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MULTIMODELING, SINGULAR PERTURBATIONS AND STOCHASTIC DECISION PROBLEMS

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In this chapter we analyze the interaction between model simplification and strategy design in a multimodel context and for multiple agent stochastic decision problems with decentralized information. Under quasi-classical information patterns, and using singular perturbations approach, we establish asymptotic optimality of different multimodels which involve continuous and two types of sampled measurements. Our general analysis and discussion serve to enhance our understanding of the innerrelationships between structural features of stochastic large scale systems, like time-scales and weak coupling, and strategy design.

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MULTIMODELING, SINGULAR PERTURBATIONS AND STOCHASTIC DECISION PROBLEMS*

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SUMMARY

In this chapter we analyze the interaction between model simplification and strategy design in a multimodel context and for multiple agent stochastic decision problems with decentralized information. Under quasi-classical information patterns, and using singular perturbations approach, we establish asymptotic optimality of different multimodels which involve continuous and two types of sampled measurements. Our general analysis and discussion serve to enhance our understanding of the interrelationships between structural features of stochastic large scale systems, like time-scales and weak coupling, and strategy design.

TABLE OF CONTENTS

1. INTRODUCTION

The problem of efficient management and control of large scale systems has been extremely challenging to control engineers. There are essentially two main issues of concern: the modeling issue is complicated due to the large dimension of the system, and the control design issue is complicated due to the presence of multiple decision makers having possibly different goals and possessing decentralized information. Efforts to understand the inherent complexities have led to the concept of nonclassical information patterns [1]. This concept expresses a basic fact that a decision maker has neither complete nor instantaneous access to other decision makers' measurements and decisions. A related but perhaps more basic fact is expressed by the multimodeling concept [2]. This concept accounts for the many realistic situations when different decision makers have different information about the system structure and dynamics and therefore use different simplified models of the same large scale system. These models may differ in parameter values, signal uncertainties, and, more critically, in their basic structural properties.

A strong motivation for the multimodeling approach is found in multi-area power systems. The decision maker in one area uses a detailed model of his area only and some lower order "equivalent" of the rest of the system. The decision makers in other areas behave in a similar way and as a result each has his own view of the same large scale system. The main advantage of such an empirical decomposition is that it leads to distributed computations and less communication between the controllers because each decision maker would only require measurements of the variables appearing in his own reduced order model. Many crucial problems (instability, suboptimality, etc.) arise because the strategies designed with such inconsistent models are then applied to the actual system.

We investigate, in this chapter, the effect of multimodeling inconsistencies on the design and implementation of multicontroller strategies under certain quasi-classical information patterns. The approach taken is perturbational. If the model inconsistencies are small, it is natural to expect that their effect on the designed strategies and on the actual system performance would be in some sense small. If this were not the case, the designed strategies would not be applicable to realistic systems whose models are never exactly known. We consider this low sensitivity property a *sine qua non* condition for any control design and, in particular, for the design of large scale systems controlled from multiple control stations.

Another fundamental property of our perturbational approach is that it concentrates on modeling errors caused by reducing the model order. Such order reductions are achieved by separating the time scales, that is, by considering slow and fast phenomena separately. A typical situation is when the decision maker in one area neglects the fast phenomena in all other areas. In geographically dispersed systems this practice is based on the experimental observation that faster phenomena propagate to shorter distances than the slower phenomena. For example, in a multimachine transient the slower oscillatory modes are observed throughout the system, while faster inter-machine oscillations are of a more local character [3].

A tool for analyzing the change in model order is the so-called singular perturbation method which converts the change of model order into a small parameter perturbation [4]. This parameter multiplies the derivatives of the fast state variables and when it is set to zero the fast phenomena are neglected. The fast phenomena are treated separately in the fast time scale where the slow variables are "frozen" at their quasi-steady state values. This two-time-scale approach is asymptotic, that is, exact in the limit as the ratio of speeds of the slow versus the fast dynamics tends to zero. When this ratio is small, approximations are obtained from reduced order models in separate time scales. This way the singular perturbation approach alleviates difficulties due to high dimensionality and ill-conditioning resulting from the interaction of slow and fast dynamic modes.

The chapter is organized as follows: In Section 2, we study the fundamental problem of modeling and control of singularly perturbed systems driven by Wiener processes under various cases of continuous and sampled observations. An extension of the single parameter model, which realistically captures the multimodeling situation, is formulated in Section 3 using multiparameter singular perturbations. In Section 4, we obtain multimodel solutions to Nash and team problems under certain quasi-classical information patterns, and establish their relationship with the solutions of the full problem. We summarize the results with some concluding remarks in Section 5.

To highlight the ideas, we have adopted an informal style for the presentation and discussion of the main results. More rigorous treatment can be found in quoted references.

2. MODELING AND CONTROL OF STOCHASTIC SINGULARLY PERTURBED SYSTEMS

2.1. Well-Posedness of Different Models

The optimal control of stochastic singularly perturbed systems with white noise inputs leads to difficulties not present in deterministic problems. This is due to the idealized behavior of white noise which "fluctuates" faster than the fast dynamic variables. To illustrate the problem of optimally controlling a stochastic fast dynamic system, consider the following standard LQG formulation

system dynamics: $$\varepsilon dz = Az\,dt + Bu\,dt + Gdw \tag{2.1a}$$

measurement process: $$dy = Cz\,dt + dv \tag{2.1b}$$

cost function: $$J = E\{z'\Gamma z + \int_0^T (z'Qz + u'u)dt\}. \tag{2.2}$$

Here, $\varepsilon > 0$ is the small singular perturbation parameter; $w(t)$ and $v(t)$ are standard Wiener processes independent of each other, and all matrices are time-invariant, with $\Gamma \geq 0$, $Q \geq 0$. We will further assume that A is a stable matrix, that is, $\text{Re}\lambda(A) < 0$.

The optimal control u^* which minimizes the cost J is obtained in the usual manner by applying the separation principle, so that

$$u^* = -B'K\hat{z} \tag{2.3}$$

where K satisfies the Riccati equation

$$\varepsilon\dot{K} = -A'K - KA - Q + KBB'K; \qquad K(T) = \frac{1}{\varepsilon}\Gamma. \tag{2.4}$$

The vector $\hat{z}(t)$ denotes the optimal estimate of $z(t)$ given the past observations, which for any given $u(t)$ is the output of the Kalman filter

$$\varepsilon d\hat{z} = A\hat{z}\,dt + Bu\,dt + PC'(dy - C\hat{z}\,dt); \qquad \hat{z}(0) = E[z(0)] \tag{2.5}$$

where $\frac{1}{\varepsilon}P(t)$ is the error covariance of $\hat{z}(t)$, satisfying

$$\varepsilon\dot{P} = AP + PA' + GG' - PC'CP; \qquad P(0) = \varepsilon\,\mathrm{Cov}(z(0)), \tag{2.6}$$

which does not depend on u(t). The resulting minimum value of the cost, J^*, is given by

$$J^* = \varepsilon\hat{z}'(0)K(0)\hat{z}(0) + \frac{1}{\varepsilon}\,\mathrm{tr}[P(T)\Gamma] + \frac{1}{\varepsilon}\int_0^T \mathrm{tr}[CPKPC' + PQ]dt. \tag{2.7}$$

Notice from (2.6) and (2.7) that $\mathrm{Cov}(z-\hat{z}) = 0(\frac{1}{\varepsilon})$ and $J^* = 0(\frac{1}{\varepsilon})$. Hence as $\varepsilon \to 0$, both the covariance of the estimation error and optimal cost diverge, even though the feedback gain of the optimal control law given by (2.3) remains finite (outside the end-point boundary-layer). This is because, in the limit as $\varepsilon \to 0$, the fast variables z themselves tend to white noise processes, thus losing their significance as physically meaningful dynamic variables. Hence the problem formulation given by (2.1) and (2.2) is ill-posed. More detailed analysis of this formulation in the filtering and control context may be found in [5,6].

One way to circumvent the difficulty encountered above is to appropriately "scale" the white noise terms in the model. Let us now investigate ramifications of the following more general formulation:

The state dynamics description is replaced by

$$\varepsilon dz = Az\,dt + Bu\,dt + \varepsilon^{\alpha}G dw; \qquad \mathrm{Re}\lambda(A) < 0 \tag{2.8a}$$

and the measurement process is

$$dy = Cz\,dt + \varepsilon^{\beta}dv \tag{2.8b}$$

where α, β are some positive constants to be chosen. The cost function J is the same as before.

Now the optimal control is given by

$$u^* = -B'K\hat{z} \tag{2.9}$$

where K(t) satisfies (2.4).

The optimal estimate z(t) is obtained from the Kalman filter

$$\varepsilon d\hat{z} = A\hat{z}\,dt + Bu\,dt + M(t)(dy - C\hat{z}\,dt); \qquad \hat{z}(0) = E[z(0)] \tag{2.10}$$

where M(t) is the filter gain given as

$$M(t) = \varepsilon^{1-2\beta}PC' \tag{2.11}$$

and P(t) is the error covariance of $\hat{z}(t)$, satisfying

$$\varepsilon\dot{P} = AP + PA' + \varepsilon^{2\alpha-1}GG' - \varepsilon^{1-2\beta}PC'CP; \qquad P(0) = \mathrm{Cov}(z(0)). \tag{2.12}$$

The minimum value of the cost, J^*, is given by

$$J^* = \varepsilon\hat{z}'(0)K(0)\hat{z}(0) + \mathrm{tr}(P(T)\Gamma) + \varepsilon^{1-2\beta}\int_0^T \mathrm{tr}(CPKPC')\,dt + \int_0^T \mathrm{tr}(PQ)\,dt. \tag{2.13}$$

Let us now examine the behavior of P(t), M(t), and J^* for various values of α and β, in the limit as $\varepsilon \to 0$. The limiting behavior of P(t) and J^* is governed primarily by the parameter α, while the limiting behavior of M(t) is governed by both parameters α and β. Notice that the behavior of K(t) is unaffected by the scaling.

A straightforward examination of (2.12) reveals that for $\alpha < \frac{1}{2}$, P(t) diverges as $\varepsilon \to 0$, which implies from (2.13) that J^* also diverges as $\varepsilon \to 0$. [Note that $\beta > 0$ by hypothesis.] When P(t) diverges, the filter gain M(t) may or may not diverge as $\varepsilon \to 0$, depending on the value of β. If $\beta > \frac{1}{2}$, however, in addition to $0 < \alpha < \frac{1}{2}$, M(t) always diverges as $\varepsilon \to 0$. This particular case ($\alpha < \frac{1}{2}$, $\beta > \frac{1}{2}$) corresponds to the situation where the observations become noise-free in the limit as $\varepsilon \to 0$, and therefore the filter gain becomes unbounded.

When $\alpha > \frac{1}{2}$ and β is any positive constant, it readily follows from (2.12) and (2.13) that P(t) and J^* go to zero as $\varepsilon \to 0$. If at the same time $\beta < \frac{1}{2}$, then M(t) also goes to zero as $\varepsilon \to 0$. This case ($\alpha > \frac{1}{2}$, $\beta < \frac{1}{2}$) corresponds to the situation when the observations become too noisy in the limit as $\varepsilon \to 0$, thus driving the filter gain to zero.

Hence the range of scaling ($\alpha, \beta > 0$; $\alpha \neq \frac{1}{2}$, $\beta \neq \frac{1}{2}$) leads to ill-posed formulations. This implies that it is not possible to give a physically meaningful interpretation to the limiting solution. [Of course for any fixed $\varepsilon > 0$, the problem is well-defined.] The only meaningful formulation is obtained when $\alpha = \beta = \frac{1}{2}$. In this case P(t), M(t), and J^* remain bounded and nonzero and yield a well-defined stochastic control problem in the limit as $\varepsilon \to 0$.

The above analysis has indicated that in order to obtain a well-defined stochastic control problem, the process and observation noise need to be scaled in an appropriate manner. To gain further insight, let us directly examine the limiting behavior of the stochastic process

$$\varepsilon dz = Az\,dt + \sqrt{\varepsilon}\,G dw; \qquad \mathrm{Re}\lambda(A) < 0, \quad GG' > 0. \tag{2.14}$$

Clearly, without the scaling term, z(t) converges to white noise in the limit as $\varepsilon \to 0$. If, with the above scaling, z(t) converges to something which is physically meaningful, then this would provide a strong justification for the model (2.8), with $\alpha = \frac{1}{2}$.

Solving for z(t) from (2.14) we obtain

$$z(t) = \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{A(t-\tau)/\varepsilon} G dw(\tau) \tag{2.15}$$

where we have assumed, without loss of generality, that $z(0) = 0$. Now

$$\mathrm{Cov}(z(t)) = E\{[\frac{1}{\sqrt{\varepsilon}}\int_0^t e^{A(t-\tau_1)/\varepsilon} G\,dw(\tau_1)][\frac{1}{\sqrt{\varepsilon}}\int_0^t e^{A(t-\tau_2)/\varepsilon} G\,dw(\tau_2)]'\}$$

$$= \frac{1}{\varepsilon}\int_0^t e^{A(t-\tau)/\varepsilon} GG' e^{A'(t-\tau)/\varepsilon} d\tau$$

$$\triangleq W_\varepsilon(t) \tag{2.16}$$

where $W_\varepsilon(t)$ satisfies, for each $\varepsilon > 0$, the linear matrix differential equation

$$\varepsilon \dot{W}_\varepsilon = AW_\varepsilon + W_\varepsilon A' + GG'.$$

Since $\mathrm{Re}\lambda(A) < 0$, we clearly have the limit (excluding boundary layers)

$$\lim_{\varepsilon \to 0} \mathrm{Cov}(z(t)) = W \tag{2.17}$$

where W is the positive definite (because $GG' > 0$) solution of the Lyapunov equation

$$AW + WA' + GG' = 0. \tag{2.18}$$

This implies that $z(t)$ converges in distribution to a zero mean constant Gaussian random vector whose covariance W satisfies (2.18) [see also [7,8]]. The above convergence is indeed physically meaningful, and therefore we are justified in using (2.14) to model a fast stochastic dynamic system.

Physically, the above analysis has indicated that in order to meaningfully estimate and control a fast dynamic system, the influence of the random disturbances has to be "limited" in some sense.

