FIG. 6 AMPLITUDE AND PHASE CHARACTERISTICS OF 3 & 5 POLE BUTTERWORTH FILTERS.

λ_1	$\overline{\theta}_1$	$\overline{\theta}_2$	g_1	g_3
3	60°	--	2.00000	0.33333
5	36°	72°	3.23607	0.41202

The non-linear part of the phase lag, (for $\lambda_2 = 1$), $(\Delta G_1^{NL}/\pi\lambda_2) = \left[G_1(x) - \lambda_2\, g_1\, x\right]/\pi\lambda_2$ is plotted in Fig. 6.

The impulse response now takes the form

$$w^{o}(T) \sim 2\int_{o}^{x_1} \cos\left[x(2\pi T - \lambda_2 g_1) - \lambda_2 g_3 x^3\right]dx \qquad (18)$$

in which Eq. 16 has been used and the approximation has been made that $F(x) \sim 1$ for $x \leqslant x_1$. When $x \approx x_1$, the "frequency" of oscillations corresponding to the cubic term is about equal to $(3g_3\,\lambda_2^{1 - 1/\lambda_1})$, which grows with λ_2. The limit x_1 may then be replaced by infinity with an error which decreases as λ_2 increases. We have then finally

$$w^{o}(T) \sim 2\int_{o}^{\infty} \cos\left[x(2\pi T - \lambda_2 g_1) - \lambda_2 g_3 x^3\right]dx \qquad (19)$$

This integral, evaluated in terms of modified Bessel functions, has been discussed by Di Toro[5]. All we need here however is the nature of the dependence of the related function $u^{o}(T)$ upon λ_2. Let us define

$$\xi = (2\pi T - g_1\lambda_2)\,(\lambda_2 g_3)^{-1/3} \qquad (20)$$

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[5] Di Toro, Proc. I.R.E. 36, 24 (1948).

Then

$$w^{o}(T) \sim 2 \int_{o}^{\infty} \cos\left[2\pi Z\xi - Z^{3}\right] dZ \; (\lambda_2 g_3)^{-1/3} \qquad (21)$$

$$u^{o}(T) \sim (1/\pi) \int_{o}^{\xi}\int_{o}^{\infty} \cos\left[2\pi Z\xi - Z^{3}\right] dZ \qquad (22)$$

The step response is thus a function of ξ alone, for large λ_2. Insofar as this is an accurate result, we should expect the following behavior of the step response (with no phase correction):

i) $(2\pi\, T_{0.1} - g_1 \lambda_2)$, $\tilde{T}$, and T_d should each be proportional to $\lambda_2^{1/3}$ for large λ_2. This agrees more or less well with the empirical formulae given in earlier sections, accordingly as the region of asymptotic behavior is reached sooner or later for differing values of λ_2.

ii) The overshoot Δu^{o} should be asymptotically independent of λ_2 and λ_1. This seems to be the case rather closely.

iii) The damping of the ringing should also be independent of λ_1. This does not seem to agree with the Illiac results, possibly because this characteristic of the step response is more sensitive than the other characteristics to the nature of the small departures of the amplitude function from unity within the pass band.

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The authors would like to acknowledge the assistance of Sanford Stein, Jack Ullman, and Jerry McCall in developing the many sub-routines that were part of the Illiac program used for these computations.

Table I

Coordinates of Six Points on the Step Response Curve

$G = G_1$ (no phase correction)

λ_1	λ_2		1	2	3	4	5	6
3	4	T	1.01	1.59	1.9288	2.5778	3.1994	3.8048
		u	0.100	0.900	1.143		1.011	0.997
3	8	T	2.22	2.94	3.3467	4.0801	4.7560	5.4023
		u	0.100	0.900	1.172		1.021	0.993
3	32	T	9.68	10.80	11.4117	12.4259	13.3094	14.1257
		u	0.100	0.900	1.220		1.054	0.974
3	128	T	39.95	41.71	42.6552	44.1554	45.4224	46.5685
		u	0.100	0.900	1.248		1.095	0.936
3	256	T						
		u						
3	512	T	161.72	164.50	165.9799	168.2781	170.1882	171.8961
		u	0.100	0.900	1.264		1.131	0.899

Table I (cont'd.)