2.2. Singularly Perturbed Systems with Continuous Measurements

Let us now consider the full (with both slow and fast variables) stochastic singularly perturbed optimal control problem

$$dx = (A_{11}x + A_{12}z + B_1u)dt + G_1dw \tag{2.19a}$$

$$\varepsilon dz = (\varepsilon^{\beta}A_{21}x + A_{22}z + B_2u)dt + \varepsilon^{\alpha}G_2dw \tag{2.19b}$$

$$dy_1 = (C_{11}x + C_{12}z)dt + dv_1 \tag{2.20a}$$

$$dy_2 = (\varepsilon^{\nu}C_{21}x + C_{22}z)dt + \varepsilon^{\nu}dv_2 \tag{2.20b}$$

$$J = E\{x'(T)\Gamma_1x(T) + 2\varepsilon x'(T)\Gamma_{12}z(T) + \varepsilon z'(T)\Gamma_2z(T)$$
$$+ \int_0^T (x'L_1'L_1x + 2\varepsilon^{\delta}x'L_1'L_2z + \varepsilon^{2\delta}z'L_2'L_2z + u'u)dt\}. \tag{2.21}$$

The parameters α, β, ν, δ represent the relative size of the small parameters within the system, with respect to the small time constants of the fast subsystem. The inclusion of a separate observation channel y_2 for the fast subsystem is essential, since otherwise for $\alpha > 0$ the fast variables cannot be estimated meaningfully from the slow observation channel (signal-to-noise ratio tends to zero). The stochastic processes $w(t)$, $v_1(t)$ and $v_2(t)$ are standard Wiener processes independent of each other and the Gaussian random vector $[x(0), z(0)]$. We also assume that $\mathrm{Re}\lambda(A_{22}) < 0$. The optimal solution to the problem posed by (2.19)-(2.21) can be obtained by invoking the separation principle:

$$u^* = -[(B_1'K_1 + B_2'K_{12}')\hat{x} + (B_2'K_2 + \varepsilon B_1'K_{12})\hat{z}] \tag{2.22}$$

$$d\hat{x} = (A_{11}\hat{x} + A_{12}\hat{z} + B_1u^*)dt + [P_1C_1' + \varepsilon^{\alpha-\nu}P_{12}C_2']d\sigma \tag{2.23a}$$

$$\varepsilon d\hat{z} = (\varepsilon^{\beta}A_{21}\hat{x} + A_{22}\hat{z} + B_2u^*)dt + \varepsilon^{\alpha}[\varepsilon P_{12}'C_1' + \varepsilon^{\alpha-\nu}P_2C_2']d\sigma \tag{2.23b}$$

where the innovations process $\sigma(t)$ is defined by

$$d\sigma(t) \triangleq \begin{bmatrix} dy_1 \\ \varepsilon^{-\nu} dy_2 \end{bmatrix} - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & \varepsilon^{-\nu} C_{22} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} dt$$

$$\triangleq \begin{bmatrix} dy_1 \\ \varepsilon^{-\nu} dy_2 \end{bmatrix} - [C_1 \quad \varepsilon^{-\nu} C_2] \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} dt. \tag{2.24}$$

The control gain matrices satisfy

$$-\dot{K}_1 = K_1A_{11} + \varepsilon^{\beta} K_{12}A_{21} + A_{11}'K_1 + \varepsilon^{\beta} A_{21}'K_{12}' + L_1'L_1 - (K_{12}B_2 + K_1B_1)(B_1'K_1 + B_2'K_{12}');$$
$$K_1(T) = \Gamma_1 \tag{2.25a}$$

$$-\varepsilon\dot{K}_{12} = K_1A_{12} + K_{12}A_{22} + \varepsilon A_{11}'K_{12} + \varepsilon^{\beta} A_{21}'K_2 + \varepsilon^{\delta} L_1'L_2 - (K_{12}B_2 + K_1B_1)(B_2'K_2 + \varepsilon B_1'K_{12});$$
$$K_{12}(T) = \Gamma_{12} \tag{2.25b}$$

$$-\varepsilon\dot{K}_2 = K_2A_{22} + A_{22}'K_2 + \varepsilon K_{12}'A_{12} + \varepsilon A_{12}'K_{12} + \varepsilon^{2\delta} L_2'L_2 - (K_2B_2 + \varepsilon K_{12}'B_1)(B_2'K_2 + \varepsilon B_1'K_{12});$$
$$K_2(T) = \Gamma_2. \tag{2.25c}$$

The filter covariances satisfy

$$\dot{P}_1 = A_{11}P_1 + P_1A_{11}' + \varepsilon^{\alpha} A_{12}P_{12}' + \varepsilon^{\alpha} P_{12}A_{12}' + G_1G_1' - (P_1C_1' + \varepsilon^{\alpha-\nu} P_{12}C_2')(C_1P_1 + \varepsilon^{\alpha-\nu} C_2P_{12}')$$
$$P_1(0) = \mathrm{Cov}(x(0)) \tag{2.26a}$$

$$\varepsilon\dot{P}_{12} = \varepsilon A_{11}P_{12} + \varepsilon^{\alpha} A_{12}P_2 + P_{12}A_2' + \varepsilon^{\beta-\alpha} P_1A_{21}' + G_1G_2' - (P_1C_1' + \varepsilon^{\alpha-\nu} P_{12}C_2')(\varepsilon C_1P_{12}$$
$$+ \varepsilon^{\alpha-\nu} C_2P_2); \quad P_{12}(0) = \varepsilon^{\alpha} \mathrm{Cov}(x(0), z(0)) \tag{2.26b}$$

$$\varepsilon\dot{P}_2 = A_{22}P_2 + P_2A_{22}' + \varepsilon^{1-\alpha+\beta} A_{21}P_{12} + \varepsilon^{1-\alpha+\beta} P_{12}'A_{21}' + G_2G_2' - (\varepsilon P_{12}'C_1' + \varepsilon^{\alpha-\nu} P_2C_2')(\varepsilon C_1P_{12}$$
$$+ \varepsilon^{\alpha-\nu} C_2P_2); \quad P_2(0) = \varepsilon^{1-2\alpha} \mathrm{Cov}(z(0)). \tag{2.26c}$$

The performance index will be finite if $\varepsilon^{2\delta}\mathrm{Cov}(z)$ is finite. But

$$\mathrm{Cov}(z) = \varepsilon^{2\alpha-1} P_2. \qquad (2.27)$$

Hence, we require that

$$\delta \geq (\frac{1}{2} - \alpha) \qquad (2.28)$$

in order to have a finite cost. Furthermore, a well-defined formulation also requires that

$$0 \leq \alpha = \nu \leq \beta \leq \tfrac{1}{2}. \qquad (2.29)$$

The restriction $\alpha = \nu$ is crucial, otherwise, either the fast variables are not observed due to very noisy observations ($\alpha > \nu$), or they are observed noiselessly ($\alpha < \nu$) in the limit as $\varepsilon \to 0$. If $\alpha > \frac{1}{2}$, the problem becomes deterministic as $\varepsilon \to 0$, and if $\beta > \frac{1}{2}$, the coupling between x and z becomes negligible. The constraint $\beta \geq \alpha$ insures that the state z is predominantly fast, and relaxing it causes no conceptual difficulties.

Note that when $\alpha = \beta = \nu = 0$, it is required that $\delta = \frac{1}{2}$ to yield a finite cost. In this case the fast variables are of no interest as far as the control is concerned, and serve only as a model for a wide-band disturbance to the slow variables. The important case is when $\alpha = \nu = \frac{1}{2}$ and $\delta=\beta=0$, since this results in a full weighting of the fast variable. For this problem, it can be shown that [9],

$$\lim_{\varepsilon \to 0} u^* = u_s + u_f; \qquad 0 < t < T \qquad (2.30)$$

where

$$u_s = -R_o^{-1}(N_o'L_o + B_o'K_o)\hat{x}_s \qquad (2.31)$$

$$d\hat{x}_s = (A_o\hat{x}_s + B_o u_s)dt + (P_o C_o' + G_o D_o')V_o^{-1}[dy - C_o\hat{x}_s dt + \frac{1}{\sqrt{\varepsilon}} C_2 A_{22}^{-1} B_2 u_s dt] \qquad (2.32)$$

$$-\dot{K}_o = K_o(A_o - B_oR_o^{-1}N_o'L_o) + (A_o - B_oR_o^{-1}N_o'L_o)'K_o + L_o'(I - N_oR_o^{-1}N_o')L_o$$

$$-K_oB_oR_o^{-1}B_o'K_o; \qquad K_o(T) = \Gamma \tag{2.33}$$

$$\dot{P}_o = A_oP_o + P_oA_o' - (P_oC_o' + G_oD_o')V_o^{-1}(C_oP_o + D_oG_o') + G_oG_o'; \qquad P_o(0) = \mathrm{Cov}(x(0)) \tag{2.34}$$

$$A_o \triangleq A_{11}, \quad B_o \triangleq B_1 - A_{12}A_{22}^{-1}B_2, \quad N_o \triangleq -L_2A_{22}^{-1}B_2, \quad L_o \triangleq L_1, \quad R_o \triangleq N_o'N_o$$

$$C_o \triangleq C_1 - C_2A_{22}^{-1}A_{21}, \quad D_o \triangleq -C_2A_{22}^{-1}G_2, \quad G_o \triangleq G_1, \quad V_o \triangleq I + D_oD_o' \tag{2.35}$$

$$u_f = -B_2'\bar{K}_2\hat{z}_f \tag{2.36}$$

$$\varepsilon d\hat{z}_f = (A_{22}\hat{z}_f + B_2u_f)dt + \bar{P}_2C_{22}'\{dy_2 - C_{22}\hat{z}_f dt - \sqrt{\varepsilon}[C_{21} - C_{22}A_{22}^{-1}A_{21}]\hat{x}_s dt$$

$$+ C_{22}A_{22}^{-1}B_2u_s dt\} \tag{2.37}$$

$$\bar{K}_2A_{22} + A_{22}'\bar{K}_2 + L_2'L_2 - \bar{K}_2B_2B_2'\bar{K}_2 = 0 \tag{2.38}$$

$$A_{22}\bar{P}_2 + \bar{P}_2A_{22}' + G_2G_2' - \bar{P}_2C_2'C_2\bar{P}_2 = 0. \tag{2.39}$$

Notice that u_s and u_f are obtained on solving a reduced-order slow control problem and an infinite-time fast control problem, respectively. These problems can be solved independently of each other. It is interesting to note that the fast filter is driven by the slow variables as well. Hence the implementation of the filters is not independent, but sequential in nature. The near-optimality result (2.30) is valid only for $t \in (0,T)$, because the boundary-layer terms have been neglected.

2.3. Singularly Perturbed Systems with Sampled Measurements

So far we have examined the modeling and control aspects of stochastic singularly perturbed systems when the measurement process is continuous in time. We shall now examine the same aspects when the measurement process consists of discrete samples. Two types of sampled observations will be considered. In the first case, sampled values of the state in additive noise are observed, and in the second case sampled values of a continuous-time measurement process are observed. These types of observations play an important role in multi-agent decision problems as we shall see later.

It is a well-known fact that the open-loop dynamics of any system of the form (2.19) with $\alpha=\frac{1}{2}$, $\beta=0$, can be transformed into a block-diagonal form where the pure slow and fast variables are explicitly displayed [10]. Hence, without loss of generality, we shall assume that the system to be controlled is given by

$$dx = (A_1 x + B_1 u)dt + G_1 dw \tag{2.40a}$$

$$\varepsilon dz = (A_2 z + B_2 u)dt + \sqrt{\varepsilon}\, G_2 dw; \quad \mathrm{Re}\lambda(A_2) < 0. \tag{2.40b}$$

The performance index will be given by†

$$J = E\{x(T)\Gamma_1 x(T) + \varepsilon z'(T)\Gamma_2 z(T) + \int_0^T (x'Q_1 x + z'Q_2 z + u'u)dt\}. \tag{2.41}$$

We now consider two cases of sampled observations

2.3.1. Case 1: Noisy measurements of sampled values of state

The observations consist of sampled noisy measurements of the state. Specifically, the observations

$$y(j) = C_1 x(t_j) + C_2 z(t_j) + v(j) \tag{2.42}$$

are available at sampled time instant t_j where $j = 0,1,\ldots,N-1$ and

†A more general formulation would include cross terms involving slow and fast variables. Here we are avoiding this in order not to obscure the essentials of the following analysis by notational complexity. We should note, though, that such a restriction leads to no conceptual loss of generality.

$0 = t_o < t_1 < \cdots < t_{N-1} = T$. Let $\theta = \{0,1,\ldots,N-1\}$. Then the random vectors $\{v(j), j \in \theta\}$ are assumed to have independent Gaussian statistics with $v(j) \sim N(0,R_j)$, $R_j > 0$, $j \in \theta$. Their statistics are also assumed to be independent of the Wiener process $w(t)$ and the Gaussian vector $[x(0),z(0)]$.

A near-optimal solution to the problem defined by (2.40)-(2.42) can be shown to be given by

$$u_o = u_s + u_f \tag{2.43}$$

where

$$u_s(t) = -B_1'K_1\Psi_1(t,t_j)\hat{x}_s(t_j); \qquad t \in [t_j,t_{j+1}), \quad j \in \theta \tag{2.44}$$

$$-\dot{K}_1 = A_1'K_1 + K_1A_1 + Q_1 - K_1B_1B_1'K_1; \qquad K_1(T) = \Gamma, \tag{2.45}$$

$$\dot{\Psi}_1(t,t_j) = (A_1 - B_1B_1'K_1)\Psi_1(t,t_j); \qquad \Psi_1(t_j,t_j) = I$$

$$t \in [t_j,t_{j+1}), \quad j \in \theta \tag{2.46}$$

$$\left.\begin{aligned} &\dot{\hat{x}}_s(t) = A_1\hat{x}_s(t) + B_1u_s(t); \qquad \hat{x}_s(0) = E[x(0)] \\ &\qquad\qquad t \in [t_{j-1},t_j); \quad j = 1,2,\ldots,N \\ &\hat{x}_s(t_j) = \hat{x}_s(t_j^-) + S_1(j)[y(j) - C_1\hat{x}_s(t_j^-) + C_2A_2^{-1}B_2u_s(t_j^-)] \end{aligned}\right\} \tag{2.47}$$

$$\left.\begin{aligned} &\dot{\Sigma}_s = A_1\Sigma_s + \Sigma_sA_1' + G_1G_1'; \qquad \Sigma_s(0) = \mathrm{Cov}[x(0)] \\ &\qquad\qquad t \in [t_{j-1},t_j); \quad j = 1,2,\ldots,N \\ &\Sigma_s(t_j) = \Sigma_s(t_j^-) - S_1(j)C_1\Sigma_s(t_j^-) \end{aligned}\right\} \tag{2.48}$$

$$S_1(j) = \Sigma_s(t_j^-)C_1'[C_1\Sigma_s(t_j^-)C_1' + C_2\Sigma_fC_2' + R_j]^{-1} \tag{2.49}$$

$$A_2\Sigma_f + \Sigma_fA_2' + G_2G_2' = 0 \tag{2.50}$$

$$u_f(t) = -B_2'K_2\Psi_2(t,t_j)\hat{z}_f(t_j); \quad t\in[t_j,t_{j+1}), \quad j\in\theta \tag{2.51}$$

$$A_2'K_2 + K_2A_2 + Q_2 - K_2B_2B_2'K_2 = 0 \tag{2.52}$$

$$\varepsilon\dot{\Psi}_2(t,t_j) = (A_2-B_2B_2'K_2)\Psi_2(t,t_j); \quad \Psi_2(t_j,t_j) = I$$
$$t\in[t_j,t_{j+1}), \quad j\in\theta \tag{2.53}$$

$$\left.\begin{aligned} &\varepsilon\dot{\hat{z}}_f(t) = A_2\hat{z}_f(t) + B_2u_f(t); \quad \hat{z}_f(0) = E[z(0)] \\ &\qquad\qquad t\in[t_{j-1},t_j); \quad j=1,2,\ldots,N \\ &\hat{z}_f(t_j) = \hat{z}_f(t_j^-) + S_2(j)[y(j)-C_1\hat{x}_s(t_j^-)-C_2\hat{z}_f(t_j^-)+C_2A_2^{-1}B_2u_s(t_j^-)] \end{aligned}\right\} \tag{2.54}$$

$$S_2(j) = \Sigma_f C_2'[C_1\Sigma_s(t_j^-)C_1' + C_2\Sigma_f C_2' + R_j]^{-1}. \tag{2.55}$$

It u^* is the optimal solution to the problem (2.40)-(2.42), then it can be shown that

$$\left.\begin{aligned} &\lim_{\varepsilon\to0} u^* = u_0; \qquad 0<t<T \\ &\lim_{\varepsilon\to0} (J(u^*)-J(u_o)) = 0. \end{aligned}\right\} \tag{2.56}$$

2.3.2. Case 2: Sampled values of continuous noisy measurements

The measurement process is a continuous-time stochastic process described by

$$\bar{y}(t) = \int_0^t [C_1x(s) + C_2z(s)]ds + q(t) \tag{2.57}$$

where $q(t)$ is a standard Wiener process independent of $w(t)$ and the Gaussian vector $[x(0),z(0)]$.

Let $0 = t_o < t_1 < \cdots < t_{N-1} < t_N = T$ and $\theta = \{0,1,\ldots,N-1\}$. The measurement process is not observed on the entire time interval $[0,T]$, but only its sampled values at time instants $t_1, t_2, \ldots, t_N$ are observed. Therefore, the only observation in the subinterval $[t_j, t_{j+1})$ is

$$\bar{y}(t_j) = \int_0^{t_j} [C_1 x(s) + C_2 z(s)]ds + q(t_j) \tag{2.58}$$

which is made at the beginning of that subinterval. In the time interval $[0,t_1)$, no observations are made and only the prior statistics of the random quantities are available.

Let

$$\begin{aligned} y(j) &= \bar{y}(t_j) - \bar{y}(t_{j-1}) \\ &= \int_{t_{j-1}}^{t_j} [C_1 x(s) + C_2 z(s)]ds + v(j) \end{aligned} \tag{2.59}$$

where $v(j) = q(t_j) - q(t_{j-1})$ is a discrete-time Gaussian white noise process with mean zero and variance $R_j = (t_j - t_{j-1})I$. Clearly the sigma-algebras generated by $\{\bar{y}(t_i), i=1,2,\ldots,j\}$ and $\{y(i)\ , i=1,2,\ldots,j\}$ are equivalent.

A near-optimal solution to the problem defined by (2.40), (2.41), (2.59) can be obtained as follows:

$$u_o = u_s + u_f \tag{2.60}$$

where

$$u_s(t) = -B_1' K_1 \Psi_1(t,t_j) \hat{x}_s(t_j); \qquad t \in [t_j, t_{j+1}), \qquad j \in \theta \tag{2.61}$$

$$-\dot{K}_1 = A_1' K_1 + K_1 A_1 + Q_1 - K_1 B_1 B_1' K_1; \qquad K_1(T) = \Gamma_1 \tag{2.62}$$

$$\dot{\Psi}_1(t,t_j) = (A_1 - B_1 B_1' K_1)\Psi_1(t,t_j); \qquad \Psi_1(t_j,t_j) = I$$

$$t \in [t_j, t_{j+1}), \qquad j \in \theta \tag{2.63}$$

$$\left.\begin{aligned}
&\dot{\hat{x}}_s(t) = A_1\hat{x}_s(t) + B_1u_s(t); \qquad \hat{x}_s(0) = E[x(0)] \\
&\qquad\qquad\qquad\qquad t \in [t_{j-1}, t_j); \qquad j = 1,2,\ldots,N \\
&\hat{x}_s(t_j) = \hat{x}_s(t_j^-) + S_1(j)[y(j) - \int_{t_{j-1}}^{t_j} [C_1\hat{x}_s(r) - C_2A_2^{-1}B_2u_s(r)]dr]
\end{aligned}\right\} \quad (2.64)$$

$$\left.\begin{aligned}
&\dot{\Sigma}_s = A_1\Sigma_s + \Sigma_s A_1' + G_1G_1'; \qquad \Sigma_s(0) = \mathrm{Cov}[x(0)] \\
&\qquad\qquad\qquad\qquad t \in [t_{j-1}, t_j); \qquad j = 1,2,\ldots,N \\
&\Sigma_s(t_j) = \Sigma_s(t_j^-) - S_1(j)[\int_{t_{j-1}}^{t_j} C_1\phi_s(r,t_{j-1})dr\ \Sigma_s(t_{j-1})\phi_s'(t_j,t_{j-1}) \\
&\qquad + \int_{t_{j-1}}^{t_j} C_1 \int_r^{t_j} \phi_s(r,p)G_1G_1'\phi_s'(t_j,p)\,dp\,dr]
\end{aligned}\right\} \quad (2.65)$$

$$\begin{aligned}
\hat{R}_j = &\int_{t_{j-1}}^{t_j} C_1\phi_s(p,t_{j-1})dp\ \Sigma_s(t_{j-1}) \int_{t_{j-1}}^{t_j} \phi_s'(r,t_{j-1})C_1'dr \\
&+ \int_{t_{j-1}}^{t_j} C_1 \int_{t_{j-1}}^{p} \phi_s(p,r)G_1G_1' \int_r^{t_j} \phi_s'(\ell,r)C_1'\,d\ell\,dr\,dp \\
&+ \int_{t_{j-1}}^{t_j} C_2\phi_f(p,t_{j-1})dp\ \Sigma_f \int_{t_{j-1}}^{t_j} \phi_f'(r,t_{j-1})C_2'dr \\
&+ \int_{t_{j-1}}^{t_j} C_2 \int_{t_{j-1}}^{p} \phi_f(p,r)G_2G_2' \int_r^{t_j} \phi_f'(\ell,r)C_2'\,d\ell\,dr\,dp + R_j
\end{aligned} \quad (2.66)$$

$$\begin{aligned}
S_1(j) = &[\phi_s(t_j,t_{j-1})\Sigma_s(t_{j-1}) \int_{t_{j-1}}^{t_j} \phi_s'(r,t_{j-1})C_1'dr \\
&+ \int_{t_{j-1}}^{t_j} \phi_s(t_j,p)G_1G_1' \int_p^{t_j} \phi_s'(r,p)C_1'dr\,dp]\ \hat{R}_j^{-1}
\end{aligned} \quad (2.67)$$

$$\dot{\phi}_s(t,t_j) = A_1\phi_s(t,t_j); \quad \phi_s(t_j,t_j) = I$$
$$t \in [t_j,t_{j+1}), \quad j \in \theta \tag{2.68}$$

$$\varepsilon\dot{\phi}_f(t,t_j) = A_2\phi_f(t,t_j); \quad \phi_f(t_j,t_j) = I$$
$$t \in [t_j,t_{j+1}), \quad j \in \theta \tag{2.69}$$

$$A_2\Sigma_f + \Sigma_f A_2' + G_2G_2' = 0 \tag{2.70}$$

$$u_f(t) = -B_2'K_2\Psi_2(t,t_j)\hat{z}_f(t_j); \quad t \in [t_j,t_{j+1}), \quad j \in \theta \tag{2.71}$$

$$A_2'K_2 + K_2A_2 + Q_2 - K_2B_2B_2'K_2 = 0 \tag{2.72}$$

$$\varepsilon\dot{\Psi}_2(t,t_j) = (A_2-B_2B_2'K_2)\Psi_2(t,t_j); \quad \Psi_2(t_j,t_j) = I$$
$$t \in [t_j,t_{j+1}), \quad j \in \theta \tag{2.73}$$

$$\left.\begin{aligned}
&\varepsilon\dot{\hat{z}}_f(t) = A_2\hat{z}_f(t) + B_2u_f(t); \quad \hat{z}_f(0) = E[z(0)] \\
&\qquad\qquad t \in [t_{j-1},t_j); \quad j = 1,2,\ldots,N \\
&\hat{z}_f(t_j) = \hat{z}_f(t_j^-) + S_2(j)[y(j) - \int_{t_{j-1}}^{t_j} [C_1\hat{x}_s(r)+C_2\hat{z}_f(r)-C_2A_2^{-1}B_2u_s(r)]dr]
\end{aligned}\right\} \tag{2.74}$$

$$\begin{aligned}
S_2(j) = [&\phi_f(t_j,t_{j-1})\Sigma_f \int_{t_{j-1}}^{t_j} \phi_f'(r,t_{j-1})C_2'dr \\
&+ \int_{t_{j-1}}^{t_j} \phi_f(t_j,p)G_2G_2' \int_p^{t_j} \phi_f'(r,p)C_2'dr\,dp]\,\hat{R}_j^{-1}.
\end{aligned} \tag{2.75}$$

If u^* is the optimal solution to the problem defined by (2.40), (2.41), (2.59), then it can be shown that

$$\left.\begin{aligned} &\lim_{\varepsilon\to 0} u^* = u_o; \qquad 0 < t < T \\ &\lim_{\varepsilon\to 0}(J(u^*)-J(u_o)) = 0 \end{aligned}\right\} \tag{2.76}$$

We should point out that the near-optimality of the composite control u_o in both cases 1 and 2 is valid only in the open interval (0,T) because the boundary-layer terms have been neglected.

An important distinction between the above formulations involving discrete observations and the earlier formulation involving continuous observations is that, in the discrete observations cases, there is no need to scale the measurement noise and it is not necessary to have a separate observation channel for the fast variables. This is because the sampling interval is fixed and independent of ε, and hence there is no interaction between the dynamics of the observation process and the input noise process.

Now that we understand the subtleties involved in the modeling and control of stochastic singularly perturbed systems under various observation patterns, the next step is to study multi-agent decision problems. But before we do this, we shall introduce, in the next section, the important concept of multimodeling of large scale systems within the framework of time-scales and singular perturbations. This concept plays a crucial role in the near-optimal design of multi-agent decision policies for stochastic singularly perturbed systems.

3. MULTIMODELING BY SINGULAR PERTURBATIONS

The need for model simplification with a reduction (or distribution) of computational effort is particularly acute for large scale systems involving hundreds or thousands of state variables, often at different geographical locations. Some form of decentralized modeling and control which exploits the weak interactions between subsystems is then required. While there are a number of approaches to the study of large scale systems [1], the success of any proposed decentralized scheme critically depends upon the choice of subsystems [11].

A fundamental relationship between time-scales and weak-coupling has been developed for power systems, Markov chains, and other classes of large scale networks [12-15]. If the interactions of N "local" subsystems are treated as $O(\varepsilon)$, and if each subsystem has an equilibrium manifold (null space), then the local subsystems are decoupled in the fast time scale. However, they strongly interact in a slow time scale and form an aggregate model whose dimension is equal to the number (N) of the local subsystems. The system is thus decomposed into N+1 subsystems (N in the fast and one in the slow time scale).

To elucidate this relationship, consider the following class of interconnected subsystems

$$\frac{d\bar{x}_i}{dt} = \frac{1}{\varepsilon_i} A_{ii}\, \bar{x}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} A_{ij}\, \bar{x}_j \; ; \; i=1,2,\ldots,N \tag{3.1}$$

where $\varepsilon_i > 0$ and A_{ii} is a stable matrix with one zero eigenvalue. Assuming that $\bar{x}_i(0)$ is not in the null space of A_{ii}, the first term dominates the second term on the right hand side of (3.1), and therefore the interconnections can be neglected initially. As the fast transients draw $\bar{x}_i(t)$ towards the equilibrium manifold (the null space of A_{ii}), the two terms on the right hand side of (3.1) become the same order of magnitude, and therefore from this time onwards the interconnections can no longer be neglected. Hence, the dynamic behavior of (3.1) can be characterized by two separate motions: an initial fast transient within each isolated subsystem, followed by a slow motion around the equilibrium manifold obtained on neglecting the interconnections. Therefore, in the short term the subsystems can be treated in isolation, while in the longer term they become strongly-coupled.