Coordinates of Six Points on the Step Response Curve

$G = G_1$ (no phase correction)

λ_1	λ_2		1	2	3	4	5	6
5	4	T	1.780	2.403	2.7596	3.3918	3.9779	4.5449
		u	0.100	0.900	1.196		1.049	0.975
5	8	T	3.773	4.544	4.9748	5.7032	6.3561	6.9746
		u	0.100	0.900	1.221		1.0738	0.956
5	32	T	15.94	17.139	17.7881	18.8235	19.7087	20.5194
		u	0.100	0.900	1.253		1.120	0.908
5	128	T	65.078	66.957	67.9630	69.5241	70.8256	71.9938
		u	0.100	0.900	1.266		1.143	0.880
5	256	T			134.4056	136.3467	137.9541	139.3888
		u			1.269		1.148	0.874
5	512	T	262.36	265.329	266.9098	269.3350	271.3348	273.1131
		u	0.100	0.900	1.271		1.151	0.871

Table I (cont'd.)

Coordinates of Six Points on the Step Response Curve

$G = G_o$ (complete phase correction)

λ_1	λ_2		1	2*	3	4	5	6
3	4	T	-0.2614	0.2614	0.5778	1.1723	1.7722	2.3606
		u	0.1000	0.9000	1.0637		1.0028	0.9994
3	8	T	-0.3004	0.3004	0.6615	1.3188	1.9605	2.5786
		u	0.1000	0.9000	1.0674		1.0041	0.9991
3	32	T	-0.3849	0.3849	0.8470	1.6739	2.4607	3.2029
		u	0.1000	0.9000	1.0698		1.0056	0.9984
3	128	T	-0.4869	0.4869	1.0716	2.1145	3.1022	4.0298
		u	0.1000	0.9000	1.0704		1.0060	0.9982
3	256	T						
		u						
3	512	T	-0.6140	0.6140	1.3515	2.6660	3.9096	5.0764
		u	0.1000	0.9000	1.0705		1.0061	0.9981

* T = 0 at u = 0.5000 for $G = G_o$

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Table I (concl'd.)

Coordinates of Six Points on the Step Response Curve

$G = G_o$ (complete phase correction)

λ_1	λ_2		1	2*	3	4	5	6
5	4	T	-0.2456	0.2456	0.5466	1.0932	1.6384	2.1803
		u	0.1000	0.9000	1.0790		1.0130	0.9946
5	8	T	-0.2670	0.2670	0.5943	1.1844	1.7667	2.3393
		u	0.1000	0.9000	1.0807		1.0152	0.9931
5	32	T	-0.3098	0.3098	0.6899	1.3730	2.0437	2.6996
		u	0.1000	0.9000	1.0817		1.0168	0.9917
5	128	T	-0.3568	0.3568	0.7945	1.5808	2.3524	3.1061
		u	0.1000	0.9000	1.0819		1.0172	0.9914
5	256	T			0.8519	1.6949	2.5221	3.3300
		u			1.0820		1.0173	0.9913
5	512	T	-0.4101	0.4101	0.9132	1.8169	2.7036	3.5696
		u	0.1000	0.9000	1.0820		1.0174	0.9913

* $T = 0$ at $u = 0.5000$ for $G = G_o$

Table II

Characterisitcs of the Response to Unit Step

$G = G_1$ (no phase correction)

λ_1	λ_2	$T_{0.1}$	$\tilde{T}$	T_D	Δu	σ_D
3	4	1.01	0.58	1.21	14.3	2.8
	8	2.22	0.72	1.29	17.2	2.3
	32	9.68	1.12	1.63	22.0	1.5
	128	39.95	1.76	2.29	24.8	0.8
	256					
	512	161.72	2.78	3.42	26.4	0.5
5	4	1.779	0.623	1.134	19.6	1.3
	8	3.772	0.771	1.237	22.1	1.0
	32	15.94	1.198	1.621	24.8	0.54
	128	65.078	1.879	2.336	26.6	0.35
	256			2.869	26.9	0.33
	512	262.36	2.97	3.556	27.1	0.31

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Table II (concl'd.)

<u>Characteristics of the Response to Unit Step</u>

$G = G_0$ (complete phase correction)

λ_1	λ_2	$T_{0.1}$	$\tilde{T}$	T_D	Δu	σ_D
3	4		0.5229	1.1768	6.37	3.3
	8		0.6008	1.2363	6.74	2.9
	32		0.7697	1.4844	6.98	2.5
	128		0.9737	1.8552	7.04	2.4
	256					
	512		1.2281	2.3337	7.06	2.4
5	4		0.4913	1.0838	7.90	1.77
	8		0.5340	1.1453	8.07	1.58
	32		0.6197	1.3116	8.17	1.42
	128		0.7136	1.5075	8.20	1.38
	256			1.6159	8.20	1.37
	512		0.8202	1.7380	8.20	1.37

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