We now introduce a transformation to make the slow and fast parts of $\bar{x}_i(t)$ explicit. Let

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \vdots\\ \bar{x}_N\end{bmatrix} = \begin{bmatrix}\varepsilon_1 & & & 0\\ & \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \varepsilon_N\end{bmatrix}^{-1}\begin{bmatrix}A_{11} & & & 0\\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{NN}\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \vdots\\ \bar{x}_N\end{bmatrix} + \begin{bmatrix}0 & A_{12} & \cdots & A_{1N}\\ A_{21} & 0 & \cdots & A_{2N}\\ & & & \\ A_{N1} & A_{N2} & \cdots & 0\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \vdots\\ \bar{x}_N\end{bmatrix} \tag{3.2}$$

or

$$\dot{\bar{x}} = (\Omega^{-1}A_o + A_1)\bar{x}. \tag{3.3}$$

Define the left and right eigenvectors of A_o for the zero eigenvalue as

$$A_oT = 0, \quad VA_o = 0, \quad VT = I_N$$

where

$$A_{ii}t_i = 0, \quad v_iA_{ii} = 0, \quad v_it_i = 1; \qquad i = 1,2,\ldots,N$$

$$T = \text{block diag}[t_1,t_2,\ldots,t_N]$$

$$V = \text{block diag}[v_1,v_2,\ldots,v_N]. \tag{3.4}$$

We also define block-diagonal matrices W and S as follows

$$WT = 0, \quad VS = 0, \quad WS = I_{n-N}. \tag{3.5}$$

Now, using the following transformation

$$x = V\bar{x}, \quad x \in R^N$$

$$z = W\bar{x}, \quad z \in R^{n-N} \tag{3.6a}$$

and its inverse

$$\bar{x} = Tx + Sz, \tag{3.6b}$$

the interconnected system (3.3) can be transformed into

$$\dot{x} = VA_1Tx + VA_1Sz$$

$$\Omega\dot{z} = \Omega WA_1Tx + W(A_o + \Omega A_1)Sz \tag{3.7}$$

For sufficiently small ε_i, (3.7) can be approximated by the model

$$\dot{x} = VA_1Tx + \sum_{j=1}^{N} \hat{A}_j z_j$$

$$\varepsilon_i\dot{z}_i = w_iA_{ii}s_iz_i; \qquad i = 1,2,\ldots,N \tag{3.8}$$

where

$$\hat{A}_j = \begin{bmatrix} v_1A_{1j} \\ v_2A_{2j} \\ \vdots \\ v_NA_{Nj} \end{bmatrix} s_j.$$

Notice that the fast transients within the subsystems are decoupled, and they interact only through the slow core. A long term aggregate model is obtained by letting $\varepsilon_i \to 0$, and is given by

$$\dot{x}_s = VA_1Tx_s. \tag{3.9}$$

The previous analysis has shown that for a wide class of large scale systems, the notions of subsystems, their coupling and time scales are interrelated and lead to a multiparameter singularly perturbed model with a strongly-coupled slow "core" representing the long term system-wide behavior, and weakly-coupled fast subsystems representing the short-term local behavior.

With the presence of control and stochastic disturbance inputs, a generalization of (3.8) can be obtained as

$$dx = A_{oo}x\,dt + \sum_{j=1}^{N}(A_{oj}z_j\,dt + B_{oj}u_j\,dt + G_{oj}\,dw_j) \tag{3.10a}$$

$$\varepsilon_i dz_i = (A_{io}x + A_{ii}z_i + \sum_{\substack{j=1 \\ j\neq i}}^{N}\varepsilon_{ij}A_{ij}z_j + B_{ii}u_i)dt + \sqrt{\varepsilon_i}\,G_{ii}\,dw_i$$

$$i = 1,2,\ldots,N \tag{3.10b}$$

where $\{u_i(t);\ i=1,2,\ldots,N\}$ are the control inputs, and $\{w_i(t);\ i=1,2,\ldots,N\}$ are standard Wiener processes independent of each other. Each fast subsystem has its own singular perturbation parameter ε_i, and is weakly-coupled to other fast subsystems through ε_{ij}. The fast subsystem i is affected by its own control input $u_i(t)$ and disturbance input $w_i(t)$. The slow subsystem, being the common "core", is affected, in general, by all the subsystem controls and disturbances.

In a situation like this, it is rational for a subsystem controller to neglect all other fast subsystems and to concentrate on its own subsystem, plus, of course, the slow interaction with others through the "core." For the i-th controller "to neglect all other subsystems" simply means to set all ε-parameters equal to zero, except for ε_i, which is to be kept at its true value. The i-th controller's simplified model is then

$$dx^i = A_i x^i dt + A_{oi} z_i dt + B_{oi} u_i dt + \sum_{\substack{j=1 \\ j\neq i}}^{N} (B_{ij} u_j dt + G_{oj} dw_j) + G_{oi} dw_i \tag{3.11a}$$

$$\varepsilon_i dz_i = A_{io} x^i dt + A_{ii} z_i dt + B_{ii} u_i dt + \sqrt{\varepsilon_i}\, G_{ii} dw_i \tag{3.11b}$$

where

$$A_i = A_{oo} - \sum_{j\neq i} A_{oj} A_{jj}^{-1} A_{jo}, \qquad B_{ij} = B_{oj} - A_{oj} A_{jj}^{-1} B_{jj}.$$

We denote x^i with a superscript rather than a subscript to stress the fact that x^i is not a component of x, but the i-th controller's view of x. In reality, the model (3.11) is often all that i-th controller knows about the whole system. The k-th controller, on the other hand, has a different model of the same large scale system. This situation, called multimodeling, was first formulated and investigated in [2] in a deterministic setup (with no disturbance inputs).

In the next section we shall study the impact of multimodel assumptions on the design of multi-agent control strategies in the presence of disturbance inputs and noisy observations.

4. MULTI-AGENT DECISION PROBLEMS

We shall restrict our discussion in this section to the case of two decision makers, as this will keep the notation simple and ease the exposition of the principle ideas. All the results that we shall present here extend to the case of more than two agents in a fairly straightforward fashion. Furthermore, we shall present and discuss only the main results; the proofs of the various propositions shall be omitted, but they can be found in the references cited.

It is well-known that a system of the form (3.10) can be transformed into a system with purely slow and fast variables [2]. Hence, without loss of generality, we shall consider multi-parameter singularly perturbed systems of the form

$$dz_o = A_{oo} z_o dt + \sum_{j=1}^{2} (B_{oj} u_j dt + G_{oj} dw_j) \tag{4.1a}$$

$$\varepsilon_i dz_i = (A_{ii} z_i + \varepsilon_{ij} A_{ij} z_j + B_{ii} u_i) dt + \sqrt{\varepsilon_i}\, G_{ii} dw_i, \quad i,j = 1,2; \quad i \neq j \tag{4.1b}$$

with $\dim z_i = n_i$, $i = 0,1,2$, and $\dim u_i = m_i$, $i = 1,2$. The initial conditions are assumed to be Gaussian with

$$E[z_i(0)] = \bar{z}_{io}, \quad E[z_i(0)\, z_j'(0)] = N_{ij}; \qquad i,j = 0,1,2. \tag{4.2}$$

Furthermore, we shall restrict ourselves to the case $\{\mathrm{Re}\, \lambda(A_{ii}) < 0,\ i = 1,2\}$.

In a multimodel situation, decision maker i models only z_o and z_i, but neglects z_j. Also, his observations are functions of z_o and z_i alone. This situation with decentralized observations leads to problems involving nonclassical information patterns, for which no finite-dimensional solution exists in general. In order to obtain finite-dimensional solutions which can be implemented in practice, one needs to modify the information structure. In this section we shall study three problems with quasi-classical information patterns. The first problem is a Nash problem with continuous measurements where the information available to the decision makers is restricted to the state of a finite-dimensional compensator of a specified structure. The next two problems are team problems with sampled measurements, where the decision makers exchange information with a delay of one sample period. The two types of sampled measurements are those that we have considered earlier in Section 2.

4.1. Nash Game with Continuous Measurements

The decision makers make decentralized continuous measurements which are given by

$$dy_{oi} = C_{oi} z_o dt + dv_{oi}$$

$$dy_{ii} = C_{ii} z_i dt + \sqrt{\varepsilon_i}\, dv_{ii}; \qquad i = 1,2 \tag{4.3}$$

where $\dim y_{oi} = p_{oi}$ and $\dim y_{ii} = p_{ii}$. The processes $v_{oi}(t)$ and $v_{ii}(t)$ are standard Wiener processes, independent of each other and of the process noise $w_i(t)$. Defining $x' = [z_o' \;\; z_1' \;\; z_2']$, $y_i' = [y_{oi}' \;\; \frac{1}{\sqrt{\varepsilon_i}} y_{ii}']$, $v_i' = [v_{oi}' \;\; v_{ii}']$, and $w' = [w_1' \;\; w_2']$. The system of equations (4.1)-(4.3) can be written in a composite form as

$$dx = (A(\varepsilon)x + \sum_{i=1}^{2} B_i(\varepsilon)u_i)dt + G(\varepsilon)dw \tag{4.4}$$

$$dy_i = C_i(\varepsilon)x dt + dv_i; \qquad i = 1,2 \tag{4.5}$$

$$E[x(0)] = \bar{x}_o, \qquad E[x(0)x'(0)] = N \tag{4.6}$$

where $\dim x = n = n_o + n_1 + n_2$ and $\dim y_i = p_i = p_{oi} + p_{ii}$. The matrices $A(\varepsilon)$, $B_i(\varepsilon)$, $G(\varepsilon)$, $C_i(\varepsilon)$, and N are appropriately defined.

The information available to decision maker i at time t is given by

$$\alpha_i(t) = \{\hat{x}_i(t), \bar{x}_o, N\} \tag{4.7}$$

where $\hat{x}_i(t)$ is the state of the n-dimensional compensator

$$d\hat{x}_i = (F_i \hat{x}_i + H_i u_i)dt + L_i[dy_i - C_i \hat{x}_i dt]. \tag{4.8}$$

Let $\sigma_i(t)$ denote the sigma-algebra generated by the information set $\alpha_i(t)$. Further, let H_i denote the class of second-order stochastic processes $\{u_i(t), t \geq 0\}$ which are $\sigma_i(t)$-measurable. Then, a permissible strategy for decision maker i is a mapping $\nu_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{m_i}$, such that $\nu_i(\cdot, \alpha_i) \in H_i$. Denote the class of all such strategies for decision maker i by Γ_i.

For each $\{\nu_i \in \Gamma_i;\ i=1,2\}$, the cost functionals for the two decision makers are given by

$$J_i(\nu_1,\nu_2) = E\{z_o'(T)\Gamma_{oi}z_o(T) + \varepsilon_i z_i'(T)\Gamma_{ii}z_i(T) + \int_0^T (z_o'Q_{oi}z_o + z_i'Q_{ii}z_i + u_i'u_i)dt \,|\, u_j(t) = \nu_j(t,\alpha_j),\quad j=1,2\} \qquad i=1,2 \tag{4.9a}$$

or, equivalently

$$J_i(\nu_1,\nu_2) = E\{x'(T)\Gamma_i(\varepsilon)x(T) + \int_0^T (x'Q_i x + u_i'u_i)dt \,|\, u_j(t) = \nu_j(t,\alpha_j),\ j=1,2\} \qquad i=1,2 \tag{4.9b}$$

where the expectation is taken over the underlying statistics.

The decision makers are required to select the matrices F_i^*, H_i^*, L_i^*; the initial conditions $\hat{x}_i^*(0)$ and strategies $\nu_i^*[t,\hat{x}_i(t)]$ such that

$$J_i(\nu_i^*, \nu_j^*) \leq J_i(\nu_i, \nu_j^*) \qquad \forall \nu_i \in \Gamma_i; \quad i,j = 1,2; \quad i \neq j. \tag{4.10}$$

The pair of inequalities above defines the *Nash equilibrium point*.

The optimal solution to the problem defined by (4.1)-(4.10) is obtained by extending the results of [16] to the nonzero-sum case, and is given by

$$\nu_i^* = -B_i' K_i \hat{x}_i; \qquad i = 1,2 \tag{4.11a}$$

$$F_i^* = A - B_j B_j' K_j [I + (M_{jo} - M_{ji})(M_{oo} - M_{oi})^{-1}]; \qquad i,j = 1,2; \quad i \neq j \tag{4.11b}$$

$$L_i^* = M_{ii} C_i'; \qquad i = 1,2 \tag{4.11c}$$

$$H_i^* = B_i; \qquad i = 1,2 \tag{4.11d}$$

$$\hat{x}_i^*(0) = \bar{x}_o; \qquad i = 1,2 \tag{4.11e}$$

where K_i satisfies the coupled set of Riccati equations

$$\dot{K}_i = -K_i A - A' K_i - Q_i + K_i S_i K_i + K_i S_j K_j + K_j S_j K_i; \qquad K_i(T) = \Gamma_i$$

$$S_i = B_i B_i'; \qquad i,j = 1,2; \quad i \neq j. \tag{4.12}$$

$M(t)$ is a symmetric nonnegative definite matrix satisfying the Lyapunov equation

$$\dot{M} = FM + FM' + BB'; \qquad M_{ij}(0) = \bar{x}_o \bar{x}_o' + N, \quad i = j = 0$$

$$= N \qquad , \text{otherwise} \tag{4.13a}$$

where

$$F = \begin{bmatrix} A-S_1K_1-S_2K_2 & S_1K_1 & S_2K_2 \\ A-F_1^*-S_2K_2 & F_1^*-L_1^*C_1 & S_2K_2 \\ A-F_2^*-S_1K_1 & S_1K_1 & F_2^*-L_2^*C_2 \end{bmatrix}$$

$$B = \begin{bmatrix} -G & 0 & 0 \\ -G & L_1^* & 0 \\ -G & 0 & L_2^* \end{bmatrix} \tag{4.13b}$$

The compensators are unbiased, in the sense that for all $t \in [0,T)$,

$$E\{x(t)|\hat{x}_i(t)\} = \hat{x}_i(t); \qquad i=1,2. \tag{4.14}$$

Furthermore

$$E\{[x(t)-\hat{x}_i(t)]\hat{x}_i'(t)\} = 0; \quad i=1,2. \tag{4.15}$$

Thus, each component of the error $x(t)-\hat{x}_i(t)$ is orthogonal to each component of $\hat{x}_i(t)$, and $\hat{x}_i(t)$ may be regarded in some sense an estimate of $x(t)$. Notice that the solution exhibits a unidirectional separation in estimation and control. Although the control gains are obtained independently, the optimal filter matrices and covariance $M(t)$ depend on the control gains, resulting in a "dual effect" [17].

The optimal costs are given by

$$J_i^* = \bar{x}_o'K_i(0)\bar{x}_o + \mathrm{tr}\{M_{ii}(0)K_i(0) + \int_0^T (K_iS_iK_iM_{ii} + K_iS_jK_jM_{jo} + K_jS_jK_iM_{oj})dt\}; \quad i,j=1,2; \quad i \neq j. \tag{4.16}$$

The linear strategy (4.11a) is the unique Nash strategy for this problem. Since the finite-dimensional estimators (4.8) are not Kalman filters, it is not clear, at the outset, what their limiting structure (as the small parameters go to zero) looks like. Does the full-order estimator decompose into a number of decoupled low-order estimators? Is it possible to obtain a near-equilibrium solution from low-order subproblems?

It will be shown that, in the limit as the small parameters go to zero, the full-order estimator (4.8) decomposes into an n_o-dimensional estimator in the slow time scale which has a similar structure, and two n_1- and n_2-dimensional Kalman filters in the fast time scale. Furthermore, the near-equilibrium solution is in fact the multimodel solution, i.e., the solution obtained when decision maker i neglects z_j, and models only z_o and z_i. The multimodel assumption leads to the formulation of three low-order subproblems: two independent stochastic control problems, one for each decision maker, in the fast time scale, and a stochastic Nash game in the slow time scale.

<u>The slow subproblem</u> is obtained by neglecting all the small parameters in (4.1), and is given by

$$dz_{os} = (A_{oo} z_{os} + \sum_{i=1}^{2} B_{oi} u_{is}) dt + \sum_{i=1}^{2} G_{oi} dw_i \tag{4.17}$$

$$dy_{is} = \begin{bmatrix} C_{oi} \\ \\ 0 \end{bmatrix} z_{os} dt + \begin{bmatrix} 0 \\ \\ -\frac{1}{\sqrt{\varepsilon_i}} C_{ii} A_{ii}^{-1} B_{ii} \end{bmatrix} u_{is} dt + \begin{bmatrix} dv_{oi} \\ \\ dv_{ii} - C_{ii} A_{ii}^{-1} G_{ii} dw_i \end{bmatrix}$$

$$= (C_{is} z_{os} + D_{is} u_{is}) dt + dv_{is}; \qquad i = 1,2 \tag{4.18}$$

$$E[z_{os}(0)] = \bar{z}_{oo}, \qquad E[z_{os}(0) \;\; z'_{os}(0)] = N_{oo}. \tag{4.19}$$

Each decision maker is constrained to use only an n_o-dimensional compensator of the form

$$d\hat{z}_{is} = (F_{is}\hat{z}_{is} + H_{is}u_{is})dt + L_{is}[dy_{is} - C_{is}\hat{z}_{is}dt - D_{is}u_{is}dt],$$

$$i = 1,2. \tag{4.20}$$

Let

$$\alpha_{is}(t) = \{\hat{z}_{is}(t),\ \bar{z}_{oo},\ N_{oo}\} \tag{4.21}$$

and $\sigma_{is}(t)$ denote the sigma-algebra generated by the information set $\alpha_{is}(t)$. Further, let H_{is} denote the class of second-order stochastic processes $\{u_{is}(t),\ t\geq 0\}$ which are $\sigma_{is}(t)$-measurable. Define the slow strategy ν_{is}, as the mapping $\nu_{is} : [0,T] \times \mathbb{R}^{n_o} \to \mathbb{R}^{m_i}$, such that $\nu_{is}(\cdot,\alpha_{is}) \in H_{is}$. Denote the class of all such slow strategies for decision maker i by Γ_{is}.

For each $\{\nu_{is} \in \Gamma_{is};\ i=1,2\}$, the slow cost functionals for the decision makers are given by

$$J_{is}(\nu_{1s},\nu_{2s}) = E\{z'_{os}(T)\Gamma_{oi}z_{os}(T) + \int_0^T (z'_{os}Q_{oi}z_{os} + u'_{is}R_{is}u_{is})dt$$

$$|u_{js}(t) = \nu_j(t,\alpha_{js}),\ j=1,2\}; \qquad i = 1,2 \tag{4.22}$$

where

$$R_{is} = I + (A_{ii}^{-1}B_{ii})'Q_i(A_{ii}^{-1}B_{ii}).$$

The decision makers are required to select the matrices F^*_{is}, H^*_{is}, L^*_{is}; the initial conditions $\hat{z}^*_{is}(0)$, and strategies $\nu^*_{is}[t,\hat{z}_{is}(t)]$ such that

$$J_i(\nu^*_{is},\nu^*_{js}) \leq J_i(\nu_{is},\nu^*_{js}) \qquad \forall \nu_{is} \in \Gamma_{is}; \qquad i,j = 1,2, \quad i \neq j. \tag{4.23}$$

The optimal solution to the slow subproblem defined by (4.17)-(4.23) is given by

$$\nu^*_{is} = -R_{is}^{-1}B'_{oi}K_{is}\hat{z}_{is}; \qquad i = 1,2 \tag{4.24a}$$

$$F^*_{is} = A_{oo} - B_{oj}R^{-1}_{js}B'_{oj}K_{js}[I + (\bar{M}_{jo} - \bar{M}_{ji})(\bar{M}_{oo} - \bar{M}_{oi})^{-1}]; \; i,j=1,2; \; i\neq j \tag{4.24b}$$

$$L^*_{is} = [\bar{M}_{ii}C'_{oi} \mid G_{oi}(C_{ii}A^{-1}_{ii}G_{ii})' \{I + (C_{ii}A^{-1}_{ii}G_{ii})(C_{ii}A^{-1}_{ii}G_{ii})'\}^{-1}]; \; i=1,2 \tag{4.24c}$$

$$H^*_{is} = B_{oi} \;; \; i=1,2 \tag{4.24d}$$

$$\hat{z}^*_{is}(0) = \bar{z}_{oo} \;; \; i=1,2 \tag{4.24e}$$

where K_{is} is the solution of the coupled set of Riccati equations

$$\dot{K}_{is} = - K_{is}A_{oo} - A'_{oo}K_{is} - Q_{oi} + K_{is}S_{is}K_{is} + K_{is}S_{js}K_{js} + K_{js}S_{js}K_{is};$$
$$K_{is}(T) = \Gamma_{oi} \;, \; S_{is} = B_{oi}R^{-1}_{is}B'_{oi} \;; \; i,j=1,2; \; i\neq j \tag{4.25}$$

$\bar{M}(t)$ is a symmetric nonnegative definite matrix satisfying the Lyapunov equation

$$\dot{\bar{M}} = F_s\bar{M} + \bar{M}F'_s + B_sB'_s \;; \; \bar{M}_{ij}(0) = \begin{cases} \bar{z}_{oo}\bar{z}'_{oo} + N_{oo} \;; & i=j=0 \\ N_{oo} \;, & \text{otherwise} \end{cases} \tag{4.26a}$$

where

$$F_s = \begin{bmatrix} A_{oo} - S_{1s}K_{1s} - S_{2s}K_{2s} & S_{1s}K_{1s} & S_{2s}K_{2s} \\ A_{oo} - F^*_{1s} - S_{2s}K_{2s} & F^*_{1s} - L^*_{1s}C_{1s} & S_{2s}K_{2s} \\ A_{oo} - F^*_{2s} - S_{1s}K_{1s} & S_{1s}K_{1s} & F^*_{2s} - L^*_{2s}C_{2s} \end{bmatrix}$$

$$B_s = \begin{bmatrix} - G_o & 0 & 0 \\ - G_o - L^*_{1s}P_1 & L^*_{1s} & 0 \\ - G_o - L^*_{2s}P_2 & 0 & L^*_{2s} \end{bmatrix}$$

$$G_o = [G_{o1} \;\; G_{o2}]$$

$$P_i = \begin{bmatrix} 0 & 0 \\ C_{ii}A^{-1}G_{ii} & 0 \end{bmatrix} \quad ; \ i=1,2. \tag{4.26b}$$

The optimal costs are given by

$$J^*_{is} = \bar{z}'_{oo}K_{is}(0)\bar{z}_{oo} + \mathrm{tr}\{\bar{M}_{ii}(0)K_{is}(0) + \int_0^T (K_{is}S_{is}K_{is}\bar{M}_{ii} + K_{is}S_{js}K_{js}\bar{M}_{jo}$$

$$+ K_{js}S_{js}K_{is}\bar{M}_{oj})dt\} \ ; \ i,j=1,2 \ ; \ i\neq j \tag{4.27}$$

The fast subproblems, on the other hand, are formulated 'locally' at the subsystem level. These are stochastic control problems because the decision makers do not interact in the fast time scale:

$$\varepsilon_i dz_{if} = (A_{ii}z_{if} + B_{ii}u_{if})dt + \sqrt{\varepsilon_i}\, G_{ii}dw_i \tag{4.28}$$

$$dy_{iif} = C_{ii}z_{if}dt + \sqrt{\varepsilon_i}\, dv_{ii} \tag{4.29}$$

$$E[z_{if}(0)] = \bar{z}_{io} \ , \ E[z_{if}(0)\ z'_{if}(0)] = N_{ii} \tag{4.30}$$

$$J_{if} = E\{\varepsilon_i z'_{if}(T)\Gamma_{ii}z_{if}(T) + \int_0^T (z'_{if}Q_{ii}z_{if} + u'_{if}u_{if})dt \quad . \tag{4.31}$$

Notice that this fast subproblem is exactly the one we studied in detail in Section 2. Its solution, as $\varepsilon_i \to 0$, is given by

$$u^*_{if} = -\,B'_{ii}K_{if}\hat{z}_{if} \tag{4.32}$$

where K_{if} satisfies the Riccati equation

$$K_{if}A_{ii} + A'_{ii}K_{if} + Q_{ii} - K_{if}B_{ii}B'_{ii}K_{if} = 0 \tag{4.33}$$

and $\hat{z}_{if}$ is the state of the Kalman filter

$$\varepsilon_i d\hat{z}_{if} = (A_{ii}\hat{z}_{if} + B_{ii}u^*_{if})dt + P_{if}C'_{ii}[dy_{if} - C_{ii}\hat{z}_{if}dt] \; ;$$

$$\hat{z}_{if}(0) = \bar{z}_{io} \qquad (4.34)$$

P_{if} is the error covariance of $\hat{z}_{if}$ satisfying

$$P_{if}A'_{ii} + A_{ii}P_{if} + G_{ii}G'_{ii} - P_{if}C'_{ii}C_{ii}P_{if} = 0 \quad , \qquad (4.35)$$

and the optimal cost is given by

$$J^*_{if} = T\ \mathrm{tr}\{P_{if}Q_{ii} + C_{ii}P_{if}K_{if}P_{if}C'_{ii}\} \quad . \qquad (4.36)$$

The following proposition establishes the connection between the solutions of the slow and fast subproblems and the full-order problem. Its proof may be found in [7].

<u>Proposition 4.1</u>:

i) $\nu^*_i(t,\hat{x}_i(t)) = \nu^*_{is}(t,\hat{z}_{is}(t)) + u^*_{if}(\hat{z}_{if}(t)) + O(\|\varepsilon\|)$; $\forall\ t\in(0,T)$

ii) $J^*_i = J^*_{is} + J^*_{if} + T\ \mathrm{tr}\{Q_{ii}W_i\} + O(\|\varepsilon\|)$; $i=1,2$

where

$$\varepsilon = [\varepsilon_1\ \varepsilon_2\ \varepsilon_{12}\ \varepsilon_{21}]$$

and W_i is the nonnegative definite solution of the Lyapunov equation

$$A_{ii}W_i + W_iA'_{ii} + G_{ii}G'_{ii} = 0 \; ; \; i=1,2 \quad . \qquad (4.37)$$

□

Since the multimodel strategies need only decentralized 'state estimates,' each decision maker needs to construct only two filters of dimensions n_o and n_i, respectively, instead of constructing one filter of dimension $n_o + n_1 + n_2$ as required by the optimal solution. This would result in lower implementation costs.

4.2. Team Problems with Sampled Measurements

We shall now consider problems wherein the measurement processes of the decision makers are not continuous on the entire time interval $[0,T]$, but consist of sampled values observed at time instants $t_o, t_1, \ldots, t_{N-1}$, where $0 = t_o < t_1 < \ldots < t_{N-1} < t_N = T$. Let θ denote the index set $\{0,1,\ldots,N-1\}$, and $y_i(j)$ denote the p_i-dimensional observations made by decision maker - i at time instant t_j, $j \in \theta$. Thus the only measurement of decision maker - i in the subinterval $[t_j, t_{j+1})$ is $y_i(j)$.

The quasi-classical information pattern that we shall consider here is the so-called "one-step-delay observation sharing pattern," wherein the decision makers exchange their independent sampled observations with a delay of one sampling interval. Hence, the information available to decision maker - i in the time interval $[t_j, t_{j+1})$ is

$$\alpha_i^j = \{y_i(j), \delta_{j-1}\} \tag{4.38a}$$

where, δ_{j-1} denotes the common information available to the decision makers in the same interval, i.e.;

$$\delta_{j-1} = \{y_1(j-1), y_2(j-1), \ldots, y_1(0), y_2(0)\} \quad . \tag{4.38b}$$

Let σ_i^j denote the sigma-algebra generated by the information set α_i^j, and H_i^N denote the class of stochastic processes $\{u_i(t), t \geq 0\}$ whose restriction to the interval $[t_j, t_{j+1})$ is σ_i^j-measurable for all $j \in \theta$. Then a permissible strategy for decision maker - i is a mapping ν_i: $[0,T] \times \mathbb{R}^{(p_1+p_2)N} \to \mathbb{R}^{m_i}$, such that $\nu_i(\cdot, \alpha_i) \in H_i^N$. Denote the class of all such strategies for decision maker - i by Γ_i^N. For each $\{\nu_i \in \Gamma_i^N \ ; \ i=1,2\}$, we define the quadratic strictly convex cost function as

$$J(\nu_1,\nu_2) = E\{z_o'(T)\Gamma_o z_o(T) + \sum_{i=1}^{2} \varepsilon_i z_i'(T)\Gamma_i z_i(T)$$

$$+ \int_0^T (z_o'Q_o z_o + \sum_{i=1}^{2} (z_i'Q_i z_i + u_i'u_i))dt \,|\, u_j(t) = \nu_j(t,\alpha_j),\ j=1,2\}$$

(4.39a)

where $\{\Gamma_i \geq 0, Q_i \geq 0,\ i=0,1,2\}$ and the expectation operation is taken over the underlying statistics.

Equivalently, in terms of the composite state vector $x(t)$ of (4.4), the cost function can be written as

$$J(\nu_1,\nu_2) = E\{x'(T)\Gamma(\varepsilon)x(T) + \int_0^T (x'Qx + u_1'u_1 + u_2'u_2)dt \,|\, u_j(t) = \nu_j(t,\alpha_j),\ j=1,2\}$$

(4.39b)

where $\Gamma(\varepsilon)$ and Q are appropriately defined in terms of the matrices appearing in (4.39a).

A *team optimal solution* is a pair $\{\nu_i^* \in \Gamma_i^N,\ i=1,2\}$ which satisfies

$$J(\nu_1^*,\nu_2^*) = \inf_{\Gamma_1^N} \inf_{\Gamma_2^N} J(\nu_1,\nu_2) \tag{4.40}$$

Here optimal and near-optimal strategies will be obtained for two cases of sampled observations, as delineated below.

4.2.1. Case 1: Noisy measurements of sampled values of state

At sampled time instant $t_j, j\in\theta$, the decision makers observe

$$y_i(j) = C_{io}z_o(t_j) + C_{ii}z_i(t_j) + v_i(j)$$

$$\equiv C_i x(t_j) + v_i(j)\ ;\ i=1,2 \quad . \tag{4.41}$$

The random vectors $\{v_i(j); j\in\theta, i=1,2\}$ are assumed to have independent Gaussian statistics $\{v_i(j) \sim N(0,V_{ij}), V_{ij}>0, j\in\theta, i=1,2\}$. Their statistics are also assumed to be independent of the Wiener processes $\{v_i(t)\ ;\ i=1,2\}$ and the initial state vector $x(0)$.

The optimal team solution to the problem defined by (4.4), (4.6), (4.38)-(4.41) has been derived in [18], and is given by

$$\nu_i^*(t,\alpha_i) = P_i(t)[y_i(j) - C_i\hat{\xi}(j)] - B_i'S(t)\psi(t,t_j)\hat{\xi}(j)\ ;\ i=1,2$$

$$t\in[t_j,t_{j+1})\ ;\ j\in\theta \tag{4.42a}$$

where $P_1(t)$, $P_2(t)$ are piecewise continuous functions on $[0,T]$ and satisfy the coupled set of linear integral equations

$$P_i(t) = B_i'S_i(t)\int_{t_j}^{t} \psi_{ij}(t,\tau)B_iB_i'L_{ij}(\tau)d\tau - B_i'L_{ij}(t)\ ;\ i=1,2$$

$$t\in[t_j,t_{j+1})\ ;\ j\in\theta \tag{4.42b}$$

where

$$L_{ij}(t) = S_i(t)[\phi(t,t_j) + \int_{t_j}^{t} \phi(t,\tau)B_kP_k(\tau)d\tau C_k]\Sigma_i(j) + K_{ij}(t);$$

$$i,k=1,2\ ;\ i\neq k\ ;\ t\in[t_j,t_{j+1})\ ;\ j\in\theta \tag{4.42c}$$

$$\dot{K}_{ij}(t) = -(A - B_iB_i'S_i(t))'K_{ij}(t) - S_i(t)B_kP_k(t)C_k\Sigma_{ij}\ ;\ i,k=1,2,\ i\neq k$$

$$K_{ij}(t_{j+1}) = 0\ ,\ t\in[t_j,t_{j+1}]\ ,\ j\in\theta \tag{4.42d}$$

$S(t)$ and $S_i(t)$ satisfy the Riccati equations

$$\dot{S} = -A'S - SA - Q + S[B_1B_1' + B_2B_2']S\ ;\ S(T) = \Gamma \tag{4.42e}$$

$$\dot{S}_i = -A'S_i - S_iA - Q + S_iB_iB_i'S_i\ ;\ S_i(t_j) = S(t_j)$$

$$t\in(t_{j-1},t_j]\ ,\ i=1,2\ ,\ j=N,\ldots,1\ . \tag{4.42f}$$

$\psi(t,\tau)$ is the state transition matrix satisfying

$$\dot{\psi}(t,\tau) = (A - B_1B_1'S - B_2B_2'S)\psi(t,\tau) \; ; \; \psi(\tau,\tau) = I \tag{4.43a}$$

$\psi_{ij}(t,\tau)$ is the state transition matrix satisfying

$$\dot{\psi}_{ij}(t,\tau) = (A - B_iB_i'S_i)\psi_{ij}(t,\tau) \; ; \; \psi_{ij}(\tau,\tau) = I \; ; \; t\in[t_j,t_{j+1}),$$

$$i=1,2 \; , \; j\in\theta \tag{4.43b}$$

$\phi(t,\tau)$ is the state transition matrix satisfying

$$\dot{\phi}(t,\tau) = A\phi(t,\tau) \; ; \; \phi(\tau,\tau) = I \tag{4.43c}$$

$\hat{\xi}(j) = \eta(t_j^-) = E[x(t_j)|\delta_{j-1}]$ and $\eta(t)$ satisfies

$$\left.\begin{aligned} &\dot{\eta} = A\eta + \sum_{i=1}^{2} B_i\nu_i^*(t,\alpha_i) \; ; \; \eta(0) = \bar{x}_o \\ &\qquad t\in[t_{j-1},t_j) \; , \; j=1,\ldots,N \\ &\eta(t_j) = \eta(t_j^-) + M(j)[y(j) - C\eta(t_j^-)] \end{aligned}\right\} \tag{4.44}$$

$$\Sigma_i(j) = \Sigma(t_j)C_i'[C_i\Sigma(t_j)C_i' + V_{ij}]^{-1} \; ; \; i=1,2 \; , \; j\in\theta \tag{4.45}$$

where $\Sigma(t_j^-) = E[(x(t_j) - \eta(t_j^-))(x(t_j) - \eta(t_j^-))']$ and $\Sigma(t)$ satisfies

$$\left.\begin{aligned} &\dot{\Sigma} = A\Sigma + \Sigma A' + GG' \; ; \; \Sigma(0) = N \\ &\qquad t\in[t_{j-1},t_j) \; ; \; j=1,\ldots,N \\ &\Sigma(t_j) = \Sigma(t_j^-) - M(j)C\Sigma(t_j^-) \end{aligned}\right\} \tag{4.46}$$

and

$$M(j) = \Sigma(t_j^-)C'[C\Sigma(t_j^-)C' + V_j]^{-1} \tag{4.47a}$$

$$V_j = \mathrm{diag}(V_{1j}, V_{2j}) \tag{4.47b}$$

$$y(j) = [y_1'(j) \;\; y_2'(j)]' \tag{4.47c}$$

$$C = [C_1' \;\; C_2']' \quad . \tag{4.47d}$$

Due to the presence of widely separated eigenvalues, the integro-differential equations (4.42)-(4.47) involved for computing the optimal solutions are numerically stiff. This renders the optimal solution computationally infeasible, specially when the order of the system is very large. Futhermore, when the small perturbation parameters are unknown, or when one decision maker does not have a knowledge of the fast dynamics of the other decision maker, it is not even possible to compute the optimal solution. Hence, there is a need to look for suboptimal solutions. The multimodel solution proposed below exploits the special structure of the system to yield a solution which does not require a knowledge of the small parameters, and allows the decision makers to model only their own fast dynamics. More importantly, as in the problem with continuous measurements, the multimodel solution is well-posed in the sense that it is the limit of the optimal solution as the small parameters go to zero.

The multimodel solution is obtained on solving three low-order problems: a slow team problem under the one-step-delay observation sharing pattern, and two fast stochastic control problems, one for each decision maker.

The system model for the slow subproblem is given by (4.17), (4.19) and the observations by

$$y_{is}(j) = C_{io}z_{os}(t_j) + v_i(j)$$

$$\equiv y_i(j) - C_{ii}z_{is}(t_j) \; ; \; j\in\theta \; , \; i=1,2 \quad . \tag{4.48}$$

The cost function is given by

$$J_s(\nu_{1s},\nu_{2s}) = E\{z'_{os}(T)\Gamma_o z_{os}(T) + \int_0^T (z'_{os}Q_o z_{os} + \sum_{i=1}^{2} u'_{is}R_{is}u_{is})dt \,|\, u_{js}(t)$$

$$= \nu_{js}(t,\alpha_j) \; , \; j=1,2\} \tag{4.49}$$

where

$$R_{is} = I + (A_{ii}^{-1}B_{ii})'Q_i(A_{ii}^{-1}B_{ii}) \quad .$$

The optimal solution to the slow team problem defined by (4.17), (4.19), (4.48) and (4.49) is given by

$$\nu^*_{is}(t,\alpha_i) = P_{is}(t)[y_{is}(j) - C_{io}\hat{\xi}_s(j)] - R_{is}^{-1}B'_{oi}S_s\psi_s(t,t_j)\hat{\xi}_s(j) \; ; \; i=1,2$$

$$t\in[t_j,t_{j+1}) \; ; \; j\in\theta \tag{4.50a}$$

where $P_{1s}(t)$, $P_{2s}(t)$ satisfy the coupled set of linear integral equations

$$P_{is}(t) = R_{is}^{-1}B'_{oi}S_{is}(t)\int_{t_j}^{t} \psi_{ijs}(t,\tau)B_{oi}R_{is}^{-1}B'_{oi}L_{ijs}(\tau)d\tau - R_{is}^{-1}B'_{oi}L_{ijs}(t) \; ; \; i=1,2$$

$$t\in[t_j,t_{j+1}) \; ; \; j\in\theta \tag{4.50b}$$

where

$$L_{ijs}(t) = S_{is}(t)[\phi_s(t,t_j) + \int_{t_j}^{t} \phi_s(t,\tau)B_{ok}R_{ks}^{-1}P_{ks}(\tau)d\tau C_{ko}]\Sigma_{is}(j) + K_{ijs}(t)$$

$$i,k=1,2 \; ; \; i\neq k \; ; \; t\in[t_j,t_{j+1}) \; ; \; j\in\theta \tag{4.50c}$$

$$\dot{K}_{ijs}(t) = -(A_{oo} - B_{oi}R_{is}^{-1}B_{oi}'S_{is}(t))'K_{ijs}(t) - S_{is}(t)B_{ok}R_{ks}^{-1}P_{ks}(t)C_{ko}\Sigma_{is}(j) \; ;$$

$$K_{ijs}(t_{j+1}) = 0 \; ; \; i,k=1,2 \; ; \; i\neq k \; ; \; t\in(t_j,t_{j+1}] \; ; \; j\in\theta \qquad (4.50d)$$

$S_s(t)$ and $S_{is}(t)$ satisfy the Riccati equations

$$\dot{S}_s = -A_{oo}'S_s - S_sA_{oo} - Q_o + S_s[B_{o1}R_{1s}^{-1}B_{o1}' + B_{o2}R_{2s}^{-1}B_{02}']S_s ; S_s(T) = \Gamma_o \qquad (4.50e)$$

$$\dot{S}_{is} = -A_{oo}'S_{is} - S_{is}A_{o1} - Q_o + S_{is}B_{oi}R_{is}^{-1}B_{oi}'S_{is} \; ; \; S_{is}(t_j) = S_s(t_j) \; ,$$

$$t\in(t_{j-1},t_j] \; ; \; i=1,2 \; ; \; j=N,\ldots,1 \qquad (4.50f)$$

$\psi_s(t,\tau)$ is the state transition matrix satisfying

$$\dot{\psi}_s(t,\tau) = (A_{oo} - B_{o1}R_{1s}^{-1}B_{o1}'S_s - B_{o2}R_{2s}^{-1}B_{o2}'S_s)\psi_s(t,\tau) \; ; \; \psi_s(\tau,\tau) = I \qquad (4.51a)$$

$\psi_{ijs}(t,\tau)$ is the state transition matrix satisfying

$$\dot{\psi}_{ijs}(t,\tau) = (A_{oo} - B_{oi}R_{is}^{-1}B_{oi}'S_{is})\psi_{ijs}(t,\tau) \; ; \; \psi_{ijs}(\tau,\tau) = I$$

$$t\in[t_j,t_{j+1}) \; ; \; i=1,2 \; ; \; j\in\theta \qquad (4.51b)$$

$\phi_s(t,\tau)$ is the state transition matrix satisfying

$$\dot{\phi}_s(t,\tau) = A_{oo}\phi_s(t,\tau) \; ; \; \phi_s(\tau,\tau) = I \qquad (4.51c)$$

$\hat{\xi}_s(j) = \eta_s(t_j^-) = E[z_{os}(t_j)|\delta_{j-1}]$ and $\eta_s(t)$ satisfies

$$\left.\begin{aligned}
&\dot{\eta}_s = A_{oo}\eta_s + \sum_{i=1}^{2} B_{oi}\nu_{is}^{*}(t,\alpha_i) \; ; \; \eta_s(0) = \bar{z}_{oo} \\
&t\in[t_{j-1},t_j) \; ; \; j=1,\ldots,N \\
&\eta_s(t_j) = \eta_s(t_j^-) + M_s(j)[y_s(j) - C_o\eta_s(t_j^-)]
\end{aligned}\right\} \qquad (4.52)$$

$$\Sigma_{is}(j) = \Sigma_s(t_j)C_{io}'[C_{io}\Sigma_s(t_j)C_{io}' + C_{ii}W_iC_{ii}' + V_{ij}]^{-1} \; ; \; i=1,2 \; ; \; j\in\theta \tag{4.53}$$

where W_i satisfies (4.37).

$\Sigma_s(t_j^-) = E[(z_{os}(t_j) - \eta_s(t_j^-))(z_{os}(t_j) - \eta_s(t_j^-))']$ and $\Sigma_s(t)$ satisfies

$$\left.\begin{aligned} &\dot{\Sigma}_s = A_{oo}\Sigma_s + \Sigma_s A_{oo}' + \sum_{i=1}^{2} G_{oi}G_{oi}' \; ; \; \Sigma_s(0) = N_{oo} \\ &\qquad t\in[t_{j-1},t_j) \; ; \; j=1,\ldots,N \\ &\Sigma_s(t_j) = \Sigma_s(t_j^-) - M_s(j)C_o\Sigma_s(t_j^-) \end{aligned}\right\} \tag{4.54}$$

and

$$M_s(j) = \Sigma_s(t_j^-)C_o'[C_o\Sigma_s(t_j^-)C_o' + \sum_{i=1}^{2} \bar{C}_{ii}W_i\bar{C}_{ii}' + V_j]^{-1} \tag{4.55a}$$

$$y_s(j) = [y_{1s}'(j) \quad y_{2s}'(j)]' \tag{4.55b}$$

$$C_o = [C_{1o}' \quad C_{2o}']' \tag{4.55c}$$

$$\bar{C}_{11} = [C_{11}' \quad 0]' \tag{4.55d}$$

$$\bar{C}_{22} = [0 \quad C_{22}']' \quad . \tag{4.55e}$$

<u>The fast subproblem</u> for decision maker - i is defined by the system equations (4.28), (4.30), the observations

$$\begin{aligned} y_{if}(j) &= C_{ii}z_{if}(t_j) + v_i(j) \\ &\equiv y_i(j) - C_{io}z_{os}(t_j) - C_{ii}z_{is}(t_j) \; ; \; j\in\theta \end{aligned} \tag{4.56}$$

and the cost function

$$J_{if} = E\{\varepsilon_i z_{if}'(T)\Gamma_i z_{if}(T) + \int_0^T (z_{if}'Q_i z_{if} + u_{if}'u_{if})dt\} \quad . \tag{4.57}$$

Notice that we have studied this stochastic control problem earlier in Section 2. Its solution, as $\varepsilon_i \to 0$, is given by

$$u^*_{if} = - B'_{ii} K_{if} \psi_{if}(t,t_j) \hat{z}_{ij}(t_j) \; ; \; t \in [t_j, t_{j+1}) \; ; \; j \in \theta \tag{4.58}$$

where K_{if} satisfies the Riccati equation

$$A'_{ii} K_{if} + K_{if} A_{ii} + Q_i - K_{if} B_{ii} B'_{ii} K_{if} = 0 \tag{4.59}$$

$\psi_{if}(t,t_j)$ is the state transition matrix satisfying

$$\varepsilon_i \dot{\psi}_{if}(t,t_j) = (A_{ii} - B_{ii} B'_{ii} K_{if}) \psi_{if}(t,t_j) \; ; \; \psi_{if}(t_j,t_j) = I$$

$$t \in [t_j, t_{j+1}) \; ; \; j \in \theta \quad . \tag{4.60}$$

$\hat{z}_{if}$ is the output of the filter

$$\left.\begin{aligned} &\varepsilon_i \dot{\hat{z}}_{if} = A_{ii} \hat{z}_{if} + B_{ii} u^*_{if} \; ; \; t \in [t_{j-1}, t_j) \; ; \; j=1,2,\ldots,N \\ &\hat{z}_{if}(0) = \bar{z}_{io} \\ &\hat{z}_{if}(t_j) = \hat{z}_{if}(t_j^-) + M_{if}(j)[y_{if}(j) - C_{ii} \hat{z}_{if}(t_j^-)] \end{aligned}\right\} \tag{4.61}$$

and

$$M_{if}(j) = W_i C'_{ii} [C_{io} \Sigma_s(t_j^-) C'_{io} + C_{ii} W_i C'_{ii} + V_{ij}]^{-1} \quad . \tag{4.62}$$

The following proposition establishes the near-optimality of the multimodel solution. Its proof may be found in [19].

<u>Proposition 4.2</u>:

i) $\nu^*_i(t,\alpha_i) = \nu^*_{is}(t,\alpha_i) + u^*_{if}(t) + O(\|\varepsilon\|) \; ; \; \forall t \in (0,T) \; ; \; i=1,2$

ii) $J(\nu^*_1, \nu^*_2) = J_s(\nu^*_{1s}, \nu^*_{2s}) + \sum_{i=1}^{2} [J_{if}(u^*_{if}) + T\,\mathrm{tr}(Q_i W_i)] + O(\|\varepsilon\|)$

□

4.2.2. Case 2: Sampled values of continuous noisy measurements

At sampled time instant t_j, $j \in \theta - \{0\}$, the decision makers observe

$$y_i(j) = \int_0^{t_j} [C_{io} z(\tau) + C_{ii} z_i(\tau)] d\tau + q_i(t_j)$$

$$= \int_0^{t_j} C_i x(\tau) d\tau + q_i(t_j) \; ; \; i=1,2 \quad . \tag{4.63}$$

Note that in the time interval $[t_o, t_1)$ no observations are made and the decision makers have access only to the prior statistics of the random quanities involved. Here, $\{q_i(t) \; ; \; i=1,2\}$ are standard Wiener processes independent of each other. Furthermore, their statistics are also assumed to be independent of the Wiener processes $\{v_i(t) \; ; \; i=1,2\}$ and the initial state vector $x(o)$.

Let

$$\bar{y}_i(j) = y_i(j) - y_i(j-1)$$

$$= \int_{t_{j-1}}^{t_j} C_i x(\tau) d\tau + v_i(j) \; ; \; i=1,2 \tag{4.64}$$

where $v_i(j) = q_i(t_j) - q_i(t_{j-1})$ is a discrete-time Gaussian white noise process with zero mean and variance $V_{ij} = (t_j - t_{j-1})I$.

Let $\bar{\alpha}_i^j$ be given by (4.38) with $y_i(j)$ replaced by $\bar{y}_i(j)$, and let $\bar{\sigma}_i^j$ denote the sigma-algebra generated by $\bar{\alpha}_i^j$. Then clearly, σ_i^j and $\bar{\sigma}_i^j$ are equivalent.

The optimal team solution to the problem defined by (4.4), (4.6), (4.38)-(4.40) and (4.64) can be obtained in a manner analogous to Case 1, and is given by [20]

$$\nu_i^*(t,\bar{\alpha}_i) = P_i(t)[\bar{y}_i(j) - \int_{t_{j-1}}^{t_j} C_i\eta(\tau)d\tau] - B_i'S(t)\psi(t,t_j)\hat{\xi}(j) \;;\; i=1,2$$

$$t\in[t_j,t_{j+1}) \;;\; j\in\theta \tag{4.65a}$$

where $P_1(t)$, $P_2(t)$ satisfy the coupled set of linear integral equations

$$P_i(t) = B_i'S_i(t) \int_{t_j}^{t} \psi_{ij}(t,\tau)B_iB_i'L_{ij}(\tau)d\tau - B_i'L_{ij}(t) \;;\; i=1,2$$

$$t\in[t_j,t_{j+1}) \;;\; j\in\theta \tag{4.65b}$$

where

$$L_{ij}(t) = S_i(t)\phi(t,t_j)\Sigma_i(j) + S_i(t) \int_{t_j}^{t} \phi(t,\tau)B_kP_k(\tau)d\tau \;\Delta_i(j) + K_{ij}(t) \;;$$

$$i,k=1,2 \;;\; i\neq k \;;\; t\in[t_j,t_{j+1}) \;;\; j\in\theta \tag{4.65c}$$

$$\dot{K}_{ij}(t) = -\,(A - B_iB_i'S_i(t))'K_{ij}(t) - S_i(t)B_kP_k(t)\Delta_i(j) \;;\; K_{ij}(t_{j+1}) = 0 \,,$$

$$i,k=1,2 \;;\; i\neq k \;;\; t\in(t_j,t_{j+1}] \;;\; j\in\theta \;. \tag{4.65d}$$

$S(t)$ and $S_i(t)$ satisfy the Riccati equations (4.42e) and (4.42f), respectively. The state transition matrices $\psi(t,\tau)$, $\psi_{ij}(t,\tau)$ and $\phi(t,\tau)$ satisfy the equations (4.43).

$\hat{\xi}(j) = \eta(t_j^-) = E[x(t_j)|\bar{\delta}_{j-1}]$ and $\eta(t)$ satisfies

$$\left.\begin{aligned} &\dot{\eta} = A\eta + \sum_{i=1}^{2} B_i\nu_i^*(t,\bar{\alpha}_i) \;;\; \eta(0) = \bar{x}_o \\ &\qquad t\in[t_{j-1},t_j) \;;\; j=1,\ldots,N \\ &\eta(t_j) = \eta(t_j^-) + M(j)[\bar{y}(j) - \int_{t_{j-1}}^{t_j} C\eta(\tau)d\tau] \end{aligned}\right\} \tag{4.66}$$

$\Sigma_i(j)$ and $\Delta_i(j)$ are appropriate dimensional matrices defined by

$$\Sigma_i(j) = [\phi(t_j,t_{j-1})\Sigma(t_{j-1}) \int_{t_{j-1}}^{t_j} \phi'(t,t_{j-1})C_i'dt + \int_{t_{j-1}}^{t_j} \phi(t_j,r)GG' \int_r^{t_j} \phi'(\tau,r) \cdot C_i'd\tau dr]\hat{V}_{ij}^{-1} \; ; \; i=1,2 \; ; \; j\in\theta \tag{4.67a}$$

$$\Delta_i(j) = [\int_{t_{j-1}}^{t_j} C_k\phi(t,t_{j-1})dt\Sigma(t_{j-1}) \int_{t_{j-1}}^{t_j} \phi'(t,t_{j-1})C_i'dt + \int_{t_{j-1}}^{t_j} C_k\phi(t_j,r)GG' \int_r^{t_j} \phi'(\tau,r)C_i'd\tau dr]\hat{V}_{ij}^{-1} \; ; \quad i,k=1,2 \; ; \; i\neq k \; ; \; j\in\theta \tag{4.67b}$$

where

$$\hat{V}_{ij} = \int_{t_{j-1}}^{t_j} C_i\phi(t,t_{j-1})dt\Sigma(t_{j-1}) \int_{t_{j-1}}^{t_j} \phi'(t,t_{j-1})C_i'dt + V_{ij} + \int_{t_{j-1}}^{t_j} C_i \int_{t_{j-1}}^{r} \phi(\tau,r)GG' \int_\tau^{t_j} \phi'(\ell,\tau)C_i'd\ell d\tau dr \; ; \; i=1,2 \; ; \; j\in\theta \tag{4.67c}$$

$\Sigma(t_j^-) = E[(x(t_j) - \eta(t_j^-))(x(t_j) - \eta(t_j^-))']$ and $\Sigma(t)$ satisfies

$$\left.\begin{aligned}
&\dot{\Sigma} = A\Sigma + \Sigma A' + GG' \; ; \; \Sigma(0) = N \\
&\qquad t\in[t_{j-1},t_j) \; ; \; j=1,\ldots,N \\
&\Sigma(t_j) = \Sigma(t_j^-) - M(j)[\int_{t_{j-1}}^{t_j} C\phi(r,t_{j-1})dr\Sigma(t_{j-1})\phi'(t_j,t_{j-1}) \\
&\qquad + \int_{t_{j-1}}^{t_j} \int_r^{t_j} C\phi(r,\tau)GG'\phi'(t_j,\tau)d\tau dr]
\end{aligned}\right\} \tag{4.68}$$

M(j) is given by

$$M(j) = [\phi(t_j,t_{j-1})\Sigma(t_{j-1})\int_{t_{j-1}}^{t_j} \phi'(r,t_{j-1})C'dr + \int_{t_{j-1}}^{t_j} \phi(t_j,\tau)GG'\int_{\tau}^{t_j} \phi'(r,\tau)C'drd\tau]\hat{V}_j^{-1},$$

$$j\in\theta \tag{4.69a}$$

where

$$\hat{V}_j = \int_{t_{j-1}}^{t_j} C\phi(\tau,t_{j-1})d\tau\Sigma(t_{j-1})\int_{t_{j-1}}^{t_j} \phi'(r,t_{j-1})C'dr + V_j$$

$$+ \int_{t_{j-1}}^{t_j} C \int_{t_{j-1}}^{r} \phi(\tau,r)GG'\int_{\tau}^{t_j}\phi'(\ell,\tau)C'd\ell d\tau dr \ ; \ j\in\theta \tag{4.69b}$$

$$V_j = \mathrm{diag}(V_{1j},V_{2j}) \tag{4.70a}$$

$$\bar{y}(j) = [\bar{y}_1'(j) \quad \bar{y}_2'(j)]' \tag{4.70b}$$

$$C = [C_1' \quad C_2']' \quad . \tag{4.70c}$$

As in Case 1, the optimal team strategies are unique and linear in the information available to the decision makers, but the expressions involved are more complicated. Hence, the computational problem worsens, making the need for suboptimal solutions more acute. Again the appealing structure of the multimodel solution makes it an attractive alternative.

As in earlier problems, the multimodel solution is obtained on solving a lower order team problem in the slow time scale and two low order decentralized control problems in the fast time scale. The system model for <u>the slow subproblem</u> is given by (4.17), (4.19), the cost function by (4.49), and the observations by

$$\bar{y}_{is}(j) = \int_{t_{j-1}}^{t_j} C_{io} z_{os}(\tau) d\tau + v_i(j)$$

$$\equiv \bar{y}_i(j) - \int_{t_{j-1}}^{t_j} C_{ii} z_{is}(\tau) d\tau \quad ; \quad i=1,2 \; ; \; j \in \theta - \{o\} \tag{4.71}$$

The optimal team solution to this slow subproblem is given by

$$\nu^*_{is}(t,\bar{\alpha}_i) = P_{is}(t)[\bar{y}_{is}(j) - \int_{t_{j-1}}^{t_j} C_{io} \eta_s(\tau) d\tau] - R^{-1}_{is} B'_{oi} S_s \psi_s(t,t_j) \hat{\xi}_s(j) \; ;$$

$$i=1,2 \; ; \; t \in [t_j, t_{j+1}) \; ; \; j \in \theta \tag{4.72a}$$

where $P_{1s}(t)$, $P_{2s}(t)$ satisfy the coupled set of linear integral equations

$$P_{is}(t) = R^{-1}_{is} B'_{oi} S_{is}(t) \int_{t_j}^{t} \psi_{ijs}(t,\tau) B_{oi} R^{-1}_{is} B'_{oi} L_{ijs}(\tau) d\tau - R^{-1}_{is} B'_{oi} L_{ijs}(t) \; ;$$

$$i=1,2 \; ; \; t \in [t_j, t_{j+1}) \; ; \; j \in \theta \tag{4.72b}$$

where

$$L_{ijs}(t) = S_{is}(t) \phi_s(t,t_j) \Sigma_{is}(j) + S_{is}(t) \int_{t_j}^{t} \phi_s(t,\tau) B_{ok} R^{-1}_{ks} P_{ks}(\tau) d\tau \Delta_{is}(j) + K_{ijs}(t)$$

$$i,k=1,2 \; ; \; i \neq k \; ; \; t \in [t_j, t_{j+1}) \; ; \; j \in \theta \tag{4.72c}$$

$$\dot{K}_{ijs}(t) = -(A_{oo} - B_{oi} R^{-1}_{is} B'_{oi} S_{is}(t))' K_{ijs}(t) - S_{is}(t) B_{ok} R^{-1}_{ks} P_{ks}(t) \Delta_{is}(j) \; ;$$

$$K_{ijs}(t_{j+1}) = 0 \; ; \; i,k=1,2 \; ; \; i \neq k \; ; \; t \in (t_j, t_{j+1}] \; ; \; j \in \theta \tag{4.72d}$$

$S_s(t)$ and $S_{is}(t)$ satisfy the Riccati equations (4.50e) and (4.50f), respectively. The state transition matrices $\psi_s(t,\tau)$, $\psi_{ijs}(t,\tau)$ and $\phi_s(t,\tau)$ satisfy the equations (4.51). Furthermore,

$\hat{\xi}_s(j) = \eta_s(t_j^-) = E[z_{os}(t_j)|\bar{\delta}_{j-1}]$ and $\eta_s(t)$ satisfies

$$\left.\begin{aligned}
&\dot{\eta}_s = A_{oo}\eta_s + \sum_{i=1}^{2} B_{oi}\nu_{is}^*(t,\bar{\alpha}_i) \; ; \; \eta_s(0) = \bar{z}_{oo} \\
&\qquad t\in[t_{j-1},t_j) \; ; \; j=1,\ldots,N \\
&\eta_s(t_j) = \eta_s(t_j^-) + M_s(j)[\bar{y}_s(j) - \int_{t_{j-1}}^{t_j} C_o\eta_s(\tau)d\tau]
\end{aligned}\right\} \tag{4.73}$$

$\Sigma_{is}(j)$ and $\Delta_{is}(j)$ are appropriate dimensional matrices defined by

$$\Sigma_{is}(j) = [\phi_s(t_j,t_{j-1})\Sigma_s(t_{j-1})\int_{t_{j-1}}^{t_j} \phi_s'(t,t_{j-1})C_{io}'dt + \int_{t_{j-1}}^{t_j} \phi_s(t_j,r) \sum_{i=1}^{2} G_{oi}G_{oi}'$$

$$\int_r^{t_j} \phi_s'(\tau,r)C_{io}'d\tau dr]\bar{V}_{ij}^{-1} \qquad i=1,2 \; ; \; j\in\theta \tag{4.74a}$$

$$\Delta_{is}(j) = [\int_{t_{j-1}}^{t_j} C_{k_o}\phi_s(t,t_{j-1})dt\Sigma_s(t_{j-1})\int_{t_{j-1}}^{t_j} \phi'(t,t_{j-1})C_{io}'dt$$

$$+ \int_{t_{j-1}}^{t_j} C_{k_o}\phi_s(t_j,r) \sum_{i=1}^{2} G_{oi}G_{oi}'\int_r^{t_j}\phi_s'(\tau,r)C_{io}'d\tau dr]\bar{V}_{ij}^{-1} \; ;$$

$$i,k=1,2 \; ; \; i\neq k \; ; \; j\in\theta \tag{4.74b}$$

where

$$\bar{V}_{ij} = V_{ij} + \int_{t_{j-1}}^{t_j} C_{io}\phi_s(t,t_{j-1})dt\Sigma_s(t_{j-1})\int_{t_{j-1}}^{t_j} \phi_s'(t,t_{j-1})C_{io}'dt$$

$$+ \int_{t_{j-1}}^{t_j} C_{ii}\phi_{if}(t,t_{j-1})dtW_i\int_{t_{j-1}}^{t_j} \phi_{if}'(t,t_{j-1})C_{ii}'dt$$

$$+\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{r} C_{io}\phi_s(\tau,r)\sum_{i=1}^{2} G_{io}G_{io}'\int_{\tau}^{t_j}\phi_s'(\ell,\tau)C_{io}'\,d\ell d\tau dr$$

$$+\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{r} C_{ii}\phi_{if}(\tau,r)G_{ii}G_{ii}'\int_{\tau}^{t_j}\phi_{if}'(\ell,\tau)C_{ii}'\,d\ell d\tau dr \; ; \; i=1,2 \; ; \; j\in\theta \tag{4.74c}$$

$\phi_{if}(t,t_j)$ is the state transition matrix satisfying

$$\varepsilon_i\dot{\phi}_{if}(t,t_j) = A_{ii}\phi_{if}(t,t_j) \; ; \; \phi_{if}(t_j,t_j) = I$$

$$t\in[t_j,t_{j+1}) \; ; \; i=1,2 \; ; \; j\in\theta \quad . \tag{4.74d}$$

and W_i satisfies (4.37). Now,

$$E[(z_{os}(t_j) - \eta_s(t_j^-))(z_{os}(t_j) - \eta_s(t_j^-))'] = \Sigma_s(t_j^-)$$

where $\Sigma_s(t)$ satisfies

$$\left.\begin{aligned} &\dot{\Sigma}_s = A_{oo}\Sigma_s + \Sigma_s A_{oo}' + \sum_{i=1}^{2} G_{oi}G_{oi}' \; ; \; \Sigma_s(0) = N_{oo} \\ &\qquad t\in[t_{j-1},t_j) \; ; \; j=1,\ldots,N \\ &\Sigma_s(t_j) = \Sigma_s(t_j^-) - M_s(j)\Big[\int_{t_{j-1}}^{t_j} C_o\phi_s(r,t_{j-1})dr\Sigma_s(t_{j-1})\phi_s'(t_j,t_{j-1}) \\ &\qquad + \int_{t_{j-1}}^{t_j}\int_{r}^{t_j} C_o\phi_s(r,\tau)\sum_{i=1}^{2} G_{oi}G_{oi}'\phi_s'(t_j,\tau)d\tau dr\Big] \end{aligned}\right\} \tag{4.75}$$

$M_s(j)$ is given by

$$M_s(j) = [\phi_s(t_j,t_{j-1})\Sigma_s(t_{j-1})\int_{t_{j-1}}^{t_j} \phi_s'(r,t_{j-1})C_o'dr$$

$$+ \int_{t_{j-1}}^{t_j} \phi_s(t_j,\tau) \sum_{i=1}^{2} G_{oi}G_{oi}'\int_{\tau}^{t_j}\phi_s'(r,\tau)C_o'drd\tau]\bar{V}_j^{-1} \;;$$

$$j\in\theta \tag{4.76a}$$

where

$$\bar{V}_j = V_j + \int_{t_{j-1}}^{t_j} C_o\phi_s(t,t_{j-1})dt\Sigma_s(t_{j-1})\int_{t_{j-1}}^{t_j} \phi_s'(t,t_{j-1})C_o'dt$$

$$+ \int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{r} C_o\phi_s(\tau,r) \sum_{i=1}^{2} G_{io}G_{io}'\int_{\tau}^{t_j}\phi_s'(\ell,\tau)C_o'd\ell d\tau dr$$

$$+ \sum_{i=1}^{2} [\int_{t_{j-1}}^{t_j} \bar{C}_{ii}\phi_{if}(t,t_{j-1})dtW_i\int_{t_{j-1}}^{t_j} \phi_{if}'(t,t_{j-1})\bar{C}_{ii}'dt$$

$$+ \int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{r} \bar{C}_{ii}\phi_{if}(\tau,r)G_{ii}G_{ii}'\int_{\tau}^{t_j}\phi_{if}'(\ell,\tau)\bar{C}_{ii}'d\ell d\tau dr] \;;\; j\in\theta \tag{4.76b}$$

V_j is defined by (4.70a); C_o, $\bar{C}_{11}$, $\bar{C}_{22}$ are defined by (4.55c-e) and

$$\bar{y}_s(j) = [\bar{y}_{1s}'(j) \quad \bar{y}_{2s}'(j)]' \quad . \tag{4.77}$$

The fast subproblem for decision maker - i is defined by the system equations (4.28), (4.30), the cost function (4.57) and the observations

$$\bar{y}_{if}(j) = \int_{t_{j-1}}^{t_j} C_{ii} z_{if}(\tau) d\tau + v_i(j)$$

$$\equiv \bar{y}_i(j) - \int_{t_{j-1}}^{t_j} [C_{io} z_{os}(\tau) + C_{ii} z_{is}(\tau)] d\tau \ ; \ j \in \theta \quad . \tag{4.78}$$

This control problem has been studied earlier in Section 2. Its solution, as $\varepsilon_i \to 0$, is given by

$$u^*_{if} = - B'_{ii} K_{if} \psi_{if}(t,t_j) z_{if}(t_j) \ ; \ t \in [t_j, t_{j+1}) \ ; \ j \in \theta \tag{4.79}$$

where K_{if} satisfies the Riccati equation (4.59) and $\psi_{if}(t,t_j)$ satisfies (4.60).

$\hat{z}_{if}$ is the output of the filter

$$\left.\begin{aligned} &\varepsilon_i \dot{\hat{z}}_{if} = A_{ii} \hat{z}_{if} + B_{ii} u^*_{if} \ ; \ t \in [t_{j-1}, t_j) \ ; \ j = 1, \ldots, N \\ &\hat{z}_{if}(0) = \bar{z}_{io} \\ &\hat{z}_{if}(t_j) = \hat{z}_{if}(t_j^-) + M_{if}(j) [\bar{y}_{if}(j) - \int_{t_{j-1}}^{t_j} C_{ii} \hat{z}_{if}(\tau) d\tau] \end{aligned}\right\} \tag{4.80}$$

and

$$M_{if}(j) = [\phi_{if}(t_j, t_{j-1}) W_i \int_{t_{j-1}}^{t_j} \phi'_{if}(r, t_{j-1}) C'_{ii} dr$$

$$+ \int_{t_{j-1}}^{t_j} \phi_{if}(t_j, \tau) G_{ii} G'_{ii} \int_{\tau}^{t_j} \phi'_{if}(r, \tau) C'_{ii} dr d\tau] \bar{V}_{ij}^{-1} \tag{4.81}$$

A near-optimality result, analogous to Proposition 4.2, can be established in this case also by following the same lines:

problems. Since each decision maker need not know the parameters associated with the fast subproblem of other decision makers, the multimodel solution is robust with respect to modeling errors; a very desirable feature in large scale system design.

Our results serve to demonstrate the richness in the modeling structure with multiparameter singular perturbations in the context of multimodeling problems. In each case, the limit of seemingly complex integro-differential equations associated with the optimal solution has a nice appealing structure when interpreted as a multimodel solution. Thus the multimodeling approach using singular perturbations is in some sense 'robust' with respect to a class of solution concepts and information patterns.

6. REFERENCES

1. N. R. Sandell, Jr., P. Varaiya, M. Athans and M. G. Safanov, "Survey of decentralized control methods for large scale systems," IEEE Trans. on Automatic Control, AC-23, 1978, pp. 108-128.

2. H. K. Khalil and P. V. Kokotovic, "Control strategies for decision makers using different models of the same system," IEEE Trans. on Automatic Control, AC-23, 1978, pp. 289-298.

3. J. R. Winkelman, J. H. Chow, B. C. Bowler, B. Avramovic and P. V. Kokotovic, "An analysis of interarea dynamics of multimachine systems," IEEE Trans. on Power App. and Systems, vol. 100, 1981, pp. 754-763.

4. P. V. Kokotovic, R. E. O'Malley, Jr. and P. Sannuti, "Singular perturbations and order reduction in control theory - an overview," Automatica, vol. 12, 1976, pp. 123-132.

5. A. H. Haddad, "Linear filtering of singularly perturbed systems," IEEE Trans. on Automatic Control, AC-31, 1976, pp. 515-519.

6. A. H. Haddad and P. V. Kokotovic, "Stochastic control of linear singularly perturbed systems," IEEE Trans. on Automatic Control, AC-22, 1977, pp. 815-821.

7. V. R. Saksena and J. B. Cruz, Jr., "A multimodel approach to stochastic Nash games," Automatica, vol. 18, 1982, pp. 295-305.

8. R. P. Singh, "The linear-quadratic-Gaussian problem for singularly perturbed systems," Int. J. System Science, vol. 13, 1982, pp. 93-100.

9. H. K. Khalil, A. H. Haddad and G. Blankenship, "Parameter scaling and well-posedness of stochastic singularly perturbed control systems," Proc. 12th Asilomar Conference on Circuits, Systems and Computers, 1978,

10. P. V. Kokotovic, "A Riccati equation for block-diagonalization of ill-conditioned systems," IEEE Trans. on Automatic Control, AC-20, 1975, pp. 812-814.

11. P. V. Kokotovic, "Subsystems, time-scales and multimodeling," Automatica, vol. 17, 1981, pp. 797-804.

12. P. V. Kokotovic, B. Avramovic, J. H. Chow and J. R. Winkelman, "Coherency-based decomposition and aggregation," Automatica, vol. 17, 1982, pp. 47-56.

13. G. Peponides, P. V. Kokotovic and J. H. Chow, "Singular perturbations and time scales in nonlinear models of power systems," IEEE Trans. on Circuits and Systems, CAS-29, 1982, pp. 758-767.

14. F. Delebecque and J. P. Quadrat, "Optimal control of Markov chains admitting strong and weak interactions," Automatica, vol. 17, 1981, pp. 281-296.

15. R. G. Phillips and P. V. Kokotovic, "A singular perturbation approach to modeling and control of Markov chains," IEEE Trans. on Automatic Control, AC-26, 1981, pp. 1087-1094.

16. I. B. Rhodes and D. G. Luenberger, "Stochastic differential games with constrained state estimators," IEEE Trans. on Automatic Control, AC-14, 1969, pp. 476-481.

17. A. A. Feldbaum, Optimal Control Systems, Academic Press, New York, 1965.

18. A. Bagchi and T. Başar, "Team decision theory for linear continuous-time systems," IEEE Trans. on Automatic Control, AC-25, 1980, pp. 1154-1161.

19. V. R. Saksena and T. Başar, "A multimodel approach to stochastic team problems," Automatica, vol. 18, 1982, pp. 713-720.

20. T. Başar and D. H. Cansever, "Team-optimal strategies for LQG continuous-time systems with sampled continuous measurements," Internal Report, Marmara Research Institute, Gebze, Kocaeli, Turkey, 1980